Czechoslovak Mathematical Journal, 64 (139) (2014), 447–458

ON THE *abc*-PROBLEM IN WEYL-HEISENBERG FRAMES

Xinggang He, Haixiong Li, Wuhan

(Received January 27, 2013)

Abstract. Let $a,b,c>0$. We investigate the characterization problem which asks for a classification of all the triples (a,b,c) such that the Weyl-Heisenberg system $\{e^{2\pi imbx}\times \chi_{[na,na+c)}\colon m,n\in\mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$. It turns out that the answer to the problem is quite complicated, see Gu and Han (2008) and Janssen (2003). Using a dilation technique, one can reduce the problem to the case where $b=1$ and only let a and c vary. In this paper, we extend the Zak transform technique and use the Fourier analysis technique to study the problem for the case of a being a rational number. We prove some special cases of values for c and a that do not produce a frame, which expands earlier works.

Keywords: *abc*-problem; Weyl-Heisenberg frame; Zak transform

MSC 2010: 42C15, 42C40

1. Introduction

Let $g\in L^2(\mathbb{R})$ and $a,b\in\mathbb{R}^+$. We use (g,a,b) to denote the *Weyl-Heisenberg system* (also called a Gabor system) $\{E_{mb}T_{na}g\colon m,n\in\mathbb{Z}\}$ generated by a *window function* g where $E_bg(t)=e^{2\pi ibt}g(t)$ is the modulation operator and $T_ag(t)=g(t-a)$ is the translation operator. We say that (g,a,b) is a *Weyl-Heisenberg frame* (WH-frame for short) for $L^2(\mathbb{R})$ if there exist two positive constants A,B such that

$$A\|f\|^2\leqslant\sum_{m,n\in\mathbb{Z}}|\langle f,E_{mb}T_{na}g\rangle|^2\leqslant B\|f\|^2 \tag{1.1}$$

holds for every $f\in L^2(\mathbb{R})$. The Gabor system (g,a,b) is called a *Bessel sequence* if the second inequality in (1.1) holds. We refer to the excellent survey papers of

The research has been supported by the National Natural Science Foundation of China 11271148 and 11401189.

Casazza [1] and Heil [5] for some background material and recent development in the WH-frame theory.

For any $g \in L^2(\mathbb{R})$, a fundamental question in the WH-frame theory is to classify all the a, b such that (g, a, b) generates a WH-frame for $L^2(\mathbb{R})$. One general restriction is the density condition which forces $ab \leqslant 1$ if (g, a, b) is a frame. Although it is a key condition for a WH-frame, the density condition is still far from providing an answer to the fundamental question. This is generally believed to be a quite difficult problem. In fact, up to the very recent time just few functions have been solved completely, such as the Gaussian $g(t) = \mathrm{e}^{-\pi t^2}$ ([9]–[11]), the hyperbolic secant $g(t) = (\mathrm{e}^t + \mathrm{e}^{-t})^{-1}$, one- and two-sided exponential functions $g(t) = \mathrm{e}^{-t}\chi_{\mathbb{R}^+}$ and $g(t) = \mathrm{e}^{-|t|}$ ([6], [8]), and the totally positive function of finite type [3].

In this paper, we are concerned with the problem when the window function g is a characteristic function χ_E of a measurable subset of $\mathbb{R}$. Without loss of generality we assume that $b = 1$ and $0 < a < 1$. For the simplest case $g = \chi_{[0,c)}$, it is slightly surprising that the classification of all $a, c \in \mathbb{R}^+$ such that $(\chi_{[0,c)}, a, 1)$ is a frame is a very difficult problem (called the *abc*-problem), which is associated with a complicated set—Janssen's tie (see Subsection 3.7 in [7]). Janssen [7] has obtained many elaborate results on a, c for $(\chi_{[0,c)}, a, 1)$ being a frame or not. Gu and Han [4] established a new way to how study this problem and got a characteristic criterion for $(\chi_{[0,c)}, a, 1)$ being a frame. Although classification has been obtained for some cases, this problem appears to be very difficult in general. In this paper, by virtue of the technique of Fourier analysis, we make some progress on this problem.

The paper is organized as follows. In Section 2, some known and new results are presented. In Section 3, we give proofs of the new results.

2. Preliminaries

Let $\lfloor c \rfloor$ be the largest integer which is less than or equal to c and let $\{c\} = c - \lfloor c \rfloor$ be the fractional part of c. For the cases $0 < c \leqslant 2$ and $0 < a \leqslant 1$, $(\chi_{[0,c)}, a, 1)$ has been solved completely as to whether it is a frame or not (see 3.3.5, 3.3.6 and 3.4.3 in [7] and also [4]). First we recall the following theorem given by Janssen [7], Propositons 3.2.2 and 3.3.4.

