

ON THE MEAN VALUE OF A SUM ANALOGOUS TO
CHARACTER SUMS OVER SHORT INTERVALS

REN GANGLIAN and ZHANG WENPENG, Shaanxi

(Received June 20, 2006)

Abstract. The main purpose of this paper is to study the mean value properties of a sum analogous to character sums over short intervals by using the mean value theorems for the Dirichlet L-functions, and to give some interesting asymptotic formulae.

Keywords: character sums, short intervals, even power mean, asymptotic formula

MSC 2000: 11L40

1. INTRODUCTION

Let $q \geq 3$ be an integer, and χ a Dirichlet character modulo q . Over the past several decades, many authors investigated various arithmetical properties of the character sums

$$\sum_{a=N+1}^{N+H} \chi(a).$$

Pólya [1] and Vinogradov [2] studied the character sums when the modulus q is equal to a prime p and obtained the following inequality

$$\left| \sum_{a=1}^x \chi(a) \right| \leq c\sqrt{p} \ln p,$$

where c is a constant. Actually, one can establish the above inequality with the constant $c = 1$. If χ is a primitive character modulo q , A. V. Sokolovskii [3] proved the existence of N with

$$\left| \sum_{N \leq n \leq N+[q/2]} \chi(n) \right| > \sqrt{1 - \frac{8 \ln q}{q}} \cdot \frac{1}{2\sqrt{2}} \cdot \sqrt{q},$$

This work is supported by the N.S.F. (60472068) of P.R. China.

where $[y]$ denotes the greatest integer which less than or equal to y . For general non-principal character, D. A. Burgess [4] obtained the mean value estimate of character sums

$$\sum_{n=1}^k \left| \sum_{m=1}^h \chi(n+m) \right|^2 < k \cdot h,$$

where h is any positive integer. Xu Zhefeng and Zhang Wenpeng [5] gave the $2k$ -th power mean of the character sums over the interval $[1, \frac{1}{4}q)$

$$\begin{aligned} \sum_{\substack{\chi(-1)=1}}^* \left| \sum_{a<q/4} \chi(a) \right|^{2k} &= \frac{J(q)q^k}{16} \left(\frac{\pi}{8} \right)^{2k-2} \prod_{p|q} \left(1 - \frac{1}{p^2} \right)^{2k-1} \\ &\quad \prod_{p \nmid 2q} \left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^2} \right) + O(q^{k+\varepsilon}). \end{aligned}$$

In this paper, we study the even power mean of analogous character sums over the short intervals

$$\sum_{\substack{\chi \text{ mod } q}}^* \left| \sum_{a<q/2} (-1)^a \chi(a) \right|^{2k} \quad \text{and} \quad \sum_{\substack{\chi \text{ mod } q}}^* \left| \sum_{a<q/4} (-1)^a \chi(a) \right|^{2k}.$$

First we transform the sum to L-functions. Then using the mean value theorems for Dirichlet L-functions, we study the mean value properties of the sums over short intervals, and obtain a sharper asymptotic formula for them. That is, we shall prove the following:

Theorem 1. *Let $q > 8$ be an odd integer. Then we have the following asymptotic formulae:*

$$\begin{aligned} &\sum_{\substack{\chi \text{ mod } q \\ \chi(-1)=1}}^* \left| \sum_{a<q/2} (-1)^a \chi(a) \right|^{2k} \\ &= \frac{J(q)q^k}{4} \left(\frac{\pi}{4} \right)^{2k-2} \prod_{p|q} \left(1 - \frac{1}{p^2} \right)^{2k-1} \prod_{p \nmid 2q} \left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^2} \right) + O(q^{k+\varepsilon}), \end{aligned}$$

and

$$\begin{aligned} &\sum_{\substack{\chi \text{ mod } q \\ \chi(-1)=1}}^* \left| \sum_{a<\frac{q}{4}} (-1)^a \chi(a) \right|^{2k} \\ &= \frac{J(q)q^k}{2^{k+1}} \left(\frac{\pi}{4} \right)^{2k-2} \prod_{p|q} \left(1 - \frac{1}{p^2} \right)^{2k-1} \prod_{p \nmid 2q} \left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^2} \right) + O(q^{k+\varepsilon}), \end{aligned}$$

where $\sum_{\chi(-1)=1}^*$ denotes the summation over all primitive characters modulo q such that $\chi(-1) = 1$, ε is any fixed positive number, $J(q)$ denotes the number of all primitive characters modulo q , $\prod_{p|q}$ denotes the product over all prime divisors p of q and $C_m^n = m!/n!(m-n)!$.

Theorem 2. Let $q > 8$ be an odd integer. Then we have the two asymptotic formulae:

$$\sum_{\substack{\chi \text{ mod } q \\ \chi(-1)=-1}}^* \left| \sum_{a < q/2} (-1)^a \chi(a) \right|^4 = \frac{3J(q)q^2}{16} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} + O(q^{2+\varepsilon}),$$

and

$$\begin{aligned} \sum_{\substack{\chi \text{ mod } q \\ \chi(-1)=-1}}^* \left| \sum_{a < q/4} (-1)^a \chi(a) \right|^4 &= \frac{15J(q)q^2}{64} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} \\ &- \frac{3\sqrt{2}J(q)q^2}{8} \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 + \frac{2}{p^2-1}\right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \frac{p^4+1}{(p^2+1)^2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2 \\ &+ \frac{J(q)q^2}{4} \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 + \frac{2}{p^2-1}\right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \frac{p^2-1}{p^2+1} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2 \\ &+ \frac{J(q)q^2}{8} \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 + \frac{2}{p^2-1}\right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \left(\frac{p^2-1}{p^2+1}\right)^3 \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2 + O(q^{2+\varepsilon}). \end{aligned}$$

