CONTRACTIBLE EDGES IN SOME k-CONNECTED GRAPHS

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Abstract. An edge e of a k-connected graph G is said to be k-contractible (or simply contractible) if the graph obtained from G by contracting e (i.e., deleting e and identifying its ends, finally, replacing each of the resulting pairs of double edges by a single edge) is still k-connected. In 2002, Kawarabayashi proved that for any odd integer $k \geqslant 5$, if G is a k-connected graph and G contains no subgraph $D = K_1 + (K_2 \cup K_{1,2})$, then G has a k-contractible edge. In this paper, by generalizing this result, we prove that for any integer $t \geqslant 3$ and any odd integer $k \geqslant 2t + 1$, if a k-connected graph G contains neither $K_1 + (K_2 \cup K_{1,t})$, nor $K_1 + (2K_2 \cup K_{1,2})$, then G has a k-contractible edge.

Keywords: component, contractible edge, k-connected graph, minimally k-connected graph

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1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. For a vertex $v \in V$, we denote the neighborhood of v by $N_G(v)$ and let $N_G[v] = N_G(v) \cup \{v\}$. For a subset $X \subseteq V$, $N_G(X) = \Big(\bigcup\limits_{x \in X} N_G(x)\Big) - X$ is the neighborhood of X in G, and $G[X]$ is the subgraph of G induced by X. We write $d_G(v)$ for the degree of the vertex $v \in V(G)$ and $\delta(G)$ for the minimum degree of G. Let $E(x)$ denote the set of edges incident to the vertex x. For disjoint nonempty subsets A and B of V, the set of edges of G joining a vertex in A to a vertex in B is denoted by $E_G(A, B)$. We denote the union of two graphs G and H by $G \cup H$, and the union of m copies of G by mG. The join $G + H$ of disjoint graphs G and H is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H. We use K_n and $K_{1,n}$ to denote the complete graphs and stars, respectively.

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A maximal connected subgraph of G is a component of G. Let G be a k-connected graph. For $T \subset V(G)$, if there are at least two components in $G-T$, then T is said to be a cutset of G. A k-cutset of G is a cutset of G with k vertices.

Let $k \geqslant 2$ be an integer. An edge e of a k-connected graph G is said to be k-contractible (or simply contractible) if the graph obtained from G by contracting e (i.e., deleting e and identifying its ends, finally, replacing each of the resulting pairs of double edges by a single edge) is still k-connected. An edge that is not k-contractible is said to be a noncontractible edge. Clearly, for a noncomplete k-connected graph G, the edge xy is a noncontractible edge of G if and only if there is a k-cutset T of G such that $\{x, y\} \subseteq T$.

A k-connected graph G is said to be minimally k-connected if $G-e$ is no longer k-connected for any $e \in E(G)$. If G is not minimally k-connected, we may delete some edges of G without changing the k connectivity of G until G becomes a minimally k-connected graph. Hence, every k-connected graph has a minimally k-connected spanning subgraph.

If any subgraph of G is not isomorphic to a given graph H, then G is H-free.

The following are some results about the contractible edges in a k-connected graph.

Theorem 1.1 ([10]). *If G is a k-connected triangle-free graph, then G contains an edge e such that the contraction of e results in a k-connected graph.*

Egawa et al. [3] proved that a k-connected triangle-free graph G contains $\min\{|V(G)| + \frac{3}{2}k^2 - 3k, |E(G)|\}$ k-contractible edges. Therefore, a k-connected graph G without triangle has many contractible edges. Hence, the condition "without triangle" is too strong. Recently, some weaker conditions "without some specified subgraphs" for a k-connected graph to have a contractible edge was obtained.

Let K_4^- denote the graph obtained from K_4 by removing just one edge.

Kawarabayashi proved the following result.

