DEGREES OF BELIEF IN PARTIALLY ORDERED SETS

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Abstract: Belief functions can be taken as an alternative to the classical probability theory, as a generalization of this theory, but also as a non-traditional and sophisticated application of the probability theory. In this paper we abandon the idea of numerically quantified degrees of belief in favour of the case when belief functions take their values in partially ordered sets, perhaps enriched to lower or upper semilattices. Such structures seem to be the most general ones to which reasonable and nontrivial parts of the theory of belief functions can be extended and generalized.

Key words: *Belief function, partially ordered set, semilattice, Dempster combination rule, nonspecificity degree*

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1. Introduction and Motivation

The degrees of belief quantified by belief functions, and the mathematical theory processing them and sometimes called the Dempster–Shafer theory, present an interesting mathematical model and tool for uncertainty quantification and processing. Belief functions can be taken, at the same time, as an alternative to the classical probability theory, as a generalization of this theory, but also as a non-traditional and sophisticated application of the probability theory. Leaving aside informal considerations concerning the intuition and motivation behind the notions of degrees of belief and belief functions, and referring the reader to [1], [11], or to other sources dealing with an informal approach to these notions, let us begin with very brief and formalized definitions applying to the simplest case of the finite basic spaces.

Let S be a finite nonempty set, sometimes called the *universe of discourse*, let $\mathcal{P}(S)$ denote the power-set of all subsets of S. A *basic probability assignment* (b.p.a.) m on S is nothing else than a probability distribution on the (obviously finite) power-set $\mathcal{P}(S)$, hence, m takes $\mathcal{P}(S)$ into the unit interval $[0, 1]$ of real

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numbers in such a way that $\sum_{A \subset S} m(A) = 1$. The *(non-normalized) belief function* defined or induced by the b.p.a. m on S is the mapping $\mathrm{bel}_m : \mathcal{P}(S) \to [0,1]$, defined for each $A \subset S$ by

$$\mathrm{bel}_m(A) = \sum_{\emptyset \neq B \subset A} m(B). \tag{1}$$

We apply the convention according to which $\mathrm{bel}_m(\emptyset) = 0$ for the empty subset $\emptyset$ of S.

Perhaps more often introduced and defined is the so called *normalized belief function* bel^*_m also taking $\mathcal{P}(S)$ into $[0,1]$ and defined, for all $A \subset S$, by

$$\mathrm{bel}^*_m(A) = (1 - m(\emptyset))^{-1}\, \mathrm{bel}_m(A) = (1 - m(\emptyset))^{-1} \sum_{\emptyset \neq B \subset A} m(B) \tag{2}$$

so that $\mathrm{bel}^*_m(S) = 1$. If $m(\emptyset) = 1$, bel^*_m is not defined. As a matter of fact, the normalized belief function can be introduced also directly, i.e., independently of a non-normalized belief function, supposing a priori that $m(\emptyset) = 0$ or, which turns to be the same, that $\mathrm{bel}^*(\emptyset) = 0$ and $\mathrm{bel}^*(S) = 1$. These conditions are obviously satisfied when the system processing the data in question is consistent. In every case, normalized belief functions can be taken as special cases of the non-normalized ones, so that we will try to begin our seeking for non-numerical belief functions just with the non-normalized numerical ones, with the aim to investigate the phenomenon of inconsistent data also in the case of non-numerical degrees of belief.

An alternative way enabling one to rewrite (1) in a more appropriate form for our purposes, and offering also an intuitive insight into the notion of the degree of belief, may read as follows. Let S be taken as the set of possible internal states of a system (medical diagnoses, answers to questions, solutions to a problem,...), just one $s_0 \in S$ being supposed to be the actual one. Our task is either to identify s_0, or at least to decide whether $s_0 \in T$ holds for some (proper, as a rule) subset T of S or not. This decision will be based on some empirical data $x \in E$, perhaps vector ones, describing the results of various observations, experiments, etc., concerning the system in question and its environment. As E may be a vector space, experimental or empirical data of different nature, as well as results of the repeated observations and experiments, can be also embedded within the framework of this model. Of course, in order to ensure at least a portion of rationality in such a decision making, the internal states of the system in question and the empirical data obtained should be bound by a relation, namely, the so called *compatibility relation* ρ will be used for these purposes. This relation is defined either by a subset of the Cartesian product $S \times E$, or, which will be easier to process in our context, by a mapping which ascribes to each $s \in S$ and $x \in E$ either the value 0 or 1. Namely, if $\rho(s,x) = 0$ for some $s \in S$ and $x \in E$, then as far as the subject or the user knows, s cannot be the actual state of the observed system supposing that x was observed. If this is not the case, we write $\rho(s,x) = 1$ and the values s and x are (mutually) compatible. Given $x \in E$, we can easily define the subset

$$U_\rho(x) = \{s \in S : \rho(s,x) = 1\} \tag{3}$$

of states of S which are compatible with x.

The phenomenon of uncertainty enters our model supposing that the empirical data are of random nature. Namely, we will suppose that x is the value taken by a random variable X defined on a fixed probability space $\langle \Omega, \mathcal{A}, P\rangle$ and with values in a measurable space $\langle E, \mathcal{E}\rangle$ generated by an appropriate nonempty σ-field of subsets of E; if E is finite, we usually take the whole power-set $\mathcal{P}(E)$. Combining the mappings U_ρ and X together, we arrive at the composed mapping $U_\rho(X(\cdot))$ which takes Ω into $\mathcal{P}(S)$, i. e., for every $\omega \in \Omega$, $U_\rho(X(\omega))$ is a subset of S, and we will suppose that this mapping is measurable in the sense that, for each $A \subset S$, the inverse image $(U_\rho(X))^{-1}(A)$ belongs to the σ-field $\mathcal{A}$ of subsets of Ω, so that the value

$$m(A) = P(\{\omega \in \Omega : U_\rho(X(\omega)) = A\}) \tag{4}$$

is defined. In other terms, the mapping $U_\rho(X(\cdot))$ is supposed to be a (generalized set-valued) random variable which takes the probability space $\langle \Omega, \mathcal{A}, P\rangle$ into the measurable space $\langle \mathcal{P}(S), \mathcal{P}(\mathcal{P}(S))\rangle$. As can be easily seen, the mapping $m : \mathcal{P}(S) \to [0,1]$ is, for each finite S, a basic probability assignment on S and for the induced non-normalized belief function we obtain that

$$\mathrm{bel}_m(A) = P(\{\omega \in \Omega : \emptyset \neq U_\rho(X(\omega)) \subset A\}) \tag{5}$$

for every $A \subset S$. Let us notice that the normalized belief function bel^*_m can be defined in a similar way using the notion of conditional probability, namely, for each $A \subset S$,

$$\mathrm{bel}^*_m(A) = P\left(\{\omega \in \Omega : U_\rho(X(\omega)) \subset A\} \Big/ \{\omega \in \Omega : U_\rho(X(\omega)) \neq \emptyset\}\right) \tag{6}$$

supposing that this conditional probability is defined.

It is just the relation (5) that all the modifications and generalizations of the notion and theory of belief functions mentioned below start from; let us list them very briefly.

(i) As the operation of summation occurring in the classical combinatoric definition (1) is eliminated, the relation (5) enables one to define the degrees of belief also for at least some subsets of an infinite universe of discourse, namely, for those subsets of S for which their inverse images in Ω are measurable, i.e., belong to the σ-field $\mathcal{A}$ of subsets of Ω.

(ii) Using the worst-case analysis (minimax principle) belief functions can be reasonably approximated in some cases when the compatibility relation is known only partially, so that for some pairs $\langle s, x\rangle \in S \times E$ the value $\rho(s,x)$ is not known or even defined.

(iii) The idea to define the degrees of belief as sizes of certain sets of elementary random events is kept, but instead of probability measure P another set function, say, a possibilistic measure Π, is used for these purposes.

(iv) The degrees of belief are quantified also by real numbers outside the unit interval of reals.

(v) The degrees of belief are not quantified by real numbers, but rather by elements of some non-numerical structures which may perhaps better reflect the nature of uncertainty in various particular cases. E.g., the degrees of belief need not be always dichotomic, i.e., some pairs of degrees of belief need not be comparable by the relation "greater than or equal to". Perhaps the first non-numerical structure arising in one's mind as a good tool for these sakes is Boolean algebra, in particular, the Boolean algebra of all subsets of a fixed space with respect to the standard set-theoretic operations.

Pursuing further the reasonings which have motivated the research sub (v), we arrive at the following question, key one in our context: which are the most general and the simplest conditions that the structure of the degrees of uncertainty should meet in order to be able to develop a non-trivial fragment of the classical theory of belief functions within this generalized and simplified framework? The aim of this paper is to argue in favour of the idea that the degrees of belief should define a partially ordered set, in the case of some particular statements enriched to an upper or lower (semi)-lattice or to a lattice. This particular nature of our investigations presented below implies that it would not be realistic to expect some qualitatively new and perhaps surprising results. On the other hand, we believe that the fragment of the theory of belief function which can be built in the investigated case is rich and interesting enough to justify our effort.

2. Preliminaries on Partially Ordered Sets

Let us recall some most elementary notions and properties from the domain of partially ordered (p.o.) sets.

Definition 2.1. *Quasi-partially ordered set* $\mathcal{T}$ is a pair $\langle T, \preceq \rangle$, where T is a nonempty set and $\preceq$ is a binary relation on T, called *quasi-partial ordering* and fulfilling, for all t_1, t_2, $t_3 \in T$, the conditions

(i) $t_1 \preceq t_1$ (reflexivity),

(ii) if $t_1 \preceq t_2$ and $t_2 \preceq t_3$, then $t_1 \preceq t_3$ (transitivity).
If $\preceq$ fulfils also, for all t_1, $t_2 \in T$, the relation

(iii) if $t_1 \preceq t_2$ and $t_2 \preceq t_1$, then $t_1 = t_2$ (antisymmetry),

then $\preceq$ is called a *partial ordering* on T and the pair $\langle T, \preceq \rangle$ is called a *partially ordered set* (p.o. set). □

Given a quasi-p.o. set $\langle T, \preceq \rangle$, let us introduce a binary relation $\approx$ on T, setting, for all t_1, $t_2 \in T$, $t_1 \approx t_2$ iff $t_1 \preceq t_2$ and $t_2 \preceq t_1$ hold simultaneously. As can be easily seen, $\approx$ is an equivalence relation on T. Indeed, $t_1 \approx t_1$ and the implication $t_1 \approx t_2 \Rightarrow t_2 \approx t_1$ follow, for all t_1, $t_2 \in T$, immediately from the reflexivity of $\preceq$ (Definition 2.1, (i)) and from the definition of $\approx$. Moreover, if $t_1 \approx t_2$ and $t_2 \approx t_3$

hold, hence, if $t_1 \preceq t_2$, $t_2 \preceq t_1$, $t_2 \preceq t_3$ and $t_3 \preceq t_2$ hold simultaneously, then $t_1 \preceq t_3$ and $t_3 \preceq t_1$ also hold due to the transitivity of $\preceq$ (Definition 2.1, (ii)). So $t_1 \approx t_3$ is valid and the relation $\approx$ is transitive; consequently, it is an equivalence relation on T.

