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ON THE PROOF OF ERDŐS' INEQUALITY

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Abstract. Using undergraduate calculus, we give a direct elementary proof of a sharp Markov-type inequality $\|p'\|_{[-1,1]} \leqslant \frac{1}{2}\|p\|_{[-1,1]}$ for a constrained polynomial p of degree at most n, initially claimed by P. Erdős, which is different from the one in the paper of T. Erdélyi (2015). Whereafter, we give the situations on which the equality holds. On the basis of this inequality, we study the monotone polynomial which has only real zeros all but one outside of the interval $(-1,1)$ and establish a new asymptotically sharp inequality.

Keywords: polynomial; Erdős' inequality; undergraduate calculus; monotone polynomial

MSC 2010: 41A17, 26D05, 42A05

1. Introduction and main results

Throughout this paper, we denote the class of all polynomials $p(x) = \sum\limits_{i=0}^{n} a_i x^i$ of degree at most n by π_n if $a_i \in \mathbb{R}$ and π_n^{c} if $a_i \in \mathbb{C}$. We also denote by $\|\cdot\|_K$ the supremum norm on a set K.

We know that the classical Bernstein's inequality

$$|p'(x)| \leqslant \frac{n}{\sqrt{1-x^2}}\|p(x)\|_{[-1,1]} \quad \text{for } -1 < x < 1$$

holds for every $p \in \pi_n^{\mathrm{c}}$ and the Markov's inequality

$$\|p'(x)\|_{[-1,1]} \leqslant n^2\|p(x)\|_{[-1,1]} \tag{1.1}$$

holds for every $p \in \pi_n^{\mathrm{c}}$. For proofs of these see [2] or [3]. In 1940, to extend the "right" Markov factor n^2 in (1.1), Erdős dealt with inequalities for the polynomials, of which

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the location of the zeros is constrained, and obtained the following inequality known as Erdős' inequality (see [5]).

Theorem E. *The inequality*

$$\|p'(x)\|_{[-1,1]} \leqslant \frac{\mathrm{e}n}{2}\|p(x)\|_{[-1,1]}$$

holds for all $p(x) \in \pi_n$ having all their zeros in $\mathbb{R} \setminus (-1,1)$.

In [5], Erdős mentioned that if we ulteriorly constrain that p' has no zero in $(-1,1)$, we have the following theorem, a more strict result.

Theorem 1.1. *Let $p \in \pi_n$. If the zeros of p and p' are all real and lie in $\mathbb{R} \setminus (-1,1)$, then*

$$|p'(x)| \leqslant \frac{n}{2}\|p(x)\|_{[-1,1]} \quad \text{for } -1 \leqslant x \leqslant 1. \tag{1.2}$$

Other inequalities for constrained polynomials can be found in e.g. [1], [6], [7], [8].

Erdős only pointed out that a slightly longer calculation would get the result of Theorem 1.1, but did not give a hint to prove it. Recently, T. Erdélyi had proved Theorem 1.1 using Lorentz representation of polynomials (see Theorem 2.5 in [4]). But he emphasized in [4] that a direct elementary proof of this using undergraduate calculus would be desirable, which is just what we have done in this paper.

Whereafter, we analyse in which situations the equality in (1.2) holds and obtain:

Theorem 1.2. *Let $p \in \pi_n$. If the zeros of p and p' are all real and lie in $\mathbb{R} \setminus (-1,1)$, then the equality in (1.2) holds if and only if*

$$p(x) = c\Big(\frac{1 \pm x}{2}\Big)^n, \quad n \neq 2 \tag{1.3}$$

and

$$p(x) = c\Big(\frac{1 \pm x}{2}\Big)^2 \quad \text{or} \quad p(x) = c\Big[1 - \Big(\frac{1 \pm x}{2}\Big)^2\Big], \quad n = 2, \tag{1.4}$$

where c is an arbitrary nonzero real number.

In this paper, we propose additional correction to Theorem 2.5 in [4] for $n = 2$.

On this basis, we study the monotone polynomial with only one zero in $(-1,1)$ and give a new inequality.

Theorem 1.3. *If $p(x) \in \pi_n$ has only real zeros all but one outside of the interval $(-1,1)$ and is monotone on $[-1,1]$, then*

$$\|p'(x)\|_{[-1,1]} \leqslant (n + \mathrm{e}^2 - 1)\|p(x)\|_{[-1,1]}. \tag{1.5}$$

2. Some auxiliary results

To prove our main results we need the following lemmas.