Theorem 1.2 ([5]). *Let $k \geqslant 3$ be an odd integer, and G be a k-connected graph. If G does not contain K_4^-, then G has a k-contractible edge.*

Theorem 1.3 ([2]). *Let $k \geqslant 4$ be an integer. If G is a k-connected graph not containing $K_1 + 2K_2$, then G contains a k-contractible edge.*

Clearly, Theorem 1.2 and Theorem 1.3 are extensions of Theorem 1.1.

For odd k, Kawarabayashi got the following stronger result.

Theorem 1.4 ([6]). *Let G be a k-connected graph with $k \geqslant 5$. If k is odd and G does not contain $D = K_1 + (K_2 \cup K_{1,2})$, then G has a k-contractible edge.*

Clearly, D contains K_4^- and $K_1 + 2K_2$. Hence, when k is odd, Theorem 1.4 is an extension of Theorem 1.2 and Theorem 1.3. Of course, it is also an extension of Theorem 1.1.

In this paper, we are going to prove that for any integer $t \geqslant 3$ and any odd integer $k \geqslant 2t+1$, if a k-connected graph G contains neither $K_1 + (K_2 \cup K_{1,t})$, nor $K_1 + (2K_2 \cup K_{1,2})$, then G has a k-contractible edge.

It is easy to see that both $K_1 + (K_2 \cup K_{1,t})$ and $K_1 + (2K_2 \cup K_{1,2})$ contain $D = K_1 + (K_2 \cup K_{1,2})$. Hence, we generalize the result of Theorem 1.4 under the condition that $k \geqslant 2t+1$ is an odd integer at least 7.

2. Several lemmas

Let G be a noncomplete k-connected graph. Let T be a k-cutset of G, and M be a component of $G - T$. For an edge $e = xy$ of the k-connected graph G, if $\{x, y\} \subseteq N_G(M) = T$ (i.e., $N_G(M)$ is a k-cutset of G), then we say that M is a component with respect to e. For a nonempty subset F of $E(G)$, if A is a component with respect to some edge $e \in F$, then A is called a component with respect to F or simply an F-component. If A is a component with respect to e with minimum cardinality, then A is called a minimum component with respect to e. The minimum F-component is defined similarly. For an edge e of a k-connected graph G, if there is a k-cutset T of G such that T contains the end vertices of e, then we denote the cardinality of a minimum component with respect to e by $\varphi(e)$. Set $J(G) = \{e \in E(G) \colon \varphi(e) \geqslant \frac{1}{2}(k+1)\}$.

Lemma 2.1 ([1], [8], [9]). *Let G be a k-connected graph with $J(G) \neq \emptyset$. If for every minimum $J(G)$-component A, we have $(E(A) \cup E_G(V(A), N_G(A))) \cap J(G) \neq \emptyset$, then G has a k-contractible edge.*

Lemma 2.2 ([4]). *Every minimally k-connected graph has a vertex of degree k.*

Lemma 2.3 ([7]). *If T is a k-cutset in a minimally k-connected graph G, then every component of $G - T$ contains a vertex x with $d_G(x) = k$.*

Lemma 2.4 ([1]). *If W is a subset of $V(G)$, then*

$$\sum_{x \in V(G)-W} |N_G(x) \cap W| = \sum_{y \in W} d_G(y) - 2|E(W)|.$$

Lemma 2.5. *Let $t \geqslant 3$ be an integer. Let H be a graph of odd order which has no isolated vertices and $|H| \geqslant 2t+1$. If H is not a star, then H contains $K_2 \cup K_{1,t}$ or $2K_2 \cup K_{1,2}$.*

Proof. Denote the cardinality of a maximum edge independent set of H by m. Then, $m \geqslant 1$. If $m = 1$, it is easy to see that H is a star. Assume that $m = 2$, and x_1x_2, x_3x_4 are two independent edges of H. Denote $X = \{x_1, x_2, x_3, x_4\}$. Then every vertex of $V(H) - X$ is adjacent to at least one vertex in X. Thus there are at least $|H| - 4$ edges between $V(H) - X$ and X. Since $m = 2$, there is a vertex in X which is adjacent to at least $t-1$ vertices in $V(H) - X$. Therefore, H contains a subgraph $K_2 \cup K_{1,t}$. Now assume that $m \geqslant 3$ and U is a maximum edge independent set of H. Since $|H|$ is odd, every vertex in $V(H) - V(U)$ is adjacent to at least one vertex of $V(U)$. Hence, H contains a subgraph $2K_2 \cup K_{1,2}$. This completes the proof. □

3. Main result

Now we prove the main result of this paper.

