

FRACTIONAL DESCRIPTOR CONTINUOUS–TIME LINEAR SYSTEMS DESCRIBED BY THE CAPUTO–FABRIZIO DERIVATIVE

TADEUSZ KACZOREK ^a, KAMIL BORAWSKI ^{a,*}

^aFaculty of Electrical Engineering
Białystok Technical University, Wiejska 45D, 15-351 Białystok, Poland
e-mail: kaczorek@ee.pw.edu.pl, kam.borawski@gmail.com

The Weierstrass–Kronecker theorem on the decomposition of the regular pencil is extended to fractional descriptor continuous-time linear systems described by the Caputo–Fabrizio derivative. A method for computing solutions of continuous-time systems is presented. Necessary and sufficient conditions for the positivity and stability of these systems are established. The discussion is illustrated with a numerical example.

Keywords: fractional system, descriptor system, continuous-time, linear system, Caputo–Fabrizio derivative.

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial condition state remains forever in the positive orthant for all nonnegative inputs. An overview of the state of the art in positive systems theory is given in the monographs of Farina and Rinaldi (2000) as well as Kaczorek (2001), and in the papers by Kaczorek (1997; 1998a; 2011a; 2014a; 2014b; 2015b). Models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

Fractional systems were investigated by Kaczorek (2012) and Ostalczyk (2016). Descriptor (singular) linear systems were considered in many papers and books (Gantmacher, 1959; Campbell *et al.*, 1976; Van Dooren, 1979; Kucera and Zagalak, 1988; Dai, 1989; Fahmy and O’Reill, 1989; Kaczorek, 1997; 1998a; Bru *et al.*, 2000; 2003; Virnik, 2008; Dodig and Stosic, 2009; Duan, 2010). Positive standard and descriptor systems and their stability were analyzed by Kaczorek (1998a; 2001; 2011b; 2014b; 2015b) and Virnik (2008). Positive linear systems with different fractional orders were addressed by Kaczorek (2011b; 2012), along with descriptor positive discrete-time and continuous-time nonlinear systems (Kaczorek, 2014a), the positivity and linearization of nonlinear discrete-time systems by state feedbacks (Kaczorek, 2014b), or new

stability tests of positive standard and fractional linear systems (Kaczorek, 2011a). The stability and robust stabilization of discrete-time switched systems were analyzed by Zhang *et al.* (2014a; 2014b), while the controllability of dynamical systems was investigated by Klamka (2013).

Recently, a new definition of the fractional derivative without a singular kernel has been proposed (Caputo and Fabrizio, 2015; Losada and Nieto, 2015). Using this new definition, the fractional descriptor continuous-time linear systems will be investigated.

The paper is organized as follows. In Section 2, the Weierstrass–Kronecker decomposition theorem is applied and the solution of the state equation of fractional descriptor continuous-time linear systems is given. Necessary and sufficient conditions for positivity are established in Section 3 and for asymptotic stability in Section 4, where also tests for checking stability are given. A numerical example is presented in Section 5. Concluding remarks are given in Section 6.

The following notation will be used: \mathbb{R} , the set of real numbers; $\mathbb{R}^{n \times m}$, the set of $n \times m$ matrices; $\mathbb{R}_+^{n \times m}$, the set of real $n \times m$ matrices with nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n \times 1}$; M_n , the set of $n \times n$ Metzler matrices (with nonnegative off-diagonal entries); I_n , the $n \times n$ identity matrix.

*Corresponding author

2. Fractional descriptor continuous-time linear systems

Consider the fractional descriptor continuous-time linear system

$$E^{CF} D^\alpha x(t) = Ax(t) + Bu(t), \quad 0 < \alpha < 1, \quad (1a)$$

$$y(t) = Cx(t), \quad (1b)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$, and

$$\begin{aligned} {}^{CF}D^\alpha x(t) &= \frac{d^\alpha x(t)}{dt^\alpha} \\ &= \frac{1}{1-\alpha} \int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right) \dot{x}(\tau) d\tau, \quad (2) \\ \dot{x}(\tau) &= \frac{dx(\tau)}{d\tau}, \quad t \geq 0, \end{aligned}$$

is the Caputo–Fabrizio fractional derivative of order α of the state vector $x(t) \in \mathbb{R}^n$ (Caputo and Fabrizio, 2015; Losada and Nieto, 2015).

It is assumed that $\det E = 0$ and

$$\det[E\lambda - A] \neq 0 \quad (3)$$

for some $\lambda \in \mathbb{C}$.

It is well-known (Kaczorek, 1998b) that if (3) holds then there exists a pair of nonsingular matrices $P, Q \in \mathbb{R}^{n \times n}$ such that

$$P[E\lambda - A]Q = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} \lambda - \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad (4)$$

$$n_1 + n_2 = n,$$

where $n_1 = \deg \det[E\lambda - A]$, $A_1 \in \mathbb{R}^{n_1 \times n_1}$ and $N \in \mathbb{R}^{n_2 \times n_2}$ is a nilpotent matrix with the index μ (i.e., $N^\mu = 0$ and $N^{\mu-1} \neq 0$). The matrices P, Q, A_1 can be found by the use of elementary row and column operations (Kaczorek, 1998b).

Premultiplying (1a) by the matrix $P \in \mathbb{R}^{n \times n}$, introducing the new state vector

$$\bar{x}(t) = Q^{-1}x(t) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix},$$

$$\bar{x}_1(t) = \begin{bmatrix} \bar{x}_{11}(t) \\ \bar{x}_{12}(t) \\ \vdots \\ \bar{x}_{1n_1}(t) \end{bmatrix}, \quad (5)$$

$$\bar{x}_2(t) = \begin{bmatrix} \bar{x}_{21}(t) \\ \bar{x}_{22}(t) \\ \vdots \\ \bar{x}_{2n_2}(t) \end{bmatrix}$$

and using (4), we obtain

$$\frac{d^\alpha \bar{x}_1(t)}{dt^\alpha} = A_1 \bar{x}_1(t) + B_1 u(t), \quad (6a)$$

$$N \frac{d^\alpha \bar{x}_2(t)}{dt^\alpha} = \bar{x}_2(t) + B_2 u(t), \quad (6b)$$

where

$$PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad B_1 \in \mathbb{R}^{n_1 \times m}, \quad B_2 \in \mathbb{R}^{n_2 \times m}. \quad (7)$$

