Int. J. Appl. Math. Comput. Sci., 2017, Vol. 27, No. 1, 33–41
DOI: 10.1515/amcs-2017-0003

MINIMUM ENERGY CONTROL OF DESCRIPTOR FRACTIONAL DISCRETE–TIME LINEAR SYSTEMS WITH TWO DIFFERENT FRACTIONAL ORDERS

ŁUKASZ SAJEWSKI [a]

[a] Faculty of Electrical Engineering
Białystok University of Technology, ul. Wiejska 45D, 15-351 Białystok, Poland
e-mail: l.sajewski@pb.edu.pl

Reachability and minimum energy control of descriptor fractional discrete-time linear systems with different fractional orders are addressed. Using the Weierstrass–Kronecker decomposition theorem of the regular pencil, a solution to the state equation of descriptor fractional discrete-time linear systems with different fractional orders is given. The reachability condition of this class of systems is presented and used for solving the minimum energy control problem. The discussion is illustrated with numerical examples.

Keywords: minimum energy control, descriptor system, fractional system, discrete-time linear system.

1. Introduction

Solutions to the minimum energy control problem for linear time-invariant systems produce an input sequence that will bring the system to a desired state with minimum energy expenditure. This subject has been considered in many papers and monographs (Kaczorek and Klamka, 1986; Kaczorek, 2014; Klamka, 1991; 2009; 2010; 2014). LTI (linear time-invariant) systems theory deals with numerous types of systems, e.g., positive (Kaczorek 2002; 2013a; 2013c; 2011d), descriptor (Campbell *et al.*, 1976; Dai 1989; Dodig and Stosic, 2009; Guang-Ren, 2010; Van Dooren, 1979; Van Dooren and Beelen, 1988) or fractional (Nishimoto, 1984; Oldham and Spanier, 1974; Podlubny, 1999). Recently fractional systems have received more attention since fractional differential equations have been used by engineers for modeling various processes (Dzieliński *et al.*, 2009; Ferreira and Machado, 2003; Popović *et al.*, 2013).

From a mathematical point of view, fractional calculus is well known (Nishimoto, 1984; Oldham and Spanier, 1974; Podlubny, 1999; Miller and Ross, 1993); however, some new results appear, e.g., a new definition of the fractional derivative (Caputo and Fabrizio, 2015) or fractional systems with different fractional orders (Kaczorek 2010; 2011a; 2011b). The resulting combination of descriptor fractional systems with discrete-time linear ones, which are commonly used for process modeling, becomes a source of interest in this paper, which deals with descriptor fractional linear systems described by difference equations of different fractional orders. A solution to the state equation of descriptor fractional discrete-time linear systems with regular pencils was given by Kaczorek (2011d; 2013b), along with that for continuous-time systems (Kaczorek, 2013a; 2013c). A solution to the descriptor fractional continuous-time linear systems with two different fractional orders was introduced by Sajewski (2015). Reduction and decomposition of descriptor fractional discrete-time linear systems were considered by Kaczorek (2011c). The reachability and minimum energy control problem for continuous-time systems with two different fractional orders was considered by Sajewski (2016a). A comparison of three different methods for finding the solution of descriptor fractional discrete-time linear systems can be found in the work of Sajewski (2016b), along with the case of fractional systems with two different fractional orders (Sajewski, 2016c).

This paper is devoted to the minimum energy control problem of descriptor fractional discrete-time linear systems with different fractional orders. In most cases, the solution for descriptor systems is accomplished by the Shuffle algorithm, which leads to more complex fractional systems with delays (see, e.g., Kaczorek, 2014). A new approach, based on the Weierstrass–Kronecker decomposition theorem, will be given. This solution is proven by an example for greater efficiency .

The organisation of this paper is as follows. In Section 2 the descriptor fractional discrete-time linear system with different fractional orders is presented and a solution to the system, with the use of the Weierstrass–Kronecker decomposition theorem, is given. An example of decomposition is also presented in Section 2. Section 3 is devoted to reachability of descriptor fractional systems, and the reachability condition is given. The minimum energy control problem of descriptor fractional discrete-time linear systems with different fractional orders is formulated and solved in Section 4, where an illustrating example is also given. Concluding remarks are included in Section 5.

2. Descriptor fractional discrete-time linear systems of different orders and their solution

Consider a descriptor fractional discrete-time linear system with two different fractional orders,

$$\begin{aligned} E_1\Delta^{\alpha}x_1(k+1) &= A_{11}x_1(k) + A_{12}x_2(k) \\ &\quad + B_1u(k), \\ E_2\Delta^{\beta}x_2(k+1) &= A_{21}x_1(k) + A_{22}x_2(k) \\ &\quad + B_2u(k), \end{aligned} \tag{1}$$

where $0 < \alpha, \beta < 2$, $k \in \mathbb{Z}_+$, $x_1(k) \in \mathbb{R}^{n_1}$ and $x_2(k) \in \mathbb{R}^{n_2}$ are the state vectors, $u(k) \in \mathbb{R}^m$ is the input vector, $E_i,\ A_{ij} \in \mathbb{R}^{n_i\times n_j},\ B_i \in \mathbb{R}^{n_i\times m};\ i,j = 1,2$; $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices and $\mathbb{Z}_+$ is the set of nonnegative integers.

