

## SWITCHING TIME ESTIMATION AND ACTIVE MODE RECOGNITION USING A DATA PROJECTION METHOD

ASSIA HAKEM <sup>a,\*</sup>, VINCENT COCQUEMPOT <sup>a</sup>, KOMI MIDZODZI PEKPE <sup>a</sup>

<sup>a</sup> CRIStAL—Research Center in Computer Science, Signals and Automatic Control of Lille  
UMR 9189, CNRS, University of Lille, F-59000 Lille, France  
e-mail: assia.hakem@gmail.com,  
{vincent.cocquempot, midzodzi.pekpe}@univ-lille1.fr

This paper proposes a data projection method (DPM) to detect a mode switching and recognize the current mode in a switching system. The main feature of this method is that the precise knowledge of the system model, i.e., the parameter values, is not needed. One direct application of this technique is fault detection and identification (FDI) when a fault produces a change in the system dynamics. Mode detection and recognition correspond to fault detection and identification, and switching time estimation to fault occurrence time estimation. The general principle of the DPM is to generate mode indicators, namely, residuals, using matrix projection techniques, where matrices are composed of input and output measured data. The DPM is presented in detail, and properties of switching detectability (fault detectability) and discernability between modes (fault identifiability) are characterized and discussed. The great advantage of this method, compared with other techniques in the literature, is that it does not need the model parameter values and thus can be applied to systems of the same type without identifying their parameters. This is particularly interesting in the design of generic embedded fault diagnosis algorithms.

**Keywords:** switching systems, mode recognition, fault detection and isolation, data-driven method, mode discernability, switching detectability, fault identifiability.

### 1. Introduction

Switching systems are characterized by the interaction between a finite state automaton and a finite number of dynamical subsystems called operating modes (Liberzon, 2005). These operating modes may be described by differential or difference equations (Lin and Antsaklis, 2009). The switching between modes could be governed by a logical switching rule called the switching law. It determines which mode is active at each time instant and is governed by (Lin and Antsaklis, 2009)

- internal features: system input, output, state variables or system parameters changing, etc.;
- external actions: human operators actions, environment conditions changing, etc.

Recently, switching systems have been the subject of intensive investigations. A motivation comes from the fact that they represent a large class of physical systems,

such as mechanical and chemical processes (Engell *et al.*, 2000), communication networks, aircraft and air traffic control systems (Livadas *et al.*, 2000), automotive systems (Antsaklis, 2000), robotics (Petroff, 2007), embedded systems (Zhang *et al.*, 2007), DC/DC converters (Ma *et al.*, 2004), oscillators (Torikai and Saito, 1998) or chaos generators (Mitsubori and Saito, 1997).

Another motivation for studying switching systems comes from the simplicity of representing complex non-linear systems using a set of simple structure models (linear time invariant subsystems, for example), where each operating zone is described by a mode (Goebel *et al.*, 2012).

Various problems for switching systems have been investigated, such as modeling (Heemels *et al.*, 2001), stability studies (Williams and Hoft, 1991), observability and controllability analysis (Daizhan, 2007), or fault detection and identification (FDI) (Akhenak *et al.*, 2008; Domlan *et al.*, 2007b), etc.

\*Corresponding author

All these studies show that, at each time instant, it is very important to know the exact active mode. Because it is not always possible to implement specialized sensors that indicate the active mode, estimation techniques have to be designed. The aims of such algorithms are to detect any switching and to accurately recognize the current mode. One direct use of such techniques is FDI.

The data projection method (DPM) is different from other methods proposed in the literature for switching detection, mode recognition and discernability characterization (Narasimhan and Biswas, 2007; Anderson *et al.*, 2001; Akhenak *et al.*, 2008; Domlan *et al.*, 2007a). The DPM is a data-driven method, which is guided by the structure of the model which has to be known. However, differently from the cited literature, the parameter values are not needed to apply the DPM. This makes this method evidently intrinsically robust to parameters values and easier to apply. In this paper, the discernability conditions between modes and the conditions of mode switching detectability are revisited. It is shown that these conditions are not equivalent: a transition between non-discernable modes can be detected in certain situations.

This original result was not obtained by Narasimhan and Biswas (2007), Anderson *et al.* (2001), Akhenak *et al.* (2008) or Domlan *et al.* (2007a). Early fault detection and identification (FDI) is crucial for human and system safety. Indeed, if a fault occurs in the system and is not detected, it may produce a severe damage in the system and in its environment. Even if the fault does not cause a severe damage in the system, it can weaken its dependability and performance. If the fault is detected, the control law can be adjusted in order to maintain the system performance (Yang *et al.*, 2010) or maintenance actions can be performed. It is thus of paramount importance to detect accurately and as fast as possible a fault, to localize precisely the faulty component and to characterize (identify) the fault.

Internal component faults modify the system dynamics (Cocquempot *et al.*, 2004). A way to deal with these faults is to consider faulty modes. Fault detection is thus equivalent to detect a faulty mode switching, and fault identification is equivalent to recognize the mode after switching.

Several approaches for switching detection and mode recognition have been studied in the literature (Narasimhan and Biswas, 2007; Anderson *et al.*, 2001), including model-based methods and data-driven ones. One difficulty in model-based methods (Narasimhan and Biswas, 2007; Anderson *et al.*, 2001) is accurate estimation of system parameters in each mode.

Moreover, even for the same kind of physical systems, the parameter values are not exactly the same and may slightly change in the system's life, which leads to model uncertainties. To cope with this problem,

robust methods are proposed in the literature. These methods are based on observers (Belkhiat, 2011), Kalman filters (Akhenak *et al.*, 2008), and analytical redundancy relations (ARRs) (Bayouhd and Travé Massuyès, 2014; El Mezyani, 2005). However, these methods are limited to given classes of uncertainties.

The DPM, which is considered in this paper, uses the collected data and the knowledge of the mode class. However, it does not need the values of the model parameters. Previous publications have introduced the DPM for linear systems (Pekpe *et al.*, 2006) to detect and to isolate sensor faults. A residual is generated by projecting the system output matrix onto the kernel of an input Hankel matrix. The proposed residual is calculated using only on-line input-output data.

The DPM is extended in this paper for switching detection and mode recognition. The condition for mode recognition is the discernability between modes. Indeed, if two modes are not discernable, it is not possible to determine which one is active. Finding the conditions for discernability between modes has been the subject of intensive studies, and several results have been reported in the literature (see, e.g., Cocquempot *et al.*, 2004; Bayouhd and Travé Massuyès, 2014).

Discernability and switching detectability will be characterized using the DPM. It is shown that these two conditions are not equivalent; in other words, a switching between two non discernable modes could be the following detected.

The main contributions of this paper are

- a method, called the DPM, to estimate the switching time by using only on-line collected measured data. This method will be extended to recognize the active mode by using on-line collected data and a database of inputs and outputs collected off-line;
- a characterization of several properties, such as discernability between modes and switching detectability.

The rest of the paper is organized as follows. In Section 2, the switching system with linear modes is described. In Section 3, the data projection method (DPM) is detailed and used for switching time estimation. In Section 4, the DPM is extended for active mode recognition. In Section 5, DPM tuning is explained. Finally, two illustrative examples are presented to show the efficiency of the proposed method.

## 2. Problem setting

Consider the dynamic switching system with linear discrete-time modes described by

$$\begin{cases} x_{k+1} = A_{\sigma_k} x_k + B_{\sigma_k} u_k, \\ y_k = C_{\sigma_k} x_k + D_{\sigma_k} u_k + w_k, \\ (x_0, \sigma_0) \in \text{Init}, \end{cases} \quad (1)$$

where  $A_{\sigma_k} \in \mathbb{R}^{n \times n}$ ,  $B_{\sigma_k} \in \mathbb{R}^{n \times m}$ ,  $C_{\sigma_k} \in \mathbb{R}^{\ell \times n}$ ,  $D_{\sigma_k} \in \mathbb{R}^{\ell \times m}$  are constant matrices, and vectors  $u_k \in \mathbb{R}^m$ ,  $x_k \in \mathbb{R}^n$  and  $y_k \in \mathbb{R}^\ell$  are respectively input, state and output signals at time-instant  $kT_e$ , with  $T_e$  being the sampling period. The system outputs are affected by a centered Gaussian noise  $w_k \in \mathbb{R}^\ell$ , where  $\text{var}(w_k^s)$  represents the variance of  $w_k^s$ ,  $w_k = (w_k^1 \ \dots \ w_k^\ell)^T$ .  $\sigma_k \in \{1, 2, \dots, Q\}$  is the mode index and 'Init' is the set of initial states  $(x_0, \sigma_0)$ ,  $Q$  being the number of modes.