Using this fact, we may define the factor-space $T/\approx$ of equivalence classes with respect to $\approx$. I.e., for each $t \in T$ we denote by $[t]$ the set $\{s \in T : s \approx t\}$ and $T/\approx = \{[t] : t \in T\}$. Let us define a binary relation $\preceq$ on $T/\approx$, setting $[t_1] \preceq [t_2]$ iff $t_1 \preceq t_2$ holds. Obviously, $\preceq$ on $T/\approx$ is independent of the choice of representers from $[t_1]$ and $[t_2]$, i.e., if $[t_1] \preceq [t_2]$, $t_3 \in [t_1]$ and $t_4 \in [t_2]$, then $t_3 \preceq t_4$ also holds. So $\preceq$ is a binary relation on classes of equivalence from $T/\approx$ and, as can be easily seen, it is a partial ordering on $T/\approx$. Consequently, the pair $\langle T/\approx, \preceq\rangle$ is a p.o. set. Here we intentionally use the symbol $\preceq$ for denoting the relations on T as well as on $T/\approx$, but we hope that the sense will be always clear from the context and that the common intuition behind both the applications of the symbol $\preceq$ at least partially legitimates this use.

Definition 2.2. Let $\langle T, \preceq\rangle$ be a p.o. set, let $\emptyset \neq A \subset T$ be given. An element $t_A \in T$ is called the *supremum of* A in $\langle T, \preceq\rangle$ or with respect to $\preceq$, and denoted by $\bigvee_{t\in A} t$, or, shortly, by $\bigvee A$, if the following conditions are fulfilled:

(i) $t \preceq t_A$ holds for all $t \in A$;

(ii) if there is $t_A^* \in T$ such that $t \preceq t_A^*$ holds for all $t \in A$, then $t_A \preceq t_A^*$ holds as well.

Dually, an element $s_A \in T$ is called the *infimum of* A in $\langle T, \preceq\rangle$ or with respect to $\preceq$, and denoted by $\bigwedge_{t\in A} t$ or, abbreviately, by $\bigwedge A$, if the following conditions are fulfilled:

(iii) $s_A \preceq t$ holds for all $t \in A$,

(iv) if there is $s_A^* \in T$ such that $s_A^* \preceq t$ holds for all $t \in A$, then $s_A^* \preceq s_A$ holds as well. □

Evidently, $\bigvee A$ and $\bigwedge A$ need not be defined for some $\emptyset \neq A \subset T$, but if they are defined, they are unique, so that the definite article (*the* supremum and *the* infimum), used in Definition 2.2 above, is legitimate. This definition could be also extended to the quasi-partially ordered sets, but in this case, if $\bigvee A$ and/or $\bigwedge A$ are defined, they are defined up to the equivalence relation $\approx$. If $A = \{t\}$ for some $t \in T$, i.e., if A is a singleton containing just one element of T, then $\bigvee\{t\}$ and $\bigwedge\{t\}$ are always defined and identical with t.

An element $\mathbf{1}_{\mathcal{T}}$ ($\mathbf{0}_{\mathcal{T}}$, resp.) of T is called the *unit* or the *maximal* (the *zero* or the *minimal*, resp.) element of a p.o. set $\mathcal{T} = \langle T, \preceq\rangle$ if $t \preceq \mathbf{1}_{\mathcal{T}}$ ($\mathbf{0}_{\mathcal{T}} \preceq t$, resp.) holds for all $t \in T$. If $\mathbf{0}_{\mathcal{T}}$ and $\mathbf{1}_{\mathcal{T}}$ exist, they are defined uniquely. If $\mathbf{0}_{\mathcal{T}}$ and $\mathbf{1}_{\mathcal{T}}$ exist, the definition of supremum and infimum can be also extended to the empty subset $\emptyset$ of T, setting $\bigvee\emptyset = \mathbf{0}_{\mathcal{T}}$ and $\bigwedge\emptyset = \mathbf{1}_{\mathcal{T}}$. These definitions can be proved to be conservative extensions of the definitions for nonempty subsets of T. Moreover, if $\mathbf{0}_{\mathcal{T}}$ and $\mathbf{1}_{\mathcal{T}}$ exist, then obviously

$$\mathbf{0}_{\mathcal{T}} = \bigwedge_{t\in T} t = \bigwedge T, \quad \mathbf{1}_{\mathcal{T}} = \bigvee_{t\in T} t = \bigvee T. \tag{7}$$

3. Set Structures over Partially Ordered Sets

In this chapter we will build a structure of partial ordering over the power-set $\mathcal{P}(T)$ of all subsets of T, which extends conservatively the properties of partial ordering in T, and which can be totally embedded into the p.o. set $\langle T, \preceq \rangle$, supposing that this p.o. set is complete in the sense that $\bigvee A$ and $\bigwedge A$ are defined for all $A \subset T$.

Definition 3.1. Let $\langle T, \preceq \rangle$ be a p.o. set, let $\sqsubset$ be the binary relation on $\mathcal{P}(T) = \{A : A \subset T\}$ defined in this way: given $A, B \subset T$, $A \sqsubset B$ holds iff, for each $S_1 \subset A$ such that $\bigvee S_1$ exists, there exists $S_2 \subset B$ such that $\bigvee S_2$ is defined and $\bigvee S_1 \preceq \bigvee S_2$ holds. □

An intuition behind this definition can read as follows. Let $A \subset T$ be the set of values describing the (degree of) uncertainty connected with an event (phenomenon) α, the description being related to a multidimensional criterion (to a multicriterial meta-criterion, in other terms). Some criteria can be represented and replaced by another, say, the dominating one in the sense that an event β is at least as certain to occur than the event α with respect to this subset of criteria, iff the value ascribed to β by the dominating criterion is at least as large as the value ascribed by this dominating criterion to event α. Considering a subset of criteria such that no criterion of this set dominates (or is dominated by) no matter which one inside this subset, event β is at least as certain to occur as α iff any of the criteria from the subset in question ascribes to β the same or even a greater value than to α. Hence, the set of all criteria can be replaced by a (smaller, as a rule) set of mutually non-dominating and non-dominated ones. E.g., if the whole set of criteria can be replaced by a dominating one, the comparison of the degrees of certainty of events α and β reduces to their comparison with respect to the dominating criterion. In the opposite extreme case, when no criterion dominates (or is dominated by) another one, event β must be at least as sure to occur as α with respect to all the criteria simultaneously so that we could to conclude that β is at least as sure to occur as α with respect to the multidimensional criterion in question.

For the general case of such a multicriterial decision making, the resulting ordering (quasi-partial one, as the following Lemma proves) is just what Definition 3.1 tries to formalize.

Lemma 3.1. The relation $\sqsubset$ is a quasi-partial ordering on $\mathcal{P}(T)$. □

Proof. If $A \subset T$, then $A \sqsubset A$ obviously holds, as for each $S_1 \subset A$ such that $\bigvee S_1$ exists we take simply $S_2 = S_1$, so that $\bigvee S_1 \preceq \bigvee S_2$ obviosly holds. Let $A \sqsubset B$ and $B \sqsubset C$ hold simultaneously for $A, B, C \subset T$. Take $S_1 \subset A$ such that $\bigvee S_1$ exists and denote by $S_2 \subset B$ (one of) the subset(s) of B such that $\bigvee S_2$ exists and $\bigvee S_1 \preceq \bigvee S_2$ holds. As $B \sqsubset C$ is valid and $\bigvee S_2$ is defined, there exists $S_3 \subset C$ such that $\bigvee S_3$ is defined and $\bigvee S_2 \preceq \bigvee S_3$ holds. Hence, due to the transitivity property of the relation $\preceq$ on T, $\bigvee S_1 \preceq \bigvee S_3$ is valid. So, for each $S_1 \subset A$ such that $\bigvee S_1$ exists, there exists $S_3 \subset C$ such that $\bigvee S_3$ is defined and $\bigvee S_1 \preceq \bigvee S_3$ holds. Consequently, $\sqsubset$ it is a transitive, as well as quasi-partial ordering relation on $\mathcal{P}(T)$. □

In particular, Definition 3.1 yields that if $A \sqsubset B$, then for each $t \in A$ there obviously exists $S_t \subset B$ such that $\bigvee S_t$ is defined and $t \preceq \bigvee S_t$ holds. Obviously, $t \in A$ defines the singleton $\{t\} \subset A$ and the supremum of $\{t\}$ is always defined and equals to t itself.

Using the same construction as in Chapter 2, we introduce the equivalence relation $\sim$ on $\mathcal{P}(T)$, setting $A \sim B$ iff $A \sqsubset B$ and $B \sqsubset A$ hold simultaneously, $A, B \subset T$. Abusing the symbol $\sqsubset$ in the same way as the symbol $\preceq$ in Chapter 2, we may extend it to the equivalence classes $[A] \in \mathcal{P}(T)/\sim$, where $[A] = \{B \subset T : B \sim A\}$, $A \subset T$. So, $[A] \sqsubset [B]$ holds iff $A \sqsubset B$ holds; the validity of this relation clearly does not depend on the choice of representatives of classes $[A]$ and $[B]$.

For each $A \subset T$, if $\bigvee A$ is defined, then the equality $[A] = [\{\bigvee A\}]$ holds. Indeed, given $S_1 \subset A$ such that $\bigvee S_1$ is defined, the relation $\bigvee S_1 \preceq \bigvee A$ holds, so that we can take $S_2 = \{\bigvee A\}$ in order to prove that $A \sqsubset \{\bigvee A\}$ holds. To prove the inverse relation, the only (nonempty) system of subsets of $\{\bigvee A\}$ is $\{\bigvee A\}$ and for this system there exists a subsystem of A, namely A itself, such that $\bigvee A$ is defined and $\bigvee A \preceq \bigvee A$ trivially holds. Consequently, if $A, B \subset T$ are such that $\bigvee A$ and $\bigvee B$ exist, then $[A] = [B]$ holds iff $\bigvee A = \bigvee B$ holds in $\langle T, \preceq \rangle$.

