



THE DECOMPOSITION THEORY OF LTI SYSTEMS

Miroslav Svátek*

Abstract: The paper presents new methodology how to decompose the high dimensional LTI (linear time invariant) system with both distinct and repeated eigenvalues of the transition matrix into a set of first-order LTI models, which could be combined to achieve approximation of the original dynamics. As a tool, the Sylvester's theorems are used to design the filter bank and parameters of the first-order models (transition values). At the end, the practical examples are shown and the next steps of research of the decomposition theory are indicated.

Key words: *Linear dynamical systems, neural networks, system decomposition, system reconstruction, digital filtering, control theory, Sylvester's theorems, state-space representation, linear time invariant systems.*

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1. Introduction

The paper combines approaches known in the signal processing area with approaches typical for neural networks. In the signal processing community, signal parameters as mean values, correlation matrix, covariance matrix, regression parameters vector, etc. are estimated through measured data. The knowledge of such parameters results in the identification of the optimal LTI (linear time invariant) model describing the main system property [1]. The model with estimated parameters could be used for time interpolation, filtering, extrapolation, etc. This approach is very good for identification of unknown system but in practice the high dimensional LTI model directly yields to the curse of dimensionality [2]. This problem is known, e.g., in control theory, decision-making, etc. [6]. On the other hand, the main idea of the artificial neural network is based on complex combining (mixture) of relatively simple dynamical components¹ (neurons) that could be easily

*Miroslav Svátek

Czech Technical University in Prague, Faculty of Transportation Sciences, Department of Control Engineering and Telematics, Konviktská 20, 110 00 Prague, E-mail: svitek@lss.fd.cvut.cz

¹This argument means that the output of each neuron in time interval n could be described by a simple function dependent only on linear combination of data vector measured in time interval $n-1$. The typical m -dimensional LTI model in time interval n combines historical data measured in time intervals $n-1, n-2, \dots, n-m$.

optimized². The appropriate mixture (synaptic weights) of dynamical components is often estimated through some computing procedures, e.g. back-propagation, etc. [3].

The presented theory of the LTI system decomposition supposes the model parameters (state-space matrices) to be identified by classical methods and the obtained high dimensional state-space model is decomposed into a set of first-order models by using Sylvester's theorem [7]. The component matrices assigned to transition matrix play the role of filter banks. The designed filter bank could be applied on the original input, output or state series in such a way that the filtered data series could be modeled by first-order LTI models. By using sufficiently enough one-dimensional models, the original system could be approximated with the predefined approximation error.

The presented method has similarities with quantum physics, where the trajectory could be modeled by easily measured state vector but with the complex transition matrix (Heisenberg representation) or, in the contrary, by complex state vector with simple transition matrix (Dirac representation). The main goal is to find the complex state representation of the LTI system (e.g. more dimensional state vector) with easy (diagonal) transition matrix. It concerns transformation from time evolution complexity into complexity of state vector.

In Chapter 2 the mathematical origin of the LTI system description, Sylvester's theorems, derivative approximation and approximation of Sylvester's theorems for repeated eigenvalues is introduced. In Chapter 3 the methodology of LTI system decomposition for distinct and repeated eigenvalues is presented together with the performance assessment. In chapter 4 the practical experiments demonstrating all outlined theories (distinct and repeated eigenvalues, Sylvester's theorem approximation, and methods of transformed or modified distinct eigenvalues) are shown and Chapter 5 summarizes the conclusions and shows the ways for next research.

2. Mathematical origin

2.1 Linear time invariant system (LTI) description

In general, the well known m -dimensional linear time invariant system (LTI) [1] could be described by the state-space model:

$$\begin{aligned}x_{n+1} &= \mathbf{A} \cdot x_n + \mathbf{B} \cdot u_n \\ y_n &= \mathbf{C} \cdot x_n + \mathbf{D} \cdot u_n\end{aligned}\quad (1)$$

where x_n, u_n, y_n are m -dimensional state, input and output vectors in time interval n and $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are state-space $m \times m$ matrices (for non-square matrices $\mathbf{B}, \mathbf{C}, \mathbf{D}$, the zero elements could be completed also in vectors u_n, y_n to achieve the general form (1)). Matrix \mathbf{A} is called a **system transition matrix**.

Assume that the process dynamics is described by an m -th order input-output ARMA model [1] with known noise variance σ_e^2 (ARMA model is an example of

²It is assumed that there exists a methodology how to combine the optimized models of neurons to achieve the optimization of their mixture. This problem is not solved in this article and the methodology is partly included in [8].

variety of applications of state-space representation (1)):

$$y_n + \sum_{i=1}^m a_i \cdot y_{n-i} = \sum_{i=1}^m b_i \cdot u_{n-i} + \varepsilon_n \quad (2)$$

where y_n, u_n are model output and input respectively. $\varepsilon_n \approx N(0, \sigma_\varepsilon^2)$ is normal white noise independent of output $y_{n-i}, i \geq 1$, and input $u_{n-i}, i \geq 1$ sequences and

$$\begin{aligned} \mathbf{a} &= [a_1, \dots, a_m] \\ \mathbf{b} &= [b_0, \dots, b_m] \end{aligned} \quad (3)$$

are vectors of parameters³. The observer canonical form of ARMA model is described by state matrices:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_1 & -a_1 & \dots & \dots & -a_m \end{pmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad (4)$$

$$\mathbf{C} = [b_m - b_0 \cdot a_m, b_{m-1} - b_0 \cdot a_{m-1}, \dots, b_1 - b_0 \cdot a_1], \quad D = b_0$$

For the equivalent system with zero noise signal ε_n the transfer function can be written (other representation of LTI systems)

$$G(e^{j\omega}) = \mathbf{C} \cdot (z\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + d|_{z=e^{j\omega}} \quad (5)$$

where ω represents the frequency space.

The impulse response of the ARMA model can be also determined (other representation of LTI systems)

$$g = [d, \mathbf{CB}, \mathbf{CAB}, \mathbf{CA}^2\mathbf{B}, \dots] \quad (6)$$

2.2 Sylvester's theorems

The decomposition methodology described in the paper is based on the application of Sylvester's theorems (both for distinct and repeated eigenvalues) on system transition matrix \mathbf{A} . Both Sylvester's theorems are presented in this chapter.