**2.1. Definition, objectives and hypotheses.** Given the switching system described by Eqn. (1), the objectives of this paper are to estimate the switching time and to recognize the current mode under the following hypotheses:

1. Matrices  $A_{\sigma_k}$ ,  $B_{\sigma_k}$ ,  $C_{\sigma_k}$ ,  $D_{\sigma_k}$  are all *unknown*.
2. State matrices  $A_{\sigma_k}$  are stable for all  $\sigma_k$ .
3.  $u_k$  and  $y_k$  are known for all values of  $k$ .
4. The time period between two successive switchings is long enough to allow mode identification, i.e., the system has a dwell-time (Hespanha and Morse, 1999). This condition will be precisely characterized later when the method is detailed.

**Definition 1.** Two modes  $(m_1, m_2)$  are *discernible* for all inputs in a time interval  $[0, T]$  ( $T$  is a positive integer), if for all initial states and for the same input applied in modes  $m_1$  and  $m_2$ , the outputs in the two modes  $m_1$  and  $m_2$  are different.

**2.2. Data projection method for sensor FDI.** For simplicity, let us consider first the dynamic linear system described by

$$\begin{cases} x_{k+1} = Ax_k + Bu_k, \\ y_k = Cx_k + Du_k + w_k + f_k, \end{cases} \quad (2)$$

where  $f_k$  represents the vector of sensor faults at time  $k$ .

The DPM framework is described here for one mode (see the work of Pekpe *et al.* (2006) for more details):

1. By stacking Eqn. (2) on a time window of size  $L$  ( $L \in \mathbb{N}$ ), we obtain

$$Y_{k-L+1:k} = CA^i X_{k-L-i+1:k-i} + H_i U_{k-L+1:k} + F_{k-L+1:k} + W_{k-L+1:k}, \quad (3)$$

with

$$\begin{aligned} H_i &= [ CA^{i-1}B \mid \dots \mid CB \mid D ], \\ U_{k-L+1:k} &= [ \bar{u}_{k-L+1, i} \quad \bar{u}_{k-L+2, i} \\ &\quad \dots \quad \bar{u}_{k, i} ], \end{aligned} \quad (4)$$

where

$$\begin{aligned} \bar{u}_{k, i} &= ( u_{k-i}^T \quad u_{k-i+1}^T \quad \dots \quad u_k^T )^T \\ &\in \mathbb{R}^{m(i+1) \times 1} \end{aligned} \quad (5)$$

and

$$\begin{aligned} Y_{k-L+1:k} &= [ y_{k-L+1} \quad \dots \quad y_{k-1} \quad y_k ] \\ &\in \mathbb{R}^{\ell \times L}. \end{aligned} \quad (6)$$

Matrices  $X_{k-L-i+1:k-i}$  and  $W_{k-L+1:k}$  are defined as  $Y_{k-L+1:k}$  (using  $x_{k-i}$  (resp.  $w_k$ ) instead of  $y_k$ ). Matrix  $F_{k-L+1:k}$  is also defined as  $Y_{k-L+1:k}$  using  $f_k$  instead of  $y_k$ , where  $f_k$  represents a sensor fault at time instant  $k$ . Moreover,

- the term  $CA^i X_{k-L-i+1:k-i}$  depends on model parameters and a set of states in a time window of size  $L$ ;
  - the term  $H_i U_{k-L+1:k}$  depends on the inputs  $U_{k-L+1:k}$  and  $H_i$  on model parameters;
  - the term  $F_{k-L+1:k}$  depends on additive sensor faults;
  - the term  $W_{k-L+1:k}$  depends on measurement noise.
2. A judicious choice of the time window (integers  $i$  and  $L$ ):
    - owing to the stability of matrix  $A$ , the term  $CA^i X_{k-L-i+1:k-i}$  becomes very small if  $i$  is large enough;
    - there exist projection matrices  $\Pi$ , such that  $U_{k-L+1:k} \Pi = 0$ ; these matrices project onto the right orthogonal space of the matrix  $U_{k-L+1:k}$  and one of them is  $\Pi_{U_{k-L+1:k}}$ :

$$\begin{aligned} \Pi_{U_{k-L+1:k}} &= I_L - U_{k-L+1:k} U_{k-L+1:k}^+ \\ &\quad \times (U_{k-L+1:k} U_{k-L+1:k}^+)^+ \bar{u}_{k, i} \end{aligned} \quad (7)$$

where  $I_L$  is an identity matrix of size  $L$  and  $M^+$  represents the Moore–Penrose pseudo-inverse of matrix  $M$ .

3. Post-multiplying (3) by  $\Pi_{U_{k-L+1:k}}$  gives the residual

$$\begin{aligned} \epsilon_k &= Y_{k-L+1:k} \Pi_{U_{k-L+1:k}} \\ &= \Delta_i + F_{k-L+1:k} \Pi_{U_{k-L+1:k}} \\ &\quad + W_{k-L+1:k} \Pi_{U_{k-L+1:k}}. \end{aligned} \quad (8)$$

Since the state matrix  $A$  is stable, the term

$$\Delta_i = CA^i X_{k-L-i+1:k-i}$$

can be neglected if  $i$  is large enough (see Pekpe et al., 2006).

4. Finally, a statistical test on  $\epsilon_k$  is used for fault detection.

### 3. Switching time estimation

**3.1. Residual generation.** In this section, the DPM is proposed for switching time estimation. It is proved that a switching can be detected under a detectability condition which will be given in Section 3.2).

We assume that integers  $i$  and  $L$  satisfy the condition  $L > m(i + 1)$  and  $L$  is an even integer such that  $\dim(\ker(U_{k-L+1:k})) > 1, \forall u_k (k \in N)$ , where  $\ker(U_{k-L+1:k})$  represents the kernel of matrix  $U_{k-L+1:k}$ .

**Theorem 1.** *Let  $i$  and  $L$  be two integers and suppose all the inputs are not null and no change occurs in time window  $[k - L - i + 1, k]$ . Then there exist  $i_s \in \mathbb{N}$  such that  $\forall i \geq i_s$  and  $L > m(i + 1)$ , and the vector  $\epsilon_k$  defined by*

$$\epsilon_k = Y_{k-L+1:k} \Pi_{U_{k-L+1:k}} \in \mathbb{R}^\ell \quad (9)$$

is a centered Gaussian noise with variance  $R_\epsilon$  ( $\epsilon_k \sim N(0, R_\epsilon)$ ), where

$$R_\epsilon = E[W_{k-L+1:k} \Pi_{U_{k-L+1:k}} \Pi_{U_{k-L+1:k}}^T W_{k-L+1:k}^T]. \quad (10)$$

*Proof.* Equation (8) applied to (1) gives

$$\epsilon_k = \Delta_i + W_{k-L+1:k} \Pi_{U_{k-L+1:k}}. \quad (11)$$

Since  $W_{k-L+1:k}$  is a zero mean Gaussian noise and the inputs are deterministic,  $W_{k-L+1:k} \Pi_{U_{k-L+1:k}}$  is also a zero mean Gaussian noise.

Consider a multiplicative norm  $\|\Delta_i\|$  of  $\Delta_i$ . We have

$$\begin{aligned} \|\Delta_i\| &= \|CA^i X_{k-L-i+1:k-i}\| \\ &\leq \|C\| \|A\|^i \|X_{k-L-i+1:k-i}\|. \end{aligned} \quad (12)$$

Since all the modes are stable, the state norm is bounded. Let  $\|X_m\|$  be the maximum of  $\|X_{k-L-i+1:k-i}\|, k \in \mathbb{N}$ . Then, the following inequality holds:

$$\|\Delta_i\| \leq \|C\| \|A\|^i \|X_m\|. \quad (13)$$

Therefore, if we choose  $i_s$  such that

$$i_s \geq \frac{\log(V_m) - \log(N \|C\| \|X_m\|)}{\log(\|A\|)}, \quad (14)$$

where  $V_m$  is the minimum of  $\text{var}(w_k^s), s \in 1, 2, \dots, \ell$ , and  $N$  is an integer which is supposed to be sufficiently large. If the inequality (14) holds, then  $\forall i > i_s$ ,

$$\|\Delta_i\| \leq \left\| \frac{V_m}{N} \right\|. \quad (15)$$

If  $N$  is a large integer, then the influence of  $\Delta_i$  is negligible before the noise and  $\epsilon_k$  is Gaussian zero mean.

