MARIAN IVAN

THEORETICAL MECHANICS

- Applications in Geosciences -

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In memoriam

***Professor* Marin DOROBANŢU**

INTRODUCTION

That book presents some theoretical problems related to the Mechanics of a Continuum Solid Body, of particular importance to Applied Geomechanics, Geological Engineering and Structural Geology. In most cases, only static aspects are discussed, but some dynamic cases are also presented.

As a rule, the modern tensorial approach is used. The linear elasticity and the homogeneity of the continuum solid body are almost thoroughly assumed to be valid, but some elements of Rheology are also presented.

In most cases, the semi-inverse method is used to solve the problems. According to it, the solution is supposed to be of a particular form, as a consequence of the simplified hypothesis previously assumed. It is verified that solution checks both the corresponding equations and the boundary conditions. Based on the Uniqueness Theorem of the Linear Elasticity, it follows the assumed particular solution is just the general solution of the problem. In all the cases discussed here, the assumed simplified hypotheses allow one to obtain simple, analytical solutions. At a first glance, the importance of such solutions is minor with respect to the real cases, where mainly the non-homogeneity of the medium plays a great role. However, the analytical solutions are the basis for deriving finite element algorithms, allowing one to model satisfactory the complex real cases. Such examples are also presented.

The lessons are mainly designed to be used as a part of the course of Mechanics followed by the students in Geophysics at the Geology and Geophysics Faculty, University of Bucharest.

The author is gratefully to his colleagues and to the referees.

CONTENTS

INTRODUCTION

A) BASIC ELEMENTS.

A.1) The displacement vector. Lagrangean (material) and Eulerian (spatial) co-ordinates.

Consider an arbitrary material point inside a continuum body, subject to a deformation process. At the initial time t_0, that point has the position vector denoted by $\vec{X}$, with respect to the origin of a co-ordinate system (Fig.A1). At a time $t \geq t_0$, the new position vector is to be $\vec{x}$. The difference $\vec{x} - \vec{X}$ represents **the displacement vector**. Taking into account that the components (x_1, x_2, x_3) of the vector $\vec{x}$ are all functions of the components (X_1, X_2, X_3) of $\vec{X}$, the displacement vector can be written as

$$\vec{U} = \vec{U}(X_1, X_2, X_3, t) \quad . \tag{a1}$$

This represents a **Lagrangean (material)** description of the deformation process. Here, (X_1, X_2, X_3) are representing the **Lagrangean (material)** co-ordinates. Alternately, the components (X_1, X_2, X_3) of $\vec{X}$ can be seen as functions of the components (x_1, x_2, x_3) of the vector $\vec{x}$. Consequently, the displacement vector can be written as

$$\vec{u} = \vec{u}(x_1, x_2, x_3, t) \quad . \tag{a2}$$

This represents a **Eulerian (spatial)** description of the deformation process, where (x_1, x_2, x_3) are the **Eulerian (spatial)** co-ordinates. A basic supposition assumed thoroughly in that notes is that the deformation process is a continuous one, i.e. all the components of $\vec{u}$ or $\vec{U}$ are continuous functions together their derivatives with respect to both their spatial co-ordinates or to time. Further conditions are discussed, for example, in (Ivan 1996).

The Lagrangean co-ordinates are usual in the Solid Mechanics, while the Eulerian co-ordinates are commonly used in Fluid Mechanics. However, in the Linear Elasticity, the distinction between these two kinds of co-ordinates is not important, as it will be seen in the next chapters. More details on such aspects can be found in (Aki and Richards 1980; Ranalli 1987).

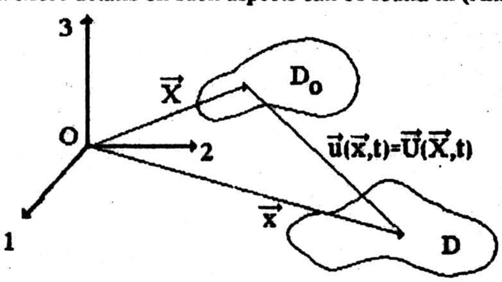

Fig.A.1. The continuum deformed body and the displacement vector.

A.2) Invariants of a tensor. Tensor deviator.

A second order tensor represents mainly a 3x3 matrix. The elements of the tensor are changing according to a certain rule with respect to a change of the co-ordinate system. Such a change with respect to a rotation will be discussed later. For simplicity, only symmetric tensors will be considered. A symmetric tensor is equal to its transpose

$$T = T^t \tag{a3}$$

(or $T_{ij} = T_{ji}$). The superscript "t" shows the transposed tensor (matrix).

Let the components of the tensor be real numbers

$$T=\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{pmatrix} \quad , \tag{a4}$$

The scalar λ and the vector $\vec{u}$ are representing the eigen-value and the eigen-vector respectively of that tensor if

$$T\vec{u}=\lambda\vec{u} \quad , \quad \vec{u}\neq\vec{0} \quad . \tag{a5}$$

It's easy to see that the eigen-values are not changing with respect to a rotation of the co-ordinates system.

Suppose now that the eigen-value λ and the components of the eigen-vector $\vec{u}$ are complex numbers. By taking the complex conjugate (denoted by an asterisk) into (a5), it follows

$$T\vec{u}^{*}=\lambda^{*}\vec{u}^{*} \tag{a6}$$

Taking into account the symmetry of the tensor, the next inner product is evaluated into two different ways

$$<T\vec{u},\vec{u}^{*}>=<\lambda\vec{u},\vec{u}^{*}>=\lambda\left|\vec{u}\right|^{2} \quad , \tag{a7}$$

and

$$<T\vec{u},\vec{u}^{*}>=<\vec{u},T^{t}\vec{u}^{*}>=<\vec{u},T\vec{u}^{*}>=<\vec{u},\lambda^{*}\vec{u}^{*}>=\lambda^{*}\left|\vec{u}\right|^{2} \tag{a8}$$

From (a7) and (a8) it follows that the eigen-values (and the components of the eigen-vectors) of a symmetric tensor are real numbers.

Eq.(a5) can be written as

$$\left(T-\lambda 1\right)\vec{u}=\vec{0} \quad , \quad \vec{u}\neq\vec{0} \quad , \tag{a9}$$

where 1 denotes the unit tensor. From (a9) it follows that the next determinant vanishes

$$\begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{12} & T_{22}-\lambda & T_{23} \\ T_{13} & T_{23} & T_{33}-\lambda \end{vmatrix}=0 \tag{a10}$$

Hence the eigen-values are the roots of the third degree equation

$$-\lambda^{3}+I_{1}\lambda^{2}-I_{2}\lambda+I_{3}=0 \quad , \tag{a11}$$

where

$$\begin{aligned} I_{1} &= T_{11}+T_{22}+T_{33}=tr\left(T\right) \quad , \\ I_{2} &= T_{11}T_{22}+T_{22}T_{33}+T_{33}T_{11}-T_{12}^{2}-T_{23}^{2}-T_{13}^{2} \quad , \\ I_{3} &= T_{11}T_{22}T_{33}+T_{12}T_{23}T_{13}+\ldots=\det\left(T\right) \end{aligned} \tag{a12}$$

With respect to a rotation of the co-ordinates system, the elements of the tensor are generally changing. Because the quantities defined by (a12) can also be expressed as functions of the roots of (a11), it follows their values are not changing with respect to a rotation. They represent the main invariants of the tensor. The first invariant is **the trace of the tensor**, while the third one represent just its determinant.

The tensor defined by

$$T^{*}=T-\frac{1}{3}tr\left(T\right)1 \quad . \tag{a13}$$

represents the tensor deviator, having its trace equal to zero. Elementary computations show its second invariant is

$$I_2^* = -\frac{1}{6}\left[\left(T_{11}-T_{22}\right)^2+\left(T_{22}-T_{33}\right)^2+\left(T_{33}-T_{11}\right)^2+6\left(T_{12}^2+T_{23}^2+T_{13}^2\right)\right] \qquad \text{(a14)}$$

That invariant is especially important to define constitutive equation for plasticity.

A.3) Strain tensor. Stress tensor. Equation of motion / equilibrium.

By using the spatial co-ordinates, the **strain tensor** is defined as (e.g. Beju, Soos and Teodorescu 1977)

$$\varepsilon = \frac{1}{2}\left(\operatorname{grad}\vec{u}+\operatorname{grad}^t\vec{u}\right) \qquad \text{(a15)}$$

where "grad" denotes the gradient. In Cartesian co-ordinates, that symmetric tensor has the elements

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right) \qquad \text{(a16)}$$

According to the CAUCHY's hypotheses there are two kinds of forces acting at an arbitrary point placed inside a body or on its boundary. The first ones are represented by **the mass forces**, characterised by a mass density $\vec{b}$. For the problems discussed in that book, such mass forces are ignored. Or, they are represented by the gravity, when $\vec{b}$ is just the gravitational acceleration $\vec{g}$. Suppose now a mechanical state of tension (stress) is present inside the deformed body, e.g. as a result of the action of a pair forces $\pm\vec{T}$. An arbitrary cross section is considered through a certain point of the body, dividing it into a part denoted by D_l at left and a pert D_r at the right respectively (Fig.A.2). A surface element dS is considered on the boundary of D_l, having the outer pointing normal vector denoted by $\vec{n}$. The material points of the boundary of D_r are acting on dS by an elementary force $\vec{df}$. It follows (e.g. Beju, Soos and Teodorescu 1977; Aki and Richards 1980; Ranalli 1987; Ivan 1996) that the next relation is valid

$$\frac{\vec{df}}{dS}=\sigma\vec{n} , \qquad \text{(a.17)}$$

where the tensor σ represents **the CAUCHY stress tensor**, spatial co-ordinates being used. According to the Principle of the Kinetic Momentum Balance, stress is a symmetric tensor. It can be shown too that the Principle of Impulse Balance leads to the next vectorial equation of motion /equilibrium

$$\operatorname{div}\sigma+\rho\vec{b}=\rho\frac{d^2\vec{u}}{dt^2} , \qquad \text{(a18)}$$

That equation is valid at an arbitrary point inside the body , where ρ is the density and $\frac{d^2\vec{u}}{dt^2}$ represents the acceleration.

By projecting eq.(a18) on the co-ordinates system axes, three scalar equations are obtained.

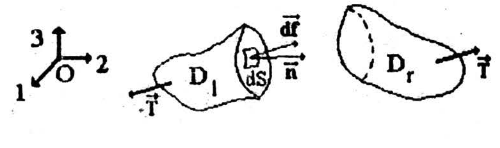

Fig.A.2. An imaginary cross section through the deformed body.

A.4) HOOKE's law.

Neglecting the initial stress (in most cases), it is further assumed a linear relation between the stress and strain tensors, i.e.

$$\sigma = H\ \varepsilon \ , \tag{a19}$$

or

$$\sigma_{ij} = H_{ijkl}\varepsilon_{kl} \ , \tag{a20}$$

where H is a fourth-order tensor. Eq.(a19) represents HOOKE's law. In the usual cases discussed here, an elastic, homogeneous, isotropic medium is considered. Then eq.(a19) takes the particular form

$$\sigma = \lambda \mathrm{tr}\,\varepsilon\ 1 + 2\mu\varepsilon \ . \tag{a21}$$

Here, tr denotes the trace of the tensor, 1 is the unit tensor(matrix) and λ, μ are the elastic coefficients of LAMÉ. Hence

$$\begin{aligned}
\sigma_{11} &= \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{11}, \\
\sigma_{22} &= \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{22}, \\
\sigma_{33} &= \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{33} \\
\sigma_{12} &= \sigma_{21} = 2\mu\varepsilon_{12}, \quad \sigma_{13} = \sigma_{31} = 2\mu\varepsilon_{13}, \quad \sigma_{23} = \sigma_{32} = 2\mu\varepsilon_{23}
\end{aligned} \quad , \tag{a.22}$$

Alternately, HOOKE's law (a21) can be reversed to give

$$\varepsilon = \frac{1+\nu}{E}\sigma - \frac{\nu}{E}\mathrm{tr}\,\sigma\ 1 \quad , \tag{a23}$$

where the modulus of YOUNG is

$$E = \mu\frac{3\lambda + 2\mu}{\lambda + \mu} \tag{a24}$$

and the transverse contraction coefficient of POISSON is

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \ . \tag{a25}$$

By reversing (a24) and (a25), it follows

$$\lambda = \frac{\nu}{(1+\nu)(1-2\nu)}E \quad , \quad \mu = \frac{1}{2(1+\nu)}E \ . \tag{a26}$$

The parameter defined by

$$\chi = \frac{3\lambda + 2\mu}{3} = \frac{E}{3(1-2\nu)} \tag{a27}$$

represents the **incompressibility** or **bulk modulus**. For (theoretical) incompressible rocks, that modulus approaches infinity. Other constitutive equations will be discussed in relation to the rheological bodies.

B) DEFORMATION OF A CYLINDRICAL BODY IN THE PRESENCE OF GRAVITY

B.1) The model.

An elastic homogeneous isotropic body is considered (Fig.B1). Its initial shape is a right, vertical, very thin cylinder of radius equal to r and height equal to H. The base of the body is placed on the horizontal, absolutely rigid, plane x_1Ox_2. The deformation of the body due to its own weight follows to be studied and the final shape of the body into the final equilibrium state will be found. The approximations of the linear theory are assumed and the variation of the density is ignored. The problem is solved by following the next steps:

i) - the equations of equilibrium are used, the unknowns here being the components of the stress tensor σ ; these equations are processed according to the simplifying hypothesis of the problem;

ii) - by using the reversed HOOKE's law, the equations of equilibrium are processed in order to have only the components of the strain tensor ε as unknowns;

iii) - by using the definition of the strain tensor, the components of the displacement vector $\vec{u}$ are obtained and the final shape of the body is found.

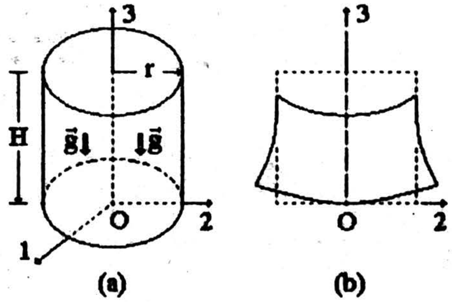

Fig.B.1. (a) A vertical cylinder lying on a rigid plane; (b) The final shape of a vertical cross section (solid line) with respect to the initial shape (dashed line). [NO SCALE]

B.2) The equations of equilibrium. Boundary conditions. Simplifying hypothesis.

A simplified approach can be derived by using cylindrical co-ordinates. However, the problem here is an introductory one. So these co-ordinates will be used later, in relation to other problems. The equations of equilibrium in Cartesian co-ordinates are

$$\begin{aligned}
&\sigma_{11,1}+\sigma_{12,2}+\sigma_{13,3}=0\\
&\sigma_{12,1}+\sigma_{22,2}+\sigma_{23,3}=0\\
&\sigma_{13,1}+\sigma_{23,2}+\sigma_{33,3}-\rho g=0
\end{aligned} \tag{b1}$$

Here, δ is the density and g is the gravitational acceleration. The forces acting upon the body are the reaction force of the horizontal plane and the gravity of the cylinder. The boundary conditions are:

-on the lateral surface of the cylinder:

$$\sigma\,\vec{n}=\vec{0}\quad ,\text{for}\quad x_3\in[0,H],\quad x_1,x_2\in\Gamma \tag{b2}$$

-on the upper base of the cylinder

$$\sigma\,\vec{n}=\vec{0}\quad ,\text{for}\quad x_3=H,\quad x_1,x_2\in\Delta \tag{b3}$$

Here, Δ is the disc of radius equal to r, having the centre at the origin of the co-ordinate system and the boundary denoted by Γ. The outer pointing normal at the lateral surface of the body is a linear combination with variable coefficients of the horizontal unit vectors, i.e.

$$\vec{n} = C_1(x_1,x_2)\vec{e}_1 + C_2(x_1,x_2)\vec{e}_2 \tag{b4}$$

For $x_3 \in [0,H]$, $x_1, x_2 \in \Gamma$ eq.(b2) becomes

$$C_1(x_1,x_2)[\sigma_{11}\vec{e}_1 + \sigma_{12}\vec{e}_2 + \sigma_{13}\vec{e}_3] + C_2(x_1,x_2)[\sigma_{12}\vec{e}_1 + \sigma_{22}\vec{e}_2 + \sigma_{23}\vec{e}_3] = \vec{0} \tag{b5}$$

The outer pointing normal at the upper base of the body is the unit vector $\vec{e}_3$. For $x_3 = H$, $x_1, x_2 \in \Delta$, eq. (b3) gives

$$\sigma_{13}\vec{e}_1 + \sigma_{23}\vec{e}_2 + \sigma_{33}\vec{e}_3 = \vec{0} \tag{b6}$$

Eq. (b5) is satisfied if the stress tensor has the form

$$[\sigma] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \tag{b7}$$

on the lateral surface of the body.

Because the cylinder is a very thin one, the stress at its inner points is approximately the same one to the stress on the lateral surface. So, it is assumed that eq.(b7) holds inside the whole volume of the body. It follows eqs.(b1a)-(b1b) are identical verified. From eq. (b1c) it follows that

$$\frac{\partial \sigma_{33}}{\partial x_3} = \rho g, \quad \sigma_{33}(x_1, x_2, x_3 = H) = 0 \tag{b8}$$

The problem represented by eq.(b8) has the next immediate solution

$$\sigma_{33}(x_1, x_2, x_3) = \rho g(x_3 - H) \tag{b9}$$

The reversed HOOKE's law is

$$\begin{aligned} \varepsilon_{11} &= \frac{1}{E}\left[(1+\nu)\sigma_{11} - \nu(\sigma_{11}+\sigma_{22}+\sigma_{33})\right] \\ \varepsilon_{22} &= \frac{1}{E}\left[(1+\nu)\sigma_{22} - \nu(\sigma_{11}+\sigma_{22}+\sigma_{33})\right] \\ \varepsilon_{33} &= \frac{1}{E}\left[(1+\nu)\sigma_{33} - \nu(\sigma_{11}+\sigma_{22}+\sigma_{33})\right] \\ \varepsilon_{12} &= \frac{1+\nu}{E}\sigma_{12}, \ \varepsilon_{23} = \frac{1+\nu}{E}\sigma_{23}, \ \varepsilon_{13} = \frac{1+\nu}{E}\sigma_{13} \end{aligned} \tag{b10}$$

Because

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{b11}$$

eqs. (b10) lead to

$$\begin{aligned} &\frac{\partial u_1}{\partial x_1} = \frac{\nu\rho g}{E}(H - x_3), \quad \frac{\partial u_2}{\partial x_2} = \frac{\nu\rho g}{E}(H - x_3), \quad \frac{\partial u_3}{\partial x_3} = \frac{\rho g}{E}(x_3 - H) \\ &\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0, \quad \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0, \quad \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = 0 \end{aligned} \tag{b12}$$

By integrating eq.(b12) it follows that

$$
\begin{aligned}
u_1 &= \frac{\nu\rho g}{E}(H - x_3)x_1 + f_1(x_2, x_3) \quad , \\
u_2 &= \frac{\nu\rho g}{E}(H - x_3)x_2 + f_2(x_1, x_3) \quad , \\
u_3 &= \frac{\rho g}{E}\left(\frac{x_3^2}{2} - Hx_3\right) + f_3(x_1, x_2) \quad , \\
&\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} = 0 \quad , \quad \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} = \frac{\nu\rho g}{E}x_2 \quad , \quad \frac{\partial f_3}{\partial x_1} + \frac{\partial f_1}{\partial x_3} = \frac{\nu\rho g}{E}x_1
\end{aligned}
\tag{b13}
$$

Hence the displacement field is found if the unknown functions f_1, f_2, f_3 are finally obtained. Differentiating (b13e) with respect to x_1 and (b13f) with respect to x_2 and adding the results, it follows

$$
2\frac{\partial^2 f_3}{\partial x_1 \partial x_2} + \frac{\partial}{\partial x_3}\left(\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x1}\right) = 0 \ . \tag{b14}
$$

So, using eq.(b13d) it follows

$$
\frac{\partial^2 f_3}{\partial x_1 \partial x_2} = 0 \tag{b15}
$$

From eq.(b15) it follows that

$$
f_3(x_1, x_2) = h_1(x_1) + h_2(x_2) \quad , \tag{b16}
$$

where h_1, h_2 are two unknown functions, following to be found. Eqs.(b13e) and (b13f) give

$$
\frac{\partial f_2(x_1, x_3)}{\partial x_3} = \frac{\nu\rho g}{E}x_2 - \frac{dh_2(x_2)}{dx_2} \quad , \quad \frac{\partial f_1(x_2, x_3)}{\partial x_3} = \frac{\nu\rho g}{E}x_1 - \frac{dh_1(x_1)}{dx_1} \tag{b17}
$$

The left side of eq.(b17a) is represented by a function depending on x_1, x_3 only, while the right side is a function of x_2. Hence both sides are equal to a constant, i.e.

$$
\frac{\partial f_2(x_1, x_3)}{\partial x_3} = -a_2 \quad , \quad \frac{dh_2}{dx_2} = \frac{\nu\rho g}{E}x_2 + a_2 \tag{b18}
$$

It follows that

$$
f_2(x_1, x_3) = -a_2 x_3 + g_2(x_1) \quad , h_2(x_2) = \frac{\nu\rho g}{2E}x_2^2 + a_2 x_2 + b_2 \tag{b19}
$$

In a similar manner, eq.(b17b) gives

$$
f_1(x_2, x_3) = -a_1 x_3 + g_1(x_2) \quad , h_1(x_1) = \frac{\nu\rho g}{2E}x_1^2 + a_1 x_1 + b_1 \tag{b20}
$$

From eq. (b13d) it follows that

$$
\frac{dg_1(x_2)}{dx_2} = -\frac{dg_2(x_1)}{dx_1} = K \tag{b21}
$$

where K is a constant. Then

$$
g_1(x_2) = Kx_2 + C_1 \quad , \quad g_2(x_1) = -Kx_1 + C_2 \tag{b22}
$$

For simplicity, material co-ordinates are used to obtain the final expression of the displacement field

$$\vec{u}(\vec{X})=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}=\frac{\rho g}{E}\begin{pmatrix}\nu(H-X_3)X_1\\ \nu(H-X_3)X_2\\ \frac{X_3^2}{2}-HX_3+\frac{\nu}{2}(X_1^2+X_2^2)\end{pmatrix}+\begin{pmatrix}-a_1X_3-KX_2\\-a_2X_3+KX_1\\a_1X_1+a_2X_2\end{pmatrix}+\begin{pmatrix}C_2\\C_1\\b_1+b_2\end{pmatrix} \tag{b23}$$

The first term in eq.(b23) is the true displacement, the last one is a translation while the second term is the rigid rotation

$$\begin{pmatrix}a_2\\-a_1\\K\end{pmatrix}\times\vec{X} \tag{b24}$$

B.3) The final shape of the body.

a) The final shape of the upper base

Consider an arbitrary point of co-ordinates equal to $(X_1, X_2, X_3 = H)$. In the initial stage, it is placed on the upper base of the cylinder. Finally, the co-ordinates of the point are

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}X_1\\X_2\\H\end{pmatrix}+\frac{\rho g}{E}\begin{pmatrix}0\\0\\ \frac{\nu}{2}(X_1^2+X_2^2)-\frac{H^2}{2}\end{pmatrix} \tag{b25}$$

Hence

$$\begin{cases}x_1=X_1\\x_2=X_2\\x_3=\frac{\nu\rho g}{2E}(X_1^2+X_2^2)-\frac{\rho g}{2E}H^2+H\end{cases} \tag{b26}$$

From eq.(b26) it follows that

$$x_1^2+x_2^2=X_1^2+X_2^2=r^2 \tag{b27}$$

Hence the circle representing the contour of the upper base remains a circle of the same radius. The plane of the circle is moving downward by a quantity equal to $\rho g(H^2-\nu r^2)/(2E)$. The surface of the disc representing the upper base of the body is no longer a plane one. It becomes a rotational parabolic surface having the equation

$$x_3=\frac{\nu\rho g}{2E}(x_1^2+x_2^2)-\frac{\rho g}{2E}H^2+H \tag{b28}$$

b) The final shape of the lower base

Consider now an arbitrary point initially placed on the lower base of the body. The point has the co-ordinates equal to $(X_1, X_2, X_3 = 0)$. The final co-ordinate of the point are

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}X_1\\X_2\\0\end{pmatrix}+\frac{\nu\rho g}{E}\begin{pmatrix}HX_1\\HX_2\\(X_1^2+X_2^2)/2\end{pmatrix} \tag{b29}$$

Hence

$$\begin{cases} x_1 = (1 + \nu\rho g H / E)X_1 \\ x_2 = (1 + \nu\rho g H / E)X_2 \\ x_3 = \dfrac{\nu\rho g}{2E}(X_1^2 + X_2^2) \end{cases} \tag{b30}$$

From eq.(b30) it follows that

$$x_1^2 + x_2^2 = (1 + \nu\rho g H / E)^2 (X_1^2 + X_2^2) = (1 + \nu\rho g H / E)^2 r^2 \quad , \tag{31}$$

i.e. the circle representing the contour of the lower base remains a circle. The new radius is increased by a quantity equal to $\nu\rho g H / E$. The initial horizontal plane of the circle is uplifted by a quantity equal to $\nu\rho g r^2 / (2E)$. The surface of the disc representing the lower base becomes a rotational paraboloid having the equation

$$x_3 = \frac{\nu\rho g}{2E(1 + \nu\rho g H / E)^2}(x_1^2 + x_2^2) \tag{b32}$$

c) The final shape of the lateral surface

Consider now a point initially placed on a generatrix line of the cylinder. Because of the cylindrical symmetry of the problem, the point having the initial co-ordinates equal to $(X_1 = 0, X_2 = r, X_3)$ is considered. Finally, that point has the position characterised by the co-ordinates

$$\begin{cases} x_1 = 0 \\ x_2 = r + \dfrac{\nu\rho g}{E}(H - X_3)r \\ x_3 = X_3 + \dfrac{\rho g}{E}\left(\dfrac{X_3^2}{2} - HX_3\right) + \dfrac{\nu\rho g}{2E}r^2 \end{cases} \tag{b33}$$

From (b33), it follows that the generatrix remains into the initial vertical plane. Its shape is changed from a straight line segment to a convex parabolic segment, having the equation

$$x_3 = \frac{E}{2\nu^2\rho g}(x_2 / r - 1)^2 - \frac{E}{\nu\rho g}(x_2 / r - 1) + H - \frac{\rho g}{2E}H^2 + \frac{\nu\rho g}{2E}r^2 \tag{b34}$$

OBSERVATION. On the lower base of the cylinder it is acting the reaction force of the rigid plane, equal to the weight of the body. When the surface of the base is decreasing, approaching the paraboloid of eq.(b32), the normal unit effort (equal to the weight divided by the contact area) is increasing. At a certain moment, its magnitude will exceed a yielding value of the material. Then, HOOKE's law, valid in the elastic domain, will be no longer appropriate here.

C) LÉVY's PROBLEM - THE TRIANGULAR DAM

C.1) The SAINT-VENANT 's equations.

Differentiating a certain element of the strain tensor

$$\varepsilon_{ij} = \left(u_{i,j} + u_{j,i}\right)/2 \quad , \tag{c1}$$

it follows, for example, that

$$\begin{aligned}\varepsilon_{11,22} + \varepsilon_{22,11} &= \left(u_{1,1}\right)_{,22} + \left(u_{2,2}\right)_{,11} = u_{1,122} + u_{2,211} \\ &= \left(u_{1,2}\right)_{,12} + \left(u_{2,1}\right)_{,12} = \left(u_{1,2} + u_{2,1}\right)_{,12} = 2\varepsilon_{12,12}\end{aligned} \tag{c2}$$

Hence

$$\varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12} \tag{c3}$$

Also,

$$\varepsilon_{22,33} + \varepsilon_{33,22} = 2\varepsilon_{23,23} \tag{c4}$$

$$\varepsilon_{33,11} + \varepsilon_{11,33} = 2\varepsilon_{31,31} \tag{c5}$$

In a similar way, it follows that

$$\begin{aligned}\left(\varepsilon_{12,3} + \varepsilon_{23,1} - \varepsilon_{31,2}\right)_{,2} &= \varepsilon_{22,31} \\ \left(\varepsilon_{23,1} + \varepsilon_{31,2} - \varepsilon_{12,3}\right)_{,3} &= \varepsilon_{33,12} \\ \left(\varepsilon_{31,2} + \varepsilon_{12,3} - \varepsilon_{23,1}\right)_{,1} &= \varepsilon_{11,23}\end{aligned} \tag{c6)-(c8}$$

The above equations (c3)-(c8) represent the SAINT-VENANT's equations of compatibility.

C.2) The model . Simplifying hypothesis. The planar deformation state.

