Czechoslovak Mathematical Journal, 58 (133) (2008), 155–172

ON A CLASS OF NONLINEAR PROBLEMS INVOLVING A $p(x)$-LAPLACE TYPE OPERATOR

Mihai Mihăilescu, Craiova

(Received January 20, 2006)

Abstract. We study the boundary value problem $-\mathrm{div}((|\nabla u|^{p_1(x)-2}+|\nabla u|^{p_2(x)-2})\nabla u) = f(x,u)$ in Ω, $u=0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$. Our attention is focused on two cases when $f(x,u) = \pm(-\lambda|u|^{m(x)-2}u + |u|^{q(x)-2}u)$, where $m(x) = \max\{p_1(x), p_2(x)\}$ for any $x \in \overline{\Omega}$ or $m(x) < q(x) < N\cdot m(x)/(N-m(x))$ for any $x \in \overline{\Omega}$. In the former case we show the existence of infinitely many weak solutions for any $\lambda > 0$. In the latter we prove that if λ is large enough then there exists a nontrivial weak solution. Our approach relies on the variable exponent theory of generalized Lebesgue-Sobolev spaces, combined with a $\mathbb{Z}_2$-symmetric version for even functionals of the Mountain Pass Theorem and some adequate variational methods.

Keywords: $p(x)$-Laplace operator, generalized Lebesgue-Sobolev space, critical point, weak solution, electrorheological fluid

MSC 2000: 35D05, 35J60, 35J70, 58E05, 68T40, 76A02

1. Introduction and preliminary results

In the recent years increasing attention has been paid to the study of differential and partial differential equations involving variable exponent conditions. The interest in studying such problems was stimulated by their applications in elastic mechanics, fluid dynamics or calculus of variations. In particular, we mention that Ruzicka developed in [26] a model of electrorheological fluid for which the essential part of the dissipative energy is given by

$$\int |\mathbf{Df}|^{p(x)}\,\mathrm{d}x,$$

where $\mathbf{Df}$ represents the symmetric part of the gradient. The same type of energy also appears in the papers of Zhikov [32], Marcellini [16] and Acerbi-Mingione [1].

For more information on modelling physical phenomena by equations involving $p(x)$-growth conditions we refer to [1], [5], [13], [24], [26], [30]. The appearance of such physical models was facilitated by the development of variable Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where $p(x)$ is a real-valued function. Variable exponent Lebesgue spaces appeared for the first time in literature as early as 1931 in an article by W. Orlicz [23]. The spaces $L^{p(x)}$ are special cases of Orlicz spaces L^{φ} originated by Nakano [22] and developed by Musielak and Orlicz [20], [21], where $f \in L^{\varphi}$ if and only if $\int \varphi(x, |f(x)|)\,\mathrm{d}x < \infty$ for a suitable φ. Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers. In that context we refer to the studies of Tsenov [29], Sharapudinov [27] and Zhikov [32], [33].

This paper is motivated by the phenomenon that can be modelled by the equations

$$\tag{1} \begin{cases} -\operatorname{div}((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u) = f(x,u) & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ is a bounded domain with smooth boundary and $1 < p_i(x)$, $p_i(x) \in C(\overline{\Omega})$ for $i \in \{1,2\}$. Our goal will be to obtain nontrivial weak solutions for problem (1) in the generalized Sobolev space $W^{1,m(x)}(\Omega)$, where $m(x) = \max\{p_1(x), p_2(x)\}$ for any $x \in \overline{\Omega}$, for some particular nonlinearities of the type $f(x,u)$. Problems of type (1) have been intensively studied in the past decades. We refer to [2], [10], [11], [31], [18], [17], [19] for some interesting results. We point out the presence in problem (1) of the operator $\Delta_{p_1(x)}u + \Delta_{p_2(x)}u$, where for a real-valued function $p(x)$ we define the $p(x)$-Laplace operator by $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. This is a natural extension of the p-Laplace operator, with p a positive constant. However, such generalizations are not trivial since the $p(x)$-Laplace operator possesses a more complicated structure than the p-Laplace operator, for example it is inhomogeneous.

We recall in what follows some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$. We refer the reader to the book of J. Musielak [20] and the papers of O. Kovacik and J. Rákosník [14], H. G. Leopold [15], D. Edmunds et al. [6], [7], [8] and X. L. Fan et al. [9], [12].

Set

$$C_+(\overline{\Omega}) = \{h;\ h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \qquad \text{and} \qquad h^- = \inf_{x\in\Omega} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \Big\{u;\ u \text{ is a measurable real-valued function such that} \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty\Big\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$|u|_{p(x)} = \inf\Big\{\mu > 0;\ \int_\Omega \Big|\frac{u(x)}{\mu}\Big|^{p(x)}\,\mathrm{d}x \leqslant 1\Big\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces [14, Theorem 2.5], the Hölder inequality holds [14, Theorem 2.1], they are reflexive if and only if $1 < p^- \leqslant p^+ < \infty$ [14, Corollary 2.7] and continuous functions are dense if $p^+ < \infty$ [14, Theorem 2.11]. The inclusion between Lebesgue spaces also generalizes naturally [14, Theorem 2.8]: if $0 < |\Omega| < \infty$ and r_1, r_2 are variable exponents so that $r_1(x) \leqslant r_2(x)$ almost everywhere in Ω then there exists a continuous embedding $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder type inequality

$$\Big|\int_\Omega uv\,\mathrm{d}x\Big| \leqslant \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)} \tag{2}$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\varrho_{p(x)}\colon L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\varrho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,\mathrm{d}x.$$

If (u_n), $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold true:

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leqslant \varrho_{p(x)}(u) \leqslant |u|_{p(x)}^{p^+}, \tag{3}$$

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leqslant \varrho_{p(x)}(u) \leqslant |u|_{p(x)}^{p^-}, \tag{4}$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \varrho_{p(x)}(u_n - u) \to 0. \tag{5}$$

Spaces with $p^+ = \infty$ have been studied by Edmunds, Lang and Nekvinda [6].

Next, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}.$$

The space $(W_0^{1,p(x)}(\Omega), \|\cdot\|_{p(x)})$ is a separable and reflexive Banach space. We note that if $q \in C_+(\overline{\Omega})$ and $q(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, where $p^\star(x) = Np(x)/(N-p(x))$ if $p(x) < N$ or $p^\star(x) = +\infty$ if $p(x) \geqslant N$ [14, Theorems 3.9 and 3.3] (see also [9, Theorems 1.3 and 1.1]).