Theorem 1. The solution $\bar{x}_1(t)$ of Eqn. (6a) for a given initial condition $\bar{x}_1(0) = \bar{x}_{10} \in \mathbb{R}^{n_1}$ and an input $u(t) \in \mathbb{R}^m$, $t \geq 0$, has the form

$$\begin{aligned} \bar{x}_1(t) &= e^{\hat{A}_1 t} (\hat{x}_{10} + \hat{B}_1 u_0) \\ &+ \int_0^t e^{\hat{A}_1(t-\tau)} \hat{B}_1 [\beta u(\tau) + \dot{u}(\tau)] d\tau, \quad (8a) \end{aligned}$$

where

$$\begin{aligned} \hat{A}_1 &= \alpha [I_{n_1} - (1-\alpha)A_1]^{-1} A_1, \\ \hat{B}_1 &= [I_{n_1} - (1-\alpha)A_1]^{-1} (1-\alpha)B_1, \\ \hat{x}_{10} &= [I_{n_1} - (1-\alpha)A_1]^{-1} \bar{x}_{10}, \quad (8b) \\ e^{\hat{A}_1 t} &= \mathcal{L}^{-1} \{ [I_{n_1} s - \hat{A}_1]^{-1} \}, \end{aligned}$$

$$\beta = \frac{\alpha}{1-\alpha}, \quad \dot{u}(\tau) = \frac{du(\tau)}{d\tau}, \quad u(0) = u_0.$$

The proof is given by Kaczorek (2015a).

Theorem 2. The solution $\bar{x}_2(t)$ of Eqn. (6b) for a given initial condition $\bar{x}_2(0) = \bar{x}_{20} \in \mathbb{R}^{n_2}$ and an input $u(t) \in \mathbb{R}^m$, $t \geq 0$, has the form

$$\begin{aligned} \bar{x}_2(t) &= e^{\hat{N}t} (\tilde{N} \bar{x}_{20} + \tilde{B}_2 u_0) \\ &+ \int_0^t e^{\hat{N}(t-\tau)} \tilde{B}_2 [\beta u(\tau) + \dot{u}(\tau)] d\tau, \quad (9a) \end{aligned}$$

where

$$\begin{aligned} \hat{N} &= \alpha [N - I_{n_2}(1-\alpha)]^{-1}, \\ \tilde{N} &= [N - I_{n_2}(1-\alpha)]^{-1} N, \\ \tilde{B}_2 &= [N - I_{n_2}(1-\alpha)]^{-1} (1-\alpha)B_2, \quad (9b) \\ \beta &= \frac{\alpha}{1-\alpha}. \end{aligned}$$

Proof. Using Laplace transform (\mathcal{L}) in (6b) as well as the convolution theorem, we obtain

$$\begin{aligned} N \mathcal{L} \left[\frac{d^\alpha \bar{x}_2(t)}{dt^\alpha} \right] &= N \frac{1}{1-\alpha} \mathcal{L} \left[\int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right) \dot{\bar{x}}_2(\tau) d\tau \right] \\ &= \mathcal{L}[\bar{x}_2(t)] + B_2 \mathcal{L}[u(t)] \quad (10) \end{aligned}$$

and

$$N \frac{1}{1-\alpha} \left\{ \frac{1}{s+\beta} [s\bar{X}_2(s) - \bar{x}_{20}] \right\} = \bar{X}_2(s) + B_2U(s), \tag{11}$$

where

$$\begin{aligned} \bar{X}_2(s) &= \mathcal{L}[\bar{x}_2(t)] = \int_0^\infty \bar{x}_2(t)e^{-st} dt, \\ U(s) &= \mathcal{L}[u(t)], \\ \beta &= \frac{\alpha}{1-\alpha}, \\ \mathcal{L}[e^{-\beta t}] &= \frac{1}{s+\beta}, \\ \mathcal{L} \left[\int_0^t \exp \left(-\frac{\alpha}{1-\alpha}(t-\tau) \right) \dot{\bar{x}}_2(\tau) d\tau \right] & \\ &= \frac{1}{s+\beta} [s\bar{X}_2(s) - \bar{x}_{20}], \\ \mathcal{L}[\dot{\bar{x}}_2(t)] &= s\bar{X}_2(s) - \bar{x}_{20}. \end{aligned} \tag{12}$$

From (10), we have

$$\begin{aligned} \{s[N - I_{n_2}(1-\alpha)] - I_{n_2}\alpha\} \bar{X}_2(s) \\ = N\bar{x}_{20} + (s+\beta)\bar{B}_2U(s), \end{aligned} \tag{13}$$

where $\bar{B}_2 = (1-\alpha)B_2$.

Note that the matrix $[N - I_{n_2}(1-\alpha)]$ is invertible. After premultiplication of (13) by $[N - I_{n_2}(1-\alpha)]^{-1}$, we obtain

$$\begin{aligned} [I_{n_2}s - \hat{N}] \bar{X}_2(s) &= [N - I_{n_2}(1-\alpha)]^{-1} N\bar{x}_{20} \\ &+ (s+\beta)[N - I_{n_2}(1-\alpha)]^{-1} \bar{B}_2U(s) \\ &= \tilde{N}\bar{x}_{20} + \beta\tilde{B}_2U(s) + \tilde{B}_2[sU(s) - u_0] + \tilde{B}_2u_0, \end{aligned} \tag{14a}$$

where

$$\begin{aligned} \hat{N} &= \alpha[N - I_{n_2}(1-\alpha)]^{-1}, \\ \tilde{N} &= [N - I_{n_2}(1-\alpha)]^{-1}N, \\ \tilde{B}_2 &= [N - I_{n_2}(1-\alpha)]^{-1}\bar{B}_2 \end{aligned} \tag{14b}$$

and

$$\begin{aligned} \bar{X}_2(s) &= [I_{n_2}s - \hat{N}]^{-1} \tilde{N}\bar{x}_{20} \\ &+ [I_{n_2}s - \hat{N}]^{-1} \tilde{B}_2u_0 \\ &+ \beta[I_{n_2}s - \hat{N}]^{-1} \tilde{B}_2U(s) \\ &+ [I_{n_2}s - \hat{N}]^{-1} \tilde{B}_2[sU(s) - u_0]. \end{aligned} \tag{15}$$

Taking into account that

$$\mathcal{L}^{-1}\{[I_{n_2}s - \hat{N}]^{-1}\} = e^{\hat{N}t} \tag{16}$$

and using the inverse Laplace transform as well as the convolution theorem, we obtain (9). ■

From (8) and (9), we can see that both solutions have a similar form, which is completely different from the standard Caputo derivative, where the subsystem (6b) has a strictly singular solution with Dirac impulses.

