APPROXIMATION METHODS FOR SOLVING THE CAUCHY PROBLEM

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Abstract. In this paper we give some new results concerning solvability of the 1-dimensional differential equation $y' = f(x, y)$ with initial conditions. We study the basic theorem due to Picard. First we prove that the existence and uniqueness result remains true if f is a Lipschitz function with respect to the first argument. In the second part we give a contractive method for the proof of Picard theorem. These considerations allow us to develop two new methods for finding an approximation sequence for the solution. Finally, some applications are given.

Keywords: Cauchy problem, Lipschitz function, Picard theorem, succesive approximations method, contractions principle

MSC 2000: 34A12, 34A34, 34A45

1. Introduction

Let $D = \{(x, y) \in \mathbb{R}^2 \colon \ |x - x_0| \leqslant a, \ |y - y_0| \leqslant b\}$ be a rectangle and $f \colon \ D \to \mathbb{R}$ a continuous function satisfying the Lipschitz condition

$$|f(x, y) - f(x, z)| \leqslant L|y - z|, \quad \forall\, (x, y), (x, z) \in D,$$

for some $L > 0$. Under these assumptions, according to the well known Picard theorem (e.g. [1], [3], [6]), the Cauchy problem

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases} \tag{1.1}$$

has (locally) a unique solution defined at least on $I = (x_0 - \varepsilon, x_0 + \varepsilon)$, where

$$\varepsilon = \min\left\{a, \frac{b}{M}\right\}, \quad M = \sup_{(x,y)\in D} |f(x, y)|.$$

Moreover, the Picard theorem gives us a method to approximate the solution, usually called the *successive approximations method*.

In this sense, let us define an operator $T\colon\ C(I) \to C(I)$ by

$$(Ty)(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,\mathrm{d}t.$$

Then the solution of problem (1.1) is the limit of the *successive approximations sequence*

$$y_0 = y(x_0), \quad y_n = Ty_{n-1},$$

that is

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,\mathrm{d}t, \quad n \in \mathbb{N}.$$

In the sequel, we will give results similar to the Picard theorem for local existence and uniqueness of the solution of the Cauchy problem (1.1). These considerations lead us to some new approximation sequences for the solution.

2. Lipschitzianity in the first argument

Assume that the continuous function $f\colon\ D \to \mathbb{R}$ satisfies the following Lipschitz condition with respect to the first argument, uniformly in y:

$$|f(x_1, y) - f(x_2, y)| \leqslant \lambda \cdot |x_1 - x_2|, \quad \forall\,(x_1, y), (x_2, y) \in D,$$

for some $\lambda > 0$. Moreover, suppose that f does not vanish on D, so let

$$\min_{(x,y)\in D} |f(x, y)| = \alpha > 0.$$

Under these assumptions, denote

$$\Delta := \{(y, x) \in \mathbb{R}^2\colon\ (x, y) \in D\}$$

and define a function $g\colon\ \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ by

$$g(y, x) := \frac{1}{f(x, y)}.$$

Obviously,

$$\max_{(y,x)\in\Delta} |g(y, x)| = \frac{1}{\alpha}.$$

We can apply the Picard theorem to the Cauchy problem

$$\begin{cases} x' = g(y,x), \\ x(y_0) = x_0. \end{cases} \tag{2.1}$$

Indeed,

$$\begin{aligned} |g(y,x_1) - g(y,x_2)| &= \frac{|f(x_1,y) - f(x_2,y)|}{f(x_1,y)\cdot f(x_2,y)} \\ &\leqslant \frac{1}{\alpha^2}\cdot |f(x_1,y) - f(x_2,y)| \\ &\leqslant \frac{\lambda}{\alpha^2}\cdot |x_1 - x_2|. \end{aligned}$$