Taking $k = 2$ in Theorem 1 and noting that

$$\begin{aligned} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^3 \prod_{p \nmid 2q} \left(1 + \frac{1}{p^2}\right) &= \prod_p \left(1 + \frac{1}{p^2}\right) \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^3 \prod_{p \nmid 2q} \frac{1}{1 + 1/p^2} \\ &= \frac{4}{5} \frac{\zeta(2)}{\zeta(4)} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)}, \end{aligned}$$

and

$$\sum_{\chi \text{ mod } q}^* = \sum_{\chi(-1)=1}^* + \sum_{\chi(-1)=-1}^*$$

we may immediately get

Corollary 1. Let $q > 8$ be an odd integer. Then we have the following asymptotic formulas

$$\sum_{\chi \bmod q}^* \left| \sum_{a < q/2} (-1)^a \chi(a) \right|^4 = \frac{3}{8} J(q) q^2 \prod_{p|q} \frac{(p^2 - 1)^3}{p^4(p^2 + 1)} + O(q^{2+\varepsilon})$$

and

$$\begin{aligned} \sum_{\chi \bmod q}^* \left| \sum_{a < \frac{q}{4}} (-1)^a \chi(a) \right|^4 &= \frac{21J(q)q^2}{64} \prod_{p|q} \frac{(p^2 - 1)^3}{p^4(p^2 + 1)} \\ &\quad - \frac{3\sqrt{2}J(q)q^2}{8} \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 + \frac{2}{p^2 - 1}\right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \frac{p^4 + 1}{(p^2 + 1)^2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2 \\ &\quad + \frac{J(q)q^2}{4} \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 + \frac{2}{p^2 - 1}\right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \frac{p^2 - 1}{p^2 + 1} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2 \\ &\quad + \frac{J(q)q^2}{8} \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 + \frac{2}{p^2 - 1}\right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \left(\frac{p^2 - 1}{p^2 + 1}\right)^3 \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2 + O(q^{2+\varepsilon}). \end{aligned}$$

2. SOME LEMMAS

To prove the theorem, we need the following lemmas.

Lemma 1. Let χ be a primitive Dirichlet character modulo q with $q \geq 3$, then for any real number $\lambda \in [0, 1]$ with $\lambda \neq r/q$, we have

$$\sum_{0 < n \leq \lambda q} \chi(n) = \begin{cases} \frac{\tau(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n) \sin(2\pi n \lambda)}{n}, & \text{if } \chi(-1) = 1; \\ \frac{\tau(\chi)}{i\pi} \left(L(1, \bar{\chi}) - \sum_{n=1}^{\infty} \bar{\chi}(n) \frac{\cos 2\pi n \lambda}{n} \right), & \text{if } \chi(-1) = -1, \end{cases}$$

where $\tau(\chi) = \sum_{a=1}^q \chi(a) e(a/q)$ is the Gauss sum, $e(y) = e^{2\pi i y}$, and $L(s, \chi)$ denotes the Dirichlet L -function corresponding to χ .

P r o o f. See Section 3.1 of [6]. □

Lemma 2. Let $q > 8$ be an odd integer and χ a primitive Dirichlet character modulo q . Then we have

- (1) $\sum_{a=1}^{[q/2]} \chi(a) = 0, \quad \text{if } \chi(-1) = 1,$
- (2) $\sum_{a=1}^{[q/2]} \chi(a) = \frac{(2 - \bar{\chi}(2))}{i\pi} \tau(\chi) L(1, \bar{\chi}), \quad \text{if } \chi(-1) = -1,$
- (3) $\sum_{a=1}^{[q/4]} \chi(a) = \frac{\tau(\chi)}{\pi} L(1, \bar{\chi}\chi_4), \quad \text{if } \chi(-1) = 1,$
- (4) $\sum_{a=1}^{[q/4]} \chi(a) = \frac{2 + \bar{\chi}(2) - \bar{\chi}(4)}{2\pi i} \tau(\chi) L(1, \bar{\chi}), \quad \text{if } \chi(-1) = -1,$
- (5) $\sum_{a=1}^{[q/8]} \chi(a) = \frac{\tau(\chi)}{\pi} \left[\frac{\bar{\chi}(2)}{2} L(1, \bar{\chi}\chi_4) + \frac{1}{\sqrt{2}} L(1, \bar{\chi}\chi_{8_1}) \right], \quad \text{if } \chi(-1) = 1,$

and

$$(6) \quad \sum_{a=1}^{[\frac{q}{8}]} \chi(a) = \frac{\tau(\chi)}{i\pi} \left[\frac{4 + \bar{\chi}(4) - \bar{\chi}(8)}{4} L(1, \bar{\chi}) - \frac{1}{\sqrt{2}} L(1, \bar{\chi}\chi_{8_2}) \right], \quad \text{if } \chi(-1) = -1,$$

where χ_4 is the primitive Dirichlet character modulo 4, and χ_{8_1} and χ_{8_2} are the two primitive Dirichlet characters modulo 8 with $\chi_{8_1}(3) = 1$ and $\chi_{8_2}(3) = -1$ respectively.

P r o o f. We only prove formula (5), the others can be obtained in the same way. Since q is an odd number, it is easily seen that $r/q \neq \frac{1}{8}$, and from Lemma 1 we can have

$$\begin{aligned} \sum_{a=1}^{[q/8]} \chi(a) &= \frac{\tau(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n) \sin(\frac{1}{4}\pi n)}{n} \\ &= \frac{\tau(\chi)}{\pi} \left(\sum_{n=1}^{\infty} \frac{\bar{\chi}(2n) \sin(\frac{1}{2}\pi n)}{2n} + \sum_{n=1}^{\infty} \frac{\bar{\chi}(2n-1) \sin(\frac{1}{4}(2n-1)\pi)}{2n-1} \right) \\ &= \frac{\tau(\chi)}{\pi} \left(\frac{\bar{\chi}(2)}{2} \sum_{n=1}^{\infty} \frac{\bar{\chi}\chi_4(n)}{n} + \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\chi_2(n)(-1)^{[(n-1)/4]}}{\sqrt{2n}} \right) \\ &= \frac{\tau(\chi)}{\pi} \left(\frac{\bar{\chi}(2)}{2} L(1, \bar{\chi}\chi_4) + \frac{1}{\sqrt{2}} L(1, \bar{\chi}\chi_{8_1}) \right). \end{aligned}$$

This proves formula (5). Similarly we can deduce the other formulas. \square

Lemma 3. Let q and r be integers with $q \geq 2$ and $(r, q) = 1$, and χ a Dirichlet character modulo q . Then we have the identities

$$\sum_{\chi \bmod q}^* \chi(r) = \sum_{d|(q,r-1)} \mu\left(\frac{q}{d}\right) \varphi(d)$$

and

$$J(q) = \sum_{d|q} \mu(d) \varphi\left(\frac{q}{d}\right),$$

where $\sum_{\chi \bmod q}^*$ denotes the summation over all primitive characters modulo q , and $J(q)$ denotes the number of all primitive characters modulo q .