3. Positive fractional linear systems

In this section, the necessary and sufficient conditions for the positivity of fractional descriptor continuous-time linear systems described by Eqns. (1) will be established.

Definition 1. The fractional descriptor continuous-time linear system (1) is called (internally) *positive* if and only if $x(t) \in \mathbb{R}_+^n$ and $y(t) \in \mathbb{R}_+^p$, $t \geq 0$, for any consistent initial conditions $x_0 \in \mathbb{R}_+^n$ and all admissible inputs $u(t) \in \mathbb{R}_+^m$, $\dot{u}(t) \in \mathbb{R}_+^m$, $t \geq 0$.

Definition 2. The matrix $Q \in \mathbb{R}_+^{n \times n}$ is called *monomial* if in each row and column only one entry is positive and the remaining entries are zero.

It is well known (Kaczorek, 2001) that $Q^{-1} \in \mathbb{R}_+^{n \times n}$ if and only if the matrix $Q \in \mathbb{R}_+^{n \times n}$ is monomial. It is assumed that for the positive system (1) the decomposition (4) is possible for the monomial matrix Q . In this case, $x(t) = Q\bar{x}(t) \in \mathbb{R}_+^n$ if and only if $\bar{x}(t) \in \mathbb{R}_+^n$, $t \geq 0$. It is also well known that premultiplication of Eqn. (1) by the matrix P does not change its solution $x(t)$.

Lemma 1. Let $\hat{A}_1 \in M_{n_1}$ and $0 < \alpha < 1$. Then

$$e^{\hat{A}_1 t} \in \mathbb{R}_+^{n_1 \times n_1} \text{ for } t \geq 0. \tag{17}$$

The proof is similar to the one given by Kaczorek (2001).

Theorem 3. Let the decomposition (4) of the system (1) be possible for a monomial matrix $Q \in \mathbb{R}_+^{n \times n}$. The subsystem (6a) for $0 < \alpha < 1$ is positive if and only if

$$\hat{A}_1 \in M_{n_1}, \quad \hat{B}_1 \in \mathbb{R}_+^{n_1 \times m}. \tag{18}$$

Proof.

(Sufficiency) If $\hat{A}_1 \in M_{n_1}$ and $\hat{B}_1 \in \mathbb{R}_+^{n_1 \times m}$ then from (8) we have $\bar{x}_1(t) \in \mathbb{R}_+^{n_1}$, $t \geq 0$ since by Lemma 1 $e^{\hat{A}_1 t} \in \mathbb{R}_+^{n_1 \times n_1}$ and $\bar{x}_{10} \in \mathbb{R}_+^{n_1}$, $u(t) \in \mathbb{R}_+^m$, $\dot{u}(t) \in \mathbb{R}_+^m$, $t \geq 0$.

(Necessity) Let $u(t) = 0$, $t \geq 0$ and $\bar{x}_{10} = e_i$ (the i -th column of the identity matrix I_{n_1}). The trajectory remains in the orthant $\mathbb{R}_+^{n_1}$ only if ${}^{CF}D^\alpha \bar{x}_1(0) = \hat{A}_1 e_i \geq 0$, which implies $\hat{A}_1 \in M_{n_1}$. If $\bar{x}_{10} = 0$, then ${}^{CF}D^\alpha \bar{x}_1(0) = \hat{B}_1 u(0) \geq 0$ and this implies $\hat{B}_1 \in \mathbb{R}_+^{n_1 \times m}$ since $u(0) \in \mathbb{R}_+^m$ is arbitrary. ■

Lemma 2. If λ_k , $k = 1, \dots, n_1$, are the eigenvalues of the matrix A_1 , then the eigenvalues of the matrix $\hat{A}_1 = \alpha[I_{n_1} - (1-\alpha)A_1]^{-1}A_1$ are given by

$$\hat{\lambda}_k = \alpha[1 - (1-\alpha)\lambda_k]^{-1}\lambda_k. \tag{19}$$

Proof. It is well known (Gantmacher, 1959) that if $f(\lambda_k)$ is well defined on the spectrum $\lambda_k, k = 1, \dots, n_1$, of the matrix A_1 then the eigenvalues of the matrix $f(A)$ are given by $f(\lambda_k), k = 1, \dots, n_1$. In this case, $f(\hat{A}) = \alpha[I_{n_1} - (1 - \alpha)A_1]^{-1}A_1$. ■

Lemma 3. *The matrix $\bar{A}_1 = (1 - \alpha)A_1 \in \mathbb{R}^{n_1 \times n_1}$ for $0 < \alpha < 1$ is asymptotically stable if and only if the matrix A_1 is asymptotically stable.*

Proof. The eigenvalues $\bar{\lambda}_k, k = 1, \dots, n_1$ of the matrix \bar{A}_1 are related with the eigenvalues $\lambda_k, k = 1, \dots, n_1$, of the matrix A_1 by

$$\bar{\lambda}_k = (1 - \alpha)\lambda_k, \quad k = 1, \dots, n_1, \quad (20)$$

since the characteristic polynomials of the matrices are related by the equality

$$\begin{aligned} \det[I_{n_1}\bar{\lambda}_k - \bar{A}_1] &= \det[I_{n_1}\bar{\lambda}_k - (1 - \alpha)A_1] \\ &= (1 - \alpha)^{n_1} \det \left[I_{n_1} \frac{\bar{\lambda}_k}{1 - \alpha} - A_1 \right] \\ &= (1 - \alpha)^{n_1} \det[I_n\lambda_k - A_1]. \end{aligned} \quad (21)$$