A horizontal dam of infinite length is considered. The cross-section is represented by a rectangular triangle OAB (Fig.C1). The length of the base is AB=l and the height is OA=h. On OA catheter is acting the hydrostatic pressure of a liquid (water) having the specific weight equal to γ. As a result, the dam is deformed. The dam is represented by an elastic homogeneous, isotropic material. Its specific weight is equal to Γ and its elastic constants are E and ν.

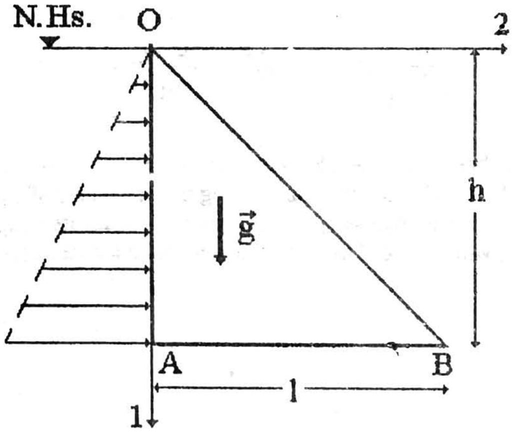

Fig.C1. A vertical cross section through the dam. N.Hs. is the free surface of the water, acting on OA side by a pressure linearly increasing with depth.

Because the shape of the dam, the displacement vector has the components like

$$\begin{cases} u_1 = u_1(x_1, x_2) \\ u_2 = u_2(x_1, x_2) \\ u_3 = 0 \end{cases} \tag{c9}$$

It follows the strain tensor components are like

$$\begin{aligned} &\varepsilon_{11} = u_{1,1} = \varepsilon_{11}(x_1, x_2) \\ &\varepsilon_{12} = \frac{1}{2}\left(u_{1,2} + u_{2,1}\right) = \varepsilon_{12}(x_1, x_2) \\ &\varepsilon_{13} = \frac{1}{2}\left(u_{1,3} + u_{3,1}\right) = 0 \\ &\varepsilon_{22} = u_{2,2} = \varepsilon_{22}(x_1, x_2) \\ &\varepsilon_{23} = \frac{1}{2}\left(u_{2,3} + u_{3,2}\right) = 0 \\ &\varepsilon_{33} = u_{3,3} = 0 \end{aligned} \tag{c10}$$

Hence the strain matrix is

$$[\varepsilon] = [\varepsilon](x_1, x_2) = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{c11}$$

It corresponds to a ***planar state of the strain*** (the plane here being 1-2).
The components of the stress tensor are

$$\begin{aligned} &\sigma_{11} = \lambda\left(\varepsilon_{11} + \varepsilon_{22}\right) + 2\mu\varepsilon_{11} \\ &\sigma_{12} = 2\mu\varepsilon_{12} \\ &\sigma_{13} = 2\mu\varepsilon_{13} = 0 \\ &\sigma_{22} = \lambda\left(\varepsilon_{11} + \varepsilon_{22}\right) + 2\mu\varepsilon_{22} \\ &\sigma_{23} = 2\mu\varepsilon_{23} = 0 \\ &\sigma_{33} = \lambda\left(\varepsilon_{11} + \varepsilon_{22}\right) + 2\mu\varepsilon_{33} = \lambda\left(\varepsilon_{11} + \varepsilon_{22}\right) = \frac{\lambda}{2(\lambda + \mu)}\left(\sigma_{11} + \sigma_{22}\right) = \nu\left(\sigma_{11} + \sigma_{22}\right) \end{aligned} \tag{c12}$$

Hence the stress matrix is

$$[\sigma] = [\sigma](x_1, x_2) = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \nu\left(\sigma_{11} + \sigma_{22}\right) \end{pmatrix} \tag{c13}$$

Because the component 33 of the stress has a non-zero value, eq.(c13) shows that the stress state corresponding to a planar state of the strain is not generally a planar one too.

C.3) Equations of equilibrium. AIRY's potential.

The only body force acting on the dam is its weight. The equations of equilibrium are

$$\begin{cases} \sigma_{11,1} + \sigma_{12,2} + \Gamma = 0 \\ \sigma_{12,1} + \sigma_{22,2} = 0 \end{cases} \tag{c14}$$

Because the presence of Γ, eqs.(c14) represent a non-homogeneous system. In the beginning, the homogeneous system is solved, i.e.

$$\begin{cases}\Sigma_{11,1}+\Sigma_{12,2}=0\\ \Sigma_{12,1}+\Sigma_{22,2}=0\end{cases} \tag{c15}$$

Using an unknown function φ, the first equation of (c15) is verified for

$$\Sigma_{11}=\frac{\partial\varphi}{\partial x_2} \quad , \quad \Sigma_{12}=-\frac{\partial\varphi}{\partial x_1} \tag{c16}$$

In the same way, the second equation of (c15) is verified for

$$\Sigma_{12}=\frac{\partial\psi}{\partial x_2} \quad , \quad \Sigma_{12}=-\frac{\partial\psi}{\partial x_1} \tag{c17}$$

It follows that

$$\frac{\partial\varphi}{\partial x_1}+\frac{\partial\psi}{\partial x_2}=0 \tag{c18}$$

i.e. the unknown functions are

$$\varphi=\frac{\partial A}{\partial x_2} \quad , \quad \psi=-\frac{\partial A}{\partial x_1} \tag{c19}$$

The unknown function $A = A(x_1, x_2)$ represents the AIRY's potential. It allows one to obtain the next expressions for the components of the stress tensor when the body force are absent:

$$\Sigma_{11}=A_{,22} \quad , \quad \Sigma_{12}=-A_{,12} \quad , \quad \Sigma_{22}=A_{,11} \tag{c20}$$

From (c13), the trace of the stress tensor can be written using LAPLACE's operator in 1-2 co-ordinates

$$\mathrm{tr}\,\Sigma = (1+\nu)\left(\Sigma_{11}+\Sigma_{22}\right)=(1+\nu)\Delta^* A \tag{c21}$$

The components of the strain tensor are obtained using the reversed HOOKE's law

$$\varepsilon_{11}=\frac{1+\nu}{E}\left(A_{,22}-\nu\Delta^* A\right) \quad ,\varepsilon_{22}=\frac{1+\nu}{E}\left(A_{,11}-\nu\Delta^* A\right) \quad ,\varepsilon_{12}=-\frac{1+\nu}{E}A_{,12} \tag{c22}$$

Using (c22) and (c3) it follows

$$A_{,2222}-\nu\left(\Delta^* A\right)_{,22}+A_{,1111}-\nu\left(\Delta^* A\right)_{,11}=-2A_{,1212} \quad , \tag{c23}$$

i.e.

$$(1-\nu)\Delta^*\Delta^* A = 0 \tag{c24}$$

Because $\nu < 0.5$, it follows that AIRY's potential is a solution of the bi-harmonic equation

$$\Delta^*\Delta^* A = 0 \tag{c25}$$

Because the trace of a tensor is an invariant, eq.(c25) holds too in the general case of the orthogonal curvilinear co-ordinates. However, eq.(c20) has to be modified.

C.4) Boundary conditions. The final shape of the dam.

On the side OA of the dam is acting the hydrostatic pressure. It follows that

$$\sigma\left(-\vec{e}_2\right)=\gamma x_1 \vec{e}_2 \tag{c26}$$

On the side OB of the dam is acting the negligible atmospheric pressure. It follows that

$$\sigma\,\vec{n}=\vec{0} \tag{c27}$$

where the outer pointing normal at the dam is

$$\vec{n}=-\sin\alpha\,\vec{e}_1+\cos\alpha\,\vec{e}_2 \tag{c28}$$

On the side OA, for $x_1 \in [0,h], x_2 = 0$, it follows that

$$\begin{cases}\sigma_{12}=0,\\ \sigma_{22}=-\gamma x_1\end{cases} \tag{c29}$$

On the side OB it follows for $x_1 \in [0,h], x_2 = x_1 \tan\alpha$ that

$$\begin{cases}\sigma_{12}-\sigma_{11}\tan\alpha=0,\\ \sigma_{22}-\sigma_{12}\tan\alpha=0\end{cases} \tag{c30}$$

Eqs.(c29)-(c30) represent 4 boundary conditions, suggesting a solution of the bi-harmonic equation (c25) which depends on 4 unknown coefficients denoted by a, b, c, d, i.e.

$$A(x_1,x_2)=\frac{a}{6}x_1^3+\frac{b}{2}x_1^2x_2+\frac{c}{2}x_1x_2^2+\frac{d}{6}x_2^3 \tag{c31}$$

Using (c20), the solution of the homogeneous system is

$$\begin{cases}\Sigma_{11}=cx_1+dx_2\\ \Sigma_{12}=-(bx_1+cx_2)\\ \Sigma_{22}=ax_1+bx_2\end{cases} \tag{c32}$$

A particular solution of the non-homogeneous system (c14) is

$$\begin{cases}\sigma_{11}=\sigma_{22}=0\\ \sigma_{12}=-\Gamma x_2\end{cases} \tag{c33}$$

It follows the general solution of (c14) is

$$\begin{cases}\sigma_{11}=cx_1+dx_2\\ \sigma_{12}=-(bx_1+cx_2)-\Gamma x_2\\ \sigma_{22}=ax_1+bx_2\end{cases} \tag{c34}$$

Replacing (c34) into (c29)-(c30) it follows that

$$\begin{cases}-(bx_1+cx_2)-\Gamma x_2=0,\ \text{for } x_1\in[0,h]\quad, x_2=0\\ ax_1+bx_2=-\gamma x_1,\ \text{for } x_1\in[0,h]\quad, x_2=0\\ -(cx_1+dx_2)\tan\alpha-(bx_1+cx_2)-\Gamma x_2=0,\ \text{for } x_1\in[0,h]\quad, x_2=x_1\tan\alpha\\ (bx_1+cx_2+\Gamma x_2)\tan\alpha+ax_1+bx_2=0,\ \text{for } x_1\in[0,h]\quad, x_2=x_1\tan\alpha\end{cases} \tag{c35}$$

It follows that

$$\begin{cases}a=-\gamma\\ b=0\\ c=-\Gamma+\gamma/\tan^2\alpha\\ d=\Gamma/\tan\alpha-2\gamma/\tan^3\alpha\end{cases} \tag{c36}$$

and

$$\begin{cases}\sigma_{11}=Ax_1+Bx_2\\ \sigma_{12}=-Cx_2\\ \sigma_{22}=-\gamma x_1\end{cases} \tag{c37}$$

where

$$A=\gamma h^2/l^2-\Gamma\quad,\quad B=\Gamma h/l-2\gamma h^3/l^3\quad,\quad C=-\gamma h^2/l^2 \tag{c38}$$

Hence

$$u_{1,1}=\varepsilon_{11}=C_1x_1+C_2x_2\ , \tag{c39}$$

where

$$C_1=(1+\nu)[A-\nu(A-\gamma)]/E \quad , \quad C_2=(1-\nu^2)B/E \tag{c40}$$

It follows that

$$u_1=C_1x_1^2/2+C_2x_1x_2+f_1(x_2) \tag{c41}$$

where the unknown function f_1 follows to be found. In the same manner,

$$u_2=C_3x_1x_2+C_4x_2^2/2+f_2(x_1) \tag{c42}$$

But

$$\varepsilon_{12}=\frac{1}{2}\left(u_{1,2}+u_{2,1}\right)=\frac{1}{2}(C_2x_1+f_1'(x_2)+C_3x_2+f_2'(x_1))=\frac{1+\nu}{E}\sigma_{12}=-\frac{1+\nu}{E}Cx_2 \tag{c43}$$

Hence

$$\begin{cases}C_2x_1+f_2'(x_1)=K\\ C_3x_2+f_1'(x_2)=-K\end{cases} , \tag{c44}$$

where K is an arbitrary constant. It follows

$$f_1(x_2)=-C_3x_2^2/2-Kx_2+K_1 \quad , \quad f_2(x_1)=-C_2x_1^2/2+Kx_1+K_2 \tag{c45}$$

Hence, the displacement field is

$$\begin{cases}u_1=C_1x_1^2/2+C_2x_1x_2-[C_3+2(1+\nu)C/E]x_2^2/2-Kx_2+K_1\\ u_2=-C_2x_1^2/2+C_3x_1x_2+C_4x_2^2/2+Kx_1+K_2\end{cases} \tag{c46}$$

The last terms into (c46) represent a rigid roto-translation.

It should be outlined that the above boundary conditions on stress values on the sides OA and OB are not complete ones. As a result, the unknown constants C_3, C_4 are present in (c46). Boundary conditions on stress values (or displacements) on the side AB are required in order to obtain an unique solution of the problem

For example, consider the case when the points A and B are fixed ones. It follows

$$\begin{cases}u_1=C_1\left(x_1^2-h^2\right)/2+C_2(x_1-h)x_2-[C_3+2(1+\nu)C/E]x_2(l-x_2)/2\\ u_2=C_2\left(h^2-x_1^2\right)/2+C_3x_2(x_1-x_2h/l)+\left\{C_2h-[C_3+2(1+\nu)C/E]l/2\right\}(x_1-h)\end{cases} \tag{c47}$$

An arbitrary point placed initially on the side AB has the initial co-ordinates $(X_1=h; X_2)$. Its final position is

$$\begin{cases}x_1=X_1+u_1(X_1,X_2)=h+[C_3+2(1+\nu)C/E]X_2(l-X_2)/2\\ x_2=X_2+u_2(X_1,X_2)=X_2+C_3hX_2(l-X_2)/l\end{cases} \tag{c48}$$

Elementary computations show that

$$C_3+\frac{2(1+\nu)}{E}C=-\frac{1+\nu}{E}\left[\gamma(1-\nu)-\nu\Gamma-(2-\nu)\gamma\frac{h^2}{l^2}\right] \tag{c49}$$

If

$$C_3+\frac{2(1+\nu)}{E}C<0 \quad , \tag{c50}$$

the final shape of the side AB is a concave parabolic segment. Because the possibility of the water to flow below the dam, that situation is not recommended in real cases. Therefore, it is asked to

$$\gamma(1-\nu)-\nu\Gamma-(2-\nu)\gamma(h/l)^2\le 0 \quad , \tag{c51}$$

i.e.

$$h/l\ge\sqrt{[\gamma(1-\nu)-\nu\Gamma]/(2-\nu)/\gamma} \tag{c52}$$

For example, assuming that $\gamma=1000\,Kgs/m^3$, $\Gamma=2400\,Kgs/m^3$, $\nu=0.25$ it follows that $h\ge 0.29l$.

EXERCISE. Obtain the final shape of the dam in the above hypothesis.

D) KIRSCH's PROBLEM - THE CIRCULAR BORE HOLE / TUNNEL

D.1) The model.

It is assumed that the whole 3-dimensional space is represented by an elastic, homogeneous, isotropic medium, having the elastic constants denoted by E and ν, respectively λ and μ. A co-ordinate system having the third axis positive upward will be used. The initial state of stress is represented by the homogeneous tensor σ^0, corresponding to a planar state of deformation, i.e.

$$\sigma^0 = \begin{pmatrix} \sigma_{11}^0 & \sigma_{12}^0 & 0 \\ \sigma_{12}^0 & \sigma_{22}^0 & 0 \\ 0 & 0 & \nu\left(\sigma_{11}^0 + \sigma_{22}^0\right) \end{pmatrix}, \quad \text{(d1)}$$

where the components σ_{ij}^0 have constant values. The mass forces are ignored, hence the equilibrium equation

$$\operatorname{div} \sigma^0 = \vec{0} \quad \text{(d2)}$$

is identically satisfied.

Suppose that a circular, infinite bore hole / tunnel is performed along the third axis, its material being instantly removed. The origin of the co-ordinate system is placed at the centre of the cavity. On the wall of the bore hole is acting now the atmospheric pressure (or the pressure of the drilling mud), denoted by p_0. Consequently, a new (non-homogeneous) stress value is obtained and the circular shape of the bore hole is changing too. It follows to obtain the new stress, denoted by σ^f, and the new shape of the bore hole in the final equilibrium stage, where

$$\operatorname{div} \sigma^f = \vec{0}\,. \quad \text{(d3)}$$

It is also assumed that the deformation is an elastic one, i.e. the stress perturbation $\sigma = \sigma^f - \sigma^0$ is related to the strain tensor by

$$\sigma = \lambda \operatorname{tr} \varepsilon \, 1 + 2\mu \varepsilon \,. \quad \text{(d4)}$$

The unknown components of the displacement vector are supposed to correspond to a planar deformation state, i.e.

$$u_1 = u_1(x_1, x_2) \quad , \quad u_2 = u_2(x_1, x_2) \quad , \quad u_3 = 0. \quad \text{(d5)}$$

Because the symmetry of the problem, the cylindrical co-ordinate system (r, θ, z) will be used, having the unit vectors denoted by $\left(\vec{e}_r, \vec{e}_\theta, \vec{e}_z\right)$ (see Fig.D1).

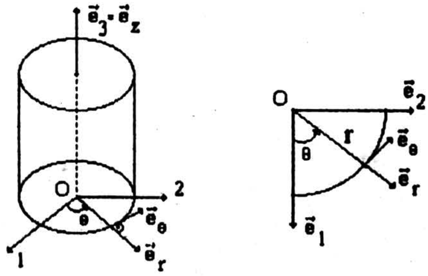

Fig.D1. The cylindrical co-ordinate system.

D.2) The planar sate of deformation in cylindrical co-ordinate system.

With respect to Fig.D1 it follows that

$$\begin{cases} \vec{e}_r = \cos\theta\, \vec{e}_1 + \sin\theta\, \vec{e}_2, \\ \vec{e}_\theta = -\sin\theta\, \vec{e}_1 + \cos\theta\, \vec{e}_2, \\ \vec{e}_z = \vec{e}_3. \end{cases} \tag{d6}$$

Hence the matrix for passing from the Cartesian co-ordinates to cylindrical co-ordinates is

$$Q = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{d7}$$

It represents a rotation of angle equal to θ in a positive (counter clockwise) sense. From (d4) and (d5) it follows that the stress matrix in Cartesian co-ordinates is

$$[\sigma]^{crt} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \nu(\sigma_{11}+\sigma_{22}) \end{pmatrix}. \tag{d8}$$

Let the stress matrix in cylindrical co-ordinates be

$$[\sigma]^{cyl} = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{r\theta} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{rz} & \sigma_{\theta z} & \sigma_{zz} \end{pmatrix}. \tag{d9}$$

It follows that

$$[\sigma]^{cyl} = Q[\sigma]^{crt} Q^t = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \nu(\sigma_{11}+\sigma_{12}) \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{d10}$$

By performing the computations in (d10), it follows

$$\sigma_{rr} = \frac{\sigma_{11}+\sigma_{22}}{2} + \sigma_{12}\sin 2\theta + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta \;, \tag{d11}$$

$$\sigma_{\theta\theta} = \frac{\sigma_{11}+\sigma_{22}}{2} - \sigma_{12}\sin 2\theta - \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta \;, \tag{d12}$$

$$\sigma_{r\theta} = -\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta \;, \tag{d13}$$

$$\sigma_{rz} = \sigma_{\theta z} = 0 \;, \quad \sigma_{zz} = \nu(\sigma_{11}+\sigma_{22}) = \nu(\sigma_{rr}+\sigma_{\theta\theta}) \,. \tag{d14}$$

D.3) The circle of MOHR.

Suppose the Cartesian co-ordinate system is selected in order its axes to be along the first two eigen vectors of the stress tensor. In that case, σ_{11} and σ_{22} are eigenvalues of the stress tensor and $\sigma_{12}=0$. From equations (d11)-(d13) it follows that

$$\left(\sigma_{rr}-\frac{\sigma_{11}+\sigma_{22}}{2}\right)^2+\sigma_{r\theta}^2=\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2, \qquad \text{(d15)}$$

and an identical relation obtained by replacing σ_{rr} with $\sigma_{\theta\theta}$. Eq.(d15) shows that σ_{rr} and $\sigma_{r\theta}$ are placed on a circle of radius equal to $\left|\sigma_{11}-\sigma_{22}\right|/2$. Suppose now that σ_{rr} (or $\sigma_{\theta\theta}$) (i.e. the radial stress component, usually denoted by σ) and $\sigma_{r\theta}$ (i.e. the tangential stress, usually denoted by τ) are obtained at various angles θ and the MOHR's circle represented by eq.(d15) is obtained. Its radius and its position of the centre allow one to obtain graphically the eigenvalues of the stress tensor. Further discussion will be presented in relation to the empirical failure criteria of materials.

D.4) AIRY's potential in cylindrical co-ordinates. The bi-harmonic equation.

Consider the representation of the stress components with the AIRY's potential in Cartesian co-ordinates, i.e.

$$\sigma_{11}=A_{,22} \quad ,\sigma_{22}=A_{,11} \quad ,\sigma_{12}=-A_{,12} \quad , \qquad \text{(d16)}$$

where the AIRY's potential verifies the bi-harmonic equation

$$\Delta^*\Delta^* A=0 \ . \qquad \text{(d17)}$$

In the beginning, the derivatives in eq.(d16) will be evaluated by using the polar co-ordinates

$$\begin{cases} x_1=r\cos\theta \\ x_2=r\sin\theta \end{cases} ,\begin{cases} r=\sqrt{x_1^2+x_2^2} \\ \theta=a\tan(x_2/x_1) \end{cases} \qquad \text{(d18)}$$

Than, a representation of the stress components in cylindrical co-ordinates with the help of the AIRY's potential will be obtained from (d11)-(d13).

But

$$\frac{\partial r}{\partial x_1}=\frac{x_1}{r}=\cos\theta \quad ,\frac{\partial r}{\partial x_2}=\frac{x_2}{r}=\sin\theta \quad ,\frac{\partial\theta}{\partial x_1}=-\frac{x_2}{r^2}=-\frac{\sin\theta}{r} \quad ,\frac{\partial\theta}{\partial x_2}=\frac{x_1}{r^2}=\frac{\cos\theta}{r} \qquad \text{(d19)}$$

It follows

$$A_{,1}=\frac{\partial A}{\partial x_1}=\frac{\partial A}{\partial r}\frac{\partial r}{\partial x_1}+\frac{\partial A}{\partial\theta}\frac{\partial\theta}{\partial x_1}=\cos\theta A_{,r}-\frac{\sin\theta}{r}A_{,\theta} \qquad \text{(d20)}$$

In the same way,

$$A_{,2}=\sin\theta A_{,r}+\frac{\cos\theta}{r}A_{,\theta} \ . \qquad \text{(d21)}$$

Also,

$$\begin{aligned} A_{,12}&=\left(A_{,1}\right)_{,2}=\left(\cos\theta A_{,r}-\frac{\sin\theta}{r}A_{,\theta}\right)_{,2} \\ &=\sin\theta\left(\cos\theta A_{,r}-\frac{\sin\theta}{r}A_{,\theta}\right)_{,r}+\frac{\cos\theta}{r}\left(\cos\theta A_{,r}-\frac{\sin\theta}{r}A_{,\theta}\right)_{,\theta} \qquad \text{(d22)} \\ &=\left(A_{,rr}-\frac{A_{,r}}{r}-\frac{A_{,\theta\theta}}{r^2}\right)\frac{\sin 2\theta}{2}+\left(\frac{A_{,r\theta}}{r}-\frac{A_{,\theta}}{r^2}\right)\cos 2\theta \end{aligned}$$

In the same way

$$A_{,11}=\cos^2\theta\, A_{,rr}+\sin^2\theta\left(\frac{A_{,r}}{r}+\frac{A_{,\theta\theta}}{r^2}\right)+2\sin\theta\cos\theta\left(-\frac{A_{,r\theta}}{r}+\frac{A_{,\theta}}{r^2}\right) \quad , \text{(d23)}$$

$$A_{,22}=\sin^2\theta\, A_{,rr}+\cos^2\theta\left(\frac{A_{,r}}{r}+\frac{A_{,\theta\theta}}{r^2}\right)-2\sin\theta\cos\theta\left(-\frac{A_{,r\theta}}{r}+\frac{A_{,\theta}}{r^2}\right) \quad , \text{(d24)}$$

Hence

$$\sigma_{11}+\sigma_{22}=A_{,22}+A_{,11}=A_{,rr}+\frac{A_{,r}}{r}+\frac{A_{,\theta\theta}}{r^2} \quad , \qquad \text{(d25)}$$

$$\sigma_{11}-\sigma_{22}=A_{,22}-A_{,11}=\left(-A_{,rr}+\frac{A_{,r}}{r}+\frac{A_{,\theta\theta}}{r^2}\right)\cos 2\theta-2\sin 2\theta\left(-\frac{A_{,r\theta}}{r}+\frac{A_{,\theta}}{r^2}\right) \quad , \text{(d26)}$$

From eqs.(d11)-(d13) it follows

$$\sigma_{rr}=\frac{A_{,r}}{r}+\frac{A_{,\theta\theta}}{r^2} \quad , \quad \sigma_{\theta\theta}=A_{,rr} \quad , \quad \sigma_{r\theta}=-\frac{A_{,r\theta}}{r}+\frac{A_{,\theta}}{r^2} \quad . \qquad \text{(d27)}$$

D.5) The divergence of a tensor in cylindrical co-ordinates.

In the case of the cylindrical co-ordinates, the square of the elementary arc is equal to

$$ds^2=dx_1^2+dx_2^2+dx_3^2=dr^2+r^2d\theta^2+dz^2 \qquad \text{(d28)}$$

Hence the differential parameters of Lamé are

$$h^1=1 \quad , \quad h^2=r \quad , \quad h^3=1 \quad , \qquad \text{(d29)}$$

the orthogonal curvilinear co-ordinates are equal to

$$c^1=1 \quad , \quad c^2=\theta \quad , \quad c^3=z \quad , \qquad \text{(d30)}$$

and the unit vectors are

$$\vec{n}^1=\vec{e}_r \quad , \quad \vec{n}^2=\vec{e}_\theta \quad , \quad \vec{n}^3=\vec{e}_z \quad . \qquad \text{(d31)}$$

It follows that

$$\frac{\partial h^q}{\partial c^\beta}=\delta^{q2}\delta^{\beta 1} \quad . \qquad \text{(d32)}$$

Substituting the above results in the formula (a2.77) (Ivan 1996) it follows the next formula for the divergence of a tensor in cylindrical co-ordinates

$$\begin{aligned}\mathrm{div}T=&\left(\frac{\partial T_{rr}}{\partial r}+\frac{\partial T_{r\theta}}{r\partial\theta}+\frac{\partial T_{rz}}{\partial z}+\frac{T_{rr}-T_{\theta\theta}}{r}\right)\vec{e}_r\\&+\left(\frac{\partial T_{r\theta}}{\partial r}+\frac{\partial T_{\theta\theta}}{r\partial\theta}+\frac{\partial T_{\theta z}}{\partial z}+2\frac{T_{r\theta}}{r}\right)\vec{e}_\theta+\left(\frac{\partial T_{rz}}{\partial r}+\frac{\partial T_{\theta z}}{r\partial\theta}+\frac{\partial T_{zz}}{\partial z}+\frac{T_{rz}}{r}\right)\vec{e}_z\end{aligned} \qquad \text{(d33)}$$

D.6) The gradient of a vector and the strain tensor in cylindrical co-ordinates.

Substituting the above results in the formula (a2.68) (Ivan 1996) it follows the next formula for the gradient of a vector in cylindrical co-ordinates

$$\operatorname{grad}\vec{u} = \frac{\partial u_r}{\partial r}\vec{e}_r\otimes\vec{e}_r + \frac{\partial u_r}{r\partial\theta}\vec{e}_r\otimes\vec{e}_\theta + \frac{\partial u_r}{\partial z}\vec{e}_r\otimes\vec{e}_z$$
$$+\frac{\partial u_\theta}{\partial r}\vec{e}_\theta\otimes\vec{e}_r + \frac{\partial u_\theta}{r\partial\theta}\vec{e}_\theta\otimes\vec{e}_\theta + \frac{\partial u_\theta}{\partial z}\vec{e}_\theta\otimes\vec{e}_z \tag{d34}$$
$$+\frac{\partial u_z}{\partial r}\vec{e}_z\otimes\vec{e}_r + \frac{\partial u_z}{r\partial\theta}\vec{e}_z\otimes\vec{e}_\theta + \frac{\partial u_z}{\partial z}\vec{e}_z\otimes\vec{e}_z - \frac{u_\theta}{r}\vec{e}_r\otimes\vec{e}_\theta + \frac{u_r}{r}\vec{e}_\theta\otimes\vec{e}_\theta$$

The components of the strain tensor $\dot{\varepsilon} = \frac{1}{2}\left[\operatorname{grad}\vec{u} + \left(\operatorname{grad}\vec{u}\right)^t\right]$ are

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}\quad,\quad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u_r}{r\partial\theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right)\quad,\quad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$
$$\varepsilon_{\theta\theta} = \frac{\partial u_\theta}{r\partial\theta} + \frac{u_r}{r}\quad,\quad \varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{\partial u_z}{r\partial\theta}\right)\quad,\quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{d35}$$

For the particular displacement field represented by (d5), the components of the vector $\vec{u}$ are

$$u_r = u_r(r,\theta)\quad,\quad u_\theta = u_\theta(r,\theta)\quad,\quad u_z = 0 \tag{d36}$$

Hence

$$\varepsilon_{rz} = \varepsilon_{\theta z} = \varepsilon_{zz} = 0 \tag{d37}$$

It is the case of a planar state of deformation, i.e.

$$\dot{\varepsilon} = \begin{pmatrix}\varepsilon_{rr} & \varepsilon_{r\theta} & 0\\ \varepsilon_{r\theta} & \varepsilon_{\theta\theta} & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{d38}$$

and all the strain elements are functions of r and θ.
Consequently, the stress is

$$\sigma = \begin{pmatrix}\sigma_{rr} & \sigma_{r\theta} & 0\\ \sigma_{r\theta} & \sigma_{\theta\theta} & 0\\ 0 & 0 & \nu(\sigma_{rr}+\sigma_{\theta\theta})\end{pmatrix}, \tag{d39}$$

all the components of the stress being too functions of r and θ, according to HOOKE's reversed law. It follows from (d33) the next equations of equilibrium are obtained in the absence of mass forces:

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{r\theta}}{r\partial\theta} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = 0 \tag{d40}$$
$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{\partial\sigma_{\theta\theta}}{r\partial\theta} + 2\frac{\sigma_{r\theta}}{r} = 0 \tag{d41}$$

D.7) The bi-harmonic equation in cylindrical co-ordinates.

