Czechoslovak Mathematical Journal, 60 (135) (2010), 1161–1169

ON THE SUM OF POWERS OF LAPLACIAN EIGENVALUES OF BIPARTITE GRAPHS

Bo Zhou, Guangzhou, Aleksandar Ilić, Niš

(Received October 16, 2009)

Abstract. For a bipartite graph G and a non-zero real α, we give bounds for the sum of the αth powers of the Laplacian eigenvalues of G using the sum of the squares of degrees, from which lower and upper bounds for the incidence energy, and lower bounds for the Kirchhoff index and the Laplacian Estrada index are deduced.

Keywords: Laplacian eigenvalues, incidence energy, Kirchhoff index, Laplacian Estrada index

MSC 2010: 05C50, 05C90

1. Introduction

Let G be a simple finite undirected graph with n vertices and m edges. Let $\mathbf{A}(G)$ be the $(0,1)$ adjacency matrix and $\mathbf{D}(G)$ the diagonal matrix of vertex degrees of G. The matrix $\mathbf{L}(G) = \mathbf{D}(G) - \mathbf{A}(G)$ is known as the Laplacian matrix of G. Denote by $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_{n-1} \geqslant \mu_n$ the Laplacian eigenvalues of G, that is the eigenvalues of $\mathbf{L}(G)$. It is known that $\mu_n = 0$ and the multiplicity of zero is equal to the number of connected components of G, see [2]. For more details on the Laplacian eigenvalues, see [3], [4].

Let α be a real number. Let $s_\alpha(G)$ be the sum of the αth powers of the non-zero Laplacian eigenvalues of G,

$$s_\alpha(G) = \sum_{i=1}^{h} \mu_i^\alpha,$$

The work was supported by the Guangdong Provincial Natural Science Foundation of China (Grant No. 8151063101000026) and Research Grant 144007 of Serbian Ministry of Science and Technological Development.

where h is the number of non-zero Laplacian eigenvalues of G. The cases $\alpha = 0, 1$ are trivial as $s_0(G) = h$ and $s_1(G) = 2m$, where m is the number of edges. Various lower and upper bounds for the sum of powers of Laplacian eigenvalues have been established in [16], [17]. For a graph G, denote by $Z(G)$ the sum of the squares of degrees of G. It is known as the first Zagreb index in mathematical chemistry [5], [15]. Let d_v be the degree of the vertex v of the graph G. Note that

$$s_2(G) = \sum_{v \in V(G)} (d_v^2 + d_v) = Z(G) + 2m,$$

see [11], and $s_{1/2}(G)$ is a kind of incidence energy [6].

For a connected graph G with n vertices, $ns_{-1}(G)$ is equal to its Kirhhoff index [7], [13]. The Laplacian Estrada index of the graph G is defined as

$$\mathrm{LEE}(G) = \sum_{i=1}^{n} \mathrm{e}^{\mu_i}.$$

For further results concerning $\mathrm{LEE}(G)$, see [17], [1], [20].

In this paper we give new bounds for s_α of bipartite graphs using the sum of the squares of degrees, where α is a non-zero real number. We discuss lower and upper bounds for the incidence energy, and lower bounds for the Kirchhoff index and the Laplacian Estrada index of bipartite graphs.

2. Preliminaries

Let $K_{a,b}$ be the complete bipartite graph with two partite sets having a and b vertices, respectively. Let $\mathcal{L}(G)$ be the line graph of G.

Lemma 1 ([3])**.** *Let G be a connected graph with diameter d. Then G has at least $d + 1$ distinct Laplacian eigenvalues.*

For a bipartite graph G, its Laplacian eigenvalues are just its signless Laplacian eigenvalues, i.e., eigenvalues of the signless Laplacian matrix $\mathbf{D}(G) + \mathbf{A}(G)$. Thus, we have

Lemma 2 ([18], [23], [19])**.** *Let G be a bipartite graph with $m \geqslant 1$ edges. Then*

$$\mu_1 \geqslant \frac{Z(G)}{m}$$

with equality if and only if $\mathcal{L}(G)$ is regular.

