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SOMMAIRE

Articles

AN ALTERNATE PROOF OF A THEOREM ON THE ASYMPTOTIC OPTIMALITY OF SOME ESTIMATORS

UMBERTO AMATO AND DAN TUDOR VUZA

Smoothing data is a major problem in data analysis due to the large amount of real world applications which need such a problem to be solved, e.g., 1D signals (speech, time series of any physical quantity), 2D and 3D signals (images, measurements). Many wavelet-based smoothing data (or denoising) schemes reside on the following common principle:

a) apply a discrete wavelet transform to the input data $x_1, \dots, x_n$ affected by noise;

b) apply statistical estimators to the collection $y_1, \dots, y_n$ produced by the preceding step with the hope to recover most of the true values $\theta_1, \dots, \theta_n$ of the wavelet transform;

c) apply the inverse wavelet transform to the collection $\eta_1, \dots, \eta_n$ produced by the preceding step.

It is hoped that the high capability of wavelets to concentrate most information in a few coefficients, while spreading noise uniformly over all coefficients, improves the performance of statistical estimators when used as above rather on the raw data $x_1, \dots, x_n$.

Donoho and Johnstone devoted many researches to the statistical aspects implied by step b) above. Suppose that the noisy data $y = (y_1, \dots, y_n)$ are spread around the true data $\theta = (\theta_1, \dots, \theta_n)$ according a certain probability distribution (depending on θ) so that $Ey_i = \theta_i$ (E means expected value). The performance of a statistical $n-$ variate estimator $\eta_i = \eta_i(y_1, \dots, y_n)$ is measured by the risk function $R(\eta, \theta) = E\sum_{i=1}^{n}(\eta_i - \theta_i)^2$. The minimum of $R(\eta, \theta)$ over all estimators of the form $\eta_i = \epsilon_i y_i$ with $\epsilon \in \{0, 1\}$ equals $\sum_{i=1}^{n} \min(\theta_i^2, \sigma_i^2)$, where $\sigma_i^2 = E(y_i - \theta_i)^2$, and it is considered by Donoho and Johnstone as a sort of "benchmark" for evaluating the performance of various estimators with respect to wavelet-based denoising.

One of their most significant results [3] asserts that the soft threshold estimator given by

$$\eta_i = \operatorname{sgn} y_i(|y_i| - \sigma\sqrt{2\log n})_+$$

has a risk at most $O(\log n)$ times the benchmark, in the situation when $y \sim N(\theta, \sigma^2 I_n)$ (normal $n-$ variate distribution with mean θ and covariance matrix equal to σ^2 times the identity). More precisely, they proved the inequality

$$R(\eta,\theta) \le (2\log n + 1)\left(\sigma^2 + \sum_{i=1}^{n}\min(\theta_i^2,\sigma^2)\right)$$

for the above defined η. The natural question which arouse was whether the factor $2\log n$ in the above inequality could be improved by using maybe another estimator. In the same paper, Donoho and Johnstone showed that the value $2\log n$ is asymptotically optimal, by proving the inequality

(1) $$\liminf_{n\to\infty}\frac{1}{2\log n}\ \inf_{\eta}\sup_{\theta}\frac{R(\eta,\theta)}{\sigma^2+\sum_{i=1}^{n}\min(\theta_i^2,\sigma^2)} \ge 1.$$

Recently, Johnstone and Silverman considered in [4] the case $y \sim N(\theta, V_n)$ for a general covariance matrix V_n. They proved the inequality

$$R(\eta,\theta) \le (2\log n + 1)\left(\overline{\sigma}^2 + \sum_{i=1}^{n}\min(\theta_i^2,\sigma_i^2)\right)$$

for the soft threshold estimator given by

$$\eta_i = \operatorname{sgn} y_i(|y_i| - \sigma_i\sqrt{2\log n})_+,$$

where σ_i^2 are the diagonal elements of V_n and $\overline{\sigma}^2 = n^{-1}\sum_{i=1}^{n}\sigma_i^2$. They also proved that the factor $2\log n$ is again asymptotically optimal in the sense that

(2) $$\liminf_{n\to\infty}\frac{1}{2\log n}\frac{\overline{\sigma}^2}{\overline{\tau}^2}\inf_{\eta}\sup_{\theta}\frac{R(\eta,\theta)}{\overline{\sigma}^2+\sum_{i=1}^{n}\min(\theta_i^2,\sigma_i^2)} \ge 1,$$

where $\overline{\tau}^2$ is a quantity depending on V_n to be defined in the following. The inequality (1) is a particular case of (2) as $\overline{\sigma}^2 = \overline{\tau}^2 = \sigma^2$ in that situation. Their proof of (2) relied on the additional hypothesis

(3) $$\frac{n^{-1}\sum_{i=1}^{n}\sigma_i^4}{(n^{-1}\sum_{i=1}^{n}\sigma_i^2)^2} \le C_1,\ \frac{\overline{\sigma}^2}{\overline{\tau}^2} \le C_2$$

with C_1 and C_2 not depending on n.

The aim of our contribution is to present an alternate proof of (2). Our proof also relies on the principle used in [3] and [4], i.e., turning θ into a random variable and using the Bayes risk with respect with a certain prior on θ. However, by applying this method in a slightly different way, our proof is somewhat simpler and does not make of hypothesis (3).

Let $\gamma_n(y_1,\dots,y_n)$ be a Gaussian distribution with covariance matrix V_n. It is a standard fact that γ_n can be written as

$$\frac{1}{\tau_1}\gamma_1\left(\frac{y_1 - m(y_2,\dots,y_n)}{\tau_1}\right)\gamma_{n-1}(y_2,\dots,y_n)$$

where m is a linear function of $y_2, \dots, y_n$, γ_{n-1} is a Gaussian distribution and τ_1 does not depend on $y_2, \dots, y_n$ (in fact τ_1^2 is the inverse of the first diagonal element of V_n^{-1}).

In a similar way one can define τ_i for $2 \le i \le n$.

Johnstone and Silverman [4] consider the numbers

$$\overline{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} \sigma_i^2, \qquad \overline{\tau}^2 = \frac{1}{n}\sum_{i=1}^{n} \tau_i^2,$$

where σ_i^2 are the diagonal elements of V_n. With these notations, our result is the following.

THEOREM 1. *Let $\eta_i(y_1, \dots, y_n)$ be an estimator of θ_i from the data $(y_1, \dots, y_n) \sim N(\theta, V_n)$ and let C_n be a constant for which*

$$R(\eta, \theta) \le C_n \left(\overline{\sigma}^2 + \sum_{i=1}^{n} \min(\theta_i^2, \sigma_i^2) \right) \tag{4}$$

is true for every $\theta_1, \dots, \theta_n$. Then

$$\liminf_{n \to \infty} \frac{1}{2 \log n} \frac{\overline{\sigma}^2}{\overline{\tau}^2} C_n \ge 1.$$

We refer to [1] for details.

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Some more abote Riemann–Stieltjes Integral

I. Bucur

The purpose of this paper is to compare the different definitions of Riemann–Stieltjes integrability of real line, one can encounter in literature. Surprisengly we remark that the terminology of "Riemann–Stieltjes integrability" is often used for two different notions. We introduce a new concept of "Darboux–Stieltjes integrability" in order to avoid this ambiguity and we study, among others, the relations of this concept with the other types of integrability.

All the functions used hear will be bounded. We remember the following definitions: If $[a,b]$ is a compact interval of the real line then any finite and increasing family of the real line $\Delta = (x_i)_{i \leq n}$, $x_i \in [a,b]$,
$a = x_0 \leq x_1 \leq x_2 \leq \dots x_n = b$ is called *a division* of $[a,b]$. The set of all division of $[a,b]$ is denoted by $\mathcal{D}[a,b]$, or simply $\mathcal{D}$. If $\Delta_1, \Delta_2 \in \mathcal{D}$ we put $\Delta_1 \leq \Delta_2$ if $\{x_0^1, x_2^1, \dots x_{n_1}^1\} \subset \{x_0^2, x_1^2, x_2^2, \dots, x_{n_2}^2\}$ where

$$\Delta_1 = (x_i^1)_{0 \leq i \leq n_1}, \quad \Delta_2 = (x_i^2)_{0 \leq i \leq n_2}.$$

If $\Delta = (x_i)_{0 \leq i \leq n}$ is an element of $\mathcal{D}$ then we call an *intermediary division* of Δ any division $\Delta' = (y_i)_{0 \leq i \leq n+1}$ of $[a,b]$ such that $y_i \in [x_{i-1}, x_i]$ for all $i \in \{1, 2, \dots n\}_i$.

We shall write simply $\Delta' \perp \Delta$. Also, if $f, g : [a,b] \to \mathbb{R}$ are two bounded functions, and Δ, Δ' as above we denote

$$\sigma(f, g, \Delta, \Delta') = \sum_{i=1}^{n} f(y_i)(g(x_i) - g(x_{i-1})).$$

and by $\|\Delta\|$ the positive real number $\sup\{(x_i - x_{i-1}) \backslash i \in \{1, 2, \dots n\}\}$. We rembember the following classical definition:

Definition A. We say that the function f *is Riemann–Stieltjes integrable with respect to the function* g if there exists a real number $I \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that $/\sigma(f, g; \Delta, \Delta') - I/ < \varepsilon$ for any $\Delta \in \mathcal{D}[a,b]$ with $\|\Delta\| < \eta_\varepsilon$ and any $\Delta' \in \mathcal{D}[a,b]$ with $\Delta' \perp \Delta$.

It is well known that the function f is Riemann–Stieltjes integrable with respect to the function g iff for any sequence $(\Delta_n)_n$ from $\mathcal{D}$ such that $\lim_{n\to\infty} \|\Delta_n\| = 0$ and any sequence $(\Delta')_n$ from $\mathcal{D}$ with $\Delta'_n \perp \Delta_n$, for any $n \in \mathbb{N}$, the sequence $(\sigma(f, g, \Delta_n, \Delta'_n))_n$ is convergent. The limit does not depend on the sequences $(\Delta_n)_n$ and $(\Delta'_n)_n$ as above and it is noted by $I = \int_a^b f dg$.

We recall also

Definition B. If $f = [a, b] \to \mathbb{R}$ is bounded any $g = [a, b] \to \mathbb{R}$ is an increasing function then we say that *f is D-Riemann–Stieltjes integrable with respect to g* if we have

$$\inf \{S(f, g, \Delta)/\Delta \in \mathcal{D}\} = \sup\{s(f, g, \Delta)/ \Delta \in \mathcal{D}\}$$

where for any $\Delta = (x_i)_{0\leq i\leq n} \in \mathcal{D}$ we have denoted

$$S(f, g, \Delta) = \sum_{i=0}^{n-1} M_i(g(x_{i+1} - g(x_i)),\ s(f, g, \Delta) = \sum_{i=0}^{n} m_i(g(x_{i+1}) - g(x_i))$$

$$M_i = \sup \{f(x)/x \in [x_i, x_{i+1}]\},\ m_i = \inf \{f(x)/x \in [x_i, x_{i+1}]\}; i = 0, 1, 2 \dots, n - 1.$$

We added expressly the particle D, before Riemann–Stieltjes..., to mark that there is a distinction between the Riemann–Stieltjes integrability given in the above definitions A and B even if in many mathematical treatises such a distinction is not made.

The following well known example shows that distinction.

Let $f, g : [0, 2] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } 1 \leq x \leq 2 \\ 0 & \text{if } 0 \leq x < 1 \end{cases} \qquad g(x) = \begin{cases} 1 & \text{if } 1 < x \leq 2 \\ 0 & \text{if } 0 \leq x \leq 1 \end{cases}$$

Obviously g is increasing and f is bounded. If we consider $\Delta \in \mathcal{D}\ [0, 2]$, $\Delta = \{0, 1, 2\}$ then we have

$$S(f, g, \Delta = 1 \cdot (g(1) - g(0)) + 1(g(2) - g(1)) = 1$$

$$s(f, g, \Delta) = 0(g(1) - g(0)) + 1(g(2) - g(1)) = 1$$

i.e. f is D-Riemann–Stieltjes integrable with respect to g. On the other hand if we consider $\Delta \in \mathcal{D}[0, 2]$, $\Delta = (x_i)_{0\leq i\leq n}$ such that $x_i \neq 1$ for any $i = 1, 2, \dots n - 1$ then there exists $i \in \{1, 2, \dots n\}$ such that $x_{i-1} < 1 < x_i$. We consider also an intermediary division Δ' of $[0, 2]$ $\Delta' = (y_j)_{0\leq j\leq n+1}$. Since $y_i \in [x_{i-1}, x_i]$ we may have either $y_i \in [x_{i-1}, 1)$ or $y_i \in [1, 2]$. Hence we shall have

$$\sigma(f, g, \Delta, \Delta') = 0, \text{respectively}\ \sigma(f, g, \Delta, \Delta') = 1$$

and therefore f is not Riemann–Stieltjes integrable with respect to g in the sens of definition A.

It is not difficult to see that if g is a real, increasing function and f is a bounded function on $[a, b]$ such that f is Riemann Stieltjes integrable with respect to g then f is also D–Riemann Stietjes integrable with respect to g. Hence the Riemann–Stieltjes integrability with respect to real increasing functions is stronger than D Riemann Stieltjes integrability.

We give now the following definition, larger than definition A:

Definition C. Let $f, g : [a,b] \to \mathbb{R}$ be two real bounded functions. We shall say that f is *Darboux-Stieltjes integrable* with respect to g if for any $\varepsilon > 0$ there exists $\Delta_\varepsilon \in \mathcal{D}[a,b]$ such that for any $\Delta', \Delta'' \in \mathcal{D}[a,b]$ with $\Delta_\varepsilon \leq \Delta', \Delta_\varepsilon \leq \Delta''$ we have

$$|\sigma(f,g,\Delta',d') - \sigma(f,g,\Delta'',d'')| < \varepsilon \quad \forall d', d'' \in \mathcal{D}_{[a,b]}, d' \perp \Delta', d'' \perp \Delta''.$$

1. Proposition. *Let $f, g : [a,b] \to \mathbb{R}$ be two real, bounded functions. Then f is Darboux-Stieltjes integrable with respect to g iff there exists an increasing sequence $(d_n)_n$ in $\mathcal{D}_{[a,b]}(d_n \leq d_{n+1}$ for any $n \in N)$ such that for any sequences $(\Delta'_n)_n \subset \mathcal{D}_{[a,b]}$ and $(d'_n)_n \subset \mathcal{D}_{[a,b]}$ with $d_n \leq \Delta'_n$ and $d'_n \perp \Delta'_n$ for any $n \in \mathbb{N}$, the sequence $(\sigma(f,g,\Delta'_n,d'_n))_n$ is convergent.*

The limit of the sequence $(\sigma(f,g,\Delta'_n,d'_n))_n$ does not depend on the chosen sequences $(\Delta'_n)_n, (d'_n)_n$ in $\mathcal{D}_{[a,b]}$ and will be denoted by $\int_a^b f dg$.

Proof. We suppose that f is Darboux-Stieltjes integrable w.r. to g. For any $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}^*$ we consider a division Δ_n of $[a,b]$ such that for any $\Delta', \Delta'', d', d'' \in \mathcal{D}_{[a,b]}$ such that $\Delta_n \leq \Delta', \Delta_n \leq \Delta''$, $d' \perp \Delta', d'' \perp \Delta''$ we have

$$|\sigma(f,g,\Delta',d') - \sigma(f,g,\Delta''d'')| < \frac{1}{n}.$$

Now, we consider a new sequence $(d_n)_n$ in $\mathcal{D}_{[a,b]}$ such that $\Delta_i \leq d_n$ for any $i \leq n$ and $d_n \leq d_{n+1}$ for any $n \in \mathbb{N}^*$.

If $(\Delta'_n)_n$ and $(d'_n)_n$ are two sequences in $\mathcal{D}_{[a,b]}$ such that $d_n \leq \Delta'_n$ and $d'_n \perp \Delta_n{}'$ for any $n \in \mathbb{N}$ we have $n, m \geq k \longrightarrow \Delta_k \leq d_n \leq \Delta'_n,\ \Delta_k \leq d_m \leq \Delta'_m$
and therefore

$$n, m \geq h \longrightarrow |\sigma(f,g,\Delta_n,d'_n) - \sigma(f,g,\Delta_m,d'_m)| < \frac{1}{k}$$

i.e. the real sequence $(\sigma(f,g,\Delta_n,d'_n))_n$ is convergent.

By a mixing procedure one can see that the limit of this sequence does not depend on the sequences $(\Delta_n)_n, (d'_n)$ in $\mathcal{D}$ chosen as above.

Conversely, we suppose that there exists an increasing sequence $(d_n)_n$ in $\mathcal{D}_{[a,b]}$ such that for any sequences $(\Delta'_n)_n, (d'_n)_n$ in $\mathcal{D}$ with $d_n \leq \Delta'_n$, $d'_n \perp \Delta'_n$ $(\forall) n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \sigma(f,g,\Delta'_n,d'_n) = I$$

where I is a real number. Let now $\varepsilon > 0$ be arbitrary. We assert that there exists $n_\varepsilon \in \mathbb{N}$ such that for any $\Delta', d' \in D$ with $d_{n_\varepsilon} \leq \Delta'$ and $d' \perp \Delta'$ we have $|\sigma(f,g,,\Delta',d') - I| < \varepsilon$.

Indeed, in the contrary case there exists $\varepsilon_0 > 0$ such that for any $n \in \mathbb{N}$ there are $\Delta'_n, d'_n \in \mathcal{D}$ such that

$$d_n \leq \Delta'_n, d'_n \perp \Delta'_n \text{ and } /\sigma(f,g,\Delta'_n,d'_n) - I/ \leq \varepsilon_0$$

The last inequality contradicts the fact than $\lim_{n\to\infty} \sigma(f, g, \Delta'_n, d'_n) = I$.

2. Corollary *If f, g are as in Proposition 1 then f is Darboux-Stieltjes integrable w.r. to g iff there exists an increasing sequence $(d^\circ_n)_n$ in $\mathcal{D}$ with $\lim_{n\to\infty} \|d^\circ_n\| = 0$ such that for any sequences $(\Delta'_n)_n, (d'_n)_n$ in $\mathcal{D}$ with $d^\circ_n \leq \Delta'_n$ and $d'_n \perp \Delta'_n$, for any $n \in \mathbb{N}$, the sequence $(\sigma(f, g, \Delta'_n, d'_n))_n$ is convergent.*

It is sufficient to consider the sequence $(d_n)_n$ from Proposition 1 and to choose a sequence $(d^\circ_n)_n$ which is increasing, with $\lim_{n\to\infty} \|d^\circ_n\| = 0$ and such that $d_n \leq d^\circ_n$ for any $n \in \mathbb{N}$.

3. Corollary. *If $f, g : [a, b] \to \mathbb{R}$ are bounded and f is Riemann–Stieltjes integrable w.r. to g (in the sens of Definition A) then f si Darboux–Stieltjes integrable w.r. to g.*

4. Corollary . *If $f, g : [a, b] \to \mathbb{R}$ are bounded then f is Darboux–Stieltjes integrable w.r. to g iff there exists a real number I such that for any $\varepsilon > 0$ there exists $\Delta_\varepsilon \in \mathcal{D}$ such that for any Δ, $d \in \mathcal{D}$ with $\Delta_\varepsilon \leq \Delta$ and $d \perp \Delta$ we have*

$$/\sigma(f, g, \Delta, d) - I/ \leq \varepsilon$$

Proof. If $/\sigma(f, g, \Delta, d) - I/ \leq \varepsilon$ for all $\Delta, d \in \mathcal{D}$ with $\Delta_\varepsilon \leq \Delta, d \perp \Delta$ then for any $\Delta', \Delta'', d', d'' \in \mathcal{D}$ with $\Delta_\varepsilon \leq \Delta', \Delta_\varepsilon \leq \Delta'', d' \perp \Delta', d'' \perp \Delta''$ we have $/\sigma(f, g, \Delta, d) - I/ \leq \varepsilon, /\sigma(f, g, \Delta'', d'') - I/ < \varepsilon$ and therefore

$$/\sigma(f, g, \Delta, d) - /\sigma(f, g, \Delta, d)/ \leq 2\varepsilon$$

i.e. f is Darboux–Stieltjes integrable w.r. to g. Conversely, if f is Darboux-Stieltjes integrable w.r. to g then for any $\varepsilon > 0$ we choose $\Delta_\varepsilon \in \mathcal{D}$ such that

$$\begin{aligned} &\Delta', \Delta', d', d'' \in \mathcal{D}, \Delta_\varepsilon \leq \Delta', \Delta_\varepsilon \leq \Delta'' d' \perp \Delta', d'' \perp \Delta'' \\ &\Longrightarrow /\sigma(f, g, \Delta, d) - /\sigma(f, g, \Delta, d)/ \leq \varepsilon \end{aligned}$$

and we consider a sequence $(d_n)_n$ in $\mathcal{D}$ as in Proposition 1 such that $\Delta_\varepsilon \leq \Delta$ and $d \perp \Delta$ we have $/\sigma(f, g, \Delta, d) - /\sigma(f, g, d_n, d'_n)/ \leq \varepsilon$ for all $d'_n \in \mathcal{D}, d'_n \perp d_n$. Passing to limit we have $/\sigma(f, g, \Delta, d) - I/ \leq \varepsilon$, where $I = \int_a^b f dg$.

5. Proposition. *$f, g : [a, b] \to \mathbb{R}$ be two bounded functions and such that g is increasing. Then f is Darboux–Stieltjes integrable w.r. to g iff f is D–Riemann–Stieltjes integrable w.r. to g (i.e. in the sens of Definition B)*

Proof. If f is D–Riemann–Stieltjes integrable w.r. to g then for any $\varepsilon > 0$ there exists $\Delta_\varepsilon \in \mathcal{D}$ such that $S(f, g, \Delta_\varepsilon) - s(f, g, \Delta_\varepsilon) < \varepsilon$. If we choose $\Delta \in \mathcal{D}$ such that $\Delta \in \mathcal{D}$ such that $\Delta_\varepsilon \leq \Delta$ we have

$$S(f, g, \Delta) \leq S(f, g, \Delta_\varepsilon), \; s(f, g, \Delta_\varepsilon) \leq s(f, g, \Delta);$$

$$S(f, g, \Delta) - s(f, g, \Delta_\varepsilon) \leq S(f, g, \Delta_\varepsilon) - s(f, g, \Delta_\varepsilon) < \varepsilon$$

If we take now $\Delta', \Delta'', d', d'' \in \mathcal{D}$ such that $\Delta_\varepsilon \leq \Delta', \Delta_\varepsilon \leq \Delta''$ and $d' \perp \Delta', d'' \perp \Delta''$ then we have

$$s(f,g,\Delta') \leq \sigma(f,g,\Delta',d') \leq S(f,g,\Delta')$$

$$s(f,g,\Delta'') \leq \sigma(f,g,\Delta'',d'') \leq S(f,g,\Delta'')$$

$$\max(s(f,g,\Delta'), s(f,g,\Delta')) \leq \min(S(f,g,\Delta'), S(f,g,\Delta''))$$

and therefore

$$/\sigma(f,g\Delta',d') - \sigma(f,g,\ \Delta'',d'')/ \leq (S(f,g,\Delta') - s(f,g,\Delta'))+ \\ (S(f,g,\Delta'') - s(f,g,\Delta')) < 2\varepsilon$$

i.e. f is Darboux–Stieltjes integrable w.r. to g.

Conversely, if f is Darboux-Stieltjes integrable w.r. to g then for any $\varepsilon > 0$ we consider $\Delta_\varepsilon \in \mathcal{D}$ such that for any $\Delta', \Delta'', d', d'' \in \mathcal{D}$ such that $\Delta_\varepsilon \leq \Delta', \Delta_\varepsilon \leq \Delta'', d' \perp \Delta', d' \perp \Delta', d'' \perp \Delta''$ we have

$$/\sigma(f,g,\Delta',d') - \sigma(f,g,\Delta'',d'')/ < \varepsilon$$

Particularly, taking $\Delta' = \Delta'' = \Delta_\varepsilon$ and $d', d'' \in \mathcal{D}$ such that $d' \perp \Delta_\varepsilon, d'' \perp \Delta_\varepsilon$, we have

$$/\sigma(f,g,\Delta_\varepsilon,d') - \sigma(f,g,\Delta_\varepsilon,d'')/ < \varepsilon$$

Taking in mind that

$$S(f,g,\Delta_\varepsilon) = \sup\ \{\sigma(f,g,\Delta_\varepsilon,d')/d' \in \mathcal{D}, d' \perp \Delta_\varepsilon\},$$

$$s(f,g,\Delta_\varepsilon) = \inf\ \{\sigma(f,g,\Delta_\varepsilon,d'')/d'' \in \mathcal{D}, d'' \perp \Delta_\varepsilon\}$$

we get

$$S(f,g,\Delta_\varepsilon) - s(f,g,\Delta_\varepsilon) \leq \varepsilon$$

i.e. f is D-Riemann-Stieltjes integrable w.r. to g.

