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A SIMPLIFIED MODEL FOR AN *N*-LAYER ELASTIC FOUNDATION

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The paper presents a model for an elastic foundation consisting of a finite number of homogeneous and isotropic layers. The simplified continuous elastic media with internal constraints introduced by R Świtka [1] has been adopted for discussion, in which it is assumed that the horizontal components of a displacement vector equal zero.
The purpose of the paper is a solution determining the displacement of the foundation surface, caused by a uniform load within a circular area. By means of the *Hankel* integral transformation and the notion of the layer transfer matrix in the transform domain, a solution to vertical displacements of the foundation surface has been obtained.
The method for computing the infinite integral specifying numerical values of displacements of the foundation surface has been presented. The solution concerning the interaction of a uniform load within a circular area presented in the paper can be used for an approximate solution (by computer methods) to the task of the contact between a beam or a raft and the surface of an elastic layer foundation.

Keywords: elastic layered half space, internal constraints, boundary value problem, computer methods, soil-structure interaction.

1. INTRODUCTION

The paper presents a model for an elastic foundation consisting of a finite number of homogeneous and isotropic layers. The continuous elastic media introduced by *Świtka* ([1] his associate professor thesis) in 1968 has been adopted for discussion. In this model it is assumed that the horizontal displacement components of the vector equal zero. The paper [1] presents complete equations for a simplified media and solves the basic problems concerning elastic half space

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with a static and a dynamic load. The basic equations for a simplified media have been presented in paragraph 2.

The purpose of the paper is to generalise the simplified foundation (with internal constraints) introduced in the paper [1], through taking into consideration the occurrence of horizontal layers with different material parameters in the foundation (fig. 1). By means of the *Hankel* integral transformation and the notion of the layer transfer matrix in the transform domain corresponding to a specified layer of the foundation, a solution for vertical displacements of the foundation surface caused by a uniform load within a circular area has been obtained. The method for computing the infinite integral specifying numerical values of displacements of the foundation surface has been presented.

The solution concerning the impact of a uniform load within a circular area presented in the paper can be used for an approximate solution (by computer methods) of the contact between a beam or a raft and the surface of an elastic layered foundation. A method for solving the problem of the contact between a beam and an elastic layered foundation by means of computers has been indicated as an example of the possibilities of using the results obtained in the paper.

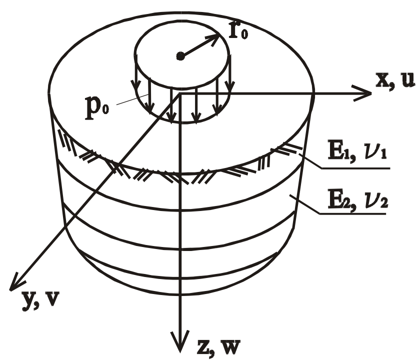

Fig. 1. The boundary value problem analysed in the paper

2. SIMPLIFIED ELASTIC FOUNDATION [1]

The new model for an elastic foundation introduced by *Świtka* in his associate professor thesis [1] was obtained on the basis of analysing the mechanical system. The mechanical system analysed was designed in such a way that it did not make horizontal displacement elements of the constraints possible. In mathematical notation it leads to kinematic assumptions:

$$u(x, y, z) = 0; \quad v(x, y, z) = 0 . \tag{2.1}$$

Considering the above assumptions, geometrical equations are:

$$\varepsilon_z = w_{,z}, \ \gamma_{xz} = w_{,x}, \ \gamma_{yz} = w_{,y}, \tag{2.2}$$

and physical equations in the form of the function of the displacements *u, v, w* are:

$$\begin{aligned} &\sigma_x = 2\mu u,_z + \lambda\theta = \lambda w,_z && \tau_{xy} = 2\mu\left(u,_y + v,_x\right) = 0 \\ &\sigma_y = 2\mu v,_z + \lambda\theta = \lambda w,_z && \tau_{zx} = 2\mu\left(u,_z + w,_x\right) = \mu w,_x \\ &\sigma_z = 2\mu w,_z + \lambda\theta = \left(\lambda + 2\mu\right) w,_z && \tau_{zy} = 2\mu\left(v,_z + w,_y\right) = \mu w,_y . \end{aligned} \tag{2.3}$$