P r o o f. See Lemma 4 in the reference [7]. □

Lemma 4. Let q be any integer with $q > 2$, let $\tau_k(n)$ denote the k -th divisor function (i.e., the number of solutions of the equation $n_1 n_2 \dots n_k = n$ in positive integers n_1, n_2, \dots, n_k) and let $\tau(n)$ be the divisor function. Then we have the identity

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\tau_k^2(n)}{n^2} = \zeta^{2k-1}(2) \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{2k-1} \prod_{p \nmid q} \left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^2}\right),$$

where $\zeta(s)$ is the Riemann zeta function.

P r o o f. See Lemma 3 in the reference [8]. □

Lemma 5. Let $q > 8$ be an odd integer, χ be a Dirichlet character modulo q , χ_4 be the primitive character modulo 4, χ_{8_1} and χ_{8_2} be the two primitive characters modulo 8. Then we have the identities

$$\begin{aligned} & \sum_{\chi(-1)=1}^* |L(1, \bar{\chi} \chi_4)|^{2k} \\ &= \frac{J(q)}{2} \left(\frac{\pi^2}{8}\right)^{2k-1} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{2k-1} \prod_{p \nmid 2q} \left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^2}\right) + O(q^\varepsilon), \end{aligned}$$

$$\begin{aligned}
& \sum_{\substack{\chi(-1)=1}}^* |L(1, \bar{\chi}\chi_{8_1})|^{2k} \\
&= J(q) \left(\frac{\pi^2}{8}\right)^{2k-1} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{2k-1} \prod_{p \nmid 2q} \left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^2}\right) + O(q^\varepsilon), \\
& \sum_{\substack{\chi(-1)=-1}}^* |L(1, \bar{\chi})|^4 = \frac{5\pi^4 J(q)}{144} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} + O(q^\varepsilon), \\
& \sum_{\substack{\chi(-1)=-1}}^* |L(1, \bar{\chi}\chi_{8_2})|^4 = \frac{3\pi^4 J(q)}{256} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} + O(q^\varepsilon), \\
& \sum_{\substack{\chi(-1)=-1}}^* \chi(2^m) |L(1, \bar{\chi})|^4 = \frac{(3m+5)\pi^4 J(q)}{72 \cdot 2^{m+1}} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} + O(q^\varepsilon),
\end{aligned}$$

and

$$\sum_{\substack{\chi(-1)=-1}}^* \bar{\chi}(2^m) |L(1, \bar{\chi})|^4 = \frac{(3m+5)\pi^4 J(q)}{72 \cdot 2^{m+1}} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} + O(q^\varepsilon),$$

where $\sum_{\substack{\chi \text{ mod } q}}^*$ denotes the summation over all primitive characters modulo q , and $J(q)$ denotes the number of all primitive characters modulo q .

Proof. Using the method from the proof of Lemma 5 in the reference [5], we can deduce the results. \square

Lemma 6. Let $q > 8$ be any odd integer, and χ the Dirichlet character modulo q , $m \geq 0$. Then we have the following asymptotic formulae:

$$\begin{aligned}
& \sum_{\substack{\chi(-1)=-1}}^* \chi(2^m) L^2(1, \bar{\chi}) L(1, \chi) L(1, \chi\chi_{8_2}) \\
&= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty} n^{-2} \tau(2^m n) \sum_{t|n} \chi_{8_2}(t) + O(q^\varepsilon), \\
& \sum_{\substack{\chi(-1)=-1}}^* \chi(2^m) L^2(1, \chi) L(1, \bar{\chi}) L(1, \bar{\chi}\chi_{8_2}) \\
&= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty} n^{-2} \tau(n) \sum_{t|2^m n} \chi_{8_2}(t) + O(q^\varepsilon), \\
& \sum_{\substack{\chi(-1)=-1}}^* \chi(2^m) L^2(1, \bar{\chi}) L^2(1, \chi\chi_{8_2}) = \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty} \frac{\tau(2^m n) \chi_{8_2}(n) \tau(n)}{n^2} + O(q^\varepsilon),
\end{aligned}$$

$$\begin{aligned}
& \sum_{\chi(-1)=-1}^* \chi(2^m) |L(1, \bar{\chi})|^2 |L(1, \bar{\chi}\chi_{8_2})|^2 \\
&= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}' n^{-2} \sum_{t|2^m n} \chi_{8_2}(t) \sum_{t|n} \chi_{8_2}(t) + O(q^\varepsilon), \\
& \sum_{\chi(-1)=-1}^* \chi(2^m) L^2(1, \chi) L^2(1, \bar{\chi}\chi_{8_2}) \\
&= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}' \frac{\tau(n)\chi_{8_2}(2^m n)\tau(2^m n)}{n^2} + O(q^\varepsilon), \\
& \sum_{\chi(-1)=-1}^* \bar{\chi}(2^m) L^2(1, \chi) L(1, \bar{\chi}) L(1, \bar{\chi}\chi_{8_2}) \\
&= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}' n^{-2} \tau(2^m n) \sum_{t|n} \chi_{8_2}(t) + O(q^\varepsilon), \\
& \sum_{\chi(-1)=-1}^* \bar{\chi}(2^m) L(1, \chi) L(1, \chi\chi_{8_2}) L^2(1, \bar{\chi}\chi_{8_2}) \\
&= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}' n^{-2} \chi_{8_2}(n) \tau(n) \sum_{t|2^m n} \chi_{8_2}(t) + O(q^\varepsilon), \\
& \sum_{\chi(-1)=-1}^* \bar{\chi}(2^m) L^2(1, \bar{\chi}) L^2(1, \chi\chi_{8_2}) = \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}' \frac{\tau(n)\chi_{8_2}(2^m n)\tau(2^m n)}{n^2} + O(q^\varepsilon), \\
& \sum_{\chi(-1)=-1}^* \bar{\chi}(2^m) |L(1, \bar{\chi})|^2 |L(1, \bar{\chi}\chi_{8_2})|^2 \\
&= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}' n^{-2} \sum_{t|2^m n} \chi_{8_2}(t) \sum_{t|n} \chi_{8_2}(t) + O(q^\varepsilon), \\
& \sum_{\chi(-1)=-1}^* \bar{\chi}(2^m) L^2(1, \chi) L^2(1, \bar{\chi}\chi_{8_2}) = \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}' \frac{\tau(2^m n)\chi_{8_2}(n)\tau(n)}{n^2} + O(q^\varepsilon), \\
& \sum_{\chi(-1)=-1}^* \chi(2^m) L(1, \bar{\chi}) L(1, \bar{\chi}\chi_{8_2}) L^2(1, \chi\chi_{8_2}) \\
&= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}' n^{-2} \chi_{8_2}(n) \tau(n) \sum_{t|2^m n} \chi_{8_2}(t) + O(q^\varepsilon),
\end{aligned}$$