6. Theorem. *Let $f, g : [a,b] \to \mathbb{R}$ be two bounded functions such that f is Darboux-Stieltjes integrable w.r. to g. If the functions f and g have no common point of discontinuity in the interval $[a,b]$ then f is Riemann-Stieltjes (see definition A) integrable w.r. to g.*

Proof. Let $\|f\| = \sup\ \{|f(x)|; x \in [a,b]\}$, $\|g(x)\| = \sup\ \{g(x)|; x \in [a,b]\}$ be the uniform norm of f, respectively g and let $\varepsilon > 0$ be arbitrary. We mark by $\Delta_\varepsilon = (x_i)_{0 \leq i \leq k}$ the division of $[a,b]$ such that for any $\Delta, d \in \mathcal{D}[a,b]$ with $\Delta_\varepsilon \leq \Delta$ and, $d \perp \Delta$ we have

$$/\sigma(f,g,\Delta,d) - \int_a^b f dg/ < \varepsilon.$$

We consider now $\eta_\varepsilon > 0$ such that for any $i \in \{1,2,\dots k\}$ we have one of the relations

$$/z - x_i/ \leq \eta_\varepsilon \Longrightarrow /f(z) - f(x_i))/ \leq \frac{\varepsilon}{m} \text{ or } /g(z) - g(x_i)/ < \frac{\varepsilon}{m}$$

where $m := 4(\|f\|\|g\| + 1) \cdot (k+1)$.

Let $\Delta' \in \mathcal{D}[a,b]$ with $\|\Delta'\| < \eta_\varepsilon$ and let Δ'_i be the division of $[a,b]$ given by $\Delta'_i = \Delta \cup \{x_i\}$. Suppose that $x_i \in [y_{j_0}, y_{j_0+1}]$, where $y_j, y_{j+1} \in \Delta'$ and we consider $d' \in \mathcal{D}, d' \perp \Delta', d' = (\xi_j)_j$, $\xi_j \in [y_j, y_{j+1}]$ for any j. Let $\xi'_{j_0} \in [x_i, y_{j_0}], \xi''_{j_0} \in [x_i, y_{j_0+1}]$ and let d'_i be the division of $[a,b]$ obtained from d' replacing ξ_{j_0} by the cuple $\xi'_{j_0}, \xi''_{j_0+1}$. We have

$$\sigma(f,g,\Delta',d') - \sigma(f,g,\Delta'_i,d'_i) = (f(\xi_{j_0}) - f(\xi'_{j_0}))(g(x_i) - g(y_{j_0}))+$$
$$(f(\xi_{j_0}) - f(\xi''_{j_0}))(g(y_{j_0+1}) - g(x_i))$$

and therefore we have

$$/\sigma(f,g,\Delta',d') - \sigma(f,g,\Delta'_i,d'_i)/ \leq 4(\|f\| + \|g\|) \cdot \frac{\varepsilon}{m}$$

We proced as follows:

We start with the divisions Δ', d' and taking $i = 1$ we construct, as above, the divisions Δ'_1, d'_1. Then starting with the divisions Δ'_1, d'_1 instead of Δ', d' and taking $i = 2$ we construct as above the divisions $\Delta'_{12}, d'_{12} \ldots$.

Finaly we obtain the divisions Δ'', d'' of $[a,b]$ such that $d'' \perp \Delta'', \Delta_\varepsilon \leq \Delta''$ and moreover

$$/\sigma(f,g,\Delta',d') - \sigma(f,g,\Delta'',d'')\| \leq k \cdot 4(\|f\| + \|g\|) \cdot \frac{\varepsilon}{m} \leq \varepsilon.$$

On the other hand we have

$$/\sigma(f,g,\Delta'',d'') - \int_a^b f dg/ \leq \varepsilon$$

and therefore

$$/\sigma(f,g,\Delta',d') - \int_a^b f dg/ \leq 2\varepsilon$$

for all $\Delta', d' \in \mathcal{D}$, $d' \perp \Delta'$ such that $\|\Delta'\| \leq \eta_\varepsilon$. Hence the function f is Riemann-Stieltjes integrable w.r. to g

7. Corollary. *If the function f is Darboux-Stieltjes integrable w.r. to g and one of the functions f or g is continuous then f is Riemann-Stieltjes integrable w.r. to g.*

Remark. The preceding results (Theorem 6 and Corollary 7) are known for the case where g is increasing.

8. Proposition . *a) If in Darboux-Stieltjes integrable with respect to g on $[a,b]$ then for any $c \in [a,b]$ the function f is Darboux-Stieltjes integrable w.r. to g on the intervals $[a,c]$ and $[c,b]$ and we have*

$$\int_a^b f dg = \int_a^c f dg + \int_c^b f dg$$

b) Conversely, if f is Darboux-Stieltjes integrable w.r. to g on the intervals $[a,c]$ and $[c,b]$ (where $c \in [a,b]$) then f is Darboux-Stieltjes integrable w.r. to g on $[a,b]$.

c) If f_1, f_2 are Darboux–Stieltjes integrable w.r. to g on $[a,b]$ then $\alpha f_1 + \beta f_2$ is Darboux–Stieltjes integrable w.r. to g, for any $\alpha, \beta \in \mathbb{R}$ and we have

$$\int_a^b (\alpha f_1 + \beta f_2) dg = \alpha \int_a^b f_1 dg + \beta \int_a^b f_2 dg.$$

d) If f is Darboux–Stieltjes integrable w.r. to g_1 and g_2 on $[a,b]$ then f is Darboux–Stieltjes integrable w.r. to $\alpha g_1 + \beta g_2$, for any $\alpha, \beta \in \mathbb{R}$ and we have

$$\int_a^b f d(\alpha g_1 + \beta g_2) = \alpha \int_a^b f dg_1 + \beta \int_a^b f dg_2$$

Proof. a) Let f be Darboux–Stieltjes integrable w.r. to g on $[a,b]$ and let $c \in [a,b]$. For any $\varepsilon > 0$ there exists $\Delta_\varepsilon \in \mathcal{D}[a,b]$ such that for any $\Delta', \Delta'', d', d'' \in \mathcal{D}[a,b]$ with $\Delta_\varepsilon \leq \Delta', \Delta_\varepsilon \leq \Delta'', d' \perp \Delta', d'' \perp \Delta''$ we have

$$/\sigma(f, g, \Delta', d') - \sigma(f, g, \Delta'', d'')/ \leq \varepsilon$$

The above inequality holds also if we replace the division Δ_ε by the division Δ'_ε finer that Δ_ε (i.e. $\Delta_\varepsilon \leq \Delta^1_\varepsilon$). So, we may suppose that $c \in \Delta_\varepsilon$. Let Δ^1_ε (resp. Δ^2_ε) the division of $[a,c]$ (resp. [c,b]) given by $\Delta^1_\varepsilon = \Delta_\varepsilon \cap [a,c]$ (resp. $\Delta^2_\varepsilon = \Delta_\varepsilon \cap [c,d]$).

Let now $\Delta'_1, \Delta''_1, d'_1, d''_1 \in \mathcal{D}[a,c]$ be such that $\Delta'_\varepsilon \leq \Delta'_1, \Delta'_\varepsilon \leq \Delta''_1$ $d'_1 \perp \Delta'_1, d''_1 \perp \Delta''_1$. If we put $\Delta' = \Delta'_1 \cup \Delta^2_\varepsilon, \Delta'' = \Delta''_1 \cup \Delta^2_\varepsilon$ and we choose $d_2 \in \mathcal{D}[c,b], d_2 \perp \Delta^2_\varepsilon$ then noting

$$d' = d'_1 \cup d_2, \; d'' = d''_1 \cup d_2$$

we have $d' \perp \Delta', d'' \perp \Delta'', \Delta_\varepsilon \leq \Delta', \Delta_\varepsilon \leq \Delta''$ and therefore

$$/\sigma(f, g, \Delta', d') - \sigma(f, g, \Delta'' d'')/ \leq \varepsilon.$$

On the other hand we have

$$\sigma(f, g, \Delta', d') = \sigma(f, g, \Delta'_1, d'_1) + \sigma(f, g, \Delta^2_\varepsilon, d_2)$$

$$\sigma(f, g, \Delta'', d'') = \sigma(f, g, \Delta''_1, d''_1) + \sigma(f, g, \Delta^2_\varepsilon, d_2)$$

and therefore

$$\sigma(f, g, \Delta'_1, d'_1) - \sigma(f, g, \Delta''_1, d''_1)/ \leq \varepsilon$$

Hence f is Darboux–Stieltjes integrable w.r. to g on $[a,c]$. We prove similarly the integrability on the interval $[c,b]$ and the assertion b). For the remainded part of assertion a) as well the statements c) and d) one can use Proposition 1.

9. Proposition. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. If g_1, g_2 are two real, increasing functions on $[a,b]$ then the following assertions are equivalent:*

a) f is Darboux–Stieltjes integrable with respect to g_1 and g_2.

b) If f is Darboux–Stieltjes integrable with respect to $g_1 + g_2$.

Proof. The relation $a) \Longrightarrow b)$ follows from Proposition 8. If f is Darboux–Stieltjes integrable w.r. to $g_1 + g_2$ then for any $\varepsilon > 0$ there exists $\Delta_\varepsilon \in \mathcal{D}[a,b]$, $\Delta_\varepsilon = (x_i)_{0 \leq i \leq n}$ such that

$$\sum_{i=0}^{n-1} (M_i - m_i)[(g_1 + g_2)(x_{i+1}) - (g_1 + g_2)(x_i)] < \varepsilon$$

where $M_i = \sup\{f(x)/ x \in [x_i, x_{i+1}]\}$, $m_i = \inf\{f(x)/x \in [x_i, x_{i+1}]\}, i \in \{0, 1, \dots n-1\}$. Since

$$\sum_{i=0}^{n-1}(M_i - m_i)(g_1(x_{i+1}) - g_1(x_i)) \geq 0, \sum_{i=0}^{n-1}(M_i - m_i)(g_2(x_{i+1}) - g_2(x_i)) \geq 0$$

we deduce $S(f, g, \Delta_\varepsilon) - s(f, g, \Delta_\varepsilon) \leq \varepsilon, S(f, g_2, \Delta_\varepsilon) - s(f, g_2, \Delta_\varepsilon) \leq \varepsilon$ and therefore (see Proposition 5) f is Darboux–Stieltjes integrable w.r. to g, as well w.r. to g_2.

Theorem 10. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. We denote by $\mathcal{I}(f)$ the set of all functions $g : [a, b]$ with bounded variation such that f is Darboux–Stieltjes integrable with respect to g and by V the set of all functions $g : [a, b] \to \mathbb{R}$ with bounded variations. The liniar subspace $\mathcal{I}(F)$ is a band of the Dedekind complete Riesz space V endowed with the following order relation*

$$g_1 \leq g_2 \Longleftrightarrow g_2 - g_1 \text{ is increasing and positive on } [a, b].$$

Proof. The fact that the set $\mathcal{I}(f)$ is an order ideal (normal subspace) of V follows from Proposition 9. Let us consider a family $(g_i)_{i\in I}$ in $(\mathcal{I}(f))^+$ which is dominated by an element g_0 of V^+. We want to show that $\bigvee_{i\in I} g_i \in \mathcal{I}(f)$ where $\bigvee_{i\in I} g_i$ is the least upper bound of the set $\{g_i/i \in I\}$ in V. Since $\mathcal{I}(f)$ is an order ideal in V, without lost of generality, we may suppose that the family $(g_i)_{i\in I}$ is upper directed. Since for any $i \in I$ there exists $g_i' \in V^+$ such that $g_i + g_i' = g_0$ we deduce the relation $h + h' = g_0$ where $h, h' : [a, b] \to \mathbb{R}_+$ are defined by

$$h(x) = \sup_{i\in I} g_i(x), \quad h(x)' = \inf_{i\in I} g_i'(x)$$

Obviously $h, h' \in V^+$ and moreover, $h - g_j \in V^+$ for any $j \in I$. Indeed, for any $i \geq j$ we have $g_i - g_j \in V^+$ for any $j \in I$. Indeed, for any $i \geq j$ we have $g_i - g_j \in V^+$ and therefore, passing to the limit, $h - g_j \in V^+$. Let $\varepsilon > 0$ be arbitrary and let $j_\varepsilon \in I$ be such that $h(b) - g_{j_\varepsilon}(b) < \dfrac{\varepsilon}{2\|f\|}$. We consider also $\Delta \in \mathcal{D}[a, b]$ such that

$$S(f, g_{j_\varepsilon}, \Delta) - s(f, g_{j_\varepsilon}, \Delta) < \frac{\varepsilon}{2}$$

and we denote by h' the element of V^+ given by $h' = h - g_{j_\varepsilon}$.

We have $h = g_{j_\varepsilon} + h'$ and therefore

$$S(f, h, \Delta - s(f, h, \Delta) = (S(f, g_{j_\varepsilon}, \Delta) - s(f, g_{j_\varepsilon}, \Delta)) + (S(f, h', \Delta) - s(f, h', \Delta)) \leq \frac{\varepsilon}{2} + \|f\| \cdot (h'(b) - h'(a)) \leq \frac{\varepsilon}{2} + \|f\| \cdot h'(b) \leq \frac{\varepsilon}{2} + \|f\| \cdot \frac{\varepsilon}{2\|f\|} = \varepsilon.$$

Hence $h \in \mathcal{I}(f)$ and from the construction of h we have $h = i \in I\, g_i$.

Remark. The above theorem enab e us to used the theory of Dedeking complete Riesz Spaces in obtaining now results n the theory of Darboux-Stieltjes integration.

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Espaces ordonnés localement convexes et opérateurs linéaires continus

Romulus Cristescu

Dans cette note on considère des espaces linéaires ordonnés munis de topologies vectorielles soumises à differentes conditions et on donne certaines propriétés des espaces ordonnés d'opérateurs linéaires et continus[1].

§1

Une topologie vectorielle τ sur un espace linéaire ordonné X est dite (ω)-continue si toute suite généralisée décroissante vers 0 dans X, (τ)-converge vers 0.

Un espace linéaire ordonné X muni d'une topologie vectorielle τ est dit (τo)-complet si pour tout sous-ensemble[2] (τ)-borné et dirigé supérieurement il existe le supremum.

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1) Les notions, dont les definitions ne sont pas donnés dans cette note, sont celles de [5]

2) On indique par τ la référence à la topologie considéré sur un espace linéaire ordonné X (par exemple: ensemble (τ)-borné) et par la lettre o la référence à l'ordre de X (par exemple : ensemble (o)-borné)

Si X et Y sont les espaces linéaires ordonnés, on désignera par $R(X,Y)$ l'ensemble des operateurs réguliers (définis sur X à valeurs dans Y). Si X et Y sont munis aussi de topologies vectorielles, on désignera par $\mathcal{L}(X,Y)$ l'ensemble des opérateurs linéaires et (τ)-continus. En particulier on pose $X_\tau^* = \mathcal{L}(X,R)$.

Théorème 1. Si X est un espace linéaire dirigé muni d'une topologie vectorielle et Y un espace linéaire ordoné muni d'une topologie vectorielle locallement pleine, alors la topologie de la convergence simple sur l'espace linéaire ordonné[3] $\mathcal{L}(X,Y)$ est localement pleine.

Démonstration. Si $\{U_\delta\}_{\delta\in\Delta}$ et $\{V_\delta\}_{\delta\in\Delta}$ sont deux suites généralisés d'éléments de l'espaces $\mathcal{L}(X,Y)$ telles que $0 \le U_\delta \le V_\delta, (\forall_\delta \in \Delta)$ et

$(\tau) - \lim_{\delta\in\Delta} V_\delta\ (x) = 0, \forall x \in X,$ alors $(\tau)\text{-}\lim_{\delta\in\Delta} U_\delta(x)=0, (\forall x \in X_+)$,

parce que la topologie de l'espace Y est localement pleine. L'espace Y etant dirigé il en résulte $(\tau)\text{-}\lim_{\delta\in\Delta} U_\delta(x)=0$, quelque soit $x \in X$.

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3) Dans l'espace $\mathcal{L}(X,Z)$ on considère l'ordre donné par le cône des opérateurs positifs

Théorème 2. Soit X une espace linéaire dirigé, muni d'une topologie localement convexe telle que X soit un sepace tonnelé. Soit Y une espace linéaire ordonné muni d'une topologie localement convexe separée et (ω)–continue. Si l'espace Y est (το)–complet, alors l'espace linéaire ordonné $\mathcal{L}(X,Y)$ est aussi (το)–complet par rapport à la topologie de la convergence simple.

Démonstration. Soit $\{U_\delta\}_{\delta\in\Delta}$ une suite généralisée croissante d'éléments de l'éspace $\mathcal{L}(X,Y)$ bornée par rapport à la topologie de la convergence simple. Donc pour chaque élément $x \in X_+$ la suite généralisée $\{U_\delta\}_{\delta\in\Delta}$ est croissante et (τ)–bornée. L'espace Y étant (το)–complet, il existe

$$V(x) = \bigvee_{\delta\in\Delta} U_\delta(x), \ (\forall x \in X_+)$$

et l'opérateur $V: X_+ \to Y$ donné par cette formule est additif et

$$V(\alpha x) = \alpha V(x), (\forall x \in X_+ ; \forall \alpha \in \mathbb{R}_+)$$

En posant

$$U(x'-x'') = V(x') - V(x''), \quad (x', x'' \in X_+)$$

on obtient donc un opérateur linéaire $U: X \to Y$. Par consequence $\{U_\delta\}_{\delta\in\Delta}$ est une suite généralisée dans l'espace $\mathcal{L}(X,Y)$ telle que

$$U(x)=(\tau)-\lim_{\delta\in\Delta} U_\delta(x), \ (\forall x \in X)$$

parce que la topologie de l'espace Y est (ω)-continue. D'apres le théorème de Banach-Steinhauss [9] on a $U \in \mathcal{L}(X, Y)$. Maintenant on vérifie aisément que $U = \bigvee_{\delta\in\Delta} U_\delta$ dans l'espace linéaire ordoné $\mathcal{L}(X, Y)$.

Corollaire. Si X est un espace linéaire dirigé muni d'une topologie localement convexe telle que X soit un espace tonnelé, alors dans l'espace linéaire ordonné X_τ^*, pour tout sous-ensemble dirigé superieurement et majoré il existe le supremum.

En effet, il est suffisant d'observer que si $\{f_\delta\}_{\delta\in\Delta}$ est une suite généralisée croissante et majorée d'éléments positifs de X_τ^*, alors pour chaque element $x \in X$, la suite généralisée $\{f_\delta(x)\}_{\delta\in\Delta}$ est bornée.

Théorème 3. Si Y est un espace linéaire réticulé archimédien muni d'une topologie localement convexe séparée (ω)-continue telle que Y soit (τ)-complet, alors Y est un espace linéaire complèment réticulé.

Démonstration. Soit $\{y_\delta\}_{\delta\in\Delta}$ une suite généralisée croissante et majorée d'éléments de Y. Posons

$$A = \{ y \in Y \mid y_\delta \leq y, \forall \delta \in \Delta \}$$

$$B = \{ y - y_\delta \mid y \in A, \delta \in \Delta \}$$

On a 0=inf B. En effet, si z est un minorant de l'ensemble B, allors

$y - z \in A, (\forall y \in A)$. Par induction il résulte que si $y_0 \in A$ alors

$$y_0 - nz \in A, (\forall n \in \mathbb{N})$$

donc pour un élément quelconque $\delta_0 \in \Delta$ on a

$$nz \leq y_0 - y_{\delta_0}, (\forall n \in \mathbb{N})$$

L'espace Y étant archimédien, il en résulte $z \leq 0$ donc $0 = \inf B$.

On vérifie aisément que l'ensemble B est dirigé supérieurement.

La topologie de l'espace Y étant (ω)-continue, pour toute semi-norme (τ)-continue q sur Y on a inf q(B)=0. Donc pour chaque seminorme (τ)-continue q sur Y et pour chaque nombre $\varepsilon > 0$ il existe $y_\varepsilon \in A$ et $\delta_\varepsilon \in \Delta$ tels que

$$q(y_\varepsilon - y_{\delta_\varepsilon}) \leq \frac{\varepsilon}{2}$$

Il en résulte que si $\delta', \delta'' \geq \delta_\varepsilon$, alors

$$q(y_{\delta'} - y_{\delta''}) \leq q(y_{\delta'} - y_\varepsilon) + q(y_\varepsilon - y_{\delta''}) \leq \varepsilon$$

donc $\{y_\delta\}_{\delta\in\Delta}$ est une suite généralisée Cauchy par rapport à la topologie de l'espace Y. Puisque l'espace Y est (τ)-complet, il existe $y = \lim_{\delta\in\Delta} y_\delta$ et on a $y = \bigvee_{\delta\in\Delta} y_\delta$ car le cône positif de l'espace Y est (τ)-fermé.

Corollaire. Si Y est un treillis de Banach à norme (ω)-continue, alors Y est un espace linéaire complètement réticulé.

Remarque. Si X est un espace linéaire ordonné et Y un espace linéaire topologique, un opérateur linéaire U: X → Y s'appelle operateur (στ)-compact si pour chaque sous-ensemble (o)-borné A de X, l'ensemble U(A) est relativement compact. Si X est un espace[4] de type (R) et Y est un espace linéaire complètement réticulé muni d'une topologie localement solide séparée (ω)-continue telle que Y soit (τ)-complet, alors l'ensemble $\mathfrak{K}(X,Y)$ des opérateurs réguliers (στ)-compacts (définis sur X à valeurs dans Y) est une composante de l'espace $\mathfrak{R}(X,Y)$, ([6], 3.4)

§ 2

Soient Y un espace linéaire dirigé et X un sous-espace linéaire majorant. Si p: X → R est une semi-norme monotone, nous posons

$$\bar{p}(y) = \inf\{p(x) \mid \pm y \leq x \in X\}, (y \in Y)$$

4) On appele espace de type (R) tout espaces linéaire dirigé qui satisfait à la conditions de Riesz [6]

La fonction $p : Y \to R$ donnó par cette formule est une semi-norme solide et

$\tilde{p}|X = p$. .

Si X est muni d'une topologie localement convexe-solide τ et si $\mathcal{P}$ est l'ensemble de toutes les semi-normes solides et (τ)-continues définies sur X, alors la topologie $\tilde{\tau}$ définie par $\tilde{\mathcal{P}} = \{p | \tilde{p} \in \mathcal{P}\}$ s'appelle <u>l'extension naturelle</u> de τ

sur Y.

Nous rappelons que si Z est un espace linéaire ordonné muni d'une topologie vectorielle τ, on dit [5] que Z posède le propriété (S) si pour chaque sous-ensemble (τ)-borné A de Z il existe un sous-ensemble (τ)-borné B d'éléments positifs tel que $A \subset B - B$.

<u>Thèoreme 4</u>. Soient Y un espace de type (R) et X un sous-espace linéaire majorant de Y. Soit τ une topologie localement convexe-solide bornologique sur X. Si Y posséde la propriété (S) par rapport à l'extension naturelle $\tilde{\tau}$ de τ sur Y, alors Y est un espace bornologique par rapport à $\tilde{\tau}$.

<u>Demonstration</u>. Soit $q : Y \to R$ une semi-norme telle que pour chaque sous-ensemble $A \subset Y$, qui est borné par rapport à la topologie $\tilde{\tau}$, l'ensemble q(A) soit borné. En posant

$$q_0(y) = \sup q([0,y]), \quad (y \in Y_+)$$

on obtient une fonctionelle sous-linéaire et monotone $q_0: Y_+ \to R_+$. En posant maintenant

$$\overline{q}(y) = \inf\{q_0(v) \mid \pm y \leq v\}, \quad (y \in Y)$$

on obtient une seminorme solide $\overline{q}$ sur Y telle que $\overline{q}|Y_+ = q_0$.

Soit maintenant A un sous-ensemble de Y, borné par rapport à la topologie $\tilde{\tau}$ et soit B un sous-ensemble de Y_+, borné par rapport à la même topologie, tel que $A \subset B - B$. En désignant par B_0 l'enveloppe pleine de B, il existe $\lambda \in R$ tel que $q(y) \leq \lambda, \forall y \in B_0$. Si $0 \leq z \leq y \in B_0$, alors $z \in B_0$ donc $q(z) \leq \lambda$, d'où il résulte $q_0(y) \leq \lambda$. Par consequence $\overline{q}(y) \leq 2\lambda$, $\forall y \in A$.

Puisque $\tilde{\tau}|X = \tau$ et l'espace X est bornologique par rapport à la topologie τ, il existe une semi-norme solide p sur X, continue par rapport à τ, telle que

$$\overline{q}(x) \leq p(x), \ (\forall x \in X)$$

Si $y \in Y$, alors il existe $x \in X$ tel que $\pm y \leq x$ et puisque $\overline{q}$ est une semi-norme solide, on a

$$\overline{q}(y) \leq \overline{q}(x) \leq p(x)$$

d'où il résulte

(1) $\qquad \overline{q}(y) \leq \widetilde{p}(y), \ (\forall y \in Y)$

Si $y \in Y_+$, allors $q(y) \leq q_0(y)$. Si y est un élément quelconque de Y et si $\pm y \leq v$, alors en posant

$$y_1 = \frac{1}{2}(v + y), \qquad y_2 = \frac{1}{2}(v - y)$$

on a $y = y_1 - y_2$, $0 \leq y_i \leq v, (i = 1,2)$, $v = y_1 + y_2$ et

$$q(y) \leq q(y_1) + q(y_2) \leq 2q_0(v)$$

d'où il résulte

$$q(y) \leq 2\overline{q}(y), \quad (\forall y \in Y)$$

Avec (1) on a

$$q(y) \leq 2\widetilde{p}(y), \quad (\forall y \in Y)$$

donc q est continue par rapport à la topo ogie $\widetilde{\tau}$.

En conclusion, Y est un espace bornologique par rapport à la topologie $\widetilde{\tau}$.

Corollaire. Si X est un espace linéaire dirigé archimédien muni d'une topologie localement convexe-solide séparée τ telle que X sont un espace bornologique, alors l'extension Dedekind $\widetilde{X}$ de X est un treillis localement convexe bornologique par rapport à l'extension naturelle $\widetilde{\tau}$ de τ sur $\widetilde{X}$.