Next, the equations of the internal equilibrium equations of the simplified elastic media are determined by the equations

$$\begin{aligned} &\mu w,_{xz} + X = 0, \\ &\mu w,_{yz} + Y = 0, \\ &\mu\left(w,_{xx} + w,_{yy} + w,_{zz}\right) + \left(\lambda + \mu\right) w,_{zz} + Z = 0. \end{aligned} \tag{2.4}$$

By putting the third of the equations (2.4) in order an differential equation is obtained

$$\mu\left(w,_{xx} + w,_{yy}\right) + \left(\lambda + 2\mu\right) w,_{zz} + Z = 0, \tag{2.5}$$

specifying the vertical displacements of a simplified foundation. By introducing the notation:

$$\beta^2 = \frac{\mu}{\lambda + 2\mu} = \frac{G}{E^*} = \frac{1-2\nu}{2-2\nu}, \quad E^* = \lambda + 2\mu = E\frac{1-\nu}{(1+\nu)(1-2\nu)}, \tag{2.6}$$

where: E* - the module of the deformation of a simplified foundation; E,ν – the deformation module and the *Poisson* coefficient of the classic foundation, the equation (2.5) can be written as:

$$\beta^2\left(w,_{xx} + w,_{yy}\right) + w,_{zz} + Z = 0. \tag{2.7}$$

The boundary condition on the foundation surface (z=0) has the form

$$\sigma_z = E^* w,_z = -q_0, \tag{2.8}$$

q_0 – the normal load on the foundation surface.

The differential equation (2.7) and the boundary condition on the foundation surface (2.8), specify the boundary value problem of simplified uniform elastic half space (with internal constraints).

In order to compare the material permanent values of a simplified foundation and the classic foundation two basic solutions have been analysed. The solution of the *Boussinesq* problem (the vertical force acting upon the foundation surface) has the form:

$$w(x,y,0)=\frac{1-\nu}{2\pi G}\frac{1}{r}, \tag{2.9}$$

and the solution obtained in the paper [1] or from the equations (2.7) and (2.8):

$$w(x,y,0)=\frac{\beta}{2\pi G}\frac{1}{r}. \tag{2.10}$$

It results from the comparison between the two formulas that after it has been assumed that

$$\beta=1-\nu, \tag{2.11}$$

identical results are obtained for the top surface of the foundation. The constants E^* and β, in this case, take the form of the formulas (the bottom index has been introduced for contrast "0"):

$$E_0^*=\lambda+2\mu=E\frac{1-\nu}{2(1+\nu)(1-\nu)^2}, \quad \beta_0=1-\nu. \tag{2.12}$$

The equality of the displacements obtained exists only on the surface of the foundation. With $z>0$ the results concerning displacements stresses are slightly different.

3. EQUATION OF THE LAYER OF A SIMPLIFIED FOUNDATION

The boundary value problem of a specified layer of a simplified foundation in cylindrical coordinates is determined by the differential equation of the simplified continuous elastic media (with internal constraints)

$$\beta^2\left(w,_{rr}+\frac{1}{r}w,_r\right)+w,_{zz}=0 \tag{3.1}$$

and boundary conditions (fig.2):

$$\begin{aligned} w(r,z_{k-1}) &= w_{k-1}(r) \\ w(r,z_k) &= w_k(r) \\ \sigma(r,z_{k-1}) &= \sigma_{k-1}(r) \\ \sigma(r,z_k) &= \sigma_k(r) \end{aligned} \qquad (3.2)$$

The values $w_{k-1}, w_k, \sigma_{k-1}$ and σ_k are considered to be known.