$$\begin{aligned}
& \sum_{\chi(-1)=-1}^* \chi(2^m) L(1, \chi) L(1, \chi \chi_{8_2}) L^2(1, \bar{\chi} \chi_{8_2}) \\
&= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}' n^{-2} \chi_{8_2}(2^m n) \tau(2^m n) \sum_{t|n} \chi_{8_2}(t) + O(q^\varepsilon), \\
& \sum_{\chi(-1)=1}^* \bar{\chi}(2^m) L(1, \bar{\chi}) L(1, \bar{\chi} \chi_{8_2}) L^2(1, \chi \chi_{8_2}) \\
&= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}' n^{-2} \chi_{8_2}(2^m n) \tau(2^m n) \sum_{t|n} \chi_{8_2}(t) + O(q^\varepsilon),
\end{aligned}$$

where $\sum_{n=1}^{\infty}'$ denotes the summation over all positive integers n with $(n, q) = 1$.

Proof. We only prove the first formula, the others can be obtained by the same method.

For convenience, we put

$$A(y, \bar{\chi}) = \sum_{2^m N < n \leqslant y} \bar{\chi}(n) \tau(n), \quad \text{and} \quad B(y, \chi) = \sum_{N < n \leqslant y} \chi(n) \sum_{t|n} \chi_{8_2}(t),$$

where N is a parameter with $q \leqslant N < q^3$. Then from Abel's identity we have

$$L^2(1, \bar{\chi}) = \sum_{n=1}^{\infty} \frac{\bar{\chi}(n) \tau(n)}{n} = \sum_{1 \leqslant n \leqslant 2^m N} \frac{\bar{\chi}(n) \tau(n)}{n} + \int_{2^m N}^{\infty} \frac{A(y, \bar{\chi})}{y^2} dy$$

and

$$\begin{aligned}
L(1, \chi) L(1, \chi \chi_{8_2}) &= \sum_{n=1}^{\infty} n^{-1} \chi(n) \sum_{t|n} \chi_{8_2}(t) \\
&= \sum_{1 \leqslant n \leqslant N} n^{-1} \chi(n) \sum_{t|n} \chi_{8_2}(t) + \int_N^{\infty} \frac{B(y, \chi)}{y^2} dy.
\end{aligned}$$

Hence, we can have

$$\begin{aligned}
& \sum_{\chi(-1)=-1}^* \chi(2^m) L^2(1, \bar{\chi}) L(1, \chi) L(1, \chi \chi_{8_2}) \\
&= \sum_{\chi(-1)=-1}^* \chi(2^m) \left(\sum_{1 \leqslant n_1 \leqslant 2^m N} \frac{\bar{\chi}(n_1) \tau(n_1)}{n_1} + \int_{2^m N}^{\infty} \frac{A(y, \bar{\chi})}{y^2} dy \right) \\
&\quad \times \left(\sum_{1 \leqslant n_2 \leqslant N} n_2^{-1} \chi(n_2) \sum_{t|n_2} \chi_{8_2}(t) + \int_N^{\infty} \frac{B(y, \chi)}{y^2} dy \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\chi(-1)=-1}^* \chi(2^m) \left(\sum_{1 \leq n_1 \leq 2^m N} \frac{\bar{\chi}(n_1) \tau(n_1)}{n_1} \right) \left(\sum_{1 \leq n_2 \leq N} n_2^{-1} \chi(n_2) \sum_{t|n_2} \chi_{8_2}(t) \right) \\
&\quad + \sum_{\chi(-1)=-1}^* \chi(2^m) \left(\sum_{1 \leq n_1 \leq 2^m N} \frac{\bar{\chi}(n_1) \tau(n_1)}{n_1} \right) \left(\int_N^\infty \frac{B(y, \chi)}{y^2} dy \right) \\
&\quad + \sum_{\chi(-1)=-1}^* \chi(2^m) \left(\sum_{1 \leq n_2 \leq N} n_2^{-1} \chi(n_2) \sum_{t|n_2} \chi_{8_2}(t) \right) \left(\int_{2^m N}^\infty \frac{A(y, \bar{\chi})}{y^2} dy \right) \\
&\quad + \sum_{\chi(-1)=-1}^* \chi(2^m) \left(\int_{2^m N}^\infty \frac{A(y, \bar{\chi})}{y^2} dy \right) \left(\int_N^\infty \frac{B(y, \chi)}{y^2} dy \right)
\end{aligned}$$

$$(7) = M_1 + M_2 + M_3 + M_4.$$

Now we shall calculate each term in the expression (7).

(i) From Lemma 3 we have

$$\begin{aligned}
M_1 &= \sum_{\chi(-1)=-1}^* \chi(2^m) \left(\sum_{1 \leq n_1 \leq 2^m N} \frac{\bar{\chi}(n_1) \tau(n_1)}{n_1} \right) \left(\sum_{1 \leq n_2 \leq N} n_2^{-1} \chi(n_2) \sum_{t|n_2} \chi_{8_2}(t) \right) \\
&= \frac{1}{2} \sum'_{1 \leq n_1 \leq 2^m N} \sum'_{1 \leq n_2 \leq N} (n_1 n_2)^{-1} \tau(n_1) \sum_{t|n_2} \chi_{8_2}(t) \sum_{d|(q, 2^m n_1 n_2 - 1)} \mu\left(\frac{q}{d}\right) \varphi(d) \\
&\quad - \frac{1}{2} \sum'_{1 \leq n_1 \leq 2^m N} \sum'_{1 \leq n_2 \leq N} (n_1 n_2)^{-1} \tau(n_1) \sum_{t|n_2} \chi_{8_2}(t) \sum_{d|(q, 2^m n_1 n_2 + 1)} \mu\left(\frac{q}{d}\right) \varphi(d) \\
(8) \quad &= \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right) \varphi(d) \sum'_{\substack{1 \leq n_1 \leq 2^m N \\ 2^m n_2 \equiv n_1 \pmod{d}}} \sum'_{1 \leq n_2 \leq N} (n_1 n_2)^{-1} \tau(n_1) \sum_{t|n_2} \chi_{8_2}(t) \\
&\quad - \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right) \varphi(d) \sum'_{\substack{1 \leq n_1 \leq 2^m N \\ 2^m n_2 \equiv -n_1 \pmod{d}}} \sum'_{1 \leq n_2 \leq N} (n_1 n_2)^{-1} \tau(n_1) \sum_{t|n_2} \chi_{8_2}(t),
\end{aligned}$$

where $\sum'_{1 \leq n \leq N}$ denotes the summation over n from 1 to N such that $(n, q) = 1$.