The fractional difference of order α (β) is defined by (Kaczorek, 2011a)

$$\begin{aligned} \Delta^{\alpha}x(k) &= \sum_{j=0}^{k} c_{\alpha}(j)x(k-j), \\ c_{\alpha}(0) &= 1, \\ c_{\alpha}(j) &= (-1)^j \binom{\alpha}{j} \\ &= (-1)^j \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!}, \\ &\qquad j = 1,2,\dots \end{aligned} \tag{2}$$

Rewriting (1) in matrix form, we have

$$E\begin{bmatrix} \Delta^{\alpha}x_1(k+1) \\ \Delta^{\beta}x_2(k+1) \end{bmatrix} = A\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + Bu(k), \tag{3}$$

where

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{4}$$

In descriptor systems it is assumed that

$$\det E = 0, \tag{5}$$

and we also assume a regular pencil,

$$\det\left[\begin{bmatrix} E_1z_1 & 0 \\ 0 & E_2z_2 \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right] \neq 0, \tag{6}$$

for some $z_1, z_2 \in \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers.

Sajewski (2016c) showed that the Weierstrass–Kronecker decomposition theorem of the regular pencil (Kaczorek, 2011a; 1998) can be used for systems with two different fractional orders.

A solution of the descriptor fractional discrete-time linear system (1) with (5) and (6) is supported by the following lemma.

Lemma 1. *There exist nonsingular matrices $P, Q \in \mathbb{R}^{n\times n}$ such that the descriptor fractional discrete-time linear systems (3) with regular pencil (6) can be decomposed as*

$$\begin{aligned} P&\left[\begin{bmatrix} E_1z_1 & 0 \\ 0 & E_2z_2 \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right]Q \\ &= \begin{bmatrix} \bar{E}_1z_1 & 0 \\ 0 & \bar{E}_2z_2 \end{bmatrix} - \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \end{aligned} \tag{7}$$

where

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \tag{8}$$

and the submatrices of (7) have the following form:

$$\begin{aligned} \bar{E}_1 &= P_1E_1Q_1 = \begin{bmatrix} I_{n_1^1} & 0 \\ 0 & N_1 \end{bmatrix}, \\ \bar{E}_2 &= P_2E_2Q_2 = \begin{bmatrix} I_{n_2^1} & 0 \\ 0 & N_2 \end{bmatrix}, \\ \bar{A}_{11} &= P_1A_{11}Q_1 = \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & I_{n_1^2} \end{bmatrix}, \\ \bar{A}_{12} &= P_1A_{12}Q_2 = \begin{bmatrix} \tilde{A}_{12} & 0 \\ 0 & 0 \end{bmatrix}, \\ \bar{A}_{21} &= P_2A_{21}Q_1 = \begin{bmatrix} \tilde{A}_{21} & 0 \\ 0 & 0 \end{bmatrix}, \\ \bar{A}_{22} &= P_2A_{22}Q_2 = \begin{bmatrix} \tilde{A}_{22} & 0 \\ 0 & I_{n_2^2} \end{bmatrix}, \\ P_1B_1 &= \begin{bmatrix} \tilde{B}_1^1 \\ \tilde{B}_1^2 \end{bmatrix}, \quad P_2B_2 = \begin{bmatrix} \tilde{B}_2^1 \\ \tilde{B}_2^2 \end{bmatrix}, \end{aligned} \tag{9}$$

where I_n is the $n \times n$ identity matrix, $N_1 \in \mathbb{R}^{n_1^2 \times n_1^2}$, $N_2 \in \mathbb{R}^{n_2^2 \times n_2^2}$ are a nilpotent matrices with the index μ_i, $i = 1, 2$ (i.e., $N_i^{\mu_i} = 0$ and $N_i^{\mu_i - 1} \neq 0$),

$$\tilde{A}_{11} \in \mathbb{R}^{n_1^1 \times n_1^1},$$

$$\tilde{A}_{22} \in \mathbb{R}^{n_2^1 \times n_2^1}, \quad \tilde{A}_{21}^1 \in \mathbb{R}^{n_2^1 \times n_1^1},$$

$$\tilde{A}_{12}^1 \in \mathbb{R}^{n_1^1 \times n_2^1}, \quad \tilde{B}_1^1 \in \mathbb{R}^{n_1^1 \times m},$$

$$\tilde{B}_1^2 \in \mathbb{R}^{n_1^2 \times m}, \tilde{B}_2^1 \in \mathbb{R}^{n_2^1 \times m},$$

$$\tilde{B}_2^2 \in \mathbb{R}^{n_2^2 \times m}$$

$$\operatorname{rank} E_1 = n_1^1, \quad \operatorname{rank} E_2 = n_2^1,$$

$$n_1^1 + n_1^2 = n_1, \quad n_2^1 + n_2^2 = n_2, \quad n_1 + n_2 = n.$$

Example 1. Let the system (3) have the pencil E, A (given by (4)) of the form

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 4 & 2 \\ 1 & 4 & 1 \end{bmatrix},$$
$$A_{11} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 4 & 11 & 4 \\ 2 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} -3 & 2 & 0 \\ 6 & 2 & 0 \\ 3 & 7 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0.8 & 1.7 & 2.8 \\ 0.4 & 0.8 & 1.4 \\ 2.2 & 4.6 & 2.2 \end{bmatrix}. \tag{10}$$

We wish to find its decomposition (9). It easy to check that the condition (5) is met since $\det E_1 = 0$, $\det E_2 = 0$ and E is diagonal. The condition (6) is also met since

$$\begin{aligned} \det & \left[\begin{bmatrix} E_1 z_1 & 0 \\ 0 & E_2 z_2 \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right] \\ &= z_1^2 z_2^2 - 0.3 z_1^2 z_2 - z_1 z_2^2 + 0.02 z_1^2 \\ &\quad - 16.7 z_1 z_2 - 8.02 z_1 + 12 z_2 - 10.2. \end{aligned}$$

Using the row and column elementary operations (Kaczorek; 1998), we obtain the matrices P and Q of the form

