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A KOMLÓS-TYPE THEOREM FOR THE SET-VALUED HENSTOCK-KURZWEIL-PETTIS INTEGRAL AND APPLICATIONS

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Abstract. This paper presents a Komlós theorem that extends to the case of the set-valued Henstock-Kurzweil-Pettis integral a result obtained by Balder and Hess (in the integrably bounded case) and also a result of Hess and Ziat (in the Pettis integrability setting). As applications, a solution to a best approximation problem is given, weak compactness results are deduced and, finally, an existence theorem for an integral inclusion involving the Henstock-Kurzweil-Pettis set-valued integral is obtained.

Keywords: Komlós convergence, Henstock-Kurzweil integral, Henstock-Kurzweil-Pettis set-valued integral, selection

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1. Introduction

Komlós's classical theorem (see [17]) yields that from any L^1-bounded sequence of real functions one can extract a subsequence such that the arithmetic averages of all its subsequences converge pointwise almost everywhere. Similar results were then obtained in the vector-valued case and, moreover, in the case of $\mathscr{P}_{\rm wkc}(X)$-valued functions, X being a separable Banach space: in Theorem 2.5 in [2] an integrable boundedness condition is imposed, while Theorem 3.1 in [16] requires Pettis integrability of the multifunctions.

Through the present work, we extend these results providing a Komlós-type theorem for $\mathscr{P}_{\rm wkc}(X)$-valued functions under Henstock-Kurzweil-Pettis integrability assumptions. The set-valued Henstock-Kurzweil-Pettis integral was introduced in [19] in the same manner as the Pettis set-valued integral (see e.g. [9]), but the support functionals are integrated in the Henstock-Kurzweil sense instead of the Lebesgue one.

Our method is based on an abstract Komlós-type result (Theorem 2.1 in [1]), which was also used to obtain a Komlós theorem for Pettis integrable (multi)functions in [3]. As a corollary, a Komlós result similar to that obtained in [16] for the Pettis set-valued integral is given.

In the second part of the work, we apply the results obtained in the first part to give a solution to a best approximation problem. Such a problem was investigated under different assumptions in [5] for integrably bounded multifunctions, as well as in [16] for Pettis integrable set-valued applications.

The third section contains several weak compactness criteria in the set-valued HKP-integration, using Komlós's results given above and a uniform integrability condition specific to the HK integrability. In particular, a weak compactness result for the family of all integrable multi-selections of an HKP-integrable weakly compact convex-valued multifunction is proved.

Recently, many authors have investigated the existence of solutions of differential (or integral) equations under Henstock-Kurzweil (e.g. [7], [10], [11] and [20]) and Henstock-Kurzweil-Pettis integrability assumptions (e.g. [8]). In that line, we obtain an existence result for a set-valued integral equation involving the Henstock-Kurzweil-Pettis integral which represents an extension of Theorem VI-7 in [6] (where the Pettis integrability is required).

2. Terminology and notation

Let us begin by introducing the basic facts on the Henstock-Kurzweil integrability, a concept that on the real line extends the classical Lebesgue one.

A positive function δ on a real interval $[0,T]$ provided with the Lebesgue σ-algebra Σ and the Lebesgue measure $\mu = \mathrm{d}s$ is called a gauge. A partition of $[0,T]$ is a finite family $(I_i, t_i)_{i=1}^k$ of nonoverlapping intervals that covers $[0,T]$ with the associated so-called tags $t_i \in I_i$. A partition is said to be δ-fine if for each i, $I_i \subset]t_i - \delta(t_i), t_i + \delta(t_i)[$.

Definition 1. A function $f\colon [0,T] \to \mathbb{R}$ is Henstock-Kurzweil (shortly, HK-) integrable if there exists a real, denoted by $(\mathrm{HK})\int_0^T f(t)\,\mathrm{d}t$, satisfying that for every $\varepsilon > 0$ one can find a gauge δ_ε such that, for every δ_ε-fine partition $(I_i, t_i)_{i=1}^k$, $\left|\sum_{i=1}^k f(t_i)\mu(I_i) - (\mathrm{HK})\int_0^T f(t)\,\mathrm{d}t\right| < \varepsilon$. The function f is HK-integrable on a measurable $E \subset [0,T]$ if $f\chi_E$ is HK-integrable on $[0,T]$.

Remark 2. Theorem 9.8 in [14] yields that an HK-integrable function is HK-integrable on any subinterval and, by Theorem 9.12 in [14], its primitive $(\mathrm{HK})\int_0^{\cdot} f(t)\,\mathrm{d}t$ is continuous.

Let us recall the properties that connect this kind of integrability with the Lebesgue one:

Proposition 3 (Theorem 9.13 in [14]). *Let $f\colon [0,T] \to \mathbb{R}$ be* HK-*integrable on* $[0,T]$. *Then*

a) f *is measurable;*
b) *if f is nonnegative on* $[0,T]$, *then it is Lebesgue integrable;*
c) f *is Lebesgue integrable on* $[0,T]$, *if and only if it is* HK-*integrable on every measurable subset of* $[0,T]$.

The Lebesgue integrability is preserved under multiplication by essentially bounded real functions. The following result states that the HK-integrability is preserved under multiplication by functions of bounded variation.

Lemma 4 (Theorem 12.21 in [14]). *Let $f\colon [0,T] \to \mathbb{R}$ be an* HK-*integrable function and let $g\colon [0,T] \to \mathbb{R}$ be of bounded variation. Then fg is* HK-*integrable.*

We will also use the following uniform integrability notion, specific to the HK-integrability, that allows to obtain a Vitali-type convergence result (Theorem 13.16 in [14]):

Definition 5. A family $\mathscr{F}$ of HK-integrable functions defined on $[0,T]$ is said to be uniformly HK-integrable if for each $\varepsilon > 0$ there exists a gauge δ_ε such that for every δ_ε-fine partition of $[0,T]$ and every $f \in \mathscr{F}$, $\left|\sum\limits_{i=1}^{k} f(t_i)\mu(I_i) - (\mathrm{HK})\int_0^T f(t)\,\mathrm{d}t\right| < \varepsilon$.

Let us note that this concept does not allow us to ignore the μ-null sets, as is shown by the following example.

Example 6 (see [14], p. 209). The sequence $(f_n)_{n\in\mathbb{N}}$, where $f_n\colon [0,1] \to \mathbb{R}$ is defined for each $n \in \mathbb{N}$ by $f_n(t) = 0$ $\forall t \in\,]0,1]$ and $f_n(0) = n$, is not uniformly HK-integrable, although all functions of this sequence differ only at one point.

Remark 7. The class of Henstock-Kurzweil integrable functions (which coincides with the class of Denjoy and Perron integrable functions, cf. [14]) is contained in the class of Khintchine integrable functions (see [14], Chapter 15). In [13] and [12], Khintchine integrability is called Denjoy integrability. This will not lead to any confusion, because we will use only the HK-integral and, when appealing to the results in [13] and [12], we will mean the integration in Khintchine sense.

Through the paper, X is a separable Banach space, X^* and X^{**} denote its topological dual and bi-dual, respectively, and $\mathscr{P}_{\mathrm{wkc}}(X)$ stands for the family of its weakly compact convex subsets. On $\mathscr{P}_{\mathrm{wkc}}(X)$ the Hausdorff distance D is considered and, for every $A \in \mathscr{P}_{\mathrm{wkc}}(X)$, we put $|A| = D(A, \{0\})$.