For simplicity, we split the sum over n_1 or n_2 into four cases: i) $2^m d \leq n_1 \leq 2^m N$ and $d \leq n_2 \leq N$; ii) $2^m d \leq n_1 \leq 2^m N$ and $1 \leq n_2 \leq d - 1$; iii) $1 \leq n_1 \leq 2^m d - 1$ and $d \leq n_2 \leq N$; iv) $1 \leq n_1 \leq 2^m d - 1$ and $1 \leq n_2 \leq d - 1$. So we have

$$\begin{aligned}
&\sum_{d|q} \mu\left(\frac{q}{d}\right) \varphi(d) \sum'_{\substack{2^m d \leq n_1 \leq 2^m N \\ 2^m n_2 \equiv n_1 \pmod{d}}} \sum'_{d \leq n_2 \leq N} (n_1 n_2)^{-1} \tau(n_1) \sum_{t|n_2} \chi_{8_2}(t) \\
&\ll \sum_{d|q} \varphi(d) \sum_{2^m \leq r_1 \leq \frac{2^m N}{d}} \sum_{1 \leq r_2 \leq \frac{N}{d}} \sum_{\substack{l_1=1 \\ l_2 \equiv l_1 \pmod{d}}}^{d-1} \sum_{l_2=1}^{d-1} \frac{\tau(r_1 d + l_1) \tau(r_2 d + l_2)}{(r_1 d + l_1)(r_2 d + l_2)}
\end{aligned}$$

$$\begin{aligned}
&\ll \sum_{d|q} \varphi(d) \sum_{2^m \leq r_1 \leq \frac{2^m N}{d}} \sum_{1 \leq r_2 \leq \frac{N}{d}} \sum_{l_1=1}^{d-1} \frac{[(r_1 d + l_1)(r_2 d + l_1)]^\varepsilon}{(r_1 d + l_1)(r_2 d + l_1)} \\
&\ll \sum_{d|q} \frac{\varphi(d)}{d} \sum_{2^m \leq r_1 \leq \frac{2^m N}{d}} \sum_{1 \leq r_2 \leq \frac{N}{d}} \frac{[(r_1 d + 1)(r_2 d + 1)]^\varepsilon}{r_1 r_2} \ll q^\varepsilon, \\
&\sum_{d|q} \mu\left(\frac{q}{d}\right) \varphi(d) \sum'_{2^m d \leq n_1 \leq 2^m N} \sum'_{\substack{1 \leq n_2 \leq d-1 \\ 2^m n_2 \equiv n_1 \pmod{d}}} (n_1 n_2)^{-1} \tau(n_1) \sum_{t|n_2} \chi_{8_2}(t) \\
&\ll \sum_{d|q} \varphi(d) \sum_{2^m \leq r_1 \leq \frac{2^m N}{d}} \sum_{1 \leq n_2 \leq d-1} (r_1 n_2 d)^{\varepsilon-1} \ll q^\varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{d|q} \mu\left(\frac{q}{d}\right) \varphi(d) \sum'_{\substack{1 \leq n_1 \leq 2^m d-1 \\ n_2 \equiv n_1 \pmod{d}}} \sum'_{d \leq n_2 \leq N} (n_1 n_2)^{-1} \tau(n_1) \sum_{t|n_2} \chi_{8_2}(t) \\
&\ll \sum_{d|q} \varphi(d) \sum_{1 \leq n_1 \leq 2^m d-1} \sum_{1 \leq r_2 \leq \frac{N}{d}} (n_1 r_2 d)^{\varepsilon-1} \ll q^\varepsilon,
\end{aligned}$$

where we have used the estimate $\tau(n) \ll n^\varepsilon$.

For the case $1 \leq n_1 \leq 2^m d-1$, $1 \leq n_2 \leq d-1$, the solution of the congruence $2^m n_2 \equiv n_1 \pmod{d}$ is $2^m n_2 = n_1$. Hence,

$$\begin{aligned}
&\sum_{d|q} \mu\left(\frac{q}{d}\right) \varphi(d) \sum'_{\substack{1 \leq n_1 \leq 2^m d-1 \\ 2^m n_2 \equiv n_1 \pmod{d}}} \sum'_{1 \leq n_2 \leq d-1} (n_1 n_2)^{-1} \tau(n_1) \sum_{t|n_2} \chi_{8_2}(t) \\
&= \sum_{d|q} \mu\left(\frac{q}{d}\right) \varphi(d) \sum'_{1 \leq n_2 \leq d-1} 2^{-m} n_2^{-2} \tau(2^m n_2) \sum_{t|n_2} \chi_{8_2}(t) \\
&= \sum_{d|q} \mu\left(\frac{q}{d}\right) \varphi(d) \sum_{\substack{n_2=1 \\ (n_2, q)=1}}^{\infty} 2^{-m} n_2^{-2} \tau(2^m n_2) \sum_{t|n_2} \chi_{8_2}(t) + O(q^\varepsilon).
\end{aligned}$$

So we have

$$\begin{aligned}
(9) \quad &\frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right) \varphi(d) \sum'_{\substack{1 \leq n_1 \leq 2^m N \\ 2^m n_2 \equiv n_1 \pmod{d}}} \sum'_{1 \leq n_2 \leq N} (n_1 n_2)^{-1} \tau(n_1) \sum_{t|n_2} \chi_{8_2}(t) \\
&= \frac{J(q)}{2^{m+1}} \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} n^{-2} \tau(2^m n) \sum_{t|n} \chi_{8_2}(t) + O(q^\varepsilon).
\end{aligned}$$

