On identities satisfied by cancellative semigroups and their groups of fractions

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Abstract. A subsemigroup S of a group G is called *generating* if elements of S generate G as a group. If S satisfies an identity, then $G = SS^{-1} = S^{-1}S$ is a group of fractions of S. We recall the results concerning the problem which was open for more then 20 years, whether each identity satisfied in S must be satisfied in its group of fractions. The groups where the answer for this question is positive for every generating semigroup we call S-R-groups. We show that varieties of S-R-groups form a sublattice in the lattice of all group varieties.

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1. Preliminaries

Let F be the free group on the set $X = \{x_1, x_2, x_3, \ldots\}$, and let $\mathcal{F} \subseteq F$ be the cancellative free semigroup with the unity, generated by X.

- A group G satisfies an identity $u(x_1, \ldots, x_m) \equiv v(x_1, \ldots, x_m)$ if for every elements $g_1, \ldots, g_m$ in G the equality $u(g_1, \ldots, g_m) = v(g_1, \ldots, g_m)$ holds.
- An identity in a semigroup has a form $u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ where the words u and v are written without inverses of variables, $u, v \in \mathcal{F}$. Such an identity in a group is called a *semigroup identity*. It is clear that abelian groups and groups of finite exponent satisfy semigroup identities.
- A semigroup S satisfies left (right) Ore condition if for arbitrary $a, b \in S$ there are $a', b' \in S$ such that $a'a = b'b$ (*resp.* $aa' = bb'$).
- A subsemigroup S of a group G is called *a generating semigroup* if elements of S generate G as a group.

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- If a generating semigroup $S \subseteq G$ satisfies a nontrivial identity, then S satisfies Ore conditions (see e.g. [12, Proposition 1]).
- If a generating semigroup $S \subseteq G$ satisfies a nontrivial identity, then G is the group of fractions of S, that is $G = SS^{-1} = S^{-1}S$ [5].
- Each congruence $\rho \subseteq \mathcal{F} \times \mathcal{F}$ defines a subset $\mathcal{A}_\rho \subseteq \mathcal{F}\mathcal{F}^{-1}$, where

  $\mathcal{A}_\rho := \{ ab^{-1} \mid (a,b) \in \rho \}$, $\mathcal{A}_\rho^F$ *denotes the normal closure of* $\mathcal{A}_\rho$ *in* F.
- Let G be a group with a generating semigroup S. If S satisfies a nontrivial identity then there exists a congruence ρ on $\mathcal{F}$ such that $S \cong \mathcal{F}/\rho$, $G \cong F/\mathcal{A}_\rho^F$ and $\mathcal{A}_\rho^F \cap \mathcal{F}\mathcal{F}^{-1} = \mathcal{A}_\rho$ [6, Construction 12.3 and Corollary 12.8].

$$\begin{array}{ccccc} & \swarrow & X & \searrow & \\ & & \downarrow & & \\ S \cong \mathcal{F}/\rho & \longleftarrow & \mathcal{F} & \longrightarrow & F \\ & \searrow & & \swarrow & \\ & & G \cong F/\mathcal{A}_\rho^F & & \end{array}$$

So we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \hookrightarrow & F \\ \downarrow & & \downarrow \\ S \cong \mathcal{F}/\rho & \hookrightarrow & F/\mathcal{A}_\rho^F \cong G \end{array}$$

2. History of the topic

Since the 2-generator free semigroup $sgp(x,y)$ contains a free semigroup of infinite rank $sgp(x_1, x_2, \ldots)$, where each x_i is a word on x, y, an identity $u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ implies a binary identity. So if a generating semigroup $S \subseteq G$ satisfies a nontrivial identity then S satisfies a 2-variable semigroup identity which, by the cancellation property, may be assumed as $xa(x,y) = yb(x,y)$ or $u(x,y)x = v(x,y)y$. It follows that S satisfies left and right Ore conditions. Then elements in S satisfy the identities

$$y^{-1}x = b(x,y)a^{-1}(x,y) \qquad \text{and} \qquad xy^{-1} = u^{-1}(x,y)v(x,y),$$

which implies that

$$G = SS^{-1} = S^{-1}S,$$

that is G is **the group of fractions** for S [13, 15, 9] and [6, Theorem 1.23].