Finally, the variance of  $\epsilon_k$  is

$$R_\epsilon = E[W_{k-L+1:k} \Pi_{U_{k-L+1:k}} \Pi_{U_{k-L+1:k}}^T W_{k-L+1:k}^T]. \quad (16)$$

**3.2. Residual analysis.** Introduce matrices

$$\begin{aligned} \Omega &= [\mathcal{H}_{(\sigma_\tau, \sigma_{\tau+1}), i, 1} - H_{(\sigma_\tau), i} | \\ &\quad \mathcal{H}_{(\sigma_\tau, \sigma_{\tau+1}), i, 2} - H_{(\sigma_\tau), i} | \dots | \\ &\quad \mathcal{H}_{(\sigma_\tau, \sigma_{\tau+1}), i, i-1} - H_{(\sigma_\tau), i} | \\ &\quad \mathcal{H}_{(\sigma_\tau, \sigma_{\tau+1}), i, i} - H_{(\sigma_\tau), i} | [0_{mi(i+1) \times \tau + L - k - 1} | \\ &\quad \check{U} | 0_{mi(i+1) \times k - \tau - i + 1} ] + (H_{(\sigma_{\tau+1}), i} \\ &\quad - H_{(\sigma_\tau), i}) [0_{m(i+1) \times L - k + \tau + i - 1} | U_{\tau+i:k} ]. \end{aligned} \quad (17)$$

$$H_{(\sigma_k), i} = [C_{\sigma_k} A_{\sigma_k}^{i-1} B_{\sigma_k} | C_{\sigma_k} A_{\sigma_k}^{i-2} B_{\sigma_k} | \dots | C_{\sigma_k} B_{\sigma_k} | D_{\sigma_k}], \quad (18)$$

where

$$\begin{aligned} \mathcal{H}_{(\sigma_\tau, \sigma_{\tau+1}), i, t-\tau+1} &= [H_{(\sigma_\tau, \sigma_{\tau+1}), i, t-\tau+1} | H_{(\sigma_{\tau+1}), t-\tau+1}] \\ &\in \mathbb{R}^{\ell \times m(i+1)} \end{aligned}$$

and  $H_{(\sigma_\tau, \sigma_{\tau+1}), i, t-\tau+1}$  is constructed as follows:

$$\begin{aligned} H_{(\sigma_\tau, \sigma_{\tau+1}), i, t-\tau+1} &= C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^{t-\tau+1} [A_{\sigma_\tau}^{i-t+\tau-2} B_{\sigma_\tau} | \\ &\quad A_{\sigma_\tau}^{i-t+\tau-3} B_{\sigma_\tau} | \dots | A_{\sigma_\tau} B_{\sigma_\tau} | B_{\sigma_\tau}] \\ &\in \mathbb{R}^{\ell \times m(i-t+\tau-1)}. \end{aligned} \quad (19)$$

A condition of switching detectability (internal fault detectability) is given by the following theorem and proposition.

**Theorem 2.** *If a switching occurs on time interval  $[k - L - i + 1, k]$  and the inputs are not identically zero, then for all  $i \in \mathbb{N}$  and  $i > i_s$  ( $i_s$  given by Theorem 1), vector  $\epsilon_k$  is not a centered Gaussian noise with variance  $R_\epsilon$  if and only if*

$$\text{span}(\Omega) \not\subset \text{span}(U_{k-L+1:k}), \quad (20)$$

where  $\text{span}(M)$  denotes the row space of matrix  $M$  and the condition (20) implies

1.  $H_{\sigma_\tau, i} \neq H_{\sigma_{\tau+1}, i}$  or
2.  $H_{\sigma_\tau, i} = H_{\sigma_{\tau+1}, i}$  and  $\exists r \in \{1, 2, \dots, i\}$  :  
 $\mathcal{H}_{(\sigma_\tau, \sigma_{\tau+1}), i, r} \neq H_{(\sigma_\tau), i}$ .

The proof of Theorem 2 is provided in Appendix.

We specify now two particular cases which may respect the previous detectability condition (20):

1. Case of  $H_{\sigma_\tau, i} \neq H_{\sigma_{\tau+1}, i}$ . This condition implies that modes  $\sigma_\tau$  and  $\sigma_{\tau+1}$  have different Markov parameters (their minimal realizations are different); in other words, these two modes are *discernible* as defined by Cocquempot *et al.* (2004), Hofbaur *et al.* (2010) or Bayouhd and Travé Massuyès (2014):

The first condition is sufficient for switching detectability but not necessary. This means that the proposed residual can detect a switching between discernible modes. In addition to that, it can detect switching between non-discernible modes under the second condition (so discernibility between modes is not necessary for switching detectability).

2. The second condition consists of two sub-conditions:

- non-discernible modes: if  $H_{\sigma_\tau, i} = H_{\sigma_{\tau+1}, i}$ , then there exists a non-singular matrix  $\Psi \in \mathbb{R}^{n \times n}$  satisfying
 
$$\begin{cases} A_{\sigma_{\tau+1}} = \Psi^{-1} A_{\sigma_\tau} \Psi, \\ B_{\sigma_{\tau+1}} = \Psi^{-1} B_{\sigma_\tau}, \\ C_{\sigma_{\tau+1}} = C_{\sigma_\tau} \Psi, \\ D_{\sigma_{\tau+1}} = D_{\sigma_\tau}, \end{cases}$$
- and there is an index  $r \in \{1, 2, \dots, i\}$  such that if  $\mathcal{H}_{(\sigma_\tau, \sigma_{\tau+1}), i, r} \neq H_{(\sigma_\tau), i}$ , then  $\Psi$  is not an identity matrix.

In other words, this condition can be derived by substituting  $A_{\sigma_{\tau+1}}$ ,  $B_{\sigma_{\tau+1}}$ ,  $C_{\sigma_{\tau+1}}$  and  $D_{\sigma_{\tau+1}}$  in Eqn. (19).

The *second condition* expresses that the transition between two non-discernible modes is transiently detectable (there exists  $r \in \{1, 2, \dots, i\}$  such that  $\mathcal{H}_{(\sigma_\tau, \sigma_{\tau+1}), i, r} \neq H_{(\sigma_\tau), i}$ ). This is due to the modification of the state by matrix  $\Psi$  after a switching.

The following proposition provides a statistical tool for switching detection.

**Proposition 1.** *If a switching occurs on time interval  $[k - L - i + 1, k]$ , the inputs are not identically zero and for all  $i \in \mathbb{N}$  and  $i > i_s$  ( $i_s$  given by Theorem 1) such that the condition (20) is satisfied. Then*

$$\varpi \geq \chi_{L, \alpha}^2, \quad (21)$$

where  $\chi_{L, \alpha}$  is the critical value with significance level  $\alpha$  and

$$\varpi = \epsilon_k^T R_{\epsilon}^{-1} \epsilon_k, \quad (22)$$

otherwise

$$\varpi < \chi_{L, \alpha}^2. \quad (23)$$

## 4. Current mode recognition

In this section, the DPM is extended to recognize the current mode and the *discernibility condition* is derived.

### 4.1. Residual generation using on-line and off-line data.

Let us consider two input matrices

$$U_{k-\frac{L}{2}+1:k} \in \mathbb{R}^{m(i+1) \times \frac{L}{2}}$$

and

$$U_{(\gamma), 1:\frac{L}{2}}^* \in \mathbb{R}^{m(i+1) \times \frac{L}{2}}$$

which are constructed using respectively inputs collected on-line in the current mode (to be identified) and collected off-line in mode  $\gamma$ .

It should be noted that the inputs collected off-line are persistently exciting (see Van Overschee and De Moor, 1996), and vary to excite all modes in each operating mode, which implies that the matrix  $U_{(\gamma), 1:\frac{L}{2}}^*$  is of full row rank. Unlike for inputs collected on-line, the persistence condition is no longer indispensable.

Let us consider the two output matrices  $Y_{k-\frac{L}{2}+1:k}$  and  $Y_{(\gamma), 1:\frac{L}{2}}^*$  constructed using respectively outputs collected on-line in the current mode (to be identified) and collected off-line in mode  $\gamma$ .

Let us construct the input and output matrices as follows:

$$U_{(\gamma, \sigma_k), k-\frac{L}{2}+1:k} = \begin{bmatrix} U_{(\gamma), 1:\frac{L}{2}}^* & U_{k-\frac{L}{2}+1:k} \end{bmatrix}, \quad (24)$$

$$Y_{(\gamma, \sigma_k), k-\frac{L}{2}+1:k} = \begin{bmatrix} Y_{(\gamma), 1:\frac{L}{2}}^* & Y_{k-\frac{L}{2}+1:k} \end{bmatrix}.$$

The following theorem presents the residual generator for the current mode recognition.