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad P_2 = \frac{1}{11} \begin{bmatrix} 1 & -2 & 5 \\ -2 & 4 & 1 \\ 4 & 3 & -2 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{11}$$

which decompose the pencil of the system (3) with the matrices (10) to the desired form,

$$\begin{aligned} \bar{E}_1 &= P_1 E_1 Q_1 \\ &= \begin{bmatrix} I_{n_1^1} & 0 \\ 0 & N_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \bar{E}_2 &= P_2 E_2 Q_2 \\ &= \begin{bmatrix} I_{n_2^1} & 0 \\ 0 & N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \bar{A}_{11} &= P_1 A_{11} Q_1 \\ &= \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & I_{n_1^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ \bar{A}_{12} &= P_1 A_{12} Q_2 \\ &= \begin{bmatrix} \tilde{A}_{12} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \bar{A}_{21} &= P_2 A_{21} Q_1 \\ &= \begin{bmatrix} \tilde{A}_{21} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \bar{A}_{22} &= P_2 A_{22} Q_2 \\ &= \begin{bmatrix} \tilde{A}_{22} & 0 \\ 0 & I_{n_2^2} \end{bmatrix} = \begin{bmatrix} 0.1 & 1 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ & n_1^1 = n_2^1 = 2, \quad n_1^2 = n_2^2 = 1, \quad n_1 = n_1^1 + n_1^2 = 3, \\ & n_2 = n_2^1 + n_2^2 = 3, \quad n = n_1 + n_2 = 6. \end{aligned} \tag{12}$$

♦

Computation methods for the matrices P and Q have been discussed, e.g., by Van Dooren (1979), Van Dooren and Beleen (1988) or Kaczorek (2011a).

Using the decomposition given by Lemma 1, the system (3) can be written as

$$\begin{aligned} & \begin{bmatrix} I_{n_1^1} & 0 & 0 & 0 \\ 0 & N_1 & 0 & 0 \\ 0 & 0 & I_{n_2^1} & 0 \\ 0 & 0 & 0 & N_2 \end{bmatrix} \begin{bmatrix} \Delta^\alpha \bar{x}_1^1(k+1) \\ \Delta^\alpha \bar{x}_1^2(k+1) \\ \Delta^\beta \bar{x}_2^1(k+1) \\ \Delta^\beta \bar{x}_2^2(k+1) \end{bmatrix} \\ &= \begin{bmatrix} \tilde{A}_{11} & 0 & \tilde{A}_{12} & 0 \\ 0 & I_{n_1^2} & 0 & 0 \\ \tilde{A}_{21} & 0 & \tilde{A}_{22} & 0 \\ 0 & 0 & 0 & I_{n_2^2} \end{bmatrix} \begin{bmatrix} \bar{x}_1^1(k) \\ \bar{x}_1^2(k) \\ \bar{x}_2^1(k) \\ \bar{x}_2^2(k) \end{bmatrix} \\ &\quad + \begin{bmatrix} \tilde{B}_1^1 \\ \tilde{B}_1^2 \\ \tilde{B}_2^1 \\ \tilde{B}_2^2 \end{bmatrix} u(k) \quad \text{for } k \in \mathbb{Z}_+. \end{aligned} \tag{13}$$

The notation (13) is possible since by premultiplying the state equation (3) by the matrix $P \in \mathbb{R}^{n\times n}$ and introducing the new state vector

$$\begin{bmatrix} \bar{x}_1^1(k) \\ \bar{x}_1^2(k) \\ \bar{x}_2^1(k) \\ \bar{x}_2^2(k) \end{bmatrix} = Q^{-1}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \quad \bar{x}_1^1(k) \in \mathbb{R}^{n_1^1},$$
$$\bar{x}_1^2(k) \in \mathbb{R}^{n_1^2}, \quad \bar{x}_2^1(k) \in \mathbb{R}^{n_2^1}, \quad \bar{x}_2^2(k) \in \mathbb{R}^{n_2^2}, \tag{14}$$

we obtain the following state equation:

$$PEQQ^{-1}\begin{bmatrix} \Delta^\alpha x_1(k+1) \\ \Delta^\beta x_2(k+1) \end{bmatrix} = PAQQ^{-1}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + PBu(k), \quad k \in \mathbb{Z}_+. \tag{15}$$

Following Lemma 1, decomposition of (3) allows us to distinguish two subsystems:
the standard one,

$$\begin{bmatrix} \Delta^\alpha \bar{x}_1^1(k+1) \\ \Delta^\beta \bar{x}_2^1(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}\begin{bmatrix} x_1^1(k) \\ x_2^1(k) \end{bmatrix} + \begin{bmatrix} \tilde{B}_1^1 \\ \tilde{B}_2^1 \end{bmatrix} u(k); \tag{16}$$

and the nilpotent one,

$$\begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}\begin{bmatrix} \Delta^\alpha \bar{x}_1^2(k+1) \\ \Delta^\beta \bar{x}_2^2(k+1) \end{bmatrix} = \begin{bmatrix} I_{n_1^2} & 0 \\ 0 & I_{n_2^2} \end{bmatrix}\begin{bmatrix} x_1^2(k) \\ x_2^2(k) \end{bmatrix} + \begin{bmatrix} \tilde{B}_1^2 \\ \tilde{B}_2^2 \end{bmatrix} u(k). \tag{17}$$