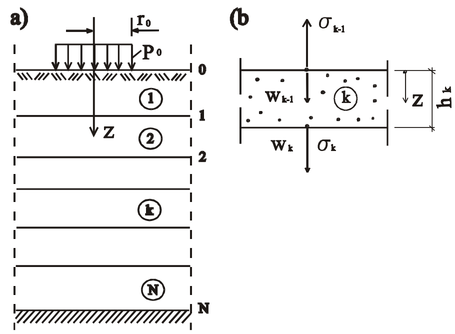

Fig.2. The layered foundation: a) the division into layers, b) the boundary conditions of a specified layer.

The *Hankel* integral transformation [2] has been used in order to solve the boundary value problem described with the equations (3.1) and (3.2). The transformed equations are specified by the formulas:

$$\tilde{w},_{zz} - \eta\,\tilde{w} = 0 \qquad (3.3)$$

and

$$\begin{aligned} \tilde{w}(r,z_k) &= \tilde{w}_k(\alpha), \\ \tilde{w}(r,z_{k+1}) &= \tilde{w}_{k+1}(\alpha), \\ \tilde{\sigma}(\alpha,z_k) &= \tilde{\sigma}_k(\alpha), \\ \tilde{\sigma}(\alpha,z_{k+1}) &= \tilde{\sigma}_{k+1}(\alpha), \end{aligned} \qquad (3.4)$$

Where: $\tilde{w}_k(\alpha,z_k) = \int_0^\infty w(r,z_k) J_0(\alpha r)\alpha d\alpha$; $\tilde{\sigma}_k(\alpha,z_k) = \int_0^\infty \sigma_z(r,z_k) J_0(\alpha r)\alpha d\alpha$; $\eta = \alpha\beta$.

The general solution of this equation (3.3) is the function

$$\tilde{w}(\alpha,z) = A_k\, exp(\eta z) + B_k\, exp(-\eta z). \qquad (3.5)$$

Constants A_k and B_k in the equation (3.5), corresponding to the layer *"k"* are calculated from the boundary conditions (3.4) with $z = 0$ and $z = h_k$

$$\begin{Bmatrix} \tilde{w}_{k-1} \\ \tilde{w}_k \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ exp(\eta h_k) & exp(\eta h_k) \end{bmatrix} \begin{Bmatrix} A_k \\ B_k \end{Bmatrix}. \tag{3.6}$$

The relation inverse to (3.6) has the form

$$\begin{Bmatrix} A_k \\ B_k \end{Bmatrix} = \frac{1}{-exp(-\eta h_k) + exp(\eta h_k)} \begin{bmatrix} -exp(-\eta h_k) & 1 \\ exp(\eta h_k) & -1 \end{bmatrix} \begin{Bmatrix} \tilde{w}_{k-1} \\ \tilde{w}_k \end{Bmatrix}. \tag{3.7}$$

By means of the transformed physical equation $\tilde{\sigma}_z = (\lambda + 2\mu)\tilde{w},_z$, the formula (3.5) and the formulas (3.7) specifying the integration constants, the relation is obtained

$$\begin{Bmatrix} \tilde{\sigma}_{k-1} \\ \tilde{\sigma}_k \end{Bmatrix} = \frac{E_k^{\bullet}\eta}{sinh(\eta h_k)} \begin{bmatrix} -cosh(\eta h_k) & 1 \\ 1 & cosh(\eta h_k) \end{bmatrix} \begin{Bmatrix} \tilde{w}_{k-1} \\ \tilde{w}_k \end{Bmatrix}. \tag{3.8}$$