Lemma 3.2. The relation $\sqsubset$ in $\mathcal{P}(T)$ is a conservative extension of the set-theoretic inclusion on T, hence, if $A \subset B \subset T$, then $A \sqsubset B$. □

Proof. If $S_1 \subset A$ is such that $\bigvee S_1$ exists, then take simply $S_2 = S_1 \subset B$, so that $\bigvee S_2 = \bigvee S_1$ is defined and $\bigvee S_1 \preceq \bigvee S_2$ obviously holds. □

Considering the p.o. set $\langle \mathcal{P}(T)/\sim, \sqsubset \rangle$, let us define the supremum operation $\sqcup$ and the infimum operation $\sqcap$ induced by $\sqsubset$, copying the standard way in which these operations are defined. As can be easily seen, $\sqcup \mathcal{A}$ and $\sqcap \mathcal{A}$ need not be always defined, hence, the operations of supremum and infimum are, in general, partial, but if $\sqcup \mathcal{A}$ and/or $\sqcap \mathcal{A}$ are defined, they are defined uniquely. Given $\mathcal{A} \subset \mathcal{P}(T)$, denoting by $\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$ the usual set-theoretic union of the sets from $\mathcal{A}$, and applying Lemma 3.2 to the trivial inclusion $A \subset \bigcup \mathcal{A}$ valid for each $A \in \mathcal{A}$, the result is that $[A] \sqsubset [\bigcup \mathcal{A}]$ holds; hence, if $\sqcup \mathcal{A}$ is defined, then $\sqcup \mathcal{A} \sqsubset [\bigcup \mathcal{A}]$ is valid.

If the p.o. set $\langle T, \preceq \rangle$ contains the unit element $\mathbf{1}_{\mathcal{T}}$, i.e., if $\bigvee T = \mathbf{1}_{\mathcal{T}}$ is defined, then the relation $\emptyset \sqsubset A \sqsubset T$ holds for each $A \subset T$. The first part $\emptyset \sqsubset A$ holds due to the trivial fact that there is no nonempty $S \subset \emptyset$, so that the antecedent of the corresponding implication is always false. Given $S_1 \subset A$ such that $\bigvee S_1$ is defined, take simply $S_2 = T$, so that $\bigvee S_1 \preceq \bigvee T = \mathbf{1}_{\mathcal{T}}$ holds (or apply Lemma 3.2) and $A \sqsubset T$ follows. Consequently, $[\emptyset] \sqsubset [A] \sqsubset [T]$ is valid in $\mathcal{P}(T)/\sim$, so that $\langle \mathcal{P}(T)/\sim, \sqsubset \rangle$ is a p.o. set with the zero element $[\emptyset]$ and the unit element $[T]$ (obviously identical with $[\{\mathbf{1}_{\mathcal{T}}\}]$).

4. Complete Upper Semilattices

Definition 4.1. A partially ordered set $\mathcal{T} = \langle T, \preceq \rangle$ is called *complete upper semil ttice*, if for all $A \subset T$ the supremum $\bigvee A$ with respect to $\preceq$ is defined. □

As an example of a complete upper semilattice take a system of all *infinite* subsets of an infinite space T with respect to the usual set-theoretic operations of union and intersection taken as supremum and infimum. Evidently, the union of any nonempty system of infinite subsets of T is an infinite subset of T, but the intersection of two or more infinite subsets can yield a finite (or even empty) subset of T which does not belong to the system of subsets in question.

Theorem 4.1. Let $\mathcal{T} = \langle T, \preceq \rangle$ be a complete upper semilattice. Then

(i) for each $t \in T$, $[\{t\}] = \{A \subset T : \bigvee A = t\}$,

(ii) for each $A \subset T$, $[A] = [\{\bigvee A\}]$, hence, $[A] = \{B \subset T : \bigvee B = \bigvee A\}$,

(iii) for each $A, B \subset T$, $[A] \sqsubset [B]$ holds iff $\bigvee A \preceq \bigvee B$ holds,

(iv) for each $A, B \subset T$, $[A] \sqcup [B] = [A \cup B]$, if $[A] \sqcup [B]$ is defined,

(v) for each $A, B \subset T$, $[A \cap B] \sqsubset [A] \sqcap [B]$, if $[A] \sqcap [B]$ is defined. □

Proof.

(i) Given $t \in T$, let $A \subset T$ be such that $\bigvee A = t$. The empty subset of $\{t\}$ can be obviously avoided, here and below, from consideration when verifying the validity of an $\sqsubset$-relation. The only $\emptyset \neq R \subset \{t\}$ is $\{t\}$ itself so that, taking $S = A$, we obtain that $\bigvee\{t\} = t \preceq \bigvee A = t$ holds, so that $\{t\} \sqsubset A$. For each $R \subset A$, $\bigvee R \preceq \bigvee A = t = \bigvee\{t\}$ is valid, so that, taking $S = \{t\}$, we arrive at the conclusion that $A \sqsubset \{t\}$ holds. Hence, for the equivalence classes we have proved that $[\{t\}] \supset \{A \subset T : \bigvee A = t\}$. If $B \subset T$ is such that $\bigvee B \neq t$, then B does not belong to $[\{t\}]$. Indeed, if $\bigvee B \succ t$, $\bigvee B \neq t$ holds, then $\{t\} \sqsubset B$, but not $B \sqsubset [\{t\}]$, if $\bigvee B \preceq t$, $\bigvee B \neq t$, then $B \sqsubset [\{t\}]$, but not $\{t\} \sqsubset B$. If t and $\bigvee B$ are incomparable with respect to $\preceq$, then neither $\{t\} \sqsubset B$ nor $B \sqsubset \{t\}$ hold. Indeed, taking $R = \{t\}$, $\bigvee R = t$ cannot be reached by $\bigvee S$ no matter which $S \subset B$ is taken, and taking $R = B$, the relation $\bigvee R = \bigvee B \preceq t$ does not hold as well. Consequently, the equality $[\{t\}] = \{A \subset T : \bigvee A = t\}$ is proven.

(ii) Take $A \subset T$ and $B \subset A$. Obviously, $\bigvee B \preceq \bigvee A$, so that $A \sqsubset \{\bigvee A\}$ follows. Inversely, taking $\{\bigvee A\}$ as the only nonempty subset of $\{\bigvee A\}$ and setting $S = A$, the relation $\bigvee A \preceq \bigvee A$ is trivial, so that $\{\bigvee A\} \sqsubset A$ and, consequently, $[A] = [\{\bigvee A\}]$ follows immediately.

(iii) Take $t_1, t_2 \in T$ such that $t_1 \preceq t_2$. For the only nonempty subset $\{t_1\}$ of $\{t_1\}$, the relation $\bigvee\{t_1\} = t_1 \preceq t_2 = \bigvee\{t_2\} = \bigvee S$ is evident, taking $S = \{t_2\}$, so that $\{t_1\} \sqsubset \{t_2\}$ follows. If $\{t_1\} \sqsubset \{t_2\}$, then the relation $t_1 \preceq t_2$ is obvious, so that $[\{t_1\}] \sqsubset [\{t_2\}]$ holds iff $t_1 \preceq t_2$. But, given $A, B \subset T$, the identities $[A] = [\{\bigvee A\}]$ and $[B] = [\{\bigvee B\}]$ follow by (ii), so that setting $t_1 = \bigvee A$ and $t_2 = \bigvee B$, (iii) is proved.

(iv) The only thing we have to prove is that $[\{t_1\}] \sqcup [\{t_2\}] = [\{t_1 \vee t_2\}]$ is valid for all $t_1, t_2 \in T$. Indeed, setting $t_1 = \bigvee A$, $t_2 = \bigvee B$, and applying (ii), we obtain that

$$
\begin{aligned}
[A] \sqcup [B] &= \left[\left\{\bigvee A\right\}\right] \sqcup \left[\left\{\bigvee B\right\}\right] = [\{t_1\}] \sqcup [\{t_2\}] = [\{t_1 \vee t_2\}] = \\
&= \left[\left\{\bigvee A \vee \bigvee B\right\}\right] = \left[\left\{\bigvee (A \cup B)\right\}\right] = [A \cup B]\,. \qquad (8)
\end{aligned}
$$

So, let $t_1,\, t_2 \in T$. As we have already proved, $t_1 \preceq t_1 \vee t_2$ and $t_2 \preceq t_1 \vee t_2$ yield that $[\{t_1\}] \sqsubset [\{t_1 \vee t_2\}]$ and $[\{t_2\}] \sqsubset [\{t_1 \vee t_2\}]$ hold, so that $[\{t_1\}] \sqcup [\{t_2\}] \sqsubset [\{t_1 \vee t_2\}]$ is also valid due to the fact that $\sqcup$ is the supremum operation with respect to the partial ordering relation $\sqsubset$ on the factor-space $\mathcal{P}(T)/\sim$. Hence, we have to prove that $[\{t_1 \vee t_2\}] \sqsubset [\{t_1\}] \sqcup [\{t_2\}]$ is valid.
In order to arrive at a contradiction, suppose that this is not the case. Then there exists $R \subset \{t_1 \vee t_2\}$ such that $\bigvee R \preceq \bigvee S$ does not hold for every $S \subset [\{t_1\}] \sqcup [\{t_2\}]$. But, $[\{t_1\}] \sqcup [\{t_2\}]$, if defined, is an element of the factor-space $\mathcal{P}(T)/\sim$, so that $[\{t_1\}] \sqcup [\{t_2\}] = [C]$ for some $C \subset T$. As $\{t_1 \vee t_2\}$ is the only possibility for R, we arrive at this conclusion: if $[\{t_1 \vee t_2\}] \sqsubset [\{t_1\}] \sqcup [\{t_2\}]$ does not hold, then for all $S \subset C$ the relation $t_1 \vee t_2 \preceq \bigvee S$ is invalid. Consequently, C cannot have a subset R such that $t_1 \preceq \bigvee R$ and $t_2 \preceq \bigvee R$ hold simultaneously; if this were the case, we could simply set $S = R$ in order to obtain $t_1 \vee t_2 \preceq \bigvee S$. Moreover, having $R_1 \subset C$ with the property $t_1 \preceq \bigvee R_1$ and $R_2 \subset C$ with the property $t_2 \preceq \bigvee R_2$, we could take $S = R_1 \cup R_2 \subset C$ to arrive at $t_1 \vee t_2 \preceq \bigvee S$. In other terms, the property "for no $R \subset C$ the property $t_i \preceq \bigvee R$ holds" is valid either for $i = 1$ or $i = 2$; let us consider the case with $i = 1$, as the other one is quite analogous. However, it follows that $\{t_1\} \sqsubset C$ does not hold, hence, also the relation $[\{t_1\}] \sqsubset [C]$ $(= [\{t_1\}] \sqcup [\{t_2\}])$ is invalid, so that we have arrived at a contradiction. Thus the relations $[\{t_1 \vee t_2\}] \sqsubset [\{t_1\}] \sqcup [\{t_2\}]$ and, consequently, $[\{t_1 \vee t_2\}] = [\{t_1\}] \sqcup [\{t_2\}]$ are proved, so that (iv) holds.