Similarly, we can also get the estimate

$$(10) \quad \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right) \varphi(d) \sum'_{\substack{1 \leq n_1 \leq 2^m N \\ 2^m n_2 \equiv -n_1 \pmod{d}}} \sum'_{1 \leq n_2 \leq N} (n_1 n_2)^{-1} \tau(n_1) \sum_{t|n_2} \chi_{8_2}(t) \ll q^\varepsilon.$$

Then from (8), (9) and (10), we have

$$(11) \quad M_1 = \frac{J(q)}{2^{m+1}} \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} n^{-2} \tau(2^m n) \sum_{t|n} \chi_{8_2}(t) + O(q^\varepsilon).$$

(ii) From Lemma 2 of [9], we have the estimate

$$\sum_{\chi \neq \chi_0} |A(y, \bar{\chi})|^2 \ll y^{1+\varepsilon} \varphi^2(q) \quad \text{and} \quad \sum_{\chi \neq \chi_0} |B(y, \chi)|^2 \ll y^{1+\varepsilon} \varphi^2(q),$$

where χ_0 denotes the principal character modulo q . Then from the Cauchy inequality we can easily get

$$\sum_{\chi(-1)=-1} |A(y, \bar{\chi})| \ll \sum_{\chi \neq \chi_0} |A(y, \bar{\chi})| \ll y^{1/2+\varepsilon} q^{3/2}$$

and

$$\sum_{\chi(-1)=-1} |B(y, \chi)| \ll \sum_{\chi \neq \chi_0} |B(y, \chi)| \ll y^{1/2+\varepsilon} q^{3/2}.$$

Using these estimates we have

$$(12) \quad \begin{aligned} M_2 &= \sum_{\chi(-1)=-1}^* \left(\sum_{1 \leq n_1 \leq 2^m N} \frac{\bar{\chi}(n_1) \tau(n_1)}{n_1} \right) \left(\int_N^\infty \frac{B(y, \chi)}{y^2} dy \right) \\ &\ll \sum_{1 \leq n_1 \leq 2^m N} n_1^{\varepsilon-1} \int_N^\infty \frac{1}{y^2} \left(\sum_{\chi(-1)=-1} |B(y, \chi)| \right) dy \\ &\ll N^\varepsilon \int_N^\infty \frac{q^{3/2} y^{1/2+\varepsilon_1}}{y^2} dy \ll \frac{q^{3/2}}{N^{\frac{1}{2}-\varepsilon}}. \end{aligned}$$

(iii) Similar by to (ii), we can also get

$$(13) \quad M_3 \ll \frac{q^{3/2}}{N^{\frac{1}{2}-\varepsilon}}.$$

(iv) By the same argument as in (ii), and noting the absolute convergence of the integrals, we can write

$$\begin{aligned}
(14) \quad M_4 &= \sum_{\chi(-1)=-1}^* \left(\int_{2^m N}^{\infty} \frac{A(y, \bar{\chi})}{y^2} dy \right) \left(\int_N^{\infty} \frac{B(y, \chi)}{y^2} dy \right) \\
&\leq \int_{2^m N}^{\infty} \int_N^{\infty} \frac{1}{y^2 z^2} \sum_{\chi(-1)=-1}^* |A(y, \bar{\chi})| |B(z, \chi)| dy dz \\
&\ll \int_{2^m N}^{\infty} \frac{1}{y^2} \int_N^{\infty} \frac{1}{z^2} \left(\sum_{\chi \neq \chi_0} |A(y, \bar{\chi})|^2 \right)^{\frac{1}{2}} \left(\sum_{\chi \neq \chi_0} |B(z, \chi)|^2 \right)^{\frac{1}{2}} dy dz \\
&\ll \int_{2^m N}^{\infty} \frac{1}{y^2} \left(\sum_{\chi \neq \chi_0} |A(y, \bar{\chi})|^2 \right)^{\frac{1}{2}} dy \times \int_N^{\infty} \frac{1}{z^2} \left(\sum_{\chi \neq \chi_0} |B(z, \chi)|^2 \right)^{\frac{1}{2}} dz \\
&\ll \int_{2^m N}^{\infty} \frac{\varphi(q)}{y^{3/2-\varepsilon}} dy \times \int_N^{\infty} \frac{\varphi(q)}{z^{3/2-\varepsilon}} dz \ll \frac{\varphi^2(q)}{N^{1-\varepsilon}}.
\end{aligned}$$

Now taking $N = q^4$ and $\varepsilon < \frac{1}{2}$, combining (7) and (11)–(14) we obtain the asymptotic formula

$$\sum_{\chi(-1)=-1}^* \chi(2^m) L^2(1, \bar{\chi}) L(1, \chi) L(1, \chi \chi_{8_2}) = \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}' n^{-2} \tau(2^m n) \sum_{t|n} \chi_{8_2}(t) + O(q^\varepsilon).$$

This completes the proof of the first formula of Lemma 6. \square

Lemma 7. *Let $q > 8$ be any odd integer, and χ the Dirichlet character modulo $q, m \geq 0$. Then we have the following asymptotic formulae:*

$$\begin{aligned}
&\sum_{n=1}^{\infty}' n^{-2} \tau(2^m n) \sum_{t|n} \chi_{8_2}(t) \\
&= \frac{(3m+4)\pi^4}{144} \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 + \frac{2}{p^2-1} \right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \frac{p^4+1}{(p^2+1)^2} \prod_{p|q} \left(1 - \frac{1}{p^2} \right)^2, \\
&\sum_{n=1}^{\infty}' \frac{\tau(n) \sum_{t|2^m n} \chi_{8_2}(t)}{n^2} \\
&= \frac{\pi^4}{36} \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 + \frac{2}{p^2-1} \right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \frac{p^4+1}{(p^2+1)^2} \prod_{p|q} \left(1 - \frac{1}{p^2} \right)^2,
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{\infty}' n^{-2} \chi_{8_2}(n) \tau(n) \sum_{t|2^m n} \chi_{8_2}(t) \\
&= \frac{\pi^4}{64} \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 + \frac{2}{p^2 - 1}\right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \frac{p^4 + 1}{(p^2 + 1)^2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2, \\
& \sum_{n=1}^{\infty}' \frac{\tau(2^m n) \chi_{8_2}(n) \tau(n)}{n^2} \\
&= \frac{(m+1)\pi^4}{64} \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 + \frac{2}{p^2 - 1}\right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \left(\frac{p^2 - 1}{p^2 + 1}\right)^3 \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2, \\
& \sum_{n=1}^{\infty}' n^{-2} \sum_{t|2^m n} \chi_{8_2}(t) \sum_{t|n} \chi_{8_2}(t) \\
&= \frac{\pi^4}{48} \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 + \frac{2}{p^2 - 1}\right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \frac{p^2 - 1}{p^2 + 1} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2, \\
& \sum_{n=1}^{\infty}' n^{-2} \chi_{8_2}(n) \tau(2^m n) \sum_{t|n} \chi_{8_2}(t) \\
&= \frac{(m+1)\pi^4}{64} \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 + \frac{2}{p^2 - 1}\right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \frac{p^4 + 1}{(p^2 + 1)^2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2.
\end{aligned}$$

