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GEOMETRIC STRUCTURES OF STABLE OUTPUT FEEDBACK SYSTEMS

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In this paper, we investigate the geometric structures of the stable time-varying and the stable static output feedback systems. Firstly, we give a parametrization of stabilizing time-varying output feedback gains subject to certain constraints, that is, the subset of stabilizing time-varying output feedback gains is diffeomorphic to the Cartesian product of the set of time-varying positive definite matrices and the set of time-varying skew symmetric matrices satisfying certain algebraic conditions. Further, we show how the Cartesian product satisfying certain algebraic conditions is imbedded into the Cartesian product of the set of time-varying positive definite matrices and the set of time-varying skew symmetric matrices. Then, we give some eigenvalue properties of the stable time-varying output feedback systems. Notice that the stable static output feedback system, which does not depend on the temporal parameter t, is just a special case of the stable time-varying output feedback system. Moreover, we use the Riemannian metric, the connections and the curvatures to describe the subset of stabilizing static output feedback gains. At last, we use a static output feedback system to illustrate our conclusions.

Keywords: diffeomorphism, geometric structure, output feedback, immersion

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1. INTRODUCTION

Some scholars have used differential geometric approaches to investigate the structures of linear (dynamical) systems(e. g. [1, 3, 5, 6, 7, 11]). In [1], S. Amari explored a parametric family of invertible linear system, and gave the Riemannian metric, the dual affine connections, and the divergence. In [6] and [7], the authors gave a deep study of the geometric structures of stable static state feedback systems. Further, the authors generalized the conclusions of the stable static state feedback systems to the stable time-varying state feedback systems in [11]. In the present paper, we mainly concern with the stable time-varying output feedback systems corresponding to certain stabilizing time-varying output feedback gains constrained by some conditions.

The set of stabilizing time-varying output feedback gains satisfying some conditions is diffeomorphic to the Cartesian product of the set of time-varying positive definite matrices and the set of time-varying skew symmetric matrices satisfying

certain algebraic conditions. Note that the Lyapunov equation plays an important role in the parametrization procedure and it gives us a criterion to verify the stable matrices. Then, from the fact that the set of time-varying stable matrices is diffeomorphic to the Cartesian product of the set of time-varying positive definite matrices and the set of time-varying skew symmetric matrices ([11]), we introduce a map to show how the Cartesian product satisfying certain algebraic conditions we consider here is imbedded into the Cartesian product of the set of time-varying positive definite matrices and the set of time-varying skew symmetric matrices. In addition, we give some eigenvalue properties of the stable time-varying output feedback systems, which are very important in classical control theory. Next, we obtain the geometric structures of the subset of stabilizing static output feedback gains through investigating its differmorphic set. Studying these structures is important, for it not only provides fundamental information of the subset of stabilizing static output feedback gains, but also gives bounds of performance in the sense of [4] and [10]. This paper provides a geometric approach to analyze the stable output feedback systems and their gains.

Notation.

i) $PD(n)$ denotes the set of $(n \times n)$ positive definite matrices.

ii) $Skew(n)$ denotes the set of $(n \times n)$ skew symmetric matrices.

iii) $Sym(n)$ denotes the set of $(n \times n)$ symmetric matrices.

iv $PD(n,t)$ denotes the set of $(n \times n)$ time-varying positive definite matrices.

v) $Skew(n,t)$ denotes the set of $(n \times n)$ time-varying skew symmetric matrices.

vi) $\mathcal{H}_s(A(t),B(t),C(t))$ denotes the subset of stabilizing time-varying output feedback gains of $\sum(A(t),\ B(t),C(t))$ satisfying (5).

vii) $\mathcal{H}_s(A,B,C)$ denotes the subset of stabilizing static output feedback gains of $\sum(A,B,C)$ satisfying (17).

viii) $\varphi(n,t)$ denotes the set of $(n \times n)$ time-varying stable matrices.

We adopt Einstein's summation convention for the indices which appear twice as sub and superscripts, e. g., $c^k = a_{ij}b^{ijk}$ automatically means $c^k = \sum_i \sum_j a_{ij}b^{ijk}$.

2. PRELIMINARIES

Lemma 2.1. (Ben-Israel and Greville [2]) Let $B^\dagger \in \mathbb{R}^{m\times n}$ be a generalized inverse matrix of B. Then $B^\dagger$ has the following properties:

i) Both $BB^\dagger$ and $I - BB^\dagger$ are symmetric matrices. Furthermore,

$$BB^\dagger B = B, \quad B^\dagger BB^\dagger = B^\dagger, \quad B^T BB^\dagger = B^T.$$

ii) $BB^\dagger$ is orthogonal projection matrix to $\mathrm{Im}B$, and $I - BB^\dagger$ is the orthogonal projection matrix to orthogonal complement of $\mathrm{Im}B$.

Lemma 2.2. (Ben-Israel and Greville [2]) Let $A_1 \in \mathbb{R}^{m\times n}$, $A_2 \in \mathbb{R}^{p\times q}$ and $A_3 \in \mathbb{R}^{m\times q}$. Then, linear matrix equation

$$A_1 X A_2 = A_3$$

can be solved if and only if

$$A_1 A_1^{\dagger} A_3 A_2^{\dagger} A_2 = A_3. \tag{1}$$

Furthermore, if (1) is satisfied, then all the solutions can be given by

$$X = A_1^{\dagger} A_3 A_2^{\dagger} + (\mathcal{Z} - A_1^{\dagger} A_1 \mathcal{Z} A_2 A_2^{\dagger}), \tag{2}$$

where, $\mathcal{Z} \in \mathbb{R}^{n\times q}$ is an arbitrary matrix.

Lemma 2.3. (Zhong, Sun and Zhang [11]) Linear time-varying continuous system

$$\dot{x}(t) = A(t)x(t)$$

is globally asymptotically stable at its equilibrium if and only if, for arbitrary time-varying positive definite matrix $Q(t)$, there exists a time-varying positive definite matrix $P(t)$, such that

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + Q(t) = 0.$$

Lemma 2.4. (Ohara and Amari [7]) The component of the Riemannian metric of $PD(n)$ at P is given by

$$g_{ij}(P) := \frac{1}{2}\mathrm{tr}(P^{-1}E_i P^{-1}E_j), \tag{3}$$

where

$$E_i := E_{\sigma(p,q)} = \begin{cases} E_{pq} & p = q, \\ E_{pq} + E_{qp} & p < q \end{cases}$$

is the basis matrix of $\frac{n(n+1)}{2}$-dimensional vector space $Sym(n)$, E_{pq} is the matrix with one at the (p,q)th element and zero otherwise, and σ is an appropriate rule to assign integers to the pairs (p,q), i. e. $\sigma(p,q) = i$, $1 \le p \le q \le n$ and $1 \le i \le N := \frac{n(n+1)}{2}$.

Two parallel displacements Π_c and Π_c^* of $TPD(n)$ are defined by

$$\Pi_c(t)X = X, \qquad \Pi_c^*(t)X = P(t)P_0^{-1}XP_0^{-1}P(t),$$

for any curve c with initial point P_0 and $X = a^i E_i \in TPD(n)$. Let ∇ and ∇^* denote the corresponding affine connections. It is easy to prove that the pair of connections (∇, ∇^*) derived from (Π_c, Π_c^*) is mutually dual.