(v) Easily follows from the inclusions $A \cap B \subset A$ and $A \cap B \subset B$.
In spite of (iv), equality in (v) does not hold in general, as the following counter-example illustrates. Indeed, let $\mathbf{0}_{\mathcal{T}}$ be the minimal element of $\mathcal{T}$ such that $\mathbf{0}_{\mathcal{T}} = \bigvee \emptyset$ for the empty subset of T (the existence and uniqueness of $\mathbf{0}_{\mathcal{T}}$ follows from the assumption that for all $A \subset T$, including the case $A = \emptyset$, $\bigvee A$ is defined). Let A, B be disjoint subsets of T such that $\bigvee A = \bigvee B \neq \mathbf{0}_{\mathcal{T}}$. E.g., if $\mathcal{T}$ is the Boolean algebra of all subsets of a fixed nonempty set, then A and B may be two decompositions of the same nonempty set, with no component in common. Then

$$[A \cap B] = \{\mathbf{0}_{\mathcal{T}}\}\,, \qquad (9)$$

but

$$
\begin{aligned}
[A] \sqcap [B] &= \left[\left\{\bigvee A\right\}\right] \sqcap \left[\left\{\bigvee B\right\}\right] = \left[\left\{\bigvee A\right\}\right] = \\
&= \left\{C \subset T : \bigvee C = \bigvee A\right\} \neq \mathbf{0}_{\mathcal{T}}\,, \qquad (10)
\end{aligned}
$$

due to the idempotence of the minimum operation $\sqcap$. The theorem is proved. □

Definition 4.2. A partially ordered set $\mathcal{T} = \langle T, \preceq \rangle$ is called *lower semilattice*, if for all $t_1, t_2 \in T$ their infimum $t_1 \wedge t_2 \in T$ is defined. $\square$

It may seem rather surprising, for the first time, that Definitions 4.1 and 4.2 are not dual. Namely, we have not defined the notion of a *complete* lower semilattice, even if such a definition might be easy to introduce, limiting ourselves to a weaker definition of lower semilattice which is closed just with respect the to infimum operation applied to *finite* subsets of its elements. The reason is that, when introducing and investigating, below, the Boolean version of the Dempster combination rule, the infimum operation is applied only to pairs of values of the Boolean algebra in question. Contrary to this fact, the supremum operation applies to the sets of all pairs of subsets of the state space under consideration, the intersection of which yields the same set of states. Evidently, the set of such pairs can be infinite for an infinite basic set of states. This non-symmetric role of infimum and supremum operations implies also different minimal conditions imposed to both the operations when shifting them to structure $\langle \mathcal{P}(T)/\sim, \sqsubset \rangle$.

Theorem 4.2. Let $\mathcal{T} = \langle T, \preceq \rangle$ be a complete upper semilattice and lower semilattice. Then, for all A, $B \subset T$, the identity

$$[A] \sqcap [B] = \left[\left\{\left(\bigvee A\right) \wedge \left(\bigvee B\right)\right\}\right] \tag{11}$$

is valid. $\square$

Proof. As $\mathcal{T}$ is a complete upper semilattice, the identities $[A] = [\{\bigvee A\}]$ and $[B] = [\{\bigvee B\}]$ follow by Theorem 4.1, (ii). As $\mathcal{T}$ is also a lower semilattice, $(\bigvee A) \wedge (\bigvee B)$ is defined and the obviously valid relations $(\bigvee A) \wedge (\bigvee B) \preceq \bigvee A$ and $(\bigvee A) \wedge (\bigvee B) \preceq \bigvee B$, combined with (iii) of Theorem 4.1, yield that

$$\left[\left\{\left(\bigvee A\right) \wedge \left(\bigvee B\right)\right\}\right] \sqsubset \left[\left\{\bigvee A\right\}\right] = [A] \tag{12}$$

and

$$\left[\left\{\left(\bigvee A\right) \wedge \left(\bigvee B\right)\right\}\right] \sqsubset \left[\left\{\bigvee B\right\}\right] = [B] \tag{13}$$

hold, so that, due to the definition of infimum $\sqcap$ in $\langle \mathcal{P}(T)/\sim, \sqsubset \rangle$ the relation

$$\left[\left\{\left(\bigvee A\right) \wedge \left(\bigvee B\right)\right\}\right] \sqsubset [A] \sqcap [B] \tag{14}$$

must be satisfied. Let $C \subset T$ be such that $[C] \sqsubset [A]$ and $[C] \sqsubset [B]$ are simultaneously valid. Applying again the assumption that $\mathcal{T}$ is a complete upper semilattice and the assertion (iii) of Theorem 4.1, we obtain that

$$[C] = \left[\left\{\bigvee C\right\}\right] \sqsubset [A] = \left[\left\{\bigvee A\right\}\right], \tag{15}$$

$$[C] = \left[\left\{\bigvee C\right\}\right] \sqsubset [B] = \left[\left\{\bigvee B\right\}\right], \tag{16}$$

hence, $\bigvee C \preceq \bigvee A$, $\bigvee C \preceq \bigvee B$, consequently $\bigvee C \preceq (\bigvee A) \wedge (\bigvee B)$ and we arrive at the conclusion that

$$[C] \sqsubset \left[\left\{\left(\bigvee A\right) \wedge \left(\bigvee B\right)\right\}\right] \tag{17}$$

is valid. Hence, $[\{(\bigvee A) \wedge (\bigvee B)\}]$ satisfies both the properties imposed by the definition of infimum in the partially ordered set $\langle \mathcal{P}(T)/\sim, \sqsubset\rangle$ to the infimum of classes $[A]$ and $[B]$, so that (11) is proven. □

5. Belief Functions with Values in Partially Ordered Sets

Let S be a nonempty set, let $\mathcal{T} = \langle T, \preceq\rangle$ be a p.o. set, let $\vee$ and $\wedge$ denote the (partial, in general) supremum and infimum operations in T induced by the partial ordering relation $\preceq$; let $\mathcal{B}_T = \langle \mathcal{P}(T), \cup, \cap, T - \cdot\rangle$ be the Boolean algebra induced in the power-set $\mathcal{P}(T)$ of all subsets of T by the standard set-theoretic operations of union ($\cup$), intersection ($\cap$) and complement ($T - \cdot$).

Definition 5.1. The *$\mathcal{B}_T$-valued basic possibilistic assignment on S* ($\mathcal{B}_T$-b.poss.a. on S, abbreviatcly) is a mapping $\pi : \mathcal{P}(S) \to \mathcal{P}(T)$, i.e., $\pi(A) \subset T$ for all $A \subset S$, such that ${}_{A\subset S}\pi(A) = T$. The *$\mathcal{B}_T$-(valued) belief function defined by a $\mathcal{B}_T$-b.poss.a. π on S* is the mapping $\mathrm{BEL}_\pi : \mathcal{P}(S) \to \mathcal{P}(T)$ ascribing, to each $\emptyset \neq A \subset S$, the subset

$$\mathrm{BEL}_\pi(A) = \bigcup_{\emptyset \neq B \subset A} \pi(B) \tag{18}$$

of T, by convention, $\mathrm{BEL}_\pi(\emptyset) = \emptyset$ for the empty subset of S. □

An intuition and motivation behind this definition can be based on the most trivial idea that the absolutely complete non-numerical characteristic of a set of no matter which elements is simply the set itself. So, given $A \subset S$, the value $\pi(A) \subset T$, ascribed to A by a $\mathcal{B}_T$-valued basic possibilistic assignment π, may be simply a collection of arguments of some non-numerical nature including, e.g. the verbal ones which are relevant when considering the problem whether the actual state of the system under consideration is in A or not (when keeping in mind the model sketched at the beginning of this paper). The simplest characteristic of the set $\pi(A)$ of relevant arguments is the set itself, and this is just what Definition 5.1 aims to catch. Subsequently, of course, we can process subset $\pi(A)$ of T somehow, e.g. ascribing real numbers to elements of this set and perhaps combining these numbers into a single real number from the unit interval of reals. If the numbers ascribed to various subsets of S meet some obvious conditions, we arrive at the standard case of basic probabilistic assignments and belief functions as sketched above. As to more or less trivial cases, however, such processing must be payed by a loss of information if compared with that contained in the original set $\pi(A)$ itself. E.g., the probability value asribed to a random event does not enable, as a rule, identifing this random event completely, if taken as a subset of the universe of elementary random events. In what follows, we will apply a similar pattern of reasoning, but we will not ascribe any reals to the elements of set $\pi(A)$ and perhaps the process these numbers somehow either, but we will rather take profit of the structural properties of set T and its power-set $\mathcal{P}(T)$ which can be expressed in the language (and processed by the tools) of partially ordered sets $\langle T, \preceq\rangle$ and $\langle \mathcal{P}(T)/\sim, \sqsubset\rangle$.

The properties of belief functions taking their values (degrees of belief) in a Boolean algebra are at a more general level, and in greater more detail, investigated in [8] and [9], so that we will refer to the corresponding results and statements without repeating their proofs. Here we will take into consideration the fact that values $\pi(A)$ and $\mathrm{BEL}_\pi(A)$, $A \subset S$, are subsets of the partially ordered set T, so that we can apply to them the quasi-partial ordering relation $\sqsubset$ defined on $\mathcal{P}(T)$ and extended to the classes of equivalence from the factor-space $\mathcal{P}(T)/\sim$.

Lemma 5.1. For every $A \subset B \subset S$ the relation

$$\mathrm{BEL}_\pi(A) \sqsubset \mathrm{BEL}_\pi(B) \tag{19}$$

holds. □

Proof. By definition

$$\mathrm{BEL}_\pi(A) = \bigcup_{\emptyset \neq C \subset A} \pi(C) \subset \bigcup_{\emptyset \neq C \subset B} \pi(C) = \mathrm{BEL}_\pi(B), \tag{20}$$

so that Lemma 3.2 yields the result. □

Lemma 5.2. For every A, $B \subset S$ the relation

$$[\mathrm{BEL}_\pi(A)] \sqcup [\mathrm{BEL}_\pi(B)] \sqsubset [\mathrm{BEL}_\pi(A \cup B)] \tag{21}$$

holds, supposing that $[\mathrm{BEL}_\pi(A)] \sqcup [\mathrm{BEL}_\pi(B)]$ is defined. □

Proof. As $A \subset A \cup B$ and $B \subset A \cup B$ hold, the relations $[\mathrm{BEL}_\pi(A)] \sqsubset [\mathrm{BEL}_\pi(A \cup B)]$ and $[\mathrm{BEL}_\pi(B)] \sqsubset [\mathrm{BEL}_\pi(A \cup B)]$ follow by (19). As $\sqcup$ is the supremum operation with respect to $\sqsubset$, (21) follows immediately. □

The mapping $\mathrm{BEL}_\pi : \mathcal{P}(S) \to \mathcal{P}(T)$, defined by (18), easily induces mapping $\mathrm{BEL}^*_\pi : \mathcal{P}(S) \to \mathcal{P}(T)/\sim$, setting simply

$$\begin{aligned} \mathrm{BEL}^*_\pi(A) &= [\mathrm{BEL}_\pi(A)] = \{R \subset T : R \sim \mathrm{BEL}_\pi(A)\} = \\ &= \{R \subset T : R \sqsubset \mathrm{BEL}_\pi(A) \text{ and } \mathrm{BEL}_\pi(A) \sqsubset R\}, \end{aligned} \tag{22}$$

for every $A \subset S$. Similarly, the basic possibilistic assignment $\pi : \mathcal{P}(S) \to \mathcal{P}(T)$ induces mapping $\pi^* : \mathcal{P}(S) \to \mathcal{P}(T)/\sim$ such that, for each $A \subset S$,

$$\pi^*(A) = [\pi(A)] = \{R \subset T : R \sqsubset \pi(A) \text{ and } \pi(A) \sqsubset R\}. \tag{23}$$

The inclusion $\pi(C) \subset \mathrm{BEL}_\pi(A)$, valid for every $\emptyset \neq C \subset A$ by (18), implies, due to Lemma 3.2, that the relation $[\pi(C)] \sqsubset [\mathrm{BEL}_\pi(A)]$ holds. Consequently, $\pi^*(C) \sqsubset \mathrm{BEL}^*_\pi(A)$, so that the relation

$$\bigsqcup_{\emptyset \neq C \subset A} \pi^*(C) \sqsubset \mathrm{BEL}^*_\pi(A) \tag{24}$$

easily follows, supposing that the supremum in (24) exists.