By using (a2.69) and (a2.43) (Ivan 1996), it follows the LAPLACE operator in cylindrical co-ordinates is

$$\Delta f = \operatorname{div}(\operatorname{grad} f) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \qquad \text{(d42)}$$

The AIRY's potential is also a function of r and θ. Hence the AIRY's potential is the solution of the bi-harmonic equation

$$\Delta^* \Delta^* A = 0 \qquad \text{(d43)}$$

where the LAPLACE operator in polar co-ordinates is

$$\Delta^* = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \qquad \text{(d44)}$$

It should be noted that the singular point $r = 0$ is avoided in (d44) because $r \geq R > 0$, where R is the radius of the bore hole.

It follows to solve (d43) by using (d44) in order to derive the AIRY's potential. The stress components will be obtained from (d27), imposing the boundary conditions on the wall of the bore hole. The components of the strain will be derived by using the HOOKE's reversed law. The displacement vector will be obtained from the definition of strain elements, allowing one to find the final shape of the deformed bore hole wall.

Consider the FOURIER expansion of the AIRY's potential, having the coefficients equal to functions of r

$$A(r,\theta) = A_0(r) + \sum_{n=1}^{\infty}\left[A_n(r)\cos n\theta + B_n(r)\sin n\theta\right] \qquad \text{(d45)}$$

It follows

$$\frac{\partial A}{\partial r} = A_0' + \sum_{n=1}^{\infty}\left(A_n'\cos n\theta + B_n'\sin n\theta\right), \qquad \text{(d46)}$$

$$\frac{\partial^2 A}{\partial r^2} = A_0'' + \sum_{n=1}^{\infty}\left(A_n''\cos n\theta + B_n''\sin n\theta\right), \qquad \text{(d47)}$$

$$\frac{\partial^2 A}{\partial \theta^2} = -\sum_{n=1}^{\infty} n^2\left(A_n\cos n\theta + B_n\sin n\theta\right). \qquad \text{(d48)}$$

Hence

$$\Delta A = \Delta A_0 + \sum_{n=1}^{\infty}\left[\left(A'' + \frac{1}{r}A' - \frac{n^2}{r^2}A\right)\cos n\theta + \left(B'' + \frac{1}{r}B' - \frac{n^2}{r^2}B\right)\sin n\theta\right] \qquad \text{(d49)}$$

and

$$\Delta\Delta A = \Delta\Delta A_0 + \sum_{n=1}^{\infty}\left[\left(A'''' + \frac{2}{r}A''' - \frac{2n^2+1}{r^2}A'' + \frac{2n^2+1}{r^3}A' + \frac{n^4-4n^2}{r^4}A\right)\cos n\theta + \left(B'''' + \frac{2}{r}B''' - \frac{2n^2+1}{r^2}B'' + \frac{2n^2+1}{r^3}B' + \frac{n^4-4n^2}{r^4}B\right)\sin n\theta\right]. \qquad \text{(d50)}$$

By using (d50), the bi-harmonic (d43) is verified if

$$\Delta\Delta A_0 = 0, \qquad \text{(d51)}$$

and if the functions A, B are the solutions of the next differential equation

$$\Phi'''' + \frac{2}{r}\Phi''' - \frac{2n^2+1}{r^2}\Phi'' + \frac{2n^2+1}{r^3}\Phi' + \frac{n^4-4n^2}{r^4}\Phi = 0. \qquad \text{(d52)}$$

Because A_0 is a function of r only, eq.(d51) is

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_0}{dr}\right)\right]\right\}=0 \quad \text{(d53)}$$

Hence

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_0}{dr}\right)\right]=a_0 \quad , \quad \frac{d}{dr}\left(r\frac{dA_0}{dr}\right)=a_0 r\ln r+b_0 r \quad \text{(d54)}$$

But

$$\int r\ln r dr=\frac{r^2}{2}\left(\ln r-\frac{1}{2}\right)+\text{const.} \quad \text{(d55)}$$

Hence

$$r\frac{dA_0}{dr}=\frac{a_0}{2}r^2\left(\ln r-\frac{1}{2}\right)+\frac{b_0}{2}r^2+c_0 \quad \text{(d56)}$$

Finally, denoting again the constants, it follows

$$A_0(r)=a_0 r^2\ln r+b_0 r^2+c_0\ln r+d_0 \quad \text{(d57)}$$

In order to solve eq.(52), a solution of the form

$$\Phi=r^m \quad \text{(d58)}$$

is considered. Substituting (d58) in (d52), it follows that the exponent m is the solution of the algebraic equation

$$m(m-1)(m-2)(m-3)+2m(m-1)(m-2)-(2n^2+1)m(m-1)+(2n^2+1)m+n^4-4n^2=0, \quad \text{(d59)}$$

having the roots

$$m_1=-n \quad , \quad m_2=-n+2 \quad , \quad m_3=n \quad , \quad m_4=n+2 \quad \text{(d60)}$$

Hence the AIRY's potential is

$$A(r,\theta)=a_0 r^2\ln r+b_0 r^2+c_0\ln r+d_0+\sum_{n=1}^{\infty}\left(a_n r^{n+2}+b_n r^n+c_n r^{-n+2}+d_n r^{-n}\right)\cos\theta n$$
$$+\sum_{n=1}^{\infty}\left(\alpha_n r^{n+2}+\beta_n r^n+\gamma_n r^{-n+2}+\delta_n r^{-n}\right)\sin\theta n \quad \text{(d61)}$$

where the unknown coefficients $a_i, b_i, c_i, d_i, \alpha_j, \beta_j, \gamma_j, \delta_j, \quad i=\overline{0,1,2,\ldots} \quad , \quad j=\overline{1,2,\ldots}$ follows to be obtained.

D.8) The stress elements. Conditions at infinity for the stress elements.

By using (d61) and (d27), it follows

$$\sigma_{\theta\theta}=A_{,rr}=a_0(2\ln r+3)+2b_0-\frac{c_0}{r^2}$$
$$+\sum_{n=1}^{\infty}\left(a_n(n+2)(n+1)r^n+b_n n(n-1)r^{n-2}+c_n(n-2)(n-1)r^{-n}+d_n n(n+1)r^{-n-2}\right)\cos\theta n$$
$$+\sum_{n=1}^{\infty}\left(\alpha_n(n+2)(n+1)r^n+\beta_n n(n-1)r^{n-2}+\gamma_n(n-2)(n-1)r^{-n}+\delta_n n(n+1)r^{-n-2}\right)\sin\theta n \quad \text{(d62)}$$

At great distances from the cylindrical cavity, the elastic perturbation has to vanish, i.e.

$$\lim_{r\to\infty}\sigma_{\theta\theta}=0 \quad \text{(d63)}$$

It follows

$$a_0 = 0 \quad , \quad 2b_0 + 2b_2 + 2\beta_2 = 0 \quad , \quad a_n = \alpha_n = 0 \quad , n = \overline{1,2,\ldots} \qquad b_n = \beta_n = 0 \quad , n = \overline{3,4,\ldots} \tag{d64}$$

Hence the AIRY's potential is

$$\mathcal{A}(r,\theta) = c_0 \ln r + d_0(b_1 + \beta_1) r + \sum_{n=1}^{\infty} (c_n \cos n\theta + \gamma_n \sin n\theta) r^{-n+2} + \sum_{n=1}^{\infty} (d_n \cos n\theta + \delta_n \sin n\theta) r^{-n} \tag{d65}$$

and

$$\sigma_{\theta\theta} = -\frac{c_0}{r^2} + \sum_{n=1}^{\infty} (n-2)(n-1)(c_n \cos n\theta + \gamma_n \sin n\theta) r^{-n} + \sum_{n=1}^{\infty} n(n+1)(d_n \cos n\theta + \delta_n \sin n\theta) r^{-n-2} \tag{d66}$$

From (d65), it follows that

$$\mathcal{A}_{,r} = \frac{c_0}{r} + b_1 + \beta_1 + \sum_{n=1}^{\infty} (-n+2)(c_n \cos n\theta + \gamma_n \sin n\theta) r^{-n+1} + \sum_{n=1}^{\infty} (-n)(d_n \cos n\theta + \delta_n \sin n\theta) r^{-n-1} , \tag{d67}$$

$$\mathcal{A}_{,\theta} = \sum_{n=1}^{\infty} n(-c_n \sin n\theta + \gamma_n \cos n\theta) r^{-n+2} + \sum_{n=1}^{\infty} n(-d_n \sin n\theta + \delta_n \cos n\theta) r^{-n} , \tag{d68}$$

$$\mathcal{A}_{,\theta\theta} = -\sum_{n=1}^{\infty} n^2 (c_n \cos n\theta + \gamma_n \sin n\theta) r^{-n+2} - \sum_{n=1}^{\infty} n^2 (d_n \cos n\theta + \delta_n \sin n\theta) r^{-n} . \tag{d69}$$

Hence

$$\sigma_{rr} = \frac{\mathcal{A}_{,r}}{r} + \frac{\mathcal{A}_{,\theta\theta}}{r^2} = \frac{c_0}{r^2} + \frac{b_1 + \beta_1}{r} - \sum_{n=1}^{\infty} \left(n^2 + n - 2\right)(c_n \cos n\theta + \gamma_n \sin n\theta) r^{-n} - \sum_{n=1}^{\infty} \left(n^2 + n\right)(d_n \cos n\theta + \delta_n \sin n\theta) r^{-n-2} \tag{d70}$$

From (d70), it follows that

$$\lim_{r \to \infty} \sigma_{rr} = 0 . \tag{d71}$$

Also,

$$\mathcal{A}_{,r\theta} = \sum_{n=1}^{\infty} n(n-2)(c_n \sin n\theta - \gamma_n \cos n\theta) r^{-n+1} + \sum_{n=1}^{\infty} n^2 (d_n \sin n\theta - \delta_n \cos n\theta) r^{-n-1} . \tag{d72}$$

Hence

$$\sigma_{r\theta} = -\frac{\mathcal{A}_{,r\theta}}{r} + \frac{\mathcal{A}_{,\theta}}{r^2} = -\sum_{n=1}^{\infty} (n^2 - n)(c_n \sin n\theta - \gamma_n \cos n\theta) r^{-n} - \sum_{n=1}^{\infty} \left(n^2 + n\right)(d_n \sin n\theta - \delta_n \cos n\theta) r^{-n-2} \tag{d73}$$

From (d73), it follows that

$$\lim_{r\to\infty} \sigma_{r\theta} = 0 \quad . \tag{d74}$$

D.9) Strain and displacement vector. Conditions at infinity.

From (d39) and HOOKE's reversed law it follows that

$$\boldsymbol{\varepsilon} = \frac{1}{E}\left[(1+\nu)\boldsymbol{\sigma} - \nu \operatorname{tr}\boldsymbol{\sigma}\, \mathbf{1}\right] = \frac{1+\nu}{E}\left[\boldsymbol{\sigma} - \nu(\sigma_{rr}+\sigma_{\theta\theta})\mathbf{1}\right] \tag{d75}$$

i.e.

$$\varepsilon_{rr} = \frac{1-\nu^2}{E}\left[\sigma_{rr} - \frac{\nu}{1-\nu}\sigma_{\theta\theta}\right] = \frac{\partial u_r}{\partial r} \quad , \tag{d76}$$

Hence

$$\frac{E}{1-\nu^2}\frac{\partial u_r}{\partial r} = \frac{c_0}{1-\nu}\frac{1}{r^2} + \frac{b_1+\beta_1}{r} - \sum_{n=2}^{\infty}(n-1)\left[n+2+\frac{\nu}{1-\nu}(n-2)\right](c_n \cos n\theta + \gamma_n \sin n\theta)r^{-n}$$
$$-\frac{1}{1-\nu}\sum_{n=1}^{\infty} n(n+1)(d_n \cos n\theta + \delta_n \sin n\theta)r^{-n-2} \tag{d77}$$

Integrating (d77) it follows

$$\frac{E}{1-\nu^2}u_r = -\frac{c_0}{1-\nu}\frac{1}{r} + (b_1+\beta_1)\ln r + \sum_{n=2}^{\infty}\left[n+2+\frac{\nu}{1-\nu}(n-2)\right](c_n \cos n\theta + \gamma_n \sin n\theta)r^{-n+1}$$
$$+\frac{1}{1-\nu}\sum_{n=1}^{\infty} n(d_n \cos n\theta + \delta_n \sin n\theta)r^{-n-1} + \varphi(\theta) \tag{d78}$$

From

$$\lim_{r\to\infty} u_r = 0 \quad , \tag{d79}$$

it follows that

$$b_1 + \beta_1 = 0 \quad , \quad \varphi(\theta) \equiv 0 \quad . \tag{d80}$$

Hence

$$\frac{E}{1-\nu^2}u_r = -\frac{c_0}{1-\nu}\frac{1}{r} + \sum_{n=2}^{\infty}\left[n+2+\frac{\nu}{1-\nu}(n-2)\right](c_n \cos n\theta + \gamma_n \sin n\theta)r^{-n+1}$$
$$+\frac{1}{1-\nu}\sum_{n=1}^{\infty} n(d_n \cos n\theta + \delta_n \sin n\theta)r^{-n-1} \quad . \tag{d81}$$

Finally

$$u_r = \frac{1+\nu}{E}\left[-\frac{c_0}{r} + \sum_{n=2}^{\infty}(n+2+-4\nu)(c_n \cos n\theta + \gamma_n \sin n\theta)r^{-n+1}\right.$$
$$\left. + \sum_{n=1}^{\infty} n(d_n \cos n\theta + \delta_n \sin n\theta)r^{-n-1}\right] \quad . \tag{d82}$$

In the same way,

$$\varepsilon_{\theta\theta} = \frac{1-\nu^2}{E}\left[\sigma_{\theta\theta} - \frac{\nu}{1-\nu}\sigma_{rr}\right] = \frac{\partial u_\theta}{r\partial\theta} + \frac{u_r}{r} \quad . \tag{d83}$$

It follows

$$\frac{E}{1-\nu^2}\frac{\partial u_\theta}{\partial \theta} = -\frac{E}{1-\nu^2}u_r + r\sigma_{\theta\theta} - \frac{\nu}{1-\nu}r\sigma_{rr} \quad . \tag{d84}$$

Substituting (d82), (d66) and (d70) into (d84) and integrating with respect to θ, it follows after some computations that

$$u_\theta = \frac{1+\nu}{E}\left[(1-\nu)\psi(r) + \sum_{n=2}^{\infty}(n-4+4\nu)(c_n \sin n\theta - \gamma_n \cos n\theta)r^{-n+1} + \sum_{n=1}^{\infty} n(d_n \sin n\theta - \delta_n \cos n\theta)r^{-n-1}\right] , \tag{d85}$$

Form the condition

$$\lim_{r\to\infty} u_\theta = 0 \ , \tag{d86}$$

it follows the unknown function $\psi = \psi(r)$ is subject to the condition

$$\lim_{r\to\infty} \psi(r) = 0 \quad . \tag{d87}$$

But

$$\varepsilon_{r\theta} = \frac{1+\nu}{E}\sigma_{r\theta} = \frac{1}{2}\left(\frac{\partial u_r}{r\partial\theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) \ , \tag{d88}$$

or

$$2r\sigma_{r\theta} = \frac{E}{1+\nu}\left(\frac{\partial u_r}{\partial\theta} - u_\theta + r\frac{\partial u_\theta}{\partial r}\right) \quad . \tag{d89}$$

Substituting (d73), (d82) and (d85) into (d89), it follows after some computations that

$$r\frac{d\psi}{dr} = \psi \ , \tag{d90}$$

i.e. $\psi = Cr$. From (d87) it follows that $\psi(r) \equiv 0$ and, finally,

$$u_\theta = \frac{1+\nu}{E}\left[\sum_{n=2}^{\infty}(n-4+4\nu)(c_n \sin n\theta - \gamma_n \cos n\theta)r^{-n+1} + \sum_{n=1}^{\infty} n(d_n \sin n\theta - \delta_n \cos n\theta)r^{-n-1}\right] \tag{d91}$$

D.10) Boundary conditions for the stress elements on the wall of the circular cavity.

Using the previous results, the final expressions of the plane elements of the stress are equal to

$$\sigma_{rr} = \frac{c_0}{r^2} - \sum_{n=1}^{\infty}\left(n^2+n-2\right)(c_n \cos n\theta + \gamma_n \sin n\theta)r^{-n} - \sum_{n=1}^{\infty}\left(n^2+n\right)(d_n \cos n\theta + \delta_n \sin n\theta)r^{-n-2} \tag{d92}$$

$$\sigma_{\theta\theta} = -\frac{c_0}{r^2} + \sum_{n=1}^{\infty}(n-2)(n-1)(c_n \cos n\theta + \gamma_n \sin n\theta)r^{-n} + \sum_{n=1}^{\infty} n(n+1)(d_n \cos n\theta + \delta_n \sin n\theta)r^{-n-2} \tag{d93}$$

and

$$\sigma_{r\theta} = -\sum_{n=1}^{\infty}(n^2-n)(c_n \sin n\theta - \gamma_n \cos n\theta)r^{-n} - \sum_{n=1}^{\infty}\left(n^2+n\right)(d_n \sin n\theta - \delta_n \cos n\theta)r^{-n-2} \tag{d94}$$

The cavity wall has the outer normal (with respect to the rock domain) equal to $-\vec{e}_r$ and radius equal to $r = R$. In the case of a bore hole, let Δp be the difference between the mud pressure and the pressure of the fluid contained by the porous rock (usually, because the atmospheric pressure is negligible, it follows in the case of a tunnel that $\Delta p = 0$). It follows the final stress σ^f satisfies the next boundary condition

$$\sigma^f\left(-\vec{e}_r\right) = \Delta p\,\vec{e}_r \quad , \quad \text{for} \quad r = R \tag{d92}$$

Hence

$$\begin{cases} \sigma_{rr}\big|_{r=R} = -\sigma^0_{rr} - \Delta p \\ \sigma_{r\theta}\big|_{r=R} = -\sigma^0_{r\theta} \end{cases} \tag{d93}$$

With no loss of generality, it can be assumed that the stress at infinity is along its main axes, i.e. $\sigma^0_{12} = 0$. Hence

$$\begin{cases} \dfrac{c_0}{R^2} - \sum\limits_{n=2}^{\infty}\left(n^2+n-2\right)\left(c_n \cos n\theta + \gamma_n \sin n\theta\right)R^{-n} - \sum\limits_{n=1}^{\infty}\left(n^2+n\right)\left(d_n \cos n\theta + \delta_n \sin n\theta\right)R^{-n-2} \\ \qquad = -\dfrac{\sigma^0_{11}+\sigma^0_{22}}{2} - \dfrac{\sigma^0_{11}-\sigma^0_{22}}{2}\cos 2\theta - \Delta p \\ \sum\limits_{n=2}^{\infty}\left(n^2-n\right)\left(-c_n \sin n\theta + \gamma_n \cos n\theta\right)R^{-n} + \sum\limits_{n=1}^{\infty}\left(n^2+n\right)\left(-d_n \sin n\theta + \delta_n \cos n\theta\right)R^{-n-2} \\ \qquad = \dfrac{\sigma^0_{11}-\sigma^0_{22}}{2}\sin 2\theta \end{cases} \tag{d94}$$

Hence

$$\begin{cases} \dfrac{c_0}{R^2} = -\dfrac{\sigma^0_{11}+\sigma^0_{22}}{2} - \Delta p \\ d_1 = \delta_1 = 0 \end{cases} \quad , \tag{d95}$$

and

$$\sum_{n=2}^{\infty}\left\{\left[\left(n^2+n-2\right)\frac{c_n}{R^n} + \left(n^2+n\right)\frac{d_n}{R^{n+2}}\right]\cos n\theta + \left[\left(n^2+n-2\right)\frac{\gamma_n}{R^n} + \left(n^2+n\right)\frac{\delta_n}{R^{n+2}}\right]\sin n\theta\right\} = \frac{\sigma^0_{11}-\sigma^0_{22}}{2}\cos 2\theta \tag{d96}$$

$$\sum_{n=2}^{\infty}\left\{-\left[\left(n^2-n\right)\frac{c_n}{R^n} + \left(n^2+n\right)\frac{d_n}{R^{n+2}}\right]\sin n\theta + \left[\left(n^2-n\right)\frac{\gamma_n}{R^n} + \left(n^2+n\right)\frac{\delta_n}{R^{n+2}}\right]\cos n\theta\right\} = \frac{\sigma^0_{11}-\sigma^0_{22}}{2}\sin 2\theta \; . \tag{d97}$$

It follows that

$$\begin{cases} \frac{c_0}{R^2} = -\frac{\sigma_{11}^0 + \sigma_{22}^0}{2} - \Delta p \\ 4\frac{c_2}{R^2} + 6\frac{d_2}{R^4} = \frac{\sigma_{11}^0 - \sigma_{22}^0}{2} \\ -2\frac{c_2}{R^2} - 6\frac{d_2}{R^4} = \frac{\sigma_{11}^0 - \sigma_{22}^0}{2} \end{cases} \quad \text{(d98}$$

and

$$\begin{cases} (n^2+n-2)\frac{c_n}{R^n} + (n^2+n)\frac{d_n}{R^{n+2}} = 0 \quad , \quad n = \overline{3,4,\ldots} \\ (n^2+n-2)\frac{\gamma_n}{R^n} + (n^2+n)\frac{\delta_n}{R^{n+2}} = 0 \quad , \quad n = \overline{2,3,\ldots} \\ (n^2-n)\frac{c_n}{R^n} + (n^2+n)\frac{d_n}{R^{n+2}} = 0 \quad , \quad n = \overline{3,4,\ldots} \\ (n^2-n)\frac{\gamma_n}{R^n} + (n^2+n)\frac{\delta_n}{R^{n+2}} = 0 \quad , \quad n = \overline{2,3,\ldots} \end{cases} \quad \text{(d99)}$$

Hence

$$\begin{cases} c_0 = -\left(\frac{\sigma_{11}^0 + \sigma_{22}^0}{2} + \Delta p\right) R^2 \\ c_2 = \frac{\sigma_{11}^0 - \sigma_{22}^0}{2} R^2 \quad , \quad c_n = 0 \quad , n = \overline{3,4,\ldots} \\ d_2 = -\left(\sigma_{11}^0 - \sigma_{22}^0\right)\frac{R^4}{4} \quad , \quad d_n = 0 \quad , n = \overline{3,4,\ldots} \\ \gamma_n = 0 \quad , \delta_n = 0 \quad , n = \overline{2,3,\ldots} \end{cases} \quad \text{(d100)}$$

By using (d11-d13) for $\sigma_{12}^0 = 0$, the expressions of the final stress are equal to

$$\sigma_{rr}^f = \frac{\sigma_{11}^0 + \sigma_{22}^0}{2} + \frac{\sigma_{11}^0 - \sigma_{22}^0}{2}\cos 2\theta - \left(\frac{\sigma_{11}^0 + \sigma_{22}^0}{2} + \Delta p\right)\frac{R^2}{r^2} - 2\left(\sigma_{11}^0 - \sigma_{22}^0\right)\frac{R^2}{r^2}\cos 2\theta + \frac{3}{2}\left(\sigma_{11}^0 - \sigma_{22}^0\right)\frac{R^4}{r^4}\cos 2\theta \quad \text{(d101)}$$

$$\sigma_{\theta\theta}^f = \frac{\sigma_{11}^0 + \sigma_{22}^0}{2} - \frac{\sigma_{11}^0 - \sigma_{22}^0}{2}\cos 2\theta + \left(\frac{\sigma_{11}^0 + \sigma_{22}^0}{2} + \Delta p\right)\frac{R^2}{r^2} - \frac{3}{2}\left(\sigma_{11}^0 - \sigma_{22}^0\right)\cos 2\theta\frac{R^4}{r^4} \quad \text{(d102)}$$

and

$$\sigma_{r\theta}^f = -\frac{\sigma_{11}^0 - \sigma_{22}^0}{2}\sin 2\theta - \left(\sigma_{11}^0 - \sigma_{22}^0\right)\frac{R^2}{r^2}\sin 2\theta + \frac{3}{2}\left(\sigma_{11}^0 - \sigma_{22}^0\right)\frac{R^4}{r^4}\sin 2\theta \quad \text{(d103)}$$

In real cases, the value of the stress at infinity are positive ones for compression.

D.11) The final shape of the wall.

Consider again the case when the direction of the horizontal axes of the co-ordinate system is along the corresponding eigen vectors of the initial stress $\overset{0}{\sigma}$. In that case, $\overset{0}{\sigma}_{12} = 0$. Using the above results, it follows the displacement vector for the points initially placed on the wall of the circular cavity is

$$\begin{cases} u_r(r=R,\theta) = \frac{1+\nu}{E} R\left[\frac{\overset{0}{\sigma}_{11}+\overset{0}{\sigma}_{22}}{2} + \Delta p + (3-4\nu)\frac{\overset{0}{\sigma}_{11}-\overset{0}{\sigma}_{22}}{2}\cos 2\theta\right] \\ u_\theta(r=R,\theta) = -\frac{1+\nu}{E} R(3-4\nu)\frac{\overset{0}{\sigma}_{11}-\overset{0}{\sigma}_{22}}{2}\sin 2\theta \end{cases} \quad \text{(d104)}$$

Consider an arbitrary point on the wall of the bore hole. In the initial state, it has the polar co-ordinates $(r = R, \theta)$. Its position vector with respect to the centre of the circle is

$$\vec{X} = R\left(\cos\theta\,\vec{e}_1 + \sin\theta\,\vec{e}_2\right) \quad \text{(d105)}$$

Using (d6) the position vector in the final stage is

$$\begin{aligned} \vec{\chi} &= \vec{X} + u_r(R,\theta)\vec{e}_r + u_\theta(R,\theta)\vec{e}_\theta = R\left(\cos\theta\,\vec{e}_1 + \sin\theta\,\vec{e}_2\right) \\ &+ \frac{1+\nu}{E} R\left[\frac{\overset{0}{\sigma}_{11}+\overset{0}{\sigma}_{22}}{2} + \Delta p + (3-4\nu)\frac{\overset{0}{\sigma}_{11}-\overset{0}{\sigma}_{22}}{2}\cos 2\theta\right]\left(\cos\theta\,\vec{e}_1 + \sin\theta\,\vec{e}_2\right) \\ &- \frac{1+\nu}{E} R(3-4\nu)\frac{\overset{0}{\sigma}_{11}-\overset{0}{\sigma}_{22}}{2}\sin 2\theta\left(-\sin\theta\,\vec{e}_1 + \cos\theta\,\vec{e}_2\right) \end{aligned} \quad \text{(d106)}$$

Hence

$$\begin{cases} x_1 = R\cos\theta + \frac{1+\nu}{E} R\left[\frac{\overset{0}{\sigma}_{11}+\overset{0}{\sigma}_{22}}{2} + \Delta p + (3-4\nu)\frac{\overset{0}{\sigma}_{11}-\overset{0}{\sigma}_{22}}{2}\cos 2\theta\right]\cos\theta \\ \qquad + \frac{1+\nu}{E} R(3-4\nu)\frac{\overset{0}{\sigma}_{11}-\overset{0}{\sigma}_{22}}{2}\sin 2\theta\sin\theta \\ x_2 = R\sin\theta + \frac{1+\nu}{E} R\left[\frac{\overset{0}{\sigma}_{11}+\overset{0}{\sigma}_{22}}{2} + \Delta p + (3-4\nu)\frac{\overset{0}{\sigma}_{11}-\overset{0}{\sigma}_{22}}{2}\cos 2\theta\right]\sin\theta \\ \qquad - \frac{1+\nu}{E} R(3-4\nu)\frac{\overset{0}{\sigma}_{11}-\overset{0}{\sigma}_{22}}{2}\sin 2\theta\cos\theta \end{cases} \quad \text{(d107)}$$

After elementary computations, it follows

$$\begin{cases} x_1 = \left[1+\frac{1+\nu}{E}\left(\frac{\sigma_{11}^0+\sigma_{22}^0}{2}+\Delta p+(3-4\nu)\frac{\sigma_{11}^0-\sigma_{22}^0}{2}\right)\right]X_1 \\ x_2 = \left[1+\frac{1+\nu}{E}\left(\frac{\sigma_{11}^0+\sigma_{22}^0}{2}+\Delta p-(3-4\nu)\frac{\sigma_{11}^0-\sigma_{22}^0}{2}\right)\right]X_2 \end{cases} \quad \text{(d108)}$$

Taking into account that $X_1^2+X_2^2=R^2$, it follows the final shape of the cavity is an ellipse of equation

$$\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=1 \quad , \quad \text{(d109)}$$

where the semi-axes of the ellipse are equal to

$$\begin{cases} a = R\left[1+\frac{1+\nu}{E}\left(\frac{\sigma_{11}^0+\sigma_{22}^0}{2}+\Delta p+(3-4\nu)\frac{\sigma_{11}^0-\sigma_{22}^0}{2}\right)\right] \\ b = R\left[1+\frac{1+\nu}{E}\left(\frac{\sigma_{11}^0+\sigma_{22}^0}{2}+\Delta p-(3-4\nu)\frac{\sigma_{11}^0-\sigma_{22}^0}{2}\right)\right] \end{cases} \quad \text{(d110)}$$

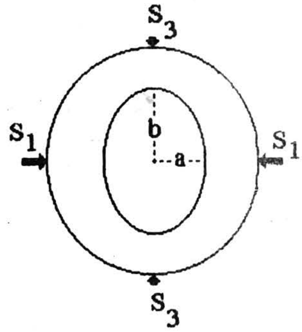

Fig.D2. The shape of the borehole (tunnel) in the initial state (the circle) and in the final state (the ellipse), corresponding to a compressive stress.