Theorem 2.1. *Let $c > 2$ and let p, q be two positive integers.*

(1) *When $a = 1/q$, then $(\chi_{[0,c)}, a, 1)$ is not a frame if and only if $\{c\} \in [0, 1/q) \cup (1 - 1/q, 1)$.*
(2) *When $a = p/q$ with $\gcd(p, q) = 1$, then $(\chi_{[0,c)}, a, 1)$ is not a frame if $\{c\} \in [0, 1/q) \cup (1 - 1/q, 1)$.*

Naturally we want to know the result of Theorem 2.1(2) at the extreme points $\{c\} = 1/q$ and $\{c\} = 1 - 1/q$. Using a new method we get the following theorem.

Theorem 2.2. *Let $c > 3$ and let p, q be two co-prime positive integers. When $\{c\} = 1/q$, then $(\chi_{[0,c)}, p/q, 1)$ is not a frame if $\gcd(p, \lfloor c \rfloor) > 1$ or p is even; when $\{c\} = (q-1)/q$, then $(\chi_{[0,c)}, a, 1)$ is not a frame if $\gcd(p, \lfloor c \rfloor + 1) > 1$ or p is even.*

We conjecture that $(\chi_{[0,c)}, p/q, 1)$ is a frame when $c > 3$ and $\gcd(p, \lfloor c \rfloor) = 1$ for any odd p with $\{c\} = 1/q$. In general, Janssen [7], Propositions 3.3.2 and 3.4.4, also showed

Theorem 2.3. *If a is rational with $0 < a \leqslant \min\{\{c\}, 1 - \{c\}\}$ or irrational with $0 < a \leqslant \max\{\{c\}, 1 - \{c\}\}$, then $(\chi_{[0,c)}, a, 1)$ is a frame.*

Next we consider the remaining case $\min\{\{c\}, 1 - \{c\}\} < a < 1$ with a being rational in two situations $\{c\} < a$ and $1 - \{c\} < a$.

Theorem 2.4. *Let $c > 3$ and let p, q be two co-prime positive integers. Then $(\chi_{[0,c)}, p/q, 1)$ is not a frame when $\lfloor q\{c\} \rfloor < \gcd(p, \lfloor c \rfloor) < \lfloor c \rfloor$ or $\lfloor q(1 - \{c\}) \rfloor < \gcd(p, \lfloor c \rfloor + 1) < \lfloor c \rfloor + 1$.*

We remark that the condition $\lfloor q\{c\} \rfloor < \gcd(p, \lfloor c \rfloor)$ yields $\{c\} < a$ and the other yields $1 - \{c\} < a$. Hence we have the following corollary.

Corollary 2.5. *Let $c > 3$ and $a = p/q$ with p, q being two co-prime positive integers. If $\{c\} < a$ and p is a proper factor of $\lfloor c \rfloor$ or $1 - \{c\} < a$ and p is a proper factor of $\lfloor c \rfloor + 1$, then $(\chi_{[0,c)}, a, 1)$ is not a frame.*

There is an example given by Janssen [7], Example (b), page 33, that $a = p/q = 4/5$ and $c = 28/5$: in this case $(\chi_{[0,c)}, a, 1)$ is a frame. Note that here $p = 4$ and $\lfloor c \rfloor = 5$, which does not satisfy the conditions of Theorem 2.4. So we conjecture that the conditions of Theorem 2.4 are the best for $(\chi_{[0,c)}, p/q, 1)$ not being a frame.

Janssen [7], Casazza and Kalton [1], [2] and others used the Zak transform to study the WH-frame for the case $a = 1$ or $a = 1/q$. In this paper, we extend the Zak transform technique and use the Fourier analysis technique to study the *abc*-problem for the case $0 < a = p/q < 1$.

3. Proofs

We begin with the definition of the Zak transform [1], Section H, page 46:

Definition 3.1. The Zak transform of a function $f \in L^2(\mathbb{R})$ is defined by

$$Z_f(x,t) = \sum_{k\in\mathbb{Z}} f(x+k)\mathrm{e}^{-2\pi\mathrm{i}kt} \tag{3.1}$$

for a.e. $x, t \in \mathbb{R}$, where the convergence of the right-hand side has to be interpreted in the $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ sense.

From the definition we have the quasi-periodicity relations

$$\begin{aligned} Z_f(x+1,t) &= \mathrm{e}^{2\pi\mathrm{i}t} Z_f(x,t),\\ Z_f(x,t+1) &= Z_f(x,t). \end{aligned}$$

It is straightforward that the Zak transform is completely determined by its values in the unit square $Q = [0,1)^2$. Note that $Z_{E_m T_n \chi_{[0,1)}}(x,t) = \mathrm{e}^{2\pi\mathrm{i}(mx+nt)}$, $(x,t) \in Q$ and $\{\mathrm{e}^{2\pi\mathrm{i}(mx+nt)}\}_{m,n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(Q)$; we see that the Zak transform is a unitary map from $L^2(\mathbb{R})$ onto $L^2(Q)$. Based on the above fact, we have the following lemma.