Therefore, from (20) it follows that $\text{Re } \bar{\lambda}_k < 0, k = 1, \dots, n_1$, if and only if $\text{Re } \lambda_k < 0, k = 1, \dots, n_1$. ■

Lemma 4. *The matrix*

$$\hat{A}_1 = \alpha[I_{n_1} - (1 - \alpha)A_1]^{-1}A_1 \in M_{n_1} \quad (22)$$

is asymptotically stable if and only if the eigenvalues $\lambda_k = -\alpha_k + j\beta_k, k = 1, \dots, n_1$, of the matrix A_1 satisfy the condition

$$[1 + (1 - \alpha)\alpha_k]\alpha_k + (1 - \alpha)\beta_k^2 = n(k) > 0.$$

Proof. From (22) for $\hat{\lambda}_k = -\hat{\alpha}_k + j\hat{\beta}_k$ and $\lambda_k = -\alpha_k + j\beta_k, k = 1, \dots, n_1$, we have

$$\begin{aligned} \hat{\lambda}_k &= -\hat{\alpha}_k + j\hat{\beta}_k = \alpha[1 - (1 - \alpha)\lambda_k]^{-1}\lambda_k \\ &= \alpha[1 - (1 - \alpha)(-\alpha_k + j\beta_k)]^{-1}(-\alpha_k + j\beta_k) \\ &= \alpha \frac{1 + (1 - \alpha)\alpha_k + j(1 - \alpha)\beta_k}{[1 + (1 - \alpha)\alpha_k]^2 + [(1 - \alpha)\beta_k]^2} (-\alpha_k + j\beta_k) \\ &= \alpha \left(\frac{-[1 + (1 - \alpha)\alpha_k]\alpha_k - (1 - \alpha)\beta_k^2}{[1 + (1 - \alpha)\alpha_k]^2 + [(1 - \alpha)\beta_k]^2} \right. \\ &\quad \left. + j \frac{[1 + (1 - \alpha)\alpha_k]\beta_k - (1 - \alpha)\alpha_k\beta_k}{[1 + (1 - \alpha)\alpha_k]^2 + [(1 - \alpha)\beta_k]^2} \right) \end{aligned} \quad (23)$$

and

$$\begin{aligned} \hat{\alpha}_k &= \alpha \left(\frac{[1 + (1 - \alpha)\alpha_k]\alpha_k + (1 - \alpha)\beta_k^2}{[1 + (1 - \alpha)\alpha_k]^2 + [(1 - \alpha)\beta_k]^2} \right) \\ &= \alpha \frac{n(k)}{d(k)}, \quad k = 1, \dots, n. \end{aligned} \quad (24)$$

From (24) it follows that $\hat{\alpha}_k > 0, k = 1, \dots, n_1$, if and only if $n(k) > 0, k = 1, \dots, n_1$. ■

Lemma 5. *The matrices*

$$\begin{aligned} \hat{A}_1 &= \alpha[I_{n_1} - (1 - \alpha)A_1]^{-1}A_1 \in M_{n_1}, \\ \hat{B} &= [I_{n_1} - (1 - \alpha)A_1]^{-1}(1 - \alpha)B_1 \in \mathbb{R}_+^{n_1 \times m} \end{aligned} \quad (25)$$

if $A_1 \in M_{n_1}$ is asymptotically stable and $B_1 \in \mathbb{R}_+^{n_1 \times m}$.

Proof. The matrix $[I_{n_1} - (1 - \alpha)A_1]^{-1} \in \mathbb{R}_+^{n_1 \times n_1}$ if the matrix $A_1 \in M_{n_1}$ is asymptotically stable (Kaczorek, 2001). Therefore, by Lemma 3 and $(1 - \alpha)B_1 \in \mathbb{R}_+^{n_1 \times m}$ for $0 < \alpha < 1$, (24) holds if $A_1 \in M_{n_1}$ is asymptotically stable. ■

From Lemma 5 and Theorem 3, we have the following.

Theorem 4. *Let the decomposition (4) of the system (1) be possible for a monomial matrix $Q \in \mathbb{R}_+^{n \times n}$. The subsystem (6a) for $0 < \alpha < 1$ is positive if $A_1 \in M_{n_1}$ is asymptotically stable and $B_1 \in \mathbb{R}_+^{n_1 \times m}$.*

From Theorem 4, we have stronger restrictions for the positivity of the subsystem (6a) described by the Caputo–Fabrizio derivative. In contrast to the standard Caputo derivative, the matrix A_1 must be asymptotically stable. The subsystem (6b) also has an additional condition which is proved in the following theorem.

Theorem 5. *Let the decomposition (4) of the system (1) be possible for a monomial matrix $Q \in \mathbb{R}_+^{n \times n}$. The subsystem (6b) for $0 < \alpha < 1$ is positive if and only if*

$$-B_2 \in \mathbb{R}_+^{n_2 \times m}, \quad v_{20} = \tilde{N}\bar{x}_{20} + \tilde{B}_2u_0 \in \mathbb{R}_+^{n_2}. \quad (26)$$

Proof.

(Sufficiency) Observe that the matrix $[N - I_{n_2}(1 - \alpha)]$ is asymptotically stable and $-\hat{N} \in \mathbb{R}_+^{n_2 \times n_2}$ (Kaczorek, 2001). If $v_{20} \in \mathbb{R}_+^{n_2}$, then $e^{\hat{N}t}v_{20} \in \mathbb{R}_+^{n_2}$. By assumption, $u(\tau) + \dot{u}(\tau) \in \mathbb{R}_+^{n_2}, t \geq 0$ and $\beta u(\tau) + \dot{u}(\tau) \in \mathbb{R}_+^{n_2}$ since $\beta > 0$ and $e^{\hat{N}(t-\tau)}\tilde{B}_2 \in \mathbb{R}_+^{n_2}, t \geq 0$, since $\tilde{B}_2 \in \mathbb{R}_+^{n_2}$. Therefore,

$$\int_0^t e^{\hat{N}(t-\tau)}\tilde{B}_2[\beta u(\tau) + \dot{u}(\tau)] d\tau \in \mathbb{R}_+^{n_2}, \quad t \geq 0.$$

(Necessity) The proof of necessity is based on Eqn. (6b). To simplify the notation, it is assumed that the matrix N has the form

$$N = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{n_2 \times n_2}. \quad (27)$$