En effet, en posant $Y=\tilde{X}$, l'espace Y possède les propriétés du théorème 4 (voir [5], 7.1.3, prop.5) donc $\tilde{X}$ est un espace bornologique. Si $0 \neq y \in \tilde{X}$ alors il existe $x \in X$ tel que $0 \leq x \leq |y|$, donc il existe une semi-norme solide p sur X, continue par rapport à τ, telle que $p(x) \neq 0$. Il en résulte $\tilde{p}(y) \neq 0$.

Dans le théorème suivant, on désigne par l(Y,Z) l'ensemble des opérateurs linéaires définis sur un espace linéaire Y, à valeurs dans un espace linéaire Z.

Ttéorème 5. Soient Y un espace linéaire dirigé, X un sous-espace linéaire majorant de Y et Z un espace linéaire ordonné. Soient τ une topologie localement convex-solide sur X, $\tilde{\tau}$ l'extension naturelle de τ sur Y et τ' une topologie localement pleine sur Z. On a :

(i) Si $0 \leq U \in \mathcal{L}(X,Z), 0 \leq V \in l(Y,Z)$ et $V|X = U$ alors $U \in \mathcal{L}(Y,Z)$.

(ii) Si Z est (o)-complet et $0 \leq U \in \mathcal{L}(X, Z)$ alors il existe un operator positif $V \in \mathcal{L}(Y,Z)$ tel que $V|X = U$.

Démonstration (i) Soient q une semi-norme absolument monotone[5)] et continue (par rapport à τ') sur Z et p une seminorme solide et continue (par rapport

5) C'est-à-dire si $\pm z \leq v$ dans Z allors $q(z) \leq q(v)$

à τ) sur X telle que

(2) $$q(U(x)) \leq p(x), \quad (\forall x \in X)$$

Si $y \in Y$ et $\pm y \leq x \in X$, alors

$$\pm V(y) \leq V(x) \leq U(x)$$

d'où, avec (2), il résulte

$$q(V(y)) \leq q(U(x)) \leq p(x)$$

et donc

$$q(V(y)) \leq \tilde{p}(y), (\forall y \in Y)$$

Par consequence $V \in \mathcal{L}(Y,Z)$.

(ii) D'après un théorème de prolongement de L. Kantorovitch (généralisé comme dans [7], 2.1) il existe un operateur positif $V \in l(Y,Z)$ tel que $V|X = U$. D'après (i) on a $V \in \mathcal{L}(Y,Z)$.

§3

Si X et Y sont des espaces linéaires ordonnés munis de topologies vectorielles, nous posons

$$\mathcal{L}_r(X,Y) = \{U \in \mathcal{L}(X,Y) \mid U = U_1 - U_2; 0 \leq U_i \in \mathcal{L}(X,Y), i = 1,2\}$$

Théorème 6. Soient X un espace de type (R), G un sous-espace linéaire, dirigé et plein de X et Y un espace linéaire complétement réticulé. Si X et Y sont munis de topologies localement solides, alors l'ensemble

$$\mathcal{G} = \{U \in \mathcal{L}_r(X,Y) \mid U(G) = \{0\}\}$$

est une composante[6] de l'espace linéaire complétement réticulé $\mathcal{L}_r(X,Y)$.

Demonstration. Il est suffisante de démontrer que l'ensemble G est une bande de l'espace $\mathcal{L}_r(X,Y)$.

Evidemment, $\mathcal{G}$ est un sous-espace linéaire de $\mathcal{L}_r(X,Y)$.

Si $U \in \mathcal{G}$ et $0 \le x \in G$, alors $[-x,x] \subset G$ puisque G est un ensemble plein. Il en résulte $|U|(x) = 0$, $\forall x \in G_+$ et donc $|U|(G) = \{0\}$, puisque G est un espace linéaire dirigé. Par consequence, $\mathcal{G}$ est un sous-espace linéaire reticulé de $\mathcal{L}_r(X,Y)$.

Evidémment, si $0 \le U_1 \in \mathcal{L}_r(X,Y)$ et $U_1 \le U_2 \in \mathcal{G}$, alors $U_1 \in \mathcal{G}$ donc $\mathcal{G}$ est un sous-espace normal de $\mathcal{L}_r(X,Y)$.

Si $0 \le U_\delta \uparrow_{\delta\in\Delta} U$ dans l'espace $\mathcal{L}_r(X,Y)$ et $U_\delta \in \mathcal{G}$, $(\forall \delta \in \Delta)$ alors $U_\delta(x) \uparrow_{\delta\in\Delta} U(x)$, $\forall x \in X_+$ donc $U(G) = \{0\}$, c'est-à-dire $U \in \mathcal{G}$.

En conclusion, $\mathcal{G}$ est une bande dans l'espace linéaire complétement réticulé

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6) C'est-à-dire $\mathcal{L}_r(X,Y) = \mathcal{G} + \mathcal{G}^{\perp}$.

$\mathcal{L}_\tau(X,Y)$ et le théorème est démontré.

Ce théorème est une géneralisation d'un théorème donné dans [6] où l'espace X était réticulé.

Remarque. Soit X un espace linéaire dirigé muni d'une topologie localement convexe-pleine séparée telle que le cône positif X_+ soit (τ)-complet. Si X posède la propriété (S) et si Y est un espace linéaire ordonnée muni d'une norme qui soit additive sur le cône positif Y_+, alors on peut démontrer que $\mathcal{L}_\tau(X,Y)=\mathcal{R}(X,Y)$. En particulier si X est un espace linéaire dirigé et un espace de Banach a norme monotone tel que la cône positif soit (τ)-fermé, alors $X_\tau^* = \mathcal{R}(X, R)$.

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THE SPACE OF REGULAR OPERATORS

Nicolae Dăneţ

Abstract

This is a short survey concerning the new results about the space of regular operators obtained in the Nineties.

1 The space of regular operators and some related spaces

Let X and Y be two vector lattices. The space of regular operators between X and Y, denoted by $L^r(X,Y)$, is the linear span of the cone of positive operators ordered by that cone, i.e.,

$$L^r(X,Y) = \{T : X \to Y \mid T = T_1 - T_2, T_1, T_2 \geq 0\}.$$

A natural question arise? What is the order structure of this space? This formulation is too general. We can reformulate the question more precisely. The space of regular operators between two vector lattices, $L^r(X,Y)$, is a *vector lattice*?

The answer of this question is well known.

Theorem 1.1(Kantorovich, 1936 [7]) *Let X and Y be two vector lattices. If the range space Y is an order complete vector lattice, then the space of regular operators, $L^r(X,Y)$, is a vector lattice (even an order complete vector lattice).*

In this case any regular operator has a positive part, $T^+ = T \vee 0$, which is given, on the positive cone X_+ of X, by the familiar Riesz-Kantorovich formula

$$T^+(x) = \sup T[0,x] = \sup\{T(z) \mid 0 \leq z \leq x\} \qquad (x \in X_+).$$

Along with the above formula for T^+ there are similar formulas for the negative part

$$T^-(x) = \sup T[-x,0] \qquad (x \in X_+),$$

and the modulus of T

$$|\, T \,|\, (x) = \sup T[-x,x] \qquad (x \in X_+).$$

It is well known that an operator has a positive part if and only if it has a modulus (and if and only if it has a negative part). If any one of this is given by Riesz-Kantorovich formula, then so are the others, since $T[0,x] = T(x) + T[-x,0] = (T[-x,x] + T(x))/2$.

If Y is not order complete then it may still happen that some regular operators have a positive part. In all cases then this does exist it is given by Riesz-Kantorovich formula, but it is an open question whether this is always the case.

A regular operators which has a positive part given by Riesz-Kantorovich formula has been termed *tame* by van Rooij in [10]. We will use the notation $L^{|r|}(X,Y)$ for the *set* of those operators from X into Y which have a *positive part*, and $L^{rt}(X,Y)$ for the subset of $L^{|r|}(X,Y)$ consisting of *tame* operators. Thus, in these notations, the above mentioned question asks whether or not we always have the equality

$$L^{rt}(X,Y) = L^{|r|}(X,Y). \tag{1}$$

Let $L^b(X,Y)$ be the space of order bounded operators. (The order bounded operators are those which map order bounded sets into order bounded sets.) Regular operators are obviously order bounded but the converse is, in general, false.

With the above notation we have

$$L^{rt}(X,Y) \subset L^{|r|}(X,Y) \subset L^r(X,Y) \subset L^b(X,Y).$$

If Y is an *order complete vector lattice*, then the above inclusion are equalities. This means that:

(1) every order bounded operators in regular;

(2) the space of regular operators, $L^r(X,Y)$, is a *vector lattice* (even an *order complete vector lattice*);

(3) every regular operator have a positive part;

(4) the positive part (modulus) is given by the Riesz-Kantorovich formula.

Question: what's happen when Y is not an order complete vector lattice?

Y.A.Abramovich and A.W.Wickstead began the study of the space or regular operators in the case when Y is not an order complete vector lattice ([2]). They studied the regular operators on or in the space l_0^∞. l_0^∞ is the space of all real sequences which are constant except a finite set. It is, in fact, the smallest possible non order complete vector lattice which is not completely trivial.

The situation when l_0^∞ is the domain of operators is simple. The order bounded operators are always regular, i.e., $L^b(l_0^\infty,Y) = L^r(l_0^\infty,Y)$, but $L^r(l_0^\infty,Y)$ is a vector lattice if and only if the range space Y is (σ)-order complete and in this case $L^r(l_0^\infty,Y)$ also becomes (σ)-order complete ([2]).

Given that $L^r(l_0^\infty, Y)$ is not always a vector lattice it is important to know which operators have a positive part. The following result gives a simple criterion for this.

Theorem 1.2([2]) *If Y is a vector lattice and $T \in L^r(l_0^\infty, Y)$ then T^+ exists if and only if the supremum*

$$\sup_{n\in N}\left(\left[\sum_{k=1}^{n}(Te_k)^+\right] \vee \left[T(1)+\sum_{k=1}^{n}(Te_k)^-\right]\right)$$

exists in Y.

The above condition seems to be much easier to work with than the condition requiring that the supremum of the image of each order interval exist, although they are actually equivalent. Using this criterion, Y.A.Abramovich and A.W.Wickstead proved the following theorem which show that for the space $X = l_0^\infty$ the problem (1) has a positive solution.

Theorem 1.3([2]) *If Y is a vector lattice and $T \in L^r(l_0^\infty, Y)$ then T^+ exists if and only if T is tame.*

In the case when the operators have the range space l_0^∞ we have:

Theorem 1.4([2]) *Let X be an uniformly complete vector lattice. Then*
(i) $L^b(X, l_0^\infty) = L^r(X, l_0^\infty)$.
(ii) $L^r(X, l_0^\infty)$ is a vector lattice.
(iii) All lattice operations in $L^r(X, l_0^\infty)$ may be computed using the Riesz-Kantorovich formula.

We recall that a vector lattice is called uniformly complete if every (ρ)-Cauchy sequence is (ρ)-convergent. The uniform completeness of the space X is not necessary for the validity of the conclusion of this theorem. If X_L is the space of Lipschitz functions on $[0, 1]$, then the three assertions of the above theorem hold, but X_L is not uniformly complete vector lattice. The authors give am example of a vector lattice X such that $L^b(X, l_0^\infty) \neq L^r(X, l_0^\infty)$ and $L^r(X, l_0^\infty)$ is not a vector lattice and ask the question: there is a vector lattice X such that $L^b(X, l_0^\infty) \neq L^r(X, l_0^\infty)$ and $L^r(X, l_0^\infty)$ is a vector lattice?

For the relation between regular operators and continuous operators on Banach lattices see [1] and [4].

2 The space of regular operators and the Riesz condition

An ordered vector space X satisfies the *Riesz condition* if for any positive elements x, z_1, z_2, with $x \leq z_1 + z_2$, there exist x_1, x_2 such that $0 \leq x_1 \leq z_1, 0 \leq x_2 \leq z_2$ and $x = x_1 + x_2$.

The ordered vector space X has the *Riesz decomposition property* if, for any positive elements x_1, x_2, z_1, z_2, the relation $x_1 + x_2 = z_1 + z_2$ holds, then there exist four elements $u_{ij} \geq 0$, $(i,j = 1,2)$ such that $x_1 = u_{11} + u_{12}$, $x_2 = u_{21} + u_{22}$, $z_1 = u_{11} + u_{21}$, $z_2 = u_{12} + u_{22}$.

The ordered vector space X has the *Riesz separation property* if $x_1, x_2 \leq z_1, z_2$ then there is y with $x_1, x_2 \leq y \leq z_1, z_2$. The Riesz separation property is sometimes called the finite interpolation property. The following result is well known.

Proposition 2.1_Let X be an ordered vector space. The following assertions are equivalent._

(i) X satisfies the Riesz condition.

(ii) X has the Riesz decomposition property.

(iii) X has the Riesz separation property.

If the range space Y is not an order complete vector lattice, the space of regular operators $L^r(X, Y)$ fails even to have the Riesz separation property.

Theorem 2.2([2]) *$L^r(l_0^\infty, l_0^\infty)$ does not have the Riesz separation property.*

The space $L^r(l_0^\infty, l_0^\infty)$ is a very good example of ordered vector space with generating cone (i.e., a directed vector space) without Riesz separation property. Using this space the author showed the difference between two notion of ideal in a directed vector space without Riesz separation property ([5]).

In the next part of this section we present the new results of A.W. Wickstead ([11]) about the spaces of regular operators between Banach lattices.

The main result is the theorem 2.5 which characterizes those Banach lattices E such that the space $L^r(\mathbf{c}, E)$ has the *Riesz separation property* as being those with *countable interpolation property*.

Definition 2.3_An ordered vector space E is said to have the countable interpolation property if, given two sequences (x_n) and (z_n) in E such that $x_n \uparrow$, $z_n \downarrow$ and $x_m \leq z_n$ for all $m, n \in \mathbb{N}$, then there is $y \in E$ such that $x_n \leq y \leq z_n$ for all $n \in \mathbb{N}$._

Every vector lattice has the Riesz separation property, but not all the vector lattices have the countable interpolation property. For example:

Seever has shown in [9] that $C(K)$ has the countable interpolation property if and only if K in an F-space, i.e., any pair of disjoint open F_σ subsets of K have disjoint closures.

C. B. Huijmans and B. de Pagter have shown in [6] that an Archimedean vector lattice E has the countable interpolation property if and only if E is uniformly complete and normal (the latter meaning $E = \{x^+\}^\perp + \{x^-\}^\perp$ for all $x \in E$).

Definition 2.4 ([11]) *A vector lattice E is said to have the strong countable interpolation property if, given two sequences (x_n) and (z_n) in E such that $x_m \leq z_n$ for all $m, n \in \mathbb{N}$, then there is $y \in E$ such that $x_n \leq y \leq z_n$ for all $n \in \mathbb{N}$.*

For vector lattices the countable interpolation property and the strong countable interpolation property are equivalent. For more general ordered vector spaces the two notions are not equivalent.

Theorem 2.5(A.W.Wickstead, 1995 [11]) *The following conditions on a Banach lattice E are equivalent:*

(i) E has the countable interpolation property.

(ii) $L^r(\mathbf{c}, E)$ has the strong countable interpolation property.

(iii) $L^r(\mathbf{c}, E)$ has the Riesz separation property.

The following example shows that there exists spaces of operators between Banach lattices which are not láttices but which have the Riesz separation property.

Example 2.6 ([11]) If X is an F-space which is not quasi-Stonean, then $C(X)$ has the countable interpolation property but is not (σ)-order complete (see [8], Proposition 2.1.5). By Theorem 2.5, $L^r(\mathbf{c}, E)$ has the Riesz separation property. On the other hand, by Theorem 3.10 of [3], $L^r(\mathbf{c}, E)$ is not a lattice.

Example 2.7([11]) If E does not have the countable interpolation property, e.g. if $E = \mathbf{c}$, then $L^r(\mathbf{c}, E)$ does not have the Riesz separation property.

If the domain of definition of the operators is allow to be an arbitrary Banach lattices, then one obtains another characterization of order complete Banach lattices.

Theorem 2.8([11]) Let *E be a Banach lattice. Then the following are equivalent:*

(i) E is order complete.

(ii) For all Banach lattices X, $L^r(X, E)$ is an order complete vector lattice.

(iii) For all Banach lattices X, $L^r(X, E)$ has the Riesz separation property.

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POINTS - EXTRÊMAUX

dans les espaces Bloch $B_0(V)$

Marinică Gavrilă

Josef A Cima et Waren R. Wogen en [1] ont donné une caractérisation des points extrêmaux de la boule unitaire de l'espace Bloch $B_0(C)$. Dans cette note, on donné les conditions suficientes pour les points-extrêmaux de la boule unitaire de l'espace Bloch $B_0(V)$, qui est un espace de fonctions a valeurs vectorielles. Les démonstrations dans le cas $B_0(V)$ sont diferente.

Si V est un espace de Banach, il arrive souvant d'utiliser la notion de point-extrêmal pour relèver la géométrie de la boule unitaire U_V de l'espace V.

Si A est un sous-ensemble nonvide et convexe d'un espace linéaire V, alors un sous-ensemble $K \subseteq A$ porte le nom de *sous-ensemble extrêmal* de A si de la relation $\alpha x_1 + (1-\alpha)x_2 \in K$ où $x_1, x_2 \in A$ et $\alpha \in (0,1)$, il resulte $x_1, x_2 \in K$.

Si le sous-ensemble K est formé d'un seul élément, $K = \{x_0\}$, alors x_0 s'appelle *point-extrêmal* de l'ensemble A. Donc x_0 est point-extrêmal de l'ensemble A si de la relation $x_0 = \alpha x_1 + (1-\alpha)x_2$, ou $x_1, x_2 \in A$ et $\alpha \in (0,1)$, il résulte l'égalité $x_1 = x_2 = x_0$. L'ensemble des points-extrêmaux de l'ensemble A sera noté $\mathrm{Ext}A$.

Remarque 1. Les points-extrêmaux de la boule unitaire U_V, d'un espace Banach V sont tous de norme 1. Alors, on a l'inclusion:

$$(1)\quad \mathrm{Ext}(U_V) \subseteq S_V$$

$$U_V = \{x \in V \,/\, \|x\| \le 1\} \quad \text{et} \quad S_V = \{x \in V \,/\, \|x\| = 1\}.$$

Il est vrai que, si $x \neq 0$ et $\|x\| < 1$, alors de l'égalité $x = \alpha x_1 + (1-\alpha)x_2$, où $\alpha = \|x\|$, $x_1 = \frac{x}{\|x\|}$, $x_2 = 0$, il résulte que x ne peut pas être un point-extrêmal de la boule unitaire U_V.

Dans la proposition suivante on donnera plusieurs variantes de la définition du point-extrêmal.

Proposition 1. Soit A un sous-ensemble non-vide et convexe d'un espace linéaire V *et* $x_0 \subset A$. Les affirmations suivantes sont équivalentes:

a) x_0 est point-extrêmal de l'ansamble A

b) Si $x_1, x_2 \in A$, $x_1 \neq x_2$ *et* $\alpha \in [0,1]$ *tel que* $x_0 = \alpha x_1 + (1-\alpha)x_2$, alors $\alpha = 0$ *ou* $\alpha = 1$;

c) L'ensamble $A \setminus \{x_0\}$ est convexe.

d) Si $x_1, x_2 \in A$ *et* $x_0 = \frac{x_1 + x_2}{2}$ *alors* $x_1 = x_2 = x_0$.

Demonstration. $d) \Rightarrow a)$ Soient $x_1, x_2 \in A, \alpha \in (0,1)$ *et* $x_0 = \alpha x_1 + (1-\alpha)x_2$. Si $\alpha = \frac{1}{2}$, alors vu l'hypothèse, on a $x_1 = x_2 = x_0$. *Si* $\alpha \in \left(0, \frac{1}{2}\right)$, alors $x_0 = \frac{x_2 + x_3}{2}$ où $x_3 = 2\alpha x_1 + (1-2\alpha)x_2$. Donc on a $x_2 = x_3 = x_0$ et alors $x_1 = x_2 = x_0$.

Si $\alpha \in \left(\frac{1}{2}, 1\right)$, *alors* $x_0 = \frac{x_1 + x_4}{2}$, où $x_4 = \lambda x_1 + (1-\lambda)x_2$ *et* $\lambda = 2\alpha - 1$.

Vu l'hypothèse il result que $x_1 = x_4 = x_0$ *et alors* $x_1 = x_2 = x_0$.

Un espace de Banach V avec la propriété $\dim V \geq 2$, a la norme uniformément convexe si pour tout $\varepsilon > 0$ il y a $\delta = \delta(\varepsilon) > 0$ et avec la propriéte que pout n'import quels éléments $x, y \subset V$ qui verifient les relation $\|x\| = \|y\| = 1$ et $\|x - y\| \geq \varepsilon$ on a $\left\|\frac{x+y}{2}\right\| < 1 - \delta$.

Proposition 2. Soit V un espace de Banach ayant la norme uniformement convexe. Si $x, x_1, x_2 \in V$ *et* $x = \frac{x_1 + x_2}{2}$, $\|x\| = \|x_1\| = \|x_2\|$, *alors* $x = x_1 = x_2$.

Démonstration. Si $x = 0$, alors $x_1 = x_2 = 0$. Si $x \neq 0$, et divisant par $\|x\|$, on obtient trois éléments $\frac{x}{\|x\|}, \frac{x_1}{\|x_1\|}$, *et* $\frac{x_2}{\|x_2\|}$ qui vérifient les hypothèses de la *proposition 2.*

Donc, on peut supposer que x, x_1 et x_2 sont des éléments de S_V. Si $x_1 \neq x_2$, alors $\|x_1 - x_2\| > 0$. Soit $\varepsilon > 0$ de sorte que $\|x_1 - x_2\| \geq \varepsilon$. La norme étant convexe, il en résulte qu'il y a $\delta = \delta(\varepsilon) > 0$ tel que $1 = \|x\| = \frac{x_1 + x_2}{2} \leq 1 - \delta$ relation qui est contradictoire.

Corollaire 1. Si V est un espace de Banach ayant la norme uniformement convexe, alors

(2) $Ext(U_V) = S_V$

Demonstration. Soit $x \in S_V$. Si $x_1, x_2 \in U_V$ tel que $x = \frac{x_1 + x_2}{2}$ il resulte $\|x_1\| = \|2x - x_2\| \geq |2\|x\| - \|x_2\|| \geq 1$. Donc $\|x_1\| = 1$. De manière analoque on démontre $\|x_2\| = 1$. En utilisant la *Proposition 1.* il resulte $x_1 = x_2 = x_0$. Donc $x \in Ext(U_V)$.

Un résultat très important et bien conuu concernant l'ensemble des points-extrêmaux d'un ensemble, est constitué par *le théoréme de Krein-Milman: "tout sous-ensemble K, non-vide, compacte et convexe d'un espace local convexe Hausdorff V est* la fermeture de la converture de l'ensemble de ses poins-etrêmaux". Donc, on a l'égalité:

(3) $K = Co(ExtK)$.

Une direction d'étude dans le domaine des points-etrêmaux est représentée par la caractérisation des points-extrêmaux de la boule unitaire des espaces Banach différents.

Proposition 3. Soient X et Y deux espaces ayant des normes $\| \|_1$ et $\| \|_2$. Soit $N:[0,\infty)x[0,\infty) \to R$ une fouction de sorte que $(x,y) \to N(|x|,|y|)$ soit une norme sur R^2. On definit la norme $\| \|$ ser $X \oplus Y$ par la relation suivante:

(4) $\|x + y\| = N(\|x\|,\|y\|), \forall x \in X$ et $y \in Y$

Alors $x + y$ est un point-extrêmal pour la boule unitaire $U_{X \oplus Y}$ si et seulement si les conditions suivantes sont remplies:

(i) x est un point-extrêmal pour la boule de rayon $\|x\|$ de X;

(ii) y est un point-extrêmal pour la boule de rayon $\|y\|$ de Y;

(iii) $(\|x\|,\|y\|)$ est un point-extrêmal pour la boule unitaire de l'espace R^2, de norme N.

Soit D la boule unitaire ouverte du plain complexe C et V un espace Banach (sur C). On rapplle qu'une founction vectorielle $f: D \to V$ est holomorphe en D si pour toute forme linéaire et continue $x^* \in V^*$, la founction complexe $< f(z), x^* >$ de variable complexe z est holomorphe en D. On utilise aussi la notation suivante $< f(z), x^* > = x^* f(z)$.

En outre, si f est une founction vectorielle holomorphe dans une boule ouverte $D_r + \alpha$, ayant le centre $\alpha \in C$, de rayon $r > 0$, à valeurs en V, alors ([6], Théorème 5, page 275) la founction f peut être développée en série de Taylor autour du point α, série qui est absolument et uniformément convergénte sur toute la boule fermée $D_\rho + \lambda$, de centre α, de rayon $\rho < r$, de la forme suivante:

(5) $f(z) = \sum_{n=0}^{\infty} \frac{(z - \alpha)^n}{n} f^{(n)}(\alpha)$.

Si $f:D \to V$ est une founction vectorielle holomorphe en D, on définit le nombre $M(f)$ par la relation:

(6) $M(f) = \sup\left\{\|f'(z)\|\left(1-|z|^2\right); z \in D\right\}$

Par *l'espace des Bloch* $B(V)$ on comprend l'espace des founctions vectorielles de variable complexe $f:D \to V$, holomorphes sur D et pour lesquelles les valeurs du nombre $M(f)$ sont finies. L'espace de Bloch B(V) devient espace de Banach (sur C) avec la norme suivante:

(7) $\|f\| = \|f(0)\|_V + M(f), \forall f \in B(V)$.