Lemma 2.5. (Ohara and Amari [7]) The covariant derivatives with respect to the parallel displacements Π_c and Π_c^* satisfy

$$\nabla_{E_i} E_j = 0 \text{ and } \nabla^*_{E_i} E_j = -E_i P^{-1} E_j - E_j P^{-1} E_i,$$

respectively.

Lemma 2.6. (Ohara and Amari [7]) The component of the fibre metric of $PD(n)\times Skew(n)$ is given by

$$f_{\mu\lambda}(P) := -\frac{1}{2}\mathrm{tr}(P^{-1}\tilde{E}_\mu P^{-1}\tilde{E}_\lambda), \tag{4}$$

where

$$\tilde{E}_\mu := \tilde{E}_{\tilde{\sigma}(p,q)} = E_{pq} - E_{qp}, \quad p < q$$

is the basis matrix of $\frac{n(n-1)}{2}$-dimensional vector space $Skew(n)$, E_{pq} is the matrix with one at the (p,q) th element and zero otherwise, and $\tilde{\sigma}$ is an appropriate rule to assign integers to the pairs (p,q), i. e., $\tilde{\sigma}(p,q) = \mu$, $1 \le p \le q \le n$ and $1 \le \mu \le \tilde{N} := \frac{n(n-1)}{2}$.

Similarly a pair of parallel displacements $(\widetilde{\Pi}_c, \widetilde{\Pi}_c^*)$ for any curve can be defined on $PD(n) \times Skew(n)$ as

$$\widetilde{\Pi}_c(t)S = S, \qquad \widetilde{\Pi}_c^*(t)S = P(t)P_0^{-1}SP_0^{-1}P(t),$$

where $S \in Skew(n)$. The pair of connections $(\widetilde{\nabla}, \widetilde{\nabla}^*)$ derived from $(\widetilde{\Pi}_c, \widetilde{\Pi}_c^*)$ is mutually dual.

Lemma 2.7. (Ohara and Amari [7]) The covariant derivatives with respect to parallel displacements $\widetilde{\Pi}_c$ and $\widetilde{\Pi}_c^*$ satisfy

$$\widetilde{\nabla}_{E_i} \widetilde{E}_\mu = 0 \text{ and } \widetilde{\nabla}^*_{E_i} \widetilde{E}_\mu = -E_i P^{-1} \widetilde{E}_\mu - \widetilde{E}_\mu P^{-1} E_i,$$

respectively.

3. PARAMETRIZATION OF STABILIZING TIME–VARYING OUTPUT FEEDBACK GAINS

Consider the following linear time-varying output feedback system

$$\begin{cases} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ y(t) &= C(t)x(t), \\ u(t) &= H(t)y(t), \end{cases}$$

i. e.,

$$\dot{x}(t) = \Big(A(t) + B(t)H(t)C(t)\Big)x(t),$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $y(t) \in \mathbb{R}^l$ is the output vector, $H(t) \in \mathbb{R}^{m\times l}$ is the output feedback gain. It is also assumed that, for any time t, $\sum(A(t), C(t))$ is observable, $\sum(A(t), B(t))$ is controllable, $B(t)$ is full column-rank and $C(t)$ is full row-rank.

Theorem 3.1. i) If the time-varying output feedback gain $H(t)$ satisfies

$$P(t)B(t)(P(t)B(t))^{\dagger}C^{T}(t)H^{T}(t)B^{T}(t)P(t)C(t)^{\dagger}C(t) = C^{T}(t)H^{T}(t)B^{T}(t)P(t), \quad (5)$$

then $H(t)$ is a stabilizing time-varying output feedback gain, i. e., $H(t)$ satisfies the following Lyapunov equation:

$$\dot{P}(t)+\Big(A(t)+B(t)H(t)C(t)\Big)^{T}P(t)+P(t)\Big(A(t)+B(t)H(t)C(t)\Big)+Q(t)=0, \quad (6)$$

for some $Q(t) \in PD(n,t)$, if and only if $P(t) \in PD(n,t)$ satisfies

$$\begin{aligned} P(t)B(t)(P(t)B(t))^{\dagger}\Big(\dot{P}(t)+A^{T}(t)P(t)+P(t)A(t)+Q(t)\Big)C(t)^{\dagger}C(t) \\ = \dot{P}(t)+A^{T}(t)P(t)+P(t)A(t)+Q(t). \end{aligned} \quad (7)$$

ii) When $P(t) \in PD(n,t)$ satisfies (7), any $H(t)$ satisfying both (5) and (6) is given by

$$\begin{aligned} H(t) = -\frac{1}{2}(P(t)B(t))^{\dagger}\Big(\dot{P}(t)+A^{T}(t)P(t)+P(t)A(t)+Q(t)\Big)C(t)^{\dagger} \\ -(P(t)B(t))^{\dagger}S(t)C(t)^{\dagger}, \end{aligned} \quad (8)$$

where $S(t) \in \mathbb{R}^{n\times n}$ is a time-varying skew symmetric matrix which satisfies

$$S(t) = P(t)B(t)(P(t)B(t))^{\dagger}S(t)C(t)^{\dagger}C(t). \quad (9)$$

P r o o f . To prove the necessity of Theorem 3.1, it is convenient to use

$$\dot{P}(t)+A^{T}(t)P(t)+P(t)A(t)+Q(t)=-\Big(C^{T}(t)H^{T}(t)B^{T}(t)P(t)+P(t)B(t)H(t)C(t)\Big) \quad (10)$$

instead of (6).

Now pre-multiply $P(t)B(t)(P(t)B(t))^{\dagger}$ and post-multiply $C(t)^{\dagger}C(t)$ on the both sides of (10), respectively, then the necessity of (7) under the existence of $H(t)$ satisfying both (5) and (6) is obvious from the properties of generalized inverse of Lemma 2.1.

Conversely, to show that when (7) holds, there exists $H(t)$ satisfying both (5) and (6), we will construct such $H(t)$ using $P(t)$ which satisfies (7). Thus, we set

$$-P(t)B(t)H(t)C(t) = \frac{1}{2}\Big(\dot{P}(t)+A^{T}(t)P(t)+P(t)A(t)+Q(t)\Big)+S(t), \quad (11)$$

where $S(t) \in Skew(n,t)$.

Furthermore, from Lemma 2.2, equation (11) has a solution $H(t)$ if and only if

$$\begin{aligned} P(t)B(t)(P(t)B(t))^{\dagger}\left(\frac{1}{2}\Big(\dot{P}(t)+A^{T}(t)P(t)+P(t)A(t)+Q(t)\Big)+S(t)\right)C(t)^{\dagger}C(t) \\ = \frac{1}{2}\Big(\dot{P}(t)+A^{T}(t)P(t)+P(t)A(t)+Q(t)\Big)+S(t), \end{aligned} \quad (12)$$

and this can be guaranteed by (7) and the assumption of (9).