A well known extension of the Lebesgue integral to the Banach-valued case is the Pettis integral (see [18]). One can generalize this notion of integrability by considering for the canonical bilinear form $\langle\cdot,\cdot\rangle$ the HK-integral instead of the Lebesgue one as follows:

Definition 8. A function $f\colon [0,T]\to X$ is said to be Henstock-Kurzweil-Pettis (shortly, HKP-) integrable if

1) f is scalarly HK-integrable, i.e. for all $x^*\in X^*$, $\langle x^*, f(\cdot)\rangle$ is HK-integrable;
2) for each $[a,b]\subset[0,T]$ there exists $x_{[a,b]}\in X$ such that

$$\langle x^*, x_{[a,b]}\rangle = (\mathrm{HK})\int_a^b \langle x^*, f(s)\rangle\,\mathrm{d}s,$$

for all $x^*\in X^*$.

We denote $x_{[a,b]}$ by (HKP) $\int_a^b f(s)\,\mathrm{d}s$ and call it the HKP-integral of f on $[a,b]$.

If in the condition 2) we require only $x_{[a,b]}\in X^{**}$, then f is called Henstock-Kurzweil-Dunford (shortly, HKD-) integrable.

Remark 9.

i) Following Remark 2, if f is HKP-integrable, then its primitive (HKP) $\int_0^{\cdot} f(t)\,\mathrm{d}t$ is weakly continuous.
ii) Obviously, any Pettis integrable function is HKP-integrable. The converse is not true: the function considered in Section 4 in [12] provides an example.

One can consider (via Lemma 4) the space of HKP-integrable X-valued functions equipped with the topology induced by the tensor product of the space of real functions of bounded variation and X^* (we call it the weak-Henstock-Kurzweil-Pettis topology and denote it by w-HKP). That is: $f_\alpha\to f$ if, for every $g\colon [0,T]\to\mathbb{R}$ of bounded variation and every $x^*\in X^*$, (HK) $\int_0^T g(s)\langle x^*, f_\alpha(s)\rangle\,\mathrm{d}s\to$ (HK) $\int_0^T g(s)\langle x^*, f(s)\rangle\,\mathrm{d}s$. Our considerations arise naturally from Pettis integrability setting, where the topology induced on the space of Pettis integrable functions by the tensor product $L^\infty([0,T])\otimes X^*$ is called the weak-Pettis topology.

Let us recall various kinds of set-valued measurability and integrability that will be used in the sequel. The support functional of $A\in\mathscr{P}_{\mathrm{wkc}}(X)$ is denoted by $\sigma(\cdot,A)$ and is defined by $\sigma(x^*,A)=\sup\{\langle x^*,x\rangle,\ x\in A\}$ for all $x^*\in X^*$. A set-valued function $F\colon [0,T]\to X$ is said to be measurable if, for every open subset $O\subset X$, the set $F^{-1}(O)=\{t\in[0,T];\ F(t)\cap O\neq\emptyset\}$ is measurable. F is called scalarly measurable if, for every $x^*\in X^*$, $\sigma(x^*,F(\cdot))$ is measurable. According to Theorem III-37 in [6], in the case when X is separable, a $\mathscr{P}_{\mathrm{wkc}}(X)$-valued multifunction is measurable if and only if it is scalarly measurable. A function $f\colon [0,T]\to X$ is called a selection of F if $f(t)\in F(t)$ a.e.

Definition 10.

i) A multifunction Γ is said to be integrably bounded if the real function $|\Gamma(\cdot)|$ is Lebesgue integrable.

ii) Γ is said to be scalarly (resp. scalarly HK-) integrable if, for every $x^* \in X^*$, $\sigma(x^*, \Gamma(\cdot))$ is Lebesgue (resp. HK-) integrable.

iii) A $\mathscr{P}_{\mathrm{wkc}}(X)$-valued function Γ is "Pettis integrable in $\mathscr{P}_{\mathrm{wkc}}(X)$" (or, simply, Pettis integrable since we will work only with $\mathscr{P}_{\mathrm{wkc}}(X)$) if it is scalarly integrable, and for every $A \in \Sigma$ there exists $I_A \in \mathscr{P}_{\mathrm{wkc}}(X)$ such that $\sigma(x^*, I_A) = \int_A \sigma(x^*, \Gamma(t))\,\mathrm{d}t$ for each $x^* \in X^*$. We denote I_A by (P) $\int_A \Gamma(t)\,\mathrm{d}t$.

iv) A $\mathscr{P}_{\mathrm{wkc}}(X)$-valued function Γ is "HKP-integrable in $\mathscr{P}_{\mathrm{wkc}}(X)$" (shortly, HKP-integrable) if it is scalarly HK-integrable, and for every $[a,b] \subset [0,T]$ there exists $I_a^b \in \mathscr{P}_{\mathrm{wkc}}(X)$, such that $\sigma(x^*, I_a^b) = (\mathrm{HK}) \int_a^b \sigma(x^*, \Gamma(t))\,\mathrm{d}t$, $\forall\, x^* \in X^*$. We denote I_a^b by (HKP) $\int_a^b \Gamma(t)\,\mathrm{d}t$.

Obviously, in the particular case of a single-valued function, these concepts coincide with those given previously in the vector case.

It is worthwhile to restate here the characterizations of HKP-integrable $\mathscr{P}_{\mathrm{wkc}}(X)$-valued multifunctions given in Theorem 1 in [19]:

Theorem 11. *Let $\Gamma\colon [0,T] \to \mathscr{P}_{\mathrm{wkc}}(X)$ be a scalarly* HK*-integrable multifunction. Then the following conditions are equivalent:*

i) Γ *is* HKP*-integrable;*

ii) Γ *has at least one* HKP*-integrable selection and for every* HKP*-integrable selection f there exists $G\colon [0,T] \to \mathscr{P}_{\mathrm{wkc}}(X)$ Pettis integrable, such that $\Gamma(t) = f(t) + G(t)$, $\forall\, t \in [0,T]$;*

iii) each measurable selection of Γ *is* HKP*-integrable.*

In the set-valued setting, we will use the following Komlós-type convergence (see 17]), involving the support functionals:

Definition 12. A sequence $(F_n)_n$ of $\mathscr{P}_{\mathrm{wkc}}(X)$-valued multifunctions is said to be Komlós-convergent (shortly, K-convergent) to a $\mathscr{P}_{\mathrm{wkc}}(X)$-valued multifunction F if for every subsequence $(F_{k_n})_n$ there exists a μ-null set $N \subset [0,T]$ (depending on the subsequence) such that for every $x^* \in X^*$ and every $t \in [0,T] \setminus N$,

$$\sigma(x^*, F(t)) = \lim_n \sigma\left(x^*, \frac{1}{n}\sum_{i=1}^{n} F_{k_i}(t)\right).$$

3. A Komlós theorem for the set-valued Henstock-Kurzweil-Pettis integral

By using an abstract Komlós-type theorem proved in [1], we obtain a Komlós-type result for the Henstock-Kurzweil-Pettis set-valued integral. For the convenience of the reader, we recall here Theorem 2.1 in [1], for the presentation of which we need some notation.