L'ensemble de founction $f \in B(V)$ pour lesquelles on a la relation:

(8) $\lim\limits_{|z| \to 1} \|f'(z)\|\left(1-|z|^2\right) = 0$

est un sous-espace fermé de $B(V)$, noté par $B_0(V)$. Joseph A. Cima et Warren R. Wogen en [1] ont doné une caractérisation des points extrêmaux de la boule unitaire de l'espace Bloch $B_0(C)$.

Dans ce paragraphe, on élargira cette caractérisation des points-extrêmaux de la boule unitaire de l'espace de Bloch $B_0(V)$.

D'abord, on fera l'analyse des founctions normalisées par $f(0) = 0$. On fera les notations suivantes:

$$(9)\quad \begin{aligned} \tilde{B}(V) &= \{f \in B(V); f(0) = 0\} \\ \tilde{B}(V) &= B_0(V) \cap \tilde{B}(V) \end{aligned}$$

La norme de tout élément f de $\tilde{B}(V)$ est égale à $M(f)$.

Premièrement, on caractérisera les points-extrêmaux de la boule unitaire de l'espace $\tilde{B}_0(V)$ et puis ceux qui appartiennent à la boule unitaire de l'espace $B_0(v)$. Pour tout élément f de la boule unitaire de l'espace $B(V)$, onconsidère l'ensemble L_f défini par la relation:

(10) $L_f = \left\{z \in D: \|f'(z)\|\left(1-|z|^2\right) = 1\right\}$

On notera par $U_{B(V)}$ la boule unitaire fermée de l'espace $B(V)$.

Théorème 1. Soit V un espace Banach à norme uniformément convexe. Si f un élément de $U_{\tilde{B}(V)}$ et il y a un nombre reél, positif $R, R < 1$, ainsi que l'ensemble $L_f \cap \{z \in C \,/\, |z| \le R\}$ est infini, alors f est un point-extrêmal pour $U_{\tilde{B}(V)}$.

Démontration: Soient g_1 et g_2, deux founctions de $U_{\tilde{B}(V)}$ de sorte que $f = \frac{g_1 + g_2}{2}$. On considére z, un élément quelque de $L_f \cap \{z \in C \,/\, |z| \le R\}$

Puisque

(11) $\|f(z)\| = \left(1-|z|^2\right)$ et $\|g_i(z)\| \le \left(1-|z|^2\right)^{-1}$, $i \in \{1,2\}$

il en résulte.

(12) $\|f'(z)\| \ge \frac{1}{2}\left[\|g_1'(z)\| + \|g_2'(z)\|\right]$

En utilisant la relation d'égalité:

$f'(z) = \frac{1}{2}\left[g_1'(z) + g_2'(z)\right]$ on obtient que:

(13) $\|f'(z)\| \le \frac{1}{2}\left[\|g_1'(z)\| + \|g_2'(z)\|\right]$

Compte tenu des relations (12) et (13), on aura:

$\|f'(z)\| = \frac{1}{2}\left[\|g_1'(z)\| + \|g_2'(z)\|\right]$ et en utilisant les relations (11), on obtient le résultat suivant:

(14) $\|f'(z)\| = \|g_1'(z)\| = \|g_2'(z)\| = \left(1-|z|^2\right)^{-1}$

Car le norme de l'espace V est uniformément convexe et considérant les relations (13) mais aussi le proposition 1. il en résulte que $f'(z) = g_1'(z) = g_2'(z)$ pour tout élément z de $L_f \cap \{z \in C / \; |z| \le R\}$. Puisque l'ensemble de ces points z est infini, vu le principe du prologement nalytique, on aura l'égalité suivante: $f' = g_1' = g_2'$.

Mais, f, g_1 et g_2 appartiennent à l'espace $\tilde{B}(V)$, et donc, $f(z) = g_1(z) = g_2(z)$. Donc, f est un point-extrèmal pour $U_{\tilde{B}(V)}$.

Corollaire 2. Si V est un espace de Banach ayant la norme uniformément convexe, f est un élément de la boule unitaire de l'espace $\tilde{B}_0(V)$ et L_f est un ensamble infini, alors f est un point-extrêmal pour la boule unitaire de l'espace $\tilde{B}_0(V)$

Théorème 3. Soit f un élément de la boule unitaire de l'espace $B_0(V)$. Alors, la founction f est un point-extrèmal pour la boule $U_{B_0(V)}$, si et seulement si une des affirmations suivantes sont remplies:

(i) f est une founction constante de norme 1.

(ii) $f(0) = 0$ et la founction f est un point-extrèmal pour la boule $\tilde{B}_0(V)$.

Démonstration. Pour toute founction f de $B_0(V)$, la founction g (qui represente la *"o-translation* de f " c'es- à-dire $g(z) = f(z) - f(0)$ est un élément de $\tilde{B}_0(V)$. En utilisant ce résultat, on a:

(15) $B_0(V) = \tilde{B}_0(V) + C$, on par C on a noté l'ensemble des founction constantes. Donc,

(16) $C = \{h : D \to V / h(z) = v_0, v_0 \in V, \forall z \in V\}$

Pour une founction f de $B_0(V)$ on a l'écriture unique $f = g + h$, où g est définie par: $g(z) = f(z) - f(0)(\forall) z \in D, g \in B_0(V)$, et $h \in C$ et on le définit par la relation $h(z) = f(0) \quad \forall z \in D$. Considérant une founction f de la boule unitaire de l'espace $B_0(V)$, la norme de f est définie par la relation

$$\|f\| = \|f_0\| + M(f),$$

donc, la nome N de la *Proposition 3.* est L^1 norme sur R^2. C'est de cette manière que $(\|f_0\|, M(f))$ est un point-extrêmal pour la boule unitaire de R^2 au sens de la norme L^1 si et seulement si $f(0) = 0 \quad M(f) = \|f\| = 1$ ou $\|f(0)\| = 1 \quad M(f) = 0$.

De ces relations, le *Théorème 3.* résulte imédiatement.

Remarque 1. Considérons les normes suivantes, équivalentés sur $B_0(V)$. Pour $1 \le p < \infty$ et $f \in B_0(V)$ soit $\|f\|_p = \left(\|f(0)\|^p + [M(f)]^p\right)^{\frac{1}{p}}$.

La *Proposition 3.* produit (engendre), dans le cas $1 < p < \infty$ une classe de points-extrêmaux essentiallement différente de celle du *Théorème 3.*(p=1). C'est analogue pour le cas où $p = \infty$.

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Central Limit Theorem and beyond it

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One of the fundamental results in Probabilty Theory is the Central Limit Theorem, which expresses the universality of normal distribution. See [2] for applications and a proof based on characteristic functions (actually the standard proof in all major textbooks on the subject).

We shall indicate a proof of this result based on Semigroup Theory. Our argument follows closely that in Goldstein [1], but we emphasize more on the role of functional calculus with distribution functions.

Recently, Voiculescu [3], [4] succeded in creating a noncommutative analogue of the Central Limit Theorem. This time, the role of normal distribution is played by the so called semicircular law. We comment on the possible existence of a unifying approach.

1. The Classical Central Limit Theorem

Let (Ω, Σ, P) be a probability space. Given a real random variable X on Ω, its *distribution function* F_X is defined as

$$F_X(t) = P\{X < t\}.$$

The *normal distribution* of mean m and variance σ^2 is

$$F_{N(m,\sigma^2)}(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} e^{-(s-m)^2/2\sigma^2} ds.$$

It is worthwhile to mention that

$$X \text{ and } Y \text{ independent implies } F_{X+Y} = F_X * F_Y$$

where the convolution of two distributions is given by the formula

$$(F * G)(t) = \int_{\mathbf{R}} F(t-s)dG(s).$$

Theorem 1. (*The Classical Central Limit Theorem*). *Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables, satisfying the normalized conditions*

$$E(X_k) = m \quad \text{and} \quad D^2(X_k) = \sigma^2.$$

Then

$$F_{\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(X_k - m)}(t) \to F_{N(m,\sigma^2)}(t)$$

for all $t \in \mathbf{R}$.

Clearly, it suffices to consider the case where $m = 0$ and $\sigma = 1$.

The proof depends on a functional calculus with distribution functions, which makes possible to infer the above result from Chernoff product formula.

Let $\mathfrak{F}$ be the set of all distribution functions and put

$$E = BUC(\mathbf{R}),$$

the Banach space (endowed with the sup norm) of all uniformly continuous bounded functions $f : \mathbf{R} \to \mathbf{R}$.

The functional calculus with elements of $\mathfrak{F}$ attaches to each $F \in \mathfrak{F}$ a contraction

$$\widetilde{F} \in L(E, E), \; \widetilde{F}(f)(s) = \int_{\mathbf{R}} f(s-t)\, dF(t).$$

Let us mention here two basic properties of this functional calculus:

$$F_n \to F \text{ at all points of continuity of } F \Leftrightarrow \widetilde{F_n} \xrightarrow{so} \widetilde{F}$$

$$\widetilde{F * G} = \widetilde{F} \cdot \widetilde{G}.$$

Consider the operator $A = \frac{1}{2} \cdot \frac{d^2 u}{dx^2}$ with domain $\mathcal{D}(A) = \{f | f, f', f'' \in E\}$. Then the C_0-semigroup $(T(t))_{t \geq 0}$ attached to the Cauchy Problem

$$\begin{aligned} \frac{du}{dt} &= Au \\ u(0) &= f \in E \end{aligned}$$

verifies the formula

$$T(t)f = \widetilde{F_{N(0,t)}}(f) \quad \text{for all } t > 0.$$

In fact,

$$T(t)f = \frac{1}{\sqrt{2\pi t}} \int_{\mathbf{R}} f(x-y) e^{-y^2/2t}\, dy \quad \text{for all } t > 0.$$

Let G be the common distribution of the X_k's and for $n \in \mathbf{N}^*$ put

$$G_n(r) = G(\sqrt{n} \cdot r).$$

Then $G_n = F_{X_k/\sqrt{n}}$ for all $k \in \{1, .., n\}$ and

$$\begin{aligned} \int_{\mathbf{R}} dG_n(r) &= 1 \\ \int_{\mathbf{R}} r\, dG_n(r) &= E\left(\frac{X_k}{\sqrt{n}}\right) = 0 \\ \int_{\mathbf{R}} r^2\, dG_n(r) &= E\left(\frac{X_k}{\sqrt{n}}\right)^2 = \frac{1}{n}. \end{aligned}$$

By the remarks above, the Central Limit Theorem is equivalent to

$$\widetilde{F_{\frac{1}{\sqrt{n}}\sum_{k=1}^{n} X_k}} \xrightarrow{so} \widetilde{F_{N(0,1)}}.$$

Or,

$$\begin{aligned} \widetilde{F_{\frac{1}{\sqrt{n}}\sum_{k=1}^{n} X_k}} &= (\widetilde{F_{X_1/\sqrt{n}} * \ldots * F_{X_n/\sqrt{n}}}) = \\ &= (\widetilde{G_n * \ldots * G_n}) = \widetilde{G_n}^{\,n} \end{aligned}$$

so what we have to show is

$$\widetilde{G_n}^{\,n} \xrightarrow{so} T(1). \qquad (1)$$

That will be done using Chernoff's Product Formula.

For, notice first that each operator $n(\widetilde{G_n} - I)$ is the generator of a contractive C_0- semigroup. In fact,

$$||e^{n(\widetilde{G_n}-I)t}|| = e^{-nt} \cdot ||e^{nt\widetilde{G_n}}|| \leq e^{-nt} \cdot e^{nt||\widetilde{G_n}||} \leq 1$$

for every $t \geq 0$.

Lemma 2. $\lim\limits_{n\to\infty} n(\widetilde{G_n}f - f) = Af$, *for every* $f \in \mathcal{D}(A)$.

Proof. We have

$$\begin{aligned} \left[n(\widetilde{G_n}f - f) - Af\right](x) &= \\ &= \int_{\mathbf{R}} n \cdot \left[f(x-r) - f(x) + rf'(x) - \frac{r^2}{2} f''(x)\right] dG_n(r) \\ &= \int_{\mathbf{R}} \frac{nr^2}{2} \cdot [f''(\theta) - f''(x)]\, dG_n(r) \end{aligned}$$

the last step being motivated by Taylor's Formula. Letting I_1 (respectively I_2) the last integral over $|r| < \delta$ (respectively $|r| \geq \delta$) we have to prove that $I_1 + I_2 \to 0$ as $n \to \infty$.

Or, from the uniform continuity of f'' we infer that

$$|I_1| \leq \frac{osc(f;\delta)}{2} \int_{|r|<\delta} nr^2 \, dG_n(r) \leq \frac{osc(f;\delta)}{2}.$$

For I_2 we have

$$\begin{aligned} |I_2| &\leq ||f''|| \cdot \int_{|r|\geq\delta} nr^2 \, dG_n(r) = \\ &= ||f''|| \cdot \int_{|\sqrt{n}\cdot r|\geq\sqrt{n}\cdot\delta} (\sqrt{n} \cdot r)^2 \, dG(\sqrt{n} \cdot r) = \\ &= ||f''|| \cdot \int_{|s|\geq\sqrt{n}\cdot\delta} s^2 \, dG(s) \to 0 \end{aligned}$$

as $n \to \infty$. ■

Now the convergence (1.1) can be deduced easily from the following two results, applied for $V_n = \widetilde{G_n}$ and $C = A$:

Lemma 3. (*Chernoff Product Formula*). *Let $V_1, V_2, \ldots$ be a sequence of contractive operators of $L(E,E)$ and suppose there exists an operator $C : \mathcal{D}(C) \to E$ generating a contractive C_0–semigroup $(T(t))_{t\geq0}$ such that*

$$C(x) = \lim_{n\to\infty} n\,[V_n(x) - x] \quad \text{for every } x \in \mathcal{D}(C).$$

Then

$$\lim_{n\to\infty} e^{n(V_n - I)}x = T(1)x \quad \text{for every } x \in E.$$

Lemma 4. *If $S \in L(E,E)$ and $||S|| \leq 1$ then*

$$||e^{n(S-I)}x - S^n x|| \leq \sqrt{n}\,||Sx - x||$$

for each $x \in E$ and each $n \in \mathbf{N}$.

We end this section by stating another form of the Central Limit Theorem:

Theorem 5. *Suppose that $X_1, X_2, \ldots$ is a sequence of independent random variables, such that $X_k \in \bigcap_{1<p<\infty} L^p(P)$, $E(X_k) = 0$ for all k and*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} D^2(X_k) = \sigma^2 > 0.$$

Then for every $k \in \mathbf{N}$,

$$\lim_{n\to\infty} E\left(\left(\frac{X_1 + \ldots + X_n}{\sqrt{n}}\right)^k\right) = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbf{R}} x^k e^{-x^2/2\sigma^2} dx$$

i.e., the kth moments of $\frac{X_1+\ldots+X_n}{\sqrt{n}}$ tends to the kth moment of a $(0,\sigma^2)-$ normal distribution.

2. Free Probability Theory

A *noncommutative probability space* is a pair $(\mathcal{A},\varphi)$, where $\mathcal{A}$ is a unital C^*- algebra and $\varphi : \mathcal{A} \to \mathbf{C}$ is a linear functional with $\varphi(1) = 1$; the elements of $\mathcal{A}$ are called *random variables.*

Let $a \in \mathcal{A}$. The *distribution* of a is defined as

$$\mu_a : \mathbf{C}[X] \to \mathbf{C}, \quad \mu_a(p(X)) = \varphi(p(a)).$$

If $a \in \mathcal{A}$, $aa^* = a^*a$, then by Riesz Representation Theorem there exists a unique probability measure ν on $\sigma(a)$ such that

$$\varphi(f(a)) = \int_{\sigma(a)} f(t)d\nu(t) \quad \text{for all } f \in C(\sigma(a)).$$

In this case, we can identify μ_a and the corresponding ν.

A sequence $(a_n)_n$ of random variables is said to be *convergent in distribution* to a random variable a if

$$\mu_{a_n}(p) \to \mu_a(p) \quad \text{for every } p \in \mathbf{C}[X].$$

Let $(\mathcal{A}_\alpha)_\alpha$ be a family of unital subalgebras of $\mathcal{A}$. The family is said to be *free* provided

$$\begin{aligned} \varphi(a_1 \ldots a_n) &= 0 \quad \text{whenever } a_i \in \mathcal{A}_{\alpha(i)}, \text{ the } \alpha(i) \text{ are distinct} \\ \text{and } \varphi(a_i) &= 0 \text{ for all } i. \end{aligned}$$

A family of elements is said to be *free* if the family of unital subalgebras they generate is free.

With this preparation we are now able to state Voiculescu's Central Limit Theorem for free random variables:

Theorem 6. (*The Noncommutative Central Limit Theorem*). *Let $(\mathcal{A},\varphi)$ be a noncommutative probability space and let $a_1, a_2, \ldots$ be a free family of random variables. Suppose that*

$$\begin{aligned} \varphi(a_k) &= 0 \quad \text{for all } k \\ \sup_{j\geq 1} \varphi(a_j^k) &< \infty \quad \text{for all } k \geq 2 \end{aligned}$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\varphi(a_j^2)=r^2/4$$

for some $r>0$. *Let*

$$S_n=\frac{a_1+\ldots+a_n}{\sqrt{n}}.$$

Then $(S_n)_n$ *converges in distribution to the semicircular distribution* γ_r, *where*

$$\gamma_r(x^k)=\frac{2}{\pi r^2}\int_{-r}^{r}x^k\sqrt{r^2-x^2}dx \quad \text{for all } k.$$

The proof of the above theorem depends upon a functional calculus with distributions, this time having values formal series.

Let a,b be free random variables. The ***additive free convolution*** of the distributions μ_a and μ_b is

$$\mu_a\boxplus\mu_b=\mu_{a+b}.$$

Additive free convolution is associative, commutative and has neutral element.

Let μ be a distribution. Then the $R-$transform of μ is the formal power series

$$R_\mu(z)=\alpha_1+\alpha_2 z+\alpha_3 z^2+\ldots$$

defined as follows: Consider the formal power series

$$G_\mu(\zeta)=\zeta^{-1}+\sum_{k=1}^{\infty}\mu(X^k)\zeta^{-k-1};$$

as a formal power series G_μ has a unique inverse (with respect to composition) of the form

$$K_\mu(z)=\frac{1}{z}+\alpha_1+\alpha_2 z+\alpha_3 z^2+\ldots$$

and we put

$$R_\mu(z)=K_\mu(z)-\frac{1}{z}.$$

Lemma 7. *The* $R-$*transform provides a semigroup homomorphism from the set* $(\Sigma,\boxplus)$ *of all distributions under additive free convolution to the set* $(f.p.s.,+)$ *of formal power series under addition, which is one-to-one and onto.*

The above lemma shows that $(\Sigma,\boxplus)$ is actually an abelian group.

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ON THE BISHOP - STONE WEIERSTRASS APPROXIMATION THEOREM

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The Stone - Weierstrass theorem is one of the principal pillars of modern analysis. Many generalizations of this important theorem have evolved since its discovery by Marshall Stone in 1947.

We recall the Stone - Weierstrass theorem for real case.

Theorem 1. *Let X be a Hausdorff compact and* $A \subset C(X,\mathbf{R})$ *an algebra with the properties:*

a) A *contains the constant functions;*

b) A *separates the points of* X *i.e., for any* $x,y \in X$, $x \neq y$ *there is* $a \in A$ *such that* $a(x) \neq a(y)$.

Then $\overline{A} = C(X,\mathbf{R})$.

A proof of this theorem, which makes use of the means of functional analysis was given by L. de Branges.

L. de Branges has established the following lemma:

Lemma 1 (L. de Branges 1959). *Let* F *be a vector subspace of* C(X,R), B *the closed unit sphere of* $M(X,\mathbf{R}) = [C(X,\mathbf{R})]'$ *and* $\mu \in \mathrm{Ext}\{F^0 \cap B\}$. *If* $g \in C(X,\mathbf{R})$ *is such that* $g\mu \in F^0$, *then* g *is constant on* S_μ *- the support of* μ.

The conclusion of Theorem 1 is false in the complex case. Let $X = D = \{z \in \mathbf{C};\ |z| \leq 1\}$ and let H be the algebra of all $h \in C(D,\mathbf{C})$ which are holomorphic in the interior of D. H is the uniform closure of the algebra of all polynomials in the complex variable z.

Obviously, H fulfils the assumptions of Theorem 1, but $H \neq C(D,\mathbf{C})$, because the function $f(z) = \mathrm{Re}\, z$, $z \in D$ is not in H.

The conclusion of Theorem 1 still holds in the complex case if A is self-adjoint i.e., $f \in A$ implies $\bar{f} \in A$.

Definition 1. *A subset* S *of* X *is called* A-*antisymmetric if every* $f \in A$, *real - valued on S, is constant on S.*

Let $f|S$ be the restriction of f to S and $A|S = \{f|S;\ \ f \in A\}$.

In the 1961 Errett Bishop extended Stone-Weierstrass theorem to the complex case.

Theorem 2 (E. Bishop). *Let* A *be a subalgebra of* $C(X,\mathbb{C})$ *that contains the constant functions. Then:*

i) *there exists a pairwise disjoint partition* $(S_i)_{i\in I}$ *of* X *formed by closed maximal* A*-antisymmetric sets;*

ii) *a function* $f \in C(X,\mathbb{C})$ *belongs to* $\overline{A}$ *iff* $f|S_i \in \overline{A|S_i}$ *for each* $i \in I$

Remark 1. *If* A *is self-adjoint and* A *separates the points of* X*, then every* A*-antisymmetric set is a singleton, and thus every* $f \in C(X,\mathbb{C})$ *trivially satisfies* $f|S_i \in A|S_i$ *for every* $i \in I$*, so* $f \in A$*; that is* $\overline{A} = C(X,\mathbb{C})$.

The original Bishop's proof of Theorem 2 is difficulty of understood, because it is based on transfinite induction.

Glickesberg is the first who has given a reasonable proof of Theorem 2. His proof is based on the following lemma:

Lemma 2 (I. Glicksberg). *If* $\mu \in \mathrm{Ext}\{A^0 \cap B\}$*, then the support of* μ *is* A*-antisymmetryc set.*

Some "elementary" proofs of Bishop's theorem were given by R.B. Burckel (1984) and T.J. Ransford (1984).

In 1971 J.Prolla generalized Bishop's theorem for weighted spaces and the following version of Theorem 2 is due to Silvio Machado in 1977.

Theorem 3 (S. Machado). *For any* $f \in C(X,\mathbb{C})$ *we have*

$$\mathrm{dist}(f, A) = \sup_{i\in I} \mathrm{dist}(f|S_i,\ A|S_i).$$

Obviously, $\mathrm{dist}(f, A) = \inf\{\|f - a\|; a \in A\} = 0$ iff $f \in \overline{A}$, and therefore. Theorem 3 is a generalisation of Theorem 2.

In 1978 G.Păltineanu extended Theorem 2 considering instead of subalgebras A of $C(X,\mathbb{C})$, the vector subspaces of $C(X,\mathbb{C})$.

Definition 2. *Let* $F \subset C(X,\mathbb{C})$ *be a vector subspace. A subset* S *of* X *is said to be* F*-antisymmetic if every* $f \in F$ *with the properties:*

a) $f|S$ *is real valued;*

b) $fg|S \in F|S$ *for any* $g \in F$*;*

is constant S.

Theorem 4. *Let* F *be a vector subspace of* $C(X,\mathbb{C})$*. Then:*

i) *there exists a pairwise disjoint partition* $(T_i)_{i\in I}$ *of* X*, formed by closed maximal* F*- antisymmetric subsets;*

ii) *for each* $f\in C(X,\mathbb{C})$ *one has :*

$$\mathrm{dist}(f,F)=\sup\{\mathrm{dist}(f|T_i, F|T_i);\quad i\in I\}$$

D.Feyel and A de la Pradelle extended in 1984 the Bishop's theorem for a convex cone.

Definition 3. *Let* $C\subset C(X,\mathbb{R})$ *be a convex cone. A subset* S *of* X *is said to be* C*-antisymmetric if every* $f\in C$ *with the properties:*

a) $0\le f|S\le 1$

b) $fg+(1-f)g|S\in C|S$ *for any* $g,h\in C$

is constant on S.

There exists a pairwise disjoint partition $(U_\alpha)_\alpha$ of X, formed by closed maximal C-antisymmetric sets.

Theorem 5. *For any* $f\in C(X,\mathbb{R})$ *we have*

$$\mathrm{dist}(f,C)=\sup_\alpha \mathrm{dist}(f|U_\alpha, C|U_\alpha).$$

It is well known that if I is a closed ideal of C(X) then there is a closed subset S of X such that $I=\{f\in C(X);\quad f|S=0\}$.

Therefore, there is a one - to - one map $I\longleftrightarrow S$ from the family of the closed ideals of C(X) onto the family of closed subsets of X.

Starting from this remark, C. Niculescu, G. Păltineanu and D.Vuza extended in 1993 Bishop's theorem for M-ideals in Banach spaces. In the sequel we shall present a version of this theorem for Banach lattices.

Let E be a Banach lattice. The centre Z(E) of E is the algebra of all bounded endomorphisms $U\in L(E,E)$, i.e those U for which it exists $\lambda>0$ such that $|U(x)|\le\lambda|x|$ for all $x\in E$. We define the real part of the centre by:

$$\mathrm{Re}\ Z(E)=Z(E)_+-Z(E)_+.$$

For each closed ideal I of E we denote by π_I the canonical map $\pi_I: E\to E/I$.