**Theorem 3.** *If no change occurs in time interval  $[k - L/2 - i + 1, k]$  and the inputs are not identically zero, then for all  $i \in \mathbb{N}$  and  $i > i_s$  ( $i_s$  given by Theorem 1) the residual  $\bar{e}_{(\gamma), k}$  defined by*

$$\bar{e}_{(\gamma), k} = Y_{(\gamma, \sigma_k), k-\frac{L}{2}+1:k} \Pi_{U_{(\gamma, \sigma_k), k-\frac{L}{2}+1:k}} \quad (25)$$

has the following evaluation form:

$$\begin{aligned} \bar{e}_{(\gamma), k} &= \delta_k^i + (H_{(\sigma_k), i} - H_{(\gamma), i}) \begin{bmatrix} 0_{m(i+1) \times \frac{L}{2}} \\ U_{k-\frac{L}{2}+1:k} \end{bmatrix} \Pi_{U_{(\gamma, \sigma_k), k-\frac{L}{2}+1:k}} \\ &+ W_k \Pi_{U_{(\gamma, \sigma_k), k-\frac{L}{2}+1:k}} \end{aligned} \quad (26)$$

where

$$\tilde{\delta}_{(\sigma_k), k-L+1-i:k-i}^i = C_{\sigma_k} A_{\sigma_k}^i X_k$$

and

$$\delta_k^i = \left[ \tilde{\delta}_{(\gamma), 1:\frac{L}{2}}^i \mid \tilde{\delta}_{(\sigma_k), k-\frac{L}{2}+1-i:k-i}^i \right] \Pi_{U_{(\gamma, \sigma_k), k-\frac{L}{2}+1:k}}$$

If  $\gamma = \sigma$  ( $\sigma_k$  is denoted by  $\sigma$  since it does not change on  $[k - L/2 - i + 1, k]$ ), then

$$\bar{\epsilon}_{(\gamma), k} = \delta_k^i + W_k \Pi_{U_{(\gamma, \sigma_k), k-\frac{L}{2}+1:k}} \quad (27)$$

and  $\forall \eta > 0, \exists i_0 \in \mathbb{N}$  such that  $\forall i > i_0 (i \in \mathbb{N}): |\delta_k^i| < \eta$ .

The proof of Theorem 3 is provided in Appendix. Using Theorem 3, one can recognize the active mode using Proposition 1.

**4.2. Residual analysis.** A condition for mode discernibility (internal faults identifiability) is given in the following theorem.

**Theorem 4.** *If no change occurs in time interval  $[k - L/2 - i + 1, k]$  and the inputs are not identically null, then for all  $i \in \mathbb{N}$  and  $i > i_s$  ( $i_s$  given by Theorem 1), a necessary and sufficient condition for mode discernibility is*

$$\text{span} \left( (H_{(\sigma_k), i} - H_{(\gamma), i}) \left[ 0_{m(i+1) \times \frac{L}{2}} \mid U_{k-\frac{L}{2}+1:k} \right] \right) \not\subseteq \text{span}(U_{(\gamma, \sigma_k), k-\frac{L}{2}+1:k}). \quad (28)$$

The proof of Theorem 4 is provided in Appendix.

### 5. DPM tuning

Consider the linear dynamic system described by

$$\begin{cases} x_{k+1} = Ax_k + Bu_k^*, \\ y_k^* = Cx_k + Du_k^* + w_k. \end{cases} \quad (29)$$

The time window size  $i$  in the DPM allows neglecting the past state influence. In order to find a good trade-off between low sensitivity to neglected terms and a reasonable complexity of on-line computation, this integer is determined using a criterion  $J(p)$  which minimizes the approximation error between the reference model of the system and the implicit model used in the projection. The integer  $i$  has to be determined in a preliminary phase executed off-line in the healthy case using a persistently excited input.

Let us define

$$\bar{u}_{k,p}^* = \begin{pmatrix} u_{k-p}^{*,T} & u_{k-p+1}^{*,T} & \dots & u_k^{*,T} \end{pmatrix}^T \in \mathbb{R}^{m(p+1) \times 1},$$

the input matrix

$$U_k^* = \begin{bmatrix} \bar{u}_{k-L+1,p}^* & \bar{u}_{k-L+2,p}^* & \dots & \bar{u}_{k,p}^* \end{bmatrix} \in \mathbb{R}^{m(p+1) \times L},$$

the output matrix

$$Y_k^* = \begin{bmatrix} y_{k-L+1}^* & \dots & y_{k-1}^* & y_k^* \end{bmatrix} \in \mathbb{R}^{\ell \times L}$$

and the state matrix

$$X_{k-L-i+1:k-i} = \begin{bmatrix} x_{k-i-L+1} & \dots & x_{k-i-1} & x_{k-i} \end{bmatrix} \in \mathbb{R}^{n \times L}.$$

**Theorem 5.** *Let  $r, p$  and  $L$  be three integers such that  $L > m(p + 1)$  and  $J(p)$  is defined by (with  $\|\cdot\|_2$ , the 2-norm)*

$$J(p) = \frac{1}{r} \sum_{k=p+1}^r \|Y_k^* \Pi_{U_k^*}\|_2^2. \quad (30)$$

Let  $\mathcal{X}_1$  be a positive real. There exists  $p_0$  such that  $\forall p > p_0$ , and the criterion  $J(p)$  defined by Eqn. (30) satisfies the following inequality:

$$J(p) \leq \text{var}(W_k) + \mathcal{X}_1. \quad (31)$$

*Proof.* From the residual expressions (9), the criterion  $J(p)$  given by (30) can be written as

$$J(p) = \frac{1}{r} \sum_{k=p+1}^r \|W_k \Pi_{U_k^*} + CA^p X_{k-L-i+1:k-i} \Pi_{U_k^*}\|_2^2. \quad (32)$$

From the 2-norm properties and from a certain rank  $p_0$ , we have

$$J(p) \leq \frac{1}{r} \sum_{k=p+1}^r \|W_k \Pi_{U_k^*}\|_2^2 + \frac{1}{r} \sum_{k=p+1}^r \|CA^p X_{k-L-p+1:k-p} \Pi_{U_k^*}\|_2^2, \quad (33)$$

where

$$\begin{aligned} \|W_k \Pi_{U_k^*}\|_2 &\leq \|W_k\|_2 \underbrace{\|\Pi_{U_k^*}\|_2}_{=1} \\ &\Rightarrow \|W_k \Pi_{U_k^*}\|_2 \leq \|W_k\|_2. \end{aligned} \quad (34)$$

The inequality (33) becomes

$$J(p) \leq \frac{1}{r} \sum_{k=p+1}^r \|W_k\|_2^2 + \mathcal{X}, \quad (35)$$

where

$$\mathcal{X} = \frac{1}{r} \sum_{k=p+1}^r \|CA^p X_{k-L-i+1:k-i}\|_2^2.$$

Since

$$\text{var}(W_k) = \lim_{r \rightarrow \infty} \frac{1}{r} \sum_{k=p+1}^r \|W_k\|_2^2,$$

under the stability hypothesis and using the same method as in the proof of Theorem 1, one can have (31).

**Remark 1.** The integer  $i$  should be chosen in the interval  $[p_0, p_x[$  ( $p_x \in \mathbb{N}$ ), where  $p_x$  is the maximum value with acceptable computational complexity and  $p_0$  the minimum integer which makes the criterion acceptable. ■

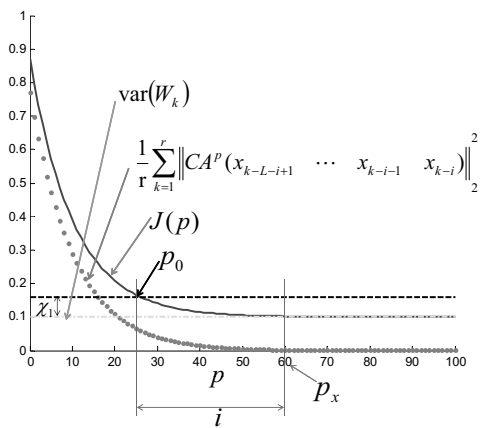


Fig. 1. Illustration of the criterion.

### 6. Illustrative examples

Consider a system with three operating modes ( $\sigma_k \in \{1, 2, 3\}$ ), where Mode 1 is a normal operating mode and the other two modes represent faulty modes resulting from two internal faults (Fig. 2). Output measurements are affected by Gaussian white noise with zero mean and variance  $\text{var}(w_k) = 0.15$ .

The switching sequence is given by Table 1.

Table 1. Simulated switching sequence.

$k \in$	$[0, 1500[$	$[1500, 2500[$	$[2500, 4000[$
mode number	1	3	2

The 3 modes are stable. Numerical values of the parameters in these modes are given below. These parameters are used to simulate the system output, but they are not used to compute the residuals.

Two examples are described below. In the first one, all modes are discernible, while in the second, Modes 2 and 3 are not discernible.

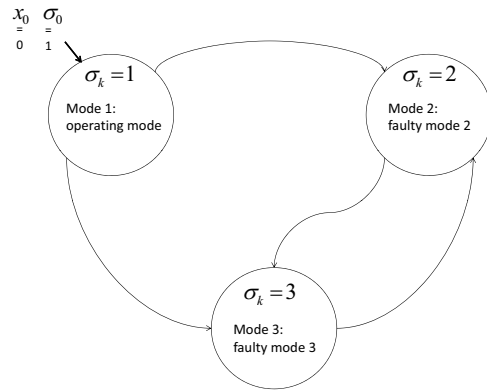


Fig. 2. Switching system.