3. Sum of powers of the Laplacian eigenvalues

Some bounds for the sum of powers of the Laplacian eigenvalues of bipartite graphs have been given in [16], [14]. The bounds obtained here improve the results in [16] and have much simpler forms than the results in [14]. As in [16], the sum of squares of degrees appears in our bounds.

Theorem 1. *Let α be a non-zero real number, and let G be a connected bipartite graph with $n \geqslant 3$ vertices, m edges and t spanning trees. Then*

$$s_\alpha(G) \geqslant \Big(\frac{Z(G)}{m}\Big)^\alpha + (n-2)\Big(\frac{tnm}{Z(G)}\Big)^{\alpha/(n-2)} \tag{1}$$

with equality if and only if $G \cong K_{1,n-1}$ or $G \cong K_{n/2,n/2}$.

P r o o f. By the matrix-tree theorem (see [4]), $\prod_{i=1}^{n-1} \mu_i = tn$. By the arithmetic-geometric mean inequality, we have

$$s_\alpha(G) \geqslant \mu_1^\alpha + (n-2)\Big(\prod_{i=2}^{n-1} \mu_i^\alpha\Big)^{1/(n-2)} = \mu_1^\alpha + (n-2)\Big(\frac{tn}{\mu_1}\Big)^{\alpha/(n-2)}$$

with equality if and only if $\mu_2 = \ldots = \mu_{n-1}$. It may be easily seen that $f(x) = x^\alpha + (n-2)(tn/x)^{\alpha/(n-2)}$ is increasing for $x \geqslant (tn)^{1/(n-1)}$ whether $\alpha > 0$ or $\alpha < 0$.

By Lemma 2 and the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality, we have $\mu_1 \geqslant Z(G)/m \geqslant 2\sqrt{Z(G)/n} \geqslant 4m/n \geqslant 2m/(n-1) \geqslant (tn)^{1/(n-1)}$. Thus, $s_\alpha(G) \geqslant f(Z(G)/m)$, from which (1) follows, with equality if and only if $\mu_2 = \ldots = \mu_{n-1}$ and $\mu_1 = Z(G)/m$.

Suppose that equality holds in (1). Then $\mathcal{L}(G)$ is regular, and G is a graph with at most three distinct Laplacian eigenvalues. By Lemma 1, G is a bipartite graph with diameter at most 2. Then G is bipartite with constant degrees on each of the two parts. Thus, it is easily seen that $G \cong K_{1,n-1}$ or $G \cong K_{n/2,n/2}$.

Conversely, it is easily seen that $\mu_2 = \ldots = \mu_{n-1}$, $\mu_1 = Z(G)/m$, and then (1) is an equality if $G \cong K_{1,n-1}$ or $G \cong K_{n/2,n/2}$. □

Let α be a non-zero real number, and let G be a connected bipartite graph with $n \geqslant 3$ vertices and t spanning trees. By the proof above, the lower bound in (1) is better than the one in [16]:

$$s_\alpha(G) \geqslant \left(2\sqrt{\frac{Z(G)}{n}}\right)^\alpha + (n-2)\left(\frac{tn}{2\sqrt{Z(G)/n}}\right)^{\alpha/(n-2)}.$$

Theorem 2. *Let G be a connected bipartite graph with $n \geqslant 3$ vertices and m edges.*

(i) *If $\alpha < 0$ or $\alpha > 1$, then*

$$s_\alpha(G) \geqslant \Big(\frac{Z(G)}{m}\Big)^\alpha + \frac{\big(2m - Z(G)/m\big)^\alpha}{(n-2)^{\alpha-1}} \tag{2}$$

with equality if and only if $G \cong K_{1,n-1}$ or $G \cong K_{n/2,n/2}$.

(ii) *If $0 < \alpha < 1$, then*

$$s_\alpha(G) \leqslant \Big(\frac{Z(G)}{m}\Big)^\alpha + \frac{\big(2m - Z(G)/m\big)^\alpha}{(n-2)^{\alpha-1}} \tag{3}$$

with equality if and only if $G \cong K_{1,n-1}$ or $G \cong K_{n/2,n/2}$.