Theorem 1: Sylvester theorem for distinct eigenvalues

If \mathbf{A} is square system transition matrix (1) and if λ_i represents one of the n distinct eigenvalues of \mathbf{A} , and if $P(\mathbf{A})$ is any polynomial of matrix \mathbf{A} , then

$$P(\mathbf{A}) = \sum_{i=1}^n \frac{P(\lambda_i) \cdot \text{Adj}(\mathbf{A} - \lambda_i \cdot \mathbf{I})}{\prod_{j \neq i} (\lambda_j - \lambda_i)} = \sum_{i=1}^n P(\lambda_i) \cdot \prod_{j \neq i} \frac{(\mathbf{A} - \lambda_j \cdot \mathbf{I})}{(\lambda_i - \lambda_j)} \quad (7)$$

where $\text{Adj}(\mathbf{A})$ means the adjoint matrix that is formed by replacing each element of matrix \mathbf{A} by its cofactor and then taking the transpose.

³In the ARMA model definition equation $a_0 = 1$ is supposed.

Theorem 2: Sylvester theorem for repeated eigenvalues

If \mathbf{A} is square system transition matrix (1) and if λ_i represents an eigenvalue of \mathbf{A} repeated s_i times, and if k is the number of all distinct eigenvalues λ_j , and if $P(\mathbf{A})$ is any polynomial of matrix \mathbf{A} , then

$$P(\mathbf{A}) = \sum_{\substack{i=1 \\ \text{(all distinct} \\ \text{eigenvalues)}}}^k \frac{(-1)^{s_i-1}}{(s_i-1)!} \left[\frac{d^{s_i-1}}{d\lambda^{s_i-1}} \left(\frac{P(\lambda) \cdot \text{Adj}(\mathbf{A} - \lambda \cdot \mathbf{I})}{\prod_{j \neq i}^k (\lambda - \lambda_j)} \right) \right]_{\lambda=\lambda_i} \quad (8)$$

where $\text{Adj}(\mathbf{A})$ means the adjoint matrix and if all eigenvalues are equal, then

$$\prod_{j \neq i}^k (\lambda - \lambda_j) = 1.$$

Definition 1:

With respect to equation (7) and with assumption of distinct eigenvalues, the special matrixes Z_1, Z_2, \dots, Z_n could be defined for matrix \mathbf{A} (in [6, 7] called **component matrices** of matrix \mathbf{A}):

$$Z_i = \prod_{j \neq i}^n \frac{(\mathbf{A} - \lambda_j \cdot \mathbf{I})}{(\lambda_i - \lambda_j)} \quad (9)$$

with the following properties:

$$\begin{aligned} Z_i \cdot Z_j &= 0 \text{ for } i \neq j, \\ Z_i \cdot Z_j &= Z_i \text{ for } i = j, \\ \sum_{i=1}^n Z_i &= 1. \end{aligned} \quad (10)$$

Proof: The proof of theorem 1 and 2 is made in [5] where the proof of matrix components for repeated eigenvalues is also presented.

2.3 Approximation of derivatives

In equation (8) the derivatives are necessary to be solved. In the following theorem the form of derivatives approximation is presented together with the approximation error.

Theorem 3: Approximation of derivatives

Let $f(x)$ be any function of x and $\frac{df(x)}{dx}, \frac{d^2f(x)}{dx^2}, \frac{d^3f(x)}{dx^3}, \dots$ first, second, third, etc. derivatives of function $f(x)$, and the approximate derivatives could be expressed

$$\begin{aligned}\frac{df(x)}{dx} &\approx \frac{1}{2h} (f(x+h) - f(x-h)), \\ \frac{d^2f(x)}{dx^2} &\approx \frac{1}{2h^2} (f(x+h) - 2f(x) + f(x-h)), \\ \frac{d^3f(x)}{dx^3} &\approx \frac{1}{2h^3} (f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)), \\ &\text{etc.}\end{aligned}\tag{11}$$

where for all derivatives the approximation error is proportional to $O(h^2)$.

For better precision of approximation other approximate forms exist, e.g. for approximation error proportional to $O(h^4)$ the following equations can be described:

$$\begin{aligned}\frac{df(x)}{dx} &\approx \frac{1}{12h} (-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)), \\ \frac{d^2f(x)}{dx^2} &\approx \frac{1}{12h^2} (-f(x+2h) + 16f(x+h) - 30f(x) + 16f(x-h) - f(x-2h)), \\ &\text{etc.}\end{aligned}\tag{12}$$

2.4 Approximation of derivatives in the Sylvester theorem for repeated eigenvalues (transformed eigenvalues method)

The approximation of derivatives (11), (12) could be applied to the Sylvester theorem for repeated eigenvalues (8) and the following theorem could be defined.

Theorem 4 (Svítek's theorem I.): Approximation of $P(A)$ with repeated eigenvalues of matrix A

If matrix A has k , all distinct eigenvalues (8) where d eigenvalues λ_d are repeated s_d times and ch eigenvalues λ_{ch} are poorly distinct, then the polynomial function $P(A)$ can be approximated by a set of ch distinct and by a set of t **transformed distinct eigenvalues** λ_t ($\lambda_1 - q_1h, \dots, \lambda_1 + q_1h, \dots, \lambda_d - q_dh, \dots, \lambda_d + q_dh$) with error proportional at least to $O(h^2)$ as follows:

$$\begin{aligned}
P(\mathbf{A}) &= \sum_{i=1}^{ch} P(\lambda_i) \cdot \prod_{\mu \neq i}^k \frac{(\mathbf{A} - \lambda_\mu \cdot \mathbf{I})}{(\lambda_i - \lambda_\mu)} + \sum_{f=1}^d \sum_{\gamma=-q_f}^{q_f} k_{f,\gamma} \cdot \\
&\quad \cdot P(\lambda_f + \gamma \cdot h) \cdot \prod_{\mu \neq f}^k \frac{(\mathbf{A} - \lambda_\mu \cdot \mathbf{I})}{(\lambda_f + \gamma \cdot h - \lambda_\mu)} = \\
&= \sum_{i=1}^{ch} P(\lambda_i) \cdot Z_i + \sum_{f=1}^d \sum_{\gamma=-q_f}^{q_f} k_{f,\gamma} \cdot P(\lambda_f + \gamma \cdot h) \cdot Z_{f,\gamma}
\end{aligned} \tag{13}$$

where q_f depends on selected approximate form (11), (12), and on the multiplicity of repeated eigenvalue λ_f , $k_{f,\gamma}$ are weight constants of approximation form (11), (12), $Z_{f,\gamma} = \prod_{\mu \neq f}^k \frac{(\mathbf{A} - \lambda_\mu \cdot \mathbf{I})}{(\lambda_f + \gamma \cdot h - \lambda_\mu)}$ are transformed component matrices assigned to transformed distinct eigenvalues and h is a selected small approximation parameter.