Thus problem (2.1) has (locally) a unique solution $x\colon (y_0-\delta, y_0+\delta)\to\mathbb{R}$, where $\delta = \min\{b, a\alpha\}$. This solution $x = x(y)$ is strictly monotone because $x' = g(y,x)\neq 0$, thus it has an inverse $y = y(x)$ defined on a neighbourhood of x_0, $y\colon (x_0-\varepsilon, x_0+\varepsilon)\to\mathbb{R}$ and $y(x_0) = y_0$. Moreover,

$$x' = \frac{\mathrm{d}x}{\mathrm{d}y} = g(y,x)$$

implies

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{g(y,x)} = f(x,y),$$

which means that y is a solution of (1.1). We can state

Theorem 2.1. *Let $f\colon D\subset\mathbb{R}^2\to\mathbb{R}$ be continuous and such that*
(i) $f(x_0,y_0)\neq 0$,
(ii) *f satisfies the Lipschitz condition with respect to the first argument:*

$$|f(x_1,y) - f(x_2,y)| \leqslant \lambda\cdot |x_1 - x_2|, \quad \forall\,(x_1,y),(x_2,y)\in D.$$

Then the Cauchy problem (1.1) *has (locally) a unique solution.*
If $D_1 = \{(x,y)\colon\ |x-x_0|\leqslant a_1,\ |y-y_0|\leqslant b_1\}\subseteq D$ is a rectangle with

$$f(x,y)\neq 0, \quad \forall\,(x,y)\in D_1,$$

then the solution of the Cauchy problem (1.1) *is defined at least on $y\colon (x_0-\varepsilon, x_0+\varepsilon)\to\mathbb{R}$, where*

$$\varepsilon := \frac{1}{M}\min\{a_1\alpha, b_1\}, \quad M := \max_{(x,y)\in D_1}|f(x,y)|, \quad \alpha := \min_{(x,y)\in D_1}|f(x,y)|.$$

P r o o f. From the continuity of f and from the fact that $f(x_0, y_0) \neq 0$, it results that there exists a non-degenerate rectangle D_1 such that f does not vanish in D_1. We can suppose that $D_1 = D$, otherwise we can repeat the proof taking D_1 instead of D.

We have proved that, under the hypotheses of Theorem 2.1, the problem (2.1) has an invertible local solution $x = x(y)$ and its inverse $y = y(x)$ is a solution of (1.1). Reciprocally, if $y = y(x)$ is a local solution of the problem (1.1), then $y' = f(x, y) \neq 0$ and $y = y(x)$ is invertible on a neighbourhood of x_0 with the inverse $x = x(y)$ being a solution of (2.1). Moreover, the problem (2.1) has the local uniqueness property, because the problem (1.1) has this property.

Let $y_1, y_2 \in (y_0 - \delta, y_0 + \delta)$, $y_1 < y_2$. Since $x(y_1), x(y_2) \in (x_0 - \varepsilon, x_0 + \varepsilon)$, we have, using the Lagrange theorem,

$$\begin{aligned} 2\varepsilon > x(y_2) - x(y_1) &= (y_2 - y_1) \cdot x'(c) = (y_2 - y_1) \cdot g(c, x(c)) \\ &= (y_2 - y_1) \cdot \frac{1}{f(x(c), c)} \geqslant \frac{1}{M}(y_2 - y_1). \end{aligned}$$

Thus

$$2\varepsilon > \frac{1}{M}(y_2 - y_1)$$

and taking $y_2 \to y_0 + \delta$, $y_1 \to y_0 - \delta$, we obtain

$$2\varepsilon \geqslant \frac{1}{M} \cdot 2\delta \ \Rightarrow \ \varepsilon \geqslant \frac{1}{M} \cdot \delta$$

with $\delta = \min\{a\alpha, b\}$. □

Now it is easy to see that if a continuous function $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}$ with $f(x_0, y_0) \neq 0$ has partial derivatives of the first order and at least one of them is bounded in D, then the Cauchy problem (1.1) has locally a unique solution.