Lemma 3.2. *Let $g \in L^2(\mathbb{R})$ and $a = p/q$ with p, q being two co-prime positive integers. If the Gabor system $(g, a, 1)$ is a Bessel sequence, then, for any $f \in L^2(\mathbb{R})$,*

$$\begin{aligned} &\sum_{m,n\in\mathbb{Z}} |\langle f, E_m T_{na} g\rangle|^2 \\ &\quad = \sum_{l=0}^{q-1} \int_{l/q}^{(l+1)/q} \mathrm{d}x \sum_{k\in\mathbb{Z}} \sum_{j=0}^{q-1} \left| \int_0^1 Z_f(x,t) \overline{Z}_g\Big(x - \frac{l}{q} + \frac{r_{lj}}{q}, t\Big) \mathrm{e}^{-2\pi\mathrm{i}(kp+d_{lj})t}\,\mathrm{d}t \right|^2, \end{aligned}$$

where $l + jp = d_{lj}q + r_{lj}$ and $0 \leqslant r_{lj} < q$ for $0 \leqslant l, j < q$.

Proof. By the definition of the Zak transform, it is easy to check that

$$Z_{E_m T_{na} g}(x,t) = Z_g(x - na, t)\mathrm{e}^{2\pi\mathrm{i}mx}.$$

Then

$$\begin{aligned}
\sum_{m,n\in\mathbb{Z}} |\langle f, E_m T_{na} g\rangle|^2 &= \sum_{m,n\in\mathbb{Z}} |\langle Z_f, Z_{E_m T_{na} g}\rangle|^2 \\
&= \sum_{m,n\in\mathbb{Z}} |\langle Z_f(x,t)\overline{Z}_g(x-na,t), \mathrm{e}^{2\pi\mathrm{i}mx}\rangle|^2 \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\left|\int_0^1 \left(\int_0^1 Z_f(x,t)\overline{Z}_g(x-na,t)\,\mathrm{d}t\right)\mathrm{e}^{-2\pi\mathrm{i}mx}\,\mathrm{d}x\right|^2 \\
&= \sum_{n\in\mathbb{Z}}\int_0^1\left|\int_0^1 Z_f(x,t)\overline{Z}_g(x-na,t)\,\mathrm{d}t\right|^2\mathrm{d}x \\
&= \sum_{n\in\mathbb{Z}}\int_0^1\left|\int_0^1 Z_f(x,t)\overline{Z}_g(x+na,t)\,\mathrm{d}t\right|^2\mathrm{d}x \\
&= \sum_{l=0}^{q-1}\int_{l/q}^{(l+1)/q}\mathrm{d}x\sum_{k\in\mathbb{Z}}\sum_{j=0}^{q-1}\left|\int_0^1 Z_f(x,t)\overline{Z}_g\Big(x+\frac{jp}{q},t\Big)\mathrm{e}^{-2\pi\mathrm{i}kpt}\,\mathrm{d}t\right|^2,
\end{aligned}$$

where we have used the decomposition $n = kq + j$ for each $n \in \mathbb{Z}$ with $0 \leqslant j \leqslant q-1$ and the quasi-periodicity properties of the Zak transform. Since there are unique nonnegative integers d_{lj} and r_{lj} with $0 \leqslant r_{lj} < q$ such that $l + jp = d_{lj}q + r_{lj}$ for $0 \leqslant l, j < q$, it follows that $x + jp/q = x - l/q + r_{lj}/q + d_{lj}$ for each $0 \leqslant l < q$. Hence the above sum is equal to

$$\sum_{l=0}^{q-1}\int_{l/q}^{(l+1)/q}\mathrm{d}x\sum_{k\in\mathbb{Z}}\sum_{j=0}^{q-1}\left|\int_0^1 Z_f(x,t)\overline{Z}_g\Big(x-\frac{\mathrm{i}}{q}+\frac{r_{lj}}{q},t\Big)\mathrm{e}^{-2\pi\mathrm{i}(kp+d_{lj})t}\,\mathrm{d}t\right|^2,$$

and the lemma follows. $\square$

It is worth noting that the Gabor system (g, a, b) is a Bessel sequence for any choice of $a, b > 0$ when $g \in L^2(\mathbb{R})$ is bounded and compactly supported. So if g is an indicator function, then Lemma 3.2 always holds.

Let $\varphi(x) = \chi_{[0,c)}(x)$ be the characteristic function of the interval $[0, c)$. Then

$$Z_\varphi(x,t) = \begin{cases} \displaystyle\sum_{n=0}^{\lfloor c\rfloor} \mathrm{e}^{-2\pi\mathrm{i}nt}, & \text{if } x \in [0, \{c\}); \\ \displaystyle\sum_{n=0}^{\lfloor c\rfloor-1} \mathrm{e}^{-2\pi\mathrm{i}nt}, & \text{if } x \in [\{c\}, 1). \end{cases} \tag{3.2}$$

From now on we write $g_1(t) = \sum\limits_{n=0}^{\lfloor c\rfloor} \mathrm{e}^{-2\pi\mathrm{i}nt}$ and $g_2(t) = \sum\limits_{n=0}^{\lfloor c\rfloor-1} \mathrm{e}^{-2\pi\mathrm{i}nt}$, and denote by $\mathcal{Z}_f$ the zero point set of a function f on $[0, 1)$, that is, $\mathcal{Z}_f = \{t \in [0, 1)\colon\ f(t) = 0\}$. Then