From (6b) and (27), we have

$$\begin{aligned} & \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \frac{d^\alpha}{dt^\alpha} \begin{bmatrix} \bar{x}_{21}(t) \\ \bar{x}_{22}(t) \\ \vdots \\ \bar{x}_{2n_2}(t) \end{bmatrix} \\ &= \begin{bmatrix} \bar{x}_{21}(t) \\ \bar{x}_{22}(t) \\ \vdots \\ \bar{x}_{2n_2}(t) \end{bmatrix} + \begin{bmatrix} B_{21} \\ B_{22} \\ \vdots \\ B_{2n_2} \end{bmatrix} u(t) \end{aligned} \quad (28)$$

and

$$\begin{aligned} \bar{x}_{2n_2}(t) &= -B_{2n_2}u(t), \\ \bar{x}_{2n_2-1}(t) &= \frac{d^\alpha \bar{x}_{2n_2}(t)}{dt^\alpha} - B_{2n_2-1}u(t), \\ &\vdots \\ \bar{x}_{21}(t) &= \frac{d^\alpha \bar{x}_{22}(t)}{dt^\alpha} - B_{21}u(t). \end{aligned} \quad (29)$$

Assuming

$$\frac{d^\alpha \bar{x}_{2n_2}}{dt^\alpha} \geq 0,$$

the subsystem (6b) is positive if and only if the conditions (26) are satisfied. ■

The considerations can be easily extended to the case when the matrix N in (6b) has the form

$$N = \text{blockdiag}[N_1 \ \cdots \ N_q], \quad q > 1 \quad (30)$$

and N_k for $k = 1, 2, \dots, q$ has the form (27).

Theorem 6. *Let the decomposition (4) of the system (1) be possible for a monomial matrix $Q \in \mathbb{R}_+^{n \times n}$. The system (1) for $0 < \alpha < 1$ is positive if and only if*

- (i) *the conditions of Theorem 4 and (26) are satisfied,*
- (ii) *$C \in \mathbb{R}_+^{p \times n}$.*

Proof. By Theorems 3–5 the solutions of Eqns. (6a) and (6b) are positive if and only if the conditions of Theorem 4 and (26) are met. From (1b) and (5), we have

$$y(t) = CQQ^{-1}x(t) = \bar{C}x(t), \quad (31)$$

where $\bar{C} = CQ$.

For a monomial matrix $Q \in \mathbb{R}_+^{n \times n}$, we have

$$\bar{C} \in \mathbb{R}_+^{p \times n} \text{ if and only if } C \in \mathbb{R}_+^{p \times n} \quad (32)$$

and

$$y(t) \in \mathbb{R}_+^p \text{ if and only if } C \in \mathbb{R}_+^{p \times n}. \quad (33)$$

4. Stability of positive systems

Consider the positive fractional descriptor continuous-time linear system (1) with $u(t) = 0$. Note that $\bar{x}_2 = 0$ and the stability of the positive system (1) depends only on the stability of the subsystem (6a) described by the equation

$$\frac{d^\alpha \bar{x}_1(t)}{dt^\alpha} = A_1 \bar{x}_1(t), \quad \bar{x}_1(t) \in \mathbb{R}_+^{n_1}, \quad A_1 \in M_{n_1}. \quad (34)$$

Definition 3. The positive fractional descriptor continuous-time linear system (1) is called *asymptotically stable* if

$$\lim_{t \rightarrow \infty} \bar{x}_1(t) = 0 \quad \text{for all } \bar{x}_{10} \in \mathbb{R}_+^{n_1}. \quad (35)$$

Theorem 7. *The positive fractional system (34) is (asymptotically) stable if and only if one of the following equivalent conditions is satisfied (the matrix \hat{A}_1 is defined by (8b)):*

- (i) *All coefficients of the polynomial*

$$\begin{aligned} & \det[I_{n_1}s - \hat{A}_1] \\ &= s^n + \hat{a}_{n-1}s^{n-1} + \cdots + \hat{a}_1s + \hat{a}_0 \end{aligned} \quad (36)$$

are positive, i.e., $\hat{a}_k > 0$ for $k = 0, 1, \dots, n_1 - 1$.

- (ii) *All principal minors $M_k, k = 1, \dots, n_1$ of the matrix $-\hat{A}_1$ are positive, i.e.,*

$$\begin{aligned} M_1 &= |-\hat{a}_{11}| > 0, \\ M_2 &= \begin{vmatrix} -\hat{a}_{11} & -\hat{a}_{12} \\ -\hat{a}_{21} & -\hat{a}_{22} \end{vmatrix} > 0, \\ &\vdots \\ M_n &= \det[-\hat{A}_1] > 0. \end{aligned} \quad (37)$$

- (iii) *The diagonal entries of the matrices*

$$\hat{A}_{1,n-k}^{(k)} \quad \text{for } k = 1, \dots, n_1 - 1 \quad (38a)$$

are negative, where $\hat{A}_{1,n-k}^{(k)}$ are defined as follows:

$$\begin{aligned} \hat{A}_{1,n}^{(0)} &= \hat{A}_1 \\ &= \begin{bmatrix} \hat{a}_{11}^{(0)} & \cdots & \hat{a}_{1,n}^{(0)} \\ \vdots & \ddots & \vdots \\ \hat{a}_{n,1}^{(0)} & \cdots & \hat{a}_{n,n}^{(0)} \end{bmatrix} \\ &= \begin{bmatrix} \hat{a}_{11}^{(0)} & \hat{b}_{n-1}^{(0)} \\ \hat{c}_{n-1}^{(0)} & \hat{A}_{1,n-1}^{(0)} \end{bmatrix}, \end{aligned}$$

$$\hat{A}_{1,n-1}^{(0)} = \begin{bmatrix} \hat{a}_{22}^{(0)} & \cdots & \hat{a}_{2,n}^{(0)} \\ \vdots & \ddots & \vdots \\ \hat{a}_{n,2}^{(0)} & \cdots & \hat{a}_{n,n}^{(0)} \end{bmatrix}, \quad (38b)$$

$$\hat{b}_{n-1}^{(0)} = [\hat{a}_{12}^{(0)} \quad \cdots \quad \hat{a}_{1,n}^{(0)}],$$