Indeed, if $\partial f/\partial y$ or $\partial f/\partial x$ is bounded, then f satisfies the Lipschitz condition with respect to the second or the first argument, respectively, and the conclusion follows from the Picard theorem or, respectively, Theorem 2.1.

Applying the successive approximations method to the problem (2.1) with $g(y, x) := 1/f(x, y)$, we can state

Theorem 2.2. *Suppose that $f \neq 0$ on D. Then the sequence*

$$x_{n+1}(y) = x_0 + \int_{y_0}^{y} \frac{\mathrm{d}t}{f(x_n(t), t)}, \quad n \in \mathbb{N}, \tag{2.2}$$

converges to an invertible function denoted by $x = x(y)$ and its inverse is the unique solution of the problem (1.1).

Indeed, $x = x(y)$ is the solution of the problem (2.1) *and we have proved that it is invertible. Its inverse is the unique solution of* (1.1).

On the other hand, we can say that the problem (1.1) *was integrated in the implicit form $x - x(y) = 0$.*

Example. Let us consider the Cauchy problem

$$\begin{cases} y' = \dfrac{y}{x + \ln y}, \\ y(1) = 1. \end{cases} \tag{2.3}$$

With

$$f(x, y) = \frac{y}{x + \ln y},$$

we have $f(1, 1) = 1 \neq 0$ and the partial derivative

$$\frac{\partial f}{\partial x} = -\frac{y}{(x + \ln y)^2}$$

is bounded, so we can apply Theorem 2.1 on a rectangle which contains (1,1). The recurrence relation (2.2) is

$$x_{n+1}(y) = 1 + \int_1^y \frac{x_n(t) + \ln t}{t}\, \mathrm{d}t = 1 + \frac{\ln^2 y}{2} + \int_1^y \frac{x_n(t)}{t}\, \mathrm{d}t$$

with $x_0(y) = 1$. It can be easily proved by induction that

$$x_n(y) = 2\Big(1 + \frac{\ln y}{1!} + \frac{\ln^2 y}{2!} + \ldots + \frac{\ln^n y}{n!}\Big) - 1 - \ln y + \frac{\ln^{n+1} y}{(n+1)!}, \quad n \in \mathbb{N}.$$

The last term tends to zero as $n \to \infty$, uniformly in y in bounded sets. It follows that $x_n(y) \to 2y - 1 - \ln y$ uniformly and consequently, the solution of (2.3) can be expressed by the implicit relation

$$2y - 1 - \ln y = x.$$

3. Contraction principle for the Cauchy problem

In this section we give another way to establish the unique local solvability of the Cauchy problem

$$\begin{cases} y' = f(x,y), \\ y(x_0) = y_0, \end{cases} \tag{3.1}$$

requiring the same conditions as in the Picard theorem, namely continuity and lipschitzianity with respect to the second argument for f. We prove that the differentiation operator $Ty = y'$ defined between two Banach spaces is invertible and then we rewrite (3.1) as a fixed point problem

$$v(x) = f(x, T^{-1}v(x))$$

with $v = Ty \iff y = T^{-1}v$ which is studied using the contraction principle of Banach. In some cases the corresponding approximation sequence is easier to compute than the sequence from the Picard theorem. Let

$$D = \{(x,y) \in \mathbb{R}^2 \colon\ |x - x_0| \leqslant a,\ |y - y_0| \leqslant b\}$$

be a rectangle and let $f\colon D \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition

$$|f(x,y) - f(x,z)| \leqslant L|y - z|$$

for each $(x,y), (x,z) \in D$ and some $L > 0$. Let us choose

$$0 < \varepsilon < \min\Big\{a, \frac{b}{M}\Big\},$$

where

$$M = \max_{(x,y)\in D} |f(x,y)|$$

and denote $I = (x_0 - \varepsilon, x_0 + \varepsilon)$. At the beginning we assume that $y_0 = 0$ without loss of generality, as we will see later. Let us consider the Cauchy problem