Definition 4. *Let* F *be a vector subspace of E. A closed ideal* I *of* E *is said to be* F*-antisymmetric if for any* $U \in \mathrm{Re}\ Z(E/I)$ *such that*

$U[\pi_I(F)] \subset \pi_I(F)$

it follows that there exists $a \in \mathbb{R}$ *such that* $U = a\, 1_{E/I}$.

We denote by $\mathcal{A}_F$ the family of all closed ideals of E, antisymmetric with respect to F.

We can prove that every $I \in \mathcal{A}_F$ contains a unique closed minimal ideal $J \in \mathcal{A}_F$.

Let $\tilde{\mathcal{A}}_F$ be the set of all minimal closed ideal of E, antisymmetric with respect to F.

Theorem 5. *Let* E *be a Banach lattice of* (AM)*- type and let* F *be a vector subspace of* E.

For each $x \in E$ *we have:*

$\mathrm{dist}(x,F) = \sup\{\mathrm{dist}(\pi_I(x), \pi_I(F));\quad I \in \tilde{\mathcal{A}}_F\}$

The proof is based on the following lemma:

Lemma 3. *Let* $B = \{f \in E'; \|f\| < 1\}$ *and* $f \in \mathrm{Ext}\{F^0 \cap B\}$. *If we denote by* $I = \{x \in E; |f|(|x|) = 0\}$, *then* $I \in \mathcal{A}_F$.

Remark 2. *Let* E=C(X), S *a closed subset of* X *and* $I_S = \{f \in C(X); f|S = 0\}$ *the corresponding closed ideal of C(X). Then,* $\pi_{I_S}(f) = f|S$ *for any* $f \in C(X)$. *We also have* $Z[C(X)] = C(X)$. Now it is clear that Theorem 4 extends Theorem 3.

A version of Theorem 5 for a convex cone in locally convex lattices of (AM)-type, was given in 1996 by C.Niculescu, G.Păltineanu and D.Vuza.

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THE DUAL SPACE OF THE HARDY SPACE GENERATED BY A REARRANGEMENT INVARIANT SPACE X AND SOME APPLICATIONS

NICOLAE POPA

Abstract

Let X be a rearrangement invariant space (in short r.i.s.) either on $\mathbb{R}$ or on $[0,1]$. First we introduce the real Hardy space H_X generated by X (respectively the dyadic Hardy space $H_X(d)$) and we show that the dual of H_X (respectively of $H_X(d)$) can be identified to $H_{X'}$ (respectively to $H_{X'}(d)$), X' being the associate space of X, in the case X a r.i.s. with an absolutely continuous norm and such that $q_X < \infty$.

We use this result in order to show that H_X (respectively $H_X(d)$) is not isomorphic to a r.i.s. Y on $[0,1]$ as Banach spaces, improving a previous result of the author [Po2].

1979 J. O. Strömberg [S] gave a description of the dual space of a Hardy-Orlicz spaces H_φ, φ being an Orlicz function. In order to do this Strömberg introduced the space $L^*_{\varphi^*}$, an extension of the well-known BMO-space.

Using the ideas contained in [S], [FJ] and [FS] as well as some techniques developped in [Po1] we extend the Strömberg's result to a r.i.s. X with an absolutely continuous norm and having the upper Boyd index $q_X < \infty$. Further on we show that the corresponding Hardy space H_X (its definition will be given later) is not isomorphic to any r.i.s. Y on $[0,1]$, improving an earlier result from [Po2].

We note that the space H_X, in its dyadic version, inherits many properties of a r.i.s. Y (see [Po1]), so it is natural to ask if H_X and Y are or not isomorphic as Banach spaces.

Now we recall some notions and definitions. For the unexplained terminology we recommend [LT] or [BS].

Lex X be a r.i.s. either on $\mathbf{R}$ or on $[0,1]$. The more natural examples of r.i.s. are the Lebesgue spaces L^p, $1 \leq p \leq \infty$.

We use the Boyd indices p_X and q_X and we recall their definition (see [LT],.[BS]) in the case X is a r.i.s. on $\mathbf{R}$.

Let $f : \mathbf{R} \to \mathbf{C}$ and $s > 0$. Put $D_s f(t) = f(t/s)$ $\forall t \in \mathbf{R}$.

Then D_s acts boundedly on X and we put:

$$p_X = \lim_{s\to\infty} \frac{\log s}{\log ||D_s||_X}, \quad q_X = \lim_{s\to 0^+} \frac{\log s}{\log ||D_s||_X}$$

and remark that $1 \leq p_X \leq q_X \leq \infty$ and $(p_X)' = q_{X'}$, $(q_X)' = p_{X'}$, where X' is the associate space of X and p' is the conjugate number for p.

If $X = L^p$ we have $p_X = q_X = p$.

Now we use thw following notations from [S]. Let $0 < r < \infty$, $0 < s \leq 1$ and Q be an interval in $\mathbf{R}$. Then put, for every Lebesgue measurable function f,

$$M_r^\sharp f(x) = \sup_{Q \ni x} \inf_{c\in \mathbf{C}} \left[\frac{1}{|Q|} \int_Q |f(y) - c|^r dy \right]^{1/r}$$

for $x \in \mathbf{R}$; and

$$M_{0,s}^\sharp f(x) = \sup_{Q\ni x} \inf_{c\in\mathbf{C}} \inf\{\alpha \geq 0 : |\{y \in Q; |f(y) - c| > \alpha\}| < s|Q|\}.$$

Our task is to show that some results from [S], [FS] and [T] can be extended taking a r.i.s. X on $\mathbf{R}$ instead of L^p or of an Orlicz space L_φ.

The following lemma is well-known:

Lemma 1 *(See [LT], [A]) Let T be a quasilinear operator bounded on $L^\infty(\mathbf{R})$ and of the weak type (p,p), where $0 < p < \infty$. If $p_X > p$ then there is a constant $C > 0$ depending only on T and X such that T maps X into X and*

$$||Tf||_X \leq C||f||_X \quad \textit{for every } f \in X.$$

Note that Lemma 1 is an extension of Lemma 2.2 -p.514-[S].

Next we extend Lemma 3.9-p.523-[S] and for the convenience of the reader we gave its proof:

Proposition 2 *Let $0 < r < \infty$, $0 < s < 1/2$ and let X be a r.i.s. on $\mathbb{R}$ such that $p_X > r$. There exists the constant $C > 0$ depending only on r and X, such that*

$$||M_r^\# f||_X \le C||M_{0,s}^\# f||_X$$

for all f satisfying $||M_{0,s}^\# f||_X < \infty$.

Moreover all the expressions $||M_{r'}^\# f||_X$, such that $0 < r' \le r$, and all $||M_{0,s}^\# f||_X$, with $0 < s \le 1/2$ are equivalent quasinorms of f if $M_1^\# f \in X$ and if $r \ge 1$.

Proof We have

$$(1) \quad M_r^\# f(x) \le C M_{r'} M_{0,s}^\# f(x)$$

for all $x \in \mathbf{R}$, $C > 0$ depending only on r with $0 < r < r'$. (See Lemma 3.7-[S].)

Now let $p_X > r' > r$ and let

$$M_p f(x) := \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q |f(y)|^p dy \right)^{1/p}$$

for $0 < p < \infty$. Since we know that M_p is bounded on $L^\infty(\mathbf{R})$ and of weak type (p,p), $0 < p < \infty$ (see for instance [S]-p.518) we apply Lemma 1 to $T = M_{r'}$ and we get

$$(2) \quad ||M_{r'}^\# M_{0,s}^\# f||_X \le C_1 ||M_{0,s}^\# f||_X$$

under the above conditions.

By (1) and (2) we get the first part of Proposition 2.

For the second part let us remark that

$$s g^\sim(x)^p \le \int_{\mathbf{R}} g(t)^p dt$$

for all $0 < s < \infty$ and all $0 \le g$ Lebesgue measurable functions on $\mathbf{R}$, $g^\sim$ being the decreasing rearrangement of g, i.e.

$$g^\sim(s) = \inf_{|E| = s} \sup_{t \in \mathbf{R} - E} |g(t)|.$$

Take now $g(y) = |f(y) - c|\chi_Q(y)$ for $y \in \mathbf{R}$ and $s = t|Q|$.

We get

$$\int_Q |f(y)-c|^p dy \geq (|f-c|^{\gamma}(t|Q|))^p \cdot t|Q|,$$

thus

$$\left(\frac{1}{|Q|}\int_Q |f(y)-c|^p dy\right)^{1/p} \geq t^{1/p}\inf\{\alpha \geq 0; |\{y \in Q : |f(y)-c| > \alpha\}| < t|Q|\}$$

and

$$t^{1/p}M^{\#}_{0,p}f(x) \leq M^{\#}_p f(x) \quad \forall x \in \mathbf{R} \text{ and } t \in (0,1).$$

Finally we get $||M^{\#}_r f||_X \geq t^{1/r}||M^{\#}_{0,t}f||_X$ for all $0 < t \leq 1/2$ and $0 < r < \infty$. Consequently $||M^{\#}_r f||_X \sim ||M^{\#}_{0,s}f||_X$ for all $0 < r < p_X$, $0 < s \leq 1/2$ (with constants depending on r and s) and all f such that $M^{\#}_{0,s}f \in X$.

If $r \geq 1$ and $||M^{\#}_1||_X < \infty$, then $||M^{\#}_{r'}f||_X < \infty$ for all $r' \leq r$. So Proposition 2 is proved. ∎

Now we have the following version of Theorem 3.1 -[S]. (See also [FS].)

Theorem 3 *Let X be a r.i.s. on $\mathbf{R}$ such that $q_X < \infty$. Let $0 < s \leq 1/2$ and let $0 < r < \infty$. There is a constant $C > 0$ depending on X and r such that*

$$||M_r f||_X \leq C||M^{\#}_r f||_X \quad \textit{for all } f \in X,$$

if the right side of the above inequality is finite.

Proof We use the techniques developped in [S] pp.526-527 and we get:

$$(1) \quad |\{x \in \mathbf{R}; M_r(f - a_f)(x) > \alpha'\}| \leq$$

$$\leq C\sum_{k=0}^{\infty}(CB^{-r})^k|\{y \in \mathbf{R}; M^{\#}_r f(x) > \frac{b^k\alpha'}{CB}\}|$$

for all $\alpha' > 0$, $b = 2^{-2/r-3/2}$, where

$$a_f = \lim_{|Q|\to\infty} m_f(Q),$$

m_f being the median value of f, i.e. $m_f(Q)$ is a real number (if f is real) such that

$$|\{x \in Q; f(x) > m_f(Q)\}| \leq \frac{1}{2}|Q|$$

and

$$|\{x \in Q, f(x) < m_f(Q)\}| \leq \frac{1}{2}|Q|.$$

But (1) can be written also like this:

$$\mu_{M_r(f-a_f)}(\alpha') \leq \sum_{k=0}^{\infty}(CB^{-r})^k \mu_{CBb^{-k}M_r^! f}(\alpha')$$

for all $\alpha' > 0$, using the notation $\mu_f(\lambda)$ for $|\{x \in \mathbf{R}; |f(x)| > \lambda\}|$ and by the well-known relation between $f^{\sim}$ and μ_f we have:

$$(1') \quad [M_r(f - a_f)]^{\sim}(\lambda) \leq \sum_{k=0}^{\infty}(CBb^{-k}M_r^!)^{\sim}\left(\frac{\lambda}{2^k(CB^{-r})^k}\right) \quad \forall\lambda > 0.$$

(1') is equivalent to:

$$(2) \quad [M_r(f - a_f)]^{\sim}(\lambda) \leq \sum_{k=0}^{\infty} CBb^{-k}D_{(2CB^{-r})^{-k}}(M_r^! f)^{\sim}(\lambda)$$

for all $\lambda > 0$.

Since $q_X < \infty$, $(\exists)\, a > 0$ and $s_0 > 0$ such that $||D_s||_X \leq s^{1/a}$ for all $s \leq s_0$.

Then, by (2), we get

$$||M_r(f - a_f)||_X \leq \sum_{k=0}^{\infty} CBb^{-k}(2CB^{-r})^{k/a}||M_r^! f||_X \leq C_1||M_r^! f||_X$$

for a sufficiently large $B > 0$.

If

$$|a_f| = \lim_{|Q|\to\infty} |m_f(Q)| > 0$$

then, since $||f - a_f||_X \leq C_1||M_r^! f||_X < \infty$, it follows that $f - a_f \in X$, i.e. $a_f \in X$ if $f \in X$.

Then the function $1 \in X$ and $1 \leq ||D_s||_X \leq s^{1/a}$ for $0 < s \leq s_0$, which is impossible.

Thus $a_f = 0$ for $f \in X$ and Theorem 3.1 is proved. ∎

Let $u(x,t)$ be a harmonic function on $\mathbf{R}_+^2$, Γ the cone $\Gamma(x) = \{(x',t); |x' - x| \leq ct\}$ and

$$u^*(x) = \sup_{(x',t)\in\Gamma(x)} |u(x',t)|.$$

Now for X a r.i.s. on $\mathbf{R}$ with $q_X < \infty$, let us put

$$H_X := \{u(x,t) := (P_t * f)(x);\ \text{for } x \in \mathbf{R}; t > 0;\ \text{where } f = f_1 + iHf_1$$

$$\text{with } ||f_1||_X, ||Hf_1||_X < \infty\},$$

where $P_t(x)$ is the Poisson kernel on $\mathbf{R}$, Hf is the Hilbert transform of f:

$$Hf(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{\mathbf{R}-[\epsilon,\epsilon]} \frac{f(t)\,dt}{x-t}.$$

On H_X we introduce the norm

$$||u||_{H_X} := ||f_1||_X + ||Hf_1||_X.$$

Then arguing as in [FS]pp.185-187 we have the following extension of Theorem 11-[FS].

Theorem 4 *Let X be a r.i.s. on $\mathbf{R}$ such that $q_X < \infty$. Then for each function f we have the equivalences:*

$$(A) \quad u^+(x) := \sup_{t>0} |\varphi_t * f(x)| \in X$$

for some φ a fast decreasing function such that $\int_{\mathbf{R}} \varphi(x)dx = 1$.

$$(B) \quad u^* \in X.$$

$$(C) \quad f(x) = \lim_{t \to 0} u(x,t), \quad \textit{where } u \in H_X.$$

Moreover

$$||u^+||_X \sim ||u||_{H_X} \sim ||u^*||_X.$$

Recall now the definition of Triebel-Lizorkin spaces, however in our, more general, situation.

Let X be a r.i.s. on X with $q_X < \infty$. Then we denote by $F_X^{0,2}$, and call it *the Triebel-Lizorkin space generated by X* the following:

$$F_X^{0,2} = \{f : \mathbf{R} \to \mathbf{C}; ||f||_{F_X^{0,2}} := \inf_{P \text{ polynome}} ||\sup_{t>0} \left(\sum_{k=-\infty}^{\infty} |F^{-1}(\varphi_k \cdot F(f+P))|^2 \right)^{1/2} ||_X$$

$$< \infty\}$$

where $\varphi_k(x) = 2^k\varphi(2^k x)$, $x \in \mathbb{R}$, $\varphi \in \mathcal{S}$ ($\mathcal{S}$ is the Schwartz spaces), with $supp\, F\varphi = \{2^{-1} \leq |\xi| \leq 2\}$, $F\varphi$ being the Fourier transform of φ.

The definition of $F_X^{0,2}$ in the case $X = L^p$, was given in [Pe], [T], [FJ]. If we identify $u = P_t * f \in H_X$ with $f : \mathbb{R} \to \mathbb{C}$ such that $f_1 + iHf_1 = f$, and consequently $H_X = \{u = P_t * f\}$ with $H_X = \{f : \mathbb{R} \to \mathbb{C}$ such that $f_1 + iHf_1 = f$, with $||f_1||_X + ||Hf_1||_X < \infty\}$, we have: (Such a identification is a known fact; see for instance [G], chap.II.)

Corollary 5 *Let X be a r.i.s. on* $\mathbb{R}$ *with $q_X < \infty$. Then, with equivalent norms, we have*

$$H_X = F_X^{0,2}.$$

The proof is essentially the same to that given in [T] or [Pe] and uses the equivalence $(A) \Leftrightarrow (C)$ in Theorem 4. ∎

In the next corollary H_X is also considered as a space of functions on $\mathbb{R}$;

Corollary 6 *Let X be a r.i.s. on* $\mathbb{R}$ *such that $q_X < \infty$. Then we have the equivalences:*

$$||f||_{H_X} \sim ||M_1 f||_X \sim ||M_1^{\sharp} f||_X + ||f||_X.$$

Proof Since M_1 is the well-known Hardy-Littlewood maximal operator we have $u^* \leq CM_1 f$ if $u(x,t) = (P_t * f)(x)$. (See [G], [BS].)

Thus by Theorem 4 and remark before Corollary 5, we have

$$||f||_{H_X} \sim ||u^*||_X \leq C||M_1 f||_X.$$

The converse of the above inequality follows from Riesz factorisation Theorem: *If $F \in H_X$, where X is a r.i.s. on* $\mathbb{R}$ *with $q_X < \infty$, then $F = BG$, where B is a Blaschke product and $G(z) \neq 0$ for all $z = x + it$, $t > 0$ and moreover $||F||_{H_X} = ||G||_{H_X}$.*

Take now $G_1 = BG^{1/2}$, $G_2 = G^{1/2}$ and by Riesz theorem we have $||G_i||_{H_{X^2}} = ||F||_{H_X}^{1/2}$, $i = 1,2$ and $F = G_1G_2$. (Here $X^2 = \{f : |f|^2 \in X\}$ with the norm $||f||_{X^2} = || \, |f|^2 ||_X^{1/2}$.)

Let f be the boundary value of $F \in H_X$. Then $f = g_1 g_2$, where g_i are the boundary values of G_i and $M_1 f \leq M_2 g_1 \cdot M_2 g_2$.

Consequently

$$||M_1 f||_X \leq ||M_2 g_1||_{X^2} \cdot ||M_2 g_2||_{X^2} \leq (\text{ since } p_{X^2} = 2p_X \geq 2 > 1) \leq$$

$$\leq C||g_1||_{X^2} \cdot ||g_2||_{X^2} = C||G_1||_{H_X}^{1/2} \cdot ||G_2||_{H_X}^{1/2} = C||F||_{H_X} \leq C_1||f||_{H_X}.$$

Hence the first equivalence is proved. In order to prove the second let us note that $M_1^\sharp f \leq M_1 f$, thus

$$||M_1^\sharp f||_X + ||f||_X \leq 2||M_1 f||_X$$

and we should show only that

$$||M_1 f||_X \leq C(||M_1^\sharp f||_X + ||f||_X).$$

But this is precisely the statement of Theorem 3. ∎

Now we can give an extension of Lemma 6.1 -[S]-p.542:

Lemma 7 *Let X be a r.i.s. on $\mathbb{R}$ such that $q_X < \infty$ and moreover assume that the norm of X be absolutely continuous (i.e. if $E_n \downarrow \emptyset$ then $||\chi_{E_n} f||_X \downarrow 0$ for all $f \in X$).*

Define

$$H_X^0 = \{f \in H_X : \text{ such that } Ff \in C_0^\infty \text{ and } Ff = 0 \text{ on some neighbourhood of } 0\}.$$

Then H_X^0 is dense in H_X.

Proof By Lemma 2.1-[FJ] and by Corollary 5 it follows that for each $f \in H_X$ we have

$$f = \sum_Q < f, \varphi_Q > \psi_Q,$$

where Q runs into the set of all dyadic intervals in $\mathbb{R}$, $\varphi_Q(x) = |Q|^{-1/2}\varphi(2^\nu x - k)$, (and similarly for ψ_Q), where

$$Q = Q_{\nu,k} := \{x \in \mathbb{R}; k \leq 2^\nu x < k+1\}, \quad k, \nu \in \mathbb{Z}.$$

Here φ, $\psi \in \mathcal{S}(\mathbb{R})$, $supp\, F\varphi$, $F\psi \subset \{\xi \in \mathbb{R}; 2^{-1} \leq |\xi| \leq 2\}$; $|F\varphi(\xi)|, |F\psi(\xi)| \geq c > 0$ for $3/5 \leq |\xi| \leq 5/3$; and

$$\sum_{\nu\in\mathbb{Z}} \overline{F\varphi(2^\nu\xi)} F\psi(2^\nu\xi) = 1 \quad \text{if } \xi \neq 0.$$

Using the estimate for the Peetre functional given in par.6-[FJ] for the couple

$$(F_1^{0,2}, F_\infty^{0,2})$$

and the similar estimate for the spaces:

$$f_X^{0,2} := \{f : \mathbf{R} \to \mathbf{C};\ f = \sum_Q s_Q h_Q;\ \text{such that } ||f||_{f_X^{0,2}} := ||(\sum_Q \frac{|s_Q|^2}{|Q|}\chi_Q)^{1/2}||_X < \infty\},$$

where h_Q is the L^2-normalized Haar function supported by Q, we get that $T_\psi : f_X^{0,2} \to F_X^{0,2}$ given by

$$T_\psi(s) = \sum_Q s_Q h_Q,$$

acts boundedly.

(Here, of course, we identified $f \in f_X^{0,2}$ with the sequence $s = (s_Q)_Q$ such that $f = \sum_Q s_Q h_Q$.)

X having an absolutely continuous norm it follows that $\sum_{Q\in A} s_Q h_Q \to \sum_Q s_Q h_Q$, if $A \uparrow \{Q; Q$ runs over dyadic intervals $\}$, in the norm of $f_X^{0,2}$. By the boundedness of T_ψ and the absolute continuity of the norm of X it follows that $\sum_Q < f, \varphi_Q > \psi_Q$ converges to f in the norm of H_X. Since $\sum_{Q\in A} < f, \varphi_Q > \psi_Q \in H_X^0$ for A a finite set, we proved Lemma 7.∎

Now we describe the dual of H_X, under rather mild restrictions about X, extending Theorem 5.1-[S]-p.533.

Note first that there are r.i.s. X on $\mathbf{R}$ (or, equivalently, on $\mathbf{R}_+$) such that $q_X < \infty$ without an absolutely continuous norm. For instance take $X = (L_{w,1})'(0,\infty)$ where $w(t) = t^{-1/2}$ and $L_{w,1}(0,\infty) := \{f : \mathbf{R}_+ \to \mathbf{C}; \int_0^\infty f^\sim(t)w(t)dt < \infty\}$.

Then it is known (see [LT]) that $p_X = q_X = 2$ and X contains a sublattice isomorphic to ℓ^∞, consequently the norm of X cannot be absolutely continuous.

Before giving the dual of H_X we need to extend the definition of H_X for a general r.i.s. X on $\mathbf{R}$.

Definition 8 *Let X be an arbitrary r.i.s. on $\mathbb{R}$. Then H_X is the set of all classes $[f]$, modulo the constants, of Lebesgue measurable functions on $\mathbb{R}$ such that $||M_1^\sharp f||_X < \infty$. On H_X we put $||f||_{H_X} := ||M_1^\sharp f||_X$.*

The definition 8 extends the previous definition of H_X for an r.i.s. X with $q_X < \infty$, view Corollary 6.

Indeed let us put $[H_X]$ the set of all classes $[f]$, for $f \in H_X$, where $q_X < \infty$. Then the formal identity map

$$I : (H_X, ||f||_X + ||M_1^\sharp f||_X) \to ([H_X], ||M_1^\sharp f||_X)$$

given by $I(f) = [f]$ is a bijective and bounded map, as is easily seen. (The injectivity of I follows from the fact that a constant function c belongs to X, with $q_X < \infty$ iff $c = 0$.)

But $([H_X], ||M_1^\sharp f||_X)$ is a Banach space and by the open mapping theorem I is an isomorphism and we may identify H_X with H_X given by Definition 8 if $q_X < \infty$.

From now on we consider H_X as a set of classes of functions rather than a set of functions.

Theorem 9 *Let X be a r.i.s. on $\mathbb{R}$ with an absolutely continuous norm and such that $1 \le p_X \le q_X < \infty$. Let X' be its associate space and $H_X^0 \subset H_X$ the dense subspace given by Lemma 7.*

Then we have: (1) for every linear and bounded functional ℓ on H_X, there is $g \in H_{X'}$ such that

$$\ell(f) = \int_{\mathbb{R}} f(x)\overline{g(x)}dx$$

for each $f \in H_X^0$ and moreover

$$||g||_{H_{X'}} \le C||\ell||.$$

(2) If $g \in H_{X'}$, then

$$\ell_g(f) = \int_{\mathbb{R}} f(x)\overline{g(x)}dx$$

for $f \in H_X^0$, can be extended to a linear and bounded functional on H_X and

$$||\ell_g|| \le C||g||_{H_{X'}}.$$

The representation given by (1) and (2) is unique, i.e. $\ell_g = 0$ iff g is constant.

Proof(1) it follows exactly in the same way as in [S]-531 since the analogon of Lemma 4.2-[S] follows from Proposition 2 and Definition 8, using the fact that $p_{X'} = (q_X)' > 1$.

(2) It is sufficient to show that

$$|\int_{\mathbb{R}} f(x)g(x)dx| \leq C||f||_{H_X} \cdot ||g||_{H_{X'}}$$

for $f \in H_X^0$ and $g \in H_{X'}$.