$$\hat{c}_{n-1}^{(0)} = \begin{bmatrix} \hat{a}_{21}^{(0)} \\ \vdots \\ \hat{a}_{n,1}^{(0)} \end{bmatrix}$$

and

$$\hat{A}_{1,n-k}^{(k)} = \hat{A}_{1,n-k}^{(k-1)} - \frac{\hat{c}_{n-k}^{(k-1)} \hat{b}_{n-k}^{(k-1)}}{\hat{a}_{k+1,k+1}^{(k-1)}}$$

$$= \begin{bmatrix} \hat{a}_{k+1,k+1}^{(k)} & \cdots & \hat{a}_{k+1,n}^{(0)} \\ \vdots & \ddots & \vdots \\ \hat{a}_{n,k+1}^{(0)} & \cdots & \hat{a}_{n,n}^{(0)} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{a}_{k+1,k+1}^{(k)} & \hat{b}_{n-k-1}^{(k)} \\ \hat{c}_{n-k-1}^{(k)} & \hat{A}_{n-k-1}^{(k)} \end{bmatrix},$$

$$\hat{A}_{1,n-k-1}^{(k)} = \begin{bmatrix} \hat{a}_{k+2,k+2}^{(k)} & \cdots & \hat{a}_{k+2,n}^{(0)} \\ \vdots & \ddots & \vdots \\ \hat{a}_{n,k+2}^{(0)} & \cdots & \hat{a}_{n,n}^{(0)} \end{bmatrix},$$

$$\hat{b}_{n-k-1}^{(0)} = [\hat{a}_{k+1,k+2}^{(k)} \quad \cdots \quad \hat{a}_{k+1,n}^{(k)}],$$

$$\hat{c}_{n-k-1}^{(k)} = \begin{bmatrix} \hat{a}_{k+2,k+1}^{(k)} \\ \vdots \\ \hat{a}_{n,k+1}^{(k)} \end{bmatrix}, \quad (38c)$$

for $k = 1, \dots, n_1 - 1$.

(iv) All diagonal entries of the upper (lower) triangular matrix

$$\tilde{A}_{1,u} = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1,n} \\ 0 & \tilde{a}_{22} & \cdots & \tilde{a}_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{a}_{n,n} \end{bmatrix}, \quad (39)$$

$$\tilde{A}_{1,l} = \begin{bmatrix} \tilde{a}_{11} & 0 & \cdots & 0 \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n,1} & \tilde{a}_{n,2} & \cdots & \tilde{a}_{n,n} \end{bmatrix}$$

are negative, i.e.,

$$\tilde{a}_{kk} < 0 \quad \text{for } k = 1, \dots, n_1,$$

and matrices \tilde{A}_1 have been obtained from the matrix \hat{A}_1 with the use of an elementary row operation.

(v) There exists a strictly positive vector

$$\lambda = [\lambda_1 \quad \cdots \quad \lambda_n]^T,$$

$$\lambda_k > 0, \quad k = 1, \dots, n_1$$

such that

$$\hat{A}_1 \lambda < 0. \quad (40)$$

Proof. Substituting in (8) $u(t) = 0, t \geq 0$, we obtain the solution of Eqn. (34) in the form

$$\bar{x}_1(t) = e^{\hat{A}_1 t} \bar{x}_{10}. \quad (41)$$

The system (34) is stable if and only if

$$\lim_{t \rightarrow \infty} e^{\hat{A}_1 t} = 0 \quad \text{for all } \bar{x}_{10} \in \mathbb{R}_+^n. \quad (42)$$

The condition (42) is satisfied if and only if $\hat{A}_1 \in M_{n_1}$. In the work of Kaczorek (2001), it is shown that the system (34) with $\hat{A}_1 \in M_{n_1}$ is asymptotically stable if and only if one of the conditions (1)–(4) is satisfied. If the system is asymptotically stable then from the condition one we have $\hat{a}_0 = \det[-\hat{A}_1] > 0$ and $-\hat{A}_1^{-1} \in \mathbb{R}_+^{n \times n}$ (Kaczorek, 2001). Then, using (40), we obtain $(-\hat{A}_1^{-1})(-\hat{A}_1)\lambda > 0$ and $\lambda > 0$ if and only if the system is asymptotically stable. ■

5. Numerical example

Consider the fractional descriptor continuous-time system described by Eqn. (1a) for $\alpha = 0.5$ and

$$E = \begin{bmatrix} -0.4 & 0 & -0.5 & 0 \\ -0.2 & 0 & 0 & 0 \\ 0.4 & 1 & 0.5 & 0 \\ 0.2 & 0 & 0 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} -0.2 & 1.8 & 0.5 & 0 \\ 0.4 & 0.4 & 0 & 0 \\ 0.2 & -1.8 & -0.5 & 0.5 \\ -0.4 & 0.6 & 0 & 0 \end{bmatrix}, \quad (43)$$

$$B = \begin{bmatrix} -1 & -3.6 \\ 0 & -0.8 \\ -1 & 2.6 \\ 0 & -0.2 \end{bmatrix},$$

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} 1(t) \\ \sin(t) + 1(t) \end{bmatrix},$$

$$1(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geq 0. \end{cases}$$

The pencil is regular since

$$\det[E\lambda - A] = -0.05(\lambda + 1)(\lambda + 2) \neq 0. \quad (44)$$

In this case,

$$P = \begin{bmatrix} -1 & 3 & 0 & 1 \\ 0 & -3 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad (45)$$

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} = PEQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} = PAQ = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (46)$$

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ -2 & -1 \\ 0 & -1 \end{bmatrix}.$$

The matrix

$$\begin{aligned} \hat{A}_1 &= \alpha[I_2 - (1 - \alpha)A_1]^{-1}A_1 \\ &= \begin{bmatrix} 1.5 & -0.5 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -0.5 & 0.5 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -0.3333 & 0.1667 \\ 0 & -0.5 \end{bmatrix} \end{aligned} \quad (47)$$

is an asymptotically stable Metzler matrix since its eigenvalues are $\lambda_1 = -0.3333, \lambda_2 = -0.5$.