**6.1. Example 1 with all discernible modes.** The parameters of Mode 1 are given by

$$A_1 = \begin{bmatrix} -0.7 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \\ 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}.$$

The parameters of Mode 2 are given by

$$A_2 = \begin{bmatrix} -0.5 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.3 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.1 & 0.1 \\ 1.2 & 1.2 \\ 0.5 & 0.5 \\ 0.8 & 0.8 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.3 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.3 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.3 & 0.3 \\ 0.3 & 0.3 \\ 0.3 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}.$$

The parameters of Mode 3 are given by

$$\begin{aligned}
 A_3 &= \begin{bmatrix} -0.4 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.6 \end{bmatrix}, \\
 B_3 &= \begin{bmatrix} 0.3 & 0.3 \\ 0.1 & 0.1 \\ 0.7 & 0.7 \\ 0.9 & 0.9 \end{bmatrix}, \\
 C_3 &= \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}, \\
 D_3 &= \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \\ 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}.
 \end{aligned}$$

The three modes are discernible since they do not have the same Markov parameters.

**DPM tuning.** The first step of DMP is to tune parameters  $i$  and  $L$ .

The integers  $i = 15$  and  $L = 68$  are chosen to calculate the proposed residual. This choice is based on the proposed criterion detailed in Section 5.

**Input and output generation.** Figures 3 and 4 represent the system inputs and outputs, which are the only data used for residual computation to estimate the switching time and to recognize the active mode on-line.

**Switching time estimation.** As shown in Fig. 5, the switching times  $\tau = 1500$  and  $\tau = 2500$  are well estimated by the proposed residuals. All residual components are sensitive to all switching times.

**Active mode recognition.** The residual  $\bar{\epsilon}_{(1), k}$  is calculated using input-output data collected on-line and off-line in Mode 1. The residual components allow Mode 1 recognition during interval  $[0, 1500]$ , as is shown in Fig. 6.

The residual  $\bar{\epsilon}_{(2), k}$  is calculated using input-output data collected on-line and off-line in Mode 2. The residual components allow Mode 2 recognition during interval  $[2501, 4000]$ , as shown in Fig. 7.

The residual  $\bar{\epsilon}_{(3), k}$  is calculated using input-output data collected on-line and off-line in Mode 3. The residual components allow Mode 3 recognition during interval  $[1501, 2500]$ , as shown in Fig. 8.

**6.2. Example 2 with non-discernible modes.** The parameters of Mode 1 are

$$\begin{aligned}
 A_1 &= \begin{bmatrix} 1.0792 & 1.9072 & 0.9395 & 0.5389 \\ -0.1542 & -0.3322 & -0.2895 & -0.2139 \\ -0.3538 & -0.7776 & -0.3280 & -0.2391 \\ -0.6090 & -1.4554 & -0.8149 & -0.1190 \end{bmatrix}, \\
 B_1 &= \begin{bmatrix} -0.1152 & -0.1152 \\ 0.1152 & 0.1152 \\ 0.0419 & 0.0419 \\ 0.1859 & 0.1859 \end{bmatrix}, \\
 C_1 &= \begin{bmatrix} 0.1 & 0.3 & 0.2 & 0.1 \\ 0.5 & 0.7 & 0.2 & 0.1 \\ 0.3 & 0 & -0.2 & 0.5 \\ -0.2 & 0.4 & -0.9 & 0.1 \end{bmatrix}, \\
 D_1 &= \begin{bmatrix} 0.7 & 0.7 \\ 1.5 & 1.5 \\ 0.6 & 0.6 \\ -0.6 & -0.6 \end{bmatrix}.
 \end{aligned}$$

The parameters of Mode 2 are

$$\begin{aligned}
 A_2 &= \begin{bmatrix} 0.5669 & 0.9342 & 0.6730 & 0.3711 \\ -0.1919 & -0.2092 & -0.3230 & -0.1961 \\ -0.1630 & -0.5192 & -0.0584 & -0.1918 \\ -0.1653 & -0.7682 & -0.5872 & 0.1007 \end{bmatrix}, \\
 B_2 &= \begin{bmatrix} 0.1054 & 0.1054 \\ 0.0321 & 0.0321 \\ -0.0577 & -0.0577 \\ -0.0363 & -0.0363 \end{bmatrix}, \\
 C_2 &= \begin{bmatrix} 0.3 & 0.9 & 0.6 & 0.3 \\ 1.5 & 2.1 & 0.6 & 0.3 \\ 0.9 & 0 & -0.6 & 1.5 \\ -0.6 & 1.2 & -2.7 & 0.3 \end{bmatrix}, \\
 D_2 &= \begin{bmatrix} 0.7 & 0.7 \\ 1.5 & 1.5 \\ 0.6 & 0.6 \\ -0.6 & -0.6 \end{bmatrix}.
 \end{aligned}$$

The parameters of Mode 3 are

$$\begin{aligned}
 A_3 &= \begin{bmatrix} -3.1682 & -5.0342 & 1.2697 & -2.7353 \\ 1.0859 & 1.8972 & -0.4467 & 0.8729 \\ 2.0977 & 2.9268 & -0.3946 & 1.6441 \\ 2.2298 & 3.2131 & -0.8241 & 2.0657 \end{bmatrix}, \\
 B_3 &= \begin{bmatrix} -0.0418 & -0.0418 \\ 0.0234 & 0.0234 \\ 0.0264 & 0.0264 \\ 0.0241 & 0.0241 \end{bmatrix},
 \end{aligned}$$



$$C_3 = \begin{bmatrix} 6 & 8.4 & -1.5 & 4.5 \\ 13.2 & 20.4 & 3.3 & 6.9 \\ -3.9 & 8.7 & -10.5 & -0.6 \\ -3.3 & 7.8 & 3.9 & -12.6 \end{bmatrix},$$

$$D_3 = \begin{bmatrix} 0.7 & 0.7 \\ 1.5 & 1.5 \\ 0.6 & 0.6 \\ -0.6 & -0.6 \end{bmatrix}.$$

The matrix  $\Psi$  such that

$$\begin{cases} A_3 = \Psi^{-1}A_2\Psi, \\ B_3 = \Psi^{-1}B_2, \\ C_3 = C_2\Psi, \\ D_3 = D_2 \end{cases}$$

is

$$\Psi = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 5 & 7 & 2 & 1 \\ 3 & 0 & -2 & 5 \\ -2 & 4 & -9 & 1 \end{bmatrix}.$$

Modes 2 and 3 are not discernible (i.e., they have the same Markov parameters  $H_2 = H_3$ ) and  $\exists r \in \{1, 2, \dots, i\} : \mathcal{H}_{(2,3),i,r} \neq H_{(2),i}$ .

**DPM tuning.** The two integers  $i = 20$  and  $L = 88$  are chosen for the three modes by calculating the criterion given by (30).

**Input and output generation.** Figures 9 and 10 represent the system inputs and outputs, which are the only data used for residual computation to estimate switching times and to recognize the active mode on-line.

#### Estimation of switching times.

- Switching between discernible modes: A switching at time instant  $\tau = 1500$  between Modes 1 and 3 is well detected as shown in Fig. 11.
- Switching between non discernible modes: A switching at time instant  $\tau = 2500$  between Modes 3 and 2 is also well detected despite the fact that these modes are not discernible,  $H_2 = H_3$  this means that  $\exists r \in \{1, 2, \dots, i\} : \mathcal{H}_{(2,3),i,r} \neq H_{(2),i}$ .

The switching times  $\tau = 1500$  and  $\tau = 2500$  have been well estimated by the proposed residual, as shown in Fig. 11. All residuals are sensitive to these switchings.

**Active mode recognition.** The bank of residuals  $\bar{e}_{(1),k}$ ,  $\bar{e}_{(2),k}$  and  $\bar{e}_{(3),k}$  (cf. Figs. 12–14) shows the following:

- From Fig. 12: Mode 1 is active during the time interval  $[0, 1500]$ .

- From Figs. 13 and 14: Modes 2 and 3 are not discernible and one of them is active. The exact active mode cannot be determined, but one can just conclude that Mode 2 or 3 is active during the time interval  $[1501, 2500]$ . This non-discernibility is due to the fact that the condition of discernibility of Theorem 4 is not satisfied.
- Since the switching occurs during these two modes' activity at time instant  $\tau = 2500$ : if Mode 2 (resp. 3) is active during the time interval  $[1501, 2500]$ , then Mode 3 (resp. 2) is active during the time interval  $[2501, 4000]$ .