Then, from (2), the representation of $H(t)$ is given by

$$H(t) = -\tfrac{1}{2}(P(t)B(t))^{\dagger}\Big(\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + Q(t)\Big)C(t)^{\dagger} - (P(t)B(t))^{\dagger}S(t)C(t)^{\dagger}.$$

Substituting (8) into the left-hand side of (5), and combining (7), (9),

$$(C(t)^{\dagger}C(t))^T\Big(\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + Q(t)\Big)((P(t)B(t))^{\dagger})^T(P(t)B(t))^T = \dot{P}(t) + A^T(t)P(t) + P(t)A(t) + Q(t),$$

and
$$S(t) = (C(t)^{\dagger}C(t))^T S(t)((P(t)B(t))^{\dagger})^T(P(t)B(t))^T,$$

we have

$$P(t)B(t)(P(t)B(t))^{\dagger}C^T(t)H^T(t)B^T(t)P(t)C(t)^{\dagger}C(t) = -\frac{1}{2}\Big(\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + Q(t)\Big) + S(t).$$

Then substituting (8) into the right-hand side of (5), we have

$$C^T(t)H^T(t)B^T(t)P(t) = -\frac{1}{2}\Big(\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + Q(t)\Big) + S(t).$$

From the above, we see that this $H(t)$ satisfies (5).

This finishes the proof of the sufficiency and part (ii) of Theorem 3.1. □

Notation. Throughout this paper we have

i) $PD(n,t;A(t),B(t),C(t),Q(t))$ denotes the set of $(n\times n)$ time-varying positive definite matrices satisfying (7).

ii) $Skew(n,t;B(t),P(t),C(t))$ denotes the set of $(n\times n)$ time-varying skew symmetric matrices satisfying (9).

4. IMMERSION

Theorem 3.1 shows that, for a given time-varying positive definite matrix $Q(t)$, any stabilizing time-varying output feedback gain $H(t)$ of $\sum(A(t),B(t),C(t))$, which satisfies (5), can be represented as (8) in terms of $P(t)\in PD(n,t;A(t),B(t),C(t),Q(t))$ and $S(t)\in Skew(n,t;B(t),P(t),C(t))$. In this section, firstly, we will show that $\mathcal{H}_s(A(t),B(t),C(t))$ is diffeomorphic to the Cartesian product $PD(n,t;A(t),B(t),C(t),Q(t)\times Skew(n,t;B(t),P(t),C(t))$.

Theorem 4.1. For a given time-varying positive definite matrix $Q(t)$, there exists a bijective mapping between $\mathcal{H}_s(A(t),B(t),C(t))$ and $PD(n,t;A(t),B(t),C(t),Q(t))\times Skew(n,t;B(t),P(t),C(t))$. Here the symbol $\times$ means the Cartesian product of two sets.

P r o o f . We show that (8) defines a bijective mapping

$$\psi_{Q(t)}: \ PD(n,t;A(t),B(t),C(t),Q(t)) \times Skew(n,t;B(t),P(t),C(t)) \\ \to \mathcal{H}_s(A(t),B(t),C(t)).$$

First of all, $H(t) = \psi_{Q(t)}(P(t),S(t))$ belongs to $\mathcal{H}_s(A(t),B(t),C(t))$ for any $(P(t),S(t)) \in PD(n,t;A(t),\ B(t),C(t),Q(t)) \times Skew(n,t;B(t),P(t),C(t))$ due to Theorem 3.1.

Thus, we should only need to assert that for any $H(t) \in \mathcal{H}_s(A(t),B(t),C(t))$, there exists a unique pair $(P(t),S(t)) \in PD(n,t;A(t),B(t),C(t),Q(t)) \times Skew(n,t;B(t),P(t),C(t))$ such that (8) holds, i. e., there exists a unique inverse of $\psi_{Q(t)}$.

It is easy to see that there exists a unique solution of (6), under the assumption of $t_0 = 0$ and $P(t_0) = 0$ without loss of generality:

$$P(t) = \int_0^t \exp\Big\{(A(t)+B(t)H(t)C(t))^T\tau\Big\}Q(t)\exp\Big\{(A(t)+B(t)H(t)C(t))\tau\Big\}\mathrm{d}\tau, \quad (13)$$

then

$$S(t) = -P(t)B(t)H(t)C(t) - \frac{1}{2}\Big(\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + Q(t)\Big). \quad (14)$$

Therefore, (13) and (14) define the inverse mapping $\psi^{-1}_{Q(t)}$. □

It can be easily seen that both $\psi_{Q(t)}$ and $\psi^{-1}_{Q(t)}$ are of C^∞ class since $\psi_{Q(t)}$ and $\psi^{-1}_{Q(t)}$ are both polynomial functions. So we get the following

Corollary 4.2. The set $\mathcal{H}_s(A(t),B(t),C(t))$ is diffeomorphic to the set $PD(n,t;A(t),B(t),C(t),Q(t)) \times Skew(n,t;B(t),P(t),C(t))$, i. e., $\psi_{Q(t)}$ is a diffeomorphism (bijective and differentiable mapping).

Diffeomorphism preserves topological properties. Hence, this corollary means that the differential geometric structures of $\mathcal{H}_s(A(t),B(t),C(t))$ can be studied by analyzing those of $PD(n,t;A(t),B(t),C(t),Q(t)) \times Skew(n,t;B(t),P(t),C(t))$.

It is obvious that $PD(n,t;A(t),B(t),C(t),Q(t))$ is a subset of $PD(n,t)$, and $Skew(n,t;B(t),P(t),C(t))$ is a subset of $Skew(n,t)$. From [11], we know that for a given time-varying positive definite matrix $Q(t)$, any time-varying stable matrix $A_S(t) \in \varphi(n,t)$ has the form of

$$A_S(t) = -\frac{1}{2}P(t)^{-1}(\dot{P}(t) + Q(t)) + P(t)^{-1}S(t),$$

where $P(t) \in PD(n,t)$ and $S(t) \in Skew(n,t)$. Such a representation defines a diffeomorphism $\Phi_{Q(t)}$ from $PD(n,t) \times Skew(n,t)$ to $\varphi(n,t)$, i. e.,

$$\Phi_{Q(t)} : PD(n,t) \times Skew(n,t) \to \varphi(n,t).$$

Next we will show how $PD(n,t;A(t),B(t),C(t),Q(t)) \times Skew(n,t;B(t),P(t),C(t))$ is imbedded into $PD(n,t) \times Skew(n,t)$.

In fact, for any $(P(t), S(t)) \in PD(n,t;A(t),B(t),C(t),Q(t)) \times Skew(n,t;B(t),P(t),C(t)), H(t)$ can be written as the form of (8), so we have

$$\begin{aligned} A(t)+B(t)H(t)C(t) = & A(t)-\frac{1}{2}B(t)(P(t)B(t))^{\dagger}\Big(\dot{P}(t)+A^{T}(t)P(t)+P(t)A(t) \\ & +Q(t)\Big)C(t)^{\dagger}C(t)-B(t)(P(t)B(t))^{\dagger}S(t)C(t)^{\dagger}C(t). \end{aligned} \tag{15}$$