It is clear that if S is abelian then the group G is necessary abelian. However, if S is a free semigroup, then G is not necessary free, because by result of A.I. Mal'tsev [13], F/F'' has a free generating semigroup.

The problem how far properties of a generating semigroup S define the properties of the group G attracted attention of many authors.

A.I. Mal'tsev considered semigroup identities of the form $u_n = v_n$, where u_n, v_n are words on letters x, y, $z_1, \ldots, z_n$, defined inductively as follows

$$u_0 = x, \quad v_0 = y, \quad u_1 = u_0 z_1 v_0, \quad v_1 = v_0 z_1 u_0, \qquad \text{and for} \quad n > 0$$

$$u_{n+1} = u_n z_{n+1} v_n, \qquad v_{n+1} = v_n z_{n+1} u_n.$$

In [13] A.I. Mal'tsev proved that a group is nilpotent of class at most n if and only if it satisfies the identity $u_n = v_n$. Moreover, if the identity $u_n = v_n$ is satisfied in a generating semigroup $S \subseteq G$ then it is satisfied in G which must be nilpotent of class at most n. We shall call the laws with such a property transferable.

2.1. Transferable identities

The following well known problems are due to G. Bergman [1, 2].

GB-Problem. *Let G be a group with a generating semigroup S. Must each identity satisfied in S be satisfied in G?*

Another formulation of this problem is whether every proper variety of semigroups is closed with respect to groups of fractions [17, Question 11.1].

The following question was posed in [10, page 95]. Let a semigroup identity $a = b$ imply a semigroup identity $u = v$ in groups. Does the same implication hold in semigroups? The equivalence of this question with the GB-Problem was proved in [11].

Definition 2.1. *Let S be a generating semigroup in a group G. We call an identity $u = v$* **transferable** *if being satisfied in S, it must be satisfied in G.*

For example, the nilpotent identities $u_n = v_n$ found by A.I. Mal'tsev [13] are transferable. In this terminology the GB-Problem asks:

Is every semigroup identity transferable?

Another weaker problem posed by G. Bergman was: *Must the group G satisfy some group identity if its generating semigroup S satisfies a nontrivial semigroup identity?*

In 2005 S.V. Ivanov and A.M. Storozhev [7] gave negative answer to both questions. Their counterexample-group G (in fact a family of them) with a generating semigroup S contains a free subgroup (hence satisfies no group identity) while S satisfies a semigroup identity similar to that introduced by A.Yu. Ol'shanskii in [16]. Since the problems have negative answers in general, the questions arise:

1. **Which semigroup identities are transferable?**
2. **In which groups all semigroup identities are transferable?**

In 1986 when the problems were discussed in G. Bergman's "Problem Seminar" in Berkeley, it was shown that the identity $x^2y^2 = y^2x^2$ is transferable. In 1992 it was shown that the identities $x^n y^n = y^n x^n$ are transferable for all natural n [8]. Besides these identities, Mal'tsev identities and the identity $x^n \equiv 1$ no other examples are described.

As to the second of the above question, Theorem C in [3] says that Bergman's question has an affirmative answer for soluble groups: if G is a soluble group (or, slightly more generally, an extension of a soluble group by a locally finite group of finite exponent), and $S \subseteq G$ is any generating subsemigroup satisfying a positive law, then that law holds in G.

2.2. Semigroup respecting groups

We define now so called S-R groups, the groups where all semigroup identities are transferable.