In real cases, the initial stress σ^0 is usually a compressive one, i.e. (see Fig.D2)

$$\sigma_{11}^0=-S_1 \quad , \quad \sigma_{22}^0=-S_3 \quad \text{(d111)}$$

where the maximum compressive stress S_1 and the minimum compressive stress S_3 have positive values $0 \le S_3 \le S_1$. It follows here that the major semi-axis **a** corresponds to the minimum stress and $a \le b$.

E) BOUSSINESQ'S PROBLEM - the concentrated force acting on the elastic semi-space

E.1) The equations of BELTRAMI and MITCHELL.

It follows to obtain the partial derivative equations for the stress tensor σ in the particular case of an elastic, homogeneous, isotropic media. In the beginning, the next symmetric tensor is evaluated

$$S = -\left[\operatorname{grad}(\rho \vec{b}) + \operatorname{grad}^t(\rho \vec{b})\right] , \tag{e1}$$

Using the equilibrium equation, it follows that

$$\operatorname{div}\sigma = -\rho \vec{b} \tag{e2}$$

Hence

$$S = \operatorname{grad}\left(\operatorname{div}\sigma\right) + \operatorname{grad}^t\left(\operatorname{div}\sigma\right) \tag{e3}$$

The HOOKE's reversed law gives

$$\varepsilon = \frac{1+\nu}{E}\sigma - \frac{\nu}{E}\Theta 1 , \tag{e4}$$

so

$$\sigma = \frac{E}{1+\nu}\varepsilon + \frac{\nu}{1+\nu}\Theta 1 , \tag{e5}$$

where

$$\Theta = \operatorname{tr}\sigma , \quad \vartheta = \operatorname{tr}\varepsilon = \frac{1-2\nu}{E}\Theta . \tag{e6}$$

But

$$\operatorname{div}(f 1) = \operatorname{grad} f . \tag{e7}$$

Hence

$$\operatorname{div}\sigma = \frac{E}{1+\nu}\operatorname{div}\varepsilon + \frac{\nu}{1+\nu}\operatorname{grad}\Theta \tag{e8}$$

Expression (e1) becomes

$$S = \frac{E}{1+\nu}\left[\operatorname{grad}(\operatorname{div}\varepsilon) + \operatorname{grad}^t(\operatorname{div}\varepsilon)\right] + \frac{\nu}{1+\nu}\left[\operatorname{grad}(\operatorname{grad}\Theta) + \operatorname{grad}^t(\operatorname{grad}\Theta)\right], \tag{e9}$$

Because the tensor $\operatorname{grad}(\operatorname{grad}\Theta)$ is a symmetric one, it follows that

$$S = \frac{E}{1+\nu}\left[\operatorname{grad}(\operatorname{div}\varepsilon) + \operatorname{grad}^t(\operatorname{div}\varepsilon)\right] + \frac{2\nu}{1+\nu}\operatorname{grad}(\operatorname{grad}\Theta) . \tag{e10}$$

The tensor in the first parenthesis of (e10) has the ij - component equal to

$$\left[\operatorname{grad}(\operatorname{div}\varepsilon) + \operatorname{grad}^t(\operatorname{div}\varepsilon)\right]_{ij} = (\operatorname{div}\varepsilon)_{i,j} + (\operatorname{div}\varepsilon)_{j,i} = \varepsilon_{iq,qj} + \varepsilon_{jq,qi} =$$

$$\frac{1}{2}\left(u_{i,qqj} + u_{q,iqj} + u_{j,qqi} + u_{q,jqi}\right) = \frac{1}{2}\left[\left(u_{i,j} + u_{j,i}\right)_{,qq} + 2\left(u_{q,q}\right)_{,ij}\right] = \tag{e11}$$

$$\left(\Delta\varepsilon\right)_{ij} + \left(\operatorname{tr}\varepsilon\right)_{,ij} = \left(\Delta\varepsilon\right)_{ij} + \frac{1-2\nu}{E}\Theta_{,ij} = \left(\Delta\varepsilon\right)_{ij} + \frac{1-2\nu}{E}\left[\operatorname{grad}(\operatorname{grad}\Theta)\right]_{ij}$$

Hence

$$\operatorname{grad}(\operatorname{div}\varepsilon) + \operatorname{grad}^t(\operatorname{div}\varepsilon) = \Delta\varepsilon + \frac{1-2\nu}{E}\operatorname{grad}(\operatorname{grad}\Theta) \tag{e12}$$

Using (e12) and (e10), the expression (e1) becomes

$$\mathbf{S} = \frac{E}{1+\nu}\Delta\boldsymbol{\varepsilon} + \frac{1}{1+\nu}\,\mathrm{grad\,grad}\Theta \tag{e13}$$

Applying the 3-dimensional LAPLACE operator Δ in (e4), it follows

$$\Delta\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\Delta\boldsymbol{\sigma} - \frac{\nu}{E}\Delta\Theta\mathbf{1} \tag{e14}$$

Replacing (e14) into (e13) gives

$$\mathbf{S} = \Delta\boldsymbol{\sigma} - \frac{\nu}{1+\nu}\Delta\Theta\mathbf{1} + \frac{1}{1+\nu}\,\mathrm{grad\,grad}\Theta\ , \tag{e15}$$

i.e.

$$\Delta\boldsymbol{\sigma} - \frac{\nu}{1+\nu}\Delta\Theta\mathbf{1} + \frac{1}{1+\nu}\,\mathrm{grad\,grad}\Theta = -\left[\mathrm{grad}(\rho\,\vec{b}) + \mathrm{grad}^t(\rho\,\vec{b})\right] \tag{e16}$$

But

$$\begin{aligned} &\mathrm{tr}(\Delta\boldsymbol{\sigma}) = \Delta(\mathrm{tr}\,\boldsymbol{\sigma}) = \Delta\Theta, \\ &\mathrm{tr}(\mathrm{grad\,grad}\,\Theta) = \Delta\Theta, \\ &\mathrm{tr}(\Delta\Theta\mathbf{1}) = 3\Delta\Theta, \\ &\mathrm{tr}\left[\mathrm{grad}(\rho\,\vec{b}) + \mathrm{grad}^t(\rho\,\vec{b})\right] = 2\mathrm{tr}\left[\mathrm{grad}(\rho\,\vec{b})\right] = 2\,\mathrm{div}(\rho\,\vec{b}) \end{aligned} \tag{e17}$$

Applying the trace operator in (e16) and using eqs.(e17) gives

$$\Delta\Theta + \frac{1}{1+\nu}\Delta\Theta - \frac{3\nu}{1+\nu}\Delta\Theta = -2\,\mathrm{div}(\rho\,\vec{b}) \tag{e18}$$

It follows the trace of the stress tensor verifies the relation

$$\Delta\Theta = -\frac{1+\nu}{1-\nu}\mathrm{div}(\rho\,\vec{b}) \tag{e19}$$

Replacing (e19) into (e16) gives

$$\Delta\boldsymbol{\sigma} + \frac{\nu}{1-\nu}\mathrm{div}(\rho\,\vec{b})\mathbf{1} + \frac{1}{1+\nu}\,\mathrm{grad\,grad}\Theta = -\left[\mathrm{grad}(\rho\,\vec{b}) + \mathrm{grad}^t(\rho\,\vec{b})\right] \tag{e20}$$

Eqs.(e19)-(e20) represents the equations of BELTRAMI and MITCHELL, having as unknowns only the elements of the stress tensor . Together with appropriate conditions (in tensions) on the boundary of the elastic body, they allow one to solve the corresponding linear static problem.

Particular cases.

a) Suppose that

$$\mathrm{div}(\rho\,\vec{b}) = 0\ , \tag{e21}$$

i.e. a vector potential $\vec{\psi}$ exists having the property that

$$\rho\,\vec{b} = \mathrm{rot}\,\vec{\psi}\ , \tag{e22}$$

From (e19) it follows that the traces of both stress and strain tensors are harmonic functions

$$\Delta\Theta = \Delta\theta = 0\ , \tag{e23}$$

b) Suppose that

$$\rho\,\vec{b} = \mathbf{grad}\,\varphi\ ,\ \text{where}\ \ \Delta\varphi = 0\ , \tag{e24}$$

it follows that

$$\mathrm{div}(\rho\,\vec{b}) = \Delta\varphi = 0\ , \tag{e25}$$

i.e. eq. (e23) is verified and eq.(20) gives

$$\Delta\boldsymbol{\sigma} + \frac{1}{1+\nu}\,\mathrm{grad\,grad}\Theta = -2\mathrm{grad}(\mathrm{grad}\varphi) \tag{e26}$$

Applying the LAPLACE operator Δ to eq.(e26), it follows that the stress tensor is the solution of the bi-harmonic equation

$$\Delta\Delta\sigma = 0 \quad \text{(e27)}$$

In most real cases, the volume forces are neglected (or they are represented only by the weight of the body, satisfying eq.(24). It follows the elastic linear problem involve solving harmonic and bi-harmonic equations.

E.2) The model.

A co-ordinate system having the third vertical axis positive downward is used. The semi-space $x_3 \geq 0$ is represented by an elastic, homogeneous, isotropic medium having the elastic coefficients λ and μ (or E and ν respectively). In the origin of the co-ordinate system is acting a vertical force having the magnitude equal to P. It follows to find the stress and the displacements. Spherical co-ordinates will be used (Fig.E1):

$$x_1 = r\sin\vartheta\cos\lambda,\; x_2 = r\sin\vartheta\sin\lambda,\; x_3 = r\cos\vartheta \quad \text{(e28)}$$

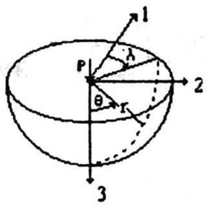

Fig.E1 The spherical co-ordinate system and a vertical force of magnitude equal to P acting at the origin.

Because symmetry, the displacement vector has the components like

$$u_r = u_r(r,\vartheta),\; u_\lambda = 0,\; u_\vartheta = u_\vartheta(r,\vartheta) \quad \text{(e29)}$$

It follows the components of both stress and strain tensors are functions of r and ϑ only.

E.3) The equations of equilibrium and strain tensor in spherical co-ordinates.

The LAMÉ differential parameters for spherical co-ordinates are

$$h^1 = 1,\; h^2 = r,\; h^3 = r\sin\vartheta \quad \text{(e30)}$$

The generalised curvilinear co-ordinates are

$$c^1 = r,\; c^2 = \vartheta,\; c^3 = \lambda \quad \text{(e31)}$$

The unit vectors of the axis are

$$\vec{n}^1 = \vec{e}_r,\; \vec{n}^2 = \vec{e}_\vartheta,\; \vec{n}^3 = \vec{e}_\lambda \quad \text{(e32)}$$

By using, for example, (IVAN, 1996), the divergence of a symmetric tensor T in spherical co-ordinates is

$$\operatorname{div} T =$$

$$\left(\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\vartheta}}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial T_{r\lambda}}{\partial\lambda} + \frac{2T_{rr}}{r} + \frac{T_{r\vartheta}}{r\tan\vartheta} - \frac{T_{\vartheta\vartheta}+T_{\lambda\lambda}}{r}\right)\vec{e}_r +$$

$$\left(\frac{\partial T_{r\vartheta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\vartheta\vartheta}}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial T_{\vartheta\lambda}}{\partial\lambda} + \frac{3T_{r\vartheta}}{r} + \frac{T_{r\vartheta}}{r\tan\vartheta} + \frac{T_{\vartheta\vartheta}-T_{\lambda\lambda}}{r\tan\vartheta}\right)\vec{e}_\vartheta + \quad \text{(e33)}$$

$$\left(\frac{\partial T_{r\lambda}}{\partial r} + \frac{1}{r}\frac{\partial T_{\vartheta\lambda}}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial T_{\lambda\lambda}}{\partial\lambda} + \frac{3T_{r\lambda}}{r} + \frac{2T_{\vartheta\lambda}}{r\tan\vartheta}\right)\vec{e}_\lambda$$

In the same way, the gradient of a vector in spherical co-ordinates is

$$\operatorname{grad}\vec{v}=\frac{\partial v_r}{\partial r}\vec{e_r}\otimes\vec{e_r}+\left(\frac{1}{r}\frac{\partial v_r}{\partial \vartheta}-\frac{v_\vartheta}{r}\right)\vec{e_r}\otimes\vec{e_\vartheta}+\left(\frac{1}{r\sin\vartheta}\frac{\partial v_r}{\partial\lambda}-\frac{v_\lambda}{r}\right)\vec{e_r}\otimes\vec{e_\lambda}+$$

$$\frac{\partial v_\vartheta}{\partial r}\vec{e_\vartheta}\otimes\vec{e_r}+\left(\frac{1}{r}\frac{\partial v_\vartheta}{\partial \vartheta}+\frac{v_r}{r}\right)\vec{e_\vartheta}\otimes\vec{e_\vartheta}+\left(\frac{1}{r\sin\vartheta}\frac{\partial v_\vartheta}{\partial\lambda}-\frac{v_\lambda}{r\tan\vartheta}\right)\vec{e_\vartheta}\otimes\vec{e_\lambda}+ \tag{e34}$$

$$\frac{\partial v_\lambda}{\partial r}\vec{e_\lambda}\otimes\vec{e_r}+\frac{1}{r}\frac{\partial v_\lambda}{\partial \vartheta}\vec{e_\lambda}\otimes\vec{e_\vartheta}+\left(\frac{1}{r\sin\vartheta}\frac{\partial v_\lambda}{\partial\lambda}+\frac{v_r}{r}+\frac{v_\vartheta}{r\tan\vartheta}\right)\vec{e_\lambda}\otimes\vec{e_\lambda}$$

The components of the strain tensor are

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r},\ \varepsilon_{r\vartheta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\vartheta}-\frac{u_\vartheta}{r}+\frac{\partial u_\vartheta}{\partial r}\right),$$

$$\varepsilon_{r\lambda}=\frac{1}{2}\left(\frac{1}{r\sin\vartheta}\frac{\partial u_r}{\partial\lambda}-\frac{u_\lambda}{r}+\frac{\partial u_\lambda}{\partial r}\right),\ \varepsilon_{\vartheta\vartheta}=\frac{1}{r}\frac{\partial u_\vartheta}{\partial\vartheta}+\frac{u_r}{r}, \tag{e35}$$

$$\varepsilon_{\vartheta\lambda}=\frac{1}{2}\left(\frac{1}{r\sin\vartheta}\frac{\partial u_\vartheta}{\partial\lambda}-\frac{u_\lambda}{r\tan\vartheta}+\frac{1}{r}\frac{\partial u_\lambda}{\partial\vartheta}\right),\ \varepsilon_{\lambda\lambda}=\frac{1}{r\sin\vartheta}\frac{\partial u_\lambda}{\partial\lambda}+\frac{u_r}{r}+\frac{u_\vartheta}{r\tan\vartheta}$$

In the particular case of eqs.(e29), it follows

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r},\quad \varepsilon_{r\vartheta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\vartheta}-\frac{u_\vartheta}{r}+\frac{\partial u_\vartheta}{\partial r}\right),\quad \varepsilon_{r\lambda}=0$$

$$\varepsilon_{\vartheta\vartheta}=\frac{1}{r}\frac{\partial u_\vartheta}{\partial\vartheta}+\frac{u_r}{r},\quad \varepsilon_{\vartheta\lambda}=0,\quad \varepsilon_{\lambda\lambda}=\frac{u_r}{r}+\frac{u_\vartheta}{r\tan\vartheta} \tag{e36}$$

Hence

$$\sigma_{r\lambda}=\sigma_{\vartheta\lambda}=0\ , \tag{e37}$$

By using (e33), the equilibrium equations in the absence of the volume forces are

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\vartheta}}{\partial\vartheta}+\frac{3\sigma_{rr}}{r}+\frac{\sigma_{r\vartheta}}{r\tan\vartheta}=\frac{\sigma_{rr}+\sigma_{\vartheta\vartheta}+\sigma_{\lambda\lambda}}{r}\ , \tag{e38}$$

$$\frac{\partial\sigma_{r\vartheta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\vartheta\vartheta}}{\partial\vartheta}+\frac{3\sigma_{r\vartheta}}{r}+\frac{\sigma_{r\vartheta}}{r\tan\vartheta}+\frac{\sigma_{\vartheta\vartheta}-\sigma_{\lambda\lambda}}{r\tan\vartheta}=0$$

i.e.

$$\frac{\partial}{\partial r}\left(r^3\sin\vartheta\,\sigma_{rr}\right)+\frac{\partial}{\partial\vartheta}\left(r^2\sin\vartheta\,\sigma_{r\vartheta}\right)=r^2\sin\vartheta\,\Theta,$$

$$\frac{\partial}{\partial r}\left(r^3\sin\vartheta\,\sigma_{r\vartheta}\right)+\frac{\partial}{\partial\vartheta}\left(r^2\sin\vartheta\,\sigma_{\vartheta\vartheta}\right)=r^2\cos\vartheta\,\sigma_{\lambda\lambda} \tag{e39}$$

E.4) LAPLACE operator in spherical co-ordinates. LEGENDRE's polynomials.

Using, for example, (IVAN, 1996) the gradient of a scalar function in spherical co-ordinates is

$$\operatorname{grad} f=\frac{\partial f}{\partial r}\vec{e_r}+\frac{1}{r}\frac{\partial f}{\partial\vartheta}\vec{e_\vartheta}+\frac{1}{r\sin\Delta}\frac{\partial f}{\partial\lambda}\vec{e_\lambda} \tag{e40}$$

Also, the LAPLACE operator is

$$\Delta f=\operatorname{div}\operatorname{grad} f=\frac{1}{r^2\sin\vartheta}\left[\sin\vartheta\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial f}{\partial\vartheta}\right)+\frac{1}{\sin\vartheta}\frac{\partial^2 f}{\partial\lambda^2}\right] \tag{e41}$$

Consider the LAPLACE equation

$$\Delta f = 0 \tag{e42}$$

for the particular case when the unknown function has the form $f = f(r, \vartheta)$. Equation (e42) becomes

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial f}{\partial \vartheta}\right) = 0 . \tag{e43}$$

By using the method of separation of the variables, a solution for eq.(43) has the form

$$f(r, \vartheta) = R(r)\, Y(\vartheta) , \tag{e44}$$

where R and Y are two unknown functions. Eq.(e43) becomes

$$\frac{\dfrac{d}{dr}\left(r^2 \dfrac{dR}{dr}\right)}{R} = -\frac{\dfrac{d}{d\vartheta}\left(\sin\vartheta \dfrac{dY}{d\vartheta}\right)}{Y \sin\vartheta} = k , \tag{e45}$$

where k is a constant. From (e45) it follows that

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} - kR = 0 \tag{e46}$$

In a general case, the function R can be developed in power series. Let's look for a particular solution having the form $R_n(r) = r^n$ where n is a natural number. It follows from (e46) that

$$k = n(n+1) \tag{e47}$$

Then the particular solution of eq. (e46) can be expressed with the aid of two arbitrary constants as

$$R_n(r) = A_n\, r^n + B_n / r^n \tag{e48}$$

Because f has to approach finite values for $r \to \infty$, it has to take $A_n = 0$.
The second relation (e45) gives

$$\sin\vartheta \frac{d^2 Y}{d\vartheta^2} + \cos\vartheta \frac{dY}{d\vartheta} + n(n+1) Y \sin\vartheta = 0 \tag{e49}$$

By performing the substitution $z = \cos\vartheta$ eq.(e49) becomes

$$\frac{d}{dz}\left[\left(1 - z^2\right)\frac{dY}{dz}\right] + n(n+1)Y = 0, \tag{e50}$$

The solution of (e50) is represented by the LEGENDRE polynomials denoted by $P_n(z), n = 0,1,2,\ldots$ So,

$$P_0(z) = 1 \quad , \quad P_1(z) = z \quad , \quad P_2(z) = (3z^2 - 1)/2 \tag{e51}$$

Because the trace Θ of the stress tensor is a solution of the harmonic equation (e23), it follows that the general solution for that trace in the case of the BOUSSINESQ problem is

$$\Theta = \sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} P_n(\cos\vartheta) . \tag{e52}$$

According to eq.(e6), a similar solution exists for the trace of strain tensor.

E.5) The displacement field.

We look for a displacement field having the form

$$u_r = \frac{1}{r}\varphi(\vartheta) \quad , \quad u_\vartheta = \frac{1}{r}\psi(\vartheta) \quad , \quad u_\lambda = 0 \quad , \tag{e53}$$

where φ and ψ are two unknown functions following to be obtained. Substituting (e53) into (e36) it follows that

$$\varepsilon_{rr} = -\varphi / r^2 \ , \ \varepsilon_{r\vartheta} = (d\varphi / d\vartheta - 2\psi)/\left(2r^2\right) \ , \ \varepsilon_{r\lambda} = 0$$
$$\varepsilon_{\vartheta\vartheta} = (\varphi + d\psi / d\vartheta)/ r^2 \ , \ \varepsilon_{\vartheta\lambda} = 0 \ , \ \varepsilon_{\lambda\lambda} = (\varphi + \psi / tg\vartheta)/ r^2 \tag{e54}$$

It follows that the trace of the strain tensor is

$$\theta = tr\,\varepsilon = \varepsilon_{rr} + \varepsilon_{\vartheta\vartheta} + \varepsilon_{\lambda\lambda} = \left(\varphi + d\psi / d\vartheta + \psi / tg\vartheta\right)/ r^2 \tag{e55}$$

A comparison of (e55) to (e52) shows that

$$\varphi + d\psi / d\vartheta + \psi / tg\vartheta = a\cos\vartheta \ , \tag{e56}$$

where a is a constant remaining to be obtained. Hence

$$\theta = a\cos\vartheta / r^2 \ , \ \Theta = (3\lambda + 2\mu)\theta = (3\lambda + 2\mu)\, a\cos\vartheta / r^2 \ ,$$
$$\varepsilon_{\lambda\lambda} = \theta - (\varepsilon_{rr} + \varepsilon_{\vartheta\vartheta}) = (a\cos\vartheta - d\psi / d\vartheta) / r^2 \tag{e57}$$

By using HOOKE's law, it follows that

$$\sigma_{rr} = (\lambda\, a\cos\vartheta - 2\mu\varphi) / r^2 \ , \ \sigma_{r\vartheta} = \mu(d\varphi / d\vartheta - 2\psi) / r^2 \ ,$$
$$\sigma_{\vartheta\vartheta} = [\lambda\, a\cos\vartheta + 2\mu(\varphi + d\psi / d\vartheta)] / r^2 \ , \tag{e58}$$
$$\sigma_{\lambda\lambda} = [(\lambda + 2\mu)\, a\cos\vartheta - 2\mu d\psi / d\vartheta] / r^2$$

Substituting (e58) into the first equilibrium equation (e39) and using (e56), it follows after elementary computations that

$$\frac{d}{d\vartheta}\left(\sin\vartheta \frac{d\varphi}{d\vartheta}\right) = a\,(2 + \lambda / \mu)\sin 2\vartheta \ . \tag{e59}$$

Hence

$$\sin\vartheta \frac{d\varphi}{d\vartheta} = -\frac{a}{2}(2 + \lambda / \mu)\cos 2\vartheta + b = b - \frac{a}{2}(2 + \lambda / \mu) + a\,(2 + \lambda / \mu)\sin^2\vartheta \ ,$$
$$\frac{d\varphi}{d\vartheta} = \frac{b - a\,(2 + \lambda / \mu) / 2}{\sin\vartheta} + a\,(2 + \lambda / \mu)\sin\vartheta \tag{e60}$$

where b is an integration constant. Because

$$\int \frac{dx}{\sin x} = \ln\left| tg\frac{x}{2} \right| + C \ , \tag{e61}$$

where C is a new integration constant, it follows that

$$\varphi = \left[b - \frac{a}{2}(2 + \lambda / \mu)\right]\ln\left| tg\frac{\vartheta}{2} \right| - a\,(2 + \lambda / \mu)\cos\vartheta + C \ . \tag{e62}$$

For $\vartheta \to \pi / 4$, the logarithmic term into (e62) leads to infinite radial displacements. That can be avoided by taking

$$b - \frac{a}{2}(2 + \lambda / \mu) = 0 \ . \tag{e63}$$

Hence

$$\varphi = -a\,(2 + \lambda / \mu)\cos\vartheta + C \quad , \quad \frac{d\varphi}{d\vartheta} = a\,(2 + \lambda / \mu)\sin\vartheta \ . \tag{e64}$$

and

$$\sigma_{rr} = [(3\lambda + 4\mu) a\cos\vartheta - 2\mu C] / r^2 \ . \tag{e65}$$

Substituting eq.(e64) into eq.(e56), it follows that

$$\frac{d\psi}{d\vartheta} + \frac{\psi}{tg\vartheta} = a\,(3 + \lambda / \mu)\cos\vartheta + C \ ,$$
$$\frac{d}{d\vartheta}(\psi\sin\vartheta) = \frac{a}{2}(3 + \lambda / \mu)\sin 2\vartheta - C\sin\vartheta \ ,$$
$$\psi\sin\vartheta = -\frac{a}{4}(3 + \lambda / \mu)\cos 2\vartheta + C\cos\vartheta + D \tag{e66}$$
$$= D - \frac{a}{4}(3 + \lambda / \mu) + \frac{a}{2}(3 + \lambda / \mu)\sin^2\vartheta + C\cos\vartheta$$

Hence

$$\psi = \frac{D - a\,(3 + \lambda / \mu) / 4}{\sin\vartheta} + \frac{a}{2}(3 + \lambda / \mu)\sin\vartheta + C\,ctg\vartheta \ . \tag{e67}$$

Because the tangential displacements has to be finite ones for $\vartheta \to 0$, it follows that

$$\psi = C\frac{\cos\vartheta - 1}{\sin\vartheta} + \frac{a}{2}(3+\lambda/\mu)\sin\vartheta = -C\,\mathrm{tg}\frac{\vartheta}{2} + \frac{a}{2}(3+\lambda/\mu)\sin\vartheta,$$
$$\frac{d\psi}{d\vartheta} = -\frac{C}{2\cos^2(\vartheta/2)} + \frac{a}{2}(3+\lambda/\mu)\cos\vartheta \tag{e68}$$

By substituting eqs.(e64) and (e68) into eq.(e58) it follows that

$$\sigma_{r\vartheta} = \mu\left(2C\,\mathrm{tg}\frac{\vartheta}{2} - a\sin\vartheta\right)/r^2 \quad , \quad \sigma_{\vartheta\vartheta} = \mu\left[C\left(1-\mathrm{tg}^2\frac{\vartheta}{2}\right) - a\cos\vartheta\right]/r^2,$$
$$\sigma_{\lambda\lambda} = \mu\left[C\left(1+\mathrm{tg}^2\frac{\vartheta}{2}\right) - a\cos\vartheta\right]/r^2 \tag{e69}$$

Substituting eqs.(e69) into the second equilibrium equation (e39) it can be seen that the last one is identically verified.