Proof: If the Sylvester theorem for repeated eigenvalues (8) is used and if the approximation is applied according to equation (11) or (12), then the part of equation (8) assigned to repeated eigenvalue λ_i can be approximated with error proportional at least to $O(h^2)$ or more (depends on used approximate form, e.g. forms (11) or (12)) as follows:

$$\begin{aligned}
&\frac{(-1)}{(s_i - 1)!} \left[\frac{d^{s_i-1}}{d\lambda^{s_i-1}} \left(\frac{P(\lambda) \cdot \text{Adj}(\mathbf{A} - \lambda \cdot \mathbf{I})}{\prod_{j \neq i}^k (\lambda - \lambda_j)} \right) \right]_{\lambda=\lambda_i} = \\
&= \frac{(-1)}{(s_i - 1)!} \cdot \left[\frac{d^{s_i-1}}{d\lambda^{s_i-1}} \cdot P(\lambda) \cdot Z(\lambda) \right]_{\lambda=\lambda_i} = \\
&= k_{-q_i} \cdot P(\lambda_i - q_i \cdot h) \cdot Z_{i,-q_i} + \dots + k_0 \cdot P(\lambda_i) \cdot \\
&\quad \cdot Z_{i,0} + \dots + k_{q_i} \cdot P(\lambda_i + q_i \cdot h) \cdot Z_{i,q_i}
\end{aligned} \tag{14}$$

where $\lambda_i - q_i \cdot h, \dots, \lambda_i, \dots, \lambda_i + q_i \cdot h$ are transformed eigenvalues assigned to s_i times repeated eigenvalue λ_i and $Z_{i,-q_i}, \dots, Z_{i,0}, \dots, Z_{i,q_i}$ are transformed component matrices assigned to transformed eigenvalues $\lambda_i - q_i \cdot h, \dots, \lambda_i, \dots, \lambda_i + q_i \cdot h$.

2.5 Approximation based on modified repeated eigenvalues (modified eigenvalues method)

The Sylvester' theorem came out from Lagrange interpolation polynomial [6] with distinct polynomial roots. The repeated roots in Lagrange polynomial can be modified and approximated by distinct roots according to Tab. I with a defined approximation error (it is easy to extend the table for higher root multiplicity).

In case the parameter h is small enough, the approximation error can be also small because of the high power of h . This methodology can be applied to Sylvester's theorem and the **modified eigenvalues** can be used instead of repeated ones and then the decomposition can be done with the help of equation (7).

modified roots/ repeated roots	$(a-2h)$	$(a-h)$	a	$(a+h)$	$(a+2h)$	approx. error
$(x+a)^2$	0	$(x+(a-h))$	0	$(x+(a+h))$	0	$\approx h^2$
$(x+a)^3$	0	$(x+(a-h))$	$(x+a)$	$(x+(a+h))$	0	$\approx h^2(x+a)$
$(x+a)^4$	$(x+(a-2h))$	$(x+(a-h))$	0	$(x+(a+h))$	$(x+(a+2h))$	$\approx 4h^4 - 5h^2(x+a)^2$
$(x+a)^5$	$(x+(a-2h))$	$(x+(a-h))$	$(x+a)$	$(x+(a+h))$	$(x+(a+2h))$	$\approx 4h^4(x+a) - 5h^2(x+a)^3$

Tab. I Modified roots in Lagrange polynomial [6].

Multiplicity of original repeated eigenvalues	Modified distinct eigenvalues
λ^2	$\lambda - h, \lambda + h$
λ^3	$\lambda - h, \lambda, \lambda + h$
λ^4	$\lambda - 2h, \lambda - h, \lambda + h, \lambda + 2h$
λ^5	$\lambda - 2h, \lambda - h, \lambda, \lambda + h, \lambda + 2h$
etc.	etc.

Tab. II Original repeated and modified distinct eigenvalues.

As an example Tab. II describes the multiplicity of original repeated eigenvalue and its modification to the distinct ones.

3. The LTI system decomposition

The LTI dynamical system decomposition is based on the application of the Sylvester's theorems or its approximation to state-space matrix **A** (4) and by using the property of components matrices to calculate the transformed input, output and state-space vectors of one-dimensional models.

3.1 The LTI system decomposition with distinct eigenvalues of transition matrix

The LTI system decomposition with distinct eigenvalues could be performed with the help of the following fundamental decomposition theorem.

Theorem 5 (Svítek's theorem II.): Fundamental decomposition of LTI dynamical systems with distinct eigenvalues of transition matrix **A**

The dynamical m -dimensional LTI dynamical system described by state-space model (1) with transition matrix **A** with distinct eigenvalues could be decomposed into m^2 one-dimensional LTI models where the component matrices assigned to transition matrix **A** (9) are used as transformation matrices of state, input and output vectors (filter banks).