Lemma 5.3. Let $\langle T, \preceq\rangle$ be a complete upper semilattice, let $\pi : \mathcal{P}(S) \to \mathcal{P}(T)$ be a $\mathcal{B}_T$-valued b.poss.a. on S. Then for each finite $A \subset S$ the equality in (24) holds, so that

$$\mathrm{BEL}^*_\pi(A) = \bigsqcup_{\emptyset \neq C \subset A} \pi^*(C), \tag{25}$$

supposing that the supremum exists. In particular, if whole the space S is finite, then

$$\bigsqcup_{A \subset S} \pi^*(A) = [T], \tag{26}$$

again, supposing that the supremum exists. □

Proof. The assertion (iv) of Theorem 4.1 can be immediately extended to any finite nonempty system of subsets $\mathcal{A} \subset \mathcal{P}(T)$, hence, if $\bigsqcup_{R \in \mathcal{A}} [R]$ is defined, then

$$\bigsqcup_{R \in \mathcal{A}} [R] = \left[\bigsqcup_{R \in \mathcal{A}} R\right]. \tag{27}$$

If $A \subset S$ is finite, then $\{C : \emptyset \neq C \subset A\}$ is finite as well, so that (25) easily follows from (27). If S is finite, then $\mathcal{P}(S)$ is also finite, so that

$$\bigsqcup_{A \subset S} \pi^*(A) = \bigsqcup_{A \in \mathcal{P}(S)} [\pi(A)] = \left[\bigcup_{A \in \mathcal{P}(S)} \pi(A)\right] = [T] \tag{28}$$

by Definition 5.1. The lemma is proved. □

Under the conditions of Lemma 5.2, $[T] = [\{\bigvee T\}]$ holds, and as $\bigvee T$ plays the role of the unit (maximal) element in $\langle T, \preceq\rangle$, relation (28) can be also taken in such a way that the mapping $\pi^* : \mathcal{P}(S) \to \mathcal{P}(T)/\sim$ is a basic possibilistic assignment on S taking its values in the factor-space $\mathcal{P}(T)/\sim$. The relation (25) then enables one to understand mapping $\mathrm{BEL}^*_\pi : \mathcal{P}(S) \to \mathcal{P}(T)/\sim$ as belief function defined by the b.poss.a. π^*. The condition that $\langle T, \preceq\rangle$ is a complete upper semilattice seems to be the weakest one imposed on the set of values of the b.poss.a. π^*, under which the basic philosophy underlying the idea of belief functions can be applied.

6. Boolean–Valued Dempster Combination Rule

Within the framework of the classical Dempster–Shafer theory of belief functions, Dempster combination rule is defined as follows. Let S be a finite nonempty set, let m_1, m_2 be two basic probability assignments on S, i.e., two probability distributions on the power-set $\mathcal{P}(S)$ of all subsets of S. Let $m_{12} : \mathcal{P}(S) \to [0,1]$ be the mapping defined by

$$m_{12}(A) = \sum_{B,C \subset S,\, B \cap C = A} m_1(B)\, m_2(C) \tag{29}$$

for each $A \subset S$. As can be easily proved, m_{12} is also a basic probability assignment on S, denoted also by $m_1 \oplus m_2$ and called the *Dempster product* of m_1 and m_2. Relation (29) is called the *Dempster combination rule* for basic probability assignments. A *sl non-normalized belief function* defined by a basic probability assignment m on S is the mapping $\mathrm{bel}_m : \mathcal{P}(S) \to [0,1]$ such that

$$\mathrm{bel}_m(A) = \sum_{\emptyset \neq B \subset A} m(B) \tag{30}$$

for all $A \subset S$, $\mathrm{bel}_m(\emptyset) = 0$ by convention. Dempster product $\oplus$ is also defined for belief functions, setting simply

$$\mathrm{bel}_{m_1} \oplus \mathrm{bel}_{m_2} =_{\mathrm{df}} \mathrm{bel}_{m_1 \oplus m_2}. \tag{31}$$

As analyzed in greater detail, e.g. in [6], the Dempster combination rule is legitimate under the two following conditions:

(I) Any piece of information allowing one to avoid a hypothesis (state, explanation, answer, solution, diagnosis, ...) from the set of possible candidates (as this hypothesis is incompatible with the empirical data them, or contradicts obtained by one of the subjects) is also accepted by the other subject. More formally, the composed compatibility relation is defined by the minimum of compatibility relations of both the subjects.

(II) The set-valued random variables defining the probability distributions m_1, m_2 on $\mathcal{P}(S)$ are statistically (stochastically) independent.

Relations (29) and (31), as well as the conditions (I) and (II) above, can be easily and immediately defined for any finite number of components (basic probability assignments or belief functions). It is almost obvious that in both the cases the Dempster combination rule is associative and commutative, so that, applying $\oplus$ many times, order and bracketing are irrelevant.

For Boolean-valued basic possibilistic assignments and belief functions induced by them, the Dempster combination rule can be rewritten in such a way that the summations are routinely replaced by suprema and the products by infima. In particular, considering the Boolean algebra $\mathcal{B}_T = \langle \mathcal{P}(T), \cup, \cap, T - \cdot \rangle$, introduced in the first paragraph of Section 5, two $\mathcal{B}_T$-valued basic possibilistic assignments π_1, π_2 on S, i.e., $\pi_1, \pi_2 : \mathcal{P}(S) \to \mathcal{P}(T)$ are such that

$$\bigcup_{A\subset S} \pi_1(A) = \bigcup_{A\subset S} \pi_2(A) = T\,, \tag{32}$$

rewriting (29) in the way just sketched, and using the notation $\pi_1 \oplus \pi_2$ for the resulting mapping, we obtain that, for each $A \subset S$,

$$(\pi_1 \oplus \pi_2)(A) = \bigcup_{B,C\subset S,\, B\cap C=A} (\pi_1(B) \cap \pi_2(C))\,. \tag{33}$$

As can be easily proved, $\pi_1 \oplus \pi_2$ is also a $\mathcal{B}_T$-valued basic possibilistic assignment on S, as the relation

$$\bigcup_{A\subset S} (\pi_1 \oplus \pi_2)(A) = T \tag{34}$$

is valid. Let us investigate, following the pattern from [6], under which conditions the rule (33) is legitimate, namely, whether the assumptions (I) and (II) from above, or some of their appropriate modifications, will do.

Let us suppose that S is a set of possible internal states of a system, only one $s_0 \in S$ being the actual one. Under different interpretations, S is a set of possible answers to a question or solutions to a problem, only one being true or correct, a set of possible technical or medical diagnoses, etc. The subject's (e.g. an observer's or decision-maker's) problem is either to identify the actual state s_0 or at least to decide whether $s_0 \in S_0 \subset S$ holds or does not hold for some (proper, to avoid trivialities) subset S_0 of S. E.g., keeping in mind the technical interpretation and terminology; S_0 can be a set of critical or dangerous, in a sense, states of the system (a nuclear power-station, e.g.) in question, when some prohibitive measures must be urgently applied. The actual state of the system cannot be directly observed, but the subject has at her/his disposal some *empirical data* (observations, measurements, results of experiments, etc.), taking their values in the empirical space E (perhaps a vector-like one). Moreover, the subject possesses a *compatibility relation* $\rho \subset S \times E$, taken as a function $\rho : S \times E \to \{0,1\}$, with this semantics: if $\rho(s,x) = 0$ for some $s \in S$ and $x \in E$, the subject knows that obtaining empirical value x, s *cannot be* the actual state of the system. E.g., if a patient is not feverish, at least some infection diseases can be eliminated from the set of possible diagnoses and a patient of a male sex cannot be pregnant. In the other case, if $\rho(s,x) = 1$, then s cannot be avoided from consideration when x is observed, so that s and x are compatible. For each $x \in E$ the set $U_\rho(x) = \{s \in S : \rho(s,x) = 1\}$ of states compatible with x is defined.

In order to introduce uncertainty into our model, with Boolean-valued uncertainty degrees, consider a complete Boolean algebra $\mathcal{B} = \langle B, \vee, \wedge, \neg \rangle$ and $\mathcal{B}$-valued complete possibilistic space $\langle \Omega, \mathcal{P}(\Omega), \Pi_0 \rangle$. Hence, Ω is a nonempty set, $\mathcal{P}(\Omega)$ is the power-set of all subsets of Ω, and Π_0 is a $\mathcal{B}$-valued complete possibilistic measure on Ω, so that $\Pi_0 : \mathcal{P}(\Omega) \to B$ is such that $\Pi_0(\emptyset) = \mathbf{0}_{\mathcal{B}}$, $\Pi_0(\Omega) = \mathbf{1}_{\mathcal{B}}$, and

$$\Pi_0\left(\dot{\bigcup_{A\in\mathcal{R}}} A\right) = \bigvee_{A\in\mathcal{R}} \Pi_0(A) \tag{35}$$

for every nonempty system $\mathcal{R}$ of subsets of Ω. The empirical value $x \in E$ is understood as a value taken by a variable (mapping) $X : \Omega \to E$, so that the composed mapping $U_\rho(X(\cdot))$ takes Ω into $\mathcal{P}(S)$, the inverse image $\{\omega \in \Omega : U_\rho(X(\omega)) = A\}$ is defined for each $A \subset S$, and the size of this subset of Ω can be quantified by a value from B, setting

$$\pi(A) = \Pi_0(\{\omega \in \Omega : U_\rho(X(\omega)) = A\}) \tag{36}$$

for each $A \subset S$. As

$$\begin{aligned} \bigvee_{A \subset S} \pi(A) &= \bigvee_{A \subset S} \Pi_0(\{\omega \in \Omega : U_\rho(X(\omega)) = A\}) = \\ &= \Pi_0 \left(\bigcup_{A \subset S} \{\omega \in \Omega : U_\rho(X(\omega)) = A\} \right) = \Pi_0(\Omega) = \mathbf{1}_{\mathcal{B}}, \end{aligned} \tag{37}$$

π is a $\mathcal{B}$-valued b.poss.a. distribution on S. If S is finite, the completeness of Π_0 is not necessary, as (37) can be proved when replacing (35) by

$$\Pi_0(A \cup B) = \Pi_0(A) \vee \Pi_0(B) \tag{38}$$

for all $A, B \subset \Omega$, in other terms, when reducing (35) to the case of systems $\mathcal{R}$ containing just two subsets of Ω.