Note that the matrix

$$\begin{aligned} \hat{B}_1 &= [I_1 - (1 - \alpha)A_1]^{-1}(1 - \alpha)B_1 \\ &= \begin{bmatrix} 1.5 & -0.5 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 & 0.5 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.3333 & 0.5 \\ 0 & 0.5 \end{bmatrix} \end{aligned} \quad (48)$$

has positive entries.

Using the Sylvester theorem, we may find the matrix

$$\begin{aligned} e^{\hat{A}_1 t} &= Z_1 e^{\lambda_1 t} + Z_2 e^{\lambda_2 t}, \\ Z_1 &= \frac{\hat{A} - I_2 \lambda_2}{\lambda_1 - \lambda_2}, \quad Z_2 = \frac{\hat{A} - I_2 \lambda_1}{\lambda_2 - \lambda_1}. \end{aligned} \quad (49)$$

From (47) and (49), we obtain

$$\begin{aligned} e^{\hat{A}_1 t} &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} e^{-0.3333t} + \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} e^{-0.5t} \\ &= \begin{bmatrix} e^{-0.3333t} & e^{-0.3333t} - e^{-0.5t} \\ 0 & e^{-0.5t} \end{bmatrix}. \end{aligned} \quad (50)$$

Using (8a) with (43), (47)–(50) and

$$\bar{x}_{10} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

we can find the desired solution of the subsystem (6a):

$$\bar{x}_1(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix}, \quad (51)$$

where

$$\begin{aligned} \xi_1(t) &= -0.7333e^{-0.3333t} - 0.2e^{-0.5t} - 0.4 \cos(t) \\ &\quad + 0.6 \sin(t) + 3, \\ \xi_2(t) &= 0.2e^{-0.5t} - 0.2 \cos(t) + 0.6 \sin(t) + 1. \end{aligned}$$

The solution (51) of the subsystem (6a) is shown in Fig. 1.

The matrices \tilde{B}_2, \hat{N} and \tilde{N} have the form

$$\begin{aligned} \tilde{B}_2 &= [N - I_2(1 - \alpha)]^{-1}(1 - \alpha)B_2 \\ &= \begin{bmatrix} -0.5 & 1 \\ 0 & -0.5 \end{bmatrix}^{-1} \begin{bmatrix} -1 & -0.5 \\ 0 & -0.5 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}, \\ \hat{N} &= \alpha[N - I_2(1 - \alpha)]^{-1} = 0.5 \begin{bmatrix} -0.5 & 1 \\ 0 & -0.5 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}, \\ \tilde{N} &= [N - I_2(1 - \alpha)]^{-1}N \\ &= \begin{bmatrix} -0.5 & 1 \\ 0 & -0.5 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}. \end{aligned} \quad (52)$$

From (52), we have $\tilde{B}_2 \in \mathbb{R}_+^{n_2 \times m}$ since $-B_2 \in \mathbb{R}_+^{n_2 \times m}$ and $-\hat{N} \in \mathbb{R}_+^{n_2 \times n_2}$. We can find the matrix $e^{\hat{N}t}$ using the inverse Laplace transform,

$$\begin{aligned} e^{\hat{N}t} &= \mathcal{L}^{-1}\{[I_{n_2}s - \hat{N}]^{-1}\} \\ &= \mathcal{L}^{-1} \begin{bmatrix} \frac{1}{s+1} & \frac{-2}{(s+1)^2} \\ 0 & \frac{1}{s+1} \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} & -2te^{-t} \\ 0 & e^{-t} \end{bmatrix}. \end{aligned} \quad (53)$$

Using (9a) with (43), (52), (53) and

$$\bar{x}_{20} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

we can find the desired solution of the subsystem (6b):

$$\bar{x}_2(t) = \begin{bmatrix} -e^{-t} + \cos(t) + 2 \sin(t) + 3 \\ \sin(t) + 1 \end{bmatrix}. \quad (54)$$

The solution (54) of the subsystem (6b) is shown in Fig. 2.

The fractional descriptor system (43) is positive since the matrix Q defined by (45) is monomial and the conditions of Theorem 6 are satisfied. The system (43) is also asymptotically stable since the matrix \hat{A}_1 is a Metzler matrix with eigenvalues $\lambda_1 = -0.3333, \lambda_2 = -0.5$.

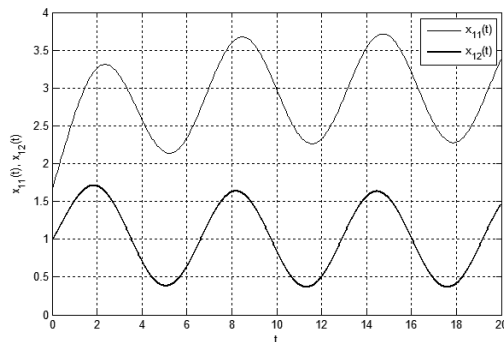


Fig. 1. State vector of the subsystem (6a).

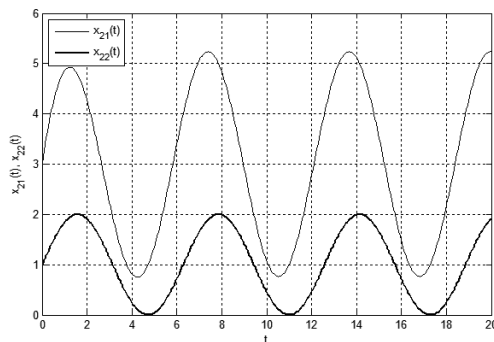


Fig. 2. State vector of the subsystem (6b).

6. Concluding remarks

The Weierstrass–Kronecker theorem on the decomposition of the regular pencil was extended to fractional descriptor continuous-time linear systems described by the Caputo–Fabrizio derivative. The solution to the state equation was given (Theorems 1 and 2). Necessary and sufficient conditions for the positivity (Theorems 3, 4 and 5) and stability of the systems were established. Tests for checking the asymptotic stability of the systems (Theorem 7) were also presented. The discussion was illustrated with a numerical example.