E.6) Boundary conditions for stress elements. The final solutions.

Eqs.(e65) and (e69) contain the unknown coefficients a and C. These constants follow to be obtained taking into account that the force P concentrated in the origin of the co-ordinate system is acting on the elastic semi-space. It can be seen that the points of the horizontal plane $x_3 = 0$ have the co-latitude $\vartheta = \pi/2$.

The unit vectors of the spherical co-ordinate system are related to the same vectors of the rectangular co-ordinate system by

$$\begin{pmatrix}\vec{e}_r\\ \vec{e}_\vartheta\\ \vec{e}_\lambda\end{pmatrix} = Q\begin{pmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{pmatrix} \quad , \quad \begin{pmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{pmatrix} = Q^t\begin{pmatrix}\vec{e}_r\\ \vec{e}_\vartheta\\ \vec{e}_\lambda\end{pmatrix}, \tag{e70}$$

where the orthogonal matrix is

$$Q = \begin{pmatrix}\sin\vartheta\cos\lambda & \sin\vartheta\sin\lambda & \cos\vartheta\\ \cos\vartheta\cos\lambda & \cos\vartheta\sin\lambda & -\sin\vartheta\\ -\sin\lambda & \cos\lambda & 0\end{pmatrix}. \tag{e71}$$

The outer pointing unit vector normal to the elastic semi-space is equal to

$$-\vec{e}_3 = -\cos\vartheta\,\vec{e}_r + \sin\vartheta\,\vec{e}_\vartheta. \tag{e72}$$

The resulting exterior force acting on the elastic semi-space is vanishing for all the points of the horizontal plane $\vartheta = \pi/2$, excepting the origin, i.e.

$$\sigma\left(-\vec{e}_3\right) = \vec{0}. \tag{e73}$$

Substituting (e72) into (e73) for $\vartheta = \pi/2$ gives

$$\sigma_{\vartheta\vartheta} = 0 \quad , \quad \sigma_{r\vartheta} = 0 . \tag{e74}$$

By using (e69), the first eq.(e74) becomes an identity and the second one leads to

$$a = 2C . \tag{e75}$$

So, eqs.(e65) and (e69) give

$$\sigma_{rr} = 2C\left[(3\lambda+4\mu)\cos\vartheta - \mu\right]/r^2 , \tag{e76}$$

$$\sigma_{r\vartheta} = -2C\mu\,\mathrm{tg}\frac{\vartheta}{2}\cos\Delta/r^2 \quad , \quad \sigma_{\vartheta\vartheta} = C\mu\cos\vartheta\left(\mathrm{tg}^2\frac{\vartheta}{2} - 1\right)/r^2. \tag{e77}$$

Consider an elastic hemisphere having the centre at the origin of the co-ordinate system. The curved surface S of the hemisphere has the outer pointing normal equal to $\vec{e}_r$. On that surface, the rest of the elastic body (i.e. the semi-space minus the hemisphere) is acting on the hemisphere with a total force equal to

$$\iint_S \sigma \ \vec{e}_r \, dA = \iint_S \left(\sigma_{rr} \vec{e}_r + \sigma_{r\vartheta} \vec{e}_\vartheta \right) dA \ , \tag{e78}$$

where dA is the surface element and the unit vectors are obtained with (e70). It follows that

$$\iint_S \sigma \ \vec{e}_r \, dA = \int_0^{2\pi} d\lambda \int_0^{\pi/2} \sigma \ \vec{e}_r \, r^2 \sin\vartheta d\vartheta$$
$$= 4\pi C \int_0^{\pi/2} \left\{ \left[(3\lambda + 4\mu)\cos\vartheta - \mu \right] \cos\vartheta \sin\vartheta + \mu \operatorname{tg}\frac{\vartheta}{2} \cos\vartheta \sin^2\vartheta \right\} d\vartheta \, \vec{e}_3 = 4\pi C(\lambda + \mu) \vec{e}_3 \tag{e79}$$

Because the hemisphere is into an equilibrium state, it follows that

$$\iint_S \sigma \ \vec{e}_r \, dA + P\vec{e}_3 = \vec{0} \ , \tag{e80}$$

Hence

$$C = -\frac{P}{4\pi(\lambda + \mu)} \ . \tag{e81}$$

Finally, the non-zero components of the stress tensor are equal to

$$\sigma_{rr} = -\frac{P}{2\pi(\lambda + \mu)} \left[(3\lambda + 4\mu)\cos\vartheta - \mu \right] / r^2 \ , \tag{e82}$$

$$\sigma_{r\Delta} = \frac{P}{2\pi(\lambda + \mu)} \mu \operatorname{tg}\frac{\vartheta}{2} \cos\vartheta \, / \, r^2 \ , \tag{e83}$$

$$\sigma_{\vartheta\vartheta} = \frac{P}{4\pi(\lambda + \mu)} \mu \cos\vartheta \left(1 - \operatorname{tg}^2 \frac{\vartheta}{2} \right) / \, r^2 . \tag{e84}$$

The non-zero components of the displacement vector are

$$u_r = \frac{P}{4\pi(\lambda + \mu)} \left[2(2 + \lambda / \mu)\cos\vartheta - 1 \right] / \, r \ ,$$
$$u_\vartheta = \frac{P}{4\pi(\lambda + \mu)} \left[\operatorname{tg}\frac{\vartheta}{2} - (3 + \lambda / \mu)\sin\vartheta \right] / \, r \tag{e85}$$

Using eq.(e71), the components of the stress tensor into the Cartesian base can be obtained as

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ & & \sigma_{33} \end{pmatrix} = Q^t \begin{pmatrix} \sigma_{rr} & \sigma_{r\vartheta} & 0 \\ \sigma_{r\vartheta} & \sigma_{\vartheta\vartheta} & 0 \\ 0 & 0 & \sigma_{\lambda\lambda} \end{pmatrix} Q \tag{e86}$$

Also, the components of the displacement vector into the Cartesian base are

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = Q^t \begin{pmatrix} u_r \\ u_\vartheta \\ 0 \end{pmatrix} \ . \tag{e87}$$

Of particular importance in real life are the components

$$\sigma_{33} = -\frac{3P}{2\pi} z^3 \, / \, r^5 \quad , \quad u_3 = \frac{1 + \nu}{E} \frac{P}{2\pi} \left[2(1 - \nu)\frac{1}{r} + \frac{z^2}{r^3} \right] \ . \tag{e88}$$

The BOUSSINESQ problem has a great importance in Geomechanics, in relation to the computation of a building foundation. The above solution derived for a concentrated vertical force can be used in the case of arbitrary vertical forces (spread on a certain domain) by assuming the principle of the superposition.

F) ELEMENTS OF THIN PLATES THEORY

F.1) The model of a thin elastic plane plate.

The thin plane plate is a cylindrical body having an arbitrary horizontal cross section in the initially non-deformed state. Its height, denoted by $H = 2h$, is much smaller than the other dimensions (usually, around 7-10 times). The material of the plate is an elastic, homogenous, isotropic one, having the constants denoted by E and ν, respectively λ and μ. The third axis of the co-ordinate system, denoted by Oz, is a vertical one, positive downward. Let be $x_1 = x$, $x_2 = y$. In the initial, non-deformed state, the plate is a plane horizontal one. The upper face has the equation $z = -h$ and the down face has $z = h$ respectively. The plane of equation $z = 0$ represents the *median plane*. After the plate is deformed, it becomes a *median surface*.

F.2) The planar state of a plate. The bending state.

Two particular situation for a deformed plate are considered (Fig.F1).

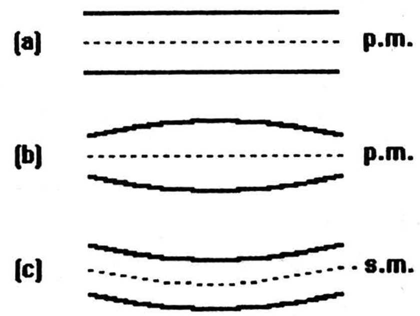

Fig.F1. (a) The non-deformed plate; (b) The plate into a planar state; (c) The plate into a bending state. Here, the median plane is denoted by **m.p.** while **m.s.** is the median surface.

In the first case, the horizontal components of the displacement vector are symmetrical ones with respect to the median plane, being even functions with respect to the z-variable, while the vertical component of the displacement vector is an anti-symmetrical one (odd function with respect to z), i.e.

$$u_k(x,y,-z) = u_k(x,y,z),\ k = 1,2$$
$$u_3(x,y,-z) = -u_3(x,y,-z). \tag{f1}$$

From eq.(f1b) it follows that

$$u_3(x,y,0) = 0 \quad , \tag{f2}$$

i.e. the material points placed initially into the median plane have no vertical displacement as a consequence of the deformation of the plate (the median plane holds a horizontal plane one). This kind of deformation represents the *planar state* of the plate.

In the second case, the horizontal components of the displacement vector are anti-symmetrical ones (odd functions with respect to z), while the vertical component of the displacement vector is a symmetrical one with respect to the median plane (even function with respect to z), i.e.

$$u_k(x,y,-z) = -u_k(x,y,z),\ k = 1,2$$
$$u_3(x,y,-z) = u_3(x,y,-z). \tag{f3}$$

By deformation, the material points initially placed into the median plane are displaced on the vertical direction, having no horizontal movement. The median plane becomes a median surface. This state represents the *bending state* of a plate.

F.3) Loads acting on the plate.

For simplicity, the forces acting on the lateral surface of the plate are neglected. On the upper surface of the plate, having the equation $z=-h$ and the outward pointing normal vector $\vec{n}=-\vec{e}_3$, it is acting the surface force $\vec{q}^{\,s}=\vec{q}^{\,s}(x,y)$. In the same way, on the down surface, having the equation $z=h$ and the normal vector $\vec{n}=\vec{e}_3$, it is acting the surface force $\vec{q}^{\,j}=\vec{q}^{\,j}(x,y)$. Hence

$$\sigma\left(-\vec{e}_3\right)=\vec{q}^{\,s}\text{, for } z=-h\quad, \tag{f4}$$

i.e.

$$\sigma_{13}(x,y,-h)=-q_1^s(x,y),\sigma_{23}(x,y,-h)=-q_2^s(x,y),\sigma_{33}(x,y,-h)=-q_3^s(x,y) \tag{f5}$$

Also

$$\sigma\,\vec{e}_3=\vec{q}^{\,j}\text{, for } z=h\quad, \tag{f6}$$

i.e.

$$\sigma_{13}(x,y,h)=q_1^j(x,y),\sigma_{23}(x,y,h)=q_2^j(x,y),\sigma_{33}(x,y,h)=q_3^j(x,y). \tag{f7}$$

We shell see that the above presented deformation states are compatible only to certain distributions of volume or surface forces applied to the plate.

F.4) Odd and even functions for the planar state and for the bending state.

Let $f=f(x,y,z)$ be a function of three variables, supposed to be smooth enough. It can be seen that

$$f(x,y,z)=\frac{f(x,y,z)+f(x,y,-z)}{2}+\frac{f(x,y,z)-f(x,y,-z)}{2} \tag{f8}$$

Let

$$f^+(x,y,z)=\frac{f(x,y,z)+f(x,y,-z)}{2}\quad, \tag{f9}$$

$$f^-(x,y,z)=\frac{f(x,y,z)-f(x,y,-z)}{2} \tag{f10}$$

It follows that

$$f^+(x,y,-z)=f^+(x,y,z)\quad,\quad f^-(x,y,-z)=-f^-(x,y,z)\ . \tag{f11}$$

The function f^+ represents the even part of f (with respect to z-variable), while the function f^- represents its odd part. It follows that

$$\left(f^+\right)^+=f^+\quad,\quad\left(f^-\right)^-=f^-\quad,\quad\left(f^+\right)^-=\left(f^-\right)^+=0\quad, \tag{f12}$$

i.e. the even part of the even part is equal to the even part too. A similar relation holds for the odd part. The even part of an odd part (and the odd part of an even part) is vanishing. For $k=1,2$, $x_1=x$, $x_2=y$, it follows that

$$\left(f^+\right)_{,k}=\frac{\partial}{\partial x_k}f^+(x,y,z)=\frac{1}{2}\left[\frac{\partial}{\partial x_k}f(x,y,z)+\frac{\partial}{\partial x_k}f(x,y,-z)\right]=\left[f_{,k}(x,y,z)\right]^+ \tag{f13}$$

Hence, the partial derivative (with respect to a horizontal co-ordinate) of the even part of a certain function is equal to the even part of that derivative. This property holds for the odd part. So,

$$\left(f^+\right)_{,k}=\left(f_{,k}\right)^+\quad,\quad\left(f^-\right)_{,k}=\left(f_{,k}\right)^- \tag{f14}$$

Also,

$$\left(f^{+}\right)_{,3}=\frac{1}{2}\left\{\frac{\partial}{\partial z}\left[f(x,y,z)+f(x,y,-z)\right]\right\}=\frac{1}{2}\left[f_{,3}(x,y,z)-f_{,3}(x,y,-z)\right]=\left(f_{,3}\right)^{-} \quad \text{(f15)}$$

Hence the partial derivative of the even part with respect to the vertical co-ordinate is equal to the odd part of the partial derivative of the function itself with respect to same co-ordinate. In the same way it follows that

$$\left(f^{-}\right)_{,3}=\left(f_{,3}\right)^{+} \quad . \quad \text{(f16)}$$

By using the HOOKE's law and the definition of the strain, it can be resumed that the planar state and the bending state are characterised by the next components:

-for the *planar state*: - the displacement vector: $u_1^{+}, u_2^{+}, u_3^{-}$,

- strain tensor: $\varepsilon_{11}^{+}, \varepsilon_{12}^{+}, \varepsilon_{13}^{-}, \varepsilon_{22}^{+}, \varepsilon_{23}^{-}, \varepsilon_{33}^{+}, \vartheta^{+}$,

- stress tensor: $\sigma_{11}^{+}, \sigma_{12}^{+}, \sigma_{13}^{-}, \sigma_{22}^{+}, \sigma_{23}^{-}, \sigma_{33}^{+}$.

-for the *bending state*: - the displacement vector: $u_1^{-}, u_2^{-}, u_3^{+}$,

- strain tensor: $\varepsilon_{11}^{-}, \varepsilon_{12}^{-}, \varepsilon_{13}^{+}, \varepsilon_{22}^{-}, \varepsilon_{23}^{+}, \varepsilon_{33}^{-}, \vartheta^{-}$,

- stress tensor: $\sigma_{11}^{-}, \sigma_{12}^{-}, \sigma_{13}^{+}, \sigma_{22}^{-}, \sigma_{23}^{+}, \sigma_{33}^{-}$.

F5) Mean value of a function. Equilibrium equations for thin plates.

Let $f=f(x,y,z)$ an integrable function with respect to z-variable. The mean value of f computed on the thickness of the plate is denoted by

$$\bar{f}(x,y)=\frac{1}{2h}\int_{-h}^{+h} f(x,y,z)dz \quad . \quad \text{(f17)}$$

For an odd function f, its mean value vanishes, i.e.

$$\overline{f^{-}}(x,y)=0 \quad , \quad \text{(f18)}$$

It follows that

$$\bar{f}=\overline{f^{+}+f^{-}}=\overline{f^{+}}+\overline{f^{-}}=\overline{f^{+}} \quad \text{(f19)}$$

For a function $C=C(x,y)$ depending only on the horizontal co-ordinates, eq.(f17) leads to

$$\overline{C}=C \quad , \quad \overline{zC}=0 \quad . \quad \text{(f20)}$$

Differentiating eq.(f17) with respect to the horizontal co-ordinates, it follows that:

$$\overline{f_{,k}}=\left(\bar{f}\right)_{,k} \quad , \quad k=1,2 \quad . \quad \text{(f21)}$$

For the vertical derivative, it follows that

$$\overline{f_{,3}}=\frac{1}{2h}\int_{-h}^{+h}\frac{\partial f}{\partial z}dz=\frac{1}{2h}\left[f(x,y,h)-f(x,y,-h)\right]=\frac{1}{h}f^{-}(x,y,z=h) \quad . \quad \text{(f22)}$$

Consider the function $zf(x,y,z)$. It follows that

$$zf^{+}=(zf)^{-} \quad , \quad zf^{-}=(zf)^{+} \quad . \quad \text{(f23)}$$

Its mean value is

$$\overline{zf}=\overline{z(f^{+}+f^{-})}=\overline{zf^{+}}+\overline{zf^{-}}=\overline{zf^{-}}=\overline{(zf)^{+}} \quad . \quad \text{(f24)}$$

Hence

$$\overline{zf_{,3}} = \frac{1}{2h}\int_{-h}^{+h} z\frac{\partial f}{\partial z}dz = \frac{1}{2h}\int_{-h}^{+h}\left[\frac{\partial}{\partial z}(zf) - f\right]dz = \frac{1}{2h}\left[(zf)_{-h}^{+h}\right] - \frac{1}{2h}\int_{-h}^{+h} f dz = \tag{f25}$$

$$\frac{f(x,y,h) + f(x,y,-h)}{2} - \bar{f}(x,y) = f^{+}(x,y,z=h) - \bar{f}$$

Neglecting the volume forces (the weight of the plate itself, for example), the equilibrium equations for the planar state are

$$\begin{aligned} \left(\sigma_{11}^{+}\right)_{,1} + \left(\sigma_{12}^{+}\right)_{,2} + \left(\sigma_{13}^{-}\right)_{,3} &= 0 \\ \left(\sigma_{12}^{+}\right)_{,1} + \left(\sigma_{22}^{+}\right)_{,2} + \left(\sigma_{23}^{-}\right)_{,3} &= 0 \\ \left(\sigma_{13}^{-}\right)_{,1} + \left(\sigma_{23}^{-}\right)_{,2} + \left(\sigma_{33}^{+}\right)_{,3} &= 0 \end{aligned} \tag{f26}$$

Let the mean values of the stress components be denoted by

$$\Sigma_{ij} = \overline{\sigma_{ij}} = \frac{1}{2h}\int_{-h}^{+h}\sigma_{ij}(x,y,z)dz \quad i,j = 1,2,3 \quad , \tag{f27}$$

Applying the mean value operator to eqs. (f26) gives

$$\begin{aligned} \left(\Sigma_{11}\right)_{,1} + \left(\Sigma_{12}\right)_{,2} + \frac{1}{2h}\left[\sigma_{13}(x,y,h) - \sigma_{13}(x,y,-h)\right] &= 0 \\ \left(\Sigma_{12}\right)_{,1} + \left(\Sigma_{22}\right)_{,2} + \frac{1}{2h}\left[\sigma_{23}(x,y,h) - \sigma_{23}(x,y,-h)\right] &= 0 \end{aligned} \quad , \tag{f28}$$

Using eqs.(f5) and (f7) it follows

$$\begin{aligned} \left(\Sigma_{11}\right)_{,1} + \left(\Sigma_{12}\right)_{,2} + \frac{1}{2h}\left[q_1^{j}(x,y) + q_1^{s}(x,y)\right] &= 0 \\ \left(\Sigma_{12}\right)_{,1} + \left(\Sigma_{22}\right)_{,2} + \frac{1}{2h}\left[q_2^{j}(x,y) + q_2^{s}(x,y)\right] &= 0 \end{aligned} \tag{f29}$$

Let

$$M_{ij} = \overline{z\sigma_{ij}} = \frac{1}{2h}\int_{-h}^{+h} z\sigma_{ij}(x,y,z)dz \quad i,j = 1,2,3 \quad . \tag{f30}$$

Multiplying eqs.(f26) by z and using again the mean value operator, it follows

$$\left(M_{13}\right)_{,1} + \left(M_{23}\right)_{,2} + \frac{\sigma_{33}(x,y,h) + \sigma_{33}(x,y,-h)}{2} - \Sigma_{33} = 0 . \tag{f31}$$

Hence, using again eqs.(f5) and (f7),

$$\left(M_{13}\right)_{,1} + \left(M_{23}\right)_{,2} + \frac{q_3^{j}(x,y) - q_3^{j}(x,y)}{2} - \Sigma_{33} = 0 , \tag{f32}$$

So, the equilibrium of a thin plate into the *planar state* leads to eqs.(f29) and (f32).
Consider the weight of the plate, the equation of equilibrium for the plate into the *bending state* are

$$\begin{aligned} \left(\sigma_{11}^{-}\right)_{,1} + \left(\sigma_{12}^{-}\right)_{,2} + \left(\sigma_{13}^{+}\right)_{,3} &= 0 \\ \left(\sigma_{12}^{-}\right)_{,1} + \left(\sigma_{22}^{-}\right)_{,2} + \left(\sigma_{23}^{+}\right)_{,3} &= 0 \\ \left(\sigma_{13}^{+}\right)_{,1} + \left(\sigma_{23}^{+}\right)_{,2} + \left(\sigma_{33}^{-}\right)_{,3} + \rho g &= 0 \end{aligned} \tag{f33}$$

Here, the density of the plate is ρ and the acceleration of gravity is g.
Proceeding in a similar manner to above, it follows the equations of equilibrium for the thin plate into a *bending state* are

$$\left(\Sigma_{13}\right)_{,1}+\left(\Sigma_{23}\right)_{,2}+\frac{1}{2h}\left[q_3^j(x,y)+q_3^s(x,y)\right]+\rho g=0,$$

$$\left(M_{11}\right)_{,1}+\left(M_{12}\right)_{,2}+\frac{1}{2}\left[q_1^j(x,y)-q_1^s(x,y)\right]-\Sigma_{13}=0 \quad . \qquad \text{(f34)}$$

$$\left(M_{12}\right)_{,1}+\left(M_{22}\right)_{,2}+\frac{1}{2}\left[q_2^j(x,y)-q_2^s(x,y)\right]-\Sigma_{23}=0$$

Eq.(f34b) is differentiated with respect to x and eq.(f34c) is differentiated with respect to y. The results are substituted in eq.(f34a). Hence

$$\left(M_{11}\right)_{,11}+2\left(M_{12}\right)_{,12}+\left(M_{22}\right)_{,22}+\frac{1}{2h}\left[q_3^j(x,y)+q_3^s(x,y)\right]+\rho g$$
$$+\frac{1}{2}\left[q_1^j(x,y)-q_1^s(x,y)\right]_{,1}+\frac{1}{2}\left[q_2^j(x,y)-q_2^s(x,y)\right]_{,2}=0 \qquad \text{(f35)}$$

F.6) Thin plane plate in the bending state.

The component u_1 of the displacement vector is developed in power series with respect to z-variable. It follows

$$u_1(x,y,z)=u_1(x,y,0)+z\frac{\partial u_1}{\partial z}(x,y,0)+\ldots \qquad \text{(f36)}$$

For the bending state, the first term in eq.(f36) vanishes. Because the thickness of the plate is a small one, only the second term is kept. So the horizontal components of the displacement vector are

$$u_1(x,y,z)=z\frac{\partial u_1}{\partial z}(x,y,0) \quad , \quad u_2(x,y,z)=z\frac{\partial u_2}{\partial z}(x,y,0) \quad . \qquad \text{(f37)}$$

The vertical displacement of the points placed into the median plane is denoted by

$$w=w(x,y)=u_3(x,y,0) \quad , \qquad \text{(f38)}$$

i.e. the mean surface has the equation $z=w(x,y)$. Here, w represents *the arrow of the plate*.

F7) BERNOULLI's hypothesis.

According to BERNOULLI, it is assumed that an arbitrary material segment of the plate, initially perpendicular on the median plane in the non-deformed state, rests perpendicular on the mean surface in the deformed state. Let $A(x,y,z)$ a certain point of the plate (not placed in the mean plane) and $A_0(x,y,0)$ its projection on the median plane. So, the segment $\overrightarrow{A_0A}$ is perpendicular on the median plane. After deformation, the material point A is moving at the point having the co-ordinates $M\left(x+u_1(x,y,z),\, y+u_2(x,y,z),\, z+u_3(x,y,z)\right)$, while the point A_0 is moving at the point $M_0\left(x+u_1(x,y,0),\, y+u_2(x,y,0),\, w(x,y)\right)$. The horizontal displacements of the points placed in the median plane are vanishing. So, writing the vector along the line, it follows that

$$\overrightarrow{M_0M}=\left(u_1(x,y,z),\, u_2(x,y,z),\, z+u_3(x,y,z)-w(x,y)\right) \qquad \text{(f39)}$$

Expanding in power series the even function u_3 with respect to z-variable, it follows

$$u_3(x,y,z)=w(x,y)+z^2\beta(x,y)+\ldots \qquad \text{(f40)}$$

Using eqs.(f37) and (f40), it follows that

$$\overrightarrow{M_0M} = \left(z\frac{\partial u_1}{\partial z}(x,y,0),\, z\frac{\partial u_2}{\partial z}(x,y,0),\, z + z^3\beta(x,y)+\ldots \right)$$
$$= z\left(\frac{\partial u_1}{\partial z}(x,y,0),\, \frac{\partial u_2}{\partial z}(x,y,0),\, 1 + z^2\beta(x,y)+\ldots \right) \qquad \text{(f41)}$$

Hence the direction is

$$\frac{\overrightarrow{M_0M}}{\|\overrightarrow{M_0M}\|} = \frac{\left(\frac{\partial u_1}{\partial z}(x,y,0),\, \frac{\partial u_2}{\partial z}(x,y,0),\, 1 + z^2\beta(x,y)+\ldots \right)}{\sqrt{\left(\frac{\partial u_1}{\partial z}\right)^2 + \left(\frac{\partial u_2}{\partial z}\right)^2 + \left(1 + z^2\beta(x,y)+\ldots\right)^2}} \qquad \text{(f42)}$$

The normal vector on the median surface is

$$\vec{n} = \left(-\frac{\partial w}{\partial x}, -\frac{\partial w}{\partial y}, 1 \right) \Big/ \sqrt{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + 1} \,. \qquad \text{(f43)}$$

Comparing eq.(f42) to eq.(f43), the supplemental hypothesis of BERNOULLI is satisfied if

$$\frac{\partial u_1}{\partial z}(x,y,0) = -\frac{\partial w}{\partial x} \quad , \quad \frac{\partial u_2}{\partial z}(x,y,0) = -\frac{\partial w}{\partial y} \quad , \quad u_3(x,y,z) = w(x,y) \quad , \qquad \text{(f44)}$$

i.e., by using eq.(f37), the displacement vector has the horizontal components equal to

$$u_1(x,y,z) = -z\frac{\partial w}{\partial x} \quad , \quad u_2(x,y,z) = -z\frac{\partial w}{\partial y} \quad , \quad u_3(x,y,z) = w(x,y) \,. \qquad \text{(f45)}$$

From eq. (f45), the components of the strain tensor are

$$\varepsilon_{11} = -z\frac{\partial^2 w}{\partial x^2} \quad ,\varepsilon_{12} = -z\frac{\partial^2 w}{\partial x \partial y} \quad ,\varepsilon_{22} = -z\frac{\partial^2 w}{\partial y^2} \quad ,\varepsilon_{33} = 0 \,. \qquad \text{(f46)}$$

The LAPLACE operator in horizontal components is denoted by

$$\Delta^* = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \,. \qquad \text{(f47)}$$

Hence the trace of the strain tensor is

$$\theta = \mathrm{tr}\,\varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = -z\Delta^* w \,. \qquad \text{(f48)}$$

F8) HOOKE's law for a thin plate.

By using eqs.(f46) and (f48) it follows that

$$\sigma_{11} = \lambda\,\mathrm{tr}\,\varepsilon + 2\mu\varepsilon_{11} = -z\left(\lambda\Delta^* w + 2\mu\frac{\partial^2 w}{\partial x^2} \right) \,. \qquad \text{(f49)}$$

But

$$\overline{z^2} = \frac{1}{2h}\int_{-h}^{+h} z^2 dz = \frac{h^2}{3} \qquad \text{(f50)}$$

Hence

$$M_{11} = \overline{z\sigma_{11}} = -\frac{h^2}{3}\left(\lambda\Delta^* w + 2\mu\frac{\partial^2 w}{\partial x^2} \right) = -\frac{H^2}{12}\left(\lambda\Delta^* w + 2\mu\frac{\partial^2 w}{\partial x^2} \right) \,. \qquad \text{(f51)}$$

In the same way,

$$M_{12}=\overline{z\sigma_{12}}=-2\mu\frac{h^2}{3}\frac{\partial^2 w}{\partial x\partial y}=-\mu\frac{H^2}{6}\frac{\partial^2 w}{\partial x\partial y}\quad, \tag{f52}$$

$$M_{22}=\overline{z\sigma_{22}}=-\frac{h^2}{3}\left(\lambda\Delta^* w+2\mu\frac{\partial^2 w}{\partial y^2}\right)=-\frac{H^2}{12}\left(\lambda\Delta^* w+2\mu\frac{\partial^2 w}{\partial y^2}\right) \tag{f53}$$

Substituting eqs.(f51)-(f53) into eq.(f35) it follows the *equation of Sophie GERMAIN*:

$$D\Delta^*\Delta^* w=\rho gH+q_3^j(x,y)+q_3^s(x,y)$$
$$+\frac{H}{2}\left\{\left[q_1^j(x,y)-q_1^s(x,y)\right]_{,1}+\left[q_2^j(x,y)-q_2^s(x,y)\right]_{,2}\right\}\quad, \tag{f54}$$

where

$$D=(\lambda+2\mu)\frac{2h^3}{3}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\frac{H^3}{12} \tag{f55}$$

represents the flexural *rigidity of the plate*.