Consider two subjects operating over the same empirical space E and possibilistic complete space $\langle \Omega, \mathcal{P}(\Omega), \Pi_0 \rangle$, using the same empirical values taken by a variable $X : \Omega \to E$, but with perhaps different compatibility relations ρ_1 and ρ_2. Let us accept the assumption (I) from above, so that the combined compatibility relation ρ_{12} is defined by

$$\rho_{12}(s, x) = \min\{\rho_1(s, x), \rho_2(s, x)\}, \tag{39}$$

so that $\rho_{12}(s, x) = 1$ iff $\rho_1(s, x) = \rho_2(s, x) = 1$; consequently, the relation

$$U_{\rho_{12}}(x) = U_{\rho_1}(x) \cap U_{\rho_2}(x) \tag{40}$$

is valid for all $x \in E$. Applying (36) to ρ_{12}, we define

$$\begin{aligned} \pi_{12}(A) &= (\pi_1 \oplus \pi_2)(A) = \Pi_0(\{\omega \in \Omega : U_{\rho_{12}}(X(\omega)) = A\}) = \\ &= \Pi_0(\{\omega \in \Omega : U_{\rho_1}(X(\omega)) \cap U_{\rho_2}(X(\omega)) = A\}) = \\ &= \Pi_0 \left(\bigcup_{B, C \subset S,\, B \cap C = A} \{\omega \in \Omega : U_{\rho_1}(X(\omega)) = B,\, U_{\rho_2}(X(\omega)) = C\} \right) = \\ &= \bigvee_{B \cap C = A} \Pi_0(\{\omega \in \Omega : U_{\rho_1}(X(\omega)) = B\} \cap \{\omega \in \Omega : U_{\rho_2}(X(\omega)) = C\}). \end{aligned} \tag{41}$$

Hence, if Π_0 is a homeomorphism which takes the Boolean algebra $\mathcal{B}_\Omega = \langle \mathcal{P}(\Omega), \cup, \cap, \Omega - \cdot \rangle$ of all subsets of Ω on $\mathcal{B}$ in such a way that

$$\begin{aligned} & \Pi_0\left(\{\omega \in \Omega : U_{\rho_1}(X(\omega)) = B\} \cap \{\omega \in \Omega : U_{\rho_2}(X(\omega)) = C\}\right) = \\ = \; & \Pi_0\left(\{\omega \in \Omega : U_{\rho_1}(X(\omega)) = B\}\right) \wedge \Pi_0\left(\{\omega \in \Omega : U_{\rho_2}(X(\omega)) = C\}\right) \quad (42) \end{aligned}$$

holds for all B, $C \subset S$; then

$$\begin{aligned} & (\pi_1 \oplus \pi_2)(A) = \\ = & \bigvee_{B \cap C = A} \left(\Pi_0(\{\omega \in \Omega : U_{\rho_1}(X(\omega)) = B\}) \wedge \Pi_0\left(\{\omega \in \Omega : U_{\rho_2}(X(\omega)) = C\}\right)\right) = \\ = & \bigvee_{B \cap C = A} \left(\pi_1(B) \wedge \pi_2(C)\right), \quad (43) \end{aligned}$$

introducing the $\mathcal{B}$-valued possibilistic assignments π_1 and π_2 just as π is defined in (36). If S is finite, the completeness of Π_0 can be weakened to (38). Hence, (42) plays the role of (II). Using the terms similar to those in the classical probabilistic case, we can say that if (42) holds, the set-valued mappings $U_{\rho_1}(X(\cdot))$ and $U_{\rho_2}(X(\cdot))$ are *possibilistically independent*.

Considering the particular case when spaces Ω and T are identical and taking the identity mapping on $\mathcal{P}(\Omega)$ as Π_0, the result is that (42) reduces to the identity

$$\begin{aligned} & \{\omega \in \Omega : U_{\rho_1}(X(\omega)) = B, U_{\rho_2}(X(\omega)) = C\} = \\ = \; & \{\omega \in \Omega : U_{\rho_1}(X(\omega)) = B\} \cap \{\omega \in \Omega : U_{\rho_2}(X(\omega)) = C\}, \quad (44) \end{aligned}$$

so that, in this particular case, the conditions (I) and (II) from above reduce to (39).

The following remark concerning the Dempster combination rule is perhaps worth being introduced explicitly. A more often presented intuition behind this rule reads that what is combined are degrees of belief of *the same subject* obtained on the ground of two or more pieces (or collections of pieces) of information. However, the interpretation with two or more subjects combining their individual belief functions into a common one can be seen to be the same. Indeed, as the universe in which empirical data take their values is supposed to be a vector space, it is well possible that this space is common for two or more subjects even when their pieces of knowledge are qualitatively different from each other. The compatibility relation $\rho_i(s, x)$ of every subject may simply depend just on several, but far not on all, items of the vector empirical value $x = \langle x_1, x_2, \ldots, x_n \rangle \in E = \mathbb{X}_{i=1}^n E_i$. This situation can be also taken in such a way that a *single* subject obtains, in two subsequent steps, two collections of data perhaps, but not necessarily, of different nature, and she/he builds her/his final belief function in two stages, first m_1 and bel_{m_1}, then $m_1 \oplus m_2$ and $\mathrm{bel}_{m_1} \oplus \mathrm{bel}_{m_2}$ ($= \mathrm{bel}_{m_1 \oplus m_2}$ by definition). From this point of view it is beyond any importance, whether the second piece of knowledge represents the results of further investigations made by a single subject or whether it is a collection of data delivered to the first subject by her/his colleague either directly as data, or processed in the form of the basic probability assignment m_2 and/or corresponding belief function m_2.

7. Dempster Combination Rule for Degrees of Belief over Partially Ordered Sets

Theorem 7.1. Let π_1, π_2 be Boolean-valued basic possibilistic assignments taking their values in the Boolean algebra $\langle \mathcal{P}(T), \cup, \cap, T - \cdot \rangle$, let their Dempster product be defined by

$$(\pi_1 \oplus \pi_2)(A) = \bigcup_{B,C \subset S,\, B \cap C = A} (\pi_1(B) \cap \pi_2(C)). \tag{45}$$

Let the Dempster product of the induced $\mathcal{P}(T)/\sim$-valued basic possibilistic assignments π_1^*, π_2^* be defined by

$$(\pi_1^* \oplus \pi_2^*)(A) = (\pi_1 \oplus \pi_2)^*(A) = [(\pi_1 \oplus \pi_2)(A)] \tag{46}$$

for all $A \subset S$. If $\mathcal{T} = \langle T, \preceq \rangle$ is a complete upper semilattice and a lower semilattice, and if

$$\bigvee (\pi_1(B) \cap \pi_2(C)) = \left(\bigvee \pi_1(B)\right) \wedge \left(\bigvee \pi_2(C)\right) \tag{47}$$

holds for each B, $C \subset S$, then the relation

$$(\pi_1^* \oplus \pi_2^*)(A) = \bigsqcup_{B \cap C = A} [\pi_1^*(B) \sqcap \pi_2^*(C)] \tag{48}$$

is valid for all $A \subset S$. □

Remark. The identity (48) can be obtained also when adapting (45) routinely to the case of partially ordered set $\langle \mathcal{P}(T)/\sim, \sqsubset \rangle$ with its supremum ($\sqcup$) and infimum ($\sqcap$) operations.

Proof. Combining (45) and (46) we obtain that, given $A \subset S$,

$$\begin{aligned}(\pi_1^* \oplus \pi_2^*)(A) &= \left[\bigcup_{B,C \subset S,\, B \cap C = A} (\pi_1(B) \cap \pi_2(C))\right] = \\ &= \bigsqcup_{B,C \subset S,\, B \cap C = A} [\pi_1(B) \cap \pi_2(C)],\end{aligned} \tag{49}$$

as $\langle T, \preceq \rangle$ is supposed to be a complete upper semilattice. Due to the same property, the subset $\pi_1(B) \cap \pi_2(C)$ of T can be replaced by the singleton containing just the supremum value (cf. Theorem 4.1), (ii), so that

$$(\pi_1^* \oplus \pi_2^*)(A) = \bigsqcup_{B,C \subset S,\, B \cap C = A} \left[\left\{\bigvee (\pi_1(B) \cap \pi_2(C))\right\}\right]. \tag{50}$$

Applying Theorem 4.2 to the assumption (47), we obtain that

$$\begin{aligned}(\pi_1^* \oplus \pi_2^*)(A) &= \bigsqcup_{B,C\subset S,\, B\cap C=A} \left[\left\{\left(\bigvee \pi_1(B)\right) \wedge \left(\bigvee \pi_2(C)\right)\right\}\right] = \\ &= \bigsqcup_{B,C\subset S,\, B\cap C=A} \left(\left[\left\{\bigvee \pi_1(B)\right\}\right] \sqcap \left[\left\{\bigvee \pi_2(C)\right\}\right]\right) = \\ &= \bigsqcup_{B,C\subset S,\, B\cap C=A} \left([\pi_1(B)] \sqcap [\pi_2(C)]\right) = \\ &= \bigsqcup_{B,C\subset S,\, B\cap C=A} \left(\pi_1^*(B) \sqcap \pi_2^*(C)\right), \end{aligned} \tag{51}$$

and Theorem 4.2 can be applied once more, but now in the opposite sense. The assertion is proved. □

Let us note that the inequality

$$\bigvee(\pi_1(B) \cap \pi_2(C)) \preceq \left(\bigvee \pi_1(B)\right) \wedge \left(\bigvee \pi_2(C)\right) \tag{52}$$

trivially holds in general, so that the weakened form of (48), namely the relation

$$(\pi_1^* \oplus \pi_2^*)(A) \preceq \bigsqcup_{B,C\subset S,\, B\cap C=A} (\pi_1^*(B) \sqcap \pi_2^*(C)), \tag{53}$$

valid for every $A \subset S$, can be proved without assumption (47).