$\mathcal{Z}_{g_1} = \{1/(\lfloor c\rfloor+1),\dots,\lfloor c\rfloor/(\lfloor c\rfloor+1)\}$, $\mathcal{Z}_{g_2} = \{1/\lfloor c\rfloor,\dots,(\lfloor c\rfloor-1)/\lfloor c\rfloor\}$ and

$$\operatorname{dist}\{\mathcal{Z}_{g_1},\mathcal{Z}_{g_2}\} = \frac{1}{\lfloor c\rfloor(\lfloor c\rfloor+1)}.$$

Write $I_u(\delta) = (u/\lfloor c\rfloor - \delta, u/\lfloor c\rfloor + \delta)$ for $1 \leqslant u \leqslant \lfloor c\rfloor - 1$. It is easily seen that none of the intervals $I_u(\delta)$ $(1 \leqslant u \leqslant \lfloor c\rfloor - 1)$ contains any points of $\mathcal{Z}_{g_1}$ if $0 < \delta < 1/(\lfloor c\rfloor(\lfloor c\rfloor+1))$.

The remainder of this section will be devoted to the proof of Theorems 2.2 and 2.4. Before we begin the proof, we will need the following lemma.

Lemma 3.3. *Let $c > 3$. Suppose that p is an even integer or $\gcd(p,\lfloor c\rfloor) > 1$. Then, for any $\delta < 1/(\lfloor c\rfloor(\lfloor c\rfloor+1))$, there is a nonzero function h with support in $\bigcup\limits_{u=1}^{\lfloor c\rfloor-1} I_u(\delta)$ such that*

$$\int_0^1 h(t)\mathrm{e}^{-2\pi\mathrm{i}kpt}\,\mathrm{d}t = 0, \quad k \in \mathbb{Z}. \tag{3.3}$$

Proof. When p is an even integer, then any nonzero function h with $h(t) = -h(t+1/2)$ for $0 \leqslant t < 1/2$ satisfies (3.3). Note that the set $\bigcup\limits_{u=1}^{\lfloor c\rfloor-1} I_u(\delta)$ is symmetric about $1/2$, which implies the existence of h with this property. When $\gcd(p,\lfloor c\rfloor) = d > 1$, one only needs to show the result for any odd p, thus $d \geqslant 3$. We will construct a nonzero function h with support in $\bigcup\limits_{u=1}^{\lfloor c\rfloor-1} I_u(\delta)$ such that $\sum\limits_{j=0}^{p-1} h(x+j/p) \equiv 0$ for $x \in [0,1/p)$; then it is the desired function by the standard argument. Note that, in this case, $1/\lfloor c\rfloor + (p/d)/p = (1+\lfloor c\rfloor/d)/\lfloor c\rfloor$, which is equivalent to $I_1(\delta) + (p/d)/p = I_\alpha(\delta)$ where $\alpha = 1 + \lfloor c\rfloor/d \leqslant \lfloor c\rfloor - 1$. Define

$$h(x) = \begin{cases} -1, & \text{if } x \in I_1(\delta); \\ 1, & \text{if } x \in I_\alpha(\delta); \\ 0, & \text{otherwise.} \end{cases}$$

Then h satisfies the assertion by the hypothesis $\delta < 1/(\lfloor c\rfloor(\lfloor c\rfloor+1))$. Hence the proof is complete. □

Having proved Lemma 3.3, we turn to the proof of the first assertion of Theorem 2.2.

Theorem 3.4. *Let $c > 3$, and let $a = p/q$ with p, q being two co-prime positive integers. If $\{c\} = 1/q$ and p is an even integer or $\gcd(p, \lfloor c \rfloor) > 1$, then $(\chi_{[0,c)}, a, 1)$ is not a frame.*

Proof. Let $l + jp = d_{lj}q + r_{lj}$, $0 \leqslant r_{lj} < q$ for all $0 \leqslant l, j < q$. Clearly $\{r_{lj}\colon j = 0, 1, \ldots, q-1\} = \{0, 1, \ldots, q-1\}$. Then, for each l, there exists a unique j^* depending on l such that $r_{lj^*} = 0$. Let $\varphi(x) = \chi_{[0,c)}(x)$, recall that $Z_\varphi(x, t) = g_1(t)$ if $x \in [0, 1/q)$ and $g_2(t)$ if $x \in [1/q, 1)$. Then, by Lemma 3.2,

$$\begin{aligned}(3.4)\qquad &\sum_{m,n\in\mathbb{Z}} |\langle f, E_m T_{na}\varphi\rangle|^2 \\ &\quad = \sum_{l=0}^{q-1} \int_{l/q}^{(l+1)/q} \mathrm{d}x \sum_{k\in\mathbb{Z}} \sum_{j\neq j^*} \biggl| \int_0^1 Z_f(x,t)\overline{g}_2(t)\mathrm{e}^{-2\pi\mathrm{i}(kp+d_{lj})t}\,\mathrm{d}t \biggr|^2 \\ &\qquad + \sum_{l=0}^{q-1} \int_{l/q}^{(l+1)/q} \mathrm{d}x \sum_{k\in\mathbb{Z}} \biggl| \int_0^1 Z_f(x,t)\overline{g}_1(t)\mathrm{e}^{-2\pi\mathrm{i}(kp+d_{lj*})t}\,\mathrm{d}t \biggr|^2.\end{aligned}$$