We can prove that the first term combining the second term in the right-hand side of (15) becomes a time-varying stable matrix, so it can be written as:

$$\begin{aligned} \varphi(n,t) \ni & A(t)-\frac{1}{2}B(t)(P(t)B(t))^{\dagger}\Big(\dot{P}(t)+A^{T}(t)P(t)+P(t)A(t)+Q(t)\Big)C(t)^{\dagger}C(t) \\ & = -\frac{1}{2}P(t)^{-1}(\dot{P}(t)+Q(t))+P(t)^{-1}S_0(P(t)), \end{aligned}$$

where

$$\begin{aligned} S_0(P(t)) = & \frac{1}{2}P(t)^{-1}(\dot{P}(t)+Q(t))+P(t)A(t)-\frac{1}{2}P(t)B(t)(P(t)B(t))^{\dagger}\Big(\dot{P}(t) \\ & +A^{T}(t)P(t)+P(t)A(t)+Q(t)\Big)C(t)^{\dagger}C(t) \end{aligned} \tag{16}$$

satisfying $S_0(P(t)) + S_0^T(P(t)) = 0$, that is $S_0(P(t)) \in Skew(n,t)$. Then

$$\begin{aligned} A(t)+B(t)H(t)C(t) = & -\frac{1}{2}P(t)^{-1}(\dot{P}(t)+Q(t))+P(t)^{-1}S_0(P(t)) \\ & -B(t)(P(t)B(t))^{\dagger}S(t)C(t)^{\dagger}C(t) \\ = & -\frac{1}{2}P(t)^{-1}(\dot{P}(t)+Q(t))+P(t)^{-1}\Big(S_0(P(t))-S(t)\Big). \end{aligned}$$

We denote $\varphi_h(n,t;A(t),B(t),C(t)) = \{A(t)+B(t)H(t)C(t) \mid H(t) \in \mathcal{H}_s(A(t),B(t),C(t))\}$, which is called as the set of stable time-varying output feedback system matrices corresponding to $\mathcal{H}_s(A(t),B(t),C(t))$.

Obviously, the linear mapping $\mathcal{X}$ defined by

$$\mathcal{X} : \mathcal{H}_s(A(t),B(t),C(t)) \ni H(t) \mapsto A(t)+B(t)H(t)C(t) \in \varphi_h(n,t;A(t),B(t),C(t))$$

induces an immersion

$$\begin{aligned} \phi_{Q(t)}^{-1} \circ \mathcal{X} \circ \psi_{Q(t)} : \ & PD(n,t;A(t),B(t),C(t),Q(t)) \times Skew(n,t;B(t),P(t),C(t)) \\ & \to PD(n,t) \times Skew(n,t), \end{aligned}$$

i. e.,

$$\phi_{Q(t)}^{-1} \circ \mathcal{X} \circ \psi_{Q(t)}\ (P(t),S(t)) = (P(t), S_0(P(t)) - S(t)),$$

for arbitrary $(P(t),S(t)) \in PD(n,t;A(t),B(t),C(t),Q(t)) \times Skew(n,t;B(t),P(t),C(t))$.

Next, we give some eigenvalue properties of the stable time-varying output feedback systems. In classical control theory, we investigate the system stability by analyzing the eigenvalue distribution of the system matrix. The following theorem provides us a method to get the expected stability by adjusting the parameters $P(t)$ and $S(t)$.

Theorem 4.3. The region in the complex plane where eigenvalues of the time-varying output feedback matrices $A(t)+B(t)H(t)C(t)$ exist is restricted by $(P(t), S(t)) \in PD(n,t;A(t),B(t),C(t),Q(t)) \times Skew(n,t;B(t),P(t),C(t))$ as

$$\begin{aligned} -\frac{1}{2}\lambda_{\max}\{(\dot{P}(t)+Q(t))P^{-1}(t)\} &\le Re(\lambda\{A(t)+B(t)H(t)C(t)\}) \\ &\le -\frac{1}{2}\lambda_{\min}\{(\dot{P}(t)+Q(t))P^{-1}(t)\}, \\ |\mathrm{Im}(\lambda\{A(t)+B(t)H(t)C(t)\})| &\le \lambda_{\max}\{i(S_0(P(t))-S(t))P^{-1}(t)\}, \end{aligned}$$

where i is the imaginary unit.

The proof of this theorem is similar with that of Theorem 5 in [11], we omit it here.

So far, we have investigated the geometric structures of the stable time-varying output feedback systems. Notice that the stable static output feedback system is just a special case of the stable time-varying output feedback system. Therefore, for the stable static output feedback systems, Theorem 3.1 can be rewritten as the following theorem for the late use.

Theorem 4.4. i) If the static output feedback gain H satisfies

$$PB(PB)^{\dagger}C^{T}H^{T}B^{T}PC^{\dagger}C = C^{T}H^{T}B^{T}P, \tag{17}$$

then H is a stabilizing static output feedback gain, i. e., H satisfies the following Lyapunov equation:

$$(A+BHC)^{T}P + P(A+BHC) + Q = 0, \tag{18}$$

for some $Q \in PD(n)$, if and only if $P \in PD(n)$ satisfies

$$PB(PB)^{\dagger}\Big(A^{T}P + PA + Q\Big)C^{\dagger}C = A^{T}P + PA + Q. \tag{19}$$

ii) When $P \in PD(n)$ satisfies (19), any H satisfying both (17) and (18) is given by

$$H = -\frac{1}{2}(PB)^{\dagger}\Big(A^{T}P + PA + Q\Big)C^{\dagger} - (PB)^{\dagger}SC^{\dagger}, \tag{20}$$

where $S \in \mathbb{R}^{n\times n}$ is a skew symmetric matrix which satisfies

$$S = PB(PB)^{\dagger}SC^{\dagger}C. \tag{21}$$

Then, we use $PD(n;A,B,C,Q)$ to denote the set of $(n \times n)$ positive definite matrices satisfying (19), and $Skew(n;B,P,C)$ to denote the set of $(n \times n)$ skew symmetric matrices satisfying (21). We see that $\mathcal{H}_s(A,B,C)$ is diffeomorphic to the Cartesian product $PD(n;A,B,C,Q) \times Skew(n;B,P,C)$, and $PD(n;A,B,C,Q) \times Skew(n;B,P,C)$ is imbedded into $PD(n) \times Skew(n)$ in the similar way obtained above. In the next section, we give the Riemannian metric, the connections and the curvatures of the parameter space for $\mathcal{H}_s(A,B,C)$.

5. GEOMETRIC STRUCTURES OF THE PARAMETER SPACE FOR $\mathcal{H}_S(A, B, C)$

In [7], A. Ohara and S. Amari defined the Riemannian metric, the connections, and other fundamental quantities for the differential geometric structures of $PD(n) \times Skew(n)$. Induced from these geometric quantities, we will exploit the geometric structures of vector bundle $PD(n; A, B, C, Q) \times Skew(n; B, P, C)$ which is diffeomorphic to $\mathcal{H}_s(A, B, C)$ and can be regarded as the parameter space for $\mathcal{H}_s(A, B, C)$.

In this section, $\{i, j, \cdots\}$, $\{a, b, \cdots\}$, $\{\lambda, \mu, \cdots\}$, $\{\alpha, \beta, \cdots\}$ describe the indices of the components of $PD(n)$, $PD(n; A, B, C, Q)$, $Skew(n)$ and $Skew(n; B, P, C)$, respectively.

Let $E_i, i = 1, 2, \ldots, N = \frac{n(n+1)}{2}$ be the linearly independent basis matrices of $Sym(n)$, then any $P \in PD(n)$ can be represented as

$$P = P(\eta) := \eta^i E_i.$$

Hence, we can regard $\eta = (\eta^i)$ as a global coordinate system for $PD(n)$ and $\partial_i := \frac{\partial}{\partial \eta^i}$ as a tangent vector field on $PD(n)$.

$PD(n; A, B, C, Q)$ is a submanifold of $PD(n)$. Denote the tangent vector space of $PD(n; A, B, C, Q)$ at a point $P \in PD(n; A, B, C, Q)$ as $T_P PD(n; A, B, C, Q)$, and $T_P^{\perp} PD(n; A, B, C, Q)$ the orthogonal complement of $T_P PD(n; A, B, C, Q)$.