The switching at time instant  $\tau = 2500$  occurs between Modes 2 and 3, which are not discernible. Despite the fact that these modes are not discernible the switching at time instant  $\tau = 2500$  is well detected in Fig. 11 and also in Figs. 13 and 14. Indeed, the detectability condition given by the third equation of Theorem 2 is satisfied.

## 7. Conclusion

A data projection method (DPM) was proposed in this paper to estimate the switching time and recognize the active mode in a switching system. This method can be used to detect and identify internal faults. The diagnosis problem may be viewed as that of estimating the switching time and recognizing the faulty mode. Two conditions, namely, those of discernibility and detectability, are established. Under the discernibility condition, the active mode can be well recognized, and under the detectability condition, the switching time can be well estimated.

The main advantage of this method, compared with others described in the literature (e.g., Akhenak *et al.*, 2008; Domlan *et al.*, 2007a; Cocquempot *et al.*, 2004; Bayouhd and Travé Massuyès, 2014), is that the DPM does not need the parameter values of the model. The residuals are generated by projecting the input-output data in a way that depends on the model structure, which is supposed to linear in the paper. As a consequence, the DPM can be directly implemented on systems of the same type (the same model structure) without identifying, for each system, the parameters. This is of great interest in practice. The main drawback of the method is what can happen if the eigenvalue of the state matrix for one mode is close to 1. One can have in this case a large size of input, output and projection matrices, and the time complexity of the algorithm will increase. This problem will be considered in our future works.

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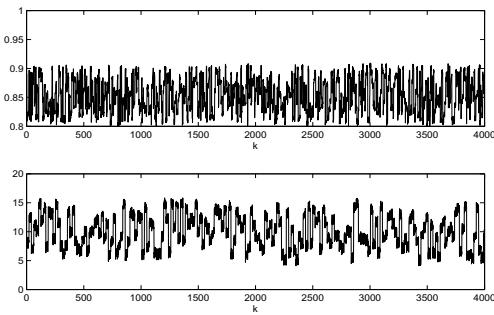


Fig. 3. System inputs  $u_k$  of Example 1.

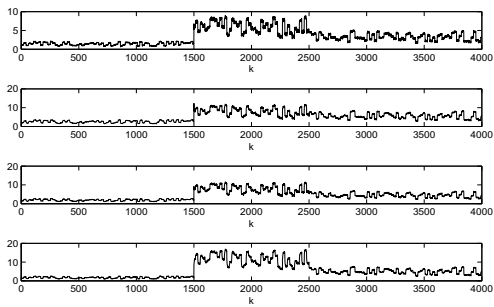


Fig. 4. System outputs  $y_k$  of Example 1.

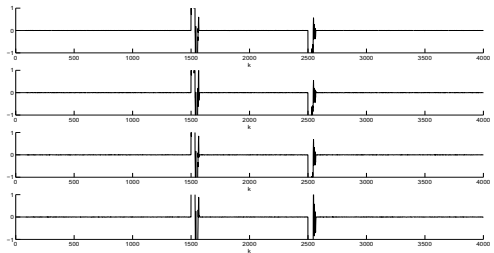


Fig. 5. Switching times estimation  $\epsilon_k$  of Example 1.

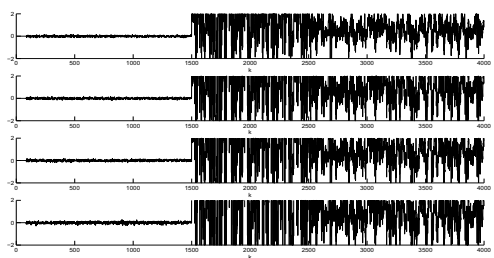


Fig. 6. First mode recognition  $\bar{\epsilon}_{(1), k}$  of Example 1.

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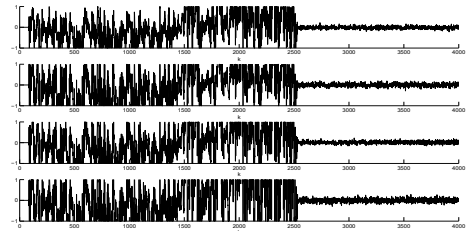


Fig. 7. Second mode recognition  $\bar{\epsilon}_{(2), k}$  of Example 1.

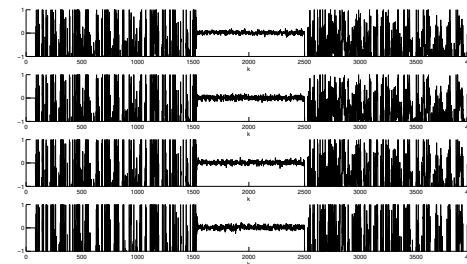


Fig. 8. Third mode recognition  $\bar{\epsilon}_{(3), k}$  of Example 1.

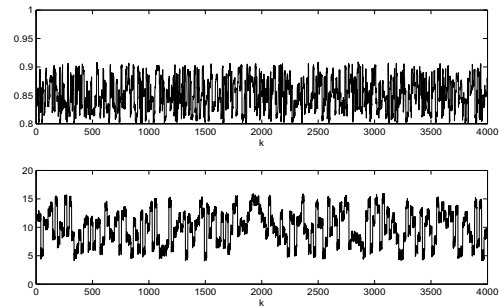


Fig. 9. System inputs  $u_k$  of Example 2.

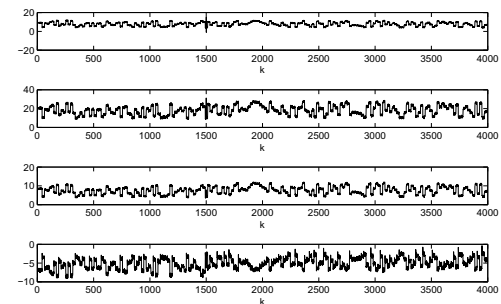
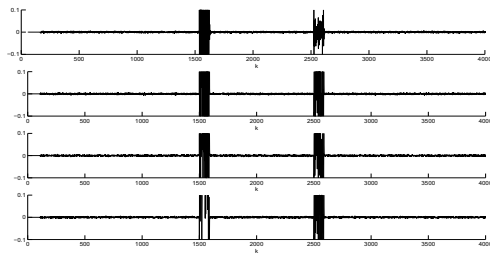
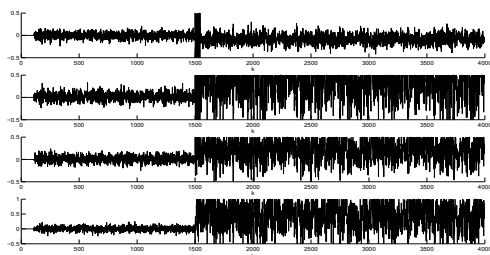
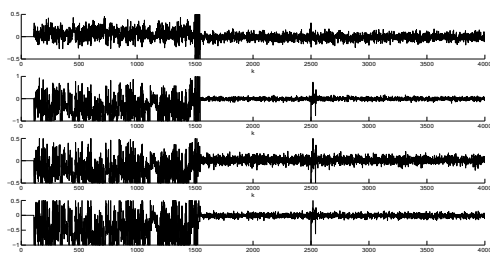
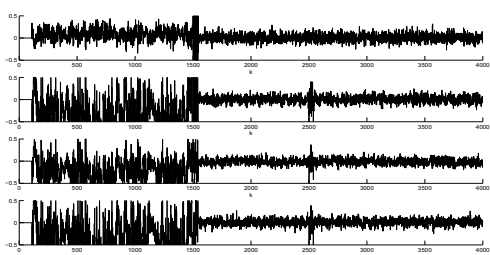


Fig. 10. System outputs  $y_k$  of Example 2.

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Fig. 11. Switching time estimation  $\epsilon_k$  of Example 2.Fig. 12. First mode recognition  $\bar{\epsilon}_{(1), k}$  of Example 2.Fig. 13. Second mode recognition  $\bar{\epsilon}_{(2), k}$  of Example 2.Fig. 14. Third mode recognition  $\bar{\epsilon}_{(3), k}$  of Example 2.

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**Assia Hakem** received her Master's degree in intelligent systems of transportation from Compiègne University (France) in 2010 and her Ph.D. degree in automatic control from the Polytechnic Institute of Lille University (France) in 2014. She is currently an assistant lecturer at the Oran University of Science and Technology (Algeria). Her research interests include system identification, as well as fault detection and isolation (FDI).



**Vincent Cocquempot** received his Ph.D. degree in automatic control from the Lille University of Sciences and Technologies in 1993. He is currently a full professor of automatic control at Lille 1 University, France. He is a researcher of the LAGIS-CNRS UMR 9189 CRISTAL Lab of Lille 1 University and heads the team on fault tolerant systems there. His research interests include robust on-line fault diagnosis and fault tolerant control for hybrid nonlinear systems.