According to the definition of d_{lj}, it is easy to get that $d_{lj} \leqslant p$ and the cardinality of $\mathcal{A}_r = \{j\colon d_{lj} \equiv r \pmod p,\ j = 0, 1, \ldots, q-1\}$ is less than or equal to $2(1 + (q-1)/p)$ for each $0 \leqslant r < p$. Since

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} \sum_{j\neq j^*} \biggl| \int_0^1 Z_f(x,t)\overline{g}_2(t)\mathrm{e}^{-2\pi\mathrm{i}(kp+d_{lj})t}\,\mathrm{d}t \biggr|^2 \\ &\quad \leqslant 2\Bigl(1 + \frac{q-1}{p}\Bigr) \sum_{k\in\mathbb{Z}} \sum_{0\leqslant r<p} \biggl| \int_0^1 Z_f(x,t)\overline{g}_2(t)\mathrm{e}^{-2\pi\mathrm{i}(kp+r)t}\,\mathrm{d}t \biggr|^2 \\ &\quad = 2\Bigl(1 + \frac{q-1}{p}\Bigr) \int_0^1 |Z_f(x,t)\overline{g}_2(t)|^2\,\mathrm{d}t,\end{aligned}$$

it follows that

$$\begin{aligned}(3.5)\qquad \sum_{m,n\in\mathbb{Z}} |\langle f, E_m T_{na}\varphi\rangle|^2 &\leqslant 2\Bigl(1 + \frac{q-1}{p}\Bigr) \int_Q |Z_f(x,t)\overline{g}_2(t)|^2\,\mathrm{d}t \\ &\quad + \sum_{l=0}^{q-1} \int_{l/q}^{(l+1)/q} \mathrm{d}x \sum_{k\in\mathbb{Z}} \biggl| \int_0^1 Z_f(x,t)\overline{g}_1(t)\mathrm{e}^{-2\pi\mathrm{i}(kp+d_{lj*})t}\,\mathrm{d}t \biggr|^2.\end{aligned}$$

For any $\varepsilon > 0$ there exists a δ less than $1/(\lfloor c \rfloor(\lfloor c \rfloor + 1))$ such that

$$|g_2(t)| < \varepsilon, \quad \text{for } t \in \bigcup_{u=1}^{\lfloor c \rfloor - 1} I_u(\delta).$$

Choosing h in Lemma 3.3, since $g_1(t) \neq 0$ on $\bigcup\limits_{u=1}^{\lfloor c\rfloor -1} I_u(\delta)$ by the restriction of δ, for each $0 \leqslant l < q$ we define functions on $[0,1]$ by

$$\psi_l(t) = \begin{cases} h(t)(\overline{g}_1(t)\mathrm{e}^{-2\pi\mathrm{i} d_{l_j*} t})^{-1}, & \text{if } t \in \bigcup\limits_{u=1}^{\lfloor c\rfloor -1} I_u(\delta); \\ 0, & \text{otherwise.} \end{cases}$$

The Fourier analysis yields that there exist complex numbers $\{a_{ln}\}_{n\in\mathbb{Z}}$ such that $\psi_l(t) = \sum\limits_{n\in\mathbb{Z}} a_{ln}\mathrm{e}^{-2\pi\mathrm{i}nt}$. Define $f(x+n) = a_{ln}$ if $x \in [l/q, (l+1)/q)$ for $0 \leqslant l < q$. Then

$$Z_f(x,t) = \sum_{n\in\mathbb{Z}} f(x+n)\mathrm{e}^{-2\pi\mathrm{i}nt} = \psi_l(t)$$

for $x \in [l/q, (l+1)/q)$, $l = 0, 1, \ldots, q-1$. For this function f, by Lemma 3.3 and above, (3.5) becomes

$$\begin{aligned} \sum_{m,n\in\mathbb{Z}} |\langle f, E_m T_{na}\varphi\rangle|^2 &\leqslant 2\Big(1 + \frac{q-1}{p}\Big) \int_Q |Z_f(x,t)\overline{g}_2(t)|^2 \,\mathrm{d}t \\ &= 2\Big(1 + \frac{q-1}{p}\Big) \sum_{l=0}^{q-1} \int_{l/q}^{(l+1)/q} \mathrm{d}x \int_{\bigcup_{u=1}^{\lfloor c\rfloor -1} I_u(\delta)} |Z_f(x,t)\overline{g}_2(t)|^2 \,\mathrm{d}t \\ &\leqslant \varepsilon^2 2\Big(1 + \frac{q-1}{p}\Big)\|f\|^2. \end{aligned}$$

By virtue of the arbitrariness of ε, we conclude that the upper condition in (1.1) is violated. Hence the result follows. □

According to Theorem 3.4 and its proof, one can show the other assertion of Theorem 2.2 similarly. Here we omit it. Before we give the proof of Theorem 2.4, we state and prove a lemma that is interesting on its own.