The Euler–Schouten(imbedding) curvature tensors of the submanifold $PD(n; A, B, C, Q)$ in $PD(n)$ with respect to ∇ and ∇^* are defined by

$$H_{abl} := (\nabla_{\partial_a} \partial_b, \partial_l), \quad H^*_{abl} := (\nabla^*_{\partial_a} \partial_b, \partial_l),$$

where ∂_a, ∂_b denote the tangent vector fields on $T_P PD(n; A, B, C, Q)$, and ∂_l denotes the tangent vector field on $T_P^{\perp} PD(n; A, B, C, Q)$. These quantities show how curve the submanifold $PD(n; A, B, C, Q)$ in $PD(n)$ in the sense of the connections ∇ and ∇^*. When $H_{abl}(H^*_{abl})$ is zero, the submanifold $PD(n; A, B, C, Q)$ is said to be ∇-autoparallel(∇^*-autoparallel).

Using (19) which specifies the submanifold $PD(n; A, B, C, Q)$ in $PD(n)$, we first construct the coordinate system (x^a) for $PD(n; A, B, C, Q)$, and then define the induced Riemannian metric and the induced connections.

Proposition 5.1. Any $P \in PD(n; A, B, C, Q)$ can be represented as

$$P(x) = E_0 + x^a E_a, \tag{22}$$

where E_0 is the certain part of P, and $E_0 = \eta_0^i E_i$, $E_a = B_a^i E_i$, $1 \le i \le N = \frac{n(n+1)}{2}$. Here, $x = (x^a)$ can be regarded as a coordinate system of the submanifold $PD(n; A, B, C, Q)$.

Proof. (19) can be considered as the non-linear equations with respect to the components of $P \in PD(n; A, B, C, Q)$. And these equations can determine some parts of P, so any $P \in PD(n; A, B, C, Q)$ has the representation of (22). □

The global coordinate η of $P(x) \in PD(n; A, B, C, Q)$, and the tangent vector field $\partial_a = \frac{\partial}{\partial x^a}$ on $PD(n; A, B, C, Q)$ are represented as

$$\eta^i(x) = \eta_0^i + B_a^i x^a, \quad \partial_a = B_a^i \partial_i,$$

where $B_a^i = \frac{\partial \eta^i}{\partial x^a}, \partial_i = \frac{\partial}{\partial \eta^i}$.

So the components of the Riemannian metric and the dual connections on $PD(n; A, B, C, Q)$ are induced from those of $PD(n)$ as

$$\begin{aligned} g_{ab}(x) &= B_a^i B_b^j g_{ij}(\eta(x)), \\ \Gamma_{abc}(x) &= B_a^i B_b^j B_c^k \Gamma_{ijk}(\eta(x)) + (\partial_a B_b^j) B_c^k g_{jk}(\eta(x)) = 0, \\ \Gamma^*_{abc}(x) &= B_a^i B_b^j B_c^k \Gamma^*_{ijk}(\eta(x)) + (\partial_a B_b^j) B_c^k g_{jk}(\eta(x)) = B_a^i B_b^j B_c^k \Gamma^*_{ijk}(\eta(x)), \end{aligned} \tag{23}$$

where the component of the connection Γ_{ijk} on $PD(n)$ is equal to 0, and B_b^j is constant.

Theorem 5.2. The submanifold $PD(n; A, B, C, Q)$ is ∇-autoparallel in $PD(n)$.

Proof. The Euler–Schouten curvature $H_{abl} := (\nabla_{\partial_a} \partial_b, \partial_l)$, where $\partial_a, \partial_b \in T_P PD(n;\ A, B, C, Q)$, $\partial_l \in T_P^{\perp} PD(n; A, B, C, Q)$, is

$$\begin{aligned} H_{abl} &= (\nabla_{\partial_a} \partial_b, \partial_l) \\ &= B_a^i B_l^k (\nabla_{\partial_i} B_b^j \partial_j, \partial_k) \\ &= \partial_a (B_b^j) B_l^k g_{jk} + B_a^i B_b^j B_l^k \Gamma_{ijk} \\ &= 0, \end{aligned}$$

for (B_b^j) is constant, and Γ_{ijk}=0 with respect to ∇ in $PD(n)$. □

It is easy to see

Corollary 5.3. The submanifold $PD(n; A, B, C, Q)$ is itself ∇-flat and ∇^*-flat.

Using (21), here, we only consider the case that S can be represented as

$$S = \widetilde{E}_0 + y^\alpha \widetilde{E}_\alpha, \tag{24}$$

where $\widetilde{E}_0$ is the certain part of S, and $\widetilde{E}_0 = \zeta_0^\lambda \widetilde{E}_\lambda$, $\widetilde{E}_\alpha = B_\alpha^\lambda \widetilde{E}_\lambda$, $\widetilde{E}_\lambda$ is the basis of $Skew(n)$, $1 \le i \le \widetilde{N} = \frac{n(n-1)}{2}$. Here, $y = (y^\alpha)$ can be regarded as a coordinate system of the submanifold $Skew(n; B, P, C)$.

The global coordinate ζ of $S \in Skew(n; B, P, C)$, and the tangent vector field $\partial_\alpha = \frac{\partial}{\partial y^\alpha}$ on $Skew(n; B,\ P, C)$ are represented as

$$\zeta^\lambda(y) = \zeta_0^\lambda + \widetilde{B}_\alpha^\lambda y^\alpha, \quad \partial_\alpha = \widetilde{B}_\alpha^\lambda \partial_\lambda,$$

where $\widetilde{B}_\alpha^\lambda = \frac{\partial \zeta^\lambda}{\partial y^\alpha}$, $\partial_\lambda = \frac{\partial}{\partial \zeta^\lambda}$.

So in this fibre case, we can induce the fibre metric and the dual connections of $PD(n;A,B,C,Q)\times Skew(n;B,P,C)$ from $PD(n)\times Skew(n)$ as

$$\begin{aligned} f_{\alpha\beta}(x) &= \widetilde{B}^{\mu}_{\alpha}\widetilde{B}^{\lambda}_{\beta}f_{\mu\lambda}(\eta(x)),\\ \widetilde{\Gamma}_{a\alpha\beta}(x) &= B^{i}_{a}\widetilde{B}^{\mu}_{\alpha}\widetilde{B}^{\lambda}_{\beta}\widetilde{\Gamma}_{i\mu\lambda}(\eta(x))+(\partial_{a}\widetilde{B}^{\mu}_{\alpha})\widetilde{B}^{\lambda}_{\beta}f_{\mu\lambda}(\eta(x))=0, \qquad (25)\\ \widetilde{\Gamma}^{*}_{a\alpha\beta}(x) &= B^{i}_{a}\widetilde{B}^{\mu}_{\alpha}\widetilde{B}^{\lambda}_{\beta}\widetilde{\Gamma}^{*}_{i\mu\lambda}(\eta(x))+(\partial_{a}\widetilde{B}^{\mu}_{\alpha})\widetilde{B}^{\lambda}_{\beta}f_{\mu\lambda}(\eta(x))=B^{i}_{a}\widetilde{B}^{\mu}_{\alpha}\widetilde{B}^{\lambda}_{\beta}\widetilde{\Gamma}^{*}_{i\mu\lambda}(\eta(x)), \end{aligned}$$

where the component of the connection $\widetilde{\Gamma}_{i\mu\lambda}$ on $PD(n)\times Skew(n)$ is equal to 0, and $\widetilde{B}^{\mu}_{\alpha}$ is constant.