Lemma 2.1. *Let $p \in \pi_n$ such that $\|p\|_{[-1,1]} \leqslant 1$ and $p(-1) = 0$, $p(1) = 1$. If the zeros of p and p' are all real and lie in $\mathbb{R} \setminus (-1, 1)$, then*

$$\frac{p(x)}{1+x} < \frac{1}{2} \exp \frac{1-x}{2} \tag{2.1}$$

holds for $x \in (-1, 1)$.

Proof. Let $x_1 = 1$ and x_j, $j = 2, 3, \ldots, n$, be n zeros of $p(x)$, then we can easily get

$$p(x) = \frac{1+x}{2} \prod_{j=2}^{n} \frac{x - x_j}{1 - x_j} \tag{2.2}$$

and

$$p'(x) = p(x) \biggl(\frac{1}{x+1} + \sum_{j=2}^{n} \frac{1}{x - x_j} \biggr). \tag{2.3}$$

Hence, from $\prod (1 + a_j) < \exp \sum a_j$, $a_j \neq 0$, and $p'(1) \geqslant 0$,

$$\frac{p(x)}{1+x} = \frac{1}{2} \prod_{j=2}^{n} \frac{x - x_j}{1 - x_j} < \frac{1}{2} \exp \sum_{j=2}^{n} \frac{x-1}{1 - x_j} = \frac{1}{2} \exp \Bigl((1-x) \Bigl(\frac{1}{2} - p'(1) \Bigr) \Bigr).$$

Then (2.1) holds. $\square$

Lemma 2.2. *Let $p \in \pi_n$, $p(-1) = 0$, $p(1) = 1$. If the zeros of p and p' are all real and lie in $\mathbb{R} \setminus (-1, 1)$, and $x_0 \in (-1, 1)$ is a zero of $p''(x)$, then*

$$\biggl(\sum_{j=1}^{n} \frac{1}{x_0 - x_j} \biggr)^2 = \sum_{j=1}^{n} \frac{1}{(x_0 - x_j)^2} \tag{2.4}$$

and

$$\sum_{j=m+1}^{n} \frac{1}{x_j - x_0} < \sum_{j=2}^{m} \frac{1}{x_0 - x_j} \tag{2.5}$$

hold, where x_j, $j = 1, 2, 3, \ldots, n$, are n zeros of $p(x)$ and $x_m \leqslant x_{m-1} \leqslant \ldots \leqslant x_1 = -1 < 1 < x_{m+1} \leqslant \ldots \leqslant x_n$.

P r o o f. Performing some simple calculations, we get

$$p''(x_0) = p(x_0)\Bigg[\Bigg(\sum_{j=1}^{n}\frac{1}{x_0-x_j}\Bigg)^2 - \sum_{j=1}^{n}\frac{1}{(x_0-x_j)^2}\Bigg].$$

Thus, for $p''(x_0)=0$, $p(x_0)>0$, (2.4) holds. Since

$$\begin{aligned} 0 < \sum_{j=2}^{n}\frac{1}{(x_0-x_j)^2} &= \frac{2}{x_0+1}\sum_{j=2}^{n}\frac{1}{x_0-x_j} + \Bigg(\sum_{j=2}^{n}\frac{1}{x_0-x_j}\Bigg)^2 \\ &= \Bigg(\sum_{j=2}^{m}\frac{1}{x_0-x_j} - \sum_{j=m+1}^{n}\frac{1}{x_j-x_0}\Bigg)\Bigg(\frac{1}{x_0+1}+\frac{p'(x_0)}{p(x_0)}\Bigg) \end{aligned}$$

(obviously $m \geqslant 2$ here) and

$$\frac{1}{x_0+1}+\frac{p'(x_0)}{p(x_0)} > 0,$$

we prove that (2.5) holds. $\square$

Lemma 2.3. *Under the conditions of Lemma* 2.2, *if*

$$\frac{1}{(x_0+1)^2} < \sum_{j=m+1}^{n}\frac{1}{(x_j-x_0)^2} \quad \textit{and} \quad \sum_{j=m+1}^{n}\frac{1}{x_j-x_0} \leqslant \frac{1}{2}\sum_{j=1}^{m}\frac{1}{x_j-x_0},$$

then for $n \geqslant 4$

$$p'(x_0) < \frac{n}{2}.$$

P r o o f. Under above conditions and by (2.5), we can find

$$\frac{1}{x_0+1} < \sum_{j=m+1}^{n}\frac{1}{x_j-x_0} < \sum_{j=2}^{m}\frac{1}{x_0-x_j}. \tag{2.6}$$