Definition 2.2 ([12]). *We call a group G* **semigroup respecting** *(S-R group) if all of the identities holding in any generating semigroup of G, hold in G.*

For example, torsion groups are S-R groups, since $S^{-1} = S = G$.

In 2008 the following properties of S-R groups were proved [12]:

- The property of a group to be S-R group is a "local property" in the sense of Mal'tsev, that is if every finitely generated subgroup of a group G is the S-R group, then so is G.
- The class of locally residually finite groups consists of S-R groups.
- Every linear group over a field is the S-R group.
- The class of locally graded groups without free noncyclic subsemigroups consists of S-R groups. We recall that a group G is called *locally graded* if every nontrivial finitely generated subgroup of G has a proper subgroup of finite index. The class of locally graded groups was introduced in 1970 by Černikov to avoid groups such as the infinite Burnside groups or Ol'shanskii-Tarski monsters. We note that all locally or residually soluble groups and all locally or residually finite groups are locally graded. The class of locally graded groups is closed for taking subgroups, extensions and cartesian products.

3. End^+ invariance and S-R-property

Let F be *the free group* on the set $X = \{x_1, x_2, x_3, \ldots\}$, and let $\mathcal{F} \subseteq F$ be the cancellative free semigroup with the unity, generated by X. By $End\, F$ we denote the set of all endomorphisms in F.

By End^+ we denote the set of so called *positive* endomorphisms in F, which map $X \to \mathcal{F}$. The set End^+ can be identified with the set $End\,\mathcal{F}$ of all endomorphisms of the semigroup $\mathcal{F}$. The inclusion $End^+ \subseteq End\, F$ is clear. We recall that $End\, F$-invariant subgroup is called **fully invariant** or verbal. Every group variety is uniquely defined by a verbal subgroup $V \subseteq F$ [14].

In view of the construction [6, Theorem 1.23] we have the following:

- A semigroup $S \cong \mathcal{F}/\rho$ is relatively free if and only if the set $\mathcal{A}_\rho$ and hence the normal subgroup $\mathcal{A}_\rho^F$ are End^+-invariant.
- The group of fractions of S, $G = F/\mathcal{A}_\rho^F$ is relatively free if and only if the normal subgroup $\mathcal{A}_\rho^F$ is End-invariant.
- Each normal End^+-invariant subgroup N in F defines a cancellative congruence ρ on $\mathcal{F}$ by $\mathcal{A}_\rho = N \cap \mathcal{F}\mathcal{F}^{-1}$. If $\mathcal{A}_\rho \neq 1$, then $\mathcal{A}_\rho^F = N$.

Hence the following two questions concerning S-R-groups are equivalent:

Must a group of fractions of a relatively free semigroup be the relatively free group?

Must each End^+-invariant normal subgroup in F be fully invariant?

We show that being invariant with respect to a special automorphism α gives a criterion for a normal End^+-invariant subgroup $N \lhd F$ to be fully invariant. We start with two Lemmas.

Lemma 3.1. *If a normal subgroup $N \lhd F$ is End^+-invariant and $N \cap \mathcal{F}\mathcal{F}^{-1} \neq 1$ then $F = \mathcal{F}\mathcal{F}^{-1}N = \mathcal{F}^{-1}\mathcal{F}N$.*

Proof. By assumption there are two different words $a', b' \in \mathcal{F}$ such that $a'b'^{-1} \in N$ hence $a' \equiv b'$ modulo N. Then by cancellation there are $a, b \in \mathcal{F}$ such that $xa(x,y) \equiv yb(x,y)$ modulo N. That is $x^{-1}y \in a(x,y)b^{-1}(x,y)N$. Then since N is normal and End^+-invariant, we get $F = \mathcal{F}\mathcal{F}^{-1}N$. The second equality follows by conjugation. □

Lemma 3.2. *If $F = \mathcal{F}\mathcal{F}^{-1}N = \mathcal{F}^{-1}\mathcal{F}N$ and $g_1, g_2, \dots, g_n$ are in F, then there are $s_1, s_2, \dots, s_n$, and r in $\mathcal{F}$ such that modulo N $g_i = s_i r^{-1}$, $i = 1, 2, \dots, n$.*

Proof. All calculations in the proof are assumed modulo N. If $n = 1$, the statement is clear. To proceed by induction, let $g_i = t_i q^{-1}$ for $i \leqslant n-1$ and $g_n = ab^{-1}$ for some $t_i, q, a, b \in \mathcal{F}$.