The relation (3.8) can be written in the form:

$$\begin{Bmatrix} \tilde{w}_{k-1} \\ \tilde{\sigma}_{k-1} \end{Bmatrix} = \begin{bmatrix} cosh(\eta h_k) & -\dfrac{sinh(\eta h_k)}{E_k^{\bullet}\eta} \\ -E_k^{\bullet}\eta \, sinh(\eta h_k) & cosh(h_k \eta) \end{bmatrix} \begin{Bmatrix} \tilde{w}_k \\ \tilde{\sigma}_k \end{Bmatrix}, \tag{3.9}$$

or in short

$$\mathbf{S}_{k-1} = \mathbf{P}_k \, \mathbf{S}_k , \tag{3.10}$$

where

$$\mathbf{S}_k = \begin{Bmatrix} \tilde{w}_k \\ \tilde{\sigma}_k \end{Bmatrix}; \quad \mathbf{S}_{k-1} = \begin{Bmatrix} \tilde{w}_{k-1} \\ \tilde{\sigma}_{k-1} \end{Bmatrix}; \quad \mathbf{P}_{k-1} = \begin{bmatrix} cosh(\eta h_k) & -\dfrac{sinh(\eta h_k)}{E_k^{\bullet}\eta} \\ -E_k^{\bullet}\eta \, sinh(\eta h_k) & cosh(h_k \eta) \end{bmatrix}. \tag{3.11}$$

The matrix $\mathbf{P_k}$ will be called the transfer matrix of the layer "*k*"in the transform domain, and the vectors $\mathbf{S}_k$ and $\mathbf{S}_{k-1}$ will be called the state vectors on the layer boundaries.

4. SOLUTION OF THE BOUNDARY VALUE PROBLEM FOR A LAYERED FOUNDATION

A foundation consisting of *N* layers is being considered. For each of the *N* layers of an elastic foundation it is possible to write an equation of the (3.10) type. Taking into account that

$$\mathbf{S}_0 = \mathbf{P}_1 \mathbf{S}_1, \quad \mathbf{S}_1 = \mathbf{P}_2 \mathbf{S}_2, \quad \mathbf{S}_2 = \mathbf{P}_3 \mathbf{S}_3, \quad \ldots \quad \mathbf{S}_{N-1} = \mathbf{P}_N \mathbf{S}_N, \tag{4.1}$$

an equation binding the vectors $\mathbf{S}_0$ and $\mathbf{S}_N$ of the state on the surface of the top layer of the foundation and on on the surface of the bottom layer is obtained.

$$\mathbf{S}_0 = \left(\mathbf{P}_1 \cdot \mathbf{P}_2 \cdot \mathbf{P}_3 \cdot \ldots \cdot \mathbf{P}_{N-1} \cdot \mathbf{P}_N\right) \mathbf{S}_N = \mathbf{\Phi}_{0N} \mathbf{S}_N . \tag{4.2}$$

The matrix $\mathbf{\Phi}_{0N}$ is the operator matrix of the transfer of the system of *N* foundation layers. The matrix components $\mathbf{\Phi}_{0N} = \mathbf{P}_1 \cdot \mathbf{P}_2 \cdot \mathbf{P}_3 \cdot \ldots \cdot \mathbf{P}_{N-1} \cdot \mathbf{P}_N$ with the dimensions 2x2 have been designated with the symbols ϕ_{ij}:

$$\mathbf{\Phi}_{0N} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}. \tag{4.3}$$

On the basis of (4.2) it is possible to write the dependence

$$\begin{Bmatrix} \tilde{w}_0(\alpha) \\ \tilde{\sigma}_0(\alpha) \end{Bmatrix} = \begin{bmatrix} \phi_{11}(\alpha) & \phi_{12}(\alpha) \\ \phi_{21}(\alpha) & \phi_{22}(\alpha) \end{bmatrix} \begin{Bmatrix} \tilde{w}_N(\alpha) \\ \tilde{\sigma}_N(\alpha) \end{Bmatrix}, \tag{4.4}$$