Proof: The m -dimensional state-space model (1) with square $m \times m$ matrices **A**, **B**, **C**, **D**⁴ could be decomposed with the help of components matrices Z_1, Z_2, \dots, Z_m into the following form:

$$\begin{aligned}
 & Z_1 x_{n+1} + Z_2 x_{n+1} + \dots + Z_m x_{n+1} = \\
 & = \lambda_1 Z_1 x_n + \lambda_2 Z_2 x_n + \dots + \lambda_m Z_m x_n + Z_1 \cdot B \cdot u_n + Z_2 \cdot B \cdot u_n + \dots + \\
 & + Z_m \cdot B \cdot u_n y_{1,n} + y_{2,n} + \dots + y_{m,n} = Z_1 \cdot y_n + Z_2 \cdot y_n + \dots + Z_m \cdot y_n = \\
 & = Z_1 \cdot C \cdot x_n + Z_2 \cdot C \cdot x_n + \dots + Z_m \cdot C \cdot x_n + Z_1 \cdot D \cdot u_n + \dots + \\
 & + Z_m \cdot D \cdot u_n
 \end{aligned} \tag{15}$$

where form $P(A)=A$ in equation (7) was used and the matrix components property $\sum_{i=1}^m Z_i = I$ (10) was taking into account. By multiplying the equations (15) by component matrix Z_j and by taking into account the property $Z_i \cdot Z_j = 0$ for $i \neq j$ and $Z_i \cdot Z_j = Z_i$ for $i = j$ (10) the LTI dynamical system could be decomposed into m following sub-systems:

$$\begin{aligned}
 Z_j \cdot x_{n+1} &= \lambda_j \cdot Z_j \cdot x_n + Z_j \cdot B \cdot u_n \\
 y_{j,n} &= Z_j \cdot y_n = Z_j \cdot C \cdot x_n + Z_j \cdot D \cdot u_n
 \end{aligned} \tag{16}$$

⁴for non-square matrices **B**, **C**, **D**, the zero elements could be completed as well as in vectors u_n, y_n to achieve the form (1)

with transition value equal to λ_j . The transformed m -dimensional state vectors $Z_j \cdot x_n, j \in \{1, 2, \dots, m\}$ were obtained through filtering the state vector by component matrices (component matrices play the role of filter banks). Each component of the transformed state vector $Z_j \cdot x_n$ could be easily described as a first-order dynamical model because the transition value λ_j is common for all m transformed states $Z_j \cdot x_n$.

3.2 The LTI system decomposition with repeated eigenvalues of transition matrix

The system with repeated eigenvalues of transition matrix **A** could not be decomposed directly according to theorem 5 because of the singularity of component matrices Z_j .

However, the decomposition could be done by means of approximation forms either through derivatives approximation (14) of equation (8) (**transformed eigenvalues method**) or through the eigenvalues modification (**modified eigenvalues method**) of **A** described in Chapter 2.5. Both approximate forms yield to a similar set of one-dimensional models like in (16) only eigenvalues, and component matrices of **A** are transformed or modified.

In the case of derivative approximation (14) of equation (8) the approximate precision is done by a selected approximate form and selected parameter h . In the case of eigenvalues modification (Chapter 2.5), the approximation precision depends on approximation errors introduced in Tab. I.

3.3 The reduction of transformed state-space model

Theorem 5 for distinct eigenvalues results in the definition of m m -dimensional state vectors $Z_j \cdot x_n, j \in \{1, 2, \dots, m\}$ (16) with simple transition values λ_j . The transformed state-space model assigned to the original system (1) has $m \times m$ state values with transformed diagonal transition matrix **A**. The advantage of the $m \times m$ state-space representation is that the simple sum of transformed state vectors $Z_j \cdot x_n, j \in \{1, 2, \dots, m\}$ yields directly to the original state vector x_n .

The number of transformed state vectors could be reduced⁵ in such a way that in model (16) only one or more state components will be used. On the other hand, this methodology means that the original state vector x_n must be computed as a linear combination of reduces transformed state components and not as a simple sum.

In the case of repeated eigenvalues of transition matrix **A**, similar reduction method could be used to decreasing the number of transformed state components. As in the former case, the construction of original state vector x_n will be more complicated (linear combination of transformed states).

⁵the dimension of the transformed state-space model will be lower than $m \times m$

4. Examples

4.1 Evolution system with distinct eigenvalues of transition matrix

Let us define the LTI evolution system with three distinct eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ as follows

$$\begin{bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \cdot \begin{bmatrix} x_{1,i-1} \\ x_{2,i-1} \\ x_{3,i-1} \end{bmatrix}, \quad \begin{bmatrix} x_{1,0} \\ x_{2,0} \\ x_{3,0} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad (17)$$

where the component matrices assigned to transition matrix could be figured out through equation (9)

$$Z_1 = \begin{bmatrix} 3 & -2.5 & 0.5 \\ 3 & -2.5 & 0.5 \\ 3 & -2.5 & 0.5 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} -3 & 4 & -1 \\ -6 & 8 & -2 \\ -12 & 16 & -4 \end{bmatrix}, \quad Z_3 = \begin{bmatrix} 1 & -1.5 & 0.5 \\ 3 & -4.5 & 1.5 \\ 9 & -13.5 & 4.5 \end{bmatrix} \quad (18)$$

where the properties of component matrices (10) could be easily proved. According to the fundamental decomposition theorem 5, the following one-dimensional systems could be defined as follows

$$\begin{bmatrix} {}^1\bar{\Phi}_{1,i} \\ {}^1\bar{\Phi}_{2,i} \\ {}^1\bar{\Phi}_{3,i} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} {}^1\bar{\Phi}_{1,i-1} \\ {}^1\bar{\Phi}_{2,i-1} \\ {}^1\bar{\Phi}_{3,i-1} \end{bmatrix},$$

$$\begin{bmatrix} {}^1\Phi_{1,i} \\ {}^1\Phi_{2,i} \\ {}^1\Phi_{3,i} \end{bmatrix} = \begin{bmatrix} 3 & -2.5 & 0.5 \\ 3 & -2.5 & 0.5 \\ 3 & -2.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \end{bmatrix} \quad (19)$$

$$\begin{bmatrix} {}^2\bar{\Phi}_{1,i} \\ {}^2\bar{\Phi}_{2,i} \\ {}^2\bar{\Phi}_{3,i} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} {}^2\bar{\Phi}_{1,i-1} \\ {}^2\bar{\Phi}_{2,i-1} \\ {}^2\bar{\Phi}_{3,i-1} \end{bmatrix},$$

$$\begin{bmatrix} {}^2\Phi_{1,i} \\ {}^2\Phi_{2,i} \\ {}^2\Phi_{3,i} \end{bmatrix} = \begin{bmatrix} -3 & 4 & -1 \\ -6 & 8 & -2 \\ -12 & 16 & -4 \end{bmatrix} \cdot \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \end{bmatrix} \quad (20)$$

$$\begin{bmatrix} {}^3\bar{\Phi}_{1,i} \\ {}^3\bar{\Phi}_{2,i} \\ {}^3\bar{\Phi}_{3,i} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} {}^3\bar{\Phi}_{1,i-1} \\ {}^3\bar{\Phi}_{2,i-1} \\ {}^3\bar{\Phi}_{3,i-1} \end{bmatrix},$$

$$\begin{bmatrix} {}^3\Phi_{1,i} \\ {}^3\Phi_{2,i} \\ {}^3\Phi_{3,i} \end{bmatrix} = \begin{bmatrix} 1 & -1.5 & 0.5 \\ 3 & -4.5 & 1.5 \\ 9 & -13.5 & 4.5 \end{bmatrix} \cdot \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \end{bmatrix} \quad (21)$$