A solution to the standard subsystem (16) is well known (Kaczorek, 2011a) and can be computed with the use of the following formula:

$$\begin{bmatrix} \bar{x}_1^1(k) \\ \bar{x}_2^1(k) \end{bmatrix} = \Phi_k \begin{bmatrix} \bar{x}_1^1(0) \\ \bar{x}_2^1(0) \end{bmatrix} + \sum_{i=0}^{k-1} \Phi_{k-i-1}\begin{bmatrix} \tilde{B}_1^1 \\ \tilde{B}_2^1 \end{bmatrix} u(i), \quad k \in \mathbb{Z}_+, \tag{18}$$

where

$$\Phi_i = \begin{cases} I_{n_1^1+n_2^1} & \text{for} \quad i = 0, \\ \tilde{A}\Phi_{i-1} - D_1\Phi_{i-2} - \cdots - D_{i-1}\Phi_0 & \\ & \text{for} \quad i = 1,\dots,k, \end{cases} \tag{19}$$

$$\Phi_i = \begin{bmatrix} \Phi_{11}^i & \Phi_{12}^i \\ \Phi_{21}^i & \Phi_{22}^i \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \tilde{A}_{1\alpha} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{2\beta} \end{bmatrix},$$
$$D_i = \begin{bmatrix} c_\alpha(i+1)I_{n_1^1} & 0 \\ 0 & c_\beta(i+1)I_{n_2^1} \end{bmatrix},$$
$$\tilde{A}_{1\alpha} = \tilde{A}_{11} + I_{n_1^1}\alpha, \quad \tilde{A}_{2\beta} = \tilde{A}_{22} + I_{n_2^1}\beta. \tag{20}$$

A solution to the nilpotent subsystem (17) depends on the max nilpotency index of matrices N_1 and N_2, that is, $\mu = \max(\mu_1, \mu_2)$.

If $N_1 = N_2 = 0$ (for which the nilpotency index $\mu = 0$), we obtain the following solution to (17):

$$\begin{bmatrix} \bar{x}_1^2(k) \\ \bar{x}_2^2(k) \end{bmatrix} = -\begin{bmatrix} \tilde{B}_1^2 \\ \tilde{B}_2^2 \end{bmatrix} u(k), \quad k \in \mathbb{Z}_+. \tag{21}$$

If

$$N_1 = N_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(for which the nilpotency index $\mu = 1$), we have two equations with two unknown elements per each element of the state vector, and this lead to the following solution of (17):

$$\begin{bmatrix} \bar{x}_1^2(k) \\ \bar{x}_2^2(k) \end{bmatrix} = \begin{bmatrix} \tilde{b}_{12}^2 \\ 0 \\ \tilde{b}_{22}^2 \\ 0 \end{bmatrix} u(k+1) - \begin{bmatrix} \tilde{B}_1^2 \\ \tilde{B}_2^2 \end{bmatrix} u(k), \quad k \in \mathbb{Z}_+, \tag{22}$$

where

$$\tilde{B}_1^2 = \begin{bmatrix} \tilde{b}_{11}^2 \\ \tilde{b}_{12}^2 \end{bmatrix}, \quad \tilde{B}_2^2 = \begin{bmatrix} \tilde{b}_{21}^2 \\ \tilde{b}_{22}^2 \end{bmatrix}.$$

Continuing for $\mu = 2,\dots,j$, we obtain

$$\begin{bmatrix} \bar{x}_1^2(k) \\ \bar{x}_2^2(k) \end{bmatrix} = (-1)^{j+1}\begin{bmatrix} \tilde{B}_{1,j}^2 \\ \tilde{B}_{2,j}^2 \end{bmatrix} u(k+j) + \cdots + \begin{bmatrix} \tilde{B}_{11}^2 \\ \tilde{B}_{21}^2 \end{bmatrix} u(k+1) - \begin{bmatrix} \tilde{B}_1^2 \\ \tilde{B}_2^2 \end{bmatrix} u(k), \quad k \in \mathbb{Z}_+. \tag{23}$$

In general, knowing the solution of the standard subsystem (16) and the solution of the nilpotent subsystem (17), we can find the desired solution of the system (3). Taking under consideration (18), (19), (20) and (23), the solution of (3) has the form

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = Q\Psi_k \begin{bmatrix} \bar{x}_1^1(0) \\ \bar{x}_1^2(0) \\ \bar{x}_2^1(0) \\ \bar{x}_2^2(0) \end{bmatrix} + \sum_{i=0}^{k+\mu} Q\Psi_{k-i-1}\tilde{B}u(i), \quad k \in \mathbb{Z}_+, \tag{24}$$

where

$$\tilde{B} = \begin{bmatrix} \tilde{B}_1^1 \\ \tilde{B}_1^2 \\ \tilde{B}_2^1 \\ \tilde{B}_2^2 \end{bmatrix}, \quad \Psi_i = \begin{bmatrix} \Phi_{11}^i & 0 & \Phi_{12}^i & 0 \\ 0 & 0 & 0 & 0 \\ \Phi_{21}^i & 0 & \Phi_{22}^i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Psi_{-\mu-1} = (-1)^{\mu+1} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \Psi_1^{-\mu-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Psi_2^{-\mu-1} \end{bmatrix}, \tag{25}$$

for $i = 0, 1, \dots, k$, $\mu = 0, 1, \dots$ and $\Psi_1^{-\mu-1} \in \mathbb{R}^{n_1^2}$, $\Psi_2^{-\mu-1} \in \mathbb{R}^{n_2^2}$ depends on shape of nilpotent matrices N_1, N_2 given by (9); e.g., for $\mu = 1$, we have

$$\Psi_1^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Psi_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\Psi_1^{-2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \Psi_2^{-2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{26}$$

3. Reachability of fractional descriptor systems

Again consider a fractional descriptor system (1) (or (3)). This system is called reachable in q steps if for any given $x_f \in \mathbb{R}^n$ there exists an input sequence $u(i) \in \mathbb{R}^m$ for $i = 0, 1, \dots, q + \mu$ that steers the state of the system from $x(0) = 0$ to $x_f \in \mathbb{R}^n$, where $\mu = \max(\mu_1, \mu_2)$ and $h = q + \mu + 1$ is the number of steps in which the state of the system is transferred from $x(0)$ to x_f.