P r o o f. Suppose first that $\alpha < 0$ or $\alpha > 1$. Then x^α is a strictly convex function, and thus by Jensen's inequality, we have

$$\Big(\sum_{i=2}^{n-1} \frac{1}{n-2}\,\mu_i\Big)^\alpha \leqslant \sum_{i=2}^{n-1} \frac{1}{n-2}\mu_i^\alpha,$$

i.e.,

$$\sum_{i=2}^{n-1} \mu_i^\alpha \geqslant \frac{1}{(n-2)^{\alpha-1}}\Big(\sum_{i=2}^{n-1} \mu_i\Big)^\alpha$$

with equality if and only if $\mu_2 = \ldots = \mu_{n-1}$. It follows that

$$\begin{aligned} s_\alpha(G) &\geqslant \mu_1^\alpha + \frac{1}{(n-2)^{\alpha-1}}\Big(\sum_{i=2}^{n-1} \mu_i\Big)^\alpha \\ &= \mu_1^\alpha + \frac{(2m-\mu_1)^\alpha}{(n-2)^{\alpha-1}}. \end{aligned}$$

It is easily seen that $g(x) = x^\alpha + (2m-x)^\alpha/(n-2)^{\alpha-1}$ is increasing for $x \geqslant 2m/(n-1)$. Note that $\mu_1 \geqslant Z(G)/m \geqslant 2\sqrt{Z(G)/n} \geqslant 4m/n \geqslant 2m/(n-1)$. Thus, $s_\alpha(G) \geqslant g(Z(G)/m)$ for $\alpha < 0$ or $\alpha > 1$, and then (2) follows.

If $0 < \alpha < 1$, then $-x^\alpha$ is a strictly convex function, and thus, by an argument similar to the above, $s_\alpha(G) \leqslant g(Z(G)/m)$, and then (3) follows.

Either equality in (2) or (3) holds if and only if $\mu_2 = \ldots = \mu_{n-1}$ and $\mu_1 = Z(G)/m$, which, by the same argument as in the proof of Theorem 2, is equivalent to $G \cong K_{1,n-1}$ or $G \cong K_{n/2,n/2}$. □

Note that for $\alpha = 1$ we have equalities in (2) and (3).

Let G be a connected bipartite graph with $n \geqslant 3$ vertices and m edges. By the proof above, we have: If $\alpha < 0$ or $\alpha > 1$, the lower bound in (2) is better than the one in [16]:

$$s_\alpha(G) \geqslant \left(2\sqrt{\frac{Z(G)}{n}}\right)^\alpha + \frac{\left(2m - 2\sqrt{Z(G)/n}\right)^\alpha}{(n-2)^{\alpha-1}},$$

and if $0 < \alpha < 1$, then the upper bound in (3) is better than the one in [16]:

$$s_\alpha(G) \leqslant \left(2\sqrt{\frac{Z(G)}{n}}\right)^\alpha + \frac{\left(2m - 2\sqrt{Z(G)/n}\right)^\alpha}{(n-2)^{\alpha-1}}.$$

Note that the bounds in Theorems 1 and 2 for $\alpha > 0$ hold also for disconnected bipartite graphs with n vertices and $m \geqslant 1$ edges. Since the multiplicity of the zero eigenvalue determines the number of connected components, the equality holds if and only if $\mu_1 = Z(G)/m$ and $\mu_2 = \ldots = \mu_{n-1} = 0$, or equivalently, $m = 1$.

4. Incidence energy

Recall the energy of a matrix is defined as the sum of its singular values [12]. The incidence energy $\mathrm{IE}(G)$ of G is defined as the energy of the vertex-edge incidence matrix $\mathbf{B}(G)$ of G [9], and the directed incidence energy $\mathrm{DIE}(G)$ of G is defined as the energy of the oriented vertex-edge incidence matrix $\mathbf{B}'(G)$ of G [6]. Note that $\mathbf{B}(G)\mathbf{B}(G)^T = \mathbf{D}(G) + \mathbf{A}(G)$ and $\mathbf{B}'(G)\mathbf{B}'(G)^T = \mathbf{L}(G)$. Thus if G is bipartite, then $\mathrm{IE}(G) = \mathrm{DIE}(G) = s_{1/2}(G)$ [6]. Here we give an immediate consequence of Theorems 1 and 2 (ii). Let G be a connected bipartite graph with $n \geqslant 3$ vertices, m edges, and t spanning trees. Then