It could be easily proved that the sum of one-dimensional state vectors (19), (20), (21) coincide in every time interval i with the state vector of three-dimensional system (17), e.g. for $i = 5$

$$\begin{bmatrix} x_{1,5} \\ x_{2,5} \\ x_{3,5} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}^5 \cdot \begin{bmatrix} x_{1,0} \\ x_{2,0} \\ x_{3,0} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}^5 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -58 \\ -237 \\ -838 \end{bmatrix} \quad (22)$$

$$\begin{aligned} & \begin{bmatrix} {}^1\Phi_{1,5} \\ {}^1\Phi_{2,5} \\ {}^1\Phi_{3,5} \end{bmatrix} + \begin{bmatrix} {}^2\Phi_{1,5} \\ {}^2\Phi_{2,5} \\ {}^2\Phi_{3,5} \end{bmatrix} + \begin{bmatrix} {}^3\Phi_{1,5} \\ {}^3\Phi_{2,5} \\ {}^3\Phi_{3,5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^5 \cdot \begin{bmatrix} {}^1\Phi_{1,0} \\ {}^1\Phi_{2,0} \\ {}^1\Phi_{3,0} \end{bmatrix} + \\ & + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}^5 \cdot \begin{bmatrix} {}^1\Phi_{1,0} \\ {}^1\Phi_{2,0} \\ {}^1\Phi_{3,0} \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}^5 \cdot \begin{bmatrix} {}^1\Phi_{1,0} \\ {}^1\Phi_{2,0} \\ {}^1\Phi_{3,0} \end{bmatrix} = \\ & = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^5 \cdot \begin{bmatrix} -0.5 \\ -0.5 \\ -0.5 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}^5 \cdot \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} + \\ & + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}^5 \cdot \begin{bmatrix} -0.5 \\ -1.5 \\ -4.5 \end{bmatrix} = \begin{bmatrix} -58 \\ -237 \\ -838 \end{bmatrix} \quad (23) \end{aligned}$$

4.2 Controlled system with repeated eigenvalues of transition matrix (transformed eigenvalues method)

Let us define the LTI control system with two repeated eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1$ as follows

$$\begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_{1,i-1} \\ x_{2,i-1} \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 7 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{1,i} \\ u_{2,i} \end{bmatrix}, \quad \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix} = \begin{bmatrix} 0.5 \\ -2 \end{bmatrix} \quad (24)$$

Let us select the precision parameter h and let us approximate equation (8) according to approximate form (11) as follows

$$\begin{aligned} A^n &= -\frac{d}{d\lambda} \left[\lambda^n \cdot \begin{bmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{bmatrix} \right]_{\lambda=1} \approx \\ &\approx (1+h)^n \cdot \left(-\frac{1}{2h} \right) \begin{bmatrix} 1-h & -1 \\ 1 & -1-h \end{bmatrix} - \\ &- (1-h)^n \cdot \left(-\frac{1}{2h} \right) \begin{bmatrix} 1+h & -1 \\ 1 & -1+h \end{bmatrix} = \\ &= (1+h)^n \cdot Z_1 + (1-h)^n \cdot Z_2 \end{aligned} \quad (25)$$

The same as in the example in Chapter 4.1, the first order models could be defined

$$\begin{aligned} \begin{bmatrix} {}^1\Phi_{1,i} \\ {}^1\Phi_{2,i} \end{bmatrix} &= \begin{bmatrix} (1+h) & 0 \\ 0 & (1+h) \end{bmatrix} \cdot \begin{bmatrix} {}^1\Phi_{1,i-1} \\ {}^1\Phi_{2,i-1} \end{bmatrix} + \begin{bmatrix} {}^1u_{1,i} \\ {}^1u_{2,i} \end{bmatrix} \\ \begin{bmatrix} {}^2\Phi_{1,i} \\ {}^2\Phi_{2,i} \end{bmatrix} &= \begin{bmatrix} (1-h) & 0 \\ 0 & (1-h) \end{bmatrix} \cdot \begin{bmatrix} {}^2\Phi_{1,i-1} \\ {}^2\Phi_{2,i-1} \end{bmatrix} + \begin{bmatrix} {}^2u_{1,i} \\ {}^2u_{2,i} \end{bmatrix} \end{aligned} \quad (26)$$

where the transformed state and input (control) vectors could be written

$$\begin{aligned} \begin{bmatrix} {}^1\Phi_{1,i} \\ {}^1\Phi_{2,i} \end{bmatrix} &= \left(-\frac{1}{2h}\right) \begin{bmatrix} 1-h & -1 \\ 1 & -1-h \end{bmatrix} \cdot \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}, \\ \begin{bmatrix} {}^1u_{1,i} \\ {}^1u_{2,i} \end{bmatrix} &= \left(-\frac{1}{2h}\right) \begin{bmatrix} 1-h & -1 \\ 1 & -1-h \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ 7 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{1,i} \\ u_{2,i} \end{bmatrix}, \\ \begin{bmatrix} {}^2\Phi_{1,i} \\ {}^2\Phi_{2,i} \end{bmatrix} &= \left(\frac{1}{2h}\right) \begin{bmatrix} 1+h & -1 \\ 1 & -1+h \end{bmatrix} \cdot \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}, \\ \begin{bmatrix} {}^2u_{1,i} \\ {}^2u_{2,i} \end{bmatrix} &= \left(\frac{1}{2h}\right) \begin{bmatrix} 1+h & -1 \\ 1 & -1+h \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ 7 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{1,i} \\ u_{2,i} \end{bmatrix}, \\ \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} &= \begin{bmatrix} {}^1\Phi_{1,i} \\ {}^1\Phi_{2,i} \end{bmatrix} + \begin{bmatrix} {}^2\Phi_{1,i} \\ {}^2\Phi_{2,i} \end{bmatrix}, \\ \begin{bmatrix} {}^1u_{1,i} \\ {}^1u_{2,i} \end{bmatrix} + \begin{bmatrix} {}^2u_{1,i} \\ {}^2u_{2,i} \end{bmatrix} &= \begin{bmatrix} 2 & -3 \\ 7 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{1,i} \\ u_{2,i} \end{bmatrix} \end{aligned} \quad (27)$$