Let (Ω, Σ, μ) be a finite measure space and Y a convex cone, provided with a topology compatible with the operations of addition and multiplication by positive scalars. $\mathscr{B}(Y)$ will denote its Borel σ-algebra. Consider a collection $\mathscr{A}$ of $\Sigma \otimes \mathscr{B}(Y)$-measurable functions $a\colon \Omega \times Y \to \mathbb{R}$ such that, for every $\omega \in \Omega$, $a(\omega, \cdot)$ is affine and continuous on Y. A function $f\colon \Omega \to Y$ is said to be $\mathscr{A}$-scalarly measurable if for every $a \in \mathscr{A}$, the real function $a(\cdot, f(\cdot))$ is Σ-measurable. Suppose that there exists a sequence $(a_j)_{j\in\mathbb{N}} \subset \mathscr{A}$ which separates the points of Y. This means that for every $\omega \in \Omega$, $y = z$ if and only if $a_j(\omega, y) = a_j(\omega, z)$, $\forall\, j \in \mathbb{N}$. Given a function $h\colon \Omega \times Y \to [0, +\infty]$, we say that $h(\omega, \cdot)$ is (sequentially) inf-compact if for every $\omega \in \Omega$ and $\alpha \in \mathbb{R}$, the set $\{y \in Y;\, h(\omega, y) \leqslant \alpha\}$ is sequentially compact.

Theorem 13 (Theorem 2.1 in [1])**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $\mathscr{A}$-scalarly measurable Y-valued functions defined on Ω and satisfying that there exists $h\colon \Omega \times Y \to [0, +\infty]$ such that $h(\omega, \cdot)$ is convex and sequentially inf-compact and*

1) $\sup_n \int_\Omega |a_j(\omega, f_n(\omega))|\mu(\mathrm{d}\omega) < +\infty$, $\forall\, j \in \mathbb{N}$;
2) $\sup_n \int_\Omega^* h(\omega, f_n(\omega))\mu(\mathrm{d}\omega) < +\infty$.

Then there exists a subsequence $(f_{k_n})_n \subset (f_n)_n$ that Komlós-converges to an $\mathscr{A}$-scalarly measurable function f such that $\int_\Omega^* h(\omega, f(\omega))\mu(\mathrm{d}\omega) < +\infty$.

In the preceding theorem, $\int_\Omega^*$ is the outer integration with respect to μ, that is, for a (possibly non-measurable) function $\overline{\varphi}\colon \Omega \to \overline{\mathbb{R}}$, we have $\int_\Omega^* \overline{\varphi}\,\mathrm{d}\mu = \inf\{\int_\Omega \varphi\,\mathrm{d}\mu,\ \varphi \in L^1(\mu),\ \varphi \geqslant \overline{\varphi}\text{ a.e.}\}$.

Applying this result to an appropriate convex cone Y and a suitable family $\mathscr{A}$ of affine continuous functions, we obtain, in the set-valued Henstock-Kurzweil-Pettis integrability setting, the following Komlós-type result:

Theorem 14. *Let X be a separable Banach space which is weakly sequentially complete and let $F_n\colon [0, T] \to \mathscr{P}_{\mathrm{wkc}}(X)$ be a sequence of* HKP*-integrable multifunctions. Suppose that*

i) *for every $x^* \in X^*$*
 ia) *there exists a real* HK*-integrable function f_{x^*} such that*

$$f_{x^*}(t) \leqslant \sigma(x^*, F_n(t)), \quad \forall\, t \in [0, T],\ \forall\, n \in \mathbb{N};$$

ib) $\sup\limits_{n\in\mathbb{N}} (\mathrm{HK}) \int_0^T \sigma(x^*, F_n(t))\,\mathrm{d}t < +\infty$;

ii) *there exist a function* $h\colon [0,T] \times \mathscr{P}_{\mathrm{wkc}}(X) \to [0,+\infty]$ *such that, for every* $t \in [0,T]$, $h(t,\cdot)$ *is convex and sequentially inf-compact, and a countable measurable partition* $(B_m)_m$ *of* $[0,T]$ *satisfying, for every* $m \in \mathbb{N}$, *the following conditions:*

iia) $\sup_n \int_{B_m} |\sigma(x^*, F_n(t))|\,\mathrm{d}t < +\infty$, $\forall\, x^* \in X^*$;

iib) $\sup_n \int_{B_m}^* h(t, F_n(t))\,\mathrm{d}t < +\infty$.

Then there exist an HKP-*integrable* $\mathscr{P}_{\mathrm{wkc}}(X)$*-valued function* F *and a subsequence of* $(F_n)_n$ *which K-converges to* F. *Moreover,* $\int_{B_m}^* h(t, F(t))\,\mathrm{d}t < +\infty$ *for each* $m \in \mathbb{N}$.

Proof. By the separability assumption on X, we can find a Mackey-dense sequence $(x_k^*)_k$ in the unit ball of X^*. Consider the convex cone $Y = \mathscr{P}_{\mathrm{wkc}}(X)$ provided with the coarsest topology with respect to which all support functionals are continuous. Consider also the family $\mathscr{A} = \{a_{x^*}\colon x^* \in X^*\}$ of functions $a_{x^*}\colon [0,T] \times Y \to \mathbb{R}$, defined as $a_{x^*}(t,C) = \sigma(x^*, C)$, which are affine and continuous on Y. Take the countable subfamily $\{a_{x_k^*}\colon k \in \mathbb{N}\}$ that, by the Mackey-density assumption, separates the points of Y. Applying Theorem 13 on each B_m, after a diagonal process we obtain a subsequence $(F_{k_n})_n$ which is Komlós-convergent to a scalarly measurable $\mathscr{P}_{\mathrm{wkc}}(X)$-valued function F. Moreover, $\int_{B_m}^* h(t, F(t))\,\mathrm{d}t < +\infty$ for each $m \in \mathbb{N}$.

In order to prove the scalar HK-integrability of the limit multifunction, fix $x^* \in X^*$ and use the hypotheses ia) and ib). For every $n \in \mathbb{N}$, the positive function $-f_{x^*} + \sigma\Big(x^*, \dfrac{1}{n}\sum\limits_{i=1}^n F_{k_i}\Big)$ is HK-integrable, therefore, by Theorem 9.13 in [14], it is Lebesgue integrable. We are now able to apply Fatou's Lemma to the sequence $\Big(-f_{x^*} + \sigma\Big(x^*, \dfrac{1}{n}\sum\limits_{i=1}^n F_{k_i}\Big)\Big)_n$ in order to obtain

$$\begin{aligned}
&\int_0^T (-f_{x^*}(t) + \sigma(x^*, F(t)))\,\mathrm{d}t \\
&\qquad\leqslant \liminf_n \int_0^T -f_{x^*}(t) + \sigma\Big(x^*, \frac{1}{n}\sum_{i=1}^n F_{k_i}(t)\Big)\,\mathrm{d}t \\
&\qquad= (\mathrm{HK})\int_0^T -f_{x^*}(t)\,\mathrm{d}t + \liminf_n (\mathrm{HK})\int_0^T \sigma\Big(x^*, \frac{1}{n}\sum_{i=1}^n F_{k_i}(t)\Big)\,\mathrm{d}t \\
&\qquad\leqslant (\mathrm{HK})\int_0^T -f_{x^*}(t)\,\mathrm{d}t + \sup_{n\in\mathbb{N}} (\mathrm{HK})\int_0^T \sigma(x^*, F_n(t))\,\mathrm{d}t < +\infty.
\end{aligned}$$

Consequently, $-f_{x^*}(\cdot) + \sigma(x^*, F(\cdot))$ is Lebesgue integrable and, since f_{x^*} is HK-integrable, the HK-integrability of $\sigma(x^*, F(\cdot))$ follows.