Taking into account the boundary conditions

$$\tilde{w}_N = 0, \qquad \tilde{\sigma}_0 = -\tilde{q}_0(\alpha) \tag{4.5}$$

the equation is obtained

$$\begin{Bmatrix} \tilde{w}_0(\alpha) \\ -\tilde{q}_0(\alpha) \end{Bmatrix} = \begin{bmatrix} \phi_{11}(\alpha) & \phi_{12}(\alpha) \\ \phi_{21}(\alpha) & \phi_{22}(\alpha) \end{bmatrix} \begin{Bmatrix} 0 \\ \tilde{\sigma}_N(\alpha) \end{Bmatrix}, \tag{4.6}$$

whose solution is the transforms of the displacements on the top surface and of the stresses on the bottom surface:

$$\tilde{w}_0(\alpha) = -\frac{\phi_{12}(\alpha)}{\phi_{22}(\alpha)} q_0(\alpha), \qquad \tilde{\sigma}_N(\alpha) = -\frac{\tilde{q}_0(\alpha)}{\phi_{22}(\alpha)}. \tag{4.7}$$

The displacements of the foundation surface are calculated from the formula (the *Hankel* inverse transformation):

$$w_0(r) = \int_0^\infty \tilde{w}_0(\alpha) J_0(\alpha r)\alpha d\alpha = -\int_0^\infty \frac{\phi_{12}(\alpha)}{\phi_{22}(\alpha)} \tilde{q}_0(\alpha) J_0(\alpha r)\alpha d\alpha \,. \tag{4.8}$$

In the case of a uniform load within a circular area with the radius r_0 (fig. 1)

$$q(r) = \begin{cases} q_0 & r \le r_0, \\ 0 & r > r_0, \end{cases} \tag{4.9}$$

and

$$\tilde{q}_0(\alpha) = \frac{q_0 r_0}{\alpha} J_1(\alpha r_0). \tag{4.10}$$

Vertical displacements in the case of the load specified by the formula (4.9) are described by the formula:

$$w_0(r) = -q_0 r_0 \int_0^\infty \frac{\phi_{12}(\alpha)}{\phi_{22}(\alpha)} J_0(\alpha r) J_1(\alpha r_0)\alpha d\alpha \,. \tag{4.11}$$

It is difficult to calculate the value $w_0(r)$ according to the formula (4.11) because of the infinite value of the top integration boundary and oscillation of the integrand diagram.

5. CALCULATING THE VALUES OF THE DISPLACEMENTS OF THE FOUNDATION SURFACE

The values of the displacements described by the formula (4.11) are calculated in the paper by means of the analytical-numerical method. By means of substitution

$$\lambda = \alpha r_0, \quad \rho = r/r_0 \tag{5.1}$$

the formula (4.11) is has the form

$$w_0(\rho) = -q_0 \int_0^\infty -\frac{\phi_{12}(\lambda/r_0)}{\phi_{22}(\lambda/r_0)} J_0(\rho\lambda) J_1(\lambda) d\lambda \,. \tag{5.2}$$

The function

$$f(\alpha) = \frac{\phi_{12}\left(r_0^{-1}\lambda\right)}{\phi_{22}\left(r_0^{-1}\lambda\right)} \tag{5.3}$$

in the integral in the formula (5.2) can be, with large values of the argument λ / r_0, approximated by the function

$$f(\alpha) \approx \frac{1}{a\lambda}, \quad r_0^{-1}\lambda > 10\,. \tag{5.4}$$

The asymptotic representation of the function (5.3) was achieved by means of numerical experiments. This approximation is very good for large values of the argument (the numerical results obtained in example No1 confirm this). As a result the expression on the right-hand side of the equation (5.2) can be calculated as the sum

$$S = S_1 + S_2 \tag{5.5}$$

where

$$S_1(\rho) = -q_0 \int_0^x \frac{\phi_{12}\left(r_0^{-1}\lambda\right)}{\phi_{22}\left(r_0^{-1}\lambda\right)} J_0(\rho\lambda) J_1(\lambda) d\lambda\,, \tag{5.6}$$

and

$$S_2(\rho) = -\frac{r_0 q_0}{a} \int_x^{\infty} \frac{1}{\lambda} J_0(\rho\lambda) J_1(\lambda) d\lambda = -\frac{r_0 q_0}{a} I(\rho)\,. \tag{5.7}$$

The value x within the integration boundaries is a "large number" ($x > 10r_0$, $r_0 > 0$). The integral S_1 is calculated by means of standard numerical integration methods (the integrand is limited within the integration limits).