In Fig. 1 and Fig. 2 the evolution of state component $x_{1,n}$ according to both original (24) and mixture models (26) is shown. The original time series is pictured as a continuous curve, the approximation by mixture models is pictured as "+" in each time interval. The approximation could be easily corrected in every time interval i in which the original state vector \vec{x}_i is observed according to equation (27) but for better understanding of approximation error Fig. 1 and Fig. 2 represent the time evolution of models (24) without any corrections.

In Fig. 3 two approximation forms (11) and (12) are compared for the zero input (control) vector and for $h = 0.01$. The figure pictures the prediction error of state component $x_{1,i}$ both with four one-dimensional systems "*" and with eight one-dimensional systems "+". The better the approximation form, the better the precision and also the more one-dimensional systems must be taken into account.

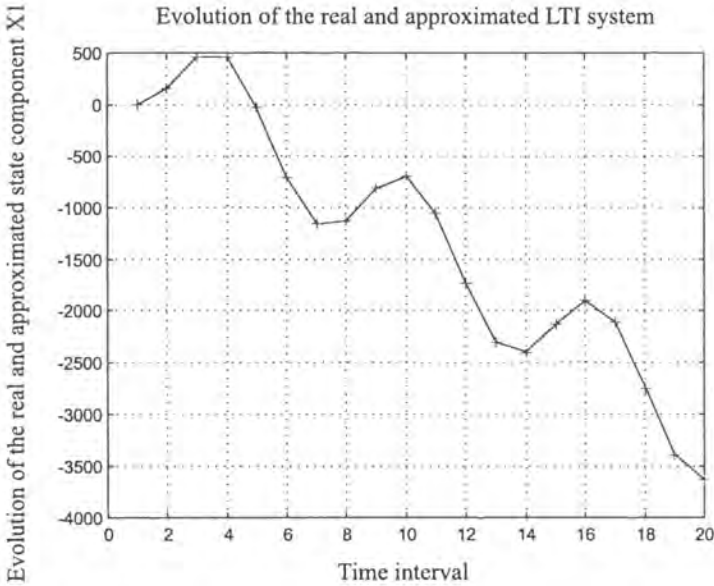


Fig. 1 Time evolution of the original (continuous curve) and approximated LTI system (+).

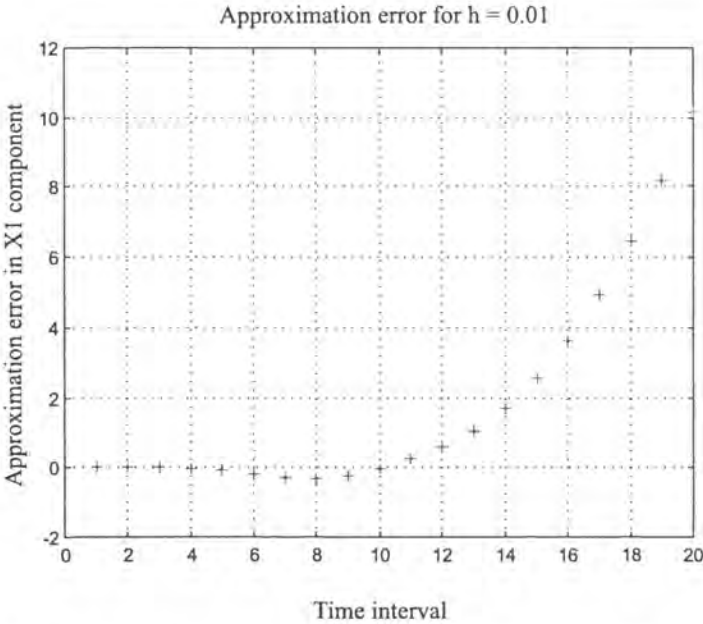


Fig. 2 Time evolution of the approximation error for $h=0.01$ of the state component in Fig. 1.

Comparison of different approximation forms

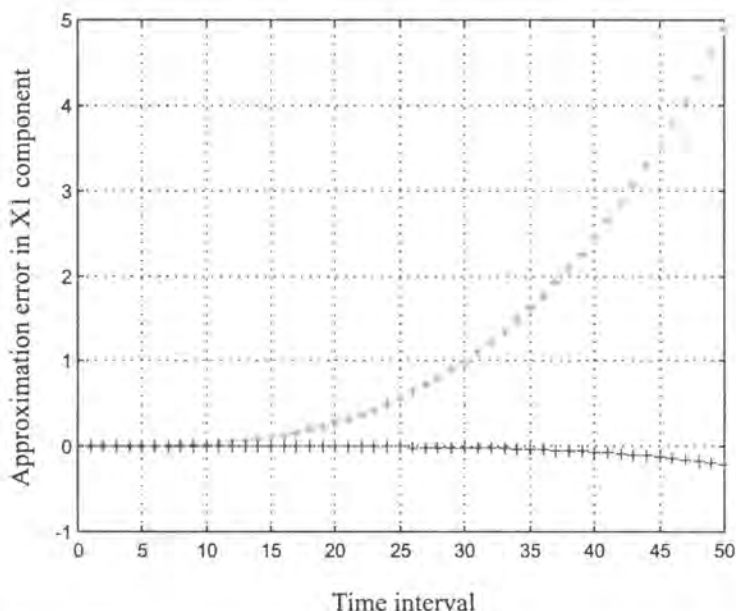


Fig. 3 Comparison of different approximation forms ("*" means approximation by form (11), "+" means approximation by form (12))