So, obtaining the median surface asks someone to solve a bi-harmonic equation, with certain boundary conditions on the upper/ lower faces of the plate. The equation (f55) will be solved in same cases of particular importance in real cases.

EXERCISE. Find the expressions of Σ_{13} and Σ_{23} for a thin plate into a bending state.

F.9) The infinite, 1-dimensional (1-D) plate. The flexure of the lithosphere.

Consider an infinite extended plate along y -co-ordinate. The component u_2 of the displacement vector is equal to zero, and the rest of the components does not depend on y -co-ordinate. In this case, the plate is assumed to be in a cylindrical bending state. Neglecting the horizontal loads, eq.(f55) becomes

$$D\frac{d^4 w}{dx^4}=\rho gH+q_3^j(x)+q_3^s(x)\quad. \tag{f56}$$

For the case presented in Fig.F2, on the upper face of the plate is acting the load P due to the relief and the lithostatic pressure, i.e.

$$q_3^s(x)=P+\rho_r g(w(x)-h)\quad, \tag{f57}$$

where ρ_r is the density of the filling sediments (assumed to be homogeneous ones) placed between the reference plane of elevation equal to zero and the upper surface of the plate. On the down face of the plate, it is acting the pressure of the liquid of density equal to ρ_m ,i.e.

$$q_3^j(x)=p_0-\rho_m g(w(x)+h)\quad, \tag{f58}$$

where p_0 is an unknown constant. Eq.(f59) becomes

$$D\frac{d^4 w}{dx^4}=P-(\rho_m-\rho_r)g\,w(x)+p_0-\rho_r gh-\rho_m gh+\rho gH\quad. \tag{f59}$$

It is assumed that in the absence of the relief (i.e. P=0), the non-deformed plate (i.e. $w(x)\equiv 0$) is in an equilibrium state under the action of its own weight and of the pressure of the liquid, i.e.

$$p_0-\rho_r gh-\rho_m gh+\rho gH=0 \tag{f60}$$

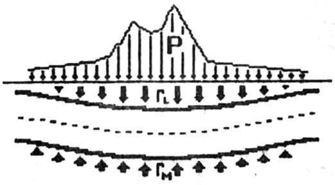

Fig.F2. The flexure of the lithosphere under the action of the relief (P), of the lithostatic pressure Γ_L and of the pressure of the liquid Γ_M.

It results the *flexure equation of the 1-D plate* :

$$\frac{d^4 w}{d x^4} = \frac{1}{D}\left[P - (\rho_m - \rho_r) g\, w(x)\right] \tag{f61}$$

Let

$$(\rho_m - \rho_r) g / D = 4/\alpha^4 \ , \tag{f62}$$

where α is the flexural *parameter* of the plate. Eq.(f61) becomes

$$\frac{d^4 w(x)}{d x^4} = \frac{P}{D} - \frac{4}{\alpha^4} w(x) \tag{f63}$$

F.10) Exterior forces on the lateral surface of the plate. Buckling.

To derive Sophie GERMAIN equation (f56) for the bending state, exterior forces acting on the lateral surface of the plate have been ignored, especially those placed into the median plane. Consider a very thin plate simply leaning like in Fig.F3. The load P is absent and the plate is infinite developed in a direction perpendicular on the plane of the figure. An element of the plate having the length equal to unit along that direction is considered too. Let h be the thickness of the plate. The forces per unit length along the above direction are denoted by $\pm \vec{N}$, being derived from the stress σ_c (positive for compression) by

$$N = \sigma_c h \tag{f64}$$

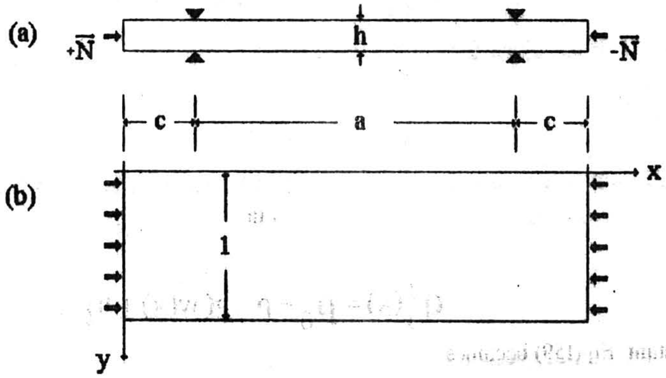

Fig.F3. A plate simply leaning, subject to the action of forces placed into the median plane. (a)A lateral view. (b) A view from above.

If the forces $\pm \vec{N}$ are small ones, the plate will be deformed according to a plane state, attempting a final configuration similar to Fig.1b. If the forces $\pm \vec{N}$ are above a certain critical value, the plate loses suddenly its equilibrium state, attending a deformation state like Fig.F4a, (or in the contrary sense, i.e. symmetrically with respect to the line of its supports. The displacement field in this case is similar to the cylindrical bending. That phenomenon represents the buckling of the plate. It characterises very thin plates or bars.

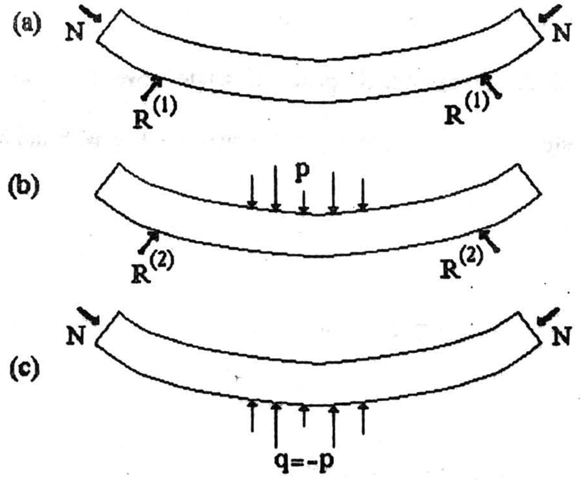

Fig.F4. (a) The equilibrium of a buckled plate acted by the forces $\pm \vec{N}$ and by the reactions of the supports.
(b) The equilibrium of a buckled plate acted by a vertical load p and by the reactions of the supports.

(c)The equilibrium of a buckled plate acted by the forces $\pm \vec{N}$ and by a load q having the same magnitude but a contrary sense with respect to the load presented in Fig.F4 (b).

In order to use the previous results, it is necessary to find when the mechanical state of stress / strain corresponding to the presence of the lateral forces is identical to the mechanical state of stress / strain corresponding to an unknown vertical load $p = p(x)$ (Fig.F4b). The equilibrium condition for the case shown in Fig.4a is

$$\sum \vec{N} + \sum \vec{R}^{(1)} = \vec{0} \quad , \tag{f65}$$

and the equilibrium condition for the case shown in Fig.F4b is

$$\sum \vec{p} + \sum \vec{R}^{(2)} = \vec{0} \quad , \tag{f66}$$

where $\sum \vec{R}^{(1)}$ and $\sum \vec{R}^{(2)}$ are the reactions of the supports in the above cases. Because the mechanical state of stress is identical in both cases, particularly in the neighbourhood of the supports, the reactions will be the same, i.e.

$$\sum \vec{R}^{(1)} = \sum \vec{R}^{(2)} . \tag{f67}$$

It follows that

$$\sum \vec{N} + \sum (-\vec{p}) = \vec{0} \quad . \tag{f68}$$

Hence for the corresponding state of stress / strain, the plate is in an equilibrium state if it is acted by the lateral forces and by a vertical load $\vec{q} = -\vec{p}$, in the absence of the supports (Fig.F4c). Consider a plate element having the horizontal length equal to dx and the ends denoted by A and B (Fig.F5).

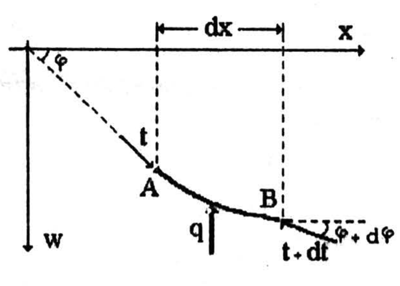

Fig.F5. The equilibrium of a plate element due to the load q and to the internal tensions.

Because the plate is a thin one, the tangential efforts are neglected. At the point A, it is acting a force (per unit length) denoted by $\vec{t}$, representing a normal effort, tangent to the plate. The angle between $\vec{t}$ and the horizontal axis is denoted by φ. At the point B, it is acting the effort $\overrightarrow{t + dt}$, making an angle equal to φ+dφ with the horizontal axis. The equilibrium conditions are

$$\begin{aligned} &t\cos\varphi - (t + dt)\cos(\varphi + d\varphi) = 0 \\ &t\sin\varphi - (t + dt)\sin(\varphi + d\varphi) - q dx = 0 \end{aligned} \tag{f69}$$

For small angles φ, it follows that

$$\begin{aligned} &\cos\varphi \cong \cos(\varphi + d\varphi) \cong 1 \\ &\sin\varphi \cong \varphi \, , \, \sin(\varphi + d\varphi) \cong \varphi + d\varphi \\ &\varphi \cong tg\varphi = -\frac{dw}{dx} \end{aligned} \tag{f70}$$

The first equation in (f69) gives

$$\frac{dt}{dx} = 0 \tag{f71}$$

Hence t has no variation along x-axis, i.e.

$$t = N \tag{f72}$$

The second equation in (f69) gives

$$t\varphi - t(\varphi + d\varphi) - q dx = 0 \tag{f73}$$

By using (f74) and (f70c) it follows that

$$q = -N\frac{d\varphi}{dx} = N\frac{d^2 w}{dx^2} \quad . \tag{f74}$$

Hence the forces on the lateral faces of the plate are mechanical equivalent to a vertical load equal to

$$p = p(x) = -N\frac{d^2 w}{dx^2} \quad . \tag{f75}$$

It follows that eq. (f63) has the next general form

$$\frac{d^4 w}{d x^4} + \frac{N}{D}\frac{d^2 w}{d x^2} + \frac{4}{\alpha^4} w = \frac{P}{D} \tag{f76}$$

F.11) The buckling of a simply leaning thin plate.

Consider the plate in Fig.F3. For simplicity, it is assumed that $c = 0$ (i.e. the plate is leaning just at its ends). The buoyancy force and the vertical loads are neglected. Equation (f76) becomes

$$\frac{d^4 w}{d x^4} + \frac{N}{D}\frac{d^2 w}{d x^2} = 0 \quad , \tag{f77}$$

together with the next conditions:

-at the end point having x=0: $\quad w = 0 \, , \, d^2 w / dx^2 = 0 \qquad$ (f78)

-at the end point having x=a: $\quad w = 0 \, , \, d^2 w / dx^2 = 0 \qquad$ (f79)

The equations (f77)-(f79) has the trivial solution $w \equiv 0$. It follows to find a *critical buckling value* $N = N^*$ in order the system (f77)-(f79) to have further non-trivial solutions. Successively, equation (f77) can be written as

$$\frac{d^2}{d x^2}\left(\frac{d^2 w}{d x^2} + \frac{N}{D} w\right) = 0 \quad , \tag{f80}$$

$$\frac{d^2 w}{d x^2} + \frac{N}{D} w = C_1 x + C_2 \quad , \tag{f81}$$

where C_1 , C_2 are two integration constants, vanishing according to (f78)-(f79). Hence (f81) is

$$\frac{d^2w}{dx^2}+\frac{N}{D}w=0 \quad , \tag{f82}$$

having the solution

$$w(x)=C_3\sin(\sqrt{N/D}x)+C_4\cos(\sqrt{N/D}x) \quad . \tag{f83}$$

From (f78) it follows that $C_4 = 0$, while from (f79) it follows the critical values

$$N_k^* = D(k\pi/a)^2 \ , \ k=1,2,\dots \tag{f84}$$

The lowest critical value is obtained for $k=1$.

EXERCISES.

(1)Perform a study for the buckling of a 1-D plate having an embedded end point, the other being free.

(2)Perform a study for the buckling of a 1-D plate having both end points free. The plate is simply leaning at 1/3 from its length with respect to its left end.

(3)Perform a study of the simply leaning 1-D plate in the presence of the buoyancy force.

(4)Modify the equation of Sophie GERMAIN for the 2-D plate in the presence of lateral forces.

(5)Perform a study for the buckling a 2-D rectangular plate, simply leaning at all its sides.

F.12) The infinite extended 1-D plate.

By integrating both sides of eq.(f76) it follows

$$D\int_{-\infty}^{+\infty}\frac{d^4w}{dx^4}dx+N\int_{-\infty}^{+\infty}\frac{d^2w}{dx^2}dx+(\rho_m-\rho)g\int_{-\infty}^{+\infty}w(x)dx=\int_{-\infty}^{+\infty}P(x)dx \quad , \tag{f85}$$

Because w and its derivatives of any order are vanishing at infinite, the first two integrals in (f85) are vanishing too. It follows that the area bounded by the median curve (the flexure) and the horizontal x-axis is proportional to the load due to the relief, irrespective the presence of the lateral forces:

$$\int_{-\infty}^{+\infty}w(x)dx=\frac{1}{(\rho_m-\rho)g}\int_{-\infty}^{+\infty}P(x)dx \quad . \tag{f86}$$

Let an approximation of the relief be a set of m steps, each one of height equal to h_j and density equal to ρ_j, i.e.

$$P(x)=\sum_{j=1}^{m}P_j(x) \quad , \quad P_j(x)=\begin{cases}\rho_j g h_j, \text{ pentru } x\in[a_j,b_j]\\ 0, \text{ in rest}\end{cases} . \tag{f87}$$

Equation (f86) becomes

$$\int_{-\infty}^{+\infty}w(x)dx=\frac{1}{\rho_m-\rho}\sum_{j=1}^{m}\rho_j h_j(b_j-a_j) \quad . \tag{f88}$$

Hence the area bounded by the flexural curve and the horizontal axis is a linear combination of the areas approximating the relief. In real cases, the flexural curve can be outlined along a finite interval denoted by $[-L, L]$, hence an upper bound for the difference of the densities can be obtained as

$$\rho_m-\rho<\sum_{j=1}^{m}\rho_j h_j(b_j-a_j)\Bigg/\int_{-L}^{+L}w(x)dx \quad . \tag{f89}$$

Equation (f76) will be solved by using FOURIER transforms.

F.13) FOURIER transforms. Properties.

The direct FOURIER transform (*t.F.d.*) of a function $f(x)$ is the new function Φ of variable u, defined as

$$\Phi[f](u)=\int_{-\infty}^{+\infty}f(x)\exp(-iux)dx, \quad i=\sqrt{-1} \tag{f90}$$

The inverse FOURIER transform (*t.F.i.*) of a function $\Phi(u)$ is the function $f(x)$ defined as

$$f(x)=\Phi^{-1}[\Phi[f](u)](x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\Phi(u)\exp(+iux)du \qquad (f91)$$

Differentiating both sides of eq. (f91) with respect to x, it follows that the *t.F.d.* of the first derivative $\frac{df}{dx}$ can be obtained by multiplying the the *t.F.d.* of $f(x)$ by iu. Hence the *t.F.d.* of the derivative $\frac{d^4w}{dx^4}$ can be obtained by multiplying the *t.F.d.* of $w(x)$ by $(iu)^4=u^4$. Consider two functions f and g of one variable. Their convolution product is

$$(f*g)(x)=\int_{-\infty}^{+\infty}f(y)g(x-y)dy \quad . \qquad (f92)$$

Permuting the integrals, it follows that the direct FOURIER transform of the convolution product is the product pf the transforms of both factors of the product, i.e.

$$\begin{aligned}\Phi[f*g](u)&=\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}f(y)g(x-y)dy\right]\exp(-iux)dx\\&=\int_{-\infty}^{+\infty}f(y)\exp(-iuy)dy\int_{-\infty}^{+\infty}g(z)\exp(-iuz)dz=\Phi[f]\ \Phi[g]\end{aligned} \quad . \qquad (f93)$$

F.14) Solution of the flexure equation by using FOURIER transforms.

The solution of eq.(f76) is the sum of two terms, a term corresponding to the homogeneous equation and a term corresponding to a particular solution, i.e.

$$w(x)=w^h(x)+w^p(x) \qquad (f94)$$

A particular solution $w^p=w^p(x)$ will be obtained applying the direct FOURIER transform to eq.(f76) and by using the above presented properties of the FOURIER transform

$$u^4\Phi[w^p](u)-\frac{N}{D}u^2\Phi[w^p](u)+\frac{4}{\alpha^4}\Phi[w^p](u)=\frac{1}{D}\Phi[P](u) \quad , \qquad (f95)$$

i.e.

$$\Phi[w^p](u)=\frac{1}{D}\frac{\Phi[P](u)}{u^4-Ku^2+4/\alpha^4} \quad . \qquad (f96)$$

It is assumed that the value of the positive constant $K=N/D$ is small enough. Consider the particular case when the load due to the relief is a load concentrated at the origin of the axes, having the magnitude equal to unit. The direct FOURIER transform of this load is equal to unit too. The corresponding solution, denoted by w_G, represents the elastostatic GREEN function. It allows one to obtain the solution corresponding to an arbitrary load of magnitude equal to P. Hence

$$\Phi[w_G](u)=\frac{1}{D}\frac{1}{\left(u^2-K/2\right)^2+4/\alpha^4-(K/2)^2} \qquad (f97)$$

It is assumed that the next condition is satisfied

$$|K|<4/\alpha^2 \qquad (f98)$$

Let

$$A=\sqrt{1/\alpha^2+K/4} \quad , \quad B=\sqrt{1/\alpha^2-K/4} \qquad (f99)$$

But

$$\frac{1}{u^4 - Ku^2 + 4/\alpha^4} = \frac{\alpha^2}{8A}\left[\frac{u}{(u+A)^2+B^2} - \frac{u}{(u-A)^2+B^2}\right] + \frac{\alpha^2}{4}\left[\frac{1}{(u+A)^2+B^2} + \frac{1}{(u-A)^2+B^2}\right] \quad \text{(f100)}$$

The next result is valid (Rîjic and Gradstein 1955)

$$\int_0^\infty \frac{z \operatorname{sgn} \operatorname{Re}(z)}{z^2 + x^2} \cos x dx = \frac{\pi}{2}\exp(-z) \quad , \quad z \neq 0 \ . \quad \text{(f101)}$$

By using (f101), the next inverse FOURIER transforms are obtained

$$\Phi^{-1}\left[\frac{1}{(u \pm A)^2 + B^2}\right](x) = \frac{1}{2B}\exp\left(-B|x| \mp iAx\right) \ . \quad \text{(f102)}$$

$$\Phi^{-1}\left[\frac{1}{(u+A)^2+B^2} + \frac{1}{(u-A)^2+B^2}\right](x) = \frac{1}{B}\exp\left(-B|x|\right)\cos\left(A|x|\right) \ . \quad \text{(f103)}$$

By using the property of the derivative, it follows

$$\Phi\left\{\frac{1}{2B}\frac{d}{dx}\left[\exp\left(-B|x| - iAx\right)\right]\right\}(u) = \frac{iu}{(u+A)^2+B^2} \ . \quad \text{(f104)}$$

In the same way

$$\Phi\left\{\frac{1}{2B}\frac{d}{dx}\left[\exp\left(-B|x| + iAx\right)\right]\right\}(u) = \frac{iu}{(u-A)^2+B^2} \ . \quad \text{(f105)}$$

Subtracting eq.(f104) from eq.(105), it follows that

$$\Phi\left\{\frac{1}{B}\frac{d}{dx}\left[\exp\left(-B|x|\right)\sin\left(Ax\right)\right]\right\}(u) = \frac{u}{(u-A)^2+B^2} - \frac{u}{(u+A)^2+B^2} \ . \quad \text{(f106)}$$

Hence

$$\Phi^{-1}\left[\frac{u}{(u-A)^2+B^2} - \frac{u}{(u+A)^2+B^2}\right] = \frac{1}{B}\frac{d}{dx}\left[\exp\left(-B|x|\right)\sin\left(Ax\right)\right] = \exp\left(-B|x|\right)\left[\frac{A}{B}\cos\left(A|x|\right) - \sin\left(A|x|\right)\right] \ . \quad \text{(f107)}$$

Using the above results, it follows after some elementary computations that

$$\Phi[w_G](u) = \frac{\alpha^2}{8D}\Phi\left\{\exp\left(-B|x|\right)\left[\frac{\sin\left(A|x|\right)}{A} + \frac{\cos\left(A|x|\right)}{B}\right]\right\} \ , \quad \text{(f108)}$$

$$w_G(x) = \frac{\alpha^2}{8D}\exp\left(-B|x|\right)\left[\frac{\sin\left(A|x|\right)}{A} + \frac{\cos\left(A|x|\right)}{B}\right] \ , \quad \text{(f109)}$$

It can be observed that $w_G \to \infty$ for $B \to 0$, corresponding to the buckling of the infinite plate in the presence of a lateral compressive stress. From (f96) and (f97) it follows that

$$\Phi[w^p] = \Phi[P]\Phi[w_G] \ . \quad \text{(f110)}$$

Hence the solution for an arbitrary load is the convolution of the load due to the relief and the function given by (f109), representing a general property of the GREEN function:

$$w^p(x) = (P * w_G)(x) = \int_{-\infty}^{+\infty} P(y) w_G(x-y) dy \ . \quad \text{(f111)}$$

For the approximation of the relief represented by eq.(f87), it follows that

$$w^{p}(x)=\frac{1}{4(\rho_m-\rho)}\sum_{j=1}^{m}\rho_j h_j\left[I\left(b_j-x\right)-I\left(a_j-x\right)\right] \quad (f112)$$

where

$$I(z)=\operatorname{sgn}(z)\left\{\exp(-B|z|)\left[-2\cos(A|z|)+\frac{K}{\sqrt{(\rho_m-\rho)g/D-K^2/4}}\sin(A|z|)\right]+2\right\} \quad (f113)$$

The solution of the homogeneous equation can be immediately derived as

$$w^{h}(x)=[C_1\cos(Ax)+C_2\sin(Ax)]\exp(-Bx)+[C_3\cos(Ax)+C_4\sin(Ax)]\exp(Bx) \quad (f114)$$

Hence the general solution is

$$w(x)=\frac{1}{4(\rho_m-\rho)}\sum_{j=1}^{m}\rho_j h_j\left[I\left(b_j-x\right)-I\left(a_j-x\right)\right]$$
$$+[C_1\cos(Ax)+C_2\sin(Ax)]\exp(-Bx)+[C_3\cos(Ax)+C_4\sin(Ax)]\exp(Bx) \quad (f115)$$

It follows to find the unknown coefficients C_1, C_2, C_3 and C_4 in some particular cases. For the infinite plate, the flexure W is subject to the next conditions:

$$\lim_{x\to\pm\infty} W(x)=0 \quad (f116)$$

Hence the coefficients C_1, C_2, C_3 and C_4 are vanishing and the general solution is just the particular solution represented by eq.(116). In the case of the semi-infinite plate the flexure W is subject, for example, to the next conditions:

$$\lim_{x\to\infty} W(x)=0 \quad (f117)$$

$$W(0+0)=W_0 \quad (f118)$$

$$\frac{d^2W}{dx^2}(0+0)=W_0''=-M_0/D \quad (f119)$$

where W_0, W_0'' and M_0 (positive when acting into a clockwise sense) are the values of the flexure, that of the second derivative of the flexure and that of the bending moment respectively at the left end of the plate where the origin of the x-axis is selected. It follows

$$C_1=W_0-\sum_{j=1}^{m}\rho_j g h_j\left[I\left(b_j\right)-I\left(a_j\right)\right] ,$$

$$C_2=\frac{1}{2AB}\left[\frac{1}{4ABD}\sum_{j=1}^{m}\rho_j g h_j\left[\exp\left(-Ba_j\right)\sin\left(Aa_j\right)-\exp\left(-Bb_j\right)\sin\left(Ab_j\right)\right]-\frac{K}{2}C_1-W_0''\right] \quad (f120)$$

and

$$C_3=C_4=0 \quad . \quad (f121)$$

A finite plate of variable thickness can be approximated in real cases by a sum of n elements having constant thickness and homogeneous elastic properties. To obtain the values of the unknown coefficients C_1, C_2, C_3 and C_4 for each element, proper conditions have to be verified at the ends of each element. A finite element algorithm based on the continuity of the values of the flexure, of its first derivative, of the bending moment and of the share force has been derived by Ivan (1997).

EXERCISE. Derive the expression of $w(x)$ for a load due to a relief having the equation

$$P=P(x)=\rho g h(x)=\begin{cases}\rho g h_0\sin(2\pi x/\lambda), \text{ pentru } x\in[-\lambda/2,+\lambda/2]\\ 0 \quad \text{, in rest}\end{cases} \quad , \quad (f122)$$

where h_0 is the amplitude of the relief and λ is its wave-length.

F.15) Finite plates.

In the case of the finite plates, the boundary conditions on the contour of the plate are essential ones in order to obtain the values of the integration constants. Some particular, most common cases will analysed in detail. As a consequence, a previous examination of the mean values Σ_{ij} and M_{ij} , $i,j=1,2,3$. it is necessary.

F.15 a) Significance of Σ_{ij} and M_{ij} for the bending state.

According to the definition of the stress tensor elements, σ_{ij} is the projection along j -axis of the surface force acting on the surface having the outer pointing normal equal to $\vec{n}_i$. Because the elements σ_{13} , σ_{23} are even functions, the mean values Σ_{13} , Σ_{23} are representing share forces, acting like in Fig.F3.

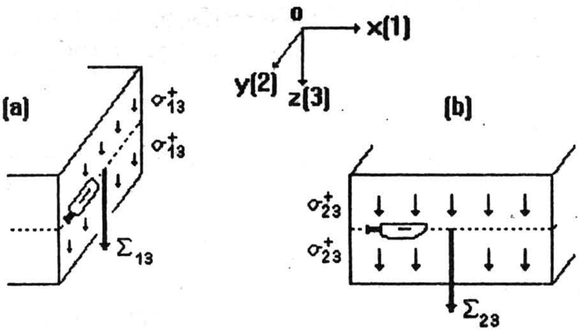

Fig.F3. The mean values Σ_{13} (a) and Σ_{23} (b). Both of them are share forces.

Similar considerations allow one to conclude that the mean values M_{11} and M_{22} are bending moments, while M_{12} , M_{21} are torsion moments (Fig.F4).

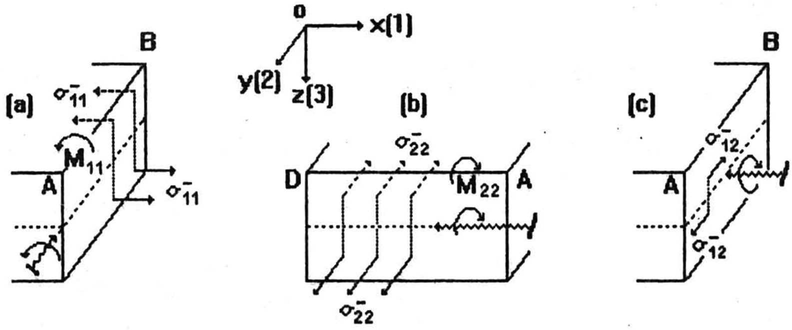

Fig.F4. The bending moments M_{11} (a) and M_{22} (b). The torsion moment M_{12} acting on the side having the outer pointing normal 1 is presented in Fig.4c. A similar torsion moment is acting on the side having the normal 2, but it is not presented in the figure.

F.15 b) The rectangular plate. Boundary conditions. LÉVY 's solution.

Let consider a rectangular plate having the sides equal to $2a$ and $2b$ respectively (Fig.F5). Consider, for example, the side AB, having the equation $x=a$, $y\in[-b,b]$). Among most commonly used boundary conditions are

-the embedded side: the flexure of the plate and the derivative of the flexure are both equal to zero

$$w(x,y)=0\,,\partial w/\partial x=0\;; \tag{f123}$$

-the rotating side: the flexure of the plate and the bending moment are both equal to zero:

$$w(x,y)=0\,,M_{11}=0\;, \tag{f124}$$

i.e.:

$$w(x,y)=0\,,\lambda\,\Delta^{*}w+2\mu\frac{\partial^2 w}{\partial x^2}=0\,. \qquad \text{(f125)}$$

-__the free side__: the share force, the bending moment and the torsion moment are all equal to zero.