Every measurable selection f of F is scalarly HK-integrable since, for each $x^* \in X^*$,

$$-\sigma(-x^*, F(t)) \leqslant \langle x^*, f(t)\rangle \leqslant \sigma(x^*, F(t)), \quad \text{a.e. } t \in [0,T].$$

By Remark 7, f is Khintchine integrable too. Theorem 3 in [12] yields that, for every $[a,b] \subset [0,T]$, there exists an element of the bi-dual $x^{**}_{[a,b]} \in X^{**}$ such that, for every $x^* \in X^*$, $\langle x^*, x^{**}_{[a,b]}\rangle = \int_a^b \langle x^*, f(s)\rangle\,\mathrm{d}s$, the integral being in the Khintchine sense. As the function to integrate is HK-integrable too, we have $\langle x^*, x^{**}_{[a,b]}\rangle = (\mathrm{HK})\int_a^b \langle x^*, f(s)\rangle\,\mathrm{d}s$. The Banach space being weakly sequentially complete by Theorem 40 in [13], we have $x^{**}_{[a,b]} \in X$ for every subinterval. Thus every measurable selection of F is HKP-integrable.

Finally, the implication iii) $\Rightarrow$ i) in Theorem 11 ensures the HKP-integrability of the limit set-valued function. $\square$

The following Blaschke-type compactness criteria (e.g. Lemma 5.1 in [15]) will allow us to obtain a useful consequence.

Lemma 15. *Let X be a separable Banach space and let $M \in \mathscr{P}_{\mathrm{wkc}}(X)$. Then the family of all weakly compact convex subsets of M is compact with respect to the coarsest topology of $\mathscr{P}_{\mathrm{wkc}}(X)$ for which $\sigma(x^*, \cdot)$ is continuous for every $x^* \in X^*$.*

Corollary 16. *Let X be a weakly sequentially complete separable Banach space and let $(F_n)_n$ be a sequence of* HKP*-integrable multifunctions $F_n\colon [0,T] \to \mathscr{P}_{\mathrm{wkc}}(X)$. Suppose that* i) *of the preceding theorem holds and that there is a $\mathscr{P}_{\mathrm{wkc}}(X)$-valued multifunction $\widetilde{F}$ such that $F_n(t) \subset \widetilde{F}(t)$ a.e. for all $n \in \mathbb{N}$. Then there exist an* HKP*-integrable $\mathscr{P}_{\mathrm{wkc}}(X)$-valued function F and a subsequence of $(F_n)_n$ which* K*-converges to F.*

P r o o f. Let us define $h\colon [0,T] \times \mathscr{P}_{\mathrm{wkc}}(X) \to [0,+\infty]$ by

$$h(t,C) = \begin{cases} 0 & \text{if } C \subset \widetilde{F}(t), \\ +\infty & \text{otherwise.} \end{cases}$$

It is convex and sequentially inf-compact with respect to the second variable. Indeed, fix $t \in [0,T]$ and $\alpha \in \mathbb{R}$. If $\alpha < 0$, then $\{C \in \mathscr{P}_{\mathrm{wkc}}(X);\ h(t,C) \leqslant \alpha\} = \emptyset$. Otherwise, $\{C \in \mathscr{P}_{\mathrm{wkc}}(X);\ h(t,C) \leqslant \alpha\} = \{C \in \mathscr{P}_{\mathrm{wkc}}(X);\ C \subset \widetilde{F}(t)\}$ which, by Lemma 15, is compact with respect to the topology of $\mathscr{P}_{\mathrm{wkc}}(X)$.

The countable measurable partition $(B_m)_m$ of the real interval given by

$$B_m = \{t \in [0,T];\ m-1 \leqslant |\widetilde{F}(t)| < m\}, \quad \forall\, m \in \mathbb{N}$$

satisfies hypothesis ii) in the preceding theorem: for every $m \in \mathbb{N}$,

$$\sup_{n\in\mathbb{N}} \int_{B_m} |\sigma(x^*, F_n(t))| \,\mathrm{d}t \leqslant \int_{B_m} |\sigma(x^*, \widetilde{F}(t))| \,\mathrm{d}t \leqslant \int_{B_m} |\widetilde{F}(t)| \,\mathrm{d}t < +\infty;$$

therefore, we are able to apply Theorem 14. □

The next consequence is a Komlós-type result similar to Theorem 3.1 in [16] for the set-valued Pettis integral:

Theorem 17. *Let X be a separable reflexive Banach space and $(F_n)_n$ a sequence of* HKP*-integrable $\mathscr{P}_{\mathrm{wkc}}(X)$-valued multifunctions satisfying hypothesis* i) *in Theorem* 14 *and*

ii′) *one can find a measurable countable partition $(B_m)_m$ of $[0,T]$ such that, for each $m \in \mathbb{N}$,*

$$\sup_{n\in\mathbb{N}} \int_{B_m} |F_n(t)| \,\mathrm{d}t < +\infty.$$

Then there exist an HKP*-integrable $\mathscr{P}_{\mathrm{wkc}}(X)$-valued function F and a subsequence of $(F_n)_n$ which* K*-converges to F. Moreover, $\int_{B_m} |F(t)| \,\mathrm{d}t < +\infty$ for every $m \in \mathbb{N}$.*

P r o o f. Alaoglu-Bourbaki's theorem yields that the function $h\colon [0,T] \times \mathscr{P}_{\mathrm{wkc}}(X) \to [0,+\infty]$ defined by $h(t,C) = |C|$ is convex and inf-compact in the second variable, whence, thanks to Theorem 14, we obtain the announced result. □

Applying Biting Lemma, we can prove a stronger property of the above mentioned subsequence and its Komlós-limit. Let us recall the Biting Lemma: for any $L^1([0,T])$-bounded sequence $(\varphi_n)_n$, there exist a subsequence $(\varphi_{k_n})_n$ and a sequence $(A_p)_p \subset \Sigma$ decreasing to $\emptyset$ such that the sequence $(\chi_{A_n^c}\varphi_{k_n})_n$ is uniformly integrable.

Proposition 18. *In the setting of Theorem 17, for every $\varepsilon > 0$, there exists $T_\varepsilon \in \Sigma$ with $\mu(T_\varepsilon) < \varepsilon$ such that for every $x^* \in X^*$ and every measurable $A \subset [0,T] \setminus T_\varepsilon$ we have*

$$\sigma\Bigl(x^*, \int_A F(t) \,\mathrm{d}t\Bigr) = \lim_n \sigma\Bigl(x^*, \int_A F_{k_n}(t) \,\mathrm{d}t\Bigr),$$

where the set-valued integrals are Aumann integrals.