Remark 1. If $p_1(x)$, $p_2(x) \in C_+(\overline{\Omega})$ then it is clear that $m(x) \in C_+(\overline{\Omega})$ where $m(x) = \max\{p_1(x), p_2(x)\}$ for any $x \in \overline{\Omega}$. On the other hand, since $p_1(x)$, $p_2(x) \leqslant m(x)$ for any $x \in \overline{\Omega}$, it follows by Theorem 2.8 in [14] that $W_0^{1,m(x)}(\Omega)$ is continuously embedded in $W_0^{1,p_i(x)}(\Omega)$ for $i \in \{1,2\}$.

2. Main results

In this paper we study problem (1) in the particular cases when

$$f(x,t) = \pm(-\lambda|t|^{m(x)-2}t + |t|^{q(x)-2}t)$$

where $m(x) = \max\{p_1(x), p_2(x)\}$ for any $x \in \overline{\Omega}$ and $q(x) \in C_+(\Omega)$ with $m(x) < q(x) < N \cdot m(x)/(N-m(x))$ for any $x \in \overline{\Omega}$ and $\lambda > 0$.

First, we consider the problem

$$\begin{cases} -\operatorname{div}((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u) & \\ \qquad = -\lambda|u|^{m(x)-2}u + |u|^{q(x)-2}u & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{6}$$

We say that $u \in W_0^{1,m(x)}(\Omega)$ is a *weak solution* of problem (6) if

$$\int_\Omega (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u \nabla v \,\mathrm{d}x + \lambda\int_\Omega |u|^{m(x)-2}uv\,\mathrm{d}x - \int_\Omega |u|^{q(x)-2}uv\,\mathrm{d}x = 0$$

for all $v \in W_0^{1,m(x)}(\Omega)$.

We will prove

Theorem 1. *For every $\lambda > 0$ problem (6) has infinitely many weak solutions provided $2 \leqslant p_i^-$ for $i \in \{1,2\}$, $m^+ < q^-$ and $q^+ < N \cdot m^-/(N-m^-)$.*

Next, we study the problem

$$\begin{cases} -\operatorname{div}((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u) & \\ \qquad = \lambda|u|^{m(x)-2}u - |u|^{q(x)-2}u & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{7}$$

We say that $u \in W_0^{1,m(x)}(\Omega)$ is a *weak solution* of problem (7) if

$$\int_\Omega (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}) \nabla u \nabla v \,\mathrm{d}x - \lambda \int_\Omega |u|^{m(x)-2} uv \,\mathrm{d}x + \int_\Omega |u|^{q(x)-2} uv \,\mathrm{d}x = 0$$

for all $v \in W_0^{1,m(x)}(\Omega)$.

We will prove

Theorem 2. *There exists $\lambda^\star > 0$ such that for any $\lambda \geqslant \lambda^\star$ problem* (7) *has a nontrivial weak solution provided $m^+ < q^-$ and $q^+ < N \cdot m^-/(N - m^-)$.*

A careful analysis of the proofs shows that Theorems 1 and 2 still remain valid for more general classes of differential operators. For example, we can replace the $p(x)$-Laplace type operator $\operatorname{div}((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u)$ by the generalized mean curvature operator $\operatorname{div}(((1 + |\nabla u|^2)^{(p(x)-2)/2}\ \nabla u))$ (see Example (ii) on page 2629 in [18]).

We remark that in the particular case corresponding to $p_1(x) = p_2(x) = m(x) = 2$ and $q(x) = q$, q being a constant, problem (6) becomes

$$\begin{cases} -\Delta u = -\lambda u + |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

This problem has been studied by Ambrosetti and Rabinowitz [3] provided $2 < q < 2^* = 2N/(N-2)$. Using the Mountain Pass Theorem combined with the observation that the operator $-\Delta + \lambda I$ ($\lambda > 0$) is coercive in $H_0^1(\Omega)$, Ambrosetti and Rabinowitz showed that problem (8) has a positive solution for any $\lambda > 0$. The result we establish in Theorem 1 establishes the existence of *infinitely many solutions* (not necessarily positive) for a related class of boundary value problems, but involving another differential operator in the class of variable exponent Sobolev spaces.

Finally, we point out the strong difference between the result of Theorem 1 and Theorem 2. While for problem (6) we find infinitely many solutions, for problem (7) we find only the existence of at least one nontrivial solution, for $\lambda > 0$ sufficiently large. This fact is connected with the method applied in order to find solutions for either of the quoted problems. For problem (6) we apply a $\mathbb{Z}_2$-symmetric version (for even functionals) of the Mountain Pass Theorem. The application of this theorem is intimately linked with the fact that under the assumptions of Theorem 1, the non-linear term $f_1(x,t) := -\lambda |t|^{m(x)-2}t + |t|^{q(x)-2}t$ satisfies the Ambrosetti-Rabinowitz condition $0 \leqslant q(x) \int_0^t f_1(x,s)\,\mathrm{d}s \leqslant t f_1(x,t)$ for all $t \geqslant 0$ with $q(x) > m(x)$ for all $x \in \overline{\Omega}$. This condition fails if $f_2(x,t) := \lambda |t|^{m(x)-2}t - |t|^{q(x)-2}t$ but, in that case, we show that the corresponding energy functional is coercive and lower semicontinuous.

The result of Theorem 2 is also in keeping with that of Theorem 2.1 in [18], where a problem of the type

$$
\begin{cases}
-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda(u^{\gamma-1} - u^{\beta-1}) & \text{for } x \in \Omega, \\
u = 0 & \text{for } x \in \partial\Omega, \\
u \geqslant 0 & \text{for } x \in \Omega.
\end{cases} \tag{9}
$$

with $1 < \beta < \gamma < \inf_{x \in \overline{\Omega}} p(x)$ is studied. By Theorem 2.1 in [18] we find at least two nontrivial solutions for problem (9) for $\lambda > 0$ large enough. Even if the problems (7) and (9) seems to have a similar nonlinear term on the right-hand side, the existence of a second solution for problem (7) cannot be stated this time, since it should be obtained by applying the Mountain Pass Theorem which cannot be used for the nonlinear term of problem (7) in accord with the information already pointed out above.