**Komi Midzodzi Pekpe** received the Ph.D. degree in automatic control from the National Polytechnic Institute of Lorraine (INPL), France, in 2004. He is currently an assistant professor with the Engineering School Polytech'Lille at the University of Lille 1 (France). His research interests include systems identification, fault detection and isolation (FDI), and vibration analysis.

## Appendix

### A1. Theorems proofs

**A1.1. Proof of Theorem 2.** The first part of the proof determines the evaluation form of the proposed residual

when a switching occurs in the time window considered ( $\tau \in [k - L - i + 1, k]$ ).

The first step is to determine the output expression in the time window  $[k - L - i + 1, k]$  that contains the switching time  $\tau$ . The output  $y_t$  is expressed differently, depending on whether  $t < \tau$ ,  $\tau \leq t \leq \tau + i - 1$  and  $t > \tau + i - 1$ , as detailed in Fig. A1.

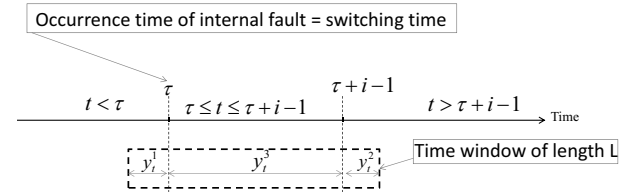


Fig. A1. Computation time window decomposition in three parts.

1. Output expression  $y_t = y_t^1$  for  $t < \tau$ :

$$\begin{aligned}
 y_t^1 &= C_{\sigma_\tau} A_{\sigma_\tau}^i x_{t-i} \\
 &+ \sum_{j=0}^{i-1} C_{\sigma_\tau} A_{\sigma_\tau}^j B_{\sigma_\tau} u_{t-1-j} \\
 &+ D_{\sigma_\tau} u_t + w_t \\
 &= C_{\sigma_\tau} A_{\sigma_\tau}^i x_{t-i} + H_{(\sigma_\tau), i} \bar{u}_{t, i} + w_t.
 \end{aligned} \tag{A1}$$

2. Output expression  $y_t = y_t^2$  for  $t > \tau + i - 1$ :

$$\begin{aligned}
 y_t^2 &= C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^i x_{t-i} \\
 &+ \sum_{j=0}^{i-1} C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^j B_{\sigma_{\tau+1}} u_{t-1-j} \\
 &+ D_{\sigma_{\tau+1}} u_t + w_t \\
 &= C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^i x_{t-i} + H_{(\sigma_{\tau+1}), i} \bar{u}_{t, i} + w_t.
 \end{aligned} \tag{A2}$$

3. Output expression  $y_t = y_t^3$  for  $\tau \leq t \leq \tau + i - 1$ :

$$\begin{aligned}
 y_t^3 &= C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^{t-\tau+1} A_{\sigma_\tau}^{i-t+\tau-1} x_{t-i} \\
 &+ \sum_{j=0}^{i-t+\tau-2} C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^{t-\tau+1} A_{\sigma_\tau}^j B_{\sigma_\tau} u_{\tau-2-j} \\
 &+ \sum_{j=0}^{t-\tau} C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^j B_{\sigma_{\tau+1}} u_{t-j-1} \\
 &+ D_{\sigma_{\tau+1}} u_t + w_t \\
 &= C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^{t-\tau+1} A_{\sigma_\tau}^{i-t+\tau-1} x_{t-i} \\
 &+ \mathcal{H}_{(\sigma_\tau, \sigma_{\tau+1}), i, t-\tau+1} \bar{u}_{t, i} + w_t.
 \end{aligned} \tag{A3}$$

Let us construct now the matrix  $Y_k = Y_{k-L+1:k}$ . For the three expressions for  $y_t$ , we have

$$Y_k = \begin{bmatrix} y_{k-L+1}^1 & \cdots & y_{\tau-2}^1 & y_{\tau-1}^1 & y_{\tau}^3 & \cdots \\ y_{\tau+i-2}^3 & y_{\tau+i-1}^3 & y_{\tau+i}^2 & \cdots & y_{k-1}^2 & y_k^2 \end{bmatrix},$$

which gives

$$\begin{aligned} Y_{k-L+1:k} &= \begin{bmatrix} \tilde{\delta}_{(\sigma_{\tau}, k-L+1-i:\tau-1-i}^i \\ \tilde{\delta}_{(\sigma_{\tau}, \sigma_{\tau+1}), \tau-i:\tau-1}^i | \tilde{\delta}_{(\sigma_{\tau+1}), \tau:k-i}^i \\ + H_{(\sigma_{\tau}), i} [U_{k-L+1:\tau-1} | 0_{m(i+1) \times k-\tau+1}] \\ + [\mathcal{H}_{(\sigma_{\tau}, \sigma_{\tau+1}), i, 1} | \mathcal{H}_{(\sigma_{\tau}, \sigma_{\tau+1}), i, 2} | \cdots | \\ \mathcal{H}_{(\sigma_{\tau}, \sigma_{\tau+1}), i, i-1} | \\ \mathcal{H}_{(\sigma_{\tau}, \sigma_{\tau+1}), i, i} [0_{mi(i+1) \times \tau+L-k-1} | \\ \check{U} | 0_{mi(i+1) \times k-\tau-i+1} | \\ + H_{(\sigma_{\tau+1}), i} [0_{m(i+1) \times L-k+\tau+i-1} | U_{\tau+i:k}] + W_k \end{bmatrix} \end{aligned} \quad (\text{A4})$$

where

$$\begin{aligned} \tilde{\delta}_{(\sigma_{\tau}, \sigma_{\tau+1}), \tau-i:\tau-1}^i &= [C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}} A_{\sigma_{\tau}}^{i-1} x_{\tau-i} | \\ C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^2 A_{\sigma_{\tau}}^{i-2} x_{\tau-i+1} | \\ \cdots | C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^{i-1} A_{\sigma_{\tau}} x_{\tau-2} | C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^i x_{\tau-1}] \end{aligned}$$

and  $\check{U} \in \mathbb{R}^{mi(i+1) \times i}$  is defined as follows:

$$\check{U} = \begin{bmatrix} \bar{u}_{\tau, i} & \mathbb{O} & \cdots & \cdots & \mathbb{O} \\ \mathbb{O} & \bar{u}_{\tau+1, i} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbb{O} \\ \mathbb{O} & \cdots & \cdots & \mathbb{O} & \bar{u}_{\tau+i-1, i} \end{bmatrix},$$

with  $\mathbb{O} = 0_{m(i+1) \times 1} \in \mathbb{R}^{m(i+1) \times 1}$  being a zero column vector.

In order to make the matrix  $U_{k-L+1:k}$  appear, we add and we subtract the term

$$H_{(\sigma_{\tau}), i} [0_{m(i+1) \times \tau-k+L-1} | U_{\tau:k}]$$

from Eqn. (A4). Thereafter, the output matrix  $Y_{k-L+1:k}$

is given by

$$\begin{aligned} Y_{k-L+1:k} &= \begin{bmatrix} \tilde{\delta}_{(\sigma_{\tau}, k-L+1-i:\tau-1-i}^i | \tilde{\delta}_{(\sigma_{\tau}, \sigma_{\tau+1}), \tau-i:\tau-1}^i \\ \tilde{\delta}_{(\sigma_{\tau+1}), \tau:k-i}^i \end{bmatrix} + H_{(\sigma_{\tau}), i} U_{k-L+1:k} \\ &+ \Omega + W_k, \end{aligned} \quad (\text{A5})$$

Post-multiplying Eqn. (A5) by  $\Pi_{U_{k-L+1:k}}$ , we get the residual evaluation form

$$\epsilon_k = \Omega \Pi_{U_{k-L+1:k}} + W_k \Pi_{U_{k-L+1:k}} + \delta_k^i, \quad (\text{A6})$$

where

$$\begin{aligned} \delta_k^i &= \begin{bmatrix} \tilde{\delta}_{(\sigma_{\tau}, k-L+1-i:\tau-1-i}^i | \tilde{\delta}_{(\sigma_{\tau}, \sigma_{\tau+1}), \tau-i:\tau-1}^i \\ \tilde{\delta}_{(\sigma_{\tau+1}), \tau:k-i}^i \end{bmatrix} \Pi_{U_{k-L+1:k}} \in \mathbb{R}^{\ell \times 1}. \end{aligned}$$

The initial state contribution  $\delta_k^i$  can be neglected for the following reasons (Kailath, 1980):

1. For  $t < \tau$  and  $t > \tau + i - 1$ :

Under stability hypothesis of  $A_{\sigma_{\tau}}$  and  $A_{\sigma_{\tau+1}}$ , the initial state contribution can be neglected for  $i$  sufficiently large, i.e.,  $\lim_{i \rightarrow \infty} C_{\sigma_{\tau}} A_{\sigma_{\tau}}^i x_{t-i} = 0$  and  $\lim_{i \rightarrow \infty} C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^i x_{t-i} = 0$ .