Theorem 3.1. *Let $t \geqslant 3$ be an integer, and $k \geqslant 2t+1$ be an odd integer. If G is a k-connected graph which contains neither $K_1 + (K_2 \cup K_{1,t})$ nor $K_1 + (2K_2 \cup K_{1,2})$, then G has a k-contractible edge.*

Proof. The proof is by contradiction. Clearly, if the conclusion is true for minimally k-connected graphs, then it is also true for general k-connected graphs. Therefore we suppose that G is a minimally k-connected graph and G contains neither $K_1 + (K_2 \cup K_{1,t})$ nor $K_1 + (2K_2 \cup K_{1,2})$ but G contains no contractible edges. Thus for every edge $e = xy$ in G, there is a k-cutset S of G such that $\{x, y\} \subseteq S$. □

Claim 1. *Let S be a k-cutset of G. If A is a component of $G - S$, then $|A| \in \{1, 2\}$, or $|A| \geqslant \frac{1}{2}(k+1)$.*

Proof. Clearly, we have $|A| \in \{1, 2\}$ or $|A| \geqslant 3$. Now, we prove that if $|A| \geqslant 3$, then $|A| \geqslant \frac{1}{2}(k+1)$.

Set $H = G[S \cup V(A)]$. It is obvious that for every $w \in V(A)$, we have $N_G(w) = N_H(w)$, and $d_G(w) = d_H(w)$. Since A is a component of $G - S$, and $|A| \geqslant 3$, A has a P_3 (i.e., a path of order 3). Suppose that $P_3 = w_1w_2w_3$, $W = \{w_1, w_2, w_3\}$ and $Q = A \cup S - W$.

Denote

$$\begin{aligned}\theta &= |\{u\colon\ u \in N_G(w_1)\cap N_G(w_2)\cap N_G(w_3)\}|,\\ \xi_1 &= |\{u\colon\ u \in N_G(w_2)\cap N_G(w_3) - N_G[w_1]\}|,\\ \xi_2 &= |\{u\colon\ u \in N_G(w_1)\cap N_G(w_3) - N_G[w_2]\}|,\\ \xi_3 &= |\{u\colon\ u \in N_G(w_1)\cap N_G(w_2) - N_G[w_3]\}|.\end{aligned}$$

Now we discuss two cases:

Case 1. $\theta \geqslant 1$. If $\theta \geqslant t+1$, then $G[W\cup(N_G(w_1)\cap N_G(w_2)\cap N_G(w_3))]$ contains a subgraph $K_1+(K_2\cup K_{1,t})$, a contradiction. Therefore, $1\leqslant\theta\leqslant t$.

Similarly, we have that $1\leqslant \theta+\xi_i\leqslant t$, $i=1,2,3$, when $w_1w_3\in E(G)$, and that $1\leqslant\theta+\xi_i\leqslant t$, $i=1,3$, when $w_1w_3\notin E(G)$.