Proof. Noting that $\tau(n)$ is a multiplicative function, we can write

$$\begin{aligned}
(15) \quad & \sum_{n=1}^{\infty}' n^{-2} \tau(2^m n) \sum_{t|n} \chi_{8_2}(t) \\
&= \tau(2^m) \sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty}' n^{-2} \tau(n) \sum_{t|n} \chi_{8_2}(t) + \sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty}' n^{-2} \tau(2^m n) \sum_{t|n} \chi_{8_2}(t) \\
&= (m+1) \sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty}' n^{-2} \tau(n) \sum_{t|n} \chi_{8_2}(t) + \sum_{r=1}^{\infty}' \sum_{\substack{j=1 \\ 2 \nmid r}}^{\infty} (r \cdot 2^j)^{-2} \tau(2^{m+j} r) \sum_{t|2^j r} \chi_{8_2}(t) \\
&= \left(m+1 + \sum_{j=1}^{\infty} \frac{m+j+1}{4^j}\right) \sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty}' n^{-2} \tau(n) \sum_{t|n} \chi_{8_2}(t).
\end{aligned}$$

By using the Euler product formula we can write

$$\begin{aligned}
(16) \quad & \sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty} n^{-2} \tau(n) \sum_{t|n} \chi_{8_2}(t) \\
& = \prod_{\substack{p \nmid q \\ p \neq 2}} \left(1 + \frac{1}{p^2} \tau(p) \sum_{t|p} \chi_{8_2}(t) + \frac{1}{p^4} \tau(p^2) \sum_{t|p^2} \chi_{8_2}(t) + \dots \right) \\
& = \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 + \frac{2^2}{p^2} + \frac{3^2}{p^4} + \dots \right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \left(1 + \frac{3}{p^4} + \frac{5}{p^8} + \frac{7}{p^{12}} + \dots \right) \\
& = \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 - \frac{1}{p^2} \right)^{-2} \left(1 + \frac{2}{p^2 - 1} \right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \left(1 - \frac{1}{p^4} \right)^{-2} \left(1 + \frac{1}{p^4} \right) \\
& = \frac{9}{16} \zeta^2(2) \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 + \frac{2}{p^2 - 1} \right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \frac{1 + 1/p^4}{(1 + 1/p^2)^2} \prod_{p|q} \left(1 - \frac{1}{p^2} \right)^2.
\end{aligned}$$

So from (15), (16) we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{-2} \tau(2^m n) \sum_{t|n} \chi_{8_2}(t) \\
& = \frac{(3m+4)\pi^4}{144} \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 + \frac{2}{p^2 - 1} \right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \frac{p^4 + 1}{(p^2 + 1)^2} \prod_{p|q} \left(1 - \frac{1}{p^2} \right)^2.
\end{aligned}$$

This proves the first formula. By the same method, one can obtain the other formulae. \square

3. PROOF OF THE THEOREM

In this section, we will complete the proof of the theorem. From Lemma 2, we can write

$$\begin{aligned}
(17) \quad & \sum_{a=1}^{[\frac{q}{2}]} (-1)^a \chi(a) = 2 \sum_{\substack{a=1, 2|a}}^{[\frac{q}{2}]} \chi(a) - \sum_{a=1}^{[\frac{q}{2}]} \chi(a) = 2\chi(2) \sum_{a=1}^{[\frac{q}{4}]} \chi(a) - \sum_{a=1}^{[\frac{q}{2}]} \chi(a) \\
& = \begin{cases} 2\chi(2) \frac{\tau(\chi)}{\pi} L(1, \bar{\chi}\chi_4) & \text{if } \chi(-1) = 1, \\ (2\chi(2) - 1) \frac{\tau(\chi)}{\pi i} L(1, \bar{\chi}) & \text{if } \chi(-1) = -1, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
(18) \quad & \sum_{a=1}^{\left[\frac{q}{4}\right]} (-1)^a \chi(a) = 2 \sum_{a=1, 2|a}^{\left[\frac{q}{4}\right]} \chi(a) - \sum_{a=1}^{\left[\frac{q}{4}\right]} \chi(a) \\
& = 2\chi(2) \sum_{a=1}^{\left[\frac{q}{8}\right]} \chi(a) - \sum_{a=1}^{\left[\frac{q}{4}\right]} \chi(a) \\
& = \begin{cases} \sqrt{2}\chi(2)\frac{\tau(\chi)}{\pi}L(1, \bar{\chi}\chi_{8_1}) & \text{if } \chi(-1) = 1, \\ \frac{\tau(\chi)}{\pi i}((2\chi(2)-1)L(1, \bar{\chi}) - \sqrt{2}\chi(2)L(1, \bar{\chi}\chi_{8_2})) & \text{if } \chi(-1) = -1. \end{cases}
\end{aligned}$$

First we prove Theorem 2. Noting that χ is a primitive character modulo q , we can get $|\tau(\chi)| = \sqrt{q}$. So from (17) and Lemma 5, we can obtain

$$\begin{aligned}
& \sum_{\chi(-1)=-1}^* \left| \sum_{a<\frac{q}{2}} (-1)^a \chi(a) \right|^4 \\
& = \sum_{\chi(-1)=-1}^* \left| (2\chi(2)-1)\frac{\tau(\chi)}{\pi}L(1, \bar{\chi}) \right|^4 \\
& = \frac{q^2}{\pi^4} \sum_{\chi(-1)=-1}^* |2\chi(2)-1|^4 |L(1, \bar{\chi})|^4 \\
& = \frac{q^2}{\pi^4} \sum_{\chi(-1)=-1}^* (33 + 4\chi(4) + 4\bar{\chi}(4) - 20\chi(2) - 20\bar{\chi}(2)) |L(1, \bar{\chi})|^4 \\
& = \frac{3J(q)q^2}{16} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} + O(q^{2+\varepsilon}).
\end{aligned}$$