The function $I(\rho)$ is specified in the following way

$$I(\rho) = \int_x^{\infty} \lambda^{-1} J_0(\rho\lambda) J_1(\lambda) d\lambda = I_1(\rho) - I_2(\rho)\,, \tag{5.8}$$

where:

$$\begin{aligned} I_1(\rho) &= \int_0^{\infty} \lambda^{-1} J_0(\rho\lambda) J_1(\lambda) d\lambda\,, \\ I_2(\rho) &= \int_0^{x} \lambda^{-1} J_0(\rho\lambda) J_1(\lambda) d\lambda = \tfrac{1}{2} \int_0^{x} \lambda^{-1} J_0(\rho\lambda) \left[J_0(\lambda) + J_2(\lambda)\right] d\lambda\,. \end{aligned} \tag{5.9}$$

The integral can be expressed as:

$$I_1(\rho)=\begin{cases} 1 & \rho=0, \\ \mathbf{E}\left(\rho^2\right) & 0<\rho<1, \\ \dfrac{2}{\pi} & \rho=1, \\ \dfrac{2\rho}{\pi}\displaystyle\int_0^1 \dfrac{\sqrt{1-t}}{\sqrt{t\left(\rho^2-t\right)}}dt & \rho>1, \end{cases} \tag{5.10}$$

and the integral specifying the function $I_1(\rho)$ where $\rho>1$ has to be calculated with special numerical methods (the method for isolating the singularities of the integrand where $t\to 0_+$ and $t\to 1_-$, described in the paper [5]).
The integral $I_2(\rho)$ is calculated by standard numerical integration methods (the integrand is limited within the integration boundaries).

6. NUMERICAL EXAMPLES

Example No 1. The uniform load p_0 acts upon the foundation surface within a circular area with the radius r_0. The results obtained for a uniform simplified foundation (elastic half space) in the paper [1] have been compared with the results obtained for a layered surface with identical physical parameters of each layer. Moreover, the result obtained for classic elastic half space has been given.

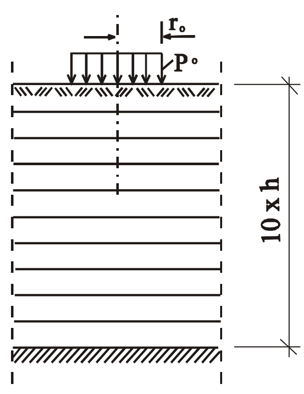

Fig. 3. The division of the foundation into layers and the designations adopted

The layered foundation adopted for the calculations consists of N=10 layers with the thickness h (fig. 3). Each layer has identical substitutive parameters E_0^* and β_0 expressed by the parameters of the classic material according to the formulas (2.12). The data has been adopted: h = 10,0m; r_0 = 1,0 m; p_0 = 40,0 kN/m^2; E = 40000 kN/m^2; ν = 0,4. The substitutive parameters of the uniform simplified foundation and layered foundation have the values $\beta_0 = 1-\nu = 0{,}6$; $E_0^* = 40000/\left[2(1+0{,}4)(1-0{,}4)^2\right] \cong 39700$ kN/m^2. The results have been presented in table 1.