P r o o f. Since the sequence of measurable sets $(B_m)_m$ covers the set of finite measure $[0,T]$ for every $\varepsilon > 0$, one can find $m_\varepsilon \in \mathbb{N}$ such that $\mu\Bigl(\bigcup_{m=m_\varepsilon+1}^{\infty} B_m\Bigr) < \frac{1}{2}\varepsilon$. By hypothesis ii′) in the preceding theorem, $\sup_{n\in\mathbb{N}} \int_{\bigcup_{m=1}^{m_\varepsilon} B_m} |F_n(t)| \,\mathrm{d}t < +\infty$, whence the Biting Lemma yields a measurable set $\widetilde{T_\varepsilon} \subset \bigcup_{m=1}^{m_\varepsilon} B_m$ such that $\mu\Bigl(\bigcup_{m=1}^{m_\varepsilon} B_m \setminus \widetilde{T_\varepsilon}\Bigr) < \frac{1}{2}\varepsilon$ and the sequence $(|F_n(\cdot)|)_n$ is uniformly integrable on $\widetilde{T_\varepsilon}$. Thus, $T_\varepsilon = \Bigl(\bigcup_{m=1}^{m_\varepsilon} B_m \setminus \widetilde{T_\varepsilon}\Bigr) \bigcup \Bigl(\bigcup_{m=m_\varepsilon+1}^{\infty} B_m\Bigr)$ has $\mu(T_\varepsilon) < \varepsilon$ and, for every $x^* \in X^*$, $(\sigma(x^*, F_n(\cdot))_n$ is uniformly integrable on $[0,T]\setminus T_\varepsilon$. Vitali's convergence theorem yields then that for every $x^* \in X^*$ and $A \subset [0,T]\setminus T_\varepsilon$ we have $\sigma(x^*, \int_A F(t)\,\mathrm{d}t) = \lim_n \sigma(x^*, \int_A F_{k_n}(t)\,\mathrm{d}t)$.

Finally, let us remark that any such measurable A is contained in $\bigcup_{m=1}^{m_\varepsilon} B_m$ and since on each B_m all F_n and F are integrably bounded, their selections are Bochner integrable on A, thus the set-valued integrals in the statement are Aumann integrals. $\square$

Remark 19. We can also prove Theorem 17 using a Komlós result for integrably bounded multifunctions (Theorem 2.5 in [2]) in a manner similar to that in which Theorem 3.1 in [16] was obtained.

4. Application to a best approximation problem

We are looking for a solution to the following best approximation problem: given two $\mathscr{P}_{\mathrm{wkc}}(X)$-valued HKP-integrable multifunctions H and F defined on $[0,T]$, we want to get a $\mathscr{P}_{\mathrm{wkc}}(X)$-valued HKP-integrable multifunction F_0 with $F_0(t) \subset F(t)$, $\forall\, t \in [0,T]$ such that

$$(1)\quad \int_0^T D(H(t), F_0(t))\,\mathrm{d}t = \inf\Bigl\{\int_0^T D(H(t), G(t))\,\mathrm{d}t;\ G \text{ HKP-integrable, } G(t)\subset F(t),\ \forall\, t \in [0,T]\Bigr\}.$$

Solutions to this problem were already found in [5] in the integrably bounded setting and in [16] in the Pettis integrable one.

If the Banach space and its topological dual have the Radon-Nikodym property, then the above problem has a solution. We use the following lower semi-continuity property of the Hausdorff distance (Lemma 5.1 in [16]):

Lemma 20. *Let $(C_n)_n \subset \mathscr{P}_{\mathrm{wkc}}(X)$ converge to $C_0 \in \mathscr{P}_{\mathrm{wkc}}(X)$ with respect to the topology of convergence of all support functionals. Then, for every $C \in \mathscr{P}_{\mathrm{wkc}}(X)$,*

$$D(C, C_0) \leqslant \liminf_n D(C, C_n).$$

Theorem 21. *Suppose that X and X^* have the Radon-Nikodym property and let H and F be two $\mathscr{P}_{\rm wkc}(X)$-valued* HKP*-integrable multifunctions defined on* $[0,T]$. *Then there is a $\mathscr{P}_{\rm wkc}(X)$-valued* HKP*-integrable multifunction F_0 with $F_0(t) \subset F(t), \forall t \in [0,T]$ such that the equality* (1) *is satisfied.*

P r o o f. By Theorem 11 there exist HKP-integrable functions f, h and $\mathscr{P}_{\rm wkc}(X)$-valued Pettis integrable multifunctions F_1, H_1 such that $F(t) = f(t) + F_1(t)$ and $H(t) = h(t) + H_1(t)$ for every $t \in [0,T]$. We can suppose that $m < \infty$, where m denotes the infimum in the equality (1), and consider a sequence $(G_n)_n$ of HKP-integrable $\mathscr{P}_{\rm wkc}(X)$-valued multifunctions contained in F such that

$$m = \lim_{n\to\infty} \int_0^T D(H(t), G_n(t))\,\mathrm{d}t.$$

Let us note that every $\mathscr{P}_{\rm wkc}(X)$-valued HKP-integrable multifunction G_n contained in F can be written as the sum of f and a $\mathscr{P}_{\rm wkc}(X)$-valued Pettis integrable multifunction G_n^1 contained in F_1. Indeed, since $G_n(t) \subset F(t) = f(t) + F_1(t)$ for every $t \in [0,T]$, we obtain that $G_n^1(t) = -f(t) + G_n(t) \subset F_1(t)$. Moreover, G_n^1 is $\mathscr{P}_{\rm wkc}(X)$-valued and thus, since F_1 is Pettis integrable, by the characterization of Pettis integrable $\mathscr{P}_{\rm wkc}(X)$-valued multifunctions (see [9]), Pettis integrability of G_n^1 follows.

We claim that $(G_n^1)_n$ satisfies the hypothesis of Theorem 3.3 in [16].

Indeed, since

$$-\sigma(-x^*, F_1(t)) \leqslant \sigma(x^*, G_n^1(t)) \leqslant \sigma(x^*, F_1(t))$$

for every $n \in \mathbb{N}$ and every $t \in [0,T]$ and, since $-\sigma(-x^*, F_1(\cdot))$ and $\sigma(x^*, F_1(\cdot))$ are Lebesgue integrable, it follows that the sequence $(\sigma(x^*, G_n^1(t)))_n$ is uniformly integrable.

Considering $B_m = \{t \in [0,T];\, m-1 < |F_1(t)| \leqslant m\}$, we obtain a countable measurable partition of the interval $[0,T]$ satisfying that $\sup\limits_{n\in\mathbb{N}} \int_{B_m} |G_n^1(t)|\,\mathrm{d}t \leqslant \int_{B_m} |F_1(t)|\,\mathrm{d}t < +\infty$ for each $m \in \mathbb{N}$, and, $\overline{\rm co}\Big(\bigcup\limits_{n\in\mathbb{N}} \int_A G_n^1(t)\,\mathrm{d}t\Big) \subset \int_A F_1(t)\,\mathrm{d}t \in \mathscr{P}_{\rm wkc}(X)$ for all $A \subset B_m$.

Then, applying Theorem 3.3 in [16] gives us a Pettis integrable $\mathscr{P}_{\rm wkc}(X)$-valued function F_0^1 and a subsequence $(G_{k_n}^1)_n$ that Komlós-converges to F_0^1.

Therefore, $(G_{k_n})_n$ Komlós-converges to $F_0 = f + F_0^1$ which is HKP-integrable and, thanks to the weak compactness and convexity of the values of F, F_0 is a.e. contained in F.