By Ore conditions, which are satisfied in F modulo N, there exist q', $b' \in \mathcal{F}$ such that modulo N $qq' = bb'$. We denote $r := qq' = bb'$, $s_i := t_i q'$ for $i \leqslant n-1$ and $s_n := ab'$. Then

$$g_i = t_i q^{-1} = t_i(q'{q'}^{-1})q^{-1} = (t_i q')({q'}^{-1}q^{-1}) = s_i r^{-1},$$

$$g_n = ab^{-1} = a(b'{b'}^{-1})b^{-1} = (ab')({b'}^{-1}b^{-1}) = s_n r^{-1}$$

which finishes the proof. □

We recall that F is the free group on the set $X = \{x_1, x_2, x_3, \dots\}$. Let $\alpha \in Aut\,F$ fix x_1 and map $x_i \to x_i x_1^{-1}$, $i \neq 1$.

Lemma 3.3. *If $N \lhd F$ is End^+-invariant and α-invariant then N is fully invariant.*

Proof. First we note that $\alpha^{-1} \in End^+$ and hence $N^{\alpha^{-1}} \subseteq N$. Hence every End^+-invariant subgroup N in F satisfies the inclusion

$$N \subseteq N^{\alpha}. \tag{1}$$

Since by assumption $N^{\alpha} \subseteq N$, we have $N^{\alpha} = N$. To show that N is fully invariant it suffices to check that if $w(x_1, \dots, x_{n-1})$ is a word in N then $w(g_2, \dots, g_n)$ also is in N for every $g_2, g_3, \dots, g_n$ in F.

So let $w(x_1, \dots, x_{n-1})$ be in N. Since the map $x_i \to x_{i+1}$ is in End^+, we have that $w = w(x_2, \dots, x_n)$ also is a word in N. Then N contains $w^{\alpha} = w(x_2 x_1^{-1}, \dots, x_n x_1^{-1})$ In view of Lemma 3.2 we can find $s_2, \dots, s_n, r \in \mathcal{F}$ such that $g_i = s_i r^{-1} mod N$. Then we map $x_1 \to r$ and $x_i \to s_i$. Since N, being End^+ invariant, is invariant to this mapping, we get $w(g_2, \dots g_n) \in N$ and hence N is fully invariant as required. □

Corollary 3.4. *Let N be a normal End^+-invariant subgroup in F. Then N is fully invariant if and only if $N = N^{\alpha}$.*

4. Varieties of S-R groups

The varieties of groups form a set partially ordered by inclusion, which is a complete lattice by means of the following definitions of greatest lower and least upper bound: $\mathfrak{M}_1 \wedge \mathfrak{M}_2$ is the variety with the set of laws defined by the verbal subgroup M_1M_2, while $\mathfrak{M}_1 \vee \mathfrak{M}_2$ has the set of laws defined by the verbal subgroup $M_1 \cap M_2$ [14]. The one-to-one correspondence between verbal subgroups in F and varieties reverses the inclusion relations.

In this section we consider group varieties consisting of S-R-groups. These varieties are defined by specific verbal subgroups in F which, as we show, form a sublattice in the lattice of all verbal subgroups in F. Because of the duality the same follows for the S-R-varieties.

Definition 4.1. *We say that a variety $\mathfrak{M}$ is semigroup respecting (S-R-variety) if it consists of S-R-groups.*

By V_N we denote the fully invariant closure of a subgroup $N \subseteq F$.