$$\mathrm{IE}(G) \geqslant \sqrt{\frac{Z(G)}{m}} + (n-2)\Big(\frac{tnm}{Z(G)}\Big)^{1/(2(n-2))},$$
$$\mathrm{IE}(G) \leqslant \sqrt{\frac{Z(G)}{m}} + \sqrt{(n-2)\Big(2m - \frac{Z(G)}{m}\Big)}$$

with either equality if and only if $G \cong K_{1,n-1}$ or $G \cong K_{n/2,n/2}$.

Note that the above bounds hold also for disconnected bipartite graphs with at least one edge, and the bounds are attained if and only if $m = 1$, see [18].

5. Kirchhoff index

Let G be a connected graph. The Kirchhoff index $\mathrm{Kf}(G)$ of G is defined as the sum of resistance distances between all unordered pairs of vertices of G [7], [10]. As mentioned above, we have $\mathrm{Kf}(G) = ns_{-1}(G)$. A recent result on the Kirchhoff index may be found in [21], [22]. Here we give an immediate consequence of Theorems 1 and 2 (i). Let G be a connected bipartite graph with $n \geqslant 3$ vertices, m edges, and t spanning trees. Then [23]

$$\mathrm{Kf}(G) \geqslant n\Big[\frac{m}{Z(G)} + (n-2)\Big(\frac{Z(G)}{tnm}\Big)^{1/(n-2)}\Big],$$
$$\mathrm{Kf}(G) \geqslant n\Big[\frac{m}{Z(G)} + \frac{(n-2)^2}{2m - Z(G)/m}\Big]$$

with either equality if and only if $G \cong K_{1,n-1}$ or $G \cong K_{n/2,n/2}$.

6. Laplacian Estrada index

We note that lower bounds for s_α with integer $\alpha \geqslant 1$ may be converted to the bounds of the Laplacian Estrada index.

Theorem 3. *Let G be a connected bipartite graph with $n \geqslant 3$ vertices, m edges, and t spanning trees. Then*

$$\mathrm{LEE}(G) \geqslant 1 + \mathrm{e}^{Z(G)/m} + (n-2)\mathrm{e}^{(tnm/Z(G))^{1/(n-2)}},$$
$$\mathrm{LEE}(G) \geqslant 1 + \mathrm{e}^{Z(G)/m} + (n-2)\mathrm{e}^{(2m-Z(G)/m)/(n-2)}$$

with either equality if and only if $G \cong K_{1,n-1}$ or $G \cong K_{n/2,n/2}$.

Proof. Using the Taylor expansion of the exponential function e^x, we have

$$\mathrm{LEE}(G) = 1 + \sum_{i=1}^{n-1} \mathrm{e}^{\mu_i} = 1 + \sum_{k \geqslant 0} \frac{s_k(G)}{k!}.$$

The proof follows from Theorems 1 and 2. □

Recall that $s_0(G) = n - 1$ and $s_1(G) = 2m$ for the graph G in Theorem 3. Then the first inequality in Theorem 3 may be improved slightly to

$$\mathrm{LEE}(G) \geqslant 1 + 2m - \frac{Z(G)}{m} - (n-2)\Big(\frac{tnm}{Z(G)}\Big)^{1/(n-2)} + \mathrm{e}^{Z(G)/m} + (n-2)\mathrm{e}^{(tnm/Z(G))^{1/(n-2)}}.$$