The arguments of [S] work also in our case. We use the notations from [S]-Theorem 5.1. Using the proof of Proposition 2 and the facts proved in [S]-p.536-538 we have that

$$||S_s^{\parallel} u_g||_{X'} \leq C||M_{0,s_2} M_1^{\parallel} g||_{X'} \leq (\text{since } M_{0,s_2} \text{ is bounded on } X') \leq C||M_1^{\parallel} g||_{X'}$$

$$= (\text{ by Definition 8}) = C||g||_{H_{X'}}.$$

Since Lemma 4.6-[S]-p.531 has an almost verbatim extension for an arbitrary r.i.s. X with $q_X < \infty$, we have

$$||S_s^{\parallel} u_f||_X \leq C||M_{0,s_1} S_\infty^* u_f||_X \leq C||S_\infty^* u_f||_X \leq C||u_f||_{H_X},$$

consequently, arguing as in [S], we have (2).∎

Remark Theorem 9 is still true if the band of elements with an absolutely continuous norm X_a is nonvoid, replacing H_X by H_{X_a}. So we can identify $(H_{X_a})^*$ with $H_{X'}$.

Now we recall the definition of dyadic Hardy space $H_X(d)$ generated by a r.i.s. X on $\mathbb{R}$. (See [Po2].)

For $f = \sum_Q s_Q h_Q$ we note by

$$m(f) = \sup_{Q \text{ dyadic interval } Q \subset \mathbb{R}} [(\sum_{P \subset Q} \frac{|s_P|^2}{|P|} \chi_P)^{1/2}]^{\sim}(\frac{|Q|}{2})\chi_Q$$

$f^{\sim}$ being the decreasing rearrangement of f.

(Note that in [Po2],[FJ], $m(f)$ has a sligthly different definition, the function which enters in definition of $m(f)$ being computed in $\frac{|Q|}{4}$, but this change is not important for ours aims.)

Then we have:

Definition 10 *Let X be a r.i.s. on $\mathbb{R}$. Then we put*

$$H_X(d) = \{f : \mathbb{R} \to \mathbb{C}; f \text{ Lebesgue measurable such that}$$

$$f = \sum_Q s_Q h_Q \text{ and } |||f|||_{H_X} := ||m(f)||_X < \infty\}.$$

We have the following interpolation result:

Theorem 11 *Let X be a r.i.s. on $\mathbb{R}$. Then there is a linear operator U independent on X such that $U : H_X \to H_X(d)$ be a Banach space isomorphism.*

Proof Since there is well-known that there is a simultaneous isomorphism $U : H_1 \to H_1(d)$, $U : BMO \to BMO(d)$. (see [M]) we have:

$$K(t, Uf; H_1(d), BMO(d)) \sim K(t, f : H_1, BMO).$$

By the known fact (see [FJ]) that

$$K(t, f : H_1(d), BMO(d)) \sim \int_0^t m(f)^{\sim}(s)ds,$$

we have

$$K(t, Uf : H_1(d), BMO(d)) \sim K(t, m(Uf); L^1, L^\infty).$$

By Corollary 5 we have that H_{L^p} coincide with $F_p^{0,2}$ in the sense of [FJ], for $1 \le p \le \infty$.

View Theorem 2.2-[FJ] it follows that

$$K(t, f; H_1, BMO) \sim K(t, f_1; H_1(d), BMO(d)) \sim K(t, m(f_1); L^1, L^\infty),$$

where $f_1 := \sum_Q (S_\varphi f)_Q h_Q$, $S_\varphi f := (< f, \varphi_Q >)_Q$, $< \cdot, \cdot >$ being the scalar product in L^2, and φ_Q being described in the proof of Lemma 7.

These results and the fact that X is a monotone interpolation space for (L^1, L^∞), in the sense that if $K(t, g : L^1, L^\infty) \le K(t, f : L^1, L^\infty)$ and $f \in X$ then it follows that $g \in X$ and $||g||_X \le ||f||_X$, give us:

$$||m(Uf)||_X \sim ||m(f_1)||_X.$$

But from [FJ]

$$||f||_{H_X} \sim ||f||_{F_X^{0,2}} \sim ||m(f_1)||_{f_X},$$

then:

$$||Uf||_{H_X(d)} \sim ||f||_{H_X}.\blacksquare$$

Now we can prove an analogon of Theorem 10 for dyadic Hardy spaces:

Theorem 12 *Let X be a r.i.s. on $\mathbb{R}$ with an absolutely continuous norm and such that $q_X < \infty$. Then its topological dual $H_X(d)$ can be identified to $H_{X'}(d)$ by a Banach space isomorphism $\tilde{V}$ non depending on X.*

Proof By Theorem 11 it follows that there is an isomorphism $U : F_Y^{0,2} \leftrightarrow f_Y^{0,2}$ (see the proof of Lemma 7 for corresponding definitions) non depending on r.i.s. Y on $\mathbb{R}$.

Then view of Corollary 5 and Theorem 9 we have

$$\begin{array}{ccccc} (H_X)^* & = & (F_X^{0,2})^* & \overset{V}{\leftrightarrow} & H_{X'} \\ \updownarrow U^* & & \updownarrow U^* & & \updownarrow U \\ (H_X(d))^* & = & (f_X^{0,2})^* & \overset{\tilde{V}}{\leftrightarrow} & H_{X'}(d) \end{array}$$

Here $\tilde{V}$ is defined by the commutativity of the diagramm before. $\tilde{V}$ is obviously an isomorphism.$\blacksquare$

Now let X be a r.i.s. on $[0,1]$ such that $q_X < \infty$ and such that its norm be absolutely continuous. Then it is known that there is a r.i.s. Y on $(0,\infty)$ such that $Y|_{[0,1]} = X$ (i.e. if $f \in Y$, its decreasing rearrangement $f^\sim$ be such that $f^\sim \cdot \chi_{(0,1)} \in X$ and $||f||_X \sim ||f^\sim \chi_{(0,1)}||_Y$.) Moreover if $f \in X$ then for

$$\overline{f}(x) = \begin{cases} f(x) & x \in (0,1) \\ 0 & x \in [1,\infty), \end{cases}$$

we have $||f||_X \sim ||\overline{f}||_Y$. (See Theorem 8.4-[JMST]).

Then if X is as before and $f = \sum_{Q \subset I = [0,1]} s_Q h_Q$, $f \in H_X(d)$ it follows that

$$\overline{f}(x) = \sum_{Q \subset I} s_Q h_Q \text{ and } S(\overline{f}) = (\sum_{Q \subset I} \frac{|s_Q|^2}{|Q|} \chi_Q)^{1/2} = S(f) \in X.$$

Obviously $[S(\overline{f})^{\sim}\chi_{(0,1)}](t) = S(f)^{\sim}(t)$ for $t \in [0,1]$, so $S(\overline{f}) \in Y(0,\infty)$, and $\|S(\overline{f})\|_Y \sim \|S(f)\|_X$, so $\overline{f} \in H_Y(d)$, i.e. the map $f \to \overline{f}$ from $H_X(d)$ into $H_Y(d)$ is an isomorphic embedding.

Moreover Y has an absolutely continuous norm, because if $\chi_{E_n} \downarrow 0$ and $f \in Y$, then $0 \le (f\chi_{E_n})^{\sim} \le f^{\sim}\chi_{(0,|E_n|)}$ which implies, for $|E_n| \le 1$, that

$$\|f\chi_{E_n}\|_Y = \|(f\chi_{E_n})^{\sim}\|_Y \le \|f^{\sim}\chi_{(0,|E_n|)}\|_X \to 0.$$

By Theorem 12 we have that

$$[H_X(d)]^{*} \sim [H_Y(d)]^{*}/(H_X(d))^{\circ} \sim \frac{H_{Y'}(d)}{[H_X(d)]^{\circ}},$$

where $[H_X(d)]^{\circ}$ is $\{f \in H_{Y'}(d)$ so that $\int_Q fh_Q dt = 0$ for all $Q \subset I\}$.

Then we have for $f = \sum_{Q\in A} s_Q h_Q \in H_{Y'}(d)$, where A is a finite set of dyadic intervals, that the class $[f] \in \frac{H_{Y'}(d)}{H_X(d)^{\circ}}$ coincides with $[f_1]$, where $f_1 = \sum_{Q\in A} s_Q h_Q$, or even to f_1.

But $supp\, m(f_1) \subset [0,1]$, so

$$|\int_0^1 g\, m(f_1)dt| \le \int_0^1 g^{\sim} m(f_1)^{\sim} dt =$$

$$= \int_0^{\infty} g^{\sim}(t) m(f_1)^{\sim} dt \le (\text{since } m(f_1) \in Y') \le$$

$$\le \|g^{\sim}\|_Y \cdot \|m(f_1)^{\sim}\|_{Y'} = \|g\|_X \cdot \|m(f_1)^{\sim}\|_{Y'} \quad \text{for all } g \in X.$$

Then $m(f_1) \in X'$ and $\|f_1\|_{H_{X'}(d)} = \|m(f_1)\|_{X'} \sim \|f_1\|_{H_{Y'}}$, which in turn implies that the dual of $[H_X(d)]^{*}$ can be identified with the completion of $\{f_1; f_1 = \sum_{Q\in A, Q\subset I} s_Q h_Q$, A an arbitrary finite set of dyadic intervals in $(0,\infty)\}$ for the norm of $H_X(d)$, that is with $H_{X'}(d)$ itself.

So we proved the following theorem which extends a previous result of [Po2].

Theorem 13 *Let X be a r.i.s. on $[0,1]$ with an absolutely continuous norm and $q_X < \infty$. Then the topological dual space $[H_X(d)]^{*}$ is isomorphic in a natural way to $H_{X'}(d)$.*

Now in the same way as in [Po2] we can use Theorem 13 in order to prove the following extension of Theorem 4.4-[Po2] and of a well-known result of Bourgain [B]:

Theorem 14 *Let X be a r.i.s. on $[0,1]$ with an absolutely continuous norm such that $q_X < 2$. Then $H_X(\ell^2)$ is not isomorphic as Banach spaces to H_X.*

We recall that

$$H_X(\ell^2) := \{f = (f_i)_{i=1}^{\infty}; f_i \in H_X; \text{such that } \|f\|_{H_X(\ell^2)} :=$$

$$= \|(\sum_i |f_i|^2)^{1/2}\|_{H_X} < \infty\},$$

where $\|(\sum_i |f_i|^2)^{1/2}$ is given by Krivine functional calculus in the Banach lattice $H_X(d)$, ordered by $f = \sum_Q s_Q h_Q \geq 0$ iff $s_Q \geq 0$ for all Q.

We remark that Bourgain proved Theorem 14 only in the case $X = L^1$.
Similarly to [Po2] we derive then:

Corollary 15 *Let X be a r.i.s. on $[0,1]$ as in Theorem 14. Then $H_X(d)$ is not isomorphic as Banach spaces to anyone r.i.s. Y on $[0,1]$.*

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DISSIPATIVE AND ACCRETIVE OPERATORS ON LOCALLY CONVEX SPACES

Mihai Voicu

R.S.Phillips [9] introduced the notion of dissipative operator on a Hilbert space in 1959 and apllied it to hyperbolic systems of partial differential equations. Later G. Lumer and R.S.Phillips [6] introduced a new notion of dissipative operator on a Banach space in 1961, with respect to a semi-inner product compatible with the norm and used it to characterize the generators of C_0 - one parameter contraction semigroups. Almost simultaneously, accretive operators on a Hilbert space were introduced and applied to differential operators.

In 1972 F. Hirsch [3] defined the notions of M-codissipative (resp. M-dissipative) operators with respect to the norm and used them to characterize the limit operators of the L_0 (resp. L_r) resolvents, including the abstract potentials and generators of C_0-semigroups.

In this paper we introduce the notions of accretive and dissipative operators on a locally convex space.

In the first section some basic properties are presented. The second section is dedicated to L_0 and L_r resolvents and their limit operators. The last section is devoted to resolvents and potentials on $\mathcal{L}_0(X)$. The accretive operators are used to characterize the limit operators of the L_0 resolvents, while the dissipative operators are used to characterize the limit operators of the L_r resolvents. Corollary 3.8 and Corollary 3.9 cover similar results obtained by Hirsch in [3].

1.Basic properties

Throughout this paper X is a real or complex Hausdorff locally convex space, whose topology is given by the directed family of seminorms $P=(p_\alpha)$, $\alpha\in A$.

Definition 1.1 *A linear operator $V:D(V)\to X$ is called dissipative (resp accretive) if for every $\alpha\in A$, there exists $M(\alpha)>0$ such that $p_\alpha(\lambda x)\le M(\alpha)p_\alpha(\lambda x-V(x))$*

(resp. $p_\alpha(\lambda V(x))\le M(\alpha)p_\alpha(x+\lambda V(x))$ for all $\lambda>0$ and $x\in D(V)$.

Remark 1.2 *If X is a normed space we will find the concepts of M-dissipativity (codissipativity) introduced by F.Hirsch in [3].*

Proposition 1.3. *Let $V:D(V)\to X$ be a linear operator. Then the following assertions are equivalent:*

1. *V is dissipative.*

2. *(−V) is accretive.*

Proof. We suppose that V is dissipative. Let $\alpha\in A$, $x\in D(V)$ and $\lambda>0$. We have $\lambda V(x)=\lambda V(x)-x+x$.

Let $\alpha\in A$, $x\in D(V)$. Then $p_\alpha(\lambda V(x))\le p_\alpha(\lambda V(x)-x)+p_\alpha(x)\le p_\alpha(\lambda V(x)-x)+$ $+M(\alpha)p_\alpha(x-\lambda V(x))=(M(\alpha)+1)p_\alpha(x-\lambda V(x))$ and consequently −V is accretive. We assume now that (−V) is accretive. Let $x\in D(V)$, $\lambda>0$ and $\alpha\in A$.

Since $x=x-\lambda V(x)+\lambda V(x)$ we get $p_\alpha(x)\le(M(\alpha)+1)p_\alpha(x-\lambda V(x))$ which is equivalent to $p_\alpha(\lambda x)\le(M(\alpha)+1)p_\alpha(\lambda x-V(x))$ and hence it follows that V is dissipative □

Proposition 1.4. *Let $V:D(V)\to X$ be a linear operator. Suppose that V is one-to-one. Then the following statements are equivalent*

1. *V is dissipative (accretive).*

2. *$(-V^{-1})$ is accretive (dissipative).*

The proof is easy and we omit it.

Proposition 1.5. *If $V:D(V)\to X$ is dissipative then $\lambda I-V$ is one-to-one and for every $\alpha\in A$, there exists $M(\alpha)>0$ such that $p_\alpha((\lambda I-V)^{-1}(y))\le\frac{1}{\lambda}M(\alpha)p_\alpha(y)$ for all $\lambda>0$ and $y\in R(\lambda I-V)$*

Proof. Let $x\in D(V)$, $\lambda>0$ such that $\lambda x-V(x)=0$ and $\alpha\in A$. Then from the dissipativity of V, we obtain $p_\alpha(\lambda x)=0$ and hence $x=0$

Let $y\in R(\lambda I-V)$ and $x=(\lambda I-V)^{-1}(y)$.

Then $p_\alpha(x)=\frac{1}{\lambda}p_\alpha(\lambda x)\le\frac{1}{\lambda}M(\alpha)p_\alpha(\lambda x-V(x))=\frac{1}{\lambda}M(\alpha)p_\alpha(y)$

Finally we have $p_\alpha((\lambda I - V)^{-1}(y)) \leq \frac{M(\alpha)}{\lambda} p_\alpha(y)$. □

Proposition 1.6. *If $V:D(V)\to X$ is accretive, then $I+\lambda V$ is one-to-one and for every $\alpha\in A$, there exists $M(\alpha)>0$ such that $p_\alpha((I+\lambda V)^{-1}(y))\leq(M(\alpha)+1)p_\alpha(y)$ for all $\lambda>0$ and $y\in R(I+\lambda V)$.*

Proof. Let $\lambda>0$, $x\in D(V)$ such that $x+\lambda V(x)=0$ and $\alpha\in A$. Then since V is accretive, we get $p_\alpha(\lambda V(x))=0$ and consequently $x=0$. Let $y\in R(x+\lambda V)$ and $x=(I+\lambda V)^{-1}(y)$. Then we have $p_\alpha(x)\leq p_\alpha(x+\lambda V(x))+p_\alpha(\lambda V(x))\leq p_\alpha(y)+M(\alpha)p_\alpha(y)= =(M(\alpha)+1)p_\alpha(y)$, hence $p_\alpha((I+\lambda V)^{-1}(y))\leq(M(\alpha)+1)p_\alpha(y)$. □

Proposition 1.7. *Let $V:D(V)\to X$ be a linear operator such that $\overline{D(V)}=X$ or $R(V)\subset\overline{D(V)}$. If V is dissipative then it is closable and its closure is dissipative.*

Proof. Let $x\in D(V)$. Suppose that V is dissipative and $\overline{D(V)}=X$. Let $\lambda>0$ and $x_\delta\in D(V)$, $\delta\in\Delta$ provided that $(x_\delta, V(x_\delta))\to(0, y)$.

For each $\alpha\in A$, $p_\alpha(\lambda(x_\delta+\frac{1}{\lambda}x))\leq M(\alpha)p_\alpha(\lambda x_\delta+x-V(x_\delta)-\frac{1}{\lambda}V(x))$. Since $\lim_{\delta\in\Delta} x_\delta=0$, from the above inequality we obtain $p_\alpha(x)\leq M(\alpha)p_\alpha(x-y-\frac{1}{\lambda}V(x))$, for all $\lambda>0$. Hence it follows that:

(1) $$p_\alpha(x)\leq M_\alpha p_\alpha(x-y)$$

for all $\alpha\in A$.

If $\overline{D(V)}=X$, the inequality (1) holds for all $x\in X$ and consequently for $x=y$. In conclusion $p_\alpha(y)=0$ for all $\alpha\in A$.

Hence it follows that $y=0$, which means that V is closable. Let us assume now that $R(V)\subset\overline{D(V)}$. Then $y\in\overline{D(V)}$, because $y=\lim_{\delta\in\Delta}V(x_\delta)$. Let $y_\delta\in D(V)$ such that $\lim_{\delta\in\Delta} y_\delta=y$. By using again (1) for each y_δ and by passing to limit over δ we conclude that $y=0$ and consequently V is closable.

Let $\hat{V}:D(\hat{V})\to X$ be its closure. Let $\hat{x}\in D(\hat{V})$, $\lambda>0$ and $x_\delta\in D(V)$, $x_\delta\to\hat{x}$ and $V(x_\delta)\to\hat{V}(\hat{x})$. On the other hand for each $\alpha\in A$, there exists $M(\alpha)>0$ such that $p_\alpha(\lambda x_\delta)\leq M(\alpha)p_\alpha(\lambda x_\delta-V(x_\delta))$ for all $\delta\in\Delta$, and finally it follows that $p_\alpha(\lambda\hat{x})\leq M(\alpha)p_\alpha(\lambda\hat{x}-\hat{V}(\hat{x}))$. Therefore the dissipativity of $\hat{V}$ is proved. By proposition 1.3, the result holds also for accretive operators. □

Corollary 1.8. *Let X be a Frechet space and $V: X\to X$ a linear operator. If V is dissipative (accretive) then V is continuous.*

Proof. Indeed V is closed and hence continuous.

2. Resolvents and limit operators

We denote by $\mathcal{L}(X)$ *the space of all linear and bounded operators on X*

Definition 2.1. *A mapping* $R: (0, \infty)\to\mathcal{L}(X)$ ***is called a resolvent** if the following condition is satisfied*

$$R(\lambda)-R(\mu)=(\mu-\lambda)R(\lambda)R(\mu), \text{ for all the } \lambda, \mu \in (0, \infty) \tag{2}$$

Definition 2.2. *Following F. Hirsch, we say that a resovent R is of type* L_0 *(resp.* L_∞*) if* $\lim_{\lambda\to 0}\lambda R(\lambda)(x)=0$, *(resp.* $\lim_{\lambda\to\infty}\lambda R(\lambda)(x)=x$ *) for all* $x\in X$.

Proposition 2.3. *Let* $R:(0, \infty)\to\mathcal{L}(X)$ *be a resolvent. The following assertions are equivalent:*

1. *R is of type* L_0.
2. $\begin{cases} a)\ \overline{R(I-\lambda R(\lambda))}=X, \quad \text{for all } \lambda\in(0,\infty). \\ b)\ \text{For each } \alpha\in A \text{ the mapping } \Phi_\alpha : X\to R \text{ defined by } \Phi_\alpha(x)=\lim_{\lambda\to 0}\sup p_\alpha(\lambda R(\lambda)(x)) \\ \quad \text{is continuous.}\end{cases}$

Proof. Let us suppose that R is a resolvent of type L_0 and $x\in X$. Then $x=\lim_{\lambda\to 0}(I-\lambda R(\lambda))(x)$ and hence a) holds.

The continuity of Φ_α is also obvious because $\Phi_\alpha(x)=0$ for all $x\in X$. Conversely, let $F=\{x\in X \mid \lim_{\lambda\to 0}\lambda R(\lambda)(x)=0\}$ and observe that $F=\bigcap_{\alpha\in A}\Phi_\alpha^{-1}(0)$. Since Φ_α is continuous, F is closed. On the other hand by rearranging the resolvent equation (2) we get for $\lambda, \mu\in(0, \infty)$ and $x\in X$ the following equality

$$\lambda R(\lambda)(x-\mu R(\mu)(x))=\frac{\lambda}{\lambda-\mu}(\lambda R(\lambda)(x)-\mu R(\mu)(x)) \tag{3}$$

By using once again the continuity of Φ_α we obtain from (3) that $\lim_{\lambda\to 0}\lambda R(\lambda)(x-\mu R(\mu)(x))=0$. This means that $R(I-\mu R(\mu))\subset F$ and consequently F=X and the proof is complete □

Proposition 2.4. *Let* $R:(0, \infty)\to\mathcal{L}(X)$ *be a resolvent. The following assertions are equivalent:*

1. *R is of type* L_∞.

$$\textbf{2.}\begin{cases} a)\ \overline{R(R(\lambda))} = X, \quad \text{for all } \lambda \in (0,\infty). \\ b)\ \text{For each } \alpha \in A \text{ the mapping } \Phi_\alpha : X \to R \text{ defined by } \Phi_\alpha(x) = \lim\limits_{\lambda\to\infty} \sup p_\alpha(\lambda R(\lambda)(x)) \\ \quad \text{is continuous.} \end{cases}$$

Proof. Let us suppose that R is a resolvent of type L_∞ and $x\in X$. Then $x = \lim\limits_{\lambda\to\infty} \lambda R(\lambda)(x)$, which proves that $\overline{R(R(\lambda))} = X$. Moreover $\Phi_\alpha(x)=p_\alpha(x)$ for all $\alpha\in A$. Conversely, let $x\in X$ and $\alpha\in A$. We have also $p_\alpha(R(\lambda)(x)) = \frac{1}{\lambda} p_\alpha(\lambda R(\lambda)(x))$, and hence $\lim\limits_{\lambda\to\infty} R(\lambda)(x) = 0$. It is easy to see from (2) that $\lim\limits_{\lambda\to\infty} \lambda R(\lambda)R(\mu)(x) = R(\mu)(x)$ for all $x\in X$. Because of the density of $R(R(\lambda))$ and the continuity of ϕ_α we conclude that $\lim\limits_{\lambda\to+\infty} \lambda R(\lambda)(x) = x$ for all $x\in X$ ☐

Definition 2.5. *A linear operator $V:D(V)\to X$ is called a cogenerator if there exists a resolvent R of type L_0 such that* $D(V) = \{x \in X \mid \lim\limits_{\lambda\to 0} R(\lambda)(x) \text{ exists}\}$ *and* $V(x) = \lim\limits_{\lambda\to 0} R(\lambda)(x)$, for all $x \in D(V)$.

Definition 2.6. *A linear operator $V:D(V)\to X$ is called a generator if there exists a resolvent R of type L, such that* $D(V) = \{x \in X \mid \lim\limits_{\lambda\to\infty} \lambda(\lambda R(\lambda)(x) - x) \text{ exists}\}$ *and* $V(x) = \lim\limits_{\lambda\to\infty} \lambda(\lambda R(\lambda)(x) - x)$, *for all* $x \in D(V)$.

Proposition 2.7. *Let $R:(0,\infty)\to\mathcal{L}(X)$ be a resolvent of type L_∞ and* $D = \{x\in X \mid \lim\limits_{\lambda\to\infty} \lambda(\lambda R(\lambda)(x) - x) \text{ exists}\}$ *Then the following statements hold:*

1. $\overline{D} = X$

2. *The linear operator $V:D\to X$ defined by* $V(x) = \lim\limits_{\lambda\to\infty} \lambda(\lambda R(\lambda)(x) - x)$ *is a generator and $R(\lambda)=(\lambda I-V)^{-1}$ for all $\lambda>0$.*

Proof First we remark that the resolvent equation (2) can be written as follows

$$\lambda R(\lambda)R(\mu)(x) - R(\mu)(x) = \mu R(\lambda)R(\mu)(x) - R(\lambda)(x)$$

for all $\lambda, \mu \in (0,\infty)$ and $x\in X$

If we replace $R(\mu)(x)$ by y in the above equality we obtain $\lambda(\lambda R(\lambda)(y)-y) = \mu\lambda R(\lambda)(y) - \lambda R(\lambda)(x)$ and by passing to the limit as $\lambda\to\infty$ we conclude that $y=R(\mu)(x)\in D$, $R(R(\mu))\subset D$ and $V(R(\mu)(x))=\mu R(\mu)(x)-x$ or equivalently

$$(\mu I-V)R(\mu)(x) = x \tag{4}$$

But $\overline{R(R(\mu))}$=X and consequently $\overline{D}$ = X.

Let x∈D and μ>0. Then by the definition one can write $R(\mu)V(x)=R(\mu)(\lim_{\lambda\to\infty}\lambda(\lambda R(\lambda)(x)-x))=\lim_{\lambda\to\infty}\lambda(\lambda R(\lambda)R(\mu)(x)-R(\mu)(x))=VR(\mu)(x)=\mu R(\mu)(x)-x$.