Lemma 3.5. *Let $c > 3$ and $1 < d = \gcd(p, \lfloor c\rfloor) < \lfloor c\rfloor$ with p being a positive integer. Then for any $\delta < 1/(\lfloor c\rfloor(\lfloor c\rfloor + 1))$ there is a nonzero function h with support in $\bigcup\limits_{u=1}^{\lfloor c\rfloor -1} I_u(\delta)$ such that*

$$\int_0^1 h(t)\mathrm{e}^{-2\pi\mathrm{i}(np+c_r)t}\,\mathrm{d}t = 0 \quad \text{for } r = 1, 2, \ldots, d-1 \text{ and } n \in \mathbb{Z}, \tag{3.6}$$

where $c_r \equiv r \pmod d$ for $1 \leqslant r \leqslant d-1$.

P r o o f. Recall that $I_u(\delta) = (u/\lfloor c\rfloor - \delta, u/\lfloor c\rfloor + \delta)$ for $1 \leqslant u \leqslant \lfloor c\rfloor - 1$. We define $h(t)$ on $\mathbb{R}$ by $h(t) = 1$ if $t \in \bigcup\limits_{\alpha=0}^{d-1} I_{1+\alpha\lfloor c\rfloor/d}(\delta)$, 0 if $t \in [0,1) \setminus \bigcup\limits_{\alpha=0}^{d-1} I_{1+\alpha\lfloor c\rfloor/d}(\delta)$

and extend its range of definition to $\mathbb{R}$ by periodic extension with period 1. Since $1/\lfloor c\rfloor+\alpha/d=(1+\alpha\lfloor c\rfloor/d)/\lfloor c\rfloor$, we have that $1/d$ is a period of $h(t)$. Note that

$$\int_0^1 h(t)\mathrm{e}^{-2\pi\mathrm{i}(np+c_r)t}\,\mathrm{d}t=\int_0^{1/d}\sum_{\alpha=0}^{d-1}h(t+\alpha/d)\mathrm{e}^{-2\pi\mathrm{i}c_r\alpha/d}\mathrm{e}^{-2\pi\mathrm{i}(np+c_r)t}\,\mathrm{d}t.$$

Consequently,

$$\sum_{\alpha=0}^{d-1}h(t+\alpha/d)\mathrm{e}^{-2\pi\mathrm{i}c_r\alpha/d}=h(t)\sum_{\alpha=0}^{d-1}\mathrm{e}^{-2\pi\mathrm{i}c_r\alpha/d}=0$$

for $t\in[0,1/d)$, thus the result follows. □

We remark that the formula (3.6) does not hold for any nonzero functions in $L^2(\mathbb{R})$ when $d=\lfloor c\rfloor$ and when $d=1$ for any r. Moreover, we conjecture the converse of Lemma 3.5 is true. We now proceed with the proof of the first assertion of Theorem 2.4.

Theorem 3.6. *Let $c>3$ and $a=p/q$ with p,q being two co-prime positive integers. If $\lfloor q\{c\}\rfloor<\gcd(p,\lfloor c\rfloor)<\lfloor c\rfloor$, then $(\chi_{[0,c)},a,1)$ is not a frame.*

Proof. Write $d=\gcd(p,\lfloor c\rfloor)$ and $s=\lfloor q\{c\}\rfloor$, then $s/q\leqslant\{c\}<(s+1)/q$ and $s<d$. From (3.2) and Lemma 3.2, it follows that

$$\sum_{m,n\in\mathbb{Z}}|\langle f,E_mT_{na}\varphi\rangle|^2 =\sum_{l=0}^{q-1}\int_{l/q}^{(l+1)/q}\mathrm{d}x\sum_{k\in\mathbb{Z}}\sum_{j=0}^{q-1}\biggl|\int_0^1 Z_f(x,t)\overline{Z}_\varphi\Bigl(x-\frac{l}{q}+\frac{r_{lj}}{q},t\Bigr)\mathrm{e}^{-2\pi\mathrm{i}(kp+d_{lj})t}\,\mathrm{d}t\biggr|^2,$$

where $l+jp=d_{lj}q+r_{lj}$, $0\leqslant r_{lj}<q$ for $0\leqslant l,j<q$. Again using the rearrangement, $r_{lj_v}=v$ for $0\leqslant v<q$, we have

$$\sum_{m,n\in\mathbb{Z}}|\langle f,E_mT_{na}\varphi\rangle|^2 =\sum_{l=0}^{q-1}\int_{l/q}^{(l+1)/q}\mathrm{d}x\sum_{k\in\mathbb{Z}}\sum_{v=0}^{q-1}\biggl|\int_0^1 Z_f(x,t)\overline{Z}_\varphi\Bigl(x-\frac{l}{q}+\frac{v}{q},t\Bigr)\mathrm{e}^{-2\pi\mathrm{i}(kp+d_{lj_v})t}\,\mathrm{d}t\biggr|^2 \tag{3.7}$$