Moreover, if we use $s_2(G) = Z(G) + 2m$, then Theorem 3 may still be improved slightly to

$$\begin{aligned}
\mathrm{LEE}(G) &\geqslant 1 + 3m + \frac{Z(G)}{2} - \frac{Z(G)}{m} - \frac{Z(G)^2}{2m^2} \\
&\quad - (n-2)\Big(\frac{tnm}{Z(G)}\Big)^{1/(n-2)} - \frac{1}{2}(n-2)\Big(\frac{tnm}{Z(G)}\Big)^{2/(n-2)} \\
&\quad + \mathrm{e}^{Z(G)/m} + (n-2)\mathrm{e}^{(tnm/Z(G))^{1/(n-2)}}, \\
\mathrm{LEE}(G) &\geqslant 1 + m + \frac{Z(G)}{2} - \frac{Z(G)^2}{2m^2} - \frac{(2m - Z(G)/m)^2}{2(n-2)} \\
&\quad + \mathrm{e}^{Z(G)/m} + (n-2)\mathrm{e}^{(2m-(Z(G)/m))/(n-2)}.
\end{aligned}$$

Note that the second bound in Theorem 3 holds also for disconnected bipartite graphs with at least one edge, and the bound is attained if and only if $m = 1$. The corresponding lower bounds from [17] are thus improved.

7. Concluding remarks

Let G be a bipartite graph with bipartition $V(G) = A \cup B$, $|A| = a$ and $|B| = b$. By Lemma 2 and the Cauchy-Schwarz inequality,

$$\mu_1 \geqslant \frac{Z(G)}{m} = \frac{1}{m}\Big(\sum_{v\in A} d_v^2 + \sum_{v\in B} d_v^2\Big) \geqslant \frac{1}{m}\Big(\frac{m^2}{a} + \frac{m^2}{b}\Big) = \frac{m(a+b)}{ab},$$

with equality if and only if the vertices from A and B have equal degrees. As in Sections 3 and 4, we may get bounds for the sum of powers of the Laplacian eigenvalues and lower bounds for the Laplacian Estrada index of the bipartite graph with a vertices in one partite set and b vertices in the other partite set and with $m \geqslant 1$ edges using a, b, and m.

We mention that our bounds in this paper depend on the numbers of vertices and edges, the sum of the squares of degrees (and sometimes, the number of the spanning trees). These graph invariants can be easily computed. We may also find bounds depending on some information on the structure of the graphs. For example, some bounds for s_α of bipartite graphs have been given in [14] by applying the lower bound for μ_1 which uses the neighborhood information:

$$\mu_1 \geqslant \sqrt{\frac{\sum_{u\in V(G)}\big[d_u(d_u^2 + t_u) + \sum_{v\in N(u)}(d_v^2 + t_v)\big]^2}{\sum_{u\in V(G)}(d_u^2 + t_u)^2}},$$

where t_u is the sum of degrees of the neighbors of u, and N_u denotes the set of neighbors of u in G.

Finally, we point out that the method in Theorem 2 may be used to estimate the sum of powers of singless Laplacian eigenvalues of graphs that are not necessarily bipartite, by applying lower bounds for the largest signless Laplacian eigenvalue eigenvalue μ_1^+. For example, the lower bound in Lemma 2 is also a lower bound for μ_1^+ of a graph with at least one edge.

By the same arguments as in [19] and using the lower bound $\sqrt{Z(G)/n}$ for the largest eigenvalue $\lambda(G)$ of a graph G with n vertices to its line graph $\mathcal{L}(G)$, we obtain an improved lower bound for μ_1^+:

$$\begin{aligned}\mu_1^+ &= 2 + \lambda(\mathcal{L}(G)) \geqslant 2 + \sqrt{\frac{1}{m}\sum_{uv\in E(G)}(d_u + d_v - 2)^2} \\ &= 2 + \sqrt{\frac{1}{m}\Big(\sum_{v\in V(G)} d_v^3 - 4\sum_{v\in V(G)} d_v^2 + 2\sum_{uv\in E(G)} d_u d_v + 4m\Big)}.\end{aligned}$$

The equality holds for a connected graph G if and only if $\mathcal{L}(G)$ is a regular or (bipartite) semi-regular, i.e., G is a regular graph, or a (bipartite) semi-regular graph or a path with four vertices [23].

References

[1] *G. H. Fath-Tabar, A. R. Ashrafi, I. Gutman*: Note on Estrada and L-Estrada indices of graphs. Bull. Cl. Sci. Math. Nat., Sci. Math. *34* (2009), 1–16.