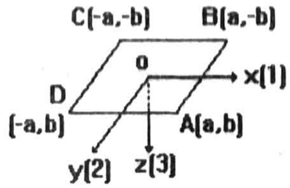

Fig.F5. The rectangular plate.

Similar conditions can be derived for plates of arbitrary shape.

As an example, consider the rectangular plate with two opposite articulated sides. Neglecting the lateral forces and the gravity, it follows to solve the simplified equation of the flexure

$$\Delta\Delta w=q(x,y)/D\quad, \qquad \text{(f126)}$$

with the boundary conditions (f125) written for $x=\pm a$ and a 2-D LAPLACE operator. Consider a particular solution having the form

$$w(x,y)=\sum_{k=1}^{\infty}f_k(y)\sin(k\pi x/a)\,, \qquad \text{(f127)}$$

it follows that

$$w(\pm a,y)=0\,,$$
$$\frac{\partial^2 w}{\partial x^2}=-\frac{\pi^2}{a^2}\sum_{k=1}^{\infty}k^2 f_k\sin(k\pi x/a) \qquad \text{(f128)}$$

Hence

$$\partial^2 w/\partial x^2=\partial^2 w/\partial y^2=0,\text{for } x=\pm a\,. \qquad \text{(f129)}$$

It follows the conditions (f125) are fulfilled for the two opposite articulated sides. Substituting (f127) into (f126), it follows that

$$\sum_{k=1}^{\infty}\left[f_k''''(y)-2\left(\frac{k\pi}{a}\right)^2 f_k''(y)+\left(\frac{k\pi}{a}\right)^4 f_k(y)\right]\sin\frac{k\pi x}{a}=q(x,y)/D\,. \qquad \text{(f130)}$$

The function $q(x,y)$ is expanded in FOURIER series and the coefficients are identified. Both sides of eq.(f130) are multiplied by $\sin\frac{j\pi x}{a}$, $j=1,2,\ldots$ The result is integrated on the interval $[-a,a]$, taking into account that

$$\int_{-a}^{+a}\sin\frac{k\pi x}{a}\sin\frac{j\pi x}{a}dx=\begin{cases}0 & ,\text{pentru } j\neq k\\ a & ,\text{pentru } j=k\end{cases} \qquad \text{(f131)}$$

Hence

$$f_k''''(y)-2\left(\frac{k\pi}{a}\right)^2 f_k''(y)+\left(\frac{k\pi}{a}\right)^4 f_k(y)=\frac{1}{Da}\int_{-a}^{+a}q(x,y)\sin\frac{k\pi x}{a}dx\,,k=1,2,\ldots \qquad \text{(f132)}$$

In the beginning, the next homogeneous equation is solved, i.e.

$$F_k''''(y)-2\left(\frac{k\pi}{a}\right)^2 F_k''(y)+\left(\frac{k\pi}{a}\right)^4 F_k(y)=0\,,k=1,2,\ldots \qquad \text{(f133)}$$

The characteristic equation is

$$r^4-2\left(\frac{k\pi}{a}\right)^2 r^2+\left(\frac{k\pi}{a}\right)^4=0\,,\text{deci}\quad r_1=r_2=-k\pi/a\quad,r_3=r_4=k\pi/a\,, \qquad \text{(f134)}$$

hence the solution of the homogeneous equation is

$$F_k(y) = A_k \cosh\frac{k\pi y}{a} + B_k \sinh\frac{k\pi y}{a} + y\left(C_k \cosh\frac{k\pi y}{a} + D_k \sinh\frac{k\pi y}{a}\right) \quad , \qquad (f135)$$

where A_k, B_k, C_k, D_k, $k = 1,2,\ldots$ are some constants following to be obtained from the boundary conditions on the other two (non-articulated) sides of the plate. The general solution of equation (f132) is the sum of (f135) and a particular solution. The last one can be obtained by the usual techniques (e.g. CAUCHY method). The above approach is due to LÉVY.

F.16) Vibrations of a plate laying on a viscous substratum.

In this chapter, the flexure is considered as a function of both spatial co-ordinates and time. The corresponding differential equation is derived, being solved in the case of the 2-dimensional plate.

a) The differential equation.

As usually, a co-ordinate system is used having the horizontal axes x and y. The z-axis is positive downward, having the unit vector denoted by $\vec{e}_z$. By applying the mean-value operator, the equations of motion for the bending state are

$$\partial M_{xx}/\partial x + \partial M_{xy}/\partial y + \left[q_x^d(x,y,t) - q_x^u(x,y,t)\right]/2 - \Sigma_{xz} = \rho \overline{z \ddot{u}_x} \quad , \qquad (f136)$$

$$\partial M_{xy}/\partial x + \partial M_{yy}/\partial y + \left[q_y^d(x,y,t) - q_y^u(x,y,t)\right]/2 - \Sigma_{yz} = \rho \overline{z \ddot{u}_y} \quad , \qquad (f137)$$

and

$$\partial \Sigma_{xz}/\partial x + \partial \Sigma_{yz}/\partial y + \frac{1}{2h}\left[q_z^d(x,y,t) + q_z^u(x,y,t)\right] + \rho g = \rho \overline{\ddot{u}_z} \quad . \qquad (f138)$$

It will be assumed that Bernoulli's hypothesis is valid for all time. Hence the displacement vector has the elements

$$u_x(x,y,z,t) = -z\frac{\partial w}{\partial x} \quad , \quad u_y(x,y,z,t) = -z\frac{\partial w}{\partial y} \quad , \quad u_z(x,y,z,t) = w(x,y,t) \quad . \qquad (f139)$$

Substituting (f139) into (f136)-(f138) it follows after elementary computations that

$$\begin{aligned} D\Delta^*\Delta^* w = \rho g H - \rho H \ddot{w} + \frac{\rho}{12}H^3\Delta^* \ddot{w} + q_z^l(x,y,t) + q_z^u(x,y,t) \\ + \frac{H}{2}\left\{\partial\left[q_x^l(x,y,t) - q_x^u(x,y,t)\right]\Big/\partial x + \partial\left[q_y^l(x,y,t) - q_y^u(x,y,t)\right]\Big/\partial y\right\} \end{aligned} \quad , \qquad (f140)$$

As usually, the horizontal loads for the upper face of the plate are neglected, the surface forces being assumed to be

$$q_x^u = 0 \quad , \quad q_y^u = 0 \quad , \quad q_z^u = \rho^F g w + P(1 - \ddot{w}/g) \quad , \qquad (f141)$$

where ρ^F is the density of the filling sediments and P is the load. The last term in (f141) is an inertial one. For the material below the plate, the next constitutive equation is assumed

$$\sigma = [p_0 - \rho^M g(H+w)]\mathbf{1} + \lambda^M \mathrm{tr}\,\varepsilon\,\mathbf{1} + 2\mu^M \varepsilon + 2\eta^M \dot{\varepsilon} \qquad (f142)$$

where p_0 is a reference pressure and ρ^M is the density of the material below the plate. The LAMÉ elastic coefficients are λ^M, μ^M and η^M is the viscosity. Hence the loads on the lower face of the plate are

$$q_x^l = \mu^M \frac{\partial w}{\partial x} + \eta^M \frac{\partial \dot{w}}{\partial x} \quad , \quad q_y^l = \mu^M \frac{\partial w}{\partial y} + \eta^M \frac{\partial \dot{w}}{\partial x} \quad , \qquad \text{(f143)}$$
$$q_z^d = p_0 - \frac{H}{2}\lambda^M \Delta^* w + \rho^M g(H+w)$$

Again, a correction due to the compressive horizontal stresses σ_x^c, σ_y^c acting along the x- and y-axes respectively at the ends of the plate follows to be considered further. The reference pressure p_0 is selected in order the flexure w to vanish in the absence of the load P. Finally, a generalisation of the Sophie GERMAIN equation for a time dependent flexure is obtained as

$$D\Delta^*\Delta^* w + \frac{H}{2}\left(\lambda^M - \mu^M\right)\Delta^* w + H\left(\sigma_x^c \frac{\partial^2 w}{\partial x^2} + \sigma_y^c \frac{\partial^2 w}{\partial y^2}\right) + (\rho^M - \rho^F) g w$$
$$= P + \eta^M \frac{H}{2}\Delta^* \dot{w} - (\rho H + P/g)\ddot{w} + \frac{\rho}{12} H^3 \Delta^* \ddot{w} \qquad \text{(f144)}$$

b) The rectangular plate with 3 embedded sides.

For usual materials $\lambda^M = \mu^M$, hence the second term on the left side of (f144) vanishes. Because the load is mainly represented by the relief, having the elevations much smaller then the thickness of the plate, the inertial term is usually negligible on the right side of (15). Let $w^e = w^e(x,y)$ be the equation of the flexure corresponding to the state of equilibrium in the presence of the load, i.e.

$$D\Delta^*\Delta^* w^e + H\left(\sigma_x^c \partial^2 w^e / \partial x^2 + \sigma_y^c \partial^2 w^e / \partial y^2\right) + (\rho^M - \rho^F) g w^e = P \qquad \text{(f145)}$$

Consider the difference

$$\delta = \delta(x,y,t) = w(x,y,t) - w^e(x,y) \qquad \text{(f146)}$$

It follows that

$$\Delta^*\Delta^*\delta + \frac{12}{\rho a^2}\frac{1}{H^2}\left(\sigma_x^c \frac{\partial^2 \delta}{\partial x^2} + \sigma_y^c \frac{\partial^2 \delta}{\partial y^2}\right) + 12\frac{\rho^M - \rho^F}{\rho}\frac{g}{H^3 a^2}\delta = \frac{1}{a^2}\left(\Delta^* \ddot{\delta} + \frac{6\eta^M}{\rho H^2}\Delta^* \dot{\delta} - \frac{12}{H^2}\ddot{\delta}\right) \qquad \text{(f147)}$$

where a denotes the velocity of P-waves through the plate.
Consider the lengths of the sides are L_x, L_y respectively. Suppose the plate is embedded according to

$$\delta(x,0,t) = 0, \quad \text{with} \quad 0 \le x \le L_x$$
$$\delta(0,y,t) = 0, \quad \text{with} \quad 0 \le y \le L_y \quad , \qquad \text{(f148)}$$
$$\delta(L_x,y,t) = 0, \quad \text{with} \quad 0 \le y \le L_y$$

at any time. A particular solution satisfying (f148) is

$$\delta_{mn}(x,y,t) = \sin\left(m\pi x / L_x\right)\sin[(n-0.5)\pi y / L_y]\,\tau_n(t) \quad , \quad m,n = 1,2,\ldots \qquad \text{(f149)}$$

Substituting (f149) in (f147) it follows the modes are damped harmonic, i.e.

$$\ddot{\tau} + 2\zeta_{mn}\dot{\tau} + \frac{4\pi^2}{T_{mn}^2}\tau = 0 \quad , \qquad \text{(f150)}$$

For $m,n = 1,2,\ldots$ the periods are

$$T_{mn} = 2\pi \frac{H}{a} \sqrt{\frac{A_{mn} + 12}{A_{mn}^2 - 12\pi^2[\sigma_x^c m^2 H_x^2 + \sigma_y^c (n-0.5)^2 H_y^2]/(\rho a^2) + 12\rho^* gH/a^2}} , \quad \text{(f151)}$$

and the decay constants are

$$\zeta_{mn} = \frac{3\eta^M}{\rho H^2} \frac{A_{mn}}{A_{mn} + 12} , \quad \text{(f152)}$$

where

$$H_x = H/L_x \quad , \quad H_y = H/L_y \quad , \quad \rho^* = (\rho^M - \rho^F)/\rho \quad \text{(f153)}$$

and

$$A_{mn} = \pi^2 \left[m^2 H_x^2 + (n-0.5)^2 H_y^2 \right] . \quad \text{(f154)}$$

According to (f139), the modes are both toroidal and spheroidal.
For each mode, a critical viscosity can be found from

$$\zeta_{mn}^{cr} = 2\pi / T_{mn} , \quad \text{(f155)}$$

i.e.

$$\eta_{mn}^{cr} = \frac{2\pi\rho H^2}{3T_{mn}} \frac{A_{mn} + 12}{A_{mn}} . \quad \text{(f156)}$$

For values of viscosity less than the critical value (f156), the motion of the plate is represented by a sum of damped oscillations. The decay constants are obtained from (f152) and the periods are

$$T_{mn}^* = T_{mn} / \sqrt{1 - \left(\zeta_{mn} / \zeta_{mn}^{cr}\right)^2} \quad \text{(f157)}$$

For values of viscosity greater than the critical value, the motion of the plate is aperiodic and the characteristic roots of (21) are $-\left(\zeta_{mn} + \sqrt{\zeta_{mn}^2 - 4\pi^2 / T_{mn}^2}\right)$ and $-\left(\zeta_{mn} - \sqrt{\zeta_{mn}^2 - 4\pi^2 / T_{mn}^2}\right)$. It follows that for large scale of times, the solution of (f150) behaves like $\exp(-Ct)$, where the decay constant is

$$C = \zeta_{11} - \sqrt{\zeta_{11}^2 - 4\pi^2 / T_{11}^2} , \quad \text{(f158)}$$

all the other terms being faster attenuated. In many real applications, an approximate value of (f159) is

$$C = \frac{4\pi^2 / T_{11}^2}{\zeta_{11} + \sqrt{\zeta_{11}^2 - 4\pi^2 / T_{11}^2}} \cong \frac{2\pi^2}{\zeta_{11} T_{11}^2} \quad \text{(f159)}$$

With regard to (f149), the decay constant (f158) or (f159) can be obtained as the ratio between the amplitude of the velocity $\partial\delta / \partial t$ and the amplitude of δ. By using eq.(f152), that ratio can be used to estimate the mean viscosity of the material below the plate

$$\eta^M = \frac{\rho H^2}{3C} \frac{2\pi^2}{T_{11}^2} \frac{A_{mn} + 12}{A_{mn}} \quad \text{(f160)}$$

A numerical application with respect to the Moesian Platform is presented by (Ivan 1997a,b).

G) THE SPHERICAL SHELL

G.1) The model. BERNOULLI's hypothesis. Displacement vector and strain tensor.

A spherical elastic, homogeneous shell having the elastic moduli equal to λ, μ is considered and the usual spherical co-ordinate system (r, θ, φ) is used. However, some of the derived results are also valid in the case of a more general, non-elastic spherical shell. In the initial state, the homogeneous density is denoted by ρ_0 and the median spherical surface of the shell has the equation

$$r = R , \tag{g1}$$

where R is the radius of the sphere. At a certain time during the deformation, the median surface will be

$$r = R - w(\theta, \varphi, t) \quad , \tag{g2}$$

where $w = w(\theta, \varphi, t)$ is the flexure of the shell, positive downward. Hence the unit vector normal to the median surface at a certain point of co-ordinates $(r = R - w, \theta, \varphi)$ is

$$\vec{n} = \left(\vec{e_r} + \frac{\partial w}{r\partial\theta}\vec{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial w}{\partial\varphi}\vec{e_\varphi}\right) / \sqrt{1 + \frac{1}{r^2}\left[\left(\frac{\partial w}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial w}{\partial\varphi}\right)^2\right]} \quad . \tag{g3}$$

Neglecting the quantities of the second order, it follows that

$$\vec{n} \cong \vec{e_r} + \frac{\partial w}{R\partial\theta}\vec{e_\theta} + \frac{1}{R\sin\theta}\frac{\partial w}{\partial\varphi}\vec{e_\varphi} \quad . \tag{g4}$$

Consider the initial, non-deformed state of the shell and two points of co-ordinates $A_0(R, \theta, \varphi)$ and $B_0(R + h, \theta, \varphi)$, where H=2h is the thickness of the shell. It follows that the segment $\overrightarrow{A_0B_0}$ has the unit vector equal to $\vec{e_r}$. At an arbitrary time, in the deformed state of the shell, the point A_0 is displaced to a point A having the position vector equal to

$$\vec{r_A} = R\vec{e_r} + \vec{u}(R, \theta, \varphi, t) \ , \tag{g5}$$

while the point B_0 is displaced to the point B having the position vector equal to

$$\vec{r_B} = (R + h)\vec{e_r} + \vec{u}(R + h, \theta, \varphi, t) \ . \tag{g6}$$

Here, $\vec{u}$ is the displacement vector at a point of certain spherical co-ordinates. Assuming the shell is thin, quantities of the order h^2 are neglected and the segment $\overrightarrow{AB}$ has the unit vector equal to

$$\left(\vec{r_B} - \vec{r_A}\right) / \left|\vec{r_B} - \vec{r_A}\right| \cong \left(1 + \frac{\partial u_r}{\partial r}\right)\vec{e_r} + \frac{\partial u_\theta}{\partial r}\vec{e_\theta} + \frac{\partial u_\varphi}{\partial r}\vec{e_\varphi} \quad . \tag{g7}$$

The partial derivatives in (g7) are computed at the point (R, θ, φ). It is assumed that Bernoulli's hypothesis is valid for all time. It follows that a segment inside the shell, which is initially normal to the median spherical surface, will be always normal to the median surface during the deformation. From (g4) and (g7), it is supposed that the displacement vector has the elements

$$\begin{cases} u_r(r, \theta, \varphi, t) = -w(\theta, \varphi, t) \\ u_\theta(r, \theta, \varphi, t) = \dfrac{r - R}{R}\dfrac{\partial w}{\partial\theta} + x(\theta, \varphi, t) \\ u_\varphi(r, \theta, \varphi, t) = \dfrac{r - R}{R}\dfrac{1}{\sin\theta}\dfrac{\partial w}{\partial\varphi} + y(\theta, \varphi, t) \end{cases} \quad , \tag{g8}$$

where x and y are two unknown functions representing the horizontal displacements of the points initially placed on the median sphere. A further hypothesis on x and y will be later considered. It follows the elements of the strain tensor are

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = 0 \ , \tag{g9}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) = -\frac{x}{2r} \ , \tag{g10}$$

$$\varepsilon_{r\varphi} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \varphi} - \frac{u_\varphi}{r} + \frac{\partial u_\varphi}{\partial r}\right) = -\frac{y}{2r} \ , \tag{g11}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) = \frac{r-R}{rR}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\left(\frac{\partial x}{\partial \theta} - w\right) \ , \tag{g12}$$

$$\varepsilon_{\theta\varphi} = \frac{1}{2r}\left(\frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial \varphi} - \frac{u_\varphi}{\tan\theta} + \frac{\partial u_\varphi}{\partial \theta}\right) = \frac{r-R}{rR}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial w}{\partial \varphi}\right) + \frac{1}{2r}\left(\frac{1}{\sin\theta}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial \theta} - \frac{y}{\tan\theta}\right) \ , \tag{g13}$$

and

$$\varepsilon_{\varphi\varphi} = \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + u_r + \frac{u_\theta}{\tan\theta}\right) = \frac{r-R}{rR}\left(\frac{1}{\sin^2\theta}\frac{\partial^2 w}{\partial \varphi^2} + \frac{1}{\tan\theta}\frac{\partial w}{\partial \theta}\right) + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial y}{\partial \varphi} - w + \frac{x}{\tan\theta}\right) \tag{g14}$$

Hence the trace of the strain tensor is

$$\mathrm{tr}\,\varepsilon = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\varphi\varphi} = \frac{R(r-R)}{r}\Delta^* w + \frac{1}{r\sin\theta}\left[-2w\sin\theta + \frac{\partial}{\partial \theta}\left(x\sin\theta\right) + \frac{\partial y}{\partial \varphi}\right] \ , \tag{g15}$$

where the two-dimensional LAPLACE operator is

$$\Delta^* w = \frac{1}{R^2\sin\theta}\left[\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial w}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 w}{\partial \varphi^2}\right] \ . \tag{g16}$$

G.2) Quasi-mean values. Equations of motion.

Consider the stress tensor in spherical co-ordinates. For each element of the tensor, the corresponding quasi-mean value is defined, for example, by

$$\Sigma_{rr} = \overline{\sigma_{rr}} = \frac{1}{H}\int_{R-h}^{R+h}\frac{r}{R}\sigma_{rr}\,dr \ . \tag{g17}$$

In the same way, the corresponding quasi-moment is defined by

$$M_{rr} = \frac{1}{H}\int_{R-h}^{R+h}\frac{r}{R}(r-R)\sigma_{rr}\,dr \ . \tag{g18}$$

The equations of motion in spherical co-ordinates are

$$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + 2\frac{\sigma_{rr}}{r} + \frac{\partial \sigma_{r\theta}}{r\partial \theta} + \frac{\sigma_{r\theta}}{r\tan\theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{r\varphi}}{\partial \varphi} - \frac{\sigma_{\theta\theta}+\sigma_{\varphi\varphi}}{r} - \rho g = \rho\frac{\partial^2 u_r}{\partial t^2} \\
&\frac{\partial \sigma_{r\theta}}{\partial r} + 3\frac{\sigma_{r\theta}}{r} + \frac{\partial \sigma_{\theta\theta}}{r\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\theta\varphi}}{\partial \varphi} + \frac{\sigma_{\theta\theta}-\sigma_{\varphi\varphi}}{r\tan\theta} = \rho\frac{\partial^2 u_\theta}{\partial t^2} \\
&\frac{\partial \sigma_{r\varphi}}{\partial r} + 3\frac{\sigma_{r\varphi}}{r} + \frac{\partial \sigma_{\theta\varphi}}{r\partial \theta} + \frac{2\sigma_{\theta\varphi}}{r\tan\theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} = \rho\frac{\partial^2 u_\varphi}{\partial t^2}
\end{aligned} \ . \tag{g19}$$

In the above equations, ρ denotes the density of the shell in the deformed state. Assumed to be a negligible second order effect, that density will be replaced by the initial density ρ_0 at the right side of (g19).

The first equation in (g19) will be multiplied by r^2 and the quasi-mean operator defined by (g17) will be applied. The next two equations in (g19) will be multiplied by r and the quasi-moment operator defined by (g18) will be applied. In order to do that, some intermediary results are necessary.

G.3) Integrals of the stress elements. Quasi-moments.

Let the stress values on the upper/lower faces of the shell be denoted as

$$\sigma^U = \sigma \ (R+h, \theta, \varphi, t) \tag{g20}$$

and, respectively, by

$$\sigma^L = \sigma \ (R-h, \theta, \varphi, t) \ . \tag{g21}$$

Elementary computations show that

$$\frac{1}{H}\frac{1}{R}\int_{R-h}^{R+h}\left(r^2\frac{\partial \sigma_{rr}}{\partial r} + 2r\sigma_{rr}\right)dr = \frac{R}{H}\left[\left(1+\frac{h}{R}\right)^2 \sigma_{rr}^U - \left(1-\frac{h}{R}\right)^2 \sigma_{rr}^L\right] \ . \tag{g22}$$

In the case of the elastic shell, the HOOKE's law leads to

$$\sigma_{r\theta} = 2\mu\,\varepsilon_{r\theta} = -\mu x / r \quad , \quad \Sigma_{r\theta} = -\mu x / R \ . \tag{g23}$$

Integrating by parts, it follows that

$$\frac{1}{H}\int_{R-h}^{R+h}\frac{r^2}{R}(r-R)\frac{\partial \sigma_{r\theta}}{\partial r}dr = \frac{1}{HR}\left\{\left[r^2(r-R)\sigma_{r\theta}\right]_{R-h}^{R+h} + \mu x\int_{R-h}^{R+h}(3r-2R)dr\right\}$$
$$= \frac{R}{2}\left[\left(1+\frac{h}{R}\right)^2 \sigma_{r\theta}^U + \left(1-\frac{h}{R}\right)^2 \sigma_{r\theta}^L\right] - R\,\Sigma_{r\theta} \ . \tag{g24}$$

Also,

$$\frac{1}{H}\int_{R-h}^{R+h}\frac{r^2}{R}(r-R)\frac{3\sigma_{r\theta}}{r}dr = 0 \ . \tag{g25}$$

Similar relations are derived for $\sigma_{r\varphi}$.

Also,

$$M_{\theta\theta} = \left(\lambda\Delta^* w + \frac{2\mu}{R^2}\frac{\partial^2 w}{\partial\theta^2}\right)\frac{1}{H}\int_{R-h}^{R+h}(r-R)^2 dr = \frac{h^2}{3}\left(\lambda\Delta^* w + \frac{2\mu}{R^2}\frac{\partial^2 w}{\partial\theta^2}\right) \ . \tag{g26}$$

In the same way, it follows that

$$M_{\theta\varphi} = \frac{h^2}{3}\frac{2\mu}{R^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial w}{\partial\varphi}\right) \ , \tag{g27}$$

and

$$M_{\varphi\varphi} = \frac{h^2}{3}\left[(\lambda+2\mu)\Delta^* w - \frac{2\mu}{R^2}\frac{\partial^2 w}{\partial\theta^2}\right] \ . \tag{g28}$$

G.4) Integrals of the displacement vector.

Using (g8), it follows

$$\frac{1}{H}\int_{R-h}^{R+h}\frac{r^2}{R}\rho_0\frac{\partial^2 u_r}{\partial t^2}dr = -\rho_0 R\left[1+\frac{1}{3}\left(\frac{h}{R}\right)^2\right]\frac{\partial^2 w}{\partial t^2} \ . \tag{g29}$$

Also,

$$\frac{1}{H}\int_{R-h}^{R+h}\frac{r^2(r-R)}{R}\rho_0\frac{\partial^2 u_\theta}{\partial t^2}dr=\rho_0 R^2\frac{\partial^2}{\partial t^2}\left[\left(\frac{h^2}{3R^2}+\frac{h^4}{5R^4}\right)\frac{\partial w}{\partial\theta}+\frac{2x}{3}\left(\frac{h}{R}\right)^2\right] \quad , \tag{g30}$$

and

$$\frac{1}{H}\int_{R-h}^{R+h}\frac{r^2(r-R)}{R}\rho_0\frac{\partial^2 u_\varphi}{\partial t^2}dr=\rho_0 R^2\frac{\partial^2}{\partial t^2}\left[\left(\frac{h^2}{3R^2}+\frac{h^4}{5R^4}\right)\frac{1}{\sin\theta}\frac{\partial w}{\partial\varphi}+\frac{2y}{3}\left(\frac{h}{R}\right)^2\right] \quad . \tag{g31}$$

G.5) Equations of motion in quasi-mean values.

Let the first equation in (g19) be multiplied by r^2. By applying the quasi-mean operator defined by (g17), it follows

$$\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,\Sigma_{r\theta}\right)+\frac{\partial\Sigma_{r\varphi}}{\partial\varphi}\right]-\left(\Sigma_{\theta\theta}+\Sigma_{\varphi\varphi}\right)-\frac{1}{RH}\int_{R-h}^{R+h}\rho r^2 g(r)dr$$
$$+\frac{R}{H}\left[(1+h/R)^2\sigma_{rr}^{U}-(1-h/R)^2\sigma_{rr}^{L}\right]=-\rho_0 R\left[1+\frac{1}{3}\left(\frac{h}{R}\right)^2\right]\frac{\partial^2 w}{\partial t^2} \tag{g32}$$

The next two equations in (g19) are multiplied by r and the quasi-moment operator defined by (g18) is applied. Hence

$$\frac{1}{R}\left(\frac{\partial M_{\theta\theta}}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial M_{\theta\varphi}}{\partial\varphi}\right)-\Sigma_{r\theta}+\frac{M_{\theta\theta}-M_{\varphi\varphi}}{R\tan\theta}+\frac{1}{2}\left[(1+h/R)^2\sigma_{r\theta}^{U}+(1-h/R)^2\sigma_{r\theta}^{L}\right]$$
$$=\rho_0 R\frac{\partial^2}{\partial t^2}\left\{\left[\frac{1}{3}\left(\frac{h}{R}\right)^2+\frac{1}{5}\left(\frac{h}{R}\right)^4\right]\frac{\partial w}{\partial\theta}+\frac{2x}{3}\left(\frac{h}{R}\right)^2\right\} \tag{g33}$$

and

$$\frac{1}{R}\left(\frac{\partial M_{\theta\varphi}}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial M_{\varphi\varphi}}{\partial\varphi}\right)-\Sigma_{r\varphi}+\frac{2M_{\theta\varphi}}{R\tan\theta}+\frac{1}{2}\left[(1+h/R)^2\sigma_{r\varphi}^{U}+(1-h/R)^2\sigma_{r\varphi}^{L}\right]$$
$$=\rho_0 R\frac{\partial^2}{\partial t^2}\left\{\left[\frac{1}{3}\left(\frac{h}{R}\right)^2+\frac{1}{5}\left(\frac{h}{R}\right)^4\right]\frac{1}{\sin\theta}\frac{\partial w}{\partial\varphi}+\frac{2y}{3}\left(\frac{h}{R}\right)^2\right\} \tag{g34}$$

G.6) Quasi-mean value of the shell density. The differential equation.