Corollary 4.2. *A verbal subgroup $M \subseteq F$ defines an S-R-variety if and only if every End^+-invariant normal subgroup N in F is fully invariant modulo M, that is $NM = V_NM$, which by Corollary 3.4 is equivalent to $NM = N^\alpha M = (NM)^\alpha$.*

For example, since by result of P. Hall [4] finitely generated nilpotent groups are residually finite, it follows (see Section 2.2) that nilpotent varieties are semigroup respecting.

We recall here the following well known result

Lemma 4.3. *Let $A, B, C \subseteq G$ and $A \subseteq C$ then $AB \cap C = A(B \cap C)$.*

Proof. The inclusion "$\supseteq$" is clear. The opposite inclusion follows since $ab = c \in AB \cap C$ implies $b = a^{-1}c \in B \cap C$, hence $ab = c \in A(B \cap C)$. □

Now we can prove

Theorem 4.4. *Semigroup respecting varieties form a modular lattice.*

Proof. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two S-R-varieties defined by verbal subgroups $M_1, M_2 \in F$. By Corollary 4.2 this is equivalent to the fact that for any normal End^+-invariant subgroup N the equalities hold:

$$NM_1 = N^\alpha M_1, \qquad NM_2 = N^\alpha M_2. \tag{2}$$

Thus we have to prove that for any normal End^+-invariant subgroup N,

$$NM_1M_2 = N^\alpha M_1M_2, \qquad N(M_1 \cap M_2) = N^\alpha(M_1 \cap M_2).$$

The first equality is clear by assumptions on M_1. For any fixed N we denote

$$D = N(M_1 \cap M_2), \text{ then } D^\alpha = N^\alpha(M_1 \cap M_2).$$

So, the only we have to prove is the equality $D = D^\alpha$. Now we note the following properties of the subgroup D.

- If a subgroup M is fully invariant then $D^\alpha \cap M = (D \cap M)^\alpha$.
- Since the subgroup D is End^+-invariant, $D \subseteq D^\alpha$ and by Lemma 4.3, for $A := D$ the following equality holds

$$DM_1 \cap D^\alpha = D(M_1 \cap D^\alpha). \tag{3}$$

- Since D is End^+-invariant and M_1 defines an S-R-variety, so by (2)

$$DM_1 = D^\alpha M_1. \tag{4}$$

- Since $D \cap M_1$ is End^+-invariant and M_2 defines an S-R-variety, also by (2)

$$(D \cap M_1)M_2 = (D \cap M_1)^\alpha M_2 \supseteq (D \cap M_1)^\alpha. \tag{5}$$

- By Lemma 4.3, for $A := (D \cap M_1) \subseteq M_1$ the following equality holds

$$(D \cap M_1)M_2 \cap M_1 = (D \cap M_1)(M_2 \cap M_1) = (D \cap M_1). \tag{6}$$

We intersect (5) with M_1 then

$$(D \cap M_1)M_2 \cap M_1 \supseteq (D \cap M_1)^\alpha \cap M_1 = (D \cap M_1)^\alpha,$$

which, in view of (6), gives $D \cap M_1 \supseteq (D \cap M_1)^\alpha$. Since $(D \cap M_1)$ is End^+-invariant, $D \cap M_1 \subseteq (D \cap M_1)^\alpha$, which implies the equalities

$$D \cap M_1 = (D \cap M_1)^\alpha = D^\alpha \cap M_1. \tag{7}$$

Now we can obtain required $D^\alpha = D$, because

$$D^\alpha = D^\alpha M_1 \cap D^\alpha \overset{(4)}{=} DM_1 \cap D^\alpha \overset{(3)}{=} D(D^\alpha \cap M_1) \overset{(7)}{=} D(D \cap M_1) = D.$$

So, by Corollary 4.2, D is fully invariant, which finishes the proof. □

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