3. Proof of theorem 1

The key argument in the proof of Theorem 1 is the following $\mathbb{Z}_2$-symmetric version (for even functionals) of the Mountain Pass Theorem (see Theorem 9.12 in [25]):

Mountain Pass Theorem. *Let X be an infinite dimensional real Banach space and let $I \in C^1(X, \mathbb{R})$ be even, satisfying the Palais-Smale condition (i.e., any sequence $\{x_n\} \subset X$ such that $\{I(x_n)\}$ is bounded and $I'(x_n) \to 0$ in $X^\star$ has a convergent subsequence) and $I(0) = 0$. Suppose that*

(I1) *there exist two constants ϱ, $a > 0$ such that $I(x) \geqslant a$ if $\|x\| = \varrho$,*

(I2) *for each finite dimensional subspace $X_1 \subset X$, the set $\{x \in X_1 ;\ I(x) \geqslant 0\}$ is bounded.*

Then I has an unbounded sequence of critical values.

Let E denote the generalized Sobolev space $W_0^{1,m(x)}(\Omega)$ and let $\lambda > 0$ be arbitrary but fixed.

The energy functional corresponding to problem (6) is defined as $J_\lambda \colon E \to \mathbb{R}$,

$$
\begin{aligned}
J_\lambda(u) = \int_\Omega \frac{1}{p_1(x)} |\nabla u|^{p_1(x)}\, \mathrm{d}x + \int_\Omega \frac{1}{p_2(x)} |\nabla u|^{p_2(x)}\, \mathrm{d}x \\
+ \lambda \int_\Omega \frac{1}{m(x)} |u|^{m(x)}\, \mathrm{d}x - \int_\Omega \frac{1}{q(x)} |u|^{q(x)}\, \mathrm{d}x.
\end{aligned}
$$

A simple calculation based on Remark 1, relations (3) and (4) and the compact embedding of E into $L^{s(x)}(\Omega)$ for all $s \in C_+(\overline{\Omega})$ with $s(x) < m^\star(x)$ on $\overline{\Omega}$ shows that

J_λ is well-defined on E and $J_\lambda \in C^1(E, \mathbb{R})$ with the derivative given by

$$\begin{aligned}\langle J'_\lambda(u), v\rangle = &\int_\Omega (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u \nabla v \, \mathrm{d}x \\ &+ \lambda \int_\Omega |u|^{m(x)-2} uv \, \mathrm{d}x - \int_\Omega |u|^{q(x)-2} uv \, \mathrm{d}x\end{aligned}$$

for any u, $v \in E$. Thus the weak solutions of (6) are exactly the critical points of J_λ.

We show now that the Mountain Pass Theorem can be applied in this case.

Lemma 1. *There exist $\eta > 0$ and $\alpha > 0$ such that $J_\lambda(u) \geqslant \alpha > 0$ for any $u \in E$ with $\|u\|_{m(x)} = \eta$.*

Proof. We first point out that since $m(x) = \max\{p_1(x), p_2(x)\}$ for any $x \in \overline{\Omega}$, we have

$$|\nabla u(x)|^{p_1(x)} + |\nabla u(x)|^{p_2(x)} \geqslant |\nabla u(x)|^{m(x)}, \ \forall x \in \overline{\Omega}. \tag{10}$$

On the other hand, we have

$$|u(x)|^{q^-} + |u(x)|^{q^+} \geqslant |u(x)|^{q(x)}, \ \forall x \in \overline{\Omega}. \tag{11}$$

Using (10) and (11) we deduce that

$$\begin{aligned} J_\lambda(u) &\geqslant \frac{1}{\max\{p_1^+, p_2^+\}} \cdot \int_\Omega |\nabla u|^{m(x)} \, \mathrm{d}x - \frac{1}{q^-} \cdot \left(\int_\Omega |u|^{q^-} \, \mathrm{d}x + \int_\Omega |u|^{q^+} \, \mathrm{d}x \right) \\ &\geqslant \frac{1}{m^+} \cdot \int_\Omega |\nabla u|^{m(x)} \, \mathrm{d}x - \frac{1}{q^-} \cdot \left(\int_\Omega |u|^{q^-} \, \mathrm{d}x + \int_\Omega |u|^{q^+} \, \mathrm{d}x \right) \end{aligned} \tag{12}$$

for any $u \in E$.

Since $m^+ < q^- \leqslant q^+ < m^\star(x)$ for any $x \in \overline{\Omega}$ and E is continuously embedded in $L^{q^-}(\Omega)$ and in $L^{q^+}(\Omega)$, it follows that there exist two positive constants C_1 and C_2 such that

$$\|u\|_{m(x)} \geqslant C_1 \cdot |u|_{q^+}, \ \|u\|_{m(x)} \geqslant C_2 \cdot |u|_{q^-}, \ \forall u \in E. \tag{13}$$

Next, we focus our attention on the case when $u \in E$ with $\|u\|_{m(x)} < 1$. For such a u by relation (4) we obtain

$$\int_\Omega |\nabla u|^{m(x)} \, \mathrm{d}x \geqslant \|u\|_{m(x)}^{m^+}. \tag{14}$$

Relations (12), (13) and (14) imply

$$\begin{aligned}J_\lambda(u) &\geqslant \frac{1}{m^+}\cdot\|u\|_{m(x)}^{m^+} - \frac{1}{q^-}\cdot\Big[\Big(\frac{1}{C_1}\cdot\|u\|_{m(x)}\Big)^{q^+} + \Big(\frac{1}{C_2}\cdot\|u\|_{m(x)}\Big)^{q^-}\Big]\\ &= (\beta-\gamma\cdot\|u\|_{m(x)}^{q^+-m^+} - \delta\cdot\|u\|_{m(x)}^{q^--m^+})\cdot\|u\|_{m(x)}^{m^+}\end{aligned}$$

for any $u\in E$ with $\|u\|_{m(x)}<1$, where β, γ and δ are positive constants.

We remark that the function $g\colon [0,1]\to\mathbb{R}$ defined by

$$g(t)=\beta-\gamma\cdot t^{q^+-m^+}-\delta\cdot t^{q^--m^+}$$

is positive in a neighborhood of the origin. We conclude that Lemma 1 holds true. $\square$

Lemma 2. *If $E_1\subset E$ is a finite dimensional subspace, the set $S=\{u\in E_1;\ J_\lambda(u)\geqslant 0\}$ is bounded in E.*

Proof. In order to prove Lemma 2, we first show that

$$\int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}\,\mathrm{d}x\leqslant K_1\cdot(\|u\|_{m(x)}^{p_1^-}+\|u\|_{m(x)}^{p_1^+}),\ \forall u\in E \tag{15}$$

where K_1 is a positive constant.