4.3 Evolution system with distinct and repeated eigenvalues of transition matrix (modified eigenvalues method)

Let us define the LTI evolution system with two repeated and one distinct eigenvalues $\lambda_1 = -\frac{1}{2}$, $\lambda_2 = -\frac{1}{2}$, $\lambda_3 = -1$ as follows

$$\begin{bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.5 & -1.25 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_{1,i-1} \\ x_{2,i-1} \\ x_{3,i-1} \end{bmatrix}, \quad \begin{bmatrix} x_{1,0} \\ x_{2,0} \\ x_{3,0} \end{bmatrix} = \begin{bmatrix} -0.1 \\ 0.1 \\ 0.1 \end{bmatrix} \quad (28)$$

The repeated eigenvalues will be modified to $\lambda_1 = -\frac{1}{2} + h$, $\lambda_2 = -\frac{1}{2} - h$. The component matrices assigned to the transition matrix with distinct and modified eigenvalues and parameter $h = 0.01$ could be figured out through equation (7)

$$Z_1 = \begin{bmatrix} 50 & 148 & 98 \\ -24.5 & -72.5 & -48 \\ 12 & 35.5 & 23.5 \end{bmatrix},$$

$$Z_2 = \begin{bmatrix} -50 & -152 & -102 \\ 25 & 77 & -52 \\ -13 & -39 & -26 \end{bmatrix},$$

$$Z_3 = \begin{bmatrix} 1 & 4 & 4 \\ -1 & -4 & -4 \\ 1 & 4 & 4 \end{bmatrix} \quad (29)$$

According to the fundamental decomposition Theorem 5 the following one-dimensional systems could be defined as follows:

$$\begin{aligned} \begin{bmatrix} {}^1\Phi_{1,i} \\ {}^1\Phi_{2,i} \\ {}^1\Phi_{3,i} \end{bmatrix} &= \begin{bmatrix} -0.5 + h & 0 & 0 \\ 0 & -0.5 + h & 0 \\ 0 & 0 & -0.5 + h \end{bmatrix} \cdot \begin{bmatrix} {}^1\Phi_{1,i-1} \\ {}^1\Phi_{2,i-1} \\ {}^1\Phi_{3,i-1} \end{bmatrix}, \quad \dots \\ \begin{bmatrix} {}^1\Phi_{1,i} \\ {}^1\Phi_{2,i} \\ {}^1\Phi_{3,i} \end{bmatrix} &= \begin{bmatrix} 50 & 148 & 98 \\ -24.5 & -72.5 & -48 \\ 12 & 35.5 & 23.5 \end{bmatrix} \cdot \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \end{bmatrix} \end{aligned} \quad (30)$$

$$\begin{aligned} \begin{bmatrix} {}^2\Phi_{1,i} \\ {}^2\Phi_{2,i} \\ {}^2\Phi_{3,i} \end{bmatrix} &= \begin{bmatrix} -0.5 - h & 0 & 0 \\ 0 & -0.5 - h & 0 \\ 0 & 0 & -0.5 - h \end{bmatrix} \cdot \begin{bmatrix} {}^2\Phi_{1,i-1} \\ {}^2\Phi_{2,i-1} \\ {}^2\Phi_{3,i-1} \end{bmatrix}, \quad \dots \\ \begin{bmatrix} {}^2\Phi_{1,i} \\ {}^2\Phi_{2,i} \\ {}^2\Phi_{3,i} \end{bmatrix} &= \begin{bmatrix} -50 & -152 & -102 \\ 25 & 77 & -52 \\ -13 & -39 & -26 \end{bmatrix} \cdot \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \end{bmatrix} \end{aligned} \quad (31)$$

$$\begin{aligned} \begin{bmatrix} {}^3\Phi_{1,i} \\ {}^3\Phi_{2,i} \\ {}^3\Phi_{3,i} \end{bmatrix} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} {}^3\Phi_{1,i-1} \\ {}^3\Phi_{2,i-1} \\ {}^3\Phi_{3,i-1} \end{bmatrix}, \quad \dots \\ \begin{bmatrix} {}^3\Phi_{1,i} \\ {}^3\Phi_{2,i} \\ {}^3\Phi_{3,i} \end{bmatrix} &= \begin{bmatrix} 1 & 4 & 4 \\ -1 & -4 & -4 \\ 1 & 4 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \end{bmatrix} \end{aligned} \quad (32)$$

In Figs. 4 and 5 time evolution of the original (28) and approximate (30), (31), (32) systems (its sum) and the approximation error for selected parameter $h = 0.01$ are presented. It could be easily shown that for sufficiently small h the mixture of one-dimensional models (30), (31), (32) (assigned to distinct and *modified eigenvalues*) well approximate the original system. On the other hand, the method of *transformed eigenvalues* (13), (14) enable finding more and more transformed eigenvalues and so the approximation can be constantly improved by adding new and new one-dimensional models. This could not be done in the case of modified eigenvalues described in this example.

5. Conclusion

The presented results have shown the theory of LTI systems decomposition for distinct and repeated eigenvalues of transition matrix \mathbf{A} of the state-space model. This

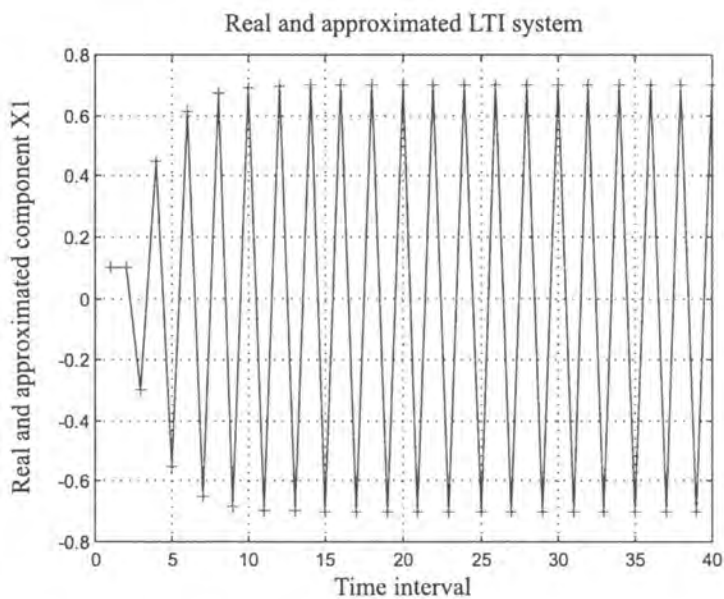


Fig. 4 Approximate ("+") and original (continues curve) evolution of state component $x_{1,n}$.