2. For  $\tau \leq t \leq \tau + i - 1$ :

The state is multiplied by a term of a general form  $A_{\sigma_{\tau+1}}^{t-\tau+1} A_{\sigma_{\tau}}^{i-t+\tau-1}$  as shown in Eqn. (A3), where the sum of powers is always equal to  $i$ . Let  $\|\cdot\|$  represent a multiplicative norm. Then we have

$$\left\| A_{\sigma_{\tau+1}}^{t-\tau+1} A_{\sigma_{\tau}}^{i-t+\tau-1} \right\| < \left\| A_{\sigma_{\tau+1}}^{t-\tau+1} \right\| \left\| A_{\sigma_{\tau}}^{i-t+\tau-1} \right\|,$$

and we also have

$$\left\| A_{\sigma_{\tau+1}}^{t-\tau+1} A_{\sigma_{\tau}}^{i-t+\tau-1} \right\| < (\max(\|A_{\sigma_{\tau+1}}\|, \|A_{\sigma_{\tau}}\|))^i.$$

For  $i \rightarrow \infty$ , the term  $(\max(\|A_{\sigma_{\tau+1}}\|, \|A_{\sigma_{\tau}}\|))^i$  is negligible. Consequently,

$$\lim_{i \rightarrow \infty} C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^{t-\tau+1} A_{\sigma_{\tau}}^{i-t+\tau-1} x_{t-i} = 0.$$

Therefore,  $\forall \eta > 0$ ,  $\exists i_0 \in \mathbb{N}$  such that  $\forall i > i_0 (i \in \mathbb{N})$  and  $\|\delta_k^i\| < \eta$ .

- *Sufficient condition:*

If  $\text{span}(\Omega) \not\subset \text{span}(U_{k-L+1:k})$ , then  $\forall \mathcal{X}$  and we have  $\Omega \neq \mathcal{X} U_{k-L+1:k}$ . Consequently, the residual mean and variance change.

- *Necessary condition:*

If the residual mean and variance change, then  $\Omega \Pi_{U_{k-L+1:k}} \neq 0$ , which implies that  $\text{span}(\Omega) \not\subset \text{span}(U_{k-L+1:k})$ .

**A1.2. Proof of Theorem 3.** If a mode  $\sigma_k$  is active in a time window  $[k - L - i + 1, k]$ , then the output matrix  $Y_{k-L+1:k}$  is given by

$$Y_{k-L+1:k} = \tilde{\delta}_{(\sigma_k), k-L+1-i:k-i}^i + H_{(\sigma_k), i} U_{k-L+1:k} + W_k. \quad (A7)$$

By replacing the first  $L/2$  columns of the matrix  $U_{k-L+1:k}$  (resp.  $Y_{k-L+1:k}$ ) by the input matrix  $U_{(\gamma), 1:L/2}^*$  (resp. the output matrix  $Y_{(\gamma), 1:L/2}^*$ ) constructed with input-output data collected off-line from mode  $\gamma$  ( $\gamma \in \{1, 2, \dots, d\}$ ), the resulting input and output matrices are given (24). From Eqn. (A7), the general expression of  $Y_{(\gamma, \sigma_k), k-L/2+1:k}$  becomes

$$\begin{aligned} Y_{(\gamma, \sigma_k), k-L/2+1:k} &= \left[ \tilde{\delta}_{(\gamma), 1:L/2}^i \mid \tilde{\delta}_{(\sigma_k), k-L/2+1-i:k-i}^i \right] \\ &+ H_{(\gamma), i} \left[ U_{(\gamma), 1:L/2}^* \mid 0_{m(i+1) \times L/2} \right] \\ &+ H_{(\sigma_k), i} \left[ 0_{m(i+1) \times L/2} \mid U_{k-L/2+1:k} \right] + W_k, \end{aligned} \quad (A8)$$

In order to make the matrix  $U_{(\gamma, \sigma_k), k-L/2+1:k}$  appear, we add and subtract the term  $H_{(\gamma), i} \left[ 0_{m(i+1) \times L/2} \mid U_{k-L/2+1:k} \right]$  from Eqn. (A8):

$$\begin{aligned} Y_{(\gamma, \sigma_k), k-L/2+1:k} &= \left[ \tilde{\delta}_{(\gamma), 1:L/2}^i \mid \tilde{\delta}_{(\sigma_k), k-L/2+1-i:k-i}^i \right] \\ &+ H_{(\gamma), i} U_{(\gamma, \sigma_k), k-L/2+1:k} \\ &+ (H_{(\sigma_k), i} - H_{(\gamma), i}) \left[ 0_{m(i+1) \times L/2} \mid U_{k-L/2+1:k} \right] \\ &+ W_k. \end{aligned} \quad (A9)$$

Post-multiplying both the sides of Eqn. (A9) by  $\Pi_{U_{(\gamma, \sigma_k), k-L/2+1:k}}$ , the evaluation form of the proposed residual yields

$$\begin{aligned} \bar{e}_{(\gamma), k} &= (H_{(\sigma_k), i} - H_{(\gamma), i}) \left[ 0_{m(i+1) \times L/2} \mid \right. \\ &\left. U_{k-L/2+1:k} \right] \Pi_{U_{(\gamma, \sigma_k), k-L/2+1:k}} + \delta_k^i \end{aligned} \quad (A10)$$

with

$$\begin{aligned} \delta_k^i &= \left[ \tilde{\delta}_{(\gamma), 1:L/2}^i \mid \tilde{\delta}_{(\sigma_k), k-L/2+1-i:k-i}^i \right] \Pi_{U_{(\gamma, \sigma_k), k-L/2+1:k}} \\ &\in \mathbb{R}^{\ell \times 1}. \end{aligned}$$

on the assumption that the modes are stable, which is equivalent to stating that eigenvalues of matrices  $A_\gamma$  and

$A_{\sigma_k}$  are inside the unit circle. Then we have (Kailath, 1980)

$$\lim_{i \rightarrow \infty} A_\gamma^i = 0, \quad \lim_{i \rightarrow \infty} A_{\sigma_k}^i = 0. \quad (A11)$$

As a consequence,  $\delta_k^i$  in Eqn. (A10) becomes negligible for  $i$  sufficiently large.

By neglecting the initial state contribution, where the approximation term is  $\delta_k^i$  and  $\forall \eta > 0, \exists i_0 \in \mathbb{N}$  such that  $\forall i > i_0 (i \in \mathbb{N}): |\delta_k^i| < \eta$ .

The evaluation form (A10) of the residual can be approximated by

$$\begin{aligned} \bar{e}_{(\gamma), k} &= (H_{(\sigma_k), i} - H_{(\gamma), i}) \left[ 0_{m(i+1) \times L/2} \mid \right. \\ &\left. U_{k-L/2+1:k} \right] \Pi_{U_{(\gamma, \sigma_k), k-L/2+1:k}}. \end{aligned} \quad (A12)$$

If the active mode is  $\sigma_k = \gamma$ , then  $H_{(\sigma_k), i} = H_{(\gamma), i}$  and we have (27).

### A1.3. Proof of Theorem 4.

- *Sufficient condition:*

If

$$\begin{aligned} &\text{span}((H_{(\sigma_k), i} - H_{(\gamma), i}) \left[ 0_{m(i+1) \times L/2} \mid U_{k-L/2+1:k} \right]) \\ &\not\subseteq \text{span}(U_{(\gamma, \sigma_k), k-L/2+1:k}), \end{aligned}$$

then for any  $\mathcal{X}$  we have

$$\begin{aligned} &(H_{(\sigma_k), i} - H_{(\gamma), i}) \left[ 0_{m(i+1) \times L/2} \mid U_{k-L/2+1:k} \right] \\ &\neq \mathcal{X} U_{(\gamma, \sigma_k), k-L/2+1:k} \end{aligned}$$

and, according to Theorem 3, the residual is not zero mean Gaussian noise while the mode is discernible.

- *Necessary condition:*

If the residual is not zero mean Gaussian noise, then, according to Theorem 3,

$$\begin{aligned} &(H_{(\sigma_k), i} - H_{(\gamma), i}) \left[ 0_{m(i+1) \times L/2} \mid U_{k-L/2+1:k} \right] \\ &\Pi_{U_{(\gamma, \sigma_k), k-L/2+1:k}} \neq 0, \end{aligned}$$

which implies that

$$\begin{aligned} &\text{span}((H_{(\sigma_k), i} - H_{(\gamma), i}) \left[ 0_{m(i+1) \times L/2} \mid U_{k-L/2+1:k} \right]) \\ &\not\subseteq \text{span}(U_{(\gamma, \sigma_k), k-L/2+1:k}). \end{aligned}$$

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