Indeed, using relations (3) and (4) we obtain

$$\int_\Omega|\nabla u|^{p_1(x)}\,\mathrm{d}x\leqslant|\nabla u|_{p_1(x)}^{p_1^-}+|\nabla u|_{p_1(x)}^{p_1^+}=\|u\|_{p_1(x)}^{p_1^-}+\|u\|_{p_1(x)}^{p_1^+},\ \forall u\in E. \tag{16}$$

On the other hand, Remark 1 implies that there exists a positive constant K_0 such that

$$\|u\|_{p_1(x)}\leqslant K_0\cdot\|u\|_{m(x)},\ \forall u\in E. \tag{17}$$

Inequalities (16) and (17) yield

$$\int_\Omega|\nabla u|^{p_1(x)}\,\mathrm{d}x\leqslant(K_0\cdot\|u\|_{m(x)})^{p_1^-}+(K_0\cdot\|u\|_{m(x)})^{p_1^+},\ \forall u\in E$$

and thus (15) holds true.

By similar arguments we conclude that there exists a positive constant K_2 such that

$$\int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\,\mathrm{d}x\leqslant K_2\cdot(\|u\|_{m(x)}^{p_2^-}+\|u\|_{m(x)}^{p_2^+}),\ \forall u\in E. \tag{18}$$

Using again (3) and (4) we arrive at

$$\int_\Omega |u|^{m(x)}\,\mathrm{d}x \leqslant |u|_{m(x)}^{m^-} + |u|_{m(x)}^{m^+},\ \forall u \in E.$$

The fact that E is continuously embedded in $L^{m(x)}(\Omega)$ ensures the existence of a positive constant $\overline{K}$ such that

$$|u|_{m(x)} \leqslant \overline{K}\cdot \|u\|_{m(x)},\ \forall u \in E.$$

The last two inequalities show that for each $\lambda > 0$ there exists a positive constant $K_3(\lambda)$ such that

$$\lambda\cdot\int_\Omega \frac{1}{m(x)}|u|^{m(x)}\,\mathrm{d}x \leqslant K_3(\lambda)\cdot(\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}),\ \forall u \in E. \tag{19}$$

By inequalities (15), (18) and (19) we get

$$\begin{aligned} J_\lambda(u) \leqslant{}& K_1\cdot(\|u\|_{m(x)}^{p_1^-} + \|u\|_{m(x)}^{p_1^+}) + K_2\cdot(\|u\|_{m(x)}^{p_2^-} + \|u\|_{m(x)}^{p_2^+}) \\ &+ K_3(\lambda)\cdot(\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}) - \frac{1}{q^+}\int_\Omega |u|^{q(x)}\,\mathrm{d}x \end{aligned}$$

for all $u \in E$.

Let $u \in E$ be arbitrary but fixed. We define

$$\Omega_< = \{x \in \Omega;\ |u(x)| < 1\},\ \Omega_\geqslant = \Omega \setminus \Omega_<.$$

Then we have

$$\begin{aligned} J_\lambda(u) \leqslant{}& K_1\cdot(\|u\|_{m(x)}^{p_1^-} + \|u\|_{m(x)}^{p_1^+}) + K_2\cdot(\|u\|_{m(x)}^{p_2^-} + \|u\|_{m(x)}^{p_2^+}) \\ &+ K_3(\lambda)\cdot(\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}) - \frac{1}{q^+}\int_\Omega |u|^{q(x)}\,\mathrm{d}x \\ \leqslant{}& K_1\cdot(\|u\|_{m(x)}^{p_1^-} + \|u\|_{m(x)}^{p_1^+}) + K_2\cdot(\|u\|_{m(x)}^{p_2^-} + \|u\|_{m(x)}^{p_2^+}) \\ &+ K_3(\lambda)\cdot(\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}) - \frac{1}{q^+}\int_{\Omega_\geqslant} |u|^{q(x)}\,\mathrm{d}x \\ \leqslant{}& K_1\cdot(\|u\|_{m(x)}^{p_1^-} + \|u\|_{m(x)}^{p_1^+}) + K_2\cdot(\|u\|_{m(x)}^{p_2^-} + \|u\|_{m(x)}^{p_2^+}) \\ &+ K_3(\lambda)\cdot(\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}) - \frac{1}{q^+}\int_{\Omega_\geqslant} |u|^{q^-}\,\mathrm{d}x \\ \leqslant{}& K_1\cdot(\|u\|_{m(x)}^{p_1^-} + \|u\|_{m(x)}^{p_1^+}) + K_2\cdot(\|u\|_{m(x)}^{p_2^-} + \|u\|_{m(x)}^{p_2^+}) \\ &+ K_3(\lambda)\cdot(\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}) - \frac{1}{q^+}\int_\Omega |u|^{q^-}\,\mathrm{d}x + \frac{1}{q^+}\int_{\Omega_<} |u|^{q^-}\,\mathrm{d}x. \end{aligned}$$

But there exists a positive constant K_4 such that

$$\frac{1}{q^+}\int_{\Omega_<} |u|^{q^-} \leqslant K_4,\ \forall u \in E.$$

Thus we deduce that

$$\begin{aligned} J_\lambda(u) \leqslant K_1 \cdot (\|u\|_{m(x)}^{p_1^-} + \|u\|_{m(x)}^{p_1^+}) + K_2 \cdot (\|u\|_{m(x)}^{p_2^-} + \|u\|_{m(x)}^{p_2^+}) \\ + K_3(\lambda) \cdot (\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}) - \frac{1}{q^+}\int_\Omega |u|^{q^-}\,\mathrm{d}x + K_4,\ \forall u \in E. \end{aligned}$$

The functional $|\ |_{q^-}\colon\ E \to \mathbb{R}$ defined by

$$|u|_{q^-} = \left(\int_\Omega |u|^{q^-}\,\mathrm{d}x\right)^{1/q^-}$$

is a norm in E. In the finite dimensional subspace E_1 the norms $|\cdot|_{q^-}$ and $\|\cdot\|_{m(x)}$ are equivalent, so there exists a positive constant $K = K(E_1)$ such that

$$\|u\|_{m(x)} \leqslant K \cdot |u|_{q^-},\ \forall u \in E_1.$$

As a consequence we have that there exists a positive constant K_5 such that

$$\begin{aligned} J_\lambda(u) \leqslant K_1 \cdot (\|u\|_{m(x)}^{p_1^-} + \|u\|_{m(x)}^{p_1^+}) + K_2 \cdot (\|u\|_{m(x)}^{p_2^-} + \|u\|_{m(x)}^{p_2^+}) \\ + K_3(\lambda) \cdot (\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}) - K_5 \cdot \|u\|_{m(x)}^{q^-} + K_4,\ \forall u \in E_1. \end{aligned}$$

Hence

$$\begin{aligned} K_1 \cdot (\|u\|_{m(x)}^{p_1^-} + \|u\|_{m(x)}^{p_1^+}) + K_2 \cdot (\|u\|_{m(x)}^{p_2^-} + \|u\|_{m(x)}^{p_2^+}) + K_3(\lambda) \cdot (\|u\|_{m(x)}^{m^-} + \|u\|_{m(x)}^{m^+}) \\ - K_5 \cdot \|u\|_{m(x)}^{q^-} + K_4 \geqslant 0,\ \forall u \in S \end{aligned}$$

and since $q^- > m^+$ we conclude that S is bounded in E.