Theorem 5.4. The vector bundle $PD(n;A,B,C,Q)\times Skew(n;B,P,C)$ is $\widetilde{\nabla}$-autoparallel in $PD(n)\times Skew(n)$.

Proof. The component of the Euler–Schouten curvature is given by

$$\widetilde{H}_{a\alpha\bar{k}}=f_{P}(\widetilde{\nabla}_{E_a}\widetilde{E}_{\alpha},\widetilde{E}_{\bar{k}})=B^{i}_{a}\widetilde{B}^{\lambda}_{\alpha}\widetilde{B}^{k}_{\bar{k}}\widetilde{\Gamma}_{i\lambda k}+\partial_{a}(\widetilde{B}^{\lambda}_{\alpha})\widetilde{B}^{k}_{\bar{k}}f_{\lambda k},$$

where $E_a\in T_PPD(n;A,B,C,Q)$, $\widetilde{E}_{\alpha}\in TSkew(n;B,P,C)$, and $\widetilde{E}_{\bar{k}}\in T^{\perp}Skew(n;B,P,C)$.

Since the component of the connection of $PD(n)\times Skew(n)$ is equal to zero([7]), that is, $\widetilde{\Gamma}_{i\lambda k}=0$, and $\widetilde{B}^{\lambda}_{\alpha}$ is constant, we get

$$\widetilde{H}_{a\alpha\tilde{k}}=0.$$

This finishes the proof of Theorem 5.4. □

It is easy to see that

Corollary 5.5. The vector bundle $PD(n;A,B,C,Q)\times Skew(n;B,P,C)$ is $\widetilde{\nabla}$-flat and $\widetilde{\nabla}^{*}$-flat vector bundle, i. e., its curvature vanishes.

6. EXAMPLE

Consider the following linear static output feedback system:

$$\begin{cases} \dot{x}(t) &= Ax(t)+Bu(t),\\ y(t) &= Cx(t),\\ u(t) &= Hy(t), \end{cases}$$

where

$$A=\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0 \end{pmatrix},\quad B=\begin{pmatrix} 1 & 0\\ 0 & 1\\ 0 & 0 \end{pmatrix},\quad C=\begin{pmatrix} 1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3 \end{pmatrix}.$$

It is easily verified that (A, C) is observable, B is full column-rank and C is inverse. Define $Q = I \in \mathbb{R}^{3\times 3}$ and consider the Lyapunov equation (18). Represent $P \in PD(3)$ and $S \in Skew(3)$ as

$$P = \begin{pmatrix} \eta^1 & \eta^2 & \eta^3 \\ \eta^2 & \eta^4 & \eta^5 \\ \eta^3 & \eta^5 & \eta^6 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -\zeta^1 & -\zeta^2 \\ \zeta^1 & 0 & -\zeta^3 \\ \zeta^2 & \zeta^3 & 0 \end{pmatrix}.$$

Using (19), (21), the pseudo-inverse matrices of B and the inverse of C:

$$B^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix},$$

we get $\eta^5 = -\eta^1, \eta^3 = -\frac{1}{2}, \eta^6 = -\eta^2$, and $\zeta^2 = \zeta^3 = 0$. Thus, any $P \in PD(3; A, B, C, I)$ and $S \in Skew(3; B, C, P)$ are of the forms:

$$P = \begin{pmatrix} \eta^1 & \eta^2 & -\frac{1}{2} \\ \eta^2 & \eta^4 & -\eta^1 \\ -\frac{1}{2} & -\eta^1 & -\eta^2 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & \zeta_1 & 0 \\ -\zeta_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{26}$$

where requires $\eta^1 > 0$, $\eta^1\eta^4 - (\eta^2)^2 > 0$, $-\eta^1\eta^2\eta^4 + \eta^1\eta^2 - \eta^4 - (\eta^1)^3 + (\eta^2)^3 > 0$.

Furthermore, the stabilizing output feedback gains matrix of $\sum(A, B, C)$ is

$$H = -\frac{1}{2|P|}\begin{pmatrix} -a_1b_1 + a_2b_2 & \frac{1}{2}a_2b_1 + a_3b_2 & a_4b_1 + a_5b_2 \\ -a_1b_3 + a_2b_4 & -\frac{1}{2}a_2b_3 + a_3b_4 & -a_4b_3 + a_5b_4 \end{pmatrix},$$

where $a_1 = \eta^2\eta^4 + (\eta^1)^2$, $a_2 = \frac{1}{2}\eta^1 + (\eta^2)^2$, $a_3 - \frac{1}{2}\eta^1\eta^2 - \frac{1}{8}$, $a_4 = -\frac{1}{3}\eta^1\eta^2 + \frac{1}{6}\eta^4$, $a_5 = \frac{1}{3}(\eta^1)^2 - \frac{1}{6}\eta^2$, $b_1 = 2\eta^2 + 1$, $b_2 = \eta^4 - \frac{1}{2} + 2\zeta^1$, $b_3 = \eta^4 - \frac{1}{2} - 2\zeta^1$, $b_4 = 1 - 2\eta^1$, and $|P| = -\eta^1\eta^2\eta^4 + \eta^1\eta^2 - \eta^4 - (\eta^1)^3 + (\eta^2)^3$.

Then we can obtain $S_0(P) \in Skew(3)$ as

$$S_0(P) = \begin{pmatrix} 0 & \frac{1}{2}\eta^4 + \frac{1}{4} & -\eta^1 \\ -\frac{1}{2}\eta^4 - \frac{1}{4} & 0 & -\eta^2 \\ \eta^1 & \eta^2 & 0 \end{pmatrix}.$$

The stable output feedback system matrix we consider here is expressed as

$$\begin{aligned} A + BHC = & \frac{1}{|P|}\begin{pmatrix} -\frac{1}{2} & \frac{1}{4}\eta^4 + \frac{1}{4} - \zeta^1 & -\eta^1 \\ -\frac{1}{4}\eta^4 - \frac{1}{4} + \zeta^1 & -\frac{1}{2} & -\eta^2 \\ \eta^1 & \eta^2 & -\frac{1}{2} \end{pmatrix} \\ & \times \begin{pmatrix} -\eta^2\eta^4 - (\eta^1)^2 & \frac{1}{2}\eta^1 + (\eta^2)^2 & -\eta^1\eta^2 + \frac{1}{2}\eta^4 \\ \frac{1}{2}\eta^1 + (\eta^2)^2 & -\eta^1\eta^2 - \frac{1}{4} & (\eta^1)^2 - \frac{1}{2}\eta^2 \\ -\eta^1\eta^2 + \frac{1}{2}\eta^4 & (\eta^1)^2 - \frac{1}{2}\eta^2 & \eta^1\eta^4 - (\eta^2)^2 \end{pmatrix}. \end{aligned}$$

The set of $PD(3; A, B, C, I) \times Skew(3; B, P, C)$ is imbedded in $PD(3) \times Skew(3)$ in this way.