On one hand,

$$\sum_{j=m+1}^{n}\frac{1}{(x_j-x_0)^2} \leqslant \Bigg(\frac{1}{2}\sum_{j=1}^{m}\frac{1}{x_j-x_0}\Bigg)^2 \leqslant \frac{m^2}{4(x_0+1)^2}$$

then

$$\sum_{j=1}^{n}\frac{1}{(x_0-x_j)^2} \leqslant \frac{m}{(x_0+1)^2} + \frac{m^2}{4(x_0+1)^2} \leqslant \frac{4m+m^2}{4(x_0+1)^2}. \tag{2.7}$$

On the other hand,

$$\sum_{j=1}^{m}\frac{1}{(x_0-x_j)^2}\leqslant\frac{m}{(1+x_0)^2}<m\sum_{j=m+1}^{n}\frac{1}{(x_j-x_0)^2}\leqslant\frac{m(n-m)}{(1-x_0)^2},$$

consequently,

$$\sum_{j=1}^{n}\frac{1}{(x_0-x_j)^2}<\frac{m(n-m)}{(1-x_0)^2}+\frac{n-m}{(1-x_0)^2}=\frac{(m+1)(n-m)}{(1-x_0)^2}. \tag{2.8}$$

Using (2.3), (2.4), (2.7) and (2.8), we obtain

$$p'(x_0)\leqslant\min\Big\{\frac{p(x_0)}{2(1+x_0)}\sqrt{4m+m^2},\ \frac{p(x_0)}{1-x_0}\sqrt{(m+1)(n-m)}\Big\}. \tag{2.9}$$

If $m=n-1$, (2.6) implies $x_0>0$. So by (2.1), (2.9), $p'(x_0)<\sqrt{\mathrm{e}}/4\times\sqrt{4m+m^2}<n/2$.

If $m\leqslant n-2$, we estimate $p'(x_0)$ separately in three intervals.

For $x_0\in(-1,-1/3)$, by (2.9) and $\sqrt{(m+1)(n-m)}\leqslant(n+1)/2$, $p'(x_0)\leqslant 3/4\times(n+1)/2<n/2$. For $x_0\in[-1/3,0)$, by (2.1), (2.9) and $\sqrt{4m+m^2}<n$, $p'(x_0)<n/2$. For $x_0\in[0,1)$, obviously $p(x_0)/(x_0+1)\leqslant 1$, and then by (2.9), $p'(x_0)<n/2$. $\square$

Lemma 2.4. *Under the conditions of Lemma* 2.2, *if*

$$\frac{1}{(x_0+1)^2}<\sum_{j=m+1}^{n}\frac{1}{(x_j-x_0)^2}\quad\text{and}\quad\sum_{j=m+1}^{n}\frac{1}{x_j-x_0}>\frac{1}{2}\sum_{j=1}^{m}\frac{1}{x_j-x_0},$$

then for $n\geqslant 4$

$$p'(x_0)<\frac{n}{2}.$$

Proof. It can be easily checked that

$$\sum_{j=1}^{m}\frac{1}{x_0-x_j}-\sum_{j=m+1}^{n}\frac{1}{x_j-x_0}<\sum_{j=m+1}^{n}\frac{1}{x_j-x_0} \tag{2.10}$$

and

$$\sum_{j=1}^{m}\frac{1}{x_0-x_j}-\sum_{j=m+1}^{n}\frac{1}{x_j-x_0}<\frac{1}{2}\sum_{j=1}^{m}\frac{1}{x_0-x_j}. \tag{2.11}$$

So by (2.3), (2.10),

$$p'(x_0) < p(x_0) \sum_{j=m+1}^{n} \frac{1}{x_j - x_0} \leqslant p(x_0) \frac{n-m}{1-x_0}, \tag{2.12}$$

and by (2.3), (2.11),

$$p'(x_0) < \frac{p(x_0)}{2} \sum_{j=1}^{m} \frac{1}{x_0 - x_j} \leqslant \frac{mp(x_0)}{2(1+x_0)}. \tag{2.13}$$

For $x_0 \in (-1, -1/3]$, if $m \leqslant n/3$, by (2.1), (2.13), $p'(x_0) < (m/2)(\mathrm{e}/2) < n/2$; and if $m > n/3$, by (2.12), $p'(x_0) < (3/4)(n-m) \leqslant n/2$.