The proof of Lemma 2 is complete. □

Lemma 3. *If $\{u_n\} \subset E$ is a sequence which satisfies the conditions*

$$|J_\lambda(u_n)| < M, \tag{20}$$

$$J'_\lambda(u_n) \to 0 \text{ as } n \to \infty \tag{21}$$

where M is a positive constant, then $\{u_n\}$ possesses a convergent subsequence.

P r o o f. First, we show that $\{u_n\}$ is bounded in E. Assume the contrary. Then, passing if necessary to a subsequence, still denoted by $\{u_n\}$, we may assume that $\|u_n\|_{m(x)} \to \infty$ as $n \to \infty$. Thus, we may assume that $\|u_n\|_{m(x)} > 1$ for any integer n.

By (21) we deduce that there exists $N_1 > 0$ such that for any $n > N_1$ we have

$$\|J'_\lambda(u_n)\| \leqslant 1.$$

On the other hand, for any $n > N_1$ fixed, the application

$$E \ni v \to \langle J'_\lambda(u_n), v\rangle$$

is linear and continuous. The above information yields that

$$|\langle J'_\lambda(u_n), v\rangle| \leqslant \|J'_\lambda(u_n)\| \cdot \|v\|_{m(x)} \leqslant \|v\|_{m(x)},\ \forall v \in E,\ n > N_1.$$

Setting $v = u_n$ we have

$$\begin{aligned} -\|u_n\|_{m(x)} \leqslant \int_\Omega |\nabla u_n|^{p_1(x)}\,\mathrm{d}x + \int_\Omega |\nabla u_n|^{p_2(x)}\,\mathrm{d}x \\ + \lambda \int_\Omega |u_n|^{m(x)}\,\mathrm{d}x - \int_\Omega |u_n|^{q(x)}\,\mathrm{d}x \leqslant \|u_n\|_{m(x)} \end{aligned}$$

for all $n > N_1$. We obtain

$$\begin{aligned} -\|u_n\|_{m(x)} - \int_\Omega |\nabla u_n|^{p_1(x)}\,\mathrm{d}x - \int_\Omega |\nabla u_n|^{p_2(x)}\,\mathrm{d}x \\ - \lambda \int_\Omega |u_n|^{m(x)}\,\mathrm{d}x \leqslant - \int_\Omega |u_n|^{q(x)}\,\mathrm{d}x \end{aligned} \tag{22}$$

for any $n > N_1$.

Provided that $\|u_n\|_{m(x)} > 1$ relations (20), (22) and (3) imply

$$\begin{aligned} M > J_\lambda(u_n) &\geqslant \Big(\frac{1}{m^+} - \frac{1}{q^-}\Big) \cdot \int_\Omega (|\nabla u_n|^{p_1(x)} + |\nabla u_n|^{p_2(x)})\,\mathrm{d}x \\ &\quad + \lambda \cdot \Big(\frac{1}{m^+} - \frac{1}{q^-}\Big) \cdot \int_\Omega |u_n|^{m(x)}\,\mathrm{d}x - \frac{1}{q^-} \cdot \|u_n\|_{m(x)} \\ &\geqslant \Big(\frac{1}{m^+} - \frac{1}{q^-}\Big) \cdot \int_\Omega |\nabla u_n|^{m(x)}\,\mathrm{d}x - \frac{1}{q^-} \cdot \|u_n\|_{m(x)} \\ &\geqslant \Big(\frac{1}{m^+} - \frac{1}{q^-}\Big) \cdot \|u_n\|^{m^-}_{m(x)} - \frac{1}{q^-} \cdot \|u_n\|_{m(x)}. \end{aligned}$$

Letting $n \to \infty$ we obtain a contradiction. It follows that $\{u_n\}$ is bounded in E.

Since $\{u_n\}$ is bounded in E we deduce that there exists a subsequence, again denoted by $\{u_n\}$, and $u_0 \in E$ such that $\{u_n\}$ converges weakly to u_0 in E. Since by Remark 1 and Theorem 1.3 in [9] E is compactly embedded in $L^{m(x)}(\Omega)$ and in $L^{q(x)}(\Omega)$ it follows that $\{u_n\}$ converges strongly to u_0 in $L^{m(x)}(\Omega)$ and $L^{q(x)}(\Omega)$. The above information and relation (21) imply

$$\langle J'_\lambda(u_n) - J'_\lambda(u_0), u_n - u_0\rangle \to 0 \text{ as } n \to \infty.$$

On the other hand, we have

$$\begin{aligned}
&\int_\Omega (|\nabla u_n|^{p_1(x)-2}\nabla u_n + |\nabla u_n|^{p_2(x)-2}\nabla u_n - |\nabla u_0|^{p_1(x)-2}\nabla u_0\\
&\quad - |\nabla u_0|^{p_2(x)-2}\nabla u_0)\cdot(\nabla u_n - \nabla u_0)\,\mathrm{d}x\\
&= \langle J'_\lambda(u_n) - J'_\lambda(u_0), u_n - u_0\rangle - \lambda\cdot\int_\Omega (|u_n|^{m(x)-2}u_n - |u_0|^{m(x)-2}u_0)(u_n - u_0)\,\mathrm{d}x\\
&\quad + \int_\Omega (|u_n|^{q(x)-2}u_n - |u_0|^{q(x)-2}u_0)(u_n - u_0)\,\mathrm{d}x.
\end{aligned}$$