Then, using Lemma 20 and Fatou's Lemma, we obtain

$$\begin{aligned} m \leqslant \int_0^T D(H(t), F_0(t))\,\mathrm{d}t &\leqslant \int_0^T \liminf_n D\Big(H(t), \frac{1}{n}\sum_{i=1}^n G_{k_i}(t)\Big)\,\mathrm{d}t \\ &\leqslant \liminf_n \int_0^T D\Big(H(t), \frac{1}{n}\sum_{i=1}^n G_{k_i}(t)\Big)\,\mathrm{d}t \\ &\leqslant \liminf_n \frac{1}{n}\sum_{i=1}^n \int_0^T D(H(t), G_{k_i}(t))\,\mathrm{d}t \\ &= \lim_{n\to\infty} \int_0^T D(H(t), G_n(t))\,\mathrm{d}t = m, \end{aligned}$$

therefore $m = \int_0^T D(H(t), F_0(t))\,\mathrm{d}t$ and thus F_0 is a solution to our minimisation problem. □

The best approximation problem (1) has a solution in the case of a weakly sequentially complete Banach space too:

Theorem 22. *Let X be weakly sequentially complete and let H, F be two $\mathscr{P}_{\mathrm{wkc}}(X)$-valued* HKP*-integrable multifunctions defined on* $[0,T]$. *There exists a $\mathscr{P}_{\mathrm{wkc}}(X)$-valued* HKP*-integrable multifunction F_0 with $F_0(t) \subset F(t)$, $\forall\, t \in [0,T]$ such that the equality* (1) *is satisfied.*

Proof. As in the proof of the preceding theorem, we can suppose that $m < \infty$ and consider a sequence $(F_n)_n$ of HKP-integrable $\mathscr{P}_{\mathrm{wkc}}(X)$-valued multifunctions contained in F such that $m = \lim\limits_{n\to\infty} \int_0^T D(H(t), F_n(t))\,\mathrm{d}t$. We claim that $(F_n)_n$ verifies the hypothesis of Corollary 16. Indeed, for every $x^* \in X^*$ there exists $-\sigma(-x^*, F)$ that is a real HK-integrable function such that $-\sigma(-x^*, F(t)) \leqslant \sigma(x^*, F_n(t))$, $\forall\, t \in [0,T]$ for every $n \in \mathbb{N}$.

Obviously, $\sup\limits_{n\in\mathbb{N}} (\mathrm{HK}) \int_0^T \sigma(x^*, F_n(t))\,\mathrm{d}t \leqslant (\mathrm{HK}) \int_0^T \sigma(x^*, F(t))\,\mathrm{d}t < +\infty$.

Then, applying Corollary 16 gives us an HKP-integrable $\mathscr{P}_{\mathrm{wkc}}(X)$-valued function F_0 and a subsequence of $(F_n)_n$ which K-converges to F_0.

Similarly to the second part of the proof of the preceding theorem, we obtain that $m = \int_0^T D(H(t), F_0(t))\,\mathrm{d}t$, so F_0 is a solution to problem (1). □

5. Application to weak compactness in the space of HKP-integrable multifunctions

Let F be a $\mathscr{P}_{\rm wkc}(X)$-valued HKP-integrable multifunction.

Definition 23. $G\colon [0,T] \to \mathscr{P}_{\rm wkc}(X)$ is said to be a multi-selection of F if $G(t) \subset F(t)$ a.e.

Obviously, every selection is a multi-selection. Consider the family of all HKP-integrable multi-selections of F and denote it by $\widetilde{S}_F^{\rm HKP}$. It is nonempty by Theorem 11.

On the space of $\mathscr{P}_{\rm wkc}(X)$-valued HKP-integrable multifunctions, by the $\widetilde{w}$-HKP topology, we will understand the coarsest one with respect to which the HK-integrals of the products of support functionals with real bounded variation functions are convergent. That is $F_\alpha \to F$ if for every $g\colon [0,T] \to \mathbb{R}$ of bounded variation and every $x^* \in X^*$,

$$(\mathrm{HK})\int_0^T g(t)\sigma(x^*, F_\alpha(t))\,\mathrm{d}t \to (\mathrm{HK})\int_0^T g(t)\sigma(x^*, F(t))\,\mathrm{d}t.$$

This is an extension of the w-HKP topology to the set-valued case.

We give now a weak compactness result.

Proposition 24. *Let X be a separable Banach space and let F be a $\mathscr{P}_{\rm wkc}(X)$-valued* HKP*-integrable multifunction. Then $\widetilde{S}_F^{\rm HKP}$ is $\widetilde{w}$-*HKP *sequentially compact.*

Proof. Let $(F_n)_n$ be a sequence of HKP-integrable multi-selections of F. Applying Theorem 11 one can find an HKP-integrable function f and a $\mathscr{P}_{\rm wkc}(X)$-valued Pettis integrable multifunction G such that, for all $t \in [0,T]$, $F(t) = f(t) + G(t)$.

As in the proof of Theorem 21 we can prove that, for every $n \in \mathbb{N}$, there exists a Pettis integrable multi-selection of G, denoted by G_n, such that $F_n(t) = f(t) + G_n(t)$, $\forall\, t \in [0,T]$.

Proposition 2.6 in [4] yields that one can find a subsequence $(G_{k_n})_n$ and a $\mathscr{P}_{\rm wkc}(X)$-valued Pettis integrable multifunction G_∞ such that, for every $g \in L^\infty([0,T])$ and any $x^* \in X^*$,

$$\lim_{n\to\infty}\int_0^T g(t)\sigma(x^*, G_{k_n}(t))\,\mathrm{d}t = \int_0^T g(t)\sigma(x^*, G_\infty(t))\,\mathrm{d}t.$$

Moreover, on every measurable A,

$$\int_A \sigma(x^*, G_\infty(t))\,\mathrm{d}t = \lim_{n\to\infty}\int_A \sigma(x^*, G_{k_n}(t))\,\mathrm{d}t \leqslant \int_A \sigma(x^*, G(t))\,\mathrm{d}t,$$

whence, for every $x^* \in X^*$, we have $\sigma(x^*, G_\infty(t)) \leqslant \sigma(x^*, G(t))$ a.e. Therefore, by passing through a Mackey-dense sequence and using the weak compactness of the values of G_∞ and G, we obtain that G_∞ is a multi-selection of G.

It follows that $(F_{k_n})_n$ $\widetilde{w}$-HKP-converges to $F_\infty = f + G_\infty$, which is a multi-selection of F, and so the $\widetilde{w}$-HKP sequential compactness of the family of multi-selections is proved. □

In particular, the family of all HKP-integrable selections is w-HKP sequentially compact.