By formula (18) we can also write

$$\begin{aligned}
& \sum_{\chi(-1)=-1}^* \left| \sum_{a<\frac{q}{4}} (-1)^a \chi(a) \right|^4 \\
& = \sum_{\chi(-1)=-1}^* \left| \frac{\tau(\chi)}{\pi i} ((2\chi(2)-1)L(1, \bar{\chi}) - \sqrt{2}\chi(2)L(1, \bar{\chi}\chi_{8_2})) \right|^4
\end{aligned}$$

$$\begin{aligned}
&= \frac{q^2}{\pi^4} \sum_{\chi(-1)=-1}^* (33 + 4\chi(4) + 4\bar{\chi}(4) - 20\chi(2) - 20\bar{\chi}(2)) |L(1, \bar{\chi})|^4 \\
&\quad + \frac{q^2}{\pi^4} \sum_{\chi(-1)=-1}^* (8 + 2\chi(4) - 8\chi(2)) L^2(1, \chi) L^2(1, \bar{\chi}\chi_{8_2}) \\
&\quad + \frac{q^2}{\pi^4} \sum_{\chi(-1)=-1}^* (8 + 2\bar{\chi}(4) - 8\bar{\chi}(2)) L^2(1, \bar{\chi}) L^2(1, \chi\chi_{8_2}) \\
&\quad - \frac{2\sqrt{2}q^2}{\pi^4} \sum_{\chi(-1)=-1}^* (12 - 9\chi(2) - 4\bar{\chi}(2) + 2\chi(4)) L^2(1, \chi) L(1, \bar{\chi}\chi_{8_2}) L(1, \bar{\chi}) \\
&\quad - \frac{2\sqrt{2}q^2}{\pi^4} \sum_{\chi(-1)=-1}^* (12 - 9\bar{\chi}(2) - 4\chi(2) + 2\bar{\chi}(4)) L^2(1, \bar{\chi}) L(1, \chi\chi_{8_2}) L(1, \chi) \\
&\quad + \frac{8q^2}{\pi^4} (5 - 2\bar{\chi}(2) - 2\chi(2)) |L(1, \bar{\chi})|^2 |L(1, \chi\bar{\chi}_{8_2})|^2 \\
&\quad - \frac{4\sqrt{2}q^2}{\pi^4} \sum_{\chi(-1)=-1}^* (2 - \bar{\chi}(2)) L(1, \bar{\chi}) L^2(1, \chi\chi_{8_2}) L(1, \bar{\chi}\chi_{8_2}) \\
&\quad - \frac{4\sqrt{2}q^2}{\pi^4} \sum_{\chi(-1)=-1}^* (2 - \chi(2)) L(1, \chi) L^2(1, \bar{\chi}\chi_{8_2}) L(1, \chi\chi_{8_2}) \\
&\quad + \frac{4q^2}{\pi^4} \sum_{\chi(-1)=-1}^* |L(1, \bar{\chi}\chi_{8_2})|^4.
\end{aligned}$$

By Lemma 5, Lemma 6 and Lemma 7, we can obtain

$$\begin{aligned}
\sum_{\chi(-1)=-1}^* \left| \sum_{a < \frac{q}{4}} (-1)^a \chi(a) \right|^4 &= \frac{15J(q)q^2}{64} \prod_{p|q} \frac{(p^2 - 1)^3}{p^4(p^2 + 1)} \\
&\quad - \frac{3\sqrt{2}J(q)q^2}{8} \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 + \frac{2}{p^2 - 1}\right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \frac{p^4 + 1}{(p^2 + 1)^2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2 \\
&\quad + \frac{J(q)q^2}{4} \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 + \frac{2}{p^2 - 1}\right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \frac{p^2 - 1}{p^2 + 1} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2 \\
&\quad + \frac{J(q)q^2}{8} \prod_{\substack{p \nmid q \\ p \equiv 1, 7 \pmod{8}}} \left(1 + \frac{2}{p^2 - 1}\right) \prod_{\substack{p \nmid q \\ p \equiv 3, 5 \pmod{8}}} \left(\frac{p^2 - 1}{p^2 + 1}\right)^3 \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2 + O(q^{2+\varepsilon}).
\end{aligned}$$

Thus we complete the proof of Theorem 2. Similarly, combining (17), (18) and Lemma 5, we can use the same method to deduce Theorem 1.

Acknowledgments. The authors express their gratitude to the referee for his very helpful comments and valuable suggestions, which greatly improved this paper.

References

- [1] *G. Pólya*: Über die Verteilung der quadratischen Reste und Nichtreste. Göttingen Nachrichten, 1918, pp. 21–29.
- [2] *I. M. Vinogradov*: On the distribution of residues and non-residues of powers. Journal of the Physico-Mathematical society of Perm 1 (1918), 94–96.
- [3] *A. V. Sokolovskii*: On a theorem of Sárkozy. Acta Arithmetica 41 (1982), 27–31.
- [4] *D. A. Burgess*: On a conjecture of Norton. Acta Arithmetica 27 (1975), 265–267.
- [5] *Zhefeng Xu and Wengen Zhang*: On the $2k$ th power mean of the character sums over short intervals. Acta Arithmetica 121 (2006), 149–160. zbl
- [6] *Juan C. Peral*: Character sums and explicit estimates for L-functions. Contemporary Mathematics 189 (1995), 449–459.
- [7] *Wenpeng Zhang*: On a Cochrane sum and its hybrid mean value formula (II). Journal of Mathematical Analysis and Applications 276 (2002), 446–457. zbl
- [8] *Wenpeng Zhang, Yuan Yi and Xiali He*: On the $2k$ -th power mean of Dirichlet L-functions with the weight of general Kloosterman sums. Journal of Number Theory 84 (2000), 199–213. zbl
- [9] *Xiali He and Wenpeng Zhang*: On the mean value of Dedekind sums. Acta Mathematica 15 (1999), 245–254. zbl

Authors' addresses: Ren Ganglian, Department of Mathematics, Xianyang normal university, Xianyang , Shaanxi,712000,China, e-mail: rg170718@sina.com; rg170819@hotmail.com Zhang Wen peng, Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R. China, e-mail: wpzhang@nwu.edu.cn.