For $x_0 \in (-1/3, 1)$, by (2.1), (2.13), $p'(x_0) < (m/2)(1/2)\mathrm{e}^{2/3} < n/2$. □

Lemma 2.5. *Under the conditions of Lemma 2.2, if*

$$\frac{1}{(x_0+1)^2} \geqslant \sum_{j=m+1}^{n} \frac{1}{(x_j - x_0)^2},$$

then for $n \geqslant 4$

$$p'(x_0) < \frac{n}{2}.$$

P r o o f. It is obvious that $m \leqslant n-1$. Then by (2.3), (2.4), we can get

$$\begin{aligned} p'(x) &= p(x_0)\biggl(\frac{1}{(x_0+1)^2} + \sum_{j=2}^{m} \frac{1}{(x_0 - x_j)^2} + \sum_{j=m+1}^{n} \frac{1}{(x_j - x_0)^2}\biggr)^{1/2} \\ &\leqslant p(x_0)\biggl(\frac{2}{(x_0+1)^2} + \sum_{j=2}^{m} \frac{1}{(x_0 - x_j)^2}\biggr)^{1/2} \\ &\leqslant \frac{p(x_0)}{x_0+1}\sqrt{m+1}. \end{aligned} \tag{2.14}$$

For $x_0 \in [-1/3, 1)$, by (2.1), (2.14), we have $p'(x) \leqslant \mathrm{e}^{2/3}\sqrt{m+1}\,/2 < n/2$.

For $x_0 \in (-1, -1/3)$ we estimate $p'(x_0)$ basing on the relation between m and n.

(i) $m = n-1$: We deduce from (2.4) that

$$\begin{aligned} 0 &\leqslant \biggl(\sum_{j=2}^{n-1} \frac{1}{x_0 - x_j}\biggr)^2 - \sum_{j=2}^{n-1} \frac{1}{(x_0 - x_j)^2} \\ &= \frac{2}{(x_0+1)(x_n - x_0)}\biggl[1 - (x_n - 2x_0 - 1)\sum_{j=2}^{n-1} \frac{1}{x_0 - x_j}\biggr], \end{aligned} \tag{2.15}$$

which implies

$$\sum_{j=2}^{n-1} \frac{1}{x_0 - x_j} \leqslant \frac{1}{x_n - 2x_0 - 1}.$$

Consequently,

$$\sum_{j=1}^{n} \frac{1}{x_0 - x_j} \leqslant \frac{1}{x_0 + 1} f(x_0), \tag{2.16}$$

where we denote

$$f(x) = \frac{x_n - 2x - 1}{x_n - x} + \frac{x + 1}{x_n - 2x - 1},$$

which is increasing for $x \in (-1, \frac{1}{3})$. By (2.1), (2.3), (2.16), for $x_0 \in (-\frac{2}{3}, -\frac{1}{3})$, $p'(x_0) < f(-\frac{1}{3})\, \frac{1}{2}\mathrm{e}^{5/6} < \frac{1}{2}n$; for $x_0 \in (-1, -\frac{2}{3}]$, $p'(x_0) \leqslant f(-\frac{2}{3})\, \frac{1}{2}\mathrm{e} < \frac{1}{2}n$.