Case 1.1. $w_1w_3\in E(G)$. By Lemma 2.4, we have

$$\begin{aligned}(1)\qquad 3k &\leqslant \sum_{w\in W} d_G(w) = 2|E(W)|+|E(W,Q)|\\ &\leqslant 3\times 2+3\theta+2(\xi_1+\xi_2+\xi_3)+(|A|+k-3-\theta-(\xi_1+\xi_2+\xi_3))\\ &= 3+2\theta+\xi_1+\xi_2+\xi_3+|A|+k.\end{aligned}$$

Suppose that $\theta+\xi_1=t$. If $\xi_2\neq 0$, then H contains $K_1+(K_2\cup K_{1,t})$, a contradiction. Thus $\xi_2=\xi_3=0$. By (1), we have

$$3k\leqslant 3+2\theta+\xi_1+|A|+k = 3+\theta+t+|A|+k.$$

This implies $|A|\geqslant 2k-t-\theta-3\geqslant 2k-2t-3\geqslant\frac{1}{2}(k+1)$.

We may suppose $1\leqslant\theta+\xi_i\leqslant t-1$ for $i=1,2,3$. By (1), we have

$$3k+\theta\leqslant 3+(\theta+\xi_1)+(\theta+\xi_2)+(\theta+\xi_3)+|A|+k\leqslant 3+3(t-1)+|A|+k.$$

Hence $|A|\geqslant 2k-3t+\theta\geqslant 2k-3t+1\geqslant\frac{1}{2}(k+1)$.

Case 1.2. $w_1w_3\notin E(G)$. First suppose that $\theta+\xi_1=t$. Then $\xi_3=0$. Consequently,

$$\begin{aligned}k+|A| &\geqslant |N_G(w_1)|+|N_G(w_2)|-|N_G(w_1)\cap N_G(w_2)|\\ &\geqslant 2k-\theta\\ &\geqslant 2k-t.\end{aligned}$$

Hence, $|A|\geqslant k-t\geqslant\frac{1}{2}(k+1)$.

Next suppose that $1\leqslant\theta+\xi_1\leqslant t-1$. Then we have

$$\begin{aligned}k+|A| &\geqslant |N_G(w_2)|+|N_G(w_3)|-|N_G(w_2)\cap N_G(w_3)|\\ &\geqslant 2k-\theta-\xi_1\\ &\geqslant 2k-t+1.\end{aligned}$$

Therefore, $|A|\geqslant k-t+1\geqslant\frac{1}{2}(k+1)$.

Case 2. $\theta = 0$. First suppose that $\xi_1 \geqslant t$. Since G contains neither $K_1 + (K_2 \cup K_{1,t})$ nor $K_1 + (2K_2 \cup K_{1,2})$, it is easy to get that $\xi_3 = 0$. Thus we have

$$|A| \geqslant |N_G(w_1)| + |N_G(w_2)| - |N_G(w_1) \cap N_G(w_2)| - k \geqslant k \geqslant \frac{k+1}{2}.$$

Next suppose that $\xi_1 \leqslant t - 1$. Then we have
$|A| \geqslant |N_G(w_2)| + |N_G(w_3)| - |N_G(w_2) \cap N_G(w_3)| - k \geqslant k - \xi_1 \geqslant k - t + 1 \geqslant \frac{1}{2}(k+1)$. □

Claim 2. *Let S be a k-cutset of G. If ab is a component of $G - S$, then $N_G(a) \cap S$ is independent.*

Proof. We suppose that, on the contrary, there are $x, y \in N_G(a) \cap S$ such that $xy \in E(G)$. Since $\delta(G) \geqslant k$, we have

$$|N_G(a) \cap N_G(b) \cap (S - \{x, y\})| \geqslant k - 4 \geqslant 2t + 1 - 4 \geqslant t\,.$$

Thus G contains $K_1 + (K_2 \cup K_{1,t})$, a contradiction. □

Claim 3. *If $e = xy$ is not contained in any triangle, then $E(x) \cap J(G) \neq \emptyset$ and $E(y) \cap J(G) \neq \emptyset$.*