Using the fact that $\{u_n\}$ converges strongly to u_0 in $L^{q(x)}(\Omega)$ and inequality (2) we have

$$\begin{aligned}
&\left|\int_\Omega (|u_n|^{q(x)-2}u_n - |u_0|^{q(x)-2}u_0)(u_n - u_0)\,\mathrm{d}x\right|\\
&\quad \leqslant \left|\int_\Omega |u_n|^{q(x)-2}u_n(u_n - u_0)\,\mathrm{d}x\right| + \left|\int_\Omega |u_0|^{q(x)-2}u_0(u_n - u_0)\,\mathrm{d}x\right|\\
&\quad \leqslant C_3\cdot \||u_n|^{q(x)-1}|_{\frac{q(x)}{q(x)-1}}\cdot|u_n - u_0|_{q(x)} + C_4\cdot\||u_0|^{q(x)-1}|_{\frac{q(x)}{q(x)-1}}\cdot|u_n - u_0|_{q(x)}
\end{aligned}$$

where C_3 and C_4 are two positive constants. Since $|u_n - u_0|_{q(x)} \to 0$ as $n \to \infty$ we deduce that

$$\lim_{n\to\infty}\int_\Omega (|u_n|^{q(x)-2}u_n - |u_0|^{q(x)-2}u_0)(u_n - u_0)\,\mathrm{d}x = 0. \tag{24}$$

By similar arguments we deduce that

$$\lim_{n\to\infty}\int_\Omega (|u_n|^{m(x)-2}u_n - |u_0|^{m(x)-2}u_0)(u_n - u_0)\,\mathrm{d}x = 0. \tag{25}$$

By (23), (24) and (25) we get

$$\begin{aligned}
\lim_{n\to\infty}\int_\Omega (&|\nabla u_n|^{p_1(x)-2}\nabla u_n + |\nabla u_n|^{p_2(x)-2}\nabla u_n - |\nabla u_0|^{p_1(x)-2}\nabla u_0\\
&- |\nabla u_0|^{p_2(x)-2}\nabla u_0)\cdot(\nabla u_n - \nabla u_0)\,\mathrm{d}x = 0.
\end{aligned} \tag{26}$$

It is known that

$$(\xi|^{r-2}\xi - |\psi|^{r-2}\psi) \cdot (\xi - \psi) \geqslant \left(\frac{1}{2}\right)^r |\xi - \psi|^r, \ \forall r \geqslant 2, \ \xi, \psi \in \mathbb{R}^N. \tag{27}$$

Relations (26) and (27) yield

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n - \nabla u_0|^{p_1(x)} \,\mathrm{d}x + \int_\Omega |\nabla u_n - \nabla u_0|^{p_2(x)} \,\mathrm{d}x = 0$$

and using relation (10) we get

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n - \nabla u_0|^{m(x)} \,\mathrm{d}x = 0.$$

This fact and relation (5) imply $\|u_n - u_0\|_{m(x)} \to 0$ as $n \to \infty$. The proof of Lemma 3 is complete. □

Proof of Theorem 1 completed. It is clear that the functional J_λ is even and verifies $J_\lambda(0) = 0$. Lemma 3 implies that J_λ satisfies the Palais-Smale condition. On the other hand, Lemmas 1 and 2 show that conditions (I1) and (I2) are satisfied. The Mountain Pass Theorem can be applied to the functional J_λ. We conclude that equation (6) has infinitely many weak solutions in E. The proof of Theorem 1 is complete.

4. Proof of theorem 2

Let E denote the generalized Sobolev space $W_0^{1,m(x)}(\Omega)$ and let $\lambda > 0$ be arbitrary but fixed.

We start by introducing the energy functional corresponding to problem (7) as $I_\lambda\colon E \to \mathbb{R}$,

$$\begin{aligned} I_\lambda(u) = &\int_\Omega \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} \,\mathrm{d}x + \int_\Omega \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} \,\mathrm{d}x \\ &- \lambda \int_\Omega \frac{1}{m(x)} |u|^{m(x)} \,\mathrm{d}x + \int_\Omega \frac{1}{q(x)} |u|^{q(x)} \,\mathrm{d}x. \end{aligned}$$

The same arguments as those used in the case of the functional J_λ show that I_λ is well-defined on E and $I_\lambda \in C^1(E, \mathbb{R})$ with the derivative given by

$$\begin{aligned} \langle I'_\lambda(u), v\rangle = &\int_\Omega (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}) \nabla u \nabla v \,\mathrm{d}x \\ &- \lambda \int_\Omega |u|^{m(x)-2} uv \,\mathrm{d}x + \int_\Omega |u|^{q(x)-2} uv \,\mathrm{d}x \end{aligned}$$

for any u, $v \in E$. We obtain that the weak solutions of (7) are the critical points of I_λ.

This time our idea is to show that I_λ possesses a nontrivial global minimum point in E. With this end in view we start by proving two auxiliary results.

Lemma 4. *The functional I_λ is coercive on E.*

Proof. In order to prove Lemma 4 we first show that for any a, $b > 0$ and $0 < k < l$ the following inequality holds:

$$a \cdot t^k - b \cdot t^l \leqslant a \cdot \Big(\frac{a}{b}\Big)^{k/(l-k)}, \ \forall\, t \geqslant 0. \tag{28}$$

Indeed, since the function

$$[0, \infty) \ni t \to t^\theta$$

is increasing for any $\theta > 0$ it follows that

$$a - b \cdot t^{l-k} < 0, \ \forall\, t > \Big(\frac{a}{b}\Big)^{1/(l-k)}$$

and

$$t^k \cdot (a - b \cdot t^{l-k}) \leqslant a \cdot t^k < a \cdot \Big(\frac{a}{b}\Big)^{k/(l-k)}, \ \forall\, t \in \Big[0, \Big(\frac{a}{b}\Big)^{1/(l-k)}\Big].$$

The above two inequalities show that (28) holds true.

Using (28) we deduce that for any $x \in \Omega$ and $u \in E$ we have

$$\begin{aligned}\frac{\lambda}{m^-}|u(x)|^{m(x)} - \frac{1}{q^+}|u(x)|^{q(x)} &\leqslant \frac{\lambda}{m^-}\Big[\frac{\lambda \cdot q^+}{m^-}\Big]^{m(x)/(q(x)-m(x))}\\ &\leqslant \frac{\lambda}{m^-}\Big[\Big(\frac{\lambda \cdot q^+}{m^-}\Big)^{m^+/(q^- - m^+)} + \Big(\frac{\lambda \cdot q^+}{m^-}\Big)^{m^-/(q^+ - m^-)}\Big] = \mathcal{C},\end{aligned}$$

where $\mathcal{C}$ is a positive constant independent of u and x. Integrating the above inequality over Ω we obtain

$$\frac{\lambda}{m^-}\int_\Omega |u|^{m(x)}\,\mathrm{d}x - \frac{1}{q^+}\int_\Omega |u|^{q(x)}\,\mathrm{d}x \leqslant \mathcal{D} \tag{29}$$

where $\mathcal{D}$ is a positive constant independent of u.