Table 1. The vertical displacements in the centre of the loaded area

Foundation model	$r = 0$	$r = r_0$
Elastic half space $w(r,0) = 4(1-\nu^2)p_0 r_0(\pi E)^{-1}\boldsymbol{E}(r/r_0)$	1,6800	1,0695
Simplified elastic half space [1] $w(r,0) = 2p_0 r_0\left(\pi\beta_0 E^*\right)^{-1}\boldsymbol{E}(r/r_0)$	1,6800	1,0695
Simplified *N*-layer elastic foundation	1,6799	1, 0695

Example No 2. The uniform load p_0 exists on the surface of a two layer foundation within a circular area with the radius r_0 – as in fig. 4. The data has been adopted: r_0 = 1,0 m; p_0 = 40,0 kN/m^2; E_1 / E_2 = 2; E_2 = 20000 kN/m^2; $\nu_1 = \nu_1$ = 0,35. The results presented in the monograph by Poulos and Davis ([4], p.161) have been compared with the results obtained in this paper. The division of a simplified foundation into layers: one layer of changeable thickness h with the parameters E_1, ν_1; the area with the parameters E_2, ν_2 has been replaced with nine layers with the thickness 10 m.

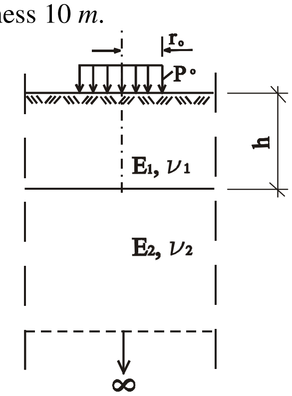

Fig. 4. The division of the foundation into layers and the designations adopted

The substitutive parameters of the simplified foundation have the value:

Table 2. The displacement in the central of the loaded area

h/r_0	Simplified foundation [mm]	Classic foundation [4] [mm]
2	2,02	2,22
4	1,88	1,99
6	1,83	1,88
8	1,80	1,86

Example No 3. The uniform load p_0 exists within a circular area with the radius r_0 on the surface of an elastic layer with the thickness h (fig.5).

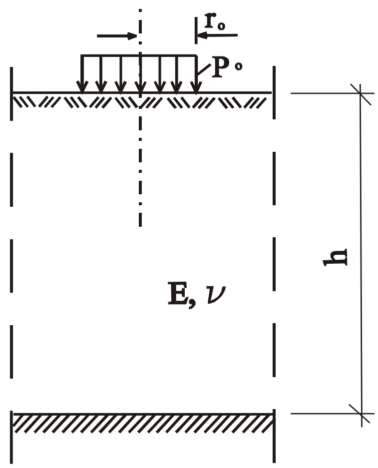

Fig. 5. An elastic layer with the thickness *h* loaded uniformly within a circular area

The results presented in the monograph by Poulos and Davis ([4], p.112) have been compared with the results obtained in this paper. The calculations have been done adopting the division of the foundation into ten layers with an identical thickness and identical substitutive parameters β_0 i $E_0^\bullet$.
The data adopted for the calculations: r_0 = 1,0m; p_0 = 40,0kN/m^2; E = 40000N/m^2; ν = 0,3. The substitutive parameters of a simplified foundation have the value: $\beta_0 = 1 - \nu = 0{,}7$; $E_0^\bullet \cong 31397\,\text{kN/m}^2$. The results obtained have been presented in table 3.

Table 3. The displacement on the edge of the loaded area

h/r_0	Simplified foundation [mm]	Classic foundation [4] [mm]
1	0,46	0,42
2	0,70	0,69
5	0,93	0,95
10	1,07	1,04

7. CONCLUSIONS

The numerical examples presented show that a simplified foundation with the use of the substitutive material constants $E_0^{\bullet} = E(1-\nu)/2(1+\nu)(1-\nu)^2$ and $\beta_0 = 1-\nu$ for each layer corresponds to a satisfactory extent to the results obtained for a classic foundation. The main purpose of the solutions presented is their use for solving problems concerning the contact between the soil and the structure where the most important problem are the differences between the displacements in adjacent points on the surface of the foundation. The solutions concerning a uniform load within a circular area presented in the paper can be used for the approximate solution (by computer methods) to the problem of the contact between a beam or a raft with the surface of an elastic foundation. An example of using the results of the paper can be the problem of the contact between a beam and an elastic foundation. After division of the beam by means of finite elements and approximation of the contact pressure with a step function (uniform interaction under a finite element) – according to fig. 6 – it is possible to formulate the dependence between the displacement of the beam base and the values of the contact pressure of the elastic foundation. In order to use the results obtained in the paper a uniform load within a rectangular area is replaced with a statically equivalent load within a circular area (fig. 6).This dependence is represented by the formula