[2] *M. Fiedler*: Algebraic conectivity of graphs. Czechoslovak Math. J. *23* (1973), 298–305. zbl

[3] *R. Merris*: Laplacian matrices of graphs: A survey. Linear Algebra Appl. *197–198* (1994), 143–176. zbl

[4] *B. Mohar*: The Laplacian spectrum of graphs. Graph Theory, Combinatorics, and Applications, Vol. 2 (Y. Alavi, G. Chartrand, O. R. Oellermann, A. J. Schwenk, eds.). Wiley, New York, 1991, pp. 871–898. zbl

[5] *I. Gutman, K. C. Das*: The first Zagreb index 30 years after. MATCH Commun. Math. Comput. Chem. *50* (2004), 83–92. zbl

[6] *I. Gutman, D. Kiani, M. Mirzakhah, B. Zhou*: On incidence energy of a graph. Linear Algebra Appl. *431* (2009), 1223–1233. zbl

[7] *I. Gutman, B. Mohar*: The quasi-Wiener and the Kirchhoff indices coincide. J. Chem. Inf. Comput. Sci. *36* (1996), 982–985.

[8] *Y. Hong, X.-D. Zhang*: Sharp upper and lower bounds for the largest eigenvalue of the Laplacian matrices of trees. Discrete Math. *296* (2005), 187–197. zbl

[9] *M. R. Jooyandeh, D. Kiani, M. Mirzakhah*: Incidence energy of a graph. MATCH Commun. Math. Comput. Chem. *62* (2009), 561–572. zbl

[10] *D. J. Klein, M. Randić*: Resistance distance. J. Math. Chem. *12* (1993), 81–95.

[11] *M. Lazić*: On the Laplacian energy of a graph. Czechoslovak Math. J. *56* (2006), 1207–1213. zbl

[12] *V. Nikiforov*: The energy of graphs and matrices. J. Math. Anal. Appl. *326* (2007), 1472–1475. zbl
[13] *J. Palacios*: Foster's formulas via probability and the Kirchhoff index. Methodol. Comput. Appl. Probab. *6* (2004), 381–387.
[14] *G. Tian, T. Huang, B. Zhou*: A note on sum of powers of the Laplacian eigenvalues of bipartite graphs. Linear Algebra Appl. *430* (2009), 2503–2510. zbl
[15] *R. Todeschini, V. Consonni*: Handbook of Molecular Descriptors. Wiley-VCH, Weinheim, 2000.
[16] *B. Zhou*: On sum of powers of the Laplacian eigenvalues of graphs. Linear Algebra Appl. *429* (2008), 2239–2246. zbl
[17] *B. Zhou*: On sum of powers of Laplacian eigenvalues and Laplacian Estrada index of graphs. MATCH Commun. Math. Comput. Chem. *62* (2009), 611–619. zbl
[18] *B. Zhou*: More upper bounds for the incidence energy. MATCH Commun. Math. Comput. Chem. *64* (2010), 123–128.
[19] *B. Zhou*: Signless Laplacian spectral radius and Hamiltonicity. Linear Algebra Appl. *432* (2010), 566–570. zbl
[20] *B. Zhou, I. Gutman*: More on the Laplacian Estrada index. Appl. Anal. Discrete Math. *3* (2009), 371–378. zbl
[21] *B. Zhou, N. Trinajstić*: A note on Kirchhoff index. Chem. Phys. Lett. *445* (2008), 120–123.
[22] *B. Zhou, N. Trinajstić*: On resistance-distance and Kirchhoff index. J. Math. Chem. *46* (2009), 283–289. zbl
[23] *B. Zhou, N. Trinajstić*: Mathematical properties of molecular descriptors based on distances. Croat. Chem. Acta *83* (2010), 227–242.

Authors' addresses: B. Zhou, Department of Mathematics, South China Normal University, Guangzhou 510631, China, e-mail: zhoubo@scnu.edu.cn; A. Ilić, Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, Niš 18000, Serbia, e-mail: aleksandari@gmail.com..