Therefore R(μ)V(x)=μR(μ)(x)−x. In other words we have

(5) $R(\mu)(\mu I-V)(x)=x$ for all x∈D and μ>0

Looking at (4) and (5) we can state that $R(\mu)=(\mu I-V)^{-1}$ for all μ∈(0, ∞). □

Remark 2.8. *Given a generator V:D(V)→X there exists a unique resolvent R:(0,∞)→L(X) of type L_∞ such that $R(\mu)=(\mu I-V)^{-1}$ for all μ∈(0, ∞). For this reason we will say that V is the generator of R.*

Proposition 2.9. *Let R:(0, ∞)→L(X) be a resolvent of type L_0 and $D=\{x\in X \mid \lim_{\lambda\to 0}R(\lambda)(x)$ exists$\}$. Then the following statements hold:*

1. $\overline{D}=X$.

2. The linear operator V:D→X defined by $V(x)=\lim_{\lambda\to 0}R(\lambda)(x)$ is a cogenerator and $R(\lambda)=V(I+\lambda V)^{-1}$ for all λ>0.

Proof. Let λ, μ∈(0, ∞) and x∈X. By the resolvent equation (2) it follows that $R(\lambda)(x-\mu R(\mu)(x))=R(\mu)(x)-\lambda R(\lambda)R(\mu)(x)$ and $\lim_{\lambda\to 0}R(\lambda)(x-\mu R(\mu)(x))=R(\mu)(x)$.

This means that $R(I-\mu R(\mu))\subset D$ and

(6) $V(I-\mu R(\mu))=R(\mu)$

which is equivalent to

(7) $(I+\mu V)(I-\mu R(\mu))=I$ for all μ>0.

Since $\overline{R(I-\mu R(\mu))}$=X we conclude that $\overline{D}=X$. Let now x∈D and λ, μ∈(0, ∞). We have from (2) $R(\lambda)(x)-R(\mu)(x)=(\mu-\lambda)R(\lambda)R(\mu)(x)$. By passing to the limit as μ→0 in the previous equality we get $R(\lambda)(x)-V(x)=-\lambda R(\lambda)V(x)$ which is equivalent to the following equality

(8) $(I-\lambda R(\lambda))(I+\lambda V)(x)=x$, for all x∈D.

From (6), (7) and (8) we deduce that $(I+\lambda V)^{-1}=I-\lambda R(\lambda)$ and $R(\lambda)=V(I+\lambda V)^{-1}$ for all λ∈(0, ∞) and the proof is complete. □

Remark 2.10. *Given a cogenerator V:D(V)→X there exists a unique resolvent R:(0, ∞)→L(X) of type L_0 such that $R(\lambda)=V(I+\lambda V)^{-1}$ for all λ∈(0, ∞)*

. For this reason we will say that V is the cogenerator of R.

3. Resolvents and potentials on $\mathcal{L}_0(X)$

Definition 3.1. *Set*

(9) $$\mathcal{L}_0(X) := \{T \in \mathcal{L}(X) : \forall \alpha \in A,\ \exists C(\alpha)>0\ \ \forall x \in X\ \ p_\alpha(T(x)) \le C(\alpha) p_\alpha(x)\ \}$$

For each $\alpha \in A$ we define $q_\alpha: \mathcal{L}_0(X) \to R$ as follows:

(10) $$q_\alpha(T) = sup\left\{ \frac{p_\alpha(T(x))}{p_\alpha(x)} \middle| x \in X, p_\alpha(x) \neq 0 \right\}.$$

Lemma 3.2. *The following assertions are true:*

1. *q_α is a seminorm for all $\alpha \in A$.*
2. *$Q = \{q_\alpha,\ \alpha \in A\}$ is a sufficient family of seminorms on $\mathcal{L}_0(X)$.*
3. *For each $T \in \mathcal{L}_0(X)$ and $\alpha \in A$ the following formula holds*
   $$p_\alpha(T(x)) \le q_\alpha(T) p_\alpha(x) \quad \text{for all } x \in X.$$

The proof is obvious.

Lemma 3.3. *If X is a sequentially complete locally convex space, then $(\mathcal{L}_0(X), Q)$ is a sequentially complete m-convex algebra.*

Proof. The inequality $q_\alpha(TU) \le q_\alpha(T) q_\alpha(U)$ is a consequence of (10). Thus, $(\mathcal{L}_0(X), Q)$ is a m-convex algebra.

Let now $T_n \in \mathcal{L}_0(X)$, $n \in N$ be a Cauchy sequence, $\varepsilon>0$, and $\alpha \in A$. Then there exists $N(\varepsilon) \in N$ such that, for any n, m $\ge N(\varepsilon)$ it follows $q_\alpha(T_n - T_m) < \varepsilon$, hence $p_\alpha(T_n(x) - T_m(x)) \le \varepsilon\ p_\alpha(x)$ for all $x \in X$. This means that for each $x \in X$ the sequence $T_n(x)$ is a Cauchy sequence in X and consequently is convergent.

Let us denote by $T(x) = \lim_{n\to\infty} T_n(x)$. Then $p_\alpha(T_n - T)(x) \le \varepsilon p_\alpha(x)$ for all $n \ge N(\varepsilon)$ and $x \in X$. Therefore $T \in \mathcal{L}_0(X)$ and $\lim_{n\to\infty} T_n = T$ in $\mathcal{L}_0(X)$. ◻

Example 3.4. Let $X=C(R)$ the space of the real contiuous functions equipped with the following family of seminorms, $p_n(f) = \sup\{|f(x)|, |x| \le n\}$, $n \in N$ and $f \in X$.

Let also $T: X \to X$ be defined by the following formula

$$T(f)(x) = \begin{cases} \int_0^x f(t)dt, & x \quad 0 \\ 0 & , x \le 0 \end{cases}$$

It is clear that T is a linear operator and $p_n(T(f)) \le n p_n(f)$ for all $f \in X$ and $n \in N$. Therefore $T \in \mathcal{L}_0(X)$ and $q_n(T) \le n$ for all $n \in N$.

Example 3.5. Let X=C(R) equipped with the same topology as above and $T:X\to X$ be defined by the following formula

$$T(f)(x)=\begin{cases} e^{-x}\int_0^x f(t)dt, & x\ 0 \\ 0 & , x\le 0\end{cases}$$

It is easy to prove that T is linear and $p_n(T(f))\le p_n(f)$ for all $f\in X$ and $n\in \mathbf{N}$. Hence it follows that $T\in \mathcal{L}_0(X)$ and $q_n(T)\le 1$ for all $n\in\mathbf{N}$.

Theorem 3.6. *Let $V:D(V)\to X$ be a linear operator. The following assertions are equivalent:*

1. V is the generator of a resolvent $R:(0,+\infty)\to\mathcal{L}_0(X)$ of type L_r verifying that: for every $\alpha\in A$, there exists $M(\alpha)>0$ such that $q_\alpha(\lambda R(\lambda))\le M(\alpha)$ for all $\lambda\in(0,\infty)$.

2. $\begin{cases} a)\ \overline{D(V)}=X, \\ b)\ V \text{ is dissipative}, \\ c)\ R(\lambda I-V)=X \quad \text{for all } \lambda\in(0,\infty).\end{cases}$

Proof. It is enough to prove the dissipativity of V, because the other two conclusions are consequences of proposition 2.7 which states that $\overline{D(V)}$=X and $R(\lambda)=(\lambda I-V)^{-1}$ for all $\lambda\in(0,\infty)$. Let $x\in D(V)$ and $\lambda>0$. Set $y=(\lambda I-V)(x)$. Then $\lambda R(\lambda)(y)=\lambda x$. Let also $\alpha\in A$ and $M(\alpha)>0$ such that $p_\alpha(\lambda R(\lambda)(y))=$ $=p_\alpha(\lambda x)\le q_\alpha(\lambda R(\lambda))p_\alpha(y)\le M(\alpha)p_\alpha(y)=M(\alpha)p_\alpha(\lambda x-V(x))$.

Hence it follows that for every $\alpha\in A$, there exists $M(\alpha)>0$ such that $p_\alpha(\lambda x)\le M(\alpha)p_\alpha(\lambda x-V(x))$ for all $x\in D(V)$ and $\lambda\in(0,\infty)$. Therefore V is dissipative. Conversely, if V is a dissipative operator then $\lambda I-V$ is one to one and by using proposition 1.5 we know that $R(\lambda)=(\lambda I-V)^{-1}\in\mathcal{L}_0(X)$, for all $\lambda\in(0,\infty)$. Moreover $R:(0,\infty)\to\mathcal{L}_0(X)$ is a resolvent because $R(\lambda)=R(\lambda, V)$.

On the other hand we know from proposition 2.4 that R is a resolvent of type L_r and $p_\alpha(\lambda R(\lambda)(y))\le M_\alpha p_\alpha(y)$ for all $\lambda\in(0,\infty)$ and $y\in X$. This means that $q_\alpha(\lambda R(\lambda))\le M_\alpha$.

In conclusion V is the generator of R and the proof is complete. □

Corollary 3.7. *Let X be a Banach space and $V:D(V)\to X$ a linear operator. The following assertions are equivalent.*

1. V is the generator of a resolvent $R:(0,+\infty)\to\mathcal{L}(X)$ of type L_r verifying that there exists $M>1$ such that $\|\lambda R(\lambda)\|\le M$ for all $\lambda\in(0,\infty)$.

2. $\begin{cases} a)\ \overline{D(V)}=X, \\ b)\ V \text{ is } M\text{-dissipative}, \\ c)\ \text{There exists } \lambda_0\in(0,\infty) \text{ such that } R(\lambda_0 I-V)=X.\end{cases}$

The proof is a consequence of Theorem 3.6 and of the following remarks.

Since X is a normed space $\mathcal{L}_0(X)=\mathcal{L}(X)$. On the other hand, to say that V is dissipative is equivalent to say that there exists M>0 such that, $\|\lambda x\| \le M\|(\lambda x - V(x)\|$ for all $x\in D(V)$ and $\lambda\in(0, \infty)$.

Moreover, in the Banach algebra $\mathcal{L}(X)$, under the conditions 2.b) and 2.c) one can prove that $R(\lambda I-V)=X$ for all $\lambda\in(0, \infty)$.

Theorem 3.8. *Let $V:D(V)\to X$ be a linear operator. The following assertions are equivalent:*

1. *V is the cogenerator of a resolvent $R:(0,\infty)\to\mathcal{L}_0(X)$ of type L_0 verifying that: for every $\alpha\in A$, there exists $M(\alpha)>0$ such that $q_\alpha(\lambda R(\lambda))\le M_\alpha$ for all $\lambda\in(0, \infty)$.*

2. $\begin{cases} a)\ \overline{D(V)} = X, \\ b)\ V \text{ is accretive}, \\ c)\ R(I+\lambda V) = X \quad \text{for all } \lambda\in(0,\infty). \end{cases}$

Proof. Proposition 2.9. shows that $\overline{D(V)}=X$ and $R(\lambda)=V(I+\lambda V)^{-1}$ for all $\lambda\in(0, \infty)$. Let $y\in D(V)$ and $\lambda>0$ and $x=(I+\lambda V)(y)$. Then $\lambda V(y)=\lambda V(I+\lambda V)^{-1}(x)=\lambda R(\lambda)(x)$. Let now $\alpha\in A$ and $M(\alpha)>0$ such that $q_\alpha(\lambda R(\lambda))\le M_\alpha$ for all $\lambda\in(0, \infty)$. Then it follows that $p_\alpha(\lambda V(y))=p_\alpha(\lambda R(\lambda)(x))\le M_\alpha p_\alpha(x)=M_\alpha p_\alpha(y+\lambda V(y))$. Therefore V is accretive. Conversely, from proposition 1.3. we know that (–V) is dissipative. According to Theorem 3.6, (–V) is the generator of a resolvent $S:(0,+\infty)\to\mathcal{L}_0(X)$ of type L. We define the following mapping $R:(0, \infty)\to\mathcal{L}_0(X)$ by the formula $R(\lambda)=\frac{1}{\lambda}(I-\frac{1}{\lambda}S(\frac{1}{\lambda}))$. It is easy to see that R is a resolvent of type L_0. We suppose that $W:D(W)\to X$ is its cogenerator. Let $x\in D(W)$. Then $W(x)=\lim_{\mu\to 0} R(\mu)(x)=\lim_{\mu\to 0}\frac{1}{\mu}(x-\frac{1}{\mu}S(\frac{1}{\mu})(x))=V(x)$. Hence it follows that V is the cogenerator of R and $R(\lambda)=V(I+\lambda V)^{-1}$ for all $\lambda\in(0, \infty)$. Let $x\in X$, $\lambda\in(0, \infty)$ and $y=(I+\lambda V)^{-1}(x)$. Let also $\alpha\in A$ and $M(\alpha)>0$ given by the accretivity of V. We have $p_\alpha(\lambda R(\lambda)(x))=p_\alpha(\lambda V(y))\le M(\alpha)p_\alpha(y+\lambda V(y))=M(\alpha)p_\alpha(x)$ and finally $q_\alpha(\lambda R(\lambda))\le M(\lambda)$ for all $\lambda\in(0, \infty)$ and the proof is complete. ◻

Corollary 3.9. *Let X be a Banach space and $V:D(V)\to X$ a linear operator. The following assertions are equivalent:*

1. *V is the cogenerator of a resolvent $R:(0, \infty)\to\mathcal{L}(X)$ of type L_0, verifying that there exists M>1 such that $\|\lambda R(\lambda)\|\le M$ for all $\lambda\in(0, \infty)$.*

$$\textbf{2.}\ \begin{cases} a)\ \overline{D(V)} = X, \\ b)\ V \text{ is } M\text{-codissipative}, \\ c)\ \text{There exists } \lambda_0 \in (0,\infty) \text{ such that } R(I+\lambda_0 V) = X. \end{cases}$$

The proof runs as in Corollary 3.7 because in this context $R(I+\lambda V)=X$ for all $\lambda\in(0, \infty)$.

Definition 3.10. *A linear operator $V:D(V)\to X$ is called an abstract potential if V is the cogenerator of a resolvent $R:(0, \infty)\to \mathcal{L}_0(X)$ of type L_0 and L_∞ verifying that: for every $\alpha\in A$, there exists $M(\alpha)>0$ such that $q_\alpha(\lambda R(\lambda))\leq M(\alpha)$ for all $\lambda\in(0, \infty)$.*

Theorem 3.11. *Let $V:D(V)\to X$ be a linear operator. The following assertions are equivalent:*

1. V is an abstract potential.

$$\textbf{2.}\ \begin{cases} a)\ \overline{D(V)} = \overline{R(V)} = X, \\ b)\ V \text{ is accretive}, \\ c)\ R(I+\lambda V) = X \quad \text{for all } \lambda > 0. \end{cases}$$

Proof. Suppose that V is an abstract potential, then the equality $R(\lambda)=V(I+\lambda V)^{-1}$ holds for all $\lambda>0$. This means that $R(R(\lambda))=R(V)$. But from proposition 2.4 it follows that $\overline{R(R(\lambda))}=X$ and consequently $\overline{R(V)}=X$. Conversely, according to theorem 3.8. V is the cogenerator of a resolvent $R:(0, \infty)\to\mathcal{L}_0(X)$ of type L_0 such that $q_\alpha(\lambda R(\lambda))\leq M(\alpha)$ for all $\lambda\in(0,\infty)$ and $x\in X$.

In addition the formula $R(\lambda)=V(I+\lambda V)^{-1}$ holds. Let $\lambda\in(0, \infty)$, $x\in X$ and $\alpha\in A$. We have $p_\alpha(\lambda R(\lambda)(x))\leq M(\alpha)p_\alpha(x)$ and hence $p_\alpha(R(\lambda)(x)) \leq \frac{M(\alpha)}{\lambda} p_\alpha(x)$, thus $\lim_{\lambda\to\infty} R(\lambda)(x) = 0$. Let us write again the resolvent equation $R(\lambda)(x)-R(\mu)(x)=(\mu-\lambda)R(\lambda)R(\mu)(x)$. By passing to the limit as $\lambda\to\infty$ we get $\lim_{\lambda\to\infty} \lambda R(\lambda)R(\mu)(x) = R(\mu)(x)$. On the other hand $\overline{R(R(\mu))}=X$, which shows that $\lim_{\lambda\to\infty} \lambda R(\lambda)y = y$ for all $y\in X$. Therefore R is a resolvent of type L_∞ and the proof is complete. □

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R É S U M É S

Théorie de Choquet pour les contractions

Gheorghe Bucur

Sort $(E, \leq)$ un espace de Riesz archimedien. On note pe E_+ l'esemble des éleéments positifs de E.

Definition. Une application linéaire $I : E \to E$ sere nommée *contraction* si pour tout element $x \in E_+$ on a

$$0 \leq Tx \leq x$$

L'ensemble de toutes les contractions sur E sera noté par $\mathcal{K}$. Le cône convexe engendré par $\mathcal{K}$, c.a.d. l'ensemble des applications $A : E \to E$ pair lesquelles il existe des nombres $\alpha \in \mathbb{R}_+$ et des contractions T sur E tels que $A \leq \alpha T$, sera noté par $\mathcal{C}$.

Un element $A \in \mathcal{C}$ est appelé *$\mathcal{C}$-extrêmale* si pour tout couple A', A'' de $\mathcal{C}$ tel que $A = A' + A''$ on a $A' = \lambda A$, $A'' = \mu A$ pour certains nombres λ, μ dans $\mathbb{R}_+$.

On remarque que l'ensemble $\mathcal{K}$ est convexe et pour tout element $A \in \mathcal{C}, A \neq 0$ il existe le plus grand nombre $\alpha(A) \in \mathbb{R}_+$ t.q. $\alpha(A) \cdot A \in \mathcal{K}$.

Proposition 1. *Pour tout élément $\mathcal{C}$-extremale $A \in \mathcal{C}$ l'élément $\alpha(A)A$ est extremale dans l'ensemble convex $\mathcal{K}$. Si A est extremale dans $\mathcal{K}$, A n'est pas toujours $\mathcal{C}$-extremale.*

Proposition 2. *Supposons que l'ensemble E^* des applications linéaireas et positives sur E sépare les points de E. Pour tout couple $(x, \mu) \in E_+ \times E^*_+$ on associe l'application $L_{x,\mu} : \mathcal{C} \to \mathbb{R}_+$ define par*

$$L_{(x,\mu)}(A) = \mu(A_x)$$

L'ensemble $\mathcal{K}$ est un compact convex par rapport à la topologie la moins fine sur $\mathcal{C}$ rendant continues les applications $L_{(x,\mu)}, x \in E_+, \mu \in E^*_+$

Proposition 3. *Un element $T \in \mathcal{C}$ (resp. $T \in \mathcal{K}$) est $\mathcal{C}$-extremale (resp. extremale) si et seulement si pour tout element $x \in E_+$ la restriction de T a l'espace $E_x := \{u \in E \mid |u| \leq \alpha x,\ \alpha \in \mathbb{R}_+\}$ est un element $\mathcal{C}$-extremal (resp. extremale).*

Théoréme 4. *Supposons que $(E, \leq)$ est un espace de Riesz completement reticulé (on σ-réticulé). Alors les éléments exremeaux de K sont exactement les projection de l'espace $(E, \leq)$.*

Théoréme 5. *Dans les conditions du théorème 4 pour toute contraction T il existe une mesure extremale μ sur K telle que*

$$T(x) = \int A(x) d\mu(A).$$

Théorème 6. *Dans les conditions du théorème 4 pour tout element $X \in E_+$ il existe une application décroissante $\alpha \to A_\alpha \mathbb{R}_+$ dans l'ensemble de points extremeaux de K telle que*

$$x = \int_{\mathbb{R}_+} A_\alpha(x) d\alpha.$$

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Opérateurs totalement (o) - bornés

Irina Cătuneanu

Je veux presenter dans cet travail une idée concernant l'adaptation de la notion d'opérateur compact pour la situation des éspaces linéaires ordonnés.

Le travail contient les premières résultats que j'ai obtenu après la constructions des définitions et je n'ai pas la pretention que le sujet soit épuisé.

Définition 1: Soit X un espace linéaire réticulé archimédien. On dit que l'ensemble $S \subset X$ est totalement (o)-borné si et seulement s'il existe $a \in X_+$ ainsi que $(\forall)$ $n \in N$, $(\exists)\mathcal{F}_n \subset X$ fini, ainsi que $(\forall)$ $x \in S$ il existe au moins un $z \in \mathcal{F}_n$ ainsi que $| x - z | \leq \frac{1}{n}a$.

Définition 2: Soient X un espace linéaire ordonné et Y un espace linéaire réticulé archimédien. Un opérateur $U:X \to Y$ est totalement (o)-borné si et seulement si $(\forall)$ $A \subset X$ ensemble (o)-borné $\Rightarrow U(A) \subset Y$ est un ensemble totalement (o)-borné.

Proposition 1: Si X est un treillis normé et Y un espace réticulé de Banach à cône fort, alors si $U:X \to Y$ est un opérateur compact il en résulte que U est un opérateur totalement (o)-borné. Si X est un treillis normé à cône fort l'ensemble des opérateurs compacts et l'ensemble des opérateurs linéaires, totalements (o)-bornés, $U:X \to Y$, coincident.

Proposition 2: Soient X et Y des espaces linéaires ordonnés et Z un espace linéaire réticulé archimédien. Si $U:X \to Y$ est un opérateur régulier et $V:Y \to Z$ un opérateur totalement (o)-borné alors VU est un opérateur totalement (o)-borné. Si Y est un espace linéaire réticulé archimédien et Z est un espace linéaire complètement réticulé, alors, si $U:X \to Y$ est un opérateur totalement (o)-borné et $V:Y \rightarrow Z$ est un opérateur régulier, il en résulte que VU est un opérateur totalement (o)-borné.

Proposition 3: Soient X et Y des espaces linéaires réticulés archimédiens. Un opérateur linéaire $U:X \rightarrow Y$ est totelement (o)-borné si et seulement si $(\forall)\ x_0 \in X_+$, $U([0, x_0])$ est un ensemble totalement (o)-borné.

Proposition 4: Soient X un espace linéaire dirigé qui satisfait à la condition de Riesz et Y un espace linéaire complètement réticulé. L'ensemble des opérateurs aditifs et totalement (o)-bornés, $U:X \rightarrow Y$, est un sous - espace linéaire de $R(X, Y)$.

Proposition 5: Soient X un espace linéaire réticulé et Y un espace linéaire complètement réticulé et σ - régulier. Si $U_n \xrightarrow{\mathcal{S}} U$ dans $\mathcal{R}(X, Y)$ et $U_n:X \rightarrow Y$ est un opérateur totalement (o)-borné, $(\forall)\ n \in N$, il en résulte que $U:X \rightarrow Y$ un opérateur totalement (o)-borné.

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Some C^*-algebras with Fourier-Parseval property

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Consider a locally compact, σ-compact Hausdorff abelian group G and let λ be the Haar measure on G. Let $\mathcal{C}(G)$ denote the C^*-algebra of bounded continuous, complex-valued functions on G. Denote by $PAP(G)$ the set of all pseudo almost periodic functions on G[1].

In this paper we prove necessary and sufficient conditions such that a function in $\mathcal{C}(G)$ belongs to $PAP(G)$.

C. Zhang defines the functions of $\mathcal{C}(\mathbb{R})$ with Fourier-Parseval property (FP, for short) in his doctoral thesis [2]. We generalize this notion from $\mathbb{R}$ to the group G. We define also a C^*-algebra of functions with Fourier- Parseval property.

We demonstrate that $PAP(G)$ is the largest among C^*-subalgebras of $\mathcal{C}(G)$ with the FP property.

Finally we present some other C^*-algebras with Fourier-Parseval property.

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ORDER HOMOMORPHISMS (II)

by Rodica-Mihaela DĂNEŢ

This paper is included in my work devoted for formulating in the directed case some notions and results valid in the lattice case (see [2] and [3]).

Here we will see the corresponding notion concerning the "lattice-homomorphisms".

We use the terminology of [1].

The lattice-homomorphisms were studied in [6].

We recall that *a (vector) lattice-homomorphism* between two vector lattices $\mathcal{X}$ and $\mathcal{Y}$ is a linear map $T:\mathcal{X} \to \mathcal{Y}$ preserving the lattice operations or equivalently satisfying the following condition: (1) $T(|x|) = |T(x)|$, for all $x \in \mathcal{X}$.

The extension of the notion of lattice homomorphisms is due to Z.Lipecki [5] (see the definition below). In the sequel, $\mathcal{X}$ and $\mathcal{Y}$ are two ordered real vector spaces, $\mathcal{X}$ being directed by its ordering and $\mathcal{Y}$ being an order complete lattice.

DEFINITION 1 A positive linear operator $T:\mathcal{X} \to \mathcal{Y}$ is *an order homomorphism* iff:

$$(2)\quad \inf\{T(y) \mid \pm x \le y\} = |T(x)|, \text{ for all } x \in \mathcal{X}.$$

It can be proved that some lattice homomorphisms are extremal points of a set of linear operators.

In [4] it is showed that this is also true for the order homomorphisms, the main result being useful in the extension problem. (see also [2] and [3])

Now we can prove that the lattice homomorphisms and the order homomorphisms have similar properties.

PROPOSITION 2 If $T:\mathcal{X} \to \mathcal{Y}$ is a positive linear operator, then the following statements are equivalent:

(i) T is an order homomorphism and the infimum in (2) is reached for each $x \in \mathcal{X}$;

(ii) a) $KerT = (KerT) \cap \mathcal{X}_+ - (KerT) \cap \mathcal{X}_+$;

b) $T(\mathcal{X}_+) = \{T(x)_+ \mid x \in \mathcal{X}\}$.