Theorem 1. *The descriptor system (1) is reachable in q steps if and only if the reachability matrix*

$$R_h = [\ Q\Psi_{k-1}\tilde{B} \ \ \dots \ \ Q\Psi_0\tilde{B} \ \ Q\Psi_{-\mu-1}\tilde{B}\ ] \tag{27}$$

contains n linearly independent columns, that is, $\operatorname{rank} R_h = n$, *where the matrices Ψ and $\tilde{B}$ are defined by (25) and the matrix Q follows from Lemma 1.*

Proof. Using (24) for $k = q$, the matrices N with index $\mu = 1$ and $x_0 = 0$, we obtain

$$\begin{aligned} x_f &= x_q \\ &= [\ Q\Psi_{-2}\tilde{B} \ \ Q\Psi_{q-2}\tilde{B} \ \ Q\Psi_{q-1}\tilde{B}\ ] \\ &\times \begin{bmatrix} u(q+1) \\ u(q) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix} = R_h \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(q+1) \end{bmatrix}, \end{aligned} \tag{28}$$

where R_h is defined by (27). From (28) it follows that there exists an input sequence $u(i) \in \mathbb{R}^m$ for $i = 0, 1, \dots, q + 1$ if and only if the matrix (27) contains n linearly independent columns. ∎

4. Minimum energy control problem

Assuming that the descriptor fractional discrete-time linear system (1) (or (3)) is reachable, this imply that there exist many input sequences that steer the state of the system from $x(0) = 0$ to the given final state $x_f \in \mathbb{R}^n$. Above all, we are looking for a sequence $u(i) \in \mathbb{R}^m$ for $i = 0, 1, \dots, q + \mu$ that minimizes the performance index

$$I(u) = \sum_{i=0}^{q+\mu} u^T(i) G_h u(i), \tag{29}$$

where $G_h \in \mathbb{R}_+^m$ is a symmetric positive defined matrix and A^T is the transpose matrix A.

The minimum energy control problem for the discussed class of fractional systems (3) can be defined as follows: for given matrices (9), fractional orders α and β, a number q, a final state $x_f \in \mathbb{R}^n$ and a matrix G_h of the performance index (29), find an input sequence $u(i) \in \mathbb{R}^m$ for $i = 0, 1, \dots, q + \mu$ that steers the state vector of the system from $x(0) = 0$ to $x_f \in \mathbb{R}^n$ and minimizes the performance index (29).

Following Kaczorek (2011a) and Klamka (1991), to solve the problem we define the matrix

$$W_h = R_h G_h^{-1} R_h^T \in \mathbb{R}^{n \times n}, \tag{30}$$

where R_h is defined by (27) and

$$G_h^{-1} = \text{blockdiag}\,[G^{-1}, \dots, G^{-1}] \in \mathbb{R}_+^{hm \times hm}. \tag{31}$$

From (30) it follows that the matrix W_h is invertible if and only if the matrix $R_h R_h^T$ is nonsingular; then the input sequence

$$\hat{u}_h = \begin{bmatrix} u(q+\mu) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix} = G_h^{-1} R_h^T W_h^{-1} x_f \in \mathbb{R}^{hm} \tag{32}$$

steers the system from $x(0) = 0$ to $x_f \in \mathbb{R}^n$ since

$$\begin{aligned} x(q) &= R_h \hat{u}_h \\ &= R_h Q_h^{-1} R_h^T W_h^{-1} x_f = x_f, \\ h &= q + \mu + 1. \end{aligned} \tag{33}$$

Theorem 2. *Let the descriptor system (3) be reachable in q steps and the conditions (30), (31) be satisfied. Let $\bar{u}(i) \in \mathbb{R}^m$, $i = 0, 1, \dots, q + \mu$, be an input sequence that steers the state of the descriptor system (3) from $x(0) = 0$ to $x_f \in \mathbb{R}^n$. Then the input sequence (32) also steers the state of the system from $x(0) = 0$ to $x_f \in \mathbb{R}^n$ and minimizes the performance index (29), i.e., $I(\hat{u}) \le I(\bar{u})$. The minimal value of the performance index (29) is given by*

$$I(\hat{u}) = x_f^T W_h^{-1} x_f. \tag{34}$$

The proof is similar to the one given by Kaczorek (2014).

Summing up the discussion, to compute the optimal input sequence (32) and the minimal value of the performance index (34) first we have to find the matrices (8) and decompose the state matrices (4) to the form (9). Then knowing the matrices $\tilde{B}$, Ψ, G_h and using (27), compute the matrix R_h and check the reachability of the system. From (30) compute the matrix W_h. Finally, using (32), we can find the input sequence $u(i) \in \mathbb{R}^m$, $i = 0, 1, \dots, q + \mu$, and using (34), the minimal value of the performance index $I(\hat{u})$.

This approach is also valid in the case when $\alpha = \beta$ and also when $\alpha = \beta = 1$, where the matrices $\tilde{B}$, Ψ, G_h are simplified.

Example 2. (*Continuation of Example 1*) Consider the descriptor fractional discrete-time linear system (1) with the fractional orders $\alpha = 0.5$, $\beta = 0.6$, described by the matrices (10). Find the input sequence $u(i) \in \mathbb{R}^m$, $i = 0, 1 \dots,$ that steer the state of the system from the zero initial conditions

$$x_1(0) = [\ 0 \quad 0 \quad 0\ ]^T, \quad x_2(0) = [\ 0 \quad 0 \quad 0\ ]^T$$

to the final state

$$x_{1f} = [\ 2 \quad 1 \quad 2\ ]^T, \quad x_{2f} = [\ 1 \quad 1 \quad 1\ ]^T$$

for $q = 3$ and minimizes the performance index (29) with $G = 1 \times 10^{-19}$.