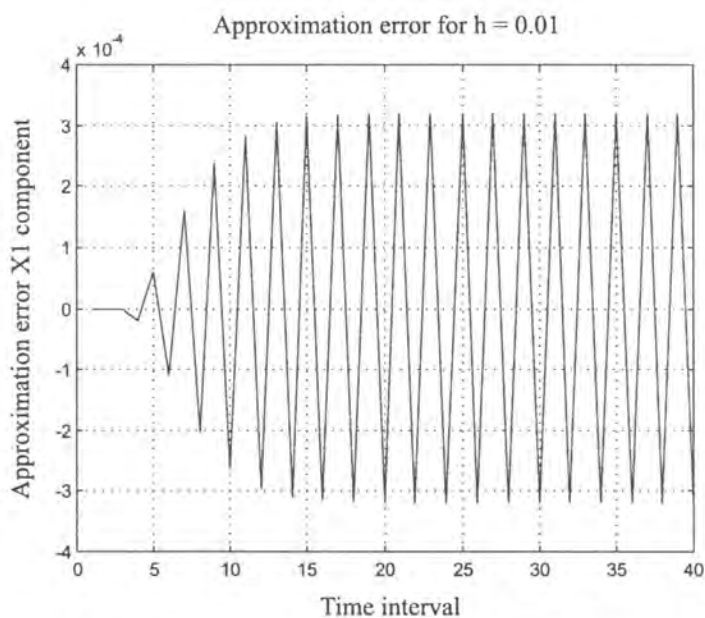


Fig. 5 Approximation error for parameter $h=0.01$.

theory vindicates the very known practice of mixtures of low dimensional dynamical models to approximate the higher order dynamical system. In the paper the direct proof of the LTI dynamical system decomposition with distinct eigenvalues of matrix \mathbf{A} was presented as a fundamental decomposition theorem. The decomposition theory was demonstrated on the three dimensional state-space model in Chapter 4.1.

For repeated eigenvalues of transition matrix \mathbf{A} the two approximation methods were designed. Firstly, the derivatives in Sylvester's theorem for repeated eigenvalues (8) were approximated with the help of approximation forms (11) and (12) - *method of transformed eigenvalues*. Secondly, the repeated eigenvalues of \mathbf{A} were replaced by modified distinct eigenvalues according to Tab. II - *method of modified eigenvalues*. The experiments have shown a good approximation stability and precision in the case of the modified eigenvalues method. The method of transformed eigenvalues was slightly sensitive to selected parameter h in (13) but the method enabled using better and better approximation forms (more and more one-dimensional models) and then the approximation error was sufficiently small. This method also poses the exact approximation performance definition.

In Chapter 3.3 the problem of reduction of the transformed state-space model versus the reconstruction complexity was identified and described. In case the reconstruction of original state-space vector was done by a linear combination of the transformed state-space vectors, the dimensionality of the transformed state-space model could be lower. If the original state-space vector reconstruction must be simple (sum of transformed state-space vectors), a high dimensionality of the transformed state-space model is necessary.

The achieved paper results could be interpreted as a way of time complexity transformation into state-space complexity (more state vectors must be observed). In the paper only LTI systems were studied but the presented methodology could be extended to another kind of systems, e.g. non-linear ones, etc.

In the next research steps, the new ways of identification of matrix eigenvalues and component matrices tailor-made to unknown transition matrix \mathbf{A} will be found to enable estimating the unknown parameters of one-dimensional models and mixture parameters directly from the measured data [1,9]. The decomposed one-dimensional models will be controlled and the mixture of control strategies will be studied [8]. It is expected that the sum or the weighted sum of control strategies of one-dimensional models will yield to more or less precise control strategy of original system. This results should be applicable for large scale systems like transport networks, electricity supply chains, telecommunication networks, etc.

References

- [1] Peterka V.: Bayesian Approach to System Identification. In: Eykhoff P. (ed.): Trends and Progress in System Identification, Pergamon Press Oxford, pp. 239-304.
- [2] Svítek M., Prehal J., Pačlík P., Kárný M.: Parameter Reduction Method for Transport Network Control, Neural Network World, 1, 2001.
- [3] Bishop Ch. M.: Neural Networks and Machine Learning, Springer, 1998.
- [4] Saridis G. N.: Entropy in Control Engineering, World Scientific, 2001.

- [5] Function of matrices - internet paper, 1996, www.ece.utexas.edu/~ari/Courses/EE380K/Functions_of_Matrices.pdf
- [6] Angot A.: Compléments de Mathématiques à l'Electrotechnique et des Télécommunications, MASSON et Cie, Paris, 1952.
- [7] System dynamics - mathematical background, 1999, http://gershwin.ens.fr/vdaniel/Doc-Local/Cours-Mirrored/Methodes_Maths/white/sdyn/s2/s2mamc/s2mamc.html
- [8] Kárný M.: Towards fully probabilistic control design, *Automatica*, **32**, 12, 1996, pp. 1719-1722.
- [9] Kárný M., Halousková A., Nedoma P.: Recursive Approximation by ARX model: a tool for Grey Box Modeling, *Int. J. Adaptive Control Sig. Process.*, **9**, 1995, pp. 525-546.
- [10] He L., Kárný M.: Bayesian Modelling and Estimation of ARMAX Model and Its Finite Mixtures. (Research Report No. 2080). ÚTIA AV ČR, Praha, 2003, 108 p.
- [11] He L., Kárný M.: Estimation and prediction with ARMMAX model: a mixture of ARMAX models with common ARX part. *International Journal of Adaptive Control and Signal Processing*, **17**, 4, 2003, pp. 265-283.