$$
\begin{aligned}
&= \sum_{l=0}^{q-1} \int_{l/q}^{l/q+\{c\}-s/q} \mathrm{d}x \sum_{k\in\mathbb{Z}} \sum_{v=0}^{s} \left| \int_0^1 Z_f(x,t)\overline{g}_1(t)\mathrm{e}^{-2\pi\mathrm{i}(kp+d_{lj_v})t}\,\mathrm{d}t \right|^2 \\
&\quad + \sum_{l=0}^{q-1} \int_{l/q+\{c\}-s/q}^{(l+1)/q} \mathrm{d}x \sum_{k\in\mathbb{Z}} \sum_{v=0}^{s-1} \left| \int_0^1 Z_f(x,t)\overline{g}_1(t)\mathrm{e}^{-2\pi\mathrm{i}(kp+d_{lj_v})t}\,\mathrm{d}t \right|^2 \\
&\quad + \sum_{l=0}^{q-1} \int_{l/q}^{l/q+\{c\}-s/q} \mathrm{d}x \sum_{k\in\mathbb{Z}} \sum_{v=s+1}^{q-1} \left| \int_0^1 Z_f(x,t)\overline{g}_2(t)\mathrm{e}^{-2\pi\mathrm{i}(kp+d_{lj_v})t}\,\mathrm{d}t \right|^2 \\
&\quad + \sum_{l=0}^{q-1} \int_{l/q+\{c\}-s/q}^{(l+1)/q} \mathrm{d}x \sum_{k\in\mathbb{Z}} \sum_{v=s}^{q-1} \left| \int_0^1 Z_f(x,t)\overline{g}_2(t)\mathrm{e}^{-2\pi\mathrm{i}(kp+d_{lj_v})t}\,\mathrm{d}t \right|^2 \\
&\leqslant \sum_{l=0}^{q-1} \int_{l/q}^{l/q+\{c\}-s/q} \mathrm{d}x \sum_{k\in\mathbb{Z}} \sum_{v=0}^{s} \left| \int_0^1 Z_f(x,t)\overline{g}_1(t)\mathrm{e}^{-2\pi\mathrm{i}(kp+d_{lj_v})t}\,\mathrm{d}t \right|^2 \\
&\quad + \sum_{l=0}^{q-1} \int_{l/q+\{c\}-s/q}^{(l+1)/q} \mathrm{d}x \sum_{k\in\mathbb{Z}} \sum_{v=0}^{s-1} \left| \int_0^1 Z_f(x,t)\overline{g}_1(t)\mathrm{e}^{-2\pi\mathrm{i}(kp+d_{lj_v})t}\,\mathrm{d}t \right|^2 \qquad (3.8) \\
&\quad + \sum_{l=0}^{q-1} \int_{l/q}^{(l+1)/q} \mathrm{d}x \sum_{k\in\mathbb{Z}} \sum_{v=0}^{q-1} \left| \int_0^1 Z_f(x,t)\overline{g}_2(t)\mathrm{e}^{-2\pi\mathrm{i}(kp+d_{lj_v})t}\,\mathrm{d}t \right|^2 .
\end{aligned}
$$

Similarly to estimating the first term of (3.4) on the right, the last term of the above satisfies

$$
\begin{aligned}
\sum_{l=0}^{q-1} \int_{l/q}^{(l+1)/q} \mathrm{d}x \sum_{k\in\mathbb{Z}} \sum_{v=0}^{q-1} \left| \int_0^1 Z_f(x,t)\overline{g}_2(t)\mathrm{e}^{-2\pi\mathrm{i}(kp+d_{lj_v})t}\,\mathrm{d}t \right|^2 \\
\leqslant 2\Big(1+\frac{q-1}{p}\Big) \int_Q |Z_f(x,t)\overline{g}_2(t)|^2\,\mathrm{d}t.
\end{aligned}
$$

Now we estimate the term (3.8). Since $s<d$, for each $0\leqslant l\leqslant q-1$ we claim that there exists m_l such that

$$
\{\mathrm{e}^{-2\pi\mathrm{i}(kp+d_{lj_v}-m_l)t}\colon\ k\in\mathbb{Z},\ 0\leqslant v<s\}\subseteq\{\mathrm{e}^{-2\pi\mathrm{i}(kp+c_\alpha)t}\colon\ k\in\mathbb{Z},\ 1\leqslant\alpha\leqslant d-1\},
$$

where $c_\alpha\equiv\alpha \pmod d$ for $1\leqslant\alpha\leqslant d-1$. In fact, if $d_{lj_v}\equiv d_{lj_{v'}} \pmod d$ for some $0\leqslant v<v'<d$, then $(j_{v'}-j_v)p=(d_{lj_v}-d_{lj_{v'}})q+v'-v$, and consequently $d\mid(v'-v)$ which is impossible. Hence the claim follows.