COROLLARY 3 If T is as in the proposition above and in addition it is "onto", then the following statements are equivalent:

(i) T is an order homomorphism and the infimum in (2) is reached for each $x \in \mathcal{X}$;

(ii) $T(\mathcal{X}_+) = \mathcal{Y}_+$.

REMARK 4 Note that the condition (ii) b) in PROPOSITION 2 is automatically satisfied in the case $\mathcal{Y} = \mathbf{R}$. But the condition ii) a) is not implied by the single assumption that T is an order homomorphism even for $\mathcal{X}$ Archimedean and $\mathcal{Y} = \mathbf{R}$.

Indeed, let $\mathcal{X}$ be the vector space of all real polynomials regarded as functions on $[-1,1]$, with the pointwise ordering. Define $T \in \mathcal{L}_+(\mathcal{X},R)$ by $T(f) = f(0)$. It follows from the Weierstrass theorem that T is a lattice homomorphism. Nevertheless if $g \in \mathcal{X}$ and $\pm x < g(x)$ for all $x \in [-1,1]$, then $g(0) > 0$.

In the sequel we use the following notion of ideal in a directed vector space $\mathcal{X}$.

DEFINITION 5 A linear subspace $G \subseteq \mathcal{X}$ is *an ideal* (or *a normal subspace*) if it is a solid set, that is:

(S_1) for all $x \in G \cap \mathcal{X}_+ \Rightarrow [-x,x] \subseteq G$;

(S_2) for all $x \in G$, there exists $y \in G \cap \mathcal{X}_+$ so that $x \in [-y,y]$;

or equivalently

(NS) for all $x \in G$ and $y \in \mathcal{X}$ such that $A_y \supseteq A_x \Rightarrow y \in G$,

where A_x denotes the set $\{z \in \mathcal{X}_+ | \pm z \le x\}$.

We also use the following notion of orthogonality.

DEFINITION 6 The elements x_1, x_2 of $\mathcal{X}$ are *orthogonal* if $\inf(A_{x_1} \cup A_{x_2}) = 0$.

THEOREM 7 If T is an order homomorphism onto $\mathcal{Y}$ then:

1) $T(\mathcal{X}_+) = \mathcal{Y}_+$;

2) If T is in addition one to one and x_1, x_2 are two orthogonal elements of $\mathcal{X}$. then $T(x_1) \perp T(x_2)$ in $\mathcal{Y}$;

3) $Ker T$ is an ideal of $\mathcal{X}$.

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CLASSICAL EXTRAPOLATION PROBLEM FOR FUNCTIONS OF SEVERAL VARIABLES

Luminiţa Lemnete Ninulescu

In this note, we give a necessary condition such that a prescribed finite sequence of complex numbers are the coefficients in the Taylor's expansion of an analytic function of several variables with values in the unit disc.

The problem is the following: Let $\{a_\alpha\}_{\alpha=0}^{\alpha_0}$, $a_\alpha \in \mathbf{C}$, $\forall \alpha \in \mathbf{N}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a n-multiindex, $n>1$. Determine a necessary condition such that there exists a function $F: D_n = \{z \in \mathbf{C}^n, |z_i| < 1, \forall\, 1 \le i \le n\} \to S_1 \subset D$, $F \in O(D_n)$ with the coefficients in the Taylor's expansion the given numbers $\{a_\alpha\}_{\alpha=0}^{\alpha_0}$.

The condition is that the Toeplitz operator associated to the solution of the problem $F: D_n \to \mathbf{C}$, $F \in S_1$ has the norm $\|T_F\| \le 1$. ($S_1 = \{f: D_n \to \mathbf{C}, f \in O(D_n)$, $\|f\|_\infty = \sup_{z \in D_n} |f(z)| \le 1\}$).

To solve this problem, let H^2 be the Hardy space of analytic functions f on D^n for which $\|f\|_{H^2} = \sup_{0<r_k<1,\, 1\le k\le n} \frac{1}{(2\pi)^n} \int_0^{2\pi} \ldots \int_0^{2\pi} |f(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})| d\theta_1 \ldots d\theta_n < 0$. H^2 is a Hilbert space included in L^2. For the solution F of the problem, let the Toeplitz operator $T_F G = P_{H^2}^{L^2} F G$. $P_{H^2}^{L^2}$ is the orthogonal projection of L^2 onto H^2. We have proved that if the solution F exists, then $\|T_F\| \le 1$.

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Two Moment Problems in a Space of Radon Measures

Octav Olteanu

We apply theorem 4[5, p.741] (or theorem 2.1. [6, p.513]) to some moment problems in the space $X = (C[a,b])^*$ of all Radon measures on $[a,b]$, where $0 \leq a < b \leq 1$.

X_+ will be the usual cone of all positive Radon measures. If $\mu \in X$ and $h \in L^1([a,b])$, we shall denote by $h(t)\mu$ the measure defined by $(h(t)\mu)(z) := \mu(hz), \quad \forall z \in C([a,b])$. By dt we denote the Lebesque measure on $[a,b]$.

Theorem 1. Let X, X_+ be as above. We denote by ν_j the measure $t^j dt$, $j \in \mathbf{N}$, $t \in [a,b]$. Let $\{\mu_j : j \in \mathbf{N}\}$ be a sequence in X. We consider the following assertions:

(a) there exists $f \in L(X,X)$, such that $f(\nu_j) = \mu_j \; \forall j \in \mathbf{N}$ and $t\mu \leq f(\mu) \leq \mu \quad \forall \mu \in X_+$;

(b) $\forall n \in \mathbf{N}, \quad \forall \{\lambda_0, ..., \lambda_n\} \subset \mathbf{R}, \quad \forall z \in (C[a,b])_+$, we have:

$$\sum_{j=0}^{n} \lambda_j \mu_j(z) \leq \sup_{0 \leq x \leq z} \left[\sum_{j=0}^{n} \lambda_j \int_a^b t^j x(t) dt \right] + \inf_{0 \leq y \leq z} \left[\sum_{j=0}^{n} \lambda_j \int_a^b t^{j+1} y(t) dt \right]$$

(c) $\forall j \in \mathbf{N}$, there exists $h_j \in L^1([a,b])$ such that $\mu_j = h_j dt$ and $t^{j+1} \leq h_j(t) \leq t^j - (dt)$ a.e. in $[a,b]$. Then (b) $\Leftrightarrow$ (a) $\Rightarrow$ (c) hold.

Theorem 2. Let X, X_+ be as above, let $\{r_j : j \in J\} \subset [a,b] \subset [0,1]$ and let us denote by ε_{r_j} the corresponding Dirac measures. Let $\{\mu_j : j \in J\} \subset X$. The following assertions are equivalent:

(a) there exists $f \in L(X,X)$ such that $f(\varepsilon_{r_j}) = \mu_j \quad \forall j \in J$ and $t\mu \leq f(\mu) \leq \mu \quad \forall \mu \in X_+$;

(b) $\forall F \subset J$, F finite subset, $\forall \{\lambda_j : j \in F\} \subset \mathbf{R}$, we have

$$\sum_{j \in F} \lambda_j \mu_j(z) \leq \sup_{0 \leq x \leq z} \left(\sum_{j \in F} \lambda_j x(r_j) \right) + \inf_{0 \leq y \leq z} \left(\sum_{j \in F} \lambda_j r_j y(r_j) \right) \quad \forall z \in (C[a,b])_+;$$

(c) $\exists \{\alpha_j : j \in J\} \subset \mathbf{R}, \quad 0 \leq r_j \leq \alpha_j \leq 1$, such that $\mu_j = \alpha_j \varepsilon_{r_j}, \quad \forall j \in J$.

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REPRESENTATIONS ASSOCIATED WITH POSITIVE DEFINITE MEASURES ON HYPERGROUPS

LILIANA PAVEL

In this paper we construct and study the representation associated with a positive definite measure on a hypergroup.

Hypergroups are locally compact spaces, whose regular complex Borel measures form an algebra which has properties similar to the convolution algebra $(M(G),*)$ of a locally compact group. We shall denote by $C_c(K)$ the set of continuous functions with compact support on K and by $M(K)$, the bounded regular Borel measures on K.

A measure $\mu \in M(K)$ is called positive definite on the hypergroup K if

$$\int_K f * f^* \, d\mu \geq 0, \quad \forall f \in C_c(K).$$

Proposition. Let $\mu \in M(K)$ positive definite measure. Then,

$$\int_K \theta * f * f^* * \theta^* d\mu \leq \| \theta \|^2 \int_K f * f^* \, d\mu, \quad \forall \theta \in M(K), \forall f \in C_c(K).$$

For $\mu \in M(K)$ positive definite measure, we construct the associated representation $T^{(\mu)}$. One regards $C_c(K)$ as a subalgebra of $L^1(K)$. By the previous Proposition, $\mathcal{N}_\mu(K) = \{f \in C_c(K) \mid \int_K f * f^* \, d\mu = 0\}$ is a left ideal in $C_c(K)$, so we obtain an inner product space $(\mathcal{H}_\mu^o(K), <,>_\mu)$ with $\mathcal{H}_\mu^o(K) = C_c(K)/\mathcal{N}_\mu(K)$ and

$$< [f] , [g] >_\mu = \int_K f * g^* \, d\mu$$

We set $\| f \|_\mu =< [f],[f] >_\mu^{\frac{1}{2}}$ for $[f] \in \mathcal{H}_\mu^o(K)$ and denote by $\mathcal{H}_\mu(K)$ the Hilbert space which is obtained from $\mathcal{H}_\mu^o(K)$ by completion with respect to $\| \cdot \|_\mu$.

For each arbitrary $x \in K$ we consider the operator $T_x^{(\mu)} : \mathcal{H}_\mu^o(K) \longrightarrow \mathcal{H}_\mu^o(K)$ by $T_x^{(\mu)}([f]) = [\delta_x * f], \quad \forall [f] \in \mathcal{H}_\mu^o(K)$. It follows that

$$< T_\nu^{(\mu)}[f], [g] >_\mu = \int_K < T_x^{(\mu)}[f], [g] >_\mu \, d\nu(x)$$

is a representation of K, called the representation associated to the positive definite measure μ.

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In [2], F. Greenleaf and M. Moskowitz, using the theory of operator algebras, have proved that for G a second countable locally compact group, $T^{(\mu)}$ is cyclic (where μ is an arbitrary positive definite measure on G). They have obtained the following result in the theory of W^*-algebras : if $\mathcal{A}$ is a W^*-algebra on a separable Hilbert space $\mathcal{H}$ such that there exists a sequence $\{\xi_n\}_n \subset \mathcal{H}$ which is assymptotically cyclic, then there is a cyclic vector in $\mathcal{H}$. By considering $\mathcal{A} = \{T_x^{(\mu)} \mid x \in G\}'' = \{T_f^{(\mu)} \mid f \in C_c(G)\}''$, they have remarked by setting $\xi_n = [e_n]$ (where $\{e_n\}_n$ is the standard approximate identity of $L^1(K)$) that $\mathcal{A}$ has the properties required, so $T^{(\mu)}$ is cyclic.

In order to do the same for the algebra $\mathcal{K} = \{T_x^{(\mu)} \mid x \in K\}'' = \{T_f^{(\mu)} \mid f \in C_c(K)\}''$, where K is a second countable hypergroup, we first have to observe that $\mathcal{K}$ is a W^*-algebra on a separable Hilbert space. Second, by taking into account that $\exists\{U_n\}_n$ is a countable decreasing basis of compact neighbourhoods of e, one can construct a standard type countable approximate identity of $L^1(K)$ ($e_n^* = e_n$, $e_n \in C_c(K), e_n \geq 0$, $\|e_n\|_1 = 1$, supp $e_n \subset U_n$). It is easily seen that $(f * e_n - f) * (f * e_n - f)^*$ converges to 0 in the inductive limit topology. The following lemma states the existence of an assymptotically cyclic sequence.

Lemma. Let $\{\xi_n\}_n \subset \mathcal{H}_\mu(K)$, $\xi_n = [e_n]$. Then, $\forall \xi \in \mathcal{H}_\mu(K)$, there exist the operators $\{A_n\}_n \subset \mathcal{K}$ such that

$$\|A_n\xi_n - \xi\|_\mu \to 0.$$

With the above, we have justified that, like in the group case, the next result holds:

Theorem. Let μ be any positive definite measure on a second countable hypergroup. Then, the associated representation $T^{(\mu)}$ has a cyclic vector.

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NEW RESULTS ON PARETO EFFICIENCY IN SEPARATED LOCALLY CONVEX SPACES

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This research paper is devoted to recent results on Pareto efficiency in Hausdorff locally convex spaces,that is,in the framework vector spaces with unknown geometries.Firstly,we present general new properties which illustrate immediate links between Strong Optimization and Vector Optimization and a dual characterization.Afterwards we offer pertinent examples and significant remarks on supernormal cones and we present our main results concerning recent properties of efficient points using (weak) supernormal cones and (weak) completeness instead of compactness.Finally,we give some considerations concerning the best approximation simultaneous and vectorial in H-locally convex spaces as particular cases ofgeneral vectorial optimization problems.

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GENERALIZED BERNSTEIN OPERATORS AND LIPSCHITZ CLASSES

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For some generalized Bernstein operators B_n there exists $c = c(\alpha) \geq 1$ such that

$$B_n(\mathrm{Lip}_1\alpha) \subset \mathrm{Lip}_c\alpha, \qquad n \geq 1.$$

Recent results of this type have been obtained by Y.Y. Feng [1] and F. Chen [2], [3].

Particularly important for applications is the case when the operators B_n are associated to an Altomare projection and when $c = 1$; this case has been investigated by the authors [4].

We shall present some results in this context, related to those contained in [1]–[4].

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Espaces vectoriels ordonnés satisfaisant la condition (C) (axiome de Cantor)

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Dans cet exposé, on présente des conditions en présence de lesquelles, un espace vectoriel ordonné, muni d'une topologie linéaire et vérifiant la condition de Riesz, satisfait aussi l'axiome de Cantor.

Définitions

Soit X un espace vectoriel ordonné et Y un sous - ensemble de X. Alors:
I. Pour une topologie linéaire τ sur X, Y satisfait la condition:

1.$C(\tau) \Leftrightarrow$ Chaque suite généralisée monotone, (o)-bornée de Y, admet une sous-suite généralisée τ-convergente en Y.

2.$C_S(\tau) \Leftrightarrow$ Chaque suite monotone, (o)-bornée de Y, admet une sous-suite τ-convergente en Y.

II. Pour τ et χ topologies linéaires sur X, Y satisfait la condition:

1. $C(\tau,\chi) \Leftrightarrow$ Chaque suite généralisée monotone, (o)- bornée et τ-Cauchy de Y, admet une sous-suite généralisée χ - convergente en Y.

2. $C_S(\tau,\chi) \Leftrightarrow$ Chaque suite monotone, (o)- bornée et τ-Cauchy de Y, admet une sous-suite χ - convergente en Y.

Théorème 1.

Soit X un espace vectoriel ordonné, muni d'une topologie linéaire Hausdorff τ et vérifiant une des hypothèses suivantes :

1. X_+ satisfait la condition $C(\tau)$.
2. X_r^* est suffisante, $X_r^* \subset X'$ et X_+ satisfait la condition $C(\sigma,\sigma)$ (σ étant la topologie faible sur X).
3. X_r^* est suffisante et chaque (o) - segment de X est faiblement compact.

Alors: X satisfait la condition de Riesz $\Leftrightarrow$ X satisfait la condition (C)

Théorème 2.

Soit X un espace vectoriel ordonné, muni d'une topologie linéaire Hausdorff τ, d'une fonctionelle réele strictement croissante sur X_+ et vérifiant une des hypothèses suivantes:

1. X_+ satisfait la condition $C_S(\tau)$.
2. X_r^* est suffisante, $X_r^* \subset X'$ et X_+ satisfait la condition $C_S(\sigma,\sigma)$.
3. X_r^* est suffisante et chaque (o) -segment de X est sequentiel faiblement compact.

Alors: X satisfait la condition de Riesz $\Leftrightarrow$ X satisfait la condition (C).

Théorème 3.

Soit X un espace vectoriel ordonné, muni d'une topologie linéaire Hausdorff τ, admettant une base τ-bornée d'éléments positifs et vérifiant une des hypothèses suivantes:

1. X^{*}_{τ} est suffisante et X_{+} satisfait la condition $C_8(\tau,\sigma)$.
2. X_{+} satisfait la condition $C_8(\tau,\tau)$.

Alors: X satisfait la condition de Riesz $\Leftrightarrow$ X satisfait la condition (C).

Théorème 4.

Soit X un espace vectoriel dirigé, muni d'une topologie localement convexe et Y_{τ} un espace localement convexe, Y_{+} satisfaisant la condition $C(\tau)$ et X vérifiant une des hypothèses suivantes :

1. X est tonnelé.
2. X est infratonnelé et satisfait la condition (S).
3. X est (o) -infratonnelé.
4. X est barillé (voir la note).

Alors: $\mathcal{L}(X,Y)$ satisfait la condition de Riesz $\Leftrightarrow$ $\mathcal{L}(X,Y)$ satisfait la condition (C).

Théorème 5.

Soit X un espace vectoriel dirigé, muni d'une topologie localement convexe τ et vérifiant une des hypothèses suivantes:

1. X est tonnelé.
2. X est infratonnelé et satisfait la condition (S).
3. X est (o) -infratonnelé.
4. X est barillé.
5. X^{*}_{τ} satisfait la condition $C(\sigma^{*},\sigma^{*})$ ($\sigma^{*}=\sigma(X^{*}_{\tau},X)$).
6. X^{*}_{τ} est muni d'une fonctionelle réele strictement croissante sur (X^{*}_{τ}), et satisfait la condition $C_5(\sigma^{*},\sigma^{*})$.
7. X est faiblement sébarable et X^{*}_{τ} satisfait la condition $C_8(\sigma^{*},\sigma^{*})$.
8. X admet un élément axial et X satisfait la condition $C_8(\sigma^{*},\sigma^{*})$.
9. X admet un élément axial, X^{*}_{τ} est suffisante et satisfait la condition $C_8(\mu^{*},\sigma^{*})$ ($\mu^{*}=\mu(X^{*}_{\tau},X)$ est la topologie de Mackey sur X).
10. $\overset{o}{X}_{+}\neq\Phi$, X^{*}_{τ} est suffisante et satisfait la condition $C_8(\beta^{*},\sigma^{*})$ ($\beta^{*}=\beta(X^{*}_{\tau},X)$ est la topologie forte sur X).

Alors: X^{*}_{τ} satisfait la condition de Riesz $\Leftrightarrow$ X^{*}_{τ} satisfait la condition (C).

Note. X est barillé $\Leftrightarrow$ Chaque sous-ensemble barillé W(i.e. W est absorbant, balancé, convexe et contient le (o)-segment [o,x] pour chaque élément x de W) de X est un voisinage de l'origine en X.

INEQUALITIES OF HORN AND KY FAN TYPE

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If $T \in L(H)$ is compact, where H is a separable Hilbert space, are well known the inequalities:

(a) $\sum_1^k s_n(S+T) \leq \sum_1^k (s_n(S) + s_n(T))$, k=1,2,3,...

(b) $\sum_1^k s_n(ST) \leq \sum_1^k s_n(S) \cdot s_n(T)$

($s_n(T) = \lambda_n (TT^*)^{\frac{1}{2}}$, (λ_n) is the sequence of the eigenvalues of $(TT^*)^{\frac{1}{2}}$ convenient ordered).

Since $s_n(T)$ coincides with $a_n(T)=\inf \{ \|T- K\| : K \in L(H), \dim K< n \}$, we remark that the above results can be extended to an operator $T \in L(X)$ where X is a Banach space.

Now are well known the inequalities :

(a') $\sum_1^k a_n (S+T) \leq 2\sum_1^k (a_n (S)+ a_n (T))$, k=1,2,3,...

(b') $\sum_1^k a_n(ST) \leq 2 \sum_1^k a_n (S) a_n (T)$

If the following we consider $(X, \|\cdot\|)$ a normed Abelian group and let S be a subgroup of X. On S we consider an other norm $\|\cdot\|^*$.

For all $f \in X$, the functions E (t,f) are defined as follows:

$E(t,f) := \inf\{\|f-g\| : g \in S, \|g\|^* < t\}, t \in R_+$.

Remark. If X=L (H) and S=FL (H) (the set of all finite rank operators, $\|T\|^* = \dim T$),

$E(n,T) = a_n(T)$, where $T \in L(H)$.

Proposition 1. The functions E (t,f) verify the inequalities:

$$(a^-) \quad \int_0^\alpha E(t, f_1 + f_2)dt \leq 2 \int_0^\alpha (E(t,f_1) + E(t,f_2))\, dt, \quad \alpha \in R_+.$$

If $(X, \|\cdot\|)$ is a normed ring; S is a subring and $\|f_1 f_2\| < \|f_1\| \cdot \|f_2\|$; $\|g_1 g_2\|^* \leq \max(\|g_1\|^*, \|g_2\|^*)$, it results:

Proposition 2. The functions E(t,f) verify the inequalities :

$$(b^*) \quad \int_0^\alpha E(t, f_1 \cdot f_2)\, dt \leq 2 \int_0^\alpha E(t, f_1) \cdot E(t, f_2)\, dt, \quad \alpha \in R_+.$$

Let be $A_{p,q}(X) = \{f \in X : (\int_0^\infty (t^{\frac{1}{p} - \frac{1}{q}} E(t, f))^q dt)^{\frac{1}{q}} < \infty\}$. This is an approximation space, quasi-normed by $\|f\|_{p,q} = \left(\int_0^\infty \left(t^{\frac{1}{p} - \frac{1}{q}} E(t,f)\right)^q dt\right)^{\frac{1}{q}}$, $0 < p \leq \infty$, $0 < q < \infty$.

Let B: X x X $\to$ X be a biliniar and bounded operator.

It is of interest to know if the induced operator B: $A_{p,q}(X) \times A_{p,q}(X) \to A_{p,q}(X)$ is also bounded. The reponses is not (for all $p,q \in (0,\infty)$). But is true that B: $A_{\infty,q}(X) \times A_{\infty,q}(X) \to A_{\infty,q}(X)$ is bounded ($0 < q < \infty$).

FIXED POINTS FOR MULTIVALUED MAPPINGS

Florica Voicu

Let X and Y two set. A multivalued mapping (briefly, m- mapping) defined an X with values in Y is a mapping $T:X\to\mathcal{P}(Y)$. We denote this mapping by T: X—o Y.

Definition 1. Let T: X—oX be a m - mapping.
Then by definition an element $x\in X$ is a fixed point of T if $x\in T(x)$.
We denote by Fix(T) the fixed points set.

Definition 2. Let T:X—oY be a m- mapping.
A single valued mapping $s:X\to Y$ is called a selection of T if $s(x)\in T(x)$ for all $x\in X$.

We suppose that X is a complete vector lattice and Z a non-empty set. Considering that $d:Z\times Z\to X_+$ is a vector metric we denote by

$\delta(A):=\sup\{ d(a,b)/ a,b\in A\}$, $A\in\mathcal{P}_b(Z)$

$\delta(A,B):=\sup\{ d(a,b) / a\in A, b\in B\}$, $A,B\in\mathcal{P}_b(Z)$

$D(A,B):=\inf\{d(a,b)/ a\in A, b\in B\}$, $A,B\in\mathcal{P}(Z)$

$H(A,B):=\sup\{D(a,B)/ a\in A\}\vee\sup\{D(b,A)/b\in B\}$, $A,B\in\mathcal{P}_{b,cl}(Z)$

LEMMA1 Let $A,B\in\mathcal{P}_{b,cl}(Z)$. Then if $q>1$, then for every $a\in A$ there exist $b\in B$ such that $d(a,b)\leq qH(A,B)$.

THEOREM1 Let X a complete vector lattice, $Z\neq\Phi$, sequential d- complete, $T:Z\to\mathcal{P}_{b,cl}(Z)$ a m -mapping, for which there exist $\alpha\in[0,1]$ such that:
$H(T(x),T(y))\leq \alpha\, d(x,y)$ for all $x,y\in Z$.
Then T has a fixed point.

LEMMA 2 Let $Z\neq\Phi$ sequential d - complete, $A\in\mathcal{P}_b(Z)$. Then for every $x\in Z$ and $0<q<1$, there exist $a\in A$ such that: $q\delta(x,A)\leq d(x,A)$.

THEOREM 2 Let X a complete vector lattice, $Z\neq\Phi$ sequential d- complete and $T:Z\to\mathcal{P}_b(Z)$ a m- mapping for which there exist $\alpha,\beta,\gamma\in R_+$, $\alpha+\beta+\gamma<1$ such that
$\delta(T(x),T(y))\leq\alpha\, d(x,y)+\beta\delta(x,T(x))+\gamma\delta(y,T(y))$ for all $x,y\in Z$.
Then T has a unique fixed point x^* and $f(x^*)=\{x^*\}$.

THEOREM 3 Let X a complete vector lattice, $Z \neq \Phi$ sequential d- complete and $T: Z \rightarrow \mathcal{P}_{b,cl}(Z)$ be m- mapping for which there exist $\alpha, \beta, \gamma \in R_+$, $\alpha+\beta+\gamma < 1$ such that:

$H(T(x),T(y)) \leq \alpha\, d(x,y) + \beta D(x,T(x)) + \gamma D(y,T(y))$ for all $x,y \in Z$.

Then T has at least a fixed point.

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