Using matrices P and Q given by (11), we obtain the decomposition (12) and

$$\begin{gathered} Q^{-1} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} \bar{x}_1^1(k) \\ \bar{x}_1^2(k) \\ \bar{x}_2^1(k) \\ \bar{x}_2^2(k) \end{bmatrix} = \begin{bmatrix} \bar{x}_{11}^1(k) \\ \bar{x}_{12}^1(k) \\ \bar{x}_{11}^2(k) \\ \bar{x}_{21}^1(k) \\ \bar{x}_{22}^1(k) \\ \bar{x}_{21}^2(k) \end{bmatrix}, \\ PB = \begin{bmatrix} \tilde{B}_1^1 \\ \tilde{B}_2^1 \\ \tilde{B}_1^2 \\ \tilde{B}_2^2 \end{bmatrix} = \begin{bmatrix} \tilde{B}_{11}^1(k) \\ \tilde{B}_{12}^1(k) \\ \tilde{B}_{11}^2(k) \\ \tilde{B}_{21}^1(k) \\ \tilde{B}_{22}^1(k) \\ \tilde{B}_{21}^2(k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0.545 \\ -0.091 \\ 0.182 \end{bmatrix}, \\ n_1^1 = n_2^1 = 2, \quad n_1^2 = n_2^2 = 1, \quad n_1 = n_2 = 3, \\ n = n_1 + n_2 = 6. \end{gathered} \tag{35}$$

Taking under consideration size n_1^1, n_2^1, the solution of the standard part (16) has the form

$$\begin{aligned} \begin{bmatrix} \bar{x}_{11}^1(k) \\ \bar{x}_{12}^1(k) \\ \bar{x}_{21}^1(k) \\ \bar{x}_{22}^1(k) \end{bmatrix} &= \Phi_k \begin{bmatrix} \bar{x}_{11}^1(0) \\ \bar{x}_{12}^1(0) \\ \bar{x}_{21}^1(0) \\ \bar{x}_{22}^1(0) \end{bmatrix} \\ &\quad + \sum_{i=0}^{k-1} \Phi_{k-i-1} \begin{bmatrix} \tilde{B}_{11}^1 \\ \tilde{B}_{12}^1 \\ \tilde{B}_{21}^1 \\ \tilde{B}_{22}^1 \end{bmatrix} u(i), \\ &\quad k \in \mathbb{Z}_+, \end{aligned} \tag{36}$$

where Φ_i is defined by (19) with

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{1\alpha} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{2\beta} \end{bmatrix} = \begin{bmatrix} 1.5 & 0 & 1 & 2 \\ 0 & 0.5 & 3 & 4 \\ 3 & 0 & 0.7 & 1 \\ 1 & 3 & 0 & 0.8 \end{bmatrix},$$

$$D_i = \begin{bmatrix} c_\alpha(i+1)I_2 & 0 \\ 0 & c_\beta(i+1)I_2 \end{bmatrix},$$

$$\begin{bmatrix} \tilde{B}_{11}^1 \\ \tilde{B}_{12}^1 \\ \tilde{B}_{21}^1 \\ \tilde{B}_{22}^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.545 \\ -0.051 \end{bmatrix}. \tag{37}$$

Similarly, taking under consideration the sizes n_1^2 and n_2^2, the solution of the nilpotent part (17) has the form

$$\begin{bmatrix} \bar{x}_{11}^2(k) \\ \bar{x}_{21}^2(k) \end{bmatrix} = -\begin{bmatrix} 1 \\ 0.182 \end{bmatrix} u(k), \quad k \in \mathbb{Z}_+, \tag{38}$$

since

$$\begin{gathered} N_1 = N_2 = 0, \quad \mu_1 = \mu_2 = 0, \\ \begin{bmatrix} \tilde{B}_{11}^2 \\ \tilde{B}_{21}^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.182 \end{bmatrix}. \end{gathered} \tag{39}$$

To compute matrices Ψ_i for $i = -1, 0, 1, 2$, first we have to compute matrices Φ_i for $i = 0, 1, 2$, which in this example have the form

$$\begin{aligned} \Phi_0 &= I_4, \\ \Phi_1 &= \begin{bmatrix} \tilde{A}_{1\alpha} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{2\beta} \end{bmatrix} \\ &= \begin{bmatrix} 1.5 & 0 & 1 & 2 \\ 0 & 0.5 & 3 & 4 \\ 3 & 0 & 0.7 & 1 \\ 1 & 3 & 0 & 0.8 \end{bmatrix}, \end{aligned}$$

$$\Phi_2 = \begin{bmatrix} \tilde{A}_{1\alpha} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{2\beta} \end{bmatrix}^2 - \begin{bmatrix} \frac{\alpha(\alpha-1)}{2!} I_2 & 0 \\ 0 & \frac{\beta(\beta-1)}{2!} I_2 \end{bmatrix} = \begin{bmatrix} 7.38 & 6 & 2.2 & 5.6 \\ 13 & 12.38 & 3.6 & 8.2 \\ 7.6 & 3 & 3.64 & 7.5 \\ 2.3 & 3.9 & 10 & 14.79 \end{bmatrix} \tag{40}$$

and

$$\Psi_{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Psi_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Psi_1 = \begin{bmatrix} 1.5 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0.5 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0.7 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Psi_2 = \begin{bmatrix} 7.28 & 6 & 0 & 2.2 & 5.6 & 0 \\ 13 & 12.38 & 0 & 3.6 & 8.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 7.6 & 3 & 0 & 3.64 & 7.5 & 0 \\ 2.3 & 3.9 & 0 & 10 & 14.79 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{41}$$