According to the definition of $I_u(\delta)$ for any $\varepsilon>0$ there exists δ less than $1/(\lfloor c\rfloor(\lfloor c\rfloor+1))$ such that

$$
|g_2(t)|<\varepsilon,\quad \text{for } t\in\bigcup_{u=1}^{\lfloor c\rfloor-1} I_u(\delta).
$$

Choosing h in Lemma 3.5, for each $0 \leqslant l < q$ the functions on $[0,1]$ are defined by

$$\psi_l(t) = \begin{cases} h(t)(\overline{g}_1(t)\mathrm{e}^{2\pi\mathrm{i}m_l t})^{-1}, & \text{if } t \in \bigcup_{u=1}^{\lfloor c\rfloor -1} I_u(\delta); \\ 0, & \text{otherwise.} \end{cases}$$

Hence it follows from the Fourier analysis that there exist complex numbers $\{a_{ln}\}_{n\in\mathbb{Z}}$ such that $\psi_l(t) = \sum\limits_{n\in\mathbb{Z}} a_{ln}\mathrm{e}^{-2\pi\mathrm{i}nt}$. Define $f(x+n) = 0$ if $x \in [l/q, l/q + \{c\} - s/q)$ and a_{ln} if $x \in [l/q + \{c\} - s/q, (l+1)/q)$ for $0 \leqslant l < q$. Then for $0 \leqslant l < q$,

$$Z_f(x,t) = \begin{cases} 0, & \text{if } x \in [l/q, l/q + \{c\} - s/q); \\ \psi_l(t), & \text{if } x \in [l/q + \{c\} - s/q, (l+1)/q). \end{cases}$$

Let f in (3.7) be the function just defined. Then we have from (3.7) and Lemma 3.5 that

$$\sum_{m,n\in\mathbb{Z}} |\langle f, E_m T_{na}\varphi\rangle|^2 \leqslant 2\Big(1 + \frac{q-1}{p}\Big)\int_Q |Z_f(x,t)\overline{g}_2(t)|^2\,\mathrm{d}t \leqslant 2\Big(1 + \frac{q-1}{p}\Big)\varepsilon^2\|f\|^2.$$

Due to the arbitrariness of ε, we conclude that the upper condition in (1.1) is violated. Hence the result follows. $\square$

Similarly, the other part of Theorem 2.4 follows from the above theorem and its proof. Therefore we omit it here.

Acknowledgement. The authors would like to thank the referee for his/her many valuable suggestions.

References

[1] *P. G. Casazza*: Modern tools for Weyl-Heisenberg (Gabor) frame theory. Adv. Imag. Elec. Phys. *115* (2000), 1–127.

[2] *P. G. Casazza, N. J. Kalton*: Roots of complex polynomials and Weyl-Heisenberg frame sets. Proc. Am. Math. Soc. *130* (2002), 2313–2318. zbl MR

[3] *K. Gröchenig, J. Stöckler*: Gabor frames and totally positive functions. Duke Math. J. *162* (2013), 1003–1031. zbl MR

[4] *Q. Gu, D. Han*: When a characteristic function generates a Gabor frame. Appl. Comput. Harmon. Anal. *24* (2008), 290–309. zbl MR

[5] *C. Heil*: History and evolution of the density theorem for Gabor frames. J. Fourier Anal. Appl. *13* (2007), 113–166. zbl MR

[6] *A. J. E. M. Janssen*: Some Weyl-Heisenberg frame bound calculations. Indag. Math., New Ser. *7* (1996), 165–183. zbl MR

[7] *A. J. E. M. Janssen*: Zak transforms with few zeros and the tie. Advances in Gabor Analysis (H. G. Feichtinger et al., eds.). Applied and Numerical Harmonic Analysis, Birkhäuser, Basel, 2003, pp. 31–70. zbl MR

[8] *A. J. E. M. Janssen, T. Strohmer*: Hyperbolic secants yield Gabor frames. Appl. Comput. Harmon. Anal. *12* (2002), 259–267. zbl MR

[9] *Y. I. Lyubarskij*: Frames in the Bargmann space of entire functions. Entire and Subharmonic Functions. Advances in Soviet Mathematics 11, American Mathematical Society, Providence, 1992, pp. 167–180. zbl MR

[10] *K. Seip*: Density theorems for sampling and interpolation in the Bargmann-Fock space I. J. Reine Angew. Math. *429* (1992), 91–106. zbl MR

[11] *K. Seip, R. Wallstén*: Density theorems for sampling and interpolation in the Bargmann-Fock space II. J. Reine Angew. Math. *429* (1992), 107–113. zbl MR

Authors' addresses: Xinggang He, School of Mathematics and Statistics, Central China Normal University, No. 152 Luoyu Road, Wuhan Hongshan Zone, 430 079, Hubei Province, P. R. China, e-mail: `xingganghe@sina.com`; Haixiong Li (corresponding author), School of Mathematics and Statistics, HuBei University of Education, No. 129 Gaoxin 2nd Road, Wuhan Hi-Tech Zone, 430 205, Hubei Province, P. R. China, e-mail: `haixiongli@sina.com`.