(ii) $m = n - 2$, i.e., $x_n \geqslant x_{n-1} > 1$: For $x_0 \in (-\frac{2}{3}, -\frac{1}{3})$, from (2.1), (2.14) we have that $p'(x_0) < \frac{1}{2}\sqrt{m+1} \exp \frac{1}{2}(1 - x_0) < \frac{1}{2}n$. For $x_0 \in (-1, -\frac{2}{3}]$, by simple calculation we can get

$$\frac{3}{2}\Big(\frac{1}{x_n - x_0} + \frac{1}{x_{n-1} - x_0}\Big) < \frac{1}{x_0 + 1} - \frac{1}{x_n - x_0} - \frac{1}{x_{n-1} - x_0}. \tag{2.17}$$

From (2.4) we deduce

$$\begin{aligned}
0 &\geqslant \sum_{j=2}^{n-2} \frac{1}{(x_0 - x_j)^2} - \Big(\sum_{j=2}^{n-2} \frac{1}{x_0 - x_j}\Big)^2 \\
&= \frac{2}{x_0 + 1} \sum_{j=2}^{n-2} \frac{1}{x_0 - x_j} - 2\Big(\frac{1}{x_n - x_0} + \frac{1}{x_{n-1} - x_0}\Big) \sum_{j=2}^{n-2} \frac{1}{x_0 - x_j} \\
&\quad - \frac{2}{x_0 + 1}\Big(\frac{1}{x_n - x_0} + \frac{1}{x_{n-1} - x_0}\Big) + \frac{2}{(x_n - x_0)(x_{n-1} - x_0)}.
\end{aligned}$$

Further,

$$\begin{aligned}
&\Big(\frac{1}{x_0 + 1} - \frac{1}{x_n - x_0} - \frac{1}{x_{n-1} - x_0}\Big) \sum_{j=2}^{n-2} \frac{1}{x_0 - x_j} \\
&\quad \leqslant \frac{1}{x_0 + 1}\Big(\frac{1}{x_n - x_0} + \frac{1}{x_{n-1} - x_0}\Big) - \frac{1}{(x_n - x_0)\,(x_{n-1} - x_0)} \\
&\quad = \Big(\frac{1}{x_0 + 1} - \frac{1}{x_n - x_0} - \frac{1}{x_{n-1} - x_0}\Big)\Big(\frac{1}{x_n - x_0} + \frac{1}{x_{n-1} - x_0}\Big) \\
&\qquad + \Big(\frac{1}{x_n - x_0} + \frac{1}{x_{n-1} - x_0}\Big)^2 - \frac{1}{(x_n - x_0)(x_{n-1} - x_0)}.
\end{aligned} \tag{2.18}$$

Combining (2.17) with (2.18), we have

$$\sum_{j=2}^{n-2}\frac{1}{x_0-x_j}\leqslant\frac{1}{x_n-x_0}+\frac{1}{x_{n-1}-x_0}+\frac{\left(\frac{1}{x_n-x_0}+\frac{1}{x_{n-1}-x_0}\right)^2-\frac{1}{(x_n-x_0)(x_{n-1}-x_0)}}{\frac{3}{2}\left(\frac{1}{x_n-x_0}+\frac{1}{x_{n-1}-x_0}\right)}.$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{n}&\frac{1}{x_0-x_j}\\ &\leqslant\frac{1}{1+x_0}+\frac{2}{3}\left(\frac{1}{x_n-x_0}+\frac{1}{x_{n-1}-x_0}\right)-\frac{2}{3}\frac{1}{x_n+x_{n-1}-2x_0}\\ &=\frac{1}{1+x_0}+\frac{2}{3}\,g(x_n-x_0,x_{n-1}-x_0)\\ &\leqslant\frac{1}{1+x_0}+\frac{2}{3}\,g(x_{n-1}-x_0,x_{n-1}-x_0)\\ &\leqslant\frac{1}{1+x_0}+\frac{1}{x_{n-1}-x_0}.\end{aligned}\tag{2.19}$$

Here we denote $g(a,b)=(a^2+b^2+ab)/(ab(a+b))$ which is a decreasing function with respect to $a>0$ and $b>0$, i.e., $g(x_n-x_0,x_{n-1}-x_0)\leqslant g(x_{n-1}-x_0,x_{n-1}-x_0)$. Now we can obtain by (2.1), (2.3), (2.19) and $1/(x_{n-1}-x_0)<3/5$, $1+x_0\leqslant 1/3$ that for $x_0\in(-1,-2/3]$

$$\begin{aligned}p'(x_0)&\leqslant p(x_0)\left(\frac{1}{1+x_0}+\frac{1}{x_{n-1}-x_0}\right)\\ &<\frac{1}{2}\exp\frac{1-x_0}{2}+\frac{3}{5}\frac{1+x_0}{2}\exp\frac{1-x_0}{2}\\ &<\frac{n}{2}.\end{aligned}$$

(iii) $m\leqslant n-3$: By (2.1) (2.14), $p'(x_0)<\mathrm{e}\sqrt{m+1}/2\leqslant n/2$, $n\geqslant 4$.

