Czechoslovak Mathematical Journal, 66 (141) (2016), 27–34

ISOMETRIC COMPOSITION OPERATORS ON WEIGHTED DIRICHLET SPACE

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(Received October 2, 2014)

Abstract. We investigate isometric composition operators on the weighted Dirichlet space $\mathcal{D}_\alpha$ with standard weights $(1-|z|^2)^\alpha$, $\alpha > -1$. The main technique used comes from Martín and Vukotić who completely characterized the isometric composition operators on the classical Dirichlet space $\mathcal{D}$. We solve some of these but not in general. We also investigate the situation when $\mathcal{D}_\alpha$ is equipped with another equivalent norm.

Keywords: composition operator; weighted Dirichlet space; isometry

MSC 2010: 46B04, 47B38, 47B33

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane. Denote by $H(\mathbb{D})$ the collection of all holomorphic functions on $\mathbb{D}$ and by $S(\mathbb{D})$ the collection of all analytic self-maps of the unit disk. For $\varphi \in S(\mathbb{D})$, the associated composition operator C_φ is defined by

$$C_\varphi(f) = (f \circ \varphi), \quad f \in H(D).$$

The study of composition operators achieved abundant results in the last four decades and has become a major driving force for the development of both complex analysis and operator theory. The main goal is to relate the operator theoretical properties of C_φ to the function theoretical properties of its symbol φ. For general information about composition operators, we refer the interested readers to two excellent books [3] and [11].

The work was supported in part by the National Natural Science Foundation of China (Grant Nos. 11371276; 11301373; 11401426).

Let X and Y be two normed vector spaces. A linear operator T mapping X into Y is called a linear isometry (or simply an isometry) if $\|Tx\|_Y = \|x\|_X$ for every $x \in X$. Classifying the form of isometries dates back to Banach who characterized the isometries on certain L^p spaces and on $C(X)$, where X is a compact metric space, in [1]. Since then, there has been much work on characterizing isometries on function spaces. A good reference book in this area is [4].

For characterizations of isometries among composition operators, it was first pointed out by Ryff [9] that every isometric composition operator on the classical Hardy space H^p is induced by an inner function vanishing at the origin. Ryff's result also appears in the work of Nordgren [8] and Shapiro [10]. For the weighted Bergman space A^2_α, the authors of [2] showed that only rotations can induce isometric composition operators. In [6], Martín and Vukotić extended this result to the general A^p_α for $1 < p < \infty$. They showed that their method also provides a new proof to Ryff's result on the Hardy space H^p. In another paper [7], Martín and Vukotić considered this question on the classical Dirichlet space $\mathcal{D}$ and developed a technique to solve this problem completely. To be specific, they showed that each isometric composition operator on $\mathcal{D}$ is induced by a univalent full self-map, where, by a full self-map, they meant an analytic self-map φ whose image has full measure, i.e. $\mathrm{Area}(\varphi(\mathbb{D})) = \mathrm{Area}(\mathbb{D})$.

Motivated by these results, we investigate the isometric composition operators acting on the weighted Dirichlet space $\mathcal{D}_\alpha$ with standard weights $(1-|z|^2)^\alpha$, $\alpha > -1$. It is well known that this family of analytic function spaces includes properly all the classical function spaces mentioned above, equipped with equivalent but not equal norms. The main technique in this paper comes from [7]. We solved some of these problems but not in general. We believe it is a nontrivial problem and a complete solution will arise some experts' interest.

2. Isometric composition operators on the weighted Dirichlet space

For $\alpha > -1$, the weighted Dirichlet space $\mathcal{D}_\alpha$ is defined as

$$\mathcal{D}_\alpha = \left\{ f \in H(\mathbb{D});\ \|f\|^2_\alpha = \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^\alpha \,\mathrm{d}A(z) < \infty \right\}$$

where $\mathrm{d}A$ is the Lebesgue measure normalized so that the area of the unit disk is 1. The quantity $\|\cdot\|_\alpha$ defines a complete semi-norm on $\mathcal{D}_\alpha$ and a norm is often given by $\|f\|_{\mathcal{D}_\alpha} = \sqrt{|f(0)|^2 + \|f\|^2_\alpha}$. The space $\mathcal{D}_\alpha$ is a Hilbert space under this norm and the inner product is obviously given by

$$\langle f, g \rangle_{\mathcal{D}_\alpha} = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}(1-|z|^2)^\alpha \,\mathrm{d}A(z).$$

Particularly, $\mathcal{D}_1$ and $\mathcal{D}_\alpha$ with $\alpha > 1$ are the classical Hardy space H^2 and the weighted Bergman space $A^2_{\alpha-2}$, respectively, both with equivalent but not equal norms.