The reachability matrix has the form

$$R_4 = \begin{bmatrix} Q\Psi_2\tilde{B} & Q\Psi_1\tilde{B} & Q\Psi_0\tilde{B} & Q\Psi_{-1}\tilde{B} \end{bmatrix} = \begin{bmatrix} 1.22 & 1.27 & 0 & 0 \\ 0.69 & 0.36 & 0 & 0 \\ -1.22 & -1.27 & 0 & 1 \\ 1.51 & -0.65 & -1.18 & -0.18 \\ 1.3 & 0.29 & 0.55 & 0 \\ 0 & 0 & 0 & 0.18 \end{bmatrix}, \tag{42}$$

and $\operatorname{rank} R_4 = 4$. Taking under consideration (31), the matrix G_4 has the form

$$G_4 = \operatorname{diag}[G, G, G, G], \tag{43}$$

and by the use of (30) with (42) and (43) we can compute

$$W_4 = R_4 G_4^{-1} R_4^T = \begin{bmatrix} 3.09 & 1.3 & -3.09 & 0.99 & 01.95 & 0 \\ 1.3 & 0.61 & -1.3 & 0.8 & 1 & 0 \\ -3.09 & -1.3 & 4.09 & -1.18 & -1.95 & 0.18 \\ 0.99 & 0.8 & -1.18 & 4.11 & 1.12 & -0.03 \\ 1.95 & 1 & -1.95 & 1.12 & 2.07 & 0 \\ 0 & 0 & 0.18 & -0.03 & 0 & 0.03 \end{bmatrix} \times 10^{19}. \tag{44}$$

Now, using (32) with (42)–(44) we obtain

$$\hat{u} = \begin{bmatrix} \hat{u}_3 \\ \hat{u}_2 \\ \hat{u}_1 \\ \hat{u}_0 \end{bmatrix} = G_4^{-1} R_4^T W_4^{-1} x_f = \begin{bmatrix} 1.06 & -0.75 & 0.07 & 0.23 & 0.53 & 1.59 \\ 1.61 & 0.22 & -0.003 & -0.19 & -0.5 & -4.77 \\ -0.52 & 0.07 & -0.09 & -0.45 & 0.89 & 0.31 \\ -0.08 & 0.05 & -0.22 & -0.004 & -0.002 & 6.67 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.88 \\ -2.02 \\ -0.39 \\ 6.11 \end{bmatrix}. \tag{45}$$

Using (34) and (44) we obtain

$$I(\hat{u}) = x_f^T W_4^{-1} x_f = 6.5 \times 10^{-4}. \tag{46}$$

♦

5. Concluding remarks

Descriptor fractional discrete-time linear systems with two different fractional orders were analyzed. The Weierstrass–Kronecker decomposition theorem of the regular pencil was used to find the solution of the state equation. Based on the proposed solution, a reachability condition was given. The minimum energy control problem for descriptor fractional discrete-time linear systems with two different fractional orders was formulated and solved. The effectiveness of the proposed method was demonstrated with numerical examples. Extension of these findings to systems consisting of more than two subsystems with different fractional orders is possible. An open problem is computation of the solution if the Weierstrass–Kronecker decomposition is impossible.

Acknowledgment

This work was supported by the National Science Centre in Poland under the work no. 2014/13/B/ST7/03467.

References

Campbell, S.L., Meyer, C.D., Rose, N.J. (1976). Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients, *SIAM Journal on Applied Mathematics* **31**(3): 411–425, DOI: 10.1137/0131035.

Caputo, M., Fabrizio, M. (2015). A new definition of fractional derivative without singular kernel, *Progress in Fractional Differentiation and Applications* **1**(2): 1–13, DOI: 10.12785/pfda/010201.

Dai, L. (1989). *Singular Control Systems*, Lectures Notes in Control and Information Sciences, Vol. 118, Springer-Verlag, Berlin, DOI: 10.1007/BFb0002475.

Dodig, M. and Stosic, M. (2009). Singular systems state feedbacks problems, *Linear Algebra and Its Applications* **431**(9): 1267–1292, DOI: 10.1016/j.laa.2009.04.024.

Dzieliński, A., Sierociuk, D. and Sarwas, G. (2009). Ultracapacitor parameters identification based on fractional order model, *European Control Conference ECC, Budapest, Hungary,* pp. 196–200.

Ferreira, N.M.F., and Machado, J.A.T. (2003). Fractional-order hybrid control of robotic manipulators, *11th International Conference on Advanced Robotics, ICAR, Coimbra, Portugal*, pp. 393–398, DOI: 10.1109/ICSMC.1998.725510.

Guang-Ren, D. (2010). *Analysis and Design of Descriptor Linear Systems*, Springer, New York, NY, DOI: 10.1007/978-1-4419-6397-0.

Kaczorek, T, and Klamka, J. (1986). Minimum energy control of 2D linear systems with variable coefficients, *International Journal of Control* **44**(3): 645–650.

Kaczorek, T. (1998). *Vectors and Matrices in Automation and Electrotechnics*, WNT, Warsaw.

Kaczorek, T. (2002). *Positive 1D and 2D Systems*, Springer-Verlag, London, DOI: 10.1007/978-1-4471-0221-2.

Kaczorek, T. (2010). Positive linear systems with different fractional orders, *Bulletin of the Polish Academy of Sciences: Technical Sciences* **58**(3): 453–458, DOI: 10.2478/v10175-010-0043-1.