□

3. Proofs of the main results

P r o o f of Theorem 1.1. Without loss of generality, we assume that $p(x) > 0$, $p'(x) > 0$ for $x \in (-1,1)$ and $\|p(x)\|_{[-1,1]} = p(1) = 1$. Then if we get

$$\|p'(x)\|_{[-1,1]} \leqslant \frac{n}{2}, \tag{3.1}$$

we have Theorem 1.1 proved because other cases can be settled by (3.1) and some linear transformations performed on them. Now we let x_j, $j = 1,2,3,\ldots,n$, be n zeros of $p(x)$ and

$$x_m \leqslant x_{m-1} \leqslant \ldots \leqslant x_1 \leqslant -1 < 1 < x_{m+1} \leqslant \ldots \leqslant x_n.$$

We divide our proof into two parts.

Part I. Firstly, we consider the case $x_1 = -1$.

When $n = 1$, we have $\|p'\|_{[-1,1]} = 1/2$, which satisfies inequality (3.1).

When $n = 2$, $p'(x)$ has no zero in $(-1,1)$ and by (2.3) we get

$$\frac{1-x_2}{2} \leqslant -1 \quad \text{or} \quad \frac{1-x_2}{2} \geqslant 1. \tag{3.2}$$

If $(1-x_2)/2 \leqslant -1$,

$$\|p\|_{[-1,1]} = p'(-1) = \frac{1+x_2}{2(x_2-1)} \leqslant 1.$$

If $(1-x_2)/2 \geqslant 1$,

$$\|p\|_{[-1,1]} = p'(1) = \frac{3-x_2}{2(1-x_2)} \leqslant 1.$$

Thus, (3.1) holds for $n = 2$.

When $n = 3$, there are three cases with respect to x_2 and x_3.

In the case $x_2 \leqslant -1$, $x_3 \leqslant -1$, by some simple calculations, we can obtain for $x \in (-1,1]$ and $j = 2,3$,

$$0 \leqslant \frac{x-x_j}{1-x_j} \leqslant 1 \quad \text{and} \quad \frac{1}{x-x_j} \leqslant \frac{1}{x+1}. \tag{3.3}$$

Thus, by (2.2), (2.3), (3.3) and becouse of the continuity of p', we gain $p'(x) \leqslant 3/2$ for $x \in [-1,1]$. And the equality holds if and only if $x_j = -1$, $j = 1,2,3$.

In the case $x_2 > 1$, $x_3 > 1$, by (2.1), (2.2), (2.3) and becouse of the continuity of $p(x)$ for $x \in [-1,1]$,

$$p'(x) \leqslant \frac{1+x}{2}\,\mathrm{e}\Big(\frac{1}{x+1} + \frac{1}{x-x_2} + \frac{1}{x-x_3}\Big) \leqslant \mathrm{e}\,\frac{1+x}{2}\,\frac{1}{1+x} < \frac{3}{2}.$$

In the case $x_2 \leqslant -1$, $x_3 > 1$, $p'(x)$ is non-negative for $x \in [-1,1]$, increasing on $(-\infty, -(1-x_2-x_3)/3]$, decreasing on $[-(1-x_2-x_3)/3, \infty)$, and concave down on $(-\infty, \infty)$. In view of $p'(1) \geqslant 0$,

$$\frac{1}{x_3-1} \leqslant \frac{1}{2} + \frac{1}{1-x_2} \leqslant 1,$$

which implies $x_3 \geqslant 2$.

If $|(1-x_2-x_3)/3| < 1$, for $(1-x_2)(x_3-1) \geqslant 2$,

$$\|p'(x)\|_{[-1,1]} = p'\Big(-\frac{1-x_2-x_3}{3}\Big) < \frac{9+3(x_2+x_3)-3x_2x_3}{6(1-x_2)(x_3-1)} \leqslant \frac{3}{2}.$$

If $(1-x_2-x_3)/3 \leqslant -1$, we have

$$\|p'(x)\|_{[-1,1]} = p'(1) \leqslant \frac{1}{2} + \frac{1}{1-x_2} < \frac{3}{2}.$$

If $(1-x_2-x_3)/3 \geqslant 1$, for $x_3 \geqslant 2$, it is obvious that $(2-x_3)x_2 + 2x_3 > 1$, which is equivalent to

$$\|p'(x)\|_{[-1,1]} = p'(-1) = \frac{1+x_2+x_3+x_2x_3}{2(1-x_2)(1-x_3)} < \frac{3}{2}.$$

Thus, (3.1) holds for $n = 3$.