Supposing φ is an analytic self-map of the unit disk, it is easy to see by a change of variables ([3], Theorem 2.32) that

$$\begin{aligned}\|f\circ\varphi\|_\alpha^2 &= \int_{\mathbb{D}} |f'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^\alpha \,\mathrm{d}A(z) \\ &= \int_{\mathbb{D}} |f'(w)|^2 n_{\varphi,\alpha}(w)\,\mathrm{d}A(w),\end{aligned}\tag{2.1}$$

where

$$n_{\varphi,\alpha}(w) = \sum_{\varphi(z_j)=w} (1-|z_j|^2)^\alpha$$

is called the generalized counting function.

Proposition 2.1. *Let $\alpha > -1$, let φ be an analytic self-map of the unit disk. Then C_φ defines an isometry on $\mathcal{D}_\alpha$ if and only if $\varphi(0) = 0$ and*

$$n_{\varphi,\alpha}(w) = (1-|w|^2)^\alpha$$

holds for every $w \in \mathbb{D}$ except a set of zero area measure.

Proof. The sufficiency is obvious by (2.1). We will only prove the necessity.

Suppose C_φ is an isometry on $\mathcal{D}_\alpha$, then C_φ preserves the inner product by the polarization identity, thus for any $f \in \mathcal{D}_\alpha$ we have

$$f(0) = \langle f, 1\rangle_{\mathcal{D}_\alpha} = \langle f\circ\varphi, 1\rangle_{\mathcal{D}_\alpha} = f(\varphi(0)).$$

Taking $f(z) = z$, we get $\varphi(0) = 0$. Therefore we have

$$\|f\circ\varphi\|_\alpha^2 = \int_{\mathbb{D}} |f'(z)|^2 n_{\varphi,\alpha}(z)\,\mathrm{d}A(z) = \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^\alpha\,\mathrm{d}A(z) = \|f\|_\alpha^2, \quad f\in\mathcal{D}_\alpha$$

where we used the formula (2.1). That is, for every function g in the weighted Bergman space A^2_α of the square integrable analytic functions in $\mathbb{D}$ with respect to the measure $(1-|z|^2)^\alpha\,\mathrm{d}A(z)$ we have

$$\int_{\mathbb{D}} |g(z)|^2(1-|z|^2)^\alpha\,\mathrm{d}A(z) = \int_{\mathbb{D}} |g(z)|^2 n_{\varphi,\alpha}(z)\,\mathrm{d}A(z).$$

This means the identity map from A^2_α into $L^2(\mathbb{D}, n_{\varphi,\alpha}\,\mathrm{d}A(z))$ is an isometry. Then the polarization identity yields

$$\int_{\mathbb{D}} f(z)\overline{g(z)}(1-|z|^2)^\alpha\,\mathrm{d}A(z) = \int_{\mathbb{D}} f(z)\overline{g(z)}n_{\varphi,\alpha}(z)\,\mathrm{d}A(z), \quad f,g \in A^2_\alpha.$$

In particular, choose $f(z) = z^m$, $m \geqslant 0$ and $g(z) = z^n$, $n \geqslant 0$ to get

$$\int_{\mathbb{D}} z^m\,\overline{z}^{\,n}(1-|z|^2)^\alpha\,\mathrm{d}A(z) = \int_{\mathbb{D}} z^m\,\overline{z}^{\,n} n_{\varphi,\alpha}(z)\,\mathrm{d}A(z).$$