$$\begin{cases} y' = f(x,y), \\ y(x_0) = 0. \end{cases} \tag{3.2}$$

Let us define

$$W := \{y \in \overline{C}^1(I)\colon\ y(x_0) = 0\}.$$

Lemma 3.1. *The operator $T\colon W \subset C(\overline{I}) \to C(\overline{I})$, $Ty = y'$ is linear, one-to-one and onto. Its inverse $T^{-1}\colon C(\overline{I}) \to W$ is linear, continuous and*

$$\|T^{-1}v\|_{\overline{C}(I)} \leqslant \varepsilon \|v\|_{\overline{C}(I)}, \quad \forall\, v \in C(\overline{I}).$$

Proof. Let $y_1, y_2 \in W$ be such that $Ty_1 = Ty_2 \Rightarrow y_1' = y_2' \Rightarrow y_1 - y_2$ is constant. But $y_1(x_0) = y_2(x_0) = 0$ and consequently, $y_1 = y_2$.

For every $v \in C(\overline{I})$ there exists $y \in W$, $y(x) := \int_{x_0}^{x} v(t)\,\mathrm{d}t$ such that $Ty = v$. Moreover,

$$|T^{-1}v(x)| = \left|\int_{x_0}^{x} v(t)\,\mathrm{d}t\right| \leqslant |x - x_0| \cdot \sup\nolimits_{t\in I}|v(t)| \leqslant \varepsilon \|v\|_{\overline{C}(I)}.$$

□

Now, the Cauchy problem (3.2) can be equivalently written as

$$Ty(x) = f(x, y(x))$$

with $y \in W$. If we put

$$Ty = v \in C(\overline{I}) \Leftrightarrow y = T^{-1}v,$$

we have

$$v(x) = f(x, T^{-1}v(x)). \tag{3.3}$$

Let us consider the operator $S\colon \overline{B}_M(0) \to \overline{B}_M(0)$ given by

$$Sv(x) := f(x, T^{-1}v(x)),$$

where

$$\overline{B}_M(0) = \{v \in C(\overline{I})\colon \|v\|_{\overline{C}(I)} \leqslant M\}.$$

S is well defined because f and T^{-1} are continuous. Moreover, if $\|v\|_{C(\overline{I})} \leqslant M$, then

$$|T^{-1}v(x)| \leqslant \varepsilon \|v\|_{C(\overline{I})} \leqslant \frac{b}{M} \cdot M = b,$$

thus $(x, T^{-1}v(x)) \in D$, $\forall\, x \in I$. Now we can see that (3.3) is equivalent to the fixed problem $v(x) = Sv(x)$.

We will prove that S is a contraction. Indeed, for $v_1, v_2 \in \overline{B}_M(0)$ we have

$$\begin{aligned}|Sv_1(x) - Sv_2(x)| &= |f(x, T^{-1}v_1(x)) - f(x, T^{-1}v_2(x))|\\ &\leqslant L \cdot |T^{-1}v_1(x) - T^{-1}v_2(x)| = L \cdot |T^{-1}(v_1(x) - v_2(x))| \leqslant L\varepsilon \|v_1 - v_2\|.\end{aligned}$$

We have obtained the inequality

$$\|Sv_1 - Sv_2\| \leqslant c\|v_1 - v_2\|, \ \ \forall\, v_1, v_2 \in \overline{B}_M(0)$$

with $c := L\varepsilon < 1$ if $\varepsilon < 1/L$. From the contraction principle of Banach it results that S has a unique fixed point denoted by $v \in \overline{B}_M(0) \subset C(\overline{I})$,

$$v(x) = f(x, D^{-1}v(x)),$$

or $y'(x) = f(x, y(x))$, with $y = D^{-1}v \in W$. Hence $y\colon (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}$ is the unique solution of the Cauchy problem (3.2). □

Now we consider the general case when $y(x_0) = y_0$:

$$\begin{cases} y' = f(x, y), \\ y(x_0) = y_0. \end{cases} \tag{3.4}$$

If we denote $z := y - y_0$, then z satisfies the Cauchy problem

$$\begin{cases} z' = g(x, z), \\ y(x_0) = 0, \end{cases} \tag{3.5}$$

where $g(x, z) := f(x, z + x_0)$. Obviously, the problem (3.5) has a unique solution as we have proved above, because g has the same properties as f. Also (3.4) has (locally) a unique solution.