Kaczorek, T. (2011a). *Selected Problems in Fractional Systems Theory*, Springer-Verlag, Berlin, DOI: 10.1007/978-3-642-20502-6.

Kaczorek, T. (2011b). Positive linear systems consisting of n subsystems with different fractional orders, *IEEE Transactions Circuits and Systems* **58**(6): 1203–1210, DOI: 10.1109/TCSI.2010.2096111.

Kaczorek, T. (2011c). Reduction and decomposition of singular fractional discrete-time linear systems, *Acta Mechanica et Automatica* **5**(5): 1–5.

Kaczorek, T. (2011d). Singular fractional discrete-time linear systems, *Control and Cybernetics* **40**(3): 1–8.

Kaczorek, T. (2013a). Descriptor fractional linear systems with regular pencils, *International Journal Applied Mathematics and Computer Science* **23**(2): 309–315, DOI: 10.2478/amcs-2013-0023.

Kaczorek, T. (2013b). Solution of the state equations of descriptor fractional discrete-time linear systems with regular pencils, *Technika Transportu Szynowego* **10**: 415–422.

Kaczorek, T. (2013c). Singular fractional continuous-time and discrete-time linear systems, *Acta Mechanica et Automatica* **7**(1): 26–33, DOI: 10.2478/ama-2013-0005.

Kaczorek, T. (2014). Minimum energy control of fractional descriptor positive discrete-time linear systems, *International Journal of Applied Mathematics and Computer Science* **24**(4): 735–743, DOI: 10.2478/amcs-2014-0054.

Klamka, J. (1991).*Controllability of Dynamical Systems*, Kluwer Academic Press, Dordrecht.

Klamka, J. (2009). Controllability and minimum energy control problem of infinite dimensional fractional discrete-time systems with delays, *1st Asian Conference on Intelligent Information and Database Systems ACIIDS, Dong Hoi, Vietnam*, DOI: 10.1109/ACIIDS.2009.53.

Klamka, J. (2010). Controllability and minimum energy control problem of fractional discrete-time systems, *in* D. Baleanu *et al.* (Eds.), *New Trends in Nanotechology and Fractional Calculus*, Springer-Verlag, New York, NY, pp. 503–509, DOI: 10.1007/978-90-481-3293-5_45.

Klamka, J. (2014). Controllability and minimum energy control of linear fractional discrete-time infinite-dimensional systems, *11th IEEE International Conference on Control & Automation ICCA, Taichung, Taiwan*, pp. 1210-1214, DOI: 10.1109/ICCA.2014.6871094.

Miller, K.S. and Ross, B. (1993). *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, NY.

Nishimoto, K. (1984). *Fractional Calculus*, Decartess Press, Koriama.

Oldham, K.B. and Spanier, J. (1974). *The Fractional Calculus*, Academic Press, New York, NY.

Podlubny, I. (1999). *Fractional Differential Equations*, Academic Press, San Diego, CA.

Popović, J.K., Pilipović S. and Atanacković, T.M. (2013). Two compartmental fractional derivative model with fractional derivatives of different order, *Communications in Nonlinear Science and Numerical Simulation* **18**(10): 2507–2514, DOI: 10.1016/j.cnsns.2013.01.004.

Sajewski, Ł. (2015). Solution of the state equation of descriptor fractional continuous-time linear systems with two different fractional, *in* R. Szewczyk *et al.* (Eds.), *Progress in Automation, Robotics and Measuring Techniques*, Advances in Intelligent Systems and Computing, Vol. 350, Springer, Cham, pp. 233–242, DOI: 10.1007/978-3-319-15796-2_24.

Sajewski, Ł. (2016a). Reachability, observability and minimum energy control of fractional positive continuous-time linear systems with two different fractional orders, *Multidimensional Systems and Signal Processing* **27**(1): 27–41, DOI: 10.1007/s11045-014-0287-2.

Sajewski, Ł. (2016b). Descriptor fractional discrete-time linear system and its solution—comparison of three different methods, *in* R. Szewczyk *et al.* (Eds.), *Challenges in Automation, Robotics and Measurement Techniques*, Advances in Intelligent Systems and Computing, Vol. 440, Springer, Cham, pp. 37–50, DOI: 10.1007/978-3-319-29357-8_4.

Sajewski, Ł. (2016c). Descriptor fractional discrete-time linear system with two different fractional orders and its solution, *Bulletin of the Polish Academy of Sciences: Technical Sciences* **64**(1): 15–20, DOI: 10.1515/bpasts-2016-0003.

Van Dooren, P. (1979). The computation of Kronecker's canonical form of a singular pencil, *Linear Algebra and Its Applications* **27**: 103–140, DOI: 10.1016/0024-3795(79)90035-1.

Van Dooren, P. and Beelen, T. (1988). An improved algorithm for the computation of Kronecker's canonical form of a singular pencil, *Linear Algebra and Its Applications* **105**: 9–65, DOI: 10.1016/0024-3795(88)90003-1.

Łukasz Sajewski (b. 1981, Poland) received his MSc degree in electrical engineering from the Białystok University of Technology in 2006 and then started his PhD studies. In 2007 he began lecturing in the field of microprocessor techniques and programmable logic devices. In 2009 he received a PhD degree in electrical engineering from the Białystok University of Technology. At present he works at the Faculty of Electrical Engineering. His main scientific interests are in control theory, especially descriptor, positive, continuous-discrete and fractional systems. He has published nearly 40 scientific papers and one book. His research interests also cover aspects of using programmable logic devices and programmable logic controllers in automatic control of industrial processes.

Received: 19 April 2016
Revised: 11 September 2016
Accepted: 10 December 2016