By the linearity of integration and the Stone-Weierstrass theorem, we know

$$\int_{\mathbb{D}} u(z)(1-|z|^2)^\alpha\,\mathrm{d}A(z) = \int_{\mathbb{D}} u(z)n_{\varphi,\alpha}(z)\,\mathrm{d}A(z)$$

holds for all continuous functions on the closed unit disk. Now, the Riesz representation theorem implies that $(1-|z|^2)^\alpha\,\mathrm{d}A(z) = n_{\varphi,\alpha}(z)\,\mathrm{d}A(z)$, which means $n_{\varphi,\alpha}(z) = (1-|z|^2)^\alpha$ almost everywhere on $\mathbb{D}$ by the Lebesgue-Radon-Nikodym theorem. $\square$

The next theorem, which is due to Martín and Vukotić [6], follows immediately as a special case ($\alpha = 0$) of Proposition 2.1.

Theorem 2.2 ([6], Theorem A). *Let φ be an analytic self-map of the unit disk $\mathbb{D}$. Then C_φ is an isometry on $\mathcal{D}$ if and only if $\varphi(0) = 0$ and φ is a univalent full map of $\mathbb{D}$.*

Theorem 2.3. *Let $-1 < \alpha < 0$ or $\alpha = 1$. Let φ be an analytic self-map of the unit disk $\mathbb{D}$. Then C_φ is an isometry on $\mathcal{D}_\alpha$ if and only if φ is a rotation of the unit disk.*

Proof. The sufficiency is obvious.

It remains to prove the necessity. We divide this into two cases:

(1) The case "$-1 < \alpha < 0$". Since φ fixes the origin by Proposition 2.1, Schwarz's lemma says $|\varphi(z)| \leqslant |z|$ for all $z \in \mathbb{D}$. From Proposition 2.1 we know that

$$\sum_{\varphi(z_j)=w} (1-|z_j|^2)^\alpha = (1-|w|^2)^\alpha, \quad \text{a.e.}$$

Since $-1 < \alpha < 0$, each term on the left hand side is no less than the right hand term, thus we have $|\varphi(z)| = |z|$ almost everywhere in $\mathbb{D}$, which implies φ must be a rotation by Schwarz's lemma.

(2) The case "$\alpha = 1$". An easy calculation shows $\|z^n\|_{\mathcal{D}_1} = 1 - 1/(n+1)$ for all $n \geqslant 1$. By [5], Theorem 2.2, φ is an inner function fixing the origin. Then [3], Lemma 3.27, says $\prod\limits_{\varphi(z_j)=w} |z_j| = |w|$ holds almost everywhere in $\mathbb{D}$. By Proposition 2.1, we have

$$\sum_{\varphi(z_j)=w} (1 - |z_j|^2) = 1 - |w|^2 = 1 - \prod_{\varphi(z_j)=w} |z_j|^2 \quad \text{a.e.} \tag{2.2}$$

If there is a point $w \in \mathbb{D}$ which has only one preimage under φ and satisfies (2.2), then the first equality implies φ is a rotation by Schwarz's lemma. Otherwise, take any point w satisfying (2.2). Since $(1-x) + (1-y) > (1-xy)$ for any $x, y \in (0,1)$, we have

$$\sum_{\varphi(z_j)=w} 1 - |z_j|^2 > 1 - \prod_{\varphi(z_j)=w} |z_j|^2,$$

a contradiction. □

Remark. We have mentioned that $\mathcal{D}_1$ coincides with the classic Hardy space H^2 with equivalent but different norms. It was pointed out by Ryff [9] that isometric composition operators on H^2 are those induced by inner functions fixing the origin (see also [8], [10]), which is quite different from our result.

For the general $\alpha > 0$, $\alpha \neq 1$, we conjecture that only rotations can induce isometric composition operators on $\mathcal{D}_\alpha$. However, we are just able to get some partial results.

Theorem 2.4. *Let $\alpha > 0$, let φ be an analytic self-map of the unit disk $\mathbb{D}$ but not a rotation. If C_φ is an isometry on $\mathcal{D}_\alpha$, then*

(1) *the area of the set $\{w\colon n_{\varphi(w),0} = 1\}$ is 0;*
(2) *0 is the unique zero of φ and is of multiplicity 1;*
(3) *$n_{\varphi,\alpha}(w) \leqslant (1 - |w|^2)^\alpha$ holds for all $z \in \mathbb{D}$.*

Proof. (1) is obvious by Schwarz's lemma.