$$\begin{bmatrix} w_{01} \\ w_{02} \\ . \\ . \\ w_{0N} \end{bmatrix} = \begin{bmatrix} d_{1N} & d_{1N} & . & . & d_{1N} \\ d_{21} & d_{22} & . & . & d_{2N} \\ . & . & . & . & . \\ . & . & . & . & . \\ d_{N1} & d_{N2} & . & . & d_{NN} \end{bmatrix} \begin{bmatrix} p_{01} \\ p_{02} \\ . \\ . \\ p_{0N} \end{bmatrix}, \tag{7.1}$$

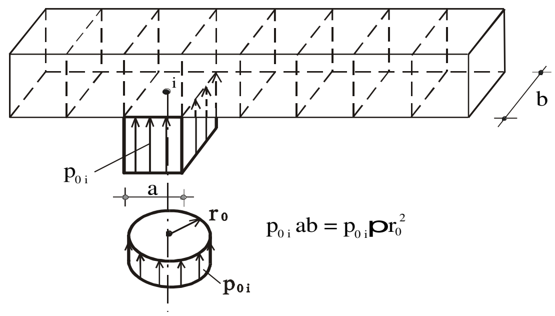

Fig. 6. The approximation of the interaction of the forces at the contact area between the beam and the foundation

where: w_{0i} - the displacements of the foundation surface at the centre of the element "i", p_{0i} - the value of the contact pressure at the centre of the element "i". The elements d_{ij} of the matrix of the "flexibility of the foundation" are defined by fig. 7.

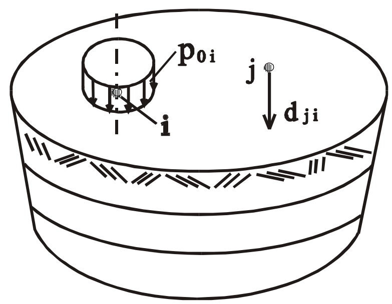

Fig. 7. The approximation of the interaction at the contact area between the beam and the foundation

The dependence (7.1) makes it possible to take into account the influence of the foundation in the equation of the beam (obtained by means of FEM – Finite Element Method) on the basis of the condition of displacement compliance.

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UPROSZCZONY MODEL *N*-WARSTWOWEGO PODŁOŻA SPRĘŻYSTEGO

S t r e s z c z e n i e

W pracy przedstawiono model podłoża sprężystego składającego się ze skończonej liczby warstw jednorodnych i izotropowych. Przyjęto do rozważań uproszczony ośrodek sprężysty z więzami wewnętrznymi wprowadzony przez R. Świtkę [1], w którym zakłada się, że składowe poziome przemieszczeń są równe zero.

Celem pracy jest podanie rozwiązania określającego przemieszczenia powierzchni podłoża, spowodowane obciążeniem równomiernie rozłożonym działającym w obszarze koła. Wykorzystując transformację całkową *Hankela* oraz pojęcie macierzy przeniesienia warstwy w dziedzinie transformat, otrzymano rozwiązanie dotyczące przemieszczeń pionowych powierzchni podłoża. Podano sposób obliczania całki niewłaściwej określającej wartości liczbowe przemieszczeń powierzchni podłoża. Podane w pracy rozwiązanie dotyczącego działania obciążenia równomiernie rozłożonego w obszarze koła może mieć zastosowanie do przybliżonego rozwiązania (metodami komputerowymi) zadania o kontakcie belki lub płyty z powierzchnią warstwowego podłoża sprężystego.