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References

- Bru, R., Coll, C., Romero-Vivo, S. and Sanchez, E. (2003). Some problems about structural properties of positive descriptor systems, in L. Benvenuti *et al.* (Eds.), *Positive Systems*, Lecture Notes in Control and Information Sciences, Vol. 294, Springer, Berlin, pp. 233–240.
- Bru, R., Coll, C. and Sanchez, E. (2000). About positively discrete-time singular systems, in M.E. Mastorakis (Ed.), *System and Control: Theory and Applications*, World Scientific and Engineering Society, Athens, pp. 44–48.
- Campbell, S.L., Meyer, C. and Rose, N. (1976). Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients, *SIAM Journal on Applied Mathematics* **31**(3): 411–425.
- Caputo, M. and Fabrizio, M. (2015). A new definition of fractional derivative without singular kernel, *Progress in Fractional Differentiation and Applications* **1**(2): 1–13.
- Dai, L. (1989). *Singular Control Systems*, Lecture Notes in Control and Information Sciences, Vol. 118, Springer-Verlag, Berlin.
- Dodig, M. and Stosic, M. (2009). Singular systems state feedbacks problems, *Linear Algebra and Its Applications* **431**(8): 1267–1292.
- Duan, G.R. (2010). *Analysis and Design of Descriptor Linear Systems*, Springer, New York, NY.
- Fahmy, M.M. and O'Reilly, J. (1989). Matrix pencil of closed-loop descriptor systems: Infinite-eigenvalues assignment, *International Journal of Control* **49**(4): 1421–1431.
- Farina, L. and Rinaldi, S. (2000). *Positive Linear Systems: Theory and Applications*, J. Wiley, New York, NY.
- Gantmacher, F.R. (1959). *The Theory of Matrices*, Chelsea Publishing Company, London.
- Kaczorek, T. (1997). Positive singular discrete time linear systems, *Bulletin of the Polish Academy of Sciences: Technical Sciences* **45**(4): 619–631.
- Kaczorek, T. (1998a). Positive descriptor discrete-time linear systems, *Problems of Nonlinear Analysis in Engineering Systems* **1**(7): 38–54.
- Kaczorek, T. (1998b). *Vectors and Matrices in Automation and Electrotechnics*, WNT, Warsaw, (in Polish).
- Kaczorek, T. (2001). *Positive 1D and 2D Systems*, Springer-Verlag, London.
- Kaczorek, T. (2011a). New stability tests of positive standard and fractional linear systems, *Circuits and Systems* **2**(4): 261–268.
- Kaczorek, T. (2011b). Positive linear systems consisting of n subsystems with different fractional orders, *IEEE Transactions on Circuits and Systems* **58**(6): 1203–1210.
- Kaczorek, T. (2012). *Selected Problems of Fractional Systems Theory*, Springer-Verlag, Berlin.
- Kaczorek, T. (2014a). Descriptor positive discrete-time and continuous-time nonlinear systems, *34 IEEE-SPIE Joint Symposium, Wilga, Poland*, DOI:10.1117/12.2074558.
- Kaczorek, T. (2014b). Positivity and linearization of a class of nonlinear discrete-time systems by state feedbacks, *Logistyka* (6): 5078–5083.
- Kaczorek, T. (2015a). Analysis of positive and stable fractional continuous-time linear systems by the use of Caputo–Fabrizio derivative, *12th International Conference on Mechatronic Systems and Materials (MSM'2016)*, Białystok, Poland.

- Kaczorek, T. (2015b). Positivity and stability of discrete-time nonlinear systems, *IEEE 2nd International Conference on Cybernetics (CYBCONF)*, Gdynia, Poland.
- Klamka, J. (2013). Controllability of dynamical systems: A survey, *Bulletin of the Polish Academy of Sciences: Technical Sciences* **61**(2): 221–229.
- Kucera, V. and Zagalak, P. (1988). Fundamental theorem of state feedback for singular systems, *Automatica* **24**(5): 653–658.
- Losada, J. and Nieto, J. (2015). Properties of a new fractional derivative without singular kernel, *Progress in Fractional Differentiation and Applications* **1**(2): 87–92.
- Ostalczyk, P. (2016). *Discrete Fractional Calculus*, World Scientific Publishing Company, River Edge, NJ.
- Van Dooren, P. (1979). The computation of Kronecker’s canonical form of a singular pencil, *Linear Algebra and Its Applications* **27**: 103–140.
- Virnik, E. (2008). Stability analysis of positive descriptor systems, *Linear Algebra and Its Applications* **429**(10): 2640–2659.
- Zhang, H., Xie, D., Zhang, H. and Wang, G. (2014a). Stability analysis for discrete-time switched systems with unstable subsystems by a mode-dependent average dwell time approach, *ISA Transactions* **53**(4): 1081–1086.
- Zhang, J., Han, Z., Wu, H. and Hung, J. (2014b). Robust stabilization of discrete-time positive switched systems with uncertainties and average dwell time switching, *Circuits, Systems and Signal Processing* **33**(1): 71–95.



Kamil Borawski received his M.Sc. degree (with honors) in electrical engineering from the Białystok University of Technology in 2015. Currently he is a Ph.D. student at the Faculty of Electrical Engineering of the same university. His main scientific interests are modern control theory, especially positive and fractional-order systems.

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Tadeusz Kaczorek received the M.Sc., Ph.D. and D.Sc. degrees in electrical engineering from the Warsaw University of Technology in 1956, 1962 and 1964, respectively. In the years 1968–1969 he was the dean of the Electrical Engineering Faculty, and in the period of 1970–1973 he was a deputy rector of the Warsaw University of Technology. In 1971 he became a professor and in 1974 a full professor at the same university. Since 2003 he has been a professor at the

Białystok University of Technology. In 1986 he was elected a corresponding member and in 1996 a full member of the Polish Academy of Sciences. In the years 1988–1991 he was the director of the Research Center of the Polish Academy of Sciences in Rome. In 2004 he was elected an honorary member of the Hungarian Academy of Sciences. He has been granted honorary doctorates by 13 universities. His research interests cover systems theory, especially singular multidimensional systems, positive multidimensional systems, singular positive 1D and 2D systems, as well as positive fractional 1D and 2D systems. He initiated research in the field of singular 2D, positive 2D and positive fractional linear systems. He has published 28 books (8 in English) and over 1100 scientific papers. He has also supervised 69 Ph.D. theses. He is the editor-in-chief of the *Bulletin of the Polish Academy of Sciences: Technical Sciences* and a member of editorial boards of ten international journals.