Before we give the Riemannian metric of $PD(3;A,B,C,I)$, we consider the metric of $PD(3)$ and $PD(3)\times Skew(3)$. Since basis vectors of $TPD(3)$ can be represented as

$$E_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad E_2=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\quad E_3=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},$$

$$E_4=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad E_5=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad E_6=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},$$

using (3), we get the components of the Riemannian metric of $PD(3)$,

$$\begin{aligned}
g_{11}&=\frac{1}{2|P|^2}(\eta^4\eta^6-(\eta_5)^2)^2,\quad g_{12}=\frac{1}{|P|^2}(\eta^4\eta^6-(\eta^5)^2)(\eta^3\eta^5-\eta^2\eta^6),\\
g_{13}&=\frac{1}{|P|^2}(\eta^4\eta^6-(\eta^5)^2)(\eta^2\eta^5-\eta^3\eta^4),\quad g_{14}=\frac{1}{2|P|^2}(\eta^3\eta^5-\eta^2\eta^6)^2,\\
g_{15}&=\frac{1}{|P|^2}(\eta^3\eta^5-\eta^2\eta^6)(\eta^2\eta^5-\eta^3\eta^4),\quad g_{16}=\frac{1}{2|P|^2}(\eta^2\eta^5-\eta^3\eta^4)^2,\\
g_{22}&=\frac{1}{|P|^2}\Big((\eta^3\eta^5-\eta^2\eta^6)^2+(\eta^1\eta^6-(\eta^3)^2)(\eta^4\eta^6-(\eta^5)^2)\Big),\\
g_{23}&=\frac{1}{|P|^2}\Big((\eta^3\eta^5-\eta^2\eta^6)(\eta^2\eta^5-\eta^3\eta^4)+(\eta^4\eta^6-(\eta^5)^2)(\eta^2\eta^3-\eta^1\eta^5)\Big),\\
g_{24}&=\frac{1}{|P|^2}(\eta^1\eta^6-(\eta^3)^2)(\eta^3\eta^5-\eta^2\eta^6),\\
g_{25}&=\frac{1}{|P|^2}\Big((\eta^1\eta^6-(\eta^3)^2)(\eta^2\eta^5-\eta^3\eta^4)+(\eta^3\eta^5-\eta^2\eta^6)(\eta^2\eta^3-\eta^1\eta^5)\Big),\\
g_{26}&=\frac{1}{|P|^2}(\eta^2\eta^3-\eta^1\eta^5)(\eta^2\eta^5-\eta^3\eta^4),\\
g_{33}&=\frac{1}{|P|^2}\Big((\eta^2\eta^5-\eta^3\eta^4)^2+(\eta^1\eta^4-(\eta^2)^2)(\eta^4\eta^6-(\eta^5)^2)\Big),\\
g_{34}&=\frac{1}{|P|^2}(\eta^2\eta^3-\eta^1\eta^5)(\eta^3\eta^5-\eta^2\eta^6),\\
g_{35}&=\frac{1}{|P|^2}\Big((\eta^2\eta^3-\eta^1\eta^5)(\eta^2\eta^5-\eta^3\eta^4)+(\eta^3\eta^5-\eta^2\eta^6)(\eta^4\eta^6-(\eta^5)^2)\Big),\\
g_{36}&=\frac{1}{|P|^2}(\eta^2\eta^3-\eta^1\eta^5)(\eta^3\eta^5-\eta^2\eta^6),\\
g_{44}&=\frac{1}{2|P|^2}(\eta^1\eta^6-(\eta^3)^2)^2,\quad g_{45}=\frac{1}{|P|^2}(\eta^1\eta^6-(\eta^3)^2)(\eta^2\eta^3-\eta^1\eta^5),\\
g_{46}&=\frac{1}{2|P|^2}(\eta^2\eta^3-\eta^1\eta^5)^2,\\
g_{55}&=\frac{1}{|P|^2}\Big((\eta^2\eta^3-\eta^1\eta^5)^2+(\eta^1\eta^6-(\eta^3)^2)(\eta^1\eta^4-(\eta^2)^2)\Big),
\end{aligned}\tag{27}$$

$$g_{56} = \frac{1}{|P|^2}(\eta^1\eta^4 - (\eta^2)^2)(\eta^2\eta^3 - \eta^1\eta^5), \quad g_{66} = \frac{1}{2|P|^2}(\eta^1\eta^4 - (\eta^2)^2)^2. \tag{28}$$

Since the basis vectors of $Skew(3)$ can be represented as

$$\widetilde{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \widetilde{E}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \widetilde{E}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

using (4), the components of the fibre metric of $PD(3) \times Skew(3)$ are given by

$$\begin{aligned}
f_{11} &= -\frac{1}{|P|^2}\Big((\eta^2\eta^6 - \eta^3\eta^5)^2 + (\eta^4\eta^6 - (\eta^5)^2)((\eta^3)^2 - \eta^1\eta^6)\Big), \\
f_{12} &= -\frac{1}{|P|^2}\Big((\eta^2\eta^6 - \eta^3\eta^5)(\eta^3\eta^4 - \eta^2\eta^5) + (\eta^4\eta^6 - (\eta^5)^2)(\eta^1\eta^5 - \eta^2\eta^3)\Big), \\
f_{13} &= -\frac{1}{|P|^2}\Big(((\eta^3)^2 - \eta^1\eta^6)(\eta^3\eta^4 - \eta^2\eta^5) + (\eta^3\eta^5 - \eta^2\eta^6)(\eta^1\eta^5 - \eta^2\eta^3)\Big), \\
f_{22} &= -\frac{1}{|P|^2}\Big((\eta^3\eta^4 - \eta^2\eta^5)^2 + (\eta^4\eta^6 - (\eta^5)^2)((\eta^2)^2 - \eta^1\eta^4)\Big), \\
f_{23} &= -\frac{1}{|P|^2}\Big((\eta^1\eta^5 - \eta^2\eta^3)(\eta^3\eta^4 - \eta^2\eta^5) + (\eta^3\eta^5 - \eta^2\eta^6)((\eta^2)^2 - \eta^1\eta^4)\Big), \\
f_{33} &= -\frac{1}{|P|^2}\Big((\eta^1\eta^5 - \eta^2\eta^3)^2 + (\eta^1\eta^6 - (\eta^3)^2)((\eta^2)^2 - \eta^1\eta^4)\Big).
\end{aligned} \tag{29}$$

From (26), we can see that any $P \in PD(3; A, B, C, I)$ can be rewritten as

$$P(x) = \begin{pmatrix} x^1 & x^2 & -\frac{1}{2} \\ x^2 & x^3 & -x^1 \\ -\frac{1}{2} & -x^1 & -x^2 \end{pmatrix},$$

where $x = (x^1, x^2, x^3)$ can be considered as a coordinate system of $PD(3; A, B, C, I)$.