Assumption (47) can be, and should be, a matter of a more detailed discussion. Let us introduce one particular case when (47) holds true. Let S, W be nonempty sets and let $\pi_0 : \mathcal{P}(S) \to \mathcal{P}(W)$ be a Boolean-valued basic possibilistic assignment, so that $\bigcup_{A\subset S} \pi_0(A) = W$. Set $\mathcal{T} = \langle \mathcal{P}(W), \subset \rangle = \langle T, \preceq \rangle$, hence, $\mathcal{T}$ is the power-set of all subsets of W partially ordered by the relation of set-theoretic inclusion. Define mapping $\lambda : \mathcal{P}(W) \to \mathcal{P}(T) = \mathcal{P}(\mathcal{P}(W))$ in this way:

$$\lambda(W_0) = \{\{w\} : w \in W_0\} \subset \mathcal{P}(W) \tag{54}$$

for each $W_0 \subset W$. In particular, given $A \subset S$, set

$$\pi(A) = \lambda(\pi_0(A)) = \{\{w\} : w \in \pi_0(A) \subset W\} \subset \mathcal{P}(W) = T. \tag{55}$$

Obviously, the supremum operation in $\langle \mathcal{P}(W), \subset \rangle$ is identical with that of the set-theoretic union and the infimum operation reduces to set-theoretic intersection, so that, for each $\mathcal{A} \subset \mathcal{P}(W)$,

$$\bigvee \mathcal{A} = \bigvee_{W_0\in\mathcal{A}} W_0 = \bigcup_{W_0\in\mathcal{A}} W_0 = \bigcup \mathcal{A} \ \text{(shortly)}, \tag{56}$$

and

$$\bigwedge \mathcal{A} = \bigwedge_{W_0\in\mathcal{A}} W_0 = \bigcap_{W_0\in\mathcal{A}} W_0 = \bigcap \mathcal{A} \ \text{(shortly)}, \tag{57}$$

In the particular case of $\pi(A)$, $A \subset S$, we obtain that

$$\bigvee \pi(A) = \bigcup_{w \in \pi_0(A)} \{w\} = \pi_0(A). \tag{58}$$

Evidently,

$$\bigwedge \pi(A) = \bigcap_{w \in \pi_0(A)} \{w\} = \pi_0(A), \tag{59}$$

if $\pi_0(A)$ is a singleton, i. e., if $\pi_0(A) = \{w_0\}$ for some $w_0 \in W$ and $\bigwedge \pi(A) = \emptyset$ (the empty subset of W) otherwise, as $\{w_1\} \cap \{w_2\} = \emptyset$ for every $w_1 \neq w_2$, w_1, $w_2 \in W$. Now,

$$\begin{aligned} \bigvee_{A \subset S} \pi(A) &= \bigvee_{A \subset S} \{\{w\} : w \in \pi_0(A)\} = \bigcup_{A \subset S} \{\{w\} : w \in \pi_0(A)\} = \\ &= \left\{\{w\} : w \in \bigcup_{A \subset S} \pi_0(A)\right) = \{\{w\} : w \in W\} = \lambda(W) = \lambda\left(\mathbf{1}_{\langle \mathcal{P}(W), \subset \rangle}\right) \end{aligned} \tag{60}$$

as W is the maximal (unit) element of the p.o. set $\langle \mathcal{P}(W), \subset \rangle$. Hence, in this sense π is a $\mathcal{P}(T)$-valued Boolean basic possibilistic assignment on S.

Let us consider two $\mathcal{P}(W)$-valued Boolean basic possibilistic assignments π_{01}, π_{02} : $\mathcal{P}(S) \to \mathcal{P}(W)$, let $\pi_1, \pi_2 : \mathcal{P}(S) \to \mathcal{P}(T) = \mathcal{P}(\mathcal{P}(W))$ be defined by

$$\pi_i(A) = \lambda(\pi_{0i}(A)) = \{\{w\} : w \in \pi_{0i}(A)\}, \quad i = 1, 2. \tag{61}$$

Given B, $C \subset S$, we get the result that

$$\begin{aligned} \bigvee(\pi_1(B) \cap \pi_2(C)) &= \bigvee(\{\{w\} : w \in \pi_{01}(B)\} \cap \{\{w\} : w \in \pi_{02}(C)\}) = \\ &= \bigvee \{\{w\} : w \in \pi_{01}(B) \cap \pi_{02}(C)\} = \\ &= \bigcup \{\{w\} : w \in \pi_{01}(B) \cap \pi_{02}(C)\} = \pi_{01}(B) \cap \pi_{02}(C). \end{aligned} \tag{62}$$

Using (58), another calculation yields that

$$\left(\bigvee \pi_1(B)\right) \wedge \left(\bigvee \pi_2(C)\right) = \pi_{01}(B) \cap \pi_{02}(C), \tag{63}$$

so that (47) holds in this particular case.

8. Nonspecifity Degrees and Dempster Rule

A quite legitimate question can arise: why, how, and in which sense is the Dempster combination rule useful? To put the question more explicitly, namely whether and in which sense and degree, the quality of a basic probability or a basic possibilistic assignment is improved when combined with another such assignment. Let us briefly analyze and illustrate this problem, first, for the case of classical numerical probabilistic assignments over a finite space S trying, later, to modify the suggested way of reasoning to the case of nonnumerical assignments with the aim to conserve as many results as possible from those achieved in the classical probabilistic case.

Let S be a finite set, let $m : \mathcal{P}(S) \rightarrow [0,1]$ be a basic probability assignment (b.p.a.) on S, i. e., $\sum_{A \subset S} m(A) = 1$. Subset $A \subset S$ is called a *focal element of* $\mathcal{P}(S)$ *with respect to* m, if $m(A)$ is nonzero, i.e. positive. Let us define the *nonspecificity degree* $W(m)$ of the b.p.a. m on S by the *expected relative size of focal elements of* $\mathcal{P}(S)$ *with respect to* m, in symbols,

$$W(m) = \sum_{A \subset S} (\|A\|/\|S\|)\, m(A), \tag{64}$$

where $\|A\|$ ($\|S\|$, resp.) denotes the cardinality, i. e., in this case simply the number of the elements of A (of S, resp.). The intuition behind reads that the smaller is the values $W(m)$ the better is m in the sense that it specifies the value s_0 of the actual state of the system under consideration within the framework of smaller subsets of S. Or, $W(m)$ takes its maximal value 1 iff m is the empty b.p.a. when $m(S) = 1$ and, consequently, $m(A) = 0$ for all $A \subset S$, $A \neq S$. Hence, m offers no more detailed specification concerning the actual value s_0 beyond the tacitly accepted assumption of the closed world according to which s_0 is supposed to belong to S. When restricting ourselves to the normalized b.p.a.'s, i. e., to b.p.a.'s such that $m(\emptyset) = 0$, $W(m)$ takes its minimum value $\|S\|^{-1}$ only when $m(A) > 0$ can hold when $\|A\| = 1$, i.e., when all focal elements are singletons. Evidently, in this case m defines a probability distribution on S. This class of b.p.a.'s also includes, as particular cases, the "degenerated" b.p.a.'s $m_{\{s\}}$ such that $m_{\{s\}}(\{s\}) = 1$, $m_{\{s\}}(\{t\}) = 0$ for each $s, t \in S$, $s \neq t$. These b.p.a.'s define the case when the obtained empirical data enable determining (with the probability one, i.e., "almost surely") the actual state of the system under investigation. Values $W(m) < 1/\|S\|$ are ascribed to partially (if $W(m) > 0$) or completely (if $W(m) = 0$) inconsistent b.p.a.'s, when $m(\emptyset) > 0$ or even $m(\emptyset) = 1$ holds for the empty subset $\emptyset$ of S.

As a matter of fact, the nonspecificity degree W defined by (64) agrees, up to the normalizing constant $\|S\|^{-1}$, with the degree proposed and investigated by Yager in [18]. Perhaps a nonspecificity degree used more often is the one suggested by Dubois and Prade in [2], when the size of a subset $A \subset S$ is not quantified by its cardinality $\|A\|$, but rather by the binary logarithm of this cardinality, so that the resulting nonspecificity degree W_0 ascribes the value $W_0(m) = \sum_{A \subset S} \log_2(\|A\|)\, m(A)$ to each b.p.a. m on S (the convention according to which $0 \cdot \log_2 0 = 0$ is adopted). This definition makes the notion of nonspecificity degrees close to that of entropy in the classical Shannon information theory (e. g., for appropriately defined stochastical independence of two or more data sources the nonspecificity degrees add together). However, for our purposes it would be difficult to distinguish, within a non-numerical partially ordered structure of quantities, sizes $\|A\|$ and $\log_2(\|A\|)$ from each other. Hence, we have a limit ourselves just to the nonspecificity degree defined by (64) aiming to "translate" it into appropriate terms of non-numerical structures. It could be a subject of further interesting investigative efforts to suggest more alternatives for non-numerical nonspecificity degrees as well as some reasonable criteria enabling one to classify the merits and the weak points of various alternatives.

Given b.p.a.'s m_1, m_2 on the same finite set S and defining their Dempster product $m_1 \oplus m_2$ in the standard combinatoric way, so that

$$(m_1 \oplus m_2)(A) = \sum_{B,C \subset S,\, B \cap C = A} m_1(B)\, m_2(C) \tag{65}$$

for each $A \subset S$, we can prove (cf. [7]) the inequality of

$$W(m_1 \oplus m_2) \leq W(m_1) \wedge W(m_2)\,, \tag{66}$$

where $\wedge$ denotes the standard infimum (i.e., minimum in this case) in $[0,1]$. Also the dual inequality holds. Defining the dual Dempster combination rule $\otimes$ by

$$(m_1 \otimes m_2)(A) = \sum_{B,C \subset S,\, B \cup C = A} m_1(B)\, m_2(C), \tag{67}$$

we can prove, dually to (66), that the inequality

$$W(m_1 \otimes m_2) \geq W(m_1) \vee W(m_2) \tag{68}$$

holds with $\vee$, denoting the standard supremum in $[0,1]$ (cf. also [7] for more details).

Analyzing the proofs of (66) and (68) we can observe that the only property of the quantitative criterion $\|A\|/\|S\|$ used throughout these proofs consists in its monotonicity with respect to the set inclusion, i. e., in the trivial fact that $A \subset B \subset S$ implies $\|A\|/\|S\| \leq \|B\|/\|S\|$. Hence, (66) and (68) can be generalized to the case when the relative cardinality of the subsets of S is replaced by a *fuzzy measure*, i. e., by mapping $\lambda : \mathcal{P}(S) \to [0,1]$ such that $\lambda(\emptyset) = 0$, $\lambda(S) = 1$, and $\lambda(A) \leq \lambda(B)$ holds for each $A \subset B \subset S$ (cf., e.g., [14] for more details on fuzzy measures). Given a b.p.a. m on a finite set S, we set

$$W_\lambda(m) = \sum_{A \subset S} \lambda(A)\, m(A), \tag{69}$$

and we can prove that for all b.p.a.'s m_1 and m_2 on S the inequalities of

$$W_\lambda(m_1 \oplus m_2) \leq W_\lambda(m_1) \wedge W_\lambda(m_2) \tag{70}$$

and

$$W_\lambda(m_1 \otimes m_2) \geq W_\lambda(m_1) \vee W_\lambda(m_2) \tag{71}$$

are valid with $\vee$ and $\wedge$ denoting the standard supremum and infimum in $[0,1]$.