Using inequalities (10) and (29) we obtain for any $u \in E$ with $\|u\|_{m(x)} > 1$ that

$$\begin{aligned}I_\lambda(u) &\geqslant \frac{1}{m^+}\int_\Omega |\nabla u|^{m(x)}\,\mathrm{d}x - \frac{\lambda}{m^-}\int_\Omega |u|^{m(x)}\,\mathrm{d}x + \frac{1}{q^+}\int_\Omega |u|^{q(x)}\,\mathrm{d}x\\ &\geqslant \frac{1}{m^+}\|u\|_{m(x)}^{m^-} - \Big(\frac{\lambda}{m^-}\int_\Omega |u|^{m(x)}\,\mathrm{d}x - \frac{1}{q^+}\int_\Omega |u|^{q(x)}\,\mathrm{d}x\Big)\\ &\geqslant \frac{1}{m^+}\|u\|_{m(x)}^{m^-} - \mathcal{D}.\end{aligned}$$

Thus I_λ is coercive and the proof of Lemma 4 is complete. $\square$

Lemma 5. *The functional I_λ is weakly lower semicontinuous.*

Proof. First we prove that the functionals $\Lambda_i\colon E \to \mathbb{R}$,

$$\Lambda_i(u) = \int_\Omega \frac{1}{p_i(x)} |\nabla u|^{p_i(x)}\, \mathrm{d}x,\ \forall\, i \in \{1,2\}$$

are convex. Indeed, since the function

$$[0,\infty) \ni t \to t^\theta$$

is convex for any $\theta > 1$, we deduce that for each $x \in \Omega$ fixed it the inequality

$$\Big|\frac{\xi+\psi}{2}\Big|^{p_i(x)} \leqslant \Big|\frac{|\xi|+|\psi|}{2}\Big|^{p_i(x)} \leqslant \frac{1}{2}|\xi|^{p_i(x)} + \frac{1}{2}|\psi|^{p_i(x)},\ \forall \xi,\psi \in \mathbb{R}^N,\ i \in \{1,2\}$$

holds. Using the above inequality we deduce that

$$\Big|\frac{\nabla u+\nabla v}{2}\Big|^{p_i(x)} \leqslant \frac{1}{2}|\nabla u|^{p_i(x)} + \frac{1}{2}|\nabla v|^{p_i(x)},\ \forall u,v \in E,\ x \in \Omega,\ i \in \{1,2\}.$$

Multiplying with $1/p_i(x)$ and integrating over Ω we obtain

$$\Lambda_i\Big(\frac{u+v}{2}\Big) \leqslant \frac{1}{2}\Lambda_i(u) + \frac{1}{2}\Lambda_i(v),\ \forall u,v \in E,\ i \in \{1,2\}.$$

Thus Λ_1 and Λ_2 are convex. It follows that $\Lambda_1 + \Lambda_2$ is convex.

Next, we show that the functional $\Lambda_1 + \Lambda_2$ is weakly lower semicontinuous on E. Taking into account that $\Lambda_1 + \Lambda_2$ is convex, by Corollary III.8 in [4] it is enough to show that $\Lambda_1 + \Lambda_2$ is strongly lower semicontinuous on E. We fix $u \in E$ and $\varepsilon > 0$. Let $v \in E$ be arbitrary. Since $\Lambda_1 + \Lambda_2$ is convex and inequality (2) holds true we have

$$\begin{aligned}
\Lambda_1(v) + \Lambda_2(v) &\geqslant \Lambda_1(u) + \Lambda_2(u) + \langle \Lambda_1'(u) + \Lambda_2'(u), v-u\rangle\\
&\geqslant \Lambda_1(u) + \Lambda_2(u) - \int_\Omega |\nabla u|^{p_1(x)-1}|\nabla(v-u)|\,\mathrm{d}x\\
&\quad - \int_\Omega |\nabla u|^{p_2(x)-1}|\nabla(v-u)|\,\mathrm{d}x\\
&\geqslant \Lambda_1(u) + \Lambda_2(u) - D_1 \cdot \| \nabla u|^{p_1(x)-1}|_{\frac{p_1(x)}{p_1(x)-1}} \cdot |\nabla(u-v)|_{p_1(x)}\\
&\quad - D_2 \cdot \| \nabla u|^{p_2(x)-1}|_{\frac{p_2(x)}{p_2(x)-1}} \cdot |\nabla(u-v)|_{p_2(x)}\\
&\geqslant \Lambda_1(u) + \Lambda_2(u) - D_3 \cdot \|u-v\|_{m(x)}\\
&\geqslant \Lambda_1(u) + \Lambda_2(u) - \varepsilon
\end{aligned}$$

for all $v \in E$ with $\|u - v\|_{m(x)} < \varepsilon / [\| |\nabla u|^{p_1(x)-1} |_{\frac{p_1(x)}{p_1(x)-1}} + \| |\nabla u|^{p_2(x)-1} |_{\frac{p_2(x)}{p_2(x)-1}}]$. We have denoted by D_1, D_2 and D_3 three positive constants. It follows that $\Lambda_1 + \Lambda_2$ is strongly lower semicontinuous and since it is convex we obtain that $\Lambda_1 + \Lambda_2$ is weakly lower semicontinuous.

Finally, we remark that if $\{u_n\} \subset E$ is a sequence which converges weakly to u in E then $\{u_n\}$ converges strongly to u in $L^{m(x)}(\Omega)$ and $L^{q(x)}(\Omega)$. Thus, I_λ is weakly lower semicontinuous. The proof of Lemma 5 is complete. $\square$

Proof of Theorem 2. By Lemmas 4 and 5 we deduce that I_λ is coercive and weakly lower semicontinuous on E. Then Theorem 1.2 in [28] implies that there exists a global minimizer $u_\lambda \in E$ of I_λ and thus a weak solution of problem (7).