When $n \geqslant 4$, there exists at least an $x_0 \in [-1,1]$ such that $\|p'(x)\|_{[-1,1]} = p(x_0)$.

We first consider the case $x_0 = -1$ which yields that $p(-1) = 0$, $p'(-1) > 0$, $p''(-1) \leqslant 0$. Then by (2.2), (2.3),

$$p''(x) = \frac{2p(x)}{x+1}\sum_{j=2}^{n}\frac{1}{x-x_j} + p(x)\Big(\sum_{j=2}^{n}\frac{1}{x-x_j}\Big)^2 - p(x)\sum_{j=2}^{n}\frac{1}{(x-x_j)^2}$$

and so

$$p''(-1) = 2p'(-1)\sum_{j=2}^{n}\frac{1}{-1-x_j} \leqslant 0.$$

Consequently, due to $\sum\limits_{j=2}^{n} 1/(x-x_j)$ being a decreasing function of x on $[-1,1]$,

$$\sum_{j=2}^{n}\frac{1}{x-x_j} \leqslant \sum_{j=2}^{n}\frac{1}{-1-x_j} \leqslant 0, \quad x \in [-1,1]. \tag{3.4}$$

Then by (2.1), (2.3), (3.4) and the continuity of $p'(x)$,

$$p'(x) \leqslant \frac{p(x)}{x+1} < \frac{1}{2}\exp\frac{1-x}{2} < \frac{n}{2}, \quad x \in [-1,1].$$

In the case $x_0 = 1$,

$$\|p'(x)\|_{[-1,1]} = p'(1) = p(1)\biggl(\frac{1}{1+1} + \sum_{j=2}^{n} \frac{1}{1-x_j}\biggr) \leqslant \frac{n}{2}$$

and the equality only holds for $x_j = -1$, $j = 1, 2, \ldots, n$.

In the case $x_0 \in (-1, 1)$, it is easy to see that $p''(x_0) = 0$. Now Lemma 2.3, Lemma 2.4 and Lemma 2.5 together prove that (3.1) holds.

Part II. Secondly, we consider the case $x_1 < -1$. Let $x(t) = (1 - x_1)(t+1)/2 + x_1$ and $r(t) = p(x) = p[(1 - x_1)(t+1)/2 + x_1]$. Then we can verify that $r(t)$ satisfies all the conditions of Part I, which yields $\|r'(t)\|_{[-1,1]} \leqslant n/2$. Thus,

$$\|p'(x)\|_{[-1,1]} \leqslant \frac{2}{1-x_1}\|r'(t)\|_{[-1,1]} \leqslant \frac{n}{1-x_1} < \frac{n}{2}. \tag{3.5}$$

The proof of Theorem 1.1 is complete. □

P r o o f of Theorem 1.2. Similarly to the preceding proof of Theorem 1.1, we only need to prove that for the conditions $p(x) > 0$, $p'(x) > 0$ for $x \in (-1, 1)$ and $\|p(x)\|_{[-1,1]} = p(1) = 1$, the equality

$$\|p'\|_{[-1,1]} = \frac{n}{2} \tag{3.6}$$

holds if and only if

$$p(x) = \Bigl(\frac{1+x}{2}\Bigr)^n, \quad n \neq 2 \tag{3.7}$$

and

$$p(x) = \Bigl(\frac{1+x}{2}\Bigr)^2 \quad \text{or} \quad p(x) = 1 - \Bigl(\frac{1-x}{2}\Bigr)^2, \quad n = 2. \tag{3.8}$$

Let x_j, $j = 1, 2, 3, \ldots, n$, be n zeros of $p(x)$ and

$$x_m \leqslant x_{m-1} \leqslant \ldots \leqslant x_1 \leqslant -1 < 1 < x_{m+1} \leqslant \ldots \leqslant x_n.$$

Firstly, in the case $x_1 = -1$, we can write

$$p(x) = \frac{1+x}{2} \prod_{j=2}^{n} \frac{x - x_j}{1 - x_j}.$$

When $n = 1$, it is evident that $p(x)$ satisfies the form of equality (3.7) with $\|p\|_{[-1,1]} = 1/2$.