For (2), suppose $a \in \mathbb{D}$, $a \neq 0$ is another zero of φ; we pick a small neighborhood $W \subseteq \varphi(\mathbb{D})$ of the origin, then the preimage of W must contain two neighborhoods of a and 0, denoted by U and V, respectively. Choose W small enough such that there are no other zeros in U and V. By the open mapping theorem for analytic functions, the intersection of images of U and V under φ will be a neighborhood of the origin, denoted still by W. Since the set of all points satisfying (2.2) is dense in $\mathbb{D}$, there exists a sequence $\{w_n\}$ in W satisfying (2.2) and $w_n \to 0$. By the continuity of φ and our assumptions on U and V, we can find two sequences $\{u_n\} \subseteq U$ and $\{v_n\} \subseteq V$

such that $\varphi(u_n) = \varphi(v_n) = w_n$ and $u_n \to a$, $v_n \to 0$. Now taking limits on both sides of the inequality $(1-|u_n|^2)^\alpha + (1-|v_n|^2)^\alpha \leqslant (1-|w_n|^2)^\alpha$, we get $(1-|a|^2)^\alpha \leqslant 0$ which is a contradiction. It is easy to check that the above argument remains valid if $a = 0$, which means 0 is a zero of multiplicity at least two. Thus, (2) is proved.

The proof of (3) is similar. Suppose (3) fails at a point $w \in \mathbb{D}$, i.e. $n_{\varphi,\alpha}(w) > (1-|w|^2)^\alpha$. By Schwarz's lemma, w has at least two preimages. Now choose an integer N such that

$$\sum_{j=1}^{N}(1-|z_j|^2)^\alpha > (1-|w|^2)^\alpha. \tag{2.3}$$

An argument similar to the proof of (2) shows that there exists $N+1$ open sets $U_1, U_2, \ldots, U_N, W$ such that $z_j \in U_j$ for $j = 1, 2, \ldots, N$ and $w \in W = \bigcap_{j=1}^{N} \varphi(U_j)$. Adjusting these open sets to be sufficiently small if necessary, we can tell (2.3) holds for all points in W, which leads to a contradiction according to Proposition 2.1. $\square$

Corollary 2.5. *Let $\alpha > 0$, $\alpha \neq 1$, let φ be an analytic self-map of the unit disk $\mathbb{D}$. Then C_φ is a surjective isometry on $\mathcal{D}_\alpha$ if and only if φ is a rotation of the unit disk.*

Proof. We prove this by showing φ is a univalent self-map, then the result follows due to (1) in the above theorem. Suppose $\varphi(a) = \varphi(b)$ for some $a, b \in \mathbb{D}$, then $f(\varphi(a)) = f(\varphi(b))$ for all $f \in \mathcal{D}_\alpha$. Since C_φ is a surjective isometry, there exists a function g in $\mathcal{D}_\alpha$ such that $z = g \circ \varphi$. Hence $a = g(\varphi(a)) = g(\varphi(b)) = b$, so φ is univalent. $\square$

3. Isometric composition operators on the weighted Dirichlet space with an alternative norm

When dealing with the weighted Dirichlet space $\mathcal{D}_\alpha$, some experts often use the weight function $\log^\alpha(1/|z|^2)$ instead of $(1-|z|^2)^\alpha$, although they both define equivalent norms. In this section, we investigate isometric composition operators on the weighted Dirichlet space under this norm, denoted still by $\|\cdot\|_{\mathcal{D}_\alpha}$. Similarly to Proposition 2.1, we have

Proposition 3.1. *Let $\alpha > -1$ and let φ be an analytic self-map of the unit disk. Then C_φ is an isometry on $\mathcal{D}_\alpha$ if and only if $\varphi(0) = 0$ and*

$$\sum_{\varphi(z_j)=w}\Big(\log\frac{1}{|z_j|^2}\Big)^\alpha = \Big(\log\frac{1}{|w|^2}\Big)^\alpha \quad \text{a.e.}$$

For $\alpha = 0$, this yields the statement of Theorem 2.2.