We show that u_λ is not trivial for λ large enough. Indeed, letting $t_0 > 1$ be a fixed real and Ω_1 an open subset of Ω with $|\Omega_1| > 0$ we deduce that there exists $u_0 \in C_0^\infty(\Omega) \subset E$ such that $u_0(x) = t_0$ for any $x \in \overline{\Omega}_1$ and $0 \leqslant u_0(x) \leqslant t_0$ in $\Omega \setminus \Omega_1$. We have

$$\begin{aligned} I_\lambda(u_0) = & \int_\Omega \frac{1}{p_1(x)} |\nabla u_0|^{p_1(x)} \,\mathrm{d}x + \int_\Omega \frac{1}{p_2(x)} |\nabla u_0|^{p_2(x)} \,\mathrm{d}x \\ & - \lambda \int_\Omega \frac{1}{m(x)} |u_0|^{m(x)} \,\mathrm{d}x + \int_\Omega \frac{1}{q(x)} |u_0|^{q(x)} \,\mathrm{d}x \\ & \leqslant L - \frac{\lambda}{m^+} \int_{\Omega_1} |u_0|^{m(x)} \,\mathrm{d}x \leqslant L - \frac{\lambda}{m^+} \cdot t_0^{m^-} \cdot |\Omega_1| \end{aligned}$$

where L is a positive constant. Thus, there exists $\lambda^\star > 0$ such that $I_\lambda(u_0) < 0$ for any $\lambda \in [\lambda^\star, \infty)$. It follows that $I_\lambda(u_\lambda) < 0$ for any $\lambda \geqslant \lambda^\star$ and thus u_λ is a nontrivial weak solution of problem (7) for λ large enough. The proof of Theorem 2 is complete. $\square$

References

[1] *E. Acerbi and G. Mingione*: Regularity results for a class of functionals with nonstandard growth. Arch. Rational Mech. Anal. *156* (2001), 121–140. zbl

[2] *C. O. Alves and M. A. S. Souto*: Existence of solutions for a class of problems involving the $p(x)$-Laplacian. Progress in Nonlinear Differential Equations and Their Applications *66* (2005), 17–32.

[3] *A. Ambrosetti and P. H. Rabinowitz*: Dual variational methods in critical point theory and applications. J. Functional Analysis *14* (1973), 349–381. zbl

[4] *H. Brezis*: Analyse fonctionnelle: théorie et applications. Masson, Paris, 1992.

[5] *L. Diening*: Theorical and numerical results for electrorheological fluids. Ph.D. thesis, University of Freiburg, Germany, 2002. zbl

[6] *D. E. Edmunds, J. Lang and A. Nekvinda*: On $L^{p(x)}$ norms. Proc. Roy. Soc. London Ser. A *455* (1999), 219–225. zbl

[7] *D. E. Edmunds and J. Rákosník*: Density of smooth functions in $W^{k,p(x)}(\Omega)$. Proc. Roy. Soc. London Ser. A *437* (1992), 229–236. zbl
[8] *D. E. Edmunds and J. Rákosník*: Sobolev embedding with variable exponent. Studia Math. *143* (2000), 267–293. zbl
[9] *X. Fan, J. Shen and D. Zhao*: Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$. J. Math. Anal. Appl. *262* (2001), 749–760. zbl
[10] *X. L. Fan and Q. H. Zhang*: Existence of solutions for $p(x)$-Laplacian Dirichlet problem. Nonlinear Anal. *52* (2003), 1843–1852. zbl
[11] *X. L. Fan, Q. H. Zhang and D. Zhao*: Eigenvalues of $p(x)$-Laplacian Dirichlet problem. J. Math. Anal. Appl. *302* (2005), 306–317. zbl
[12] *X. L. Fan and D. Zhao*: On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. J. Math. Anal. Appl. *263* (2001), 424–446. zbl
[13] *T. C. Halsey*: Electrorheological fluids. Science *258* (1992), 761–766.
[14] *O. Kováčik and J. Rákosník*: On spaces $L^{p(x)}$ and $W^{1,p(x)}$. Czech. Math. J. *41* (1991), 592–618. zbl
[15] *H. G. Leopold*: Embedding on function spaces of variable order of differentiation in function spaces of variable order of integration. Czech. Math. J. *49* (1999), 633–644. zbl
[16] *P. Marcellini*: Regularity and existence of solutions of elliptic equations with p, q growth conditions. J. Differential Equations *90* (1991), 1–30. zbl
[17] *M. Mihăilescu*: Elliptic problems in variable exponent spaces. Bull. Austral. Math. Soc. *74* (2006), 197–206. zbl
[18] *M. Mihăilescu and V. Rădulescu*: A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids. Proc. Roy. Soc. London Ser. A *462* (2006), 2625–2641.
[19] *M. Mihăilescu and V. Rădulescu*: On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent. Proceedings of the American Mathematical Society *135* (2007), no. 9, 2929–2937.
[20] *J. Musielak*: Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics, Vol. 1034, Springer, Berlin, 1983. zbl
[21] *J. Musielak and W. Orlicz*: On modular spaces. Studia Math. *18* (1959), 49–65. zbl
[22] *H. Nakano*: Modulared Semi-Ordered Linear Spaces. Maruzen Co., Ltd., Tokyo, 1950. zbl
[23] *W. Orlicz*: Über konjugierte Exponentenfolgen. Studia Math. *3* (1931), 200–211. zbl
[24] *C. Pfeiffer, C. Mavroidis, Y. Bar-Cohen and B. Dolgin*: Electrorheological fluid based force feedback device, in Proceedings of the 1999 SPIE Telemanipulator and Telepresence Technologies VI Conference (Boston, MA), Vol. 3840. 1999, pp. 88–99.
[25] *P. Rabinowitz*: Minimax methods in critical point theory with applications to differential equations, Expository Lectures from the CBMS Regional Conference held at the University of Miami, American Mathematical Society, Providence, RI. 1984.
[26] *M. Ruzicka*: Electrorheological Fluids Modeling and Mathematical Theory. Springer-Verlag, Berlin, 2002.
[27] *I. Sharapudinov*: On the topology of the space $L^{p(t)}([0;1])$. Matem. Zametki *26* (1978), 613–632.
[28] *M. Struwe*: Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. Springer, Heidelberg, 1996. zbl
[29] *I. Tsenov*: Generalization of the problem of best approximation of a function in the space L^s. Uch. Zap. Dagestan Gos. Univ. *7* (1961), 25–37.
[30] *W. M. Winslow*: Induced fibration of suspensions. J. Appl. Phys. *20* (1949), 1137–1140.
[31] *Q. Zhang*: A strong maximum principle for differential equations with nonstandard $p(x)$-growth conditions. J. Math. Anal. Appl. *312* (2005), 24–32. zbl

[32] *V. Zhikov*: Averaging of functionals in the calculus of variations and elasticity. Math. USSR Izv. *29* (1987), 33–66.
[33] *V. Zhikov*: On passing to the limit in nonlinear variational problem. Math. Sb. *183* (1992), 47–84. zbl

Author's addresses: Mihai Mihăilescu, Department of Mathematics, University of Craiova, 200 585 Craiova, Romania, e-mail: mmihailes@yahoo.com; Department of Mathematics, Central European University, 1051 Budapest, Hungary.