When $n = 2$, we assume there exists an $x_0 \in [-1,1]$ such that $p'(x_0) = \|p'(x)\|_{[-1,1]} = n/2 = 1$. Deducing from (3.2), we have $x_2 \leqslant -1$ or $x_2 \geqslant 3$. Then by (2.3), we can obtain $x_0 = (1-x_2)/2 \in \mathbb{R} \setminus (-1,1)$, which yields $x_0 = \pm 1$ and $x_2 = -1$ or 3, i.e., $p(x)$ have the form of (3.8).

When $n = 3$, in the proof of Theorem 1.1 we have shown that there is only one situation such that (3.6) holds. That is $x_j = -1$, $j = 1, 2, 3$, which meets (3.7).

When $n \geqslant 4$, we assume there exists an $x_0 \in [-1,1]$ such $p'(x_0) = \|p'(x)\|_{[-1,1]} = n/2$. If $x_0 = -1$, by (2.1), $p'(-1) = \lim\limits_{x\to -1}(p(x)/(x+1)) < n/2$ strictly. If $x_0 \in (-1,1)$, in the proof of Theorem 1.1 we have obtained $\|p'(x)\|_{[-1,1]} < n/2$ strictly. If $x_0 = 1$,

$$p'(x_0) = p'(1) = \sum_{j=1}^{n} \frac{1}{1-x_j} = \frac{n}{2}$$

and there is only one situation for $x_j = -1$, $j = 1, 2, 3, \ldots, n$ such that the equalities hold. Then $p(x)$ has the form of (3.7).

Secondly, in the case $x_1 < -1$, by (3.5) we know $\|p'(x)\|_{[-1,1]} < n/2$ strictly too.

The proof of Theorem 1.2 is complete. $\square$

Proof of Theorem 1.3. Without lost of generality, we assume that holds $\|p(x)\|_{[-1,1]} = 1$, $p'(x) \leqslant 0$ for $x \in [-1,1]$, and $p(x_i) = 0$, $-1 < x_1 \leqslant 0$, $x_i \in \mathbb{R} \setminus (-1,1)$, $i = 2, 3, \ldots, n$.

For $x \in [x_1, 1]$, performing a linear transformation on Theorem 1.1, we have

$$\|p(x)'\|_{[x_1,1]} \leqslant \frac{n}{1-x_1}\|p(x)\|_{[x_1,1]} \leqslant n\|p(x)\|_{[x_1,1]}. \tag{3.9}$$

For $x \in [-1, x_1)$, from

$$-\frac{p(1)'}{p(1)} = \frac{1}{x_1-1} + \sum_{i=2}^{n} \frac{1}{x_i-1} \leqslant 0$$

we have

$$\begin{aligned} 0 < -\frac{p(x)}{p(1)} &= \frac{x_1-x}{1-x_1}\prod_{i=2}^{n}\frac{x-x_i}{1-x_i} \leqslant \frac{x_1-x}{1-x_1}\exp\Bigl(\sum_{i=2}^{n}\frac{x-1}{1-x_i}\Bigr) \\ &\leqslant \frac{x_1-x}{1-x_1}\exp\frac{1-x}{1-x_1} \leqslant \frac{x_1-x}{1-x_1}\mathrm{e}^2. \end{aligned}$$

Consequently,

$$
\begin{aligned}
(3.10) \qquad 0 \leqslant -p'(x) &= p(x)\Big(\frac{1}{x_1 - x} + \sum_{i=2}^{n} \frac{1}{x_i - x}\Big) \\
&\leqslant \frac{-p(1)}{1 - x_1}\,\mathrm{e}^2 + p(x) \sum_{x_i \geqslant 1} \frac{1}{x_i - x} \\
&\leqslant n + \mathrm{e}^2 - 1.
\end{aligned}
$$

Combining (3.9) and (3.10), we end our proof. □

Remark. With $p(x) = x^{2m+1}$ (m is a non-negative integer), we see that the order n in Theorem 1.3 cannot be improved, i.e., (1.5) is asymptotically sharp.

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