Using the Komlós theorems obtained in the first section we can get two weak compactness criteria in the space of all $\mathscr{P}_{\rm wkc}(X)$-valued HKP-integrable multifunctions. We will use the following two lemmas:

Lemma 25. *Let $(f_n)_n$ be a uniformly* HK*-integrable, pointwise bounded sequence of real functions defined on $[0,T]$ and let $g\colon [0,T] \to \mathbb{R}$ be a function of bounded variation. Then*

i) *the sequence $\widetilde{f}_n(\cdot) = (\mathrm{HK}) \int_0^{\cdot} f_n(t)\,\mathrm{d}t$ is uniformly equicontinuous on $[0,T]$;*

ii) *$\widetilde{f}_n$ is Riemann-Stieltjes integrable with respect to g uniformly in $n \in \mathbb{N}$;*

iii) *the sequence $(gf_n)_n$ is uniformly* HK*-integrable.*

Proof. i) Let us define $\widetilde{f}\colon [0,T] \to l_\infty$ by $\widetilde{f}(t) = (\widetilde{f}_n(t))_n$, $\forall\, t \in [0,T]$. Let us first verify that $\widetilde{f}$ is l_∞-valued. Take $c \in [0,T]$. By the uniform HK-integrability hypothesis, there exists a partition of $[0,c]$ such that $\Bigl|\sum\limits_{i=1}^{k} f_n(t_i)(c_{i+1}-c_i)-\widetilde{f}_n(c)\Bigr| < 1$, $\forall\, n$. The pointwise boundedness assumption on $(f_n)_n$ allows to choose $M < \infty$ such that $|f_n(t_i)| \leqslant M$, $\forall\, i \in \{1,\ldots,k\}$, $\forall\, n \in \mathbb{N}$. Then $|\widetilde{f}_n(c)| \leqslant 1 + Mc$, $\forall\, n \in \mathbb{N}$ and so the assertion follows.

To prove the equicontinuity of the above defined sequence is equivalent to proving that the function $\widetilde{f}$ is continuous with respect to the sup-norm on l_∞ (thus uniformly continuous, since the definition domain is compact).

Fix $c \in [0,T]$ and $\varepsilon > 0$. By hypothesis, one can find $M_c < +\infty$ such that $|f_n(c)| \leqslant M_c$ for all $n \in \mathbb{N}$, and a gauge δ_ε satisfying $\Bigl|\sum\limits_{i=1}^{k} f_n(t_i)(c_{i+1}-c_i)-(\widetilde{f}_n(c_{i+1})-\widetilde{f}_n(c_i))\Bigr| < \varepsilon$ for every $n \in \mathbb{N}$ and every δ_ε-fine partition. Then every $x \in [0,T]$ with $|x-c| \leqslant \eta_{\varepsilon,c}$, where $\eta_{\varepsilon,c} = \min(\delta_\varepsilon(c), \varepsilon/M_c)$, satisfies, by Saks-Henstock's Lemma (Lemma 9.11 in [14]), the inequality

$$|\widetilde{f}_n(x) - \widetilde{f}_n(c)| \leqslant |\widetilde{f}_n(x) - \widetilde{f}_n(c) - f_n(c)(x-c)| + |f_n(c)(x-c)| \leqslant 2\varepsilon, \quad \forall\, n \in \mathbb{N},$$

since the interval (x,c) with the tag c is an element of a δ_ε-fine partition of $[0,T]$.

Consequently, $\|\widetilde{f}(x)-\widetilde{f}(c)\|_\infty \leqslant 2\varepsilon$ for every x with $|x-c| \leqslant \eta_{\varepsilon,c}$ so the continuity is proved.

ii) follows, by virtue of the equicontinuity of the sequence $(\widetilde{f}_n)_n$, by the straightforward adaptation of the proof of the fact that every continuous function is Riemann-Stieltjes integrable with respect to a function of bounded variation (e.g. Theorem 12.15 in [14]).

Finally, the assertions i) and ii) allow us to follow the same reasoning as in the proof of Lemma 4 in order to obtain iii). □

We have already noticed that the concept of uniform HK-integrability does not allow to ignore the μ-null sets (see Example 6). We have, nonetheless, the following property:

Lemma 26. *Any pointwise bounded sequence of functions $f_k\colon [0,T]\to\mathbb{R}$ which are null except on a set of null measure is uniformly* HK*-integrable.*

Proof. Let N be the μ-null set from the hypothesis.

For every $n\in\mathbb{N}$, put $N'_n=\{t\in N\colon\, 0<|f_k(t)|\leqslant n,\ \forall k\}$ and let $(N_n)_n$ be the associated pairwise disjoint sequence. By the pointwise boundedness assumption, the sequence $(N_n)_n$ covers the set N. For each n one can find an open set O_n such that $N_n\subset O_n$ and $\mu(O_n)<\varepsilon/n2^n$. Define a gauge $\delta_\varepsilon\colon [0,T]\to\mathbb{R}$ by

$$\delta_\varepsilon(t)=\begin{cases}1 & \text{if } t\in[0,T]\setminus N,\\ d(t,(O_n)^c) & \text{if } t\in N_n.\end{cases}$$

Then for every δ_ε-fine partition $\mathscr{P}$ of $[0,T]$, denote by $\mathscr{P}_n$ the subset of $\mathscr{P}$ that has tags in N_n. If I is an interval of $\mathscr{P}_n$, then $I\subset O_n$. If we denote by $f(\mathscr{P})$ the HK-integral sum associated to f and to the partition $\mathscr{P}$, then, for every k, $|f_k(\mathscr{P})|\leqslant \sum\limits_{n=1}^{\infty}|f_k(\mathscr{P}_n)|\leqslant\sum\limits_{n=1}^{\infty}n\mu(O_n)<\varepsilon$. Thus the sequence considered is uniformly HK-integrable. □

Proposition 27. *Let X be a weakly sequentially complete separable Banach space and $\mathscr{K}$ a family of $\mathscr{P}_{\rm wkc}(X)$-valued* HKP*-integrable multifunctions on $[0,T]$ satisfying*

i′) *for every $x^*\in X^*$, the family $\{\sigma(x^*,F(\cdot))\colon\, F\in\mathscr{K}\}$ is uniformly* HK*-integrable and $\mathscr{K}$ is pointwise bounded;*

ii) *there exist a function $h\colon [0,T]\times\mathscr{P}_{\rm wkc}(X)\to[0,+\infty]$ such that, for every $t\in[0,T]$, $h(t,\cdot)$ is convex and sequentially inf-compact, and a countable measurable partition $(B_m)_m$ of $[0,T]$ such that, for every $m\in\mathbb{N}$,*

iia) $\sup\bigl\{\int_{B_m}|\sigma(x^*,F(t))|\,{\rm d}t\colon\, F\in\mathscr{K}\bigr\}<+\infty$, $\forall x^*\in X^*$;

iib) $\sup\bigl\{\int_{B_m}^{*} h(t,F(t))\,\mathrm{d}t\colon\ F\in\mathscr{K}\bigr\}<+\infty$.

Then $\mathscr{K}$ *is relatively* $\widetilde{w}$-HKP *sequentially compact.*

Proof. Let $(F_n)_n$ be a sequence in $\mathscr{K}$. The existence of a subsequence $(F_{k_n})_n$ that Komlós converges to a measurable $\mathscr{P}_{\rm wkc}(X)$-valued function F follows in the same way as in the first part of the proof of Theorem 14.

The scalar HK-integrability of the limit multifunction follows from Theorem 13.16 in [14] applied, for each $x^*\in X^*$, to the sequence $\Bigl(\sigma\Bigl(x^*,\dfrac{1}{n}\sum\limits_{i=1}^{n}F_{k_i}\Bigr)\Bigr)_n$. Indeed, it is obvious that our condition i$'$) implies the uniform HK-integrability of the latter sequence and the pointwise boundedness assumption allows us (thanks to Lemma 26) to suppose that this sequence converges everywhere to $\sigma(x^*,F)$ (on the exceptional null set, we redefine all multifunctions by 0).