Assertions generalizing (70) and (71) to the case of Boolean-valued basic probabilistic assignments can be also easily proved. Let S be a nonempty set, let $\langle T, \preceq \rangle$ be a p.o. set. Let π_1, π_2 be Boolean-valued basic possibilistic assignments defined on S and taking their values in $\mathcal{P}(T)$, i.e., π_i takes $\mathcal{P}(S)$ into $\mathcal{P}(T)$ in such a way that

$$\bigcup_{A \subset S} \pi_1(A) = \bigcup_{A \subset S} \pi_2(A) = \bigvee \mathcal{P}(T) = T\,. \tag{72}$$

Let the Dempster product $\pi_1 \oplus \pi_2$ and the dual Dempster product $\pi_1 \otimes \pi_2$ be defined by

$$(\pi_1 \oplus \pi_2)(A) = \bigcup_{B,C \subset S,\, B \cap C = A} (\pi_1(B) \cap \pi_2(C)), \tag{73}$$

$$(\pi_1 \otimes \pi_2)(A) = \bigcup_{B,C \subset S,\, B \cup C = A} (\pi_1(B) \cap \pi_2(C)) \tag{74}$$

for every $A \subset S$. Let $\lambda : \mathcal{P}(S) \to \mathcal{P}(T)$ be a $\mathcal{P}(T)$*-valued Boolean fuzzy measure* on S, i.e. $\lambda(\emptyset) = \emptyset$, $\lambda(S) = T$, and $\lambda(A) \subset \lambda(B)$ holds for each $A \subset B \subset S$.

Using these notions, let us define the $\mathcal{P}(T)$*-valued Boolean nonspecificity degree* $W_\lambda^b(\pi)$ of a Boolean-valued b.poss.a. π with values in $\mathcal{P}(T)$, setting

$$W_\lambda^b(\pi) = \bigcup_{A \subset S} (\lambda(A) \cap \pi(A)). \tag{75}$$

Theorem 8.1. For each Boolean-valued b.poss.a.'s π_1, π_2 on S, the set inclusions

$$W_\lambda^b(\pi_1 \oplus \pi_2) \subset W_\lambda^b(\pi_1) \cap W_\lambda^b(\pi_2) \tag{76}$$

and, dually,

$$W_\lambda^b(\pi_1 \otimes \pi_2) \supset W_\lambda^b(\pi_1) \cup W_\lambda^b(\pi_2) \tag{77}$$

are valid. □

Proof. Evidently, when proving (76) the only thing we have to prove is the inclusion

$$W_\lambda^b(\pi_1 \oplus \pi_2) \subset W_\lambda^b(\pi_1), \tag{78}$$

as the proof for π_2 is analogous and (76) trivially follows from both these inclusions. Hence, we have to prove that the relation

$$\bigcup_{A \subset S} \left(\bigcup_{B,C \subset S,\, B \cap C = A} (\pi_1(B) \cap \pi_2(C)) \cap \lambda(A) \right) \subset \bigcup_{A \subset S} (\pi_1(A) \cap \lambda(A)) \tag{79}$$

holds. But, $B \cap C = A$ yields that $A \subset B$, so that $\lambda(A) \subset \lambda(B)$; hence,

$$\bigcup_{A \subset S} \left(\bigcup_{B,C \subset S,\, B \cap C = A} (\pi_1(B) \cap \pi_2(C)) \cap \lambda(A) \right) \subset$$
$$\bigcup_{A \subset S} \left(\bigcup_{B,C \subset S,\, B \cap C = A} (\pi_1(B) \cap \pi_2(C)) \cap \lambda(B) \right) =$$

$$
\begin{aligned}
&= \bigcup_{B,C\subset S} (\pi_1(B)\cap\pi_2(C)\cap\lambda(B)) = \bigcup_{B\subset S}\left(\pi_1(B)\cap\lambda(B)\cap\bigcup_{C\subset S}\pi_2(S)\right) = \\
&= \bigcup_{B\subset S}(\pi_1(B)\cap\lambda(B)), \qquad (80)
\end{aligned}
$$

as $\bigcup_{C\subset S}\pi_2(C) = T$ holds, so that (79) and, consequently, (76) follows. Dually, we have to prove the inclusion

$$W^b_\lambda(\pi_1\otimes\pi_2)\supset W^b_\lambda(\pi_1)\,, \qquad (81)$$

i.e., the inclusion

$$\bigcup_{A\subset S}\left(\bigcup_{B,C\subset S,\,B\cup C=A}(\pi_1(B)\cap\pi_2(C))\cap\lambda(A)\right)\supset\bigcup_{A\subset S}(\pi_1(A)\cap\lambda(A)). \qquad (82)$$

But, $B\cup C$ yields that $B\subset A$, so that $\lambda(B)\subset\lambda(A)$ holds; hence,

$$
\begin{aligned}
&\bigcup_{A\subset S}\left(\bigcup_{B,C\subset S,\,B\cup C=A}(\pi_1(B)\cap\pi_2(C))\cap\lambda(A)\right)\supset \qquad (83)\\
&\bigcup_{A\subset S}\left(\bigcup_{B,C\subset S,\,B\cup C=A}(\pi_1(B)\cap\pi_2(C))\cap\lambda(B)\right) = \\
&= \bigcup_{B,C\subset S}(\pi_1(B)\cap\pi_2(C)\cap\lambda(B)) = \\
&= \bigcup_{B\subset S}(\pi_1(B)\cap\lambda(B))
\end{aligned}
$$

as has been proved above. The assertion is proved. □

Lemma 3.2 immediately yields that, under the notations and conditions of Theorem 8.1, (76) and (77) imply that

$$\left[W^b_\lambda(\pi_1\oplus\pi_2)\right]\sqsubset\left[W^b_\lambda(\pi_1)\cap W^b_\lambda(\pi_2)\right] \qquad (84)$$

and

$$\left[W^b_\lambda(\pi_1\otimes\pi_2)\right]\sqsupset\left[W^b_\lambda(\pi_1)\cup W^b_\lambda(\pi_2)\right]\,. \qquad (85)$$

If $\langle T,\preceq\rangle$ is a complete upper semilattice, then Theorem 4.1 (iv) enables rewriting (85) in the form

$$\left[W^b_\lambda(\pi_1\otimes\pi)\right]\sqsupset\left[W^b_\lambda(\pi_1)\right]\sqcup\left[W^b_\lambda(\pi_2)\right]\,; \qquad (86)$$

if, moreover, $\langle T,\preceq\rangle$ is a lower semilattice, then Theorem 4.2 and (84) yield that

$$\left[W^b_\lambda(\pi_1\oplus\pi)\right]\sqsubset\left[W^b_\lambda(\pi_1)\right]\sqcap\left[W^b_\lambda(\pi_2)\right]\,. \qquad (87)$$

Hence, under the conditions that $\langle T, \preceq\rangle$ is a complete upper semilattice and, simultaneously, a lower semilattice, the mapping $[W_\lambda^b(\cdot)]$ seems to be a reasonable $\mathcal{P}(T)/\sim$-valued nonspecificity degree of $\mathcal{P}(T)/\sim$-valued basic possibilistic assignments, copying in a reasonable and nontrivial way some intuitive and acceptable properties of the nonspecificity degrees W, W_λ and W_λ^b as outlined above. At the same time, these conditions imposed on the partially ordered set $\langle T, \preceq\rangle$ seem to be the weakest ones under which such a modification is possible and nontrivial.

9. Conclusions

When considering some possibilities of applications of non-numerical uncertainty degrees in general, and non-numerical basic possibilistic assignments and belief functions in particular, we can modify the basic paradigma used in the case of probabilistically quantified and processed degrees of uncertainty. In this case, elementary random events, mutually disjointed and defining a composition of a certain event, are supposed to be endowed by non-negative probability values summing to one. The assumption of additivity or σ-additivity, together with the assumption of statistical (stochastical) independence of at least some random events if they occur repeatedly, enable computing probabilities for large collection of random events defining a very rich structure.

In the case of non-numerically quantified uncertainties we can start from a structure of events; the degrees of uncertainty of at least some of them can be compared by the relation "greater than" or "greater than or equal to". The degrees of uncertainty of some events can be taken, by a subject, as acceptable as far as the risk, following when taking them as surely valid, is concerned, some other degrees of uncertainty are taken as too great to accept the same decision. In both the cases the subject's feelings are immediate, not based on some numerical evaluations of these degrees of uncertainty by real numbers, in particular those from the unit interval. When taken as sets, the events, are structured by the relation of the set-theoretical inclusion, perhaps with some more demands imposed on this structure; their degrees of uncertainty are structured by a partially ordering relations, and the aim is to compute the degree of uncertainty of some more sophistically defined events. Here "to compute" means to prove that the uncertainty degrees of these more complex events are comparable with those ascribed either to the elementary events supposed to be known a priori, or with the degrees of uncertainty of events for which such a comparison has been already proved.

In particular, we can process, in this way, the non-numerical uncertainties ascribed to the events like "the actual state of the system in question is in an investigated subset of S", demanding answers of this kind: "the degree of uncertainty of this event is at least as great as the degree of uncertainty ascribed to an event A", or "the degree of uncertainty of this event is smaller than that ascribed to an event B", A and B being events from the elementary basis in both the cases so that the subject can take profit of the uncertainty degrees ascribed to them, in her/his decision making, thanks to her/his knowledge concerning the practical and extra-mathematical circumstances of the system and the decision-making problem under consideration. E.g., a solution to a problem may be taken as good and fail-proof if we know that the uncertainty describing the possibility of its failure is not greater than the danger of a strong earthquake in our region, even if we perhaps do not know the precise probability value of the occurrence of the last catastrophe.

At least the three following problems or directions of further investigation would deserve being taken into consideration.

(I) We have chosen, in this paper, a rather general approach when degrees of uncertainty are subsets of a partially ordered set. Consequently, the set of uncertainty degrees can be endowed by two structures: the Boolean one, generated on the power-set $\mathcal{P}(T)$ of the partially ordered set $\langle T, \preceq \rangle$ by the usual set-theoretic operations and relations (e.g., $\subset$, $\cap$, $\cup$), and the relations and operations defined through the partial ordering relation $\preceq$ on T (e.g., $\sqsubset$, $\sqcap$, $\sqcup$). A question arises whether it is possible to obtain a similar model either with single-valued uncertainties, even if from a larger set than T, or with set-valued uncertainties but structured only by usual set-theoretic operations and relations.

(II) In the author's opinion, the conditions imposed in this paper on the structure of the set of uncertainty degrees seem to be the weakest ones under which a non-trivial fragment of the theory of belief functions can be built up. Nevertheless, this conjecture should be re-written in a more formalized way to be either proved or rejected.

(III) It would be interesting and perhaps useful to seek for a non-artificial and rather practical structure of events charged by uncertainty such that this structure would meet the demands imposed in this paper, but would not meet some stronger demands requested by, say, probabilistic models of decision making under uncertainty.

Let us hope that at least some of these problems will be touched by further investigative efforts.

Items [3] and [12] listed below may serve as good sources of elementary knowledge concerning Boolean algebras, partial ordering and related structures. Monographs [5] and [10] then provide the basic pieces of information concerning the measure theory in general and the probability theory in particular, both in their most abstract and mathematically formalized settings. Some more references, thematically very close to the subject of this paper, are also listed below.

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