Thus, we get the relations between the coordinate system $\eta = (\eta^1, \eta^2, \eta^3, \eta^4, \eta^5, \eta^6)$ of $PD(3)$ and the coordinate system $x = (x^1, x^2, x^3)$ of $PD(3; A, B, C, I)$,

$$\begin{cases}
\eta^1 = x^1 = B_1^1 x^1, & B_1^1 = 1, \\
\eta^2 = x^2 = B_2^2 x^2, & B_2^2 = 1, \\
\eta^3 = -\dfrac{1}{2}, & \\
\eta^4 = x^3 = B_3^4 x^3, & B_3^4 = 1, \\
\eta^5 = -x^1 = B_1^5 x^1, & B_1^5 = -1, \\
\eta^6 = -x^2 = B_2^6 x^2, & B_2^6 = -1.
\end{cases}$$

Combining (23), (27) and (28), we get the components of the Riemannian metric

$G' = (g'_{ij})$ of $PD(3; A, B, C, I)$ which induces from $PD(3)$,

$$\begin{aligned}
g'_{11} =& B_1^1 B_1^1 g_{11} + 2B_1^5 B_1^1 g_{15} + B_1^5 B_1^5 g_{55} \\
=& \frac{1}{2|P|^2}((x^1)^2 + x^2x^3)^2 - \frac{2}{|P|^2}((x^2)^2 + \frac{1}{2}x^1)(\frac{1}{2}x^3 - x^1x^2) \\
&+ \frac{1}{|P|^2}\Big(((x^1)^2 - \frac{1}{2}x^2)^2 + (x^1x^2 + \frac{1}{4})((x^2)^2 - x^1x^3)\Big), \\
g'_{12} =& B_1^1 B_2^2 g_{12} + B_1^1 B_2^6 g_{16} + B_1^5 B_2^2 g_{25} + B_1^5 B_2^6 g_{56} \\
=& -\frac{1}{|P|^2}((x^1)^2 + x^2x^3)((x^2)^2 + \frac{1}{2}x^1) - \frac{1}{2|P|^2}(\frac{1}{2}x^3 - x^1x^2)^2 \\
&- \frac{1}{|P|^2}\Big((x^1x^2 + \frac{1}{4})(x^1x^2 - \frac{1}{2}x^3) + ((x^2)^2 + \frac{1}{2}x^1)((x^1)^2 - \frac{1}{2}x^2)\Big) \\
&+ \frac{1}{|P|^2}(x^1x^3 - (x^2)^2)((x^1)^2 - \frac{1}{2}x^2), \\
g'_{13} =& B_1^1 B_3^4 g_{14} + B_1^5 B_3^4 g_{45} \\
=& \frac{1}{2|P|^2}((x^2)^2 + \frac{1}{2}x^1)^2 - \frac{1}{|P|^2}(x^1x^2 + \frac{1}{4})(\frac{1}{2}x^2 - (x^1)^2), \\
g'_{22} =& B_2^2 B_2^2 g_{22} + 2B_2^2 B_2^6 g_{26} + B_2^6 B_2^6 g_{66} \\
=& \frac{1}{|P|^2}\Big(((x^2)^2 + \frac{1}{2}x^1)^2 + (x^1x^2 + \frac{1}{4})((x^1)^2 + x^2x^3)\Big) \\
&- \frac{2}{|P|^2}((x^1)^2 - \frac{1}{2}x^2)(\frac{1}{2}x^3 - x^1x^2) + \frac{1}{2|P|^2}((x^2)^2 - x^1x^3)^2, \\
g'_{23} =& B_2^2 B_3^4 g_{24} + B_2^6 B_3^4 g_{46} \\
=& -\frac{1}{|P|^2}(x^1x^2 + \frac{1}{4})((x^2)^2 + \frac{1}{2}x^1) - \frac{1}{2|P|^2}((x^1)^2 - \frac{1}{2}x^2)^2, \\
g'_{33} =& B_3^4 B_3^4 g_{44} \\
=& \frac{1}{2|P|^2}(x^1x^2 + \frac{1}{4})^2,
\end{aligned}$$

where $|P| = -(x^1)^3 + (x^2)^3 - x^1x^2x^3 + x^1x^2 - \frac{1}{4}x^3$.

Let $y = (y^1)$ be a coordinate system of $Skew(3, B, P, C)$, we get the relations between y and the coordinate system $\zeta = (\zeta^1, \zeta^2, \zeta^3)$ of $Skew(3)$ in the following

$$\begin{cases} \zeta^1 = y^1 = \widetilde{B}_1^1 x^1, & \widetilde{B}_1^1 = 1, \\ \zeta^2 = 0, \\ \zeta^3 = 0. \end{cases}$$

Combining (25) with (29), we get the the fibre metric $F' = (f'_{11})$ of $PD(3; A, B, C, I) \times Skew(3, B, P, C)$ which induces from $PD(3) \times Skew(3)$,

$$\begin{aligned}
f'_{11} =& \widetilde{B}_1^1 \widetilde{B}_1^1 f_{11} \\
=& -\frac{1}{|P|^2}\Big(((x^2)^2 + \frac{1}{2}x^1)^2 - ((x^1)^2 + x^2x^3)(x^1x^2\frac{1}{4})\Big),
\end{aligned}$$

where $|P| = -(x^1)^3 + (x^2)^3 - x^1x^2x^3 + x^1x^2 - \frac{1}{4}x^3$.

7. CONCLUSIONS

This paper gives a geometric method to investigate the stable time-varying output feedback systems corresponding to certain stabilizing time-varying output feedback gains constrained by some conditions. The present paper shows that, for the stable time-varying output feedback systems, $\mathcal{H}_s(A(t), B(t), C(t))$ is diffeomorphic to $PD(n, t; A(t), B(t), C(t), Q(t)) \times Skew(n, t; B(t), P(t), C(t))$ which also can be considered as a parametrization of $\mathcal{H}_s(A(t), B(t), C(t))$. For the stable static output feedback systems, by imbedding the set $PD(n; A, B, C, Q) \times Skew(n; B, P, C)$ into $PD(n) \times Skew(n)$, we induce the geometric structures of the subset of stabilizing static output feedback gains. In addition, we obtain some properties of eigenvalues of the stable time-varying output feedback systems, which provide us a method to get the expected stability of the stable time-varying output feedback systems by adjusting the parameters $P(t)$ and $S(t)$. However, it is a pity that we can not find the equal conditions for the set of all stabilizing time-varying output feedback gains so far. This remains as a future research.

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REFERENCES

[1] S. Amari: Differential geometry of a parametric family of invertible linear systems-Riemannian metric, dual affine connections, and divergence. Math. Systems Theory *20* (1987), 53–83.

[2] A. Ben-Israel and T. N. E Greville: Generalized Inverses. Wiley, New York 1972.

[3] D. F. Delchamps: Global structure of families of multivariable linear systems with an application to identification. Math. Systems Theory *18* (1985), 329–380.

[4] A. Hotz and R. E. Skelton: Covariance control theory. Internat. J. Control *46* (1987), 13–32.

[5] P. S. Krishnaprasad: Symplectic mechanics and rational functions. Ricerche Automat. *10* (1979), 107–135.

[6] A. Ohara and T. Kitamori: Geometric structures of stable state feedback systems. IEEE Trans. Automat. Control *38* (1993), 1579–1583.

[7] A. Ohara and S. Amari: Differential geometric structures of stable state feedback systems with dual connections. Kybernetika *30* (1994), 369–386.

[8] A. Ohara, S. Nakazumi, and N. Suda: Relations between a parametrization of Stabilizing state feedback gains and eigenvalue locations. Systems Control Lett. *16* (1991), 261–266.

[9] A. Ohara, N. Suda, and S. Amari: Dualistic Differential geometry of positive definite matrices and its applications to related problems. Linear Algebra Appl. *247* (1996), 31–53.

[10] A. Ohara and T. Kitamori: Robust stabilization for structurally perturbed plants by assigning a Lyapunov equation's solution. (In Japanese.) Trans. SICE *25* (1989), 682–689.

[11] F. Zhong, H. Sun, and Z. Zhang: Geometric structures of stable time-variant state feedback systems. J. Beijing Institute of Technology *16* (2007), 500–504.

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