Applying Lemma 25, we obtain that for any g of bounded variation,

$$\Bigl(g\sigma\Bigl(x^*,\frac{1}{n}\sum_{i=1}^{n}F_{k_i}\Bigr)\Bigr)_n$$

is uniformly HK-integrable whence, again by Theorem 13.16 in [14], we conclude that

$$(\mathrm{HK})\int_0^T g(t)\sigma(x^*,F(t))\,\mathrm{d}t=\lim_n(\mathrm{HK})\int_0^T g(t)\sigma\Bigl(x^*,\frac{1}{n}\sum_{i=1}^{n}F_{k_i}(t)\Bigr)\,\mathrm{d}t.$$

This equality can be written as

$$(\mathrm{HK})\int_0^T g(t)\sigma(x^*,F(t))\,\mathrm{d}t=\lim_n\frac{1}{n}\sum_{i=1}^{n}(\mathrm{HK})\int_0^T g(t)\sigma(x^*,F_{k_i}(t))\,\mathrm{d}t$$

and, since this is true for every subsequence of $(F_{k_n})_n$, it follows that $(F_{k_n})_n$ satisfies that for every $x^*\in X^*$ and every $g\colon\ [0,T]\to\mathbb{R}$ of bounded variation one has

$$(\mathrm{HK})\int_0^T g(t)\sigma(x^*,F(t))\,\mathrm{d}t=\lim_n(\mathrm{HK})\int_0^T g(t)\sigma(x^*,F_{k_n}(t))\,\mathrm{d}t.$$

In other words, the subsequence $(F_{k_n})_n$ $\widetilde{w}$-HKP converges, whence the relative $\widetilde{w}$-HKP sequential compactness of $\mathscr{K}$ follows. $\square$

In the same way, applying Theorem 17, we get

Proposition 28. *Let X be a separable reflexive Banach space. Let $\mathscr{K}$ be a family of* HKP*-integrable $\mathscr{P}_{\rm wkc}(X)$-valued multifunctions satisfying the following conditions:*

i′) *for every $x^* \in X^*$, the family $\{\sigma(x^*, F),\ F \in \mathscr{K}\}$ is uniformly* HK*-integrable and $\mathscr{K}$ is pointwise bounded;*

ii) *there is a countable measurable partition $(B_m)_m$ of $[0, T]$ such that, for each $m \in \mathbb{N}$, $\sup\bigl\{\int_{B_m} |F(t)|\,{\rm d}t\colon\ F \in \mathscr{K}\bigr\} < +\infty$.*

*Then $\mathscr{K}$ is relatively $\widetilde{w}$-*HKP *sequentially compact.*

6. An integral inclusion involving the Henstock-Kurzweil-Pettis set-valued integral

In the sequel, we consider the space X provided with its weak topology, denoting it by X_w, and the vector space $C([0, T], X_w)$ of all X_w-valued continuous functions on $[0, T]$ provided with the topology of uniform convergence.

The following theorem extends an existence result for solutions of a set-valued integral equation (Theorem VI-7 in [6]) that imposed a Pettis integrability condition.

Theorem 29. *Let an open subset U of X_w, an* HKP*-integrable set-valued function $\Gamma\colon\ [0, T] \to \mathscr{P}_{\rm wkc}(X)$ and $F\colon\ [0, T] \times U \to \mathscr{P}_{\rm wkc}(X)$ satisfy*

1) *$F(t, x) \subset \Gamma(t)$, $\forall\, t \in [0, T]$, $\forall x \in U$;*
2) *$F(t, \cdot)$ is upper semi-continuous for every $t \in [0, T]$;*
3) *$\sigma(x^*, F(\cdot, x))$ is measurable for every $x^* \in X^*$ and every $x \in U$.*

Then, for every fixed $\xi \in U$, there exists $T_0 \in\ ]0, T]$ such that $\xi + (\mathrm{HKP}) \int_0^{T_0} \Gamma(s)\,{\rm d}s \subset U$ and the integral inclusion

$$x(t) \in \xi + (\mathrm{HKP}) \int_0^t F(s, x(s))\,{\rm d}s$$

has a solution in $C([0, T_0], X_w)$. Moreover, the set of solutions is compact in $C([0, T_0], X_w)$.

P r o o f. Theorem 11 yields that there exist an HKP-integrable function f and a $\mathscr{P}_{\rm wkc}(X)$-valued Pettis integrable multifunction G satisfying that, for every $t \in [0, T]$, we have $\Gamma(t) = f(t) + G(t)$. By Theorem 3, f is scalarly measurable and, as the Banach space is separable, f is measurable.

Fix $\xi \in U$ and consider a weakly open subset U_1 of X and a weak neighborhood U_2 of the origin such that $\xi \in U_1$ and $U_1 + U_2 \subset U$. Since $(\mathrm{HKP}) \int_0^{\cdot} f(t)\,{\rm d}t$ is weakly continuous, there exists $T_1 \in\ ]0, T]$ such that $(\mathrm{HKP}) \int_0^t f(t)\,{\rm d}t \in U_2$ for every $t \in$

$[0,T_1]$. Then the set-valued function $\widetilde{F}\colon [0,T_1]\times U_1 \to X$ defined by $\widetilde{F}(t,x) = -f(t) + F\bigl(t, x + (\mathrm{HKP})\int_0^t f(\tau)\,\mathrm{d}\tau\bigr)$ satisfies the following conditions:

1) $\widetilde{F}(t,x) \subset G(t)$, $\forall\, t\in[0,T_1]$, $\forall\, x \in U_1$;
2) $\widetilde{F}(t,\cdot)$ is upper semi-continuous for every $t\in[0,T_1]$;
3) $\sigma(x^*,\widetilde{F}(\cdot,x))$ is measurable for every $x^*\in X^*$ and every $x\in U_1$.

Applying then Theorem VI-7 in [6] we obtain that there exists $T_0 \in\,]0,T_1]$ such that $\xi + (\mathrm{P})\int_0^{T_0} G(s)\,\mathrm{d}s \subset U_1$, the integral inclusion

$$\widetilde{x}(t) \in \xi + (\mathrm{P})\int_0^t \widetilde{F}(s,\widetilde{x}(s))\,\mathrm{d}s$$

has a solution in $C([0,T_0],X_w)$ and the set of solutions is compact in $C([0,T_0],X_w)$.

Therefore, $\xi + (\mathrm{HKP})\int_0^{T_0}\Gamma(s)\,\mathrm{d}s = \xi + (\mathrm{HKP})\int_0^{T_0} f(s)\,\mathrm{d}s + (\mathrm{P})\int_0^{T_0} G(s)\,\mathrm{d}s \subset U$ and we can find $\widetilde{x}\in C([0,T_0],X_w)$ such that

$$\widetilde{x}(t) \in \xi + (\mathrm{P})\int_0^t -f(s) + F\biggl(s, \widetilde{x}(s) + (\mathrm{HKP})\int_0^s f(\tau)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s,$$

in other words

$$\widetilde{x}(t) + (\mathrm{HKP})\int_0^t f(s)\,\mathrm{d}s \in \xi + (\mathrm{HKP})\int_0^t F\biggl(s,\widetilde{x}(s) + (\mathrm{HKP})\int_0^s f(\tau)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s.$$

Thus $x(\cdot) = \widetilde{x}(\cdot) + (\mathrm{HKP})\int_0^{\cdot} f(\tau)\,\mathrm{d}\tau$ is a continuous function mapping $[0,T_0]$ into X_w and it is a solution of our integral inclusion. $\square$

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