For $0 < \alpha < 1$, it is not clear to us whether there exists any nontrivial composition operator defining an isometry on $\mathcal{D}_\alpha$, which, we believe, is still an open question.

For $\alpha = 1$, by the famous Littlewood-Paley identity, D_1 coincides with H^2 with the same norm. We have noticed in the remark following Theorem 2.3 that the isometric composition operators are those induced by inner functions vanishing at the origin. In fact, this is an immediate consequence of Proposition 3.1 and [3], Lemma 3.27.

For $-1 < \alpha < 0$ or $\alpha > 1$, we have found that no nontrivial isometric composition operator exists on $\mathcal{D}_\alpha$. To prove this, we need the following observation.

Lemma 3.2. *Let $\alpha > 1$ and let $\{a_n\}_{n=1}^\infty$ a sequence of nonnegative numbers such that $\sum\limits_{n=1}^\infty a_n < \infty$. Then $\sum\limits_{n=1}^\infty a_n^\alpha \leqslant \Big(\sum\limits_{n=1}^\infty a_n\Big)^\alpha$, where the equality holds if and only if there is at most one nonzero element in $\{a_n\}_{n=1}^\infty$.*

Proof. If there is at most one nonzero element in $\{a_n\}_{n=1}^\infty$, the result is obvious. Otherwise, note that $0 \leqslant a_n \setminus \sum\limits_{n=1}^\infty a_n < 1$, so we have $\Big(a_n \setminus \sum\limits_{n=1}^\infty a_n\Big)^\alpha < a_n \setminus \sum\limits_{n=1}^\infty a_n$ since $\alpha > 1$. Summing both sides with respect to n, the result follows. $\square$

Theorem 3.3. *Let $-1 < \alpha < 0$ or $\alpha > 1$, let φ be an analytic self-map of $\mathbb{D}$. Then C_φ defines an isometry on $\mathcal{D}_\alpha$ if and only if φ is a rotation of $\mathbb{D}$.*

Proof. Sufficiency is obvious.

Now we prove the necessity. Supposing C_φ defines an isometry on $\mathcal{D}_\alpha$, we know from Proposition 3.1 that $\varphi(0) = 0$ and

$$\sum_{\varphi(z_j)=w} \Big(\log \frac{1}{|z_j|^2}\Big)^\alpha = \Big(\log \frac{1}{|w|^2}\Big)^\alpha \quad \text{a.e.} \tag{3.1}$$

If $-1 < \alpha < 0$, since $\varphi(0) = 0$, we have $|z_j| \geqslant |w|$ for each j by Schwarz's lemma. Hence (3.1) implies φ is a rotation.

If $\alpha > 1$, from Lemma 3.2 we get

$$\Big(\log \frac{1}{|w|^2}\Big)^\alpha = \sum_{\varphi(z_j)=w} \Big(\log \frac{1}{|z_j|^2}\Big)^\alpha \leqslant \Big(\sum_{\varphi(z_j)=w} \log \frac{1}{|z_j|^2}\Big)^\alpha \quad \text{a.e.}$$

The Littlewood inequality ([3], Theorem 2.29) reads

$$\sum_{\varphi(z_j)=w} \log \frac{1}{|z_j|^2} \leqslant \log \frac{1}{|w|^2}.$$

Therefore, we have

$$\left(\log \frac{1}{|w|^2}\right)^{\alpha} = \sum_{\varphi(z_j)=w} \left(\log \frac{1}{|z_j|^2}\right)^{\alpha} = \left(\sum_{\varphi(z_j)=w} \log \frac{1}{|z_j|^2}\right)^{\alpha}, \quad \text{a.e.} \tag{3.2}$$

The latter equality in (3.2) implies φ must be univalent in $\mathbb{D}$ by Lemma 3.2; then the former equality in (3.2) and the fact $\varphi(0) = 0$ imply that φ must be a rotation. □

Remark. For $\alpha > 1$, D_α is the weighted Bergman space $A^2_{\alpha-2}$ with equivalent norms. Our result coincides with the one provided by Carswell and Hammond [2], even under different norms.

Acknowledgment. The authors would like to thank the referee for useful comments and suggestions which improved the presentation of this paper.

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