Proof. Assume that xy is not contained in any triangle. Let S be a k-cutset of G such that $\{x, y\} \subseteq S$, and A be a minimum component of $G - S$. Since xy is not contained in any triangle, we have $|A| \geqslant 2$. If $|A| \geqslant 3$, then the conclusion holds by Claim 1. Now assume that $|A| = 2$ and $A = \{a, b\}$. Further, we may assume that $\{xa, yb\} \subseteq E(G)$, but $\{ay, bx\} \cap E(G) = \emptyset$. By Claim 2, $N_G(a) \cap S$ is independent. Thus xa is not contained in any triangle. It is obvious that any edge in $E(a) - \{xa\}$ is contained in a triangle. Let S_1 be a k-cutset such that $\{x, a\} \subseteq S_1$ and A_1 be a minimum component of $G - S_1$. Since xa is not contained in any triangle, we have $|A_1| \geqslant 2$. Now we assume that $|A_1| = 2$. Denote $N_G(a) \cap A_1 = \{p\}$. By Claim 2, $N_G(p) \cap S_1$ is independent. But this contradicts the fact that the edge ap is contained in a triangle. Therefore $|A_1| \geqslant 3$. Thus $xa \in J(G)$ by Claim 1. By the same argument, $yb \in J(G)$. This completes the proof. □

Claim 4. *If $d_G(x) = k$, then $E(x) \cap J(G) \neq \emptyset$.*

Proof. Suppose that $x \in V(G)$ with $d_G(x) = k$. Denote $H = G[N_G(x)]$.
If H contains an isolated vertex y, then xy is not contained in any triangle. By Claim 3, $E(x) \cap J(G) \neq \emptyset$. In the following, we assume that H contains no isolated vertices.

By Lemma 2.5, H is a star or H contains either $K_2 \cup K_{1,t}$ or $2K_2 \cup K_{1,2}$. Thus, if H is not a star, then $G[N_G[x]]$ contains $K_1 + (K_2 \cup K_{1,t})$ or $K_1 + (2K_2 \cup K_{1,2})$, a contradiction. Thus H is a star. So every edge in $E(x)$ is contained in a triangle.

Let v be the vertex of degree $k-1$ in H. Then $d_G(v) \geqslant k+1$, otherwise, $N_G(x) \cap N_G(v)$ is a $k-1$ cutset of G which contradicts that G is k-connected. For any vertex u of degree one in H, $xuvx$ is the only triangle containing both x and u. Let y be a vertex of degree one in H and S be a k-cutset of G such that $\{x, y\} \subseteq S$. Assume that A is a minimum component of $G - S$. If $|A| = 1$, then $A = \{v\}$ since $xyvx$ is the only triangle containing both x and y. This contradicts that $d_G(v) \geqslant k+1$. So we have that $|A| \geqslant 2$. Assume that $|A| = 2$ and $A = \{a, b\}$. Without loss of generality, we assume that $ax \in E(G)$. By Claim 2, we have $v \notin \{a, b\}$. Since $a \in N_G(x)$ and $N_G(x) \subseteq N_G[v]$, we have $v \in S$. This means that $N_G(a) \cap S$ is not independent, a contradiction by Claim 2. Thus $|A| \geqslant 3$. So $xy \in J(G)$ by Claim 1. □

Since G is a minimally k-connected graph, by Lemma 2.2, we have $\delta(G) = k$. Suppose that x is a vertex of degree k in G. Then we have $E(x) \cap J(G) \neq \emptyset$ by Claim 4. Therefore, $J(G) \neq \emptyset$.

Notice that G has no contractible edges. So by Lemma 2.1, there is a minimum $J(G)$-component A such that

$$(E(A) \cup E_G(V(A), N_G(A))) \cap J(G) = \emptyset.$$

By the definition of $J(G)$-component, we have that $N_G(A)$ is a k-cutset of G such that $E(G[N_G(A)])$ contains some edge $e \in J(G)$. By Lemma 2.3, we have that A contains a vertex s with $d_G(s) = k$. Clearly, $E(s) \subseteq (E(A) \cup E_G(V(A), N_G(A)))$. But by Claim 4, we have $E(s) \cap J(G) \neq \emptyset$, a contradiction. This completes the proof. □

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