In general, the successive approximation sequence from the Picard theorem is given by

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(s, y_n(s))\,\mathrm{d}s, \quad n \in \mathbb{N}.$$

In some cases the integral from this relation is more difficult to be computed than the integral from our method:

$$v_{n+1}(x) = f\left(x, \int_{x_0}^{x} v_n(t)\,\mathrm{d}t\right), \tag{3.6}$$

because the integral sign appears only in the second argument of f. The recurrence relation (3.6) follows from the contraction principle of Banach.

Example. Let us consider the Cauchy problem

$$\begin{cases} y' = a(x)y + b(x), \\ y(x_0) = x_0, \end{cases}$$

associated with a linear differential equation of the first order. The functions a, b are continuous on a real compact interval.

The mapping $(x,y) \overset{f}{\mapsto} a(x)y + b(x)$ is Lipschitz with respect to y. The approximation sequence from the Picard theorem is

$$y_n(x) = y_0 + \int_{x_0}^{x} (a(t)y_{n-1}(t) + b(t))\,\mathrm{d}t$$

and the approximation sequence given by (3.6) is

$$v_{n+1}(x) = a(x)\int_{x_0}^{x} v_n(t)\,\mathrm{d}t + b(x).$$

Finally, let us consider the particular case

$$\begin{cases} y' = y + x^2, \\ y(0) = 0, \end{cases} \tag{3.7}$$

which is a linear differential equation having the unique solution

$$y(x) = 2\mathrm{e}^x - x^2 - 2x - 2, \quad x \in \mathbb{R}.$$

In this case

$$f(x,y) = x^2 + y$$

and

$$|f(x,y) - f(x,z)| = |y - z|,$$

which is lipschitzianity with respect to the second argument. The operator S is now defined by

$$Sv(x) := x^2 + \int_0^x v(t)\,\mathrm{d}t.$$

Using the above theoretical results, we obtain that (3.7) has a unique solution $y = T^{-1}v$, where v is the unique fixed point of S. Moreover, v is the limit of the sequence $(v_n)_{n\in\mathbb{N}}$ recursively defined by

$$v_{n+1}(x) = x^2 + \int_0^x v_n(t)\,\mathrm{d}t,$$

where v_0 is arbitrarily chosen. If we take $v_0 = 0$, then

$$v_1(x) = x^2, \quad v_2(x) = x^2 + \int_0^x t^2\mathrm{d}t = x^2 + \frac{x^3}{3}$$
$$v_3(x) = x^2 + \int_0^x \left(t^2 + \frac{t^3}{3}\right)\mathrm{d}t = x^2 + \frac{x^3}{3} + \frac{x^4}{3\cdot 4}.$$

It is easy to see that

$$v_n(x) = x^2 + \frac{x^3}{3} + \frac{x^4}{3\cdot 4} + \ldots + \frac{x^{n+1}}{3\cdot 4\cdot \ldots \cdot (n+1)}, \quad n \geqslant 2,$$

or $v_n(x) = 2\cdot \sum\limits_{k=0}^{n+1} x^k/k! - 2x - 2$. For $n \to \infty$ we obtain $v(x) = 2\mathrm{e}^x - 2x - 2$ and the solution of (3.7) is $y = T^{-1}v$, namely $y(x) = \int_0^x v(t)\,\mathrm{d}t = 2\mathrm{e}^x - x^2 - 2x - 2$.

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