According to (g17), the quasi-mean of a constant is equal to that constant. Applying the quasi-mean operator to the mass balance equation in the linear approximation gives

$$\bar{\rho}=\rho_0(1-\overline{\mathrm{tr}\,\varepsilon}) \tag{g35}$$

It will be further assumed that the quasi-mean of the density is constant and equal to its initial value. From (g35) it follows that

$$\overline{\mathrm{tr}\,\varepsilon}=0 \quad , \tag{g36}$$

i.e., using (g15),

$$\frac{\partial}{\partial\theta}(x\sin\theta)+\frac{\partial y}{\partial\varphi}=2w\sin\theta \quad . \tag{g37}$$

For the elastic shell,

$$\sigma = \lambda \mathrm{tr}\,\varepsilon\, 1 + 2\mu\varepsilon \quad . \tag{g38}$$

By using (g9), it follows

$$\sigma_{\theta\theta} + \sigma_{\varphi\varphi} = 2(\lambda+\mu)\mathrm{tr}\,\varepsilon \quad , \tag{g39}$$

or, using (g36),

$$\Sigma_{\theta\theta} + \Sigma_{\varphi\varphi} = 0 \quad . \tag{g40}$$

The term $(h/R)^4$ will be neglected. Using (g40) and substituting $\Sigma_{r\theta}$ and $\Sigma_{r\varphi}$ between (g32)-(g34) it follows

$$\frac{1}{R^2}\left[\frac{\partial^2(\sin\theta M_{\theta\theta})}{\partial\theta^2} + \frac{2}{\sin\theta}\frac{\partial^2(\sin\theta M_{\theta\varphi})}{\partial\theta\partial\varphi} + \frac{1}{\sin\theta}\frac{\partial^2 M_{\varphi\varphi}}{\partial\varphi^2} - \frac{\partial}{\partial\theta}(\cos\theta M_{\varphi\varphi})\right]$$
$$+\frac{\sin\theta}{H}\left[(1+h/R)^2\sigma_{rr}^U - (1-h/R)^2\sigma_{rr}^L\right] + \frac{1}{2R}\frac{\partial}{\partial\theta}\left\{\sin\theta\left[(1+h/R)^2\sigma_{r\theta}^U + (1-h/R)^2\sigma_{r\theta}^L\right]\right\}$$
$$+\frac{1}{2R}\frac{\partial}{\partial\varphi}\left[(1+h/R)^2\sigma_{r\varphi}^U + (1-h/R)^2\sigma_{r\varphi}^L\right] - \frac{1}{HR}\int_{R-h}^{R+h}\rho r^2 g dr \tag{g41}$$
$$= \rho_0 \sin\theta\left\{\frac{h^2}{3}\Delta^*\left(\frac{\partial^2 w}{\partial t^2}\right) - \left[1-\left(\frac{h}{R}\right)^2\right]\frac{\partial^2 w}{\partial t^2}\right\}$$

A correction due to the compressive horizontal stresses σ^{NS}, σ^{WE} acting at the ends of the shell, approximately along the θ – and φ – axes respectively, follows to be further considered.

G.7) The buckling of a spherical shell.

Consider the element ABCD of the mean surface of the deformed shell from Fig.G1. It is centred at the point M, where the local unit vectors are $\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi$. The centres of the lateral sides are denoted by M_k, $k = 1,2,3,4$. On the meridian cross section M_4MM_2 is acting a normal compressive stress σ^{WE}, having the approximate direction from West to East, and a tangential stress τ^{NS}, having the approximate direction from North to South. On the parallel cross section M_1MM_3 is acting a normal compressive stress σ^{NS}, having the approximate direction from North to South, and a tangential stress τ^{WE}, having the approximate direction from West to East. Let Φ be the angle between the normal vector to the meridian section M_4MM_2 and the unit vector $\vec{e}_\varphi$. Also, let Θ be the angle between the normal vector to the parallel section M_1MM_3 and the unit vector $\vec{e}_\theta$. The concentrated force acting at the point M_1 is

$$\vec{\Sigma}_1 = \left\{\left(\sigma^{WE} - \frac{\partial\sigma^{WE}}{\partial\varphi}\frac{d\varphi}{2}\right)\left[\cos\left(\Phi - \frac{\partial\Phi}{\partial\varphi}\frac{d\varphi}{2}\right)\vec{e}_\varphi + \sin\left(\Phi - \frac{\partial\Phi}{\partial\varphi}\frac{d\varphi}{2}\right)\vec{e}_r\right]\right.$$
$$\left.+\left(\tau^{NS} - \frac{\partial\tau^{NS}}{\partial\varphi}\frac{d\varphi}{2}\right)\left[\cos\left(\Theta - \frac{\partial\Theta}{\partial\varphi}\frac{d\varphi}{2}\right)\vec{e}_\theta + \sin\left(\Theta - \frac{\partial\Theta}{\partial\varphi}\frac{d\varphi}{2}\right)\vec{e}_r\right]\right\} \times \left(R - w + \frac{\partial w}{\partial\varphi}\frac{d\varphi}{2}\right)H d\theta \tag{g42}$$

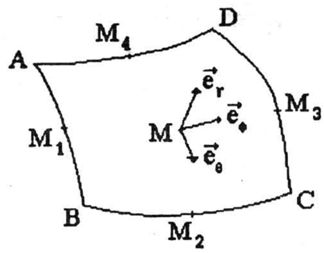

Fig.G1. A rectangular element of the mean surface of the deformed shell

In the same way, the concentrated force acting at the point M_3 is equal to

$$\vec{\Sigma}_3 = -\left\{\left(\sigma^{WE} + \frac{\partial \sigma^{WE}}{\partial \varphi}\frac{d\varphi}{2}\right)\left[\cos\left(\Phi + \frac{\partial \Phi}{\partial \varphi}\frac{d\varphi}{2}\right)\vec{e}_\varphi + \sin\left(\Phi + \frac{\partial \Phi}{\partial \varphi}\frac{d\varphi}{2}\right)\vec{e}_r\right]\right.$$
$$\left. + \left(\tau^{NS} + \frac{\partial \tau^{NS}}{\partial \varphi}\frac{d\varphi}{2}\right)\left[\cos\left(\Theta + \frac{\partial \Theta}{\partial \varphi}\frac{d\varphi}{2}\right)\vec{e}_\theta + \sin\left(\Theta + \frac{\partial \Theta}{\partial \varphi}\frac{d\varphi}{2}\right)\vec{e}_r\right]\right\} \times \left(R - w - \frac{\partial w}{\partial \varphi}\frac{d\varphi}{2}\right) H d\theta \qquad \text{(g43)}$$

The concentrated forces acting at the points M_2 and M_4 are respectively equal to

$$\vec{\Sigma}_{2,4} = \mp\left\{\left(\sigma^{NS} \pm \frac{\partial \sigma^{NS}}{\partial \theta}\frac{d\theta}{2}\right)\left[\cos\left(\Theta \pm \frac{\partial \Theta}{\partial \theta}\frac{d\theta}{2}\right)\vec{e}_\theta + \sin\left(\Theta \pm \frac{\partial \Theta}{\partial \theta}\frac{d\theta}{2}\right)\vec{e}_r\right]\right.$$
$$\left. + \left(\tau^{WE} \pm \frac{\partial \tau^{WE}}{\partial \theta}\frac{d\theta}{2}\right)\left[\cos\left(\Phi \pm \frac{\partial \Phi}{\partial \theta}\frac{d\theta}{2}\right)\vec{e}_\varphi + \sin\left(\Phi \pm \frac{\partial \Phi}{\partial \theta}\frac{d\theta}{2}\right)\vec{e}_r\right]\right\} \qquad \text{(g44)}$$
$$\times \left(R - w \mp \frac{\partial w}{\partial \theta}\frac{d\theta}{2}\right)\sin(\theta \pm d\theta) H d\varphi$$

But

$$\cos\Phi \cong 1 \quad , \quad \cos\Theta \cong 1 \quad , \quad \sin\Phi \cong \Phi \cong \frac{1}{R\sin\theta}\frac{\partial w}{\partial \varphi} \quad , \quad \sin\Theta \cong \Theta \cong \frac{1}{R}\frac{\partial w}{\partial \theta} \quad . \qquad \text{(g45)}$$

Let $p = p(\theta, \varphi)$ a surface density of forces normal to the element of the shell, having the same mechanical effect as the presence of the compressive stress. The force due to that density is equal to

$$\vec{\Sigma}_p = p(\theta, \varphi)(R - w)^2 \sin\theta d\theta d\varphi \, \vec{e}_r \quad . \qquad \text{(g46)}$$

It follows the deformed element of the shell is into an equilibrium state due to the action of the lateral stress and to the opposite force $-\vec{\Sigma}_p$, i.e.

$$\vec{\Sigma}_1 + \vec{\Sigma}_2 + \vec{\Sigma}_3 + \vec{\Sigma}_4 - \vec{\Sigma}_p = \vec{0} \quad . \qquad \text{(g47)}$$

Hence the next three equations of equilibrium are obtained:

$$\left(2\cos\theta - \frac{\sin\theta}{R}\frac{\partial w}{\partial\theta}\right)\left(\sigma^{NS}\frac{\partial w}{\partial\theta} + \frac{\tau^{WE}}{\sin\theta}\frac{\partial w}{\partial\varphi}\right)$$

$$+\sin\theta\frac{\partial}{\partial\theta}\left(\sigma^{NS}\frac{\partial w}{\partial\theta} + \frac{\tau^{WE}}{\sin\theta}\frac{\partial w}{\partial\varphi}\right) + \frac{\partial}{\partial\varphi}\left(\frac{\sigma^{WE}}{\sin\theta}\frac{\partial w}{\partial\varphi} + \tau^{NS}\frac{\partial w}{\partial\theta}\right) = -p(\theta,\varphi)\frac{R(R-w)}{H}\sin\theta \quad , \qquad \text{(g48)}$$

$$\sigma^{NS}\left(2\cos\theta - \frac{\sin\theta}{R}\frac{\partial w}{\partial\theta}\right) + \sin\theta\frac{\partial\sigma^{NS}}{\partial\theta} + \frac{\partial\tau^{NS}}{\partial\varphi} = 0 \quad , \qquad \text{(g49)}$$

and

$$\frac{\partial\sigma^{WE}}{\partial\varphi} + \tau^{WE}\left(2\cos\theta - \frac{\sin\theta}{R}\frac{\partial w}{\partial\theta}\right) + \sin\theta\frac{\partial\tau^{WE}}{\partial\theta} = 0 \quad . \qquad \text{(g50)}$$

For the particular case when σ^{WE} is a constant and $\tau^{NS} = \tau^{WE} = 0$, eqs.(g48)-(g50) show that the presence of the lateral compressive stress is equivalent to a supplemental load placed on the upper face of the shell, having the value

$$p(\theta,\varphi) = -\frac{H}{R^2}\left(\sigma^{NS}\frac{\partial^2 w}{\partial\theta^2} + \sigma^{WE}\frac{1}{\sin^2\theta}\frac{\partial^2 w}{\partial\varphi^2}\right) \quad , \qquad \text{(g51)}$$

a result similar to the case of the plane plate (Timoshenko and Woinowsky-Krieger 1959; Nowacki 1961). Quantities of the second order, like $(\partial w / \partial\theta)^2$ have been neglected again.

G.8) Load on the upper face. Stress on the lower surface of the shell.

Consider now the differential equation (g41). Usually, the horizontal loads for the upper face of the shell are neglected, and it is assumed that

$$\sigma_{rr}^{U} = -\rho^{F} g w - P\left(1 - \frac{1}{g}\frac{\partial^2 w}{\partial t^2}\right) \quad , \quad \sigma_{r\theta}^{U} = 0 \quad , \quad \sigma_{r\varphi}^{U} = 0 \quad , \qquad \text{(g52)}$$

where ρ^F is the density of the filling sediments, P is the load and an inertial term is considered. For the material below the shell, the next constitutive equation is assumed

$$\sigma^{M} = \left[p_0 - \rho^{M} g(R - h - w)\right]\mathbf{1} + \lambda^{M}\mathrm{tr}\,\varepsilon^{M}\,\mathbf{1} + 2\mu^{M}\varepsilon^{M} + 2\eta^{M}\frac{\partial\varepsilon^{M}}{\partial t} \quad , \qquad \text{(g53)}$$

where p_0 is a reference pressure, ρ^M is the density of that material and $\mathbf{1}$ is the unit tensor. The elastic coefficients are λ^M, μ^M and the viscosity is denoted by η^M. The strain tensor inside the material is ε^M and the strain rate here is $\partial\varepsilon^M / \partial t$.

The first boundary condition assumed on the lower face of the shell, having the equation $r = R - h$, is the continuity of the displacement vector. Hence the elements of the displacement vector inside the material placed immediately below the lower face of the shell are assumed to be equal to the same elements at the points of the shell placed on the lower face. By using eqs. (g8), it follows

$$\begin{cases} u_r(r,\theta,\varphi,t) = -w(\theta,\varphi,t) \\ u_\theta(r,\theta,\varphi,t) = \dfrac{-h}{R}\dfrac{\partial w}{\partial\theta} + x(\theta,\varphi,t) \\ u_\varphi(r,\theta,\varphi,t) = \dfrac{-h}{R}\dfrac{1}{\sin\theta}\dfrac{\partial w}{\partial\varphi} + y(\theta,\varphi,t) \end{cases} \qquad \text{(g54)}$$

Hence the next values for the strain tensor immediately below the shell are obtained

$$\varepsilon_{rr}^{M}=0 \quad , \quad \varepsilon_{r\theta}^{M}=-\frac{1}{2R}\frac{\partial w}{\partial \theta}-\frac{x}{2(R-h)} \quad , \quad \varepsilon_{r\varphi}^{M}=-\frac{1}{2R}\frac{1}{\sin\theta}\frac{\partial w}{\partial \varphi}-\frac{y}{2(R-h)} \quad . \tag{g55}$$

The normal vector at the lower face of the shell is $\vec{e}_r$. From Newton's third low, it follows a relation between the stress σ^S inside the shell and the stress σ^M inside the material below the shell, i.e.

$$\sigma^S\left(-\vec{e}_r\right)=-\sigma^M\left(\vec{e}_r\right) \quad , \tag{g56}$$

on the lower surface of the shell. Here, the next elements of the stress are obtained after some elementary computations

$$\sigma_{rr}^{L}=p_0-\rho^M g(R-h-w)-\frac{Rh}{R-h}\lambda^M\Delta^* w \quad , \tag{g57}$$

$$\sigma_{r\theta}^{L}=-\mu^M\left(\frac{1}{R}\frac{\partial w}{\partial \theta}+\frac{x}{R-h}\right)-\eta^M\left[\frac{1}{R}\frac{\partial}{\partial \theta}\left(\frac{\partial w}{\partial t}\right)+\frac{1}{R-h}\frac{\partial x}{\partial t}\right] \quad , \tag{g58}$$

and

$$\sigma_{r\varphi}^{L}=-\mu^M\left(\frac{1}{R\sin\theta}\frac{\partial w}{\partial \varphi}+\frac{y}{R-h}\right)-\eta^M\left[\frac{1}{R\sin\theta}\frac{\partial}{\partial \varphi}\left(\frac{\partial w}{\partial t}\right)+\frac{1}{R-h}\frac{\partial y}{\partial t}\right] \quad . \tag{g59}$$

G.9) The differential equation of time dependent flexure.

The reference pressure p_0 in (g53) is selected in order the flexure of the shell to vanish in the absence of the load. Substituting the loads on the upper/lower faces of the shell in eq.(s41) and taking into account the presence of the lateral gstress, a generalisation of the Sophie GERMAIN plain plate static equation in the case of a time dependent flexure of a spherical elastic shell is obtained as

$$\begin{aligned}
&\frac{H^3}{12}\left[(\lambda+2\mu)\Delta^*\Delta^* w+2\mu\frac{\Delta^* w}{R^2}\right]-\left(1+\frac{h}{R}\right)^2\left[\rho^F g w+P\left(1-\frac{1}{g}\frac{\partial^2 w}{\partial t^2}\right)\right]\\
&+\left(1-\frac{h}{R}\right)^2\left(\rho^M g w+\frac{Rh}{R-h}\lambda^M\Delta^* w\right)-\frac{H}{2R}\left(1-\frac{h}{R}\right)^2\mu^M\left(R\Delta^* w+\frac{2w}{R-h}\right)\\
&-\frac{H}{2R}\left(1-\frac{h}{R}\right)^2\eta^M\left[R\Delta^*\left(\frac{\partial w}{\partial t}\right)+\frac{2}{R-h}\frac{\partial w}{\partial t}\right]+\frac{H}{R^2}\left(\sigma^{NS}\frac{\partial^2 w}{\partial \theta^2}+\sigma^{WE}\frac{1}{\sin^2\theta}\frac{\partial^2 w}{\partial \varphi^2}\right)\\
&=\rho_0 H\left\{\frac{H^2}{12}\Delta^*\left(\frac{\partial^2 w}{\partial t^2}\right)-\left[1-\left(\frac{h}{R}\right)^2\right]\frac{\partial^2 w}{\partial t^2}\right\}
\end{aligned} \quad . \tag{g60}$$

If quantities of the order $(h/R)^2$ are neglected with respect to unit, it follows finally that

$$\begin{aligned}
&D\Delta^*\Delta^* w+\frac{H}{2}\left(\lambda^M-\mu^M\right)\Delta^* w+\left(\rho^M-\rho^F-\rho^*\right)g w\\
&+H\left[\left(\sigma^{NS}+\sigma^*\right)\frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2}+\left(\sigma^{WE}+\sigma^*\right)\frac{1}{R^2\sin^2\theta}\frac{\partial^2 w}{\partial \varphi^2}\right]\\
&=P^*+\frac{H}{2}\eta^M\frac{\partial}{\partial t}\left(\Delta^* w+\frac{2w}{R^2}\right)-\left(\rho H+\frac{P^*}{g}\right)\frac{\partial^2 w}{\partial t^2}+\frac{\rho}{12}H^3\frac{\partial^2}{\partial t^2}(\Delta^* w)
\end{aligned} \quad , \tag{g61}$$

where

$$D = \frac{(\lambda + 2\mu)H^3}{12} = \rho a^2 \frac{H^3}{12} \qquad \text{(g62)}$$

is the flexural rigidity and a is the velocity of the P-wave through the shell. Also,

$$\rho^* = \left(\rho^M + \rho^F + \frac{\mu^M}{gR}\right)\frac{H}{R} \quad , \quad \sigma^* = \frac{\mu}{6}\frac{H^2}{R^2} \quad , \quad P^* = P\left(1 + \frac{H}{R}\right) \quad . \qquad \text{(g63)}$$

G.10) Spherical effects with respect to the plane plate.

For the usual materials, $\lambda^M = \mu^M$. In that case, the corresponding equation for the plane plate (Ivan 1997) is.

$$D\Delta^*\Delta^* w + H\left(\sigma_x^c \frac{\partial^2 w}{\partial x^2} + \sigma_y^c \frac{\partial^2 w}{\partial y^2}\right) + \left(\rho^M - \rho^F\right)gw$$
$$= P + \frac{H}{2}\eta^M \frac{\partial}{\partial t}\left(\Delta^* w\right) - \left(\rho H + \frac{P}{g}\right)\frac{\partial^2 w}{\partial t^2} + \frac{\rho}{12}H^3 \frac{\partial^2}{\partial t^2}(\Delta^* w) \qquad \text{(g64)}$$

With respect to (g64), a change of the density difference $\rho^M - \rho^F$ according to (g63a) and a substitution of the real lateral stresses σ^{NS}, σ^{WE} by their apparent values $\sigma^{NS} + \sigma^*, \sigma^{WE} + \sigma^*$ can be observed in eq.(s61). A supplemental load is present according to (g63c). For usual values (e.g. Ivan 1997a) like $H/R \propto 1/100$, $\mu, \mu^M \propto 10^{11}$ Pa, $\rho^M, \rho^F \propto 3000 \text{kgs}/\text{m}^3, \sigma^{NS}, \sigma^{WE} \propto 30$ MPa, all these effects are usually negligible and difficult to be observed in real life. To compare $2w/R^2$ to $\Delta^* w$ in the left side of (g61), the case of a rectangular plate having the sides equal to L_x, L_y is considered. Here, the flexure is proportional to a product of sins (cosines) functions, i.e. $w \propto \sin(m\pi x/L_x)\sin(n\pi y/L_y)$. It follows the LAPLACE operator of the flexure is proportional to $\Delta^* w \propto \pi^2\left(m^2/L_x^2 + n^2/L_y^2\right)\sin(m\pi x/L_x)\sin(n\pi y/L_y)$. For the fundamental mode $(m = n = 1)$ it follows that

$$\frac{2w/R^2}{\Delta^* w} = \frac{1}{\pi^2 R^2 (1/L_x^2 + 1/L_y^2)} \quad . \qquad \text{(g65)}$$

That ratio is negligible too in the usual cases.

It can be concluded that in the usual cases, the sphericity of the crustal plates can be ignored and the equation derived for the flat plate can be used.

H) ELEMENTS OF RHEOLOGY

H.1) Introduction.

Especially for geological processes at a large time scale and great values of the stress, the internal friction of the material cannot be ignored. Consequently, the HOOKE's law has to be replaced by assuming different constitutive equations (models). Their expressions are mainly depending on the time scale of the geophysical process to be modelled. In relation to seismic or seismological applications, for example, short periods and short stresses are required (usually, seconds up to minutes, with a maximum value around one hour for the fundamental mode of the free oscillations of the Earth). Here, the non-elasticity is related to the very short period irreversible changes in the crystal defect structures of the medium (e.g. opening/closing of pre-existing cracks) and/or to the energy lost by friction at the two sides of a crack or on the non-elastic boundary coupling grain particles to the adjacent material (Aki and Richards, 1980; Ranalli, 1987; Wahr, 1996).
With respect to the mathematical relation between stress and strain, there are two kinds of constitutive equations (models).

H.2) Linear models.

Simplified models involves a linear relation between stress (and its derivatives of various orders with respect to time) and strain (together with its time derivatives).
In the beginning, only the 1-D case will be discussed. More general examples follow to be presented in relation with the dynamic aspects of the flexure of a plate (shell) and to the accretion prism. For each constitutive equation, a mechanical analogue can be considered. The elastic part will be represented by a spring, while the inelastic (viscous) behaviour is associated to a dashpot. Both parts are supposed to be linear ones, i.e. a linear relation $\sigma = 2\mu\varepsilon$ is valid for the spring and a similar linear relation holds for the dashpot $\sigma = 2\eta\dot{\varepsilon}$.

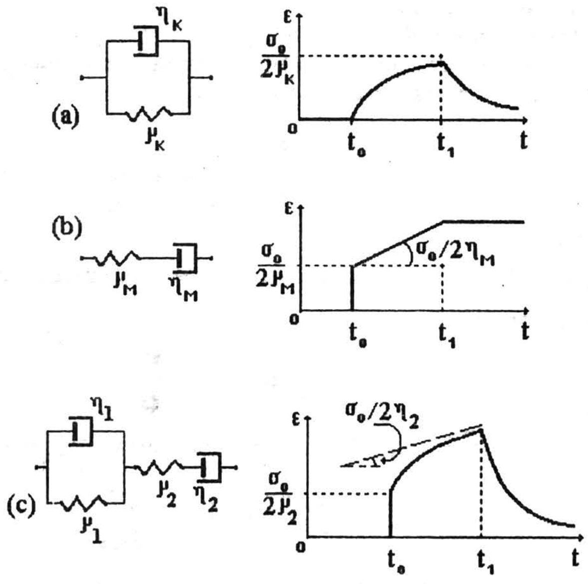

Fig.H1. (a) KELVIN-VOIGT model; (b) MAXWELL model; (c) BURGERS model

a) KELVIN-VOIGT (*strong viscous*) model.

The mechanical analogue of the first model to be considered is represented in Fig.H1a. The total stress is the sum between the stress in the spring and the stress in the dashpot, while the total deformation is equal to both the deformation of the spring and the deformation of the dashpot. It follows that KELVIN-VOIGT model has the next constitutive equation

$$\sigma = 2\mu_K\varepsilon + 2\eta_K\dot{\varepsilon}, \qquad \text{(h1)}$$

where the dot shows the (total, material) derivative with respect to time. Here, the second term is the inelastic one, η_K being the viscosity. Suppose now a constant stress equal to σ_0 is applied. Elementary computations show that the differential equation

$$\sigma_0 = 2\mu_K \varepsilon + 2\eta_K \frac{d\varepsilon}{dt} \tag{h2}$$

with the initial condition

$$\varepsilon\ (t = t_0) = 0 \tag{h3}$$

has the solution

$$\varepsilon = \frac{\sigma_0}{2\mu_K}\left[1 - \exp\left(-\frac{\mu_K}{\eta_K}(t - t_0)\right)\right], \tag{h4}$$

for $t \geq t_0$. Hence, for very great values of time, the strain approaches a limiting value equal to

$$\varepsilon_\infty = \frac{\sigma_0}{2\mu_K}. \tag{h5}$$

The "flowage function", denoted by

$$J(t) = \frac{1}{2\mu_K}\left[1 - \exp\left(-\frac{\mu_K}{\eta_K}(t - t_0)\right)\right], \tag{h6}$$

shows that for a constant stress (equal to unit here), there is a temporal variation of the strain.

Suppose now at a certain moment $t = t_1$, the constant stress σ_0 is removed, the corresponding strain at that moment being equal to ε_1. It follows now the corresponding solution decreases towards zero as

$$\varepsilon = \varepsilon_1 \exp\left(-\frac{\mu_K}{\eta_K}(t - t_1)\right).. \tag{h7}$$

b) MAXWELL (*viscous-elastic*) model.

Consider the mechanical analogue represented in Fig.H1b. The total stress is equal to both the stress of the spring and to the stress of the dashpot, while the total deformation is the sum between the deformation of the spring and the deformation of the dashpot. It follows that MAXWELL model has the next constitutive equation

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{2\mu_M} + \frac{\sigma}{2\eta_M}, \tag{h8}$$

with the initial condition represented by (h3).

Suppose again a constant stress equal to σ_0 is applied. The spring is instantly deformed to a value equal corresponding to the first term in the right hand of (h8), i.e.

$$\varepsilon_0 = \frac{\sigma_0}{2\mu_M}. \tag{h9}$$

and the solution of (h8) (for a constant stress σ_0) with the initial condition (h9) is the straight line

$$\varepsilon = \frac{\sigma_0}{2\eta_M}(t - t_0) + \frac{\sigma_0}{2\mu_M}, \tag{h10}$$

having the slope related to the stress σ_0 and to the viscosity of the dashpot. Suppose now at a certain moment $t = t_1$, the constant stress σ_0 is removed, the corresponding strain at that moment being equal to ε_1. It follows from (h8) that the strain remains constant.

c) BURGERS (*general linear*) model.

The third model to be considered has the mechanical analogue represented in Fig.H1c.

EXERCISE. Show that the corresponding differential equation is

$$2\eta_1 \ddot{\varepsilon} + 2\mu_1 \dot{\varepsilon} = \frac{\eta_1}{\mu_2}\ddot{\sigma} + \left(\frac{\eta_1}{\eta_2} + \frac{\mu_1}{\mu_2} + 1\right)\dot{\sigma} + \frac{\mu_1}{\eta_2}\sigma. \tag{h11}$$

Consider now the same initial condition (h3) and suppose again a constant stress equal to σ_0 is applied. In a similar manner to MAXWELL model, the system is instantly deformed to a value equal to

$$\varepsilon_0=\frac{\sigma_0}{2\mu_2} \tag{h12}$$

and the solution of (h11) (for a constant stress σ_0) with the initial condition (h12) is

$$\varepsilon=\frac{\sigma_0}{2\mu_2}+\frac{\sigma_0}{2\eta_2}(t-t_0)+C\left[1-\exp\left(-\frac{\mu_1}{\eta_1}(t-t_0)\right)\right] \quad , \tag{h13}$$

where C is an unknown coefficient (because (h11) is a second order differential equation, two initial conditions are required to obtain the complete solution). However, differentiating (h13) it follows

$$\dot{\varepsilon}=\frac{\sigma_0}{2\eta_2}+C\frac{\mu_1}{\eta_1}\exp\left(-\frac{\mu_1}{\eta_1}(t-t_0)\right) \quad . \tag{h14}$$

Hence, for great values of time, the solution (h13) approaches asymptotically to a straight line having the slope equal to $\sigma_0/2\eta_2$. Suppose now at a certain moment $t=t_1$, the constant stress σ_0 is removed, the corresponding strain at that moment being equal to ε_1. It follows from (h13) that the strain decreases exponentially towards zero, i.e

$$\varepsilon=\varepsilon_1\exp\left(-\frac{\mu_1}{\eta_1}(t-t_1)\right) \quad . \tag{h15}$$

EXERCISE. Show that two (or more) springs / dashpots connected in series (or parallel) sequence are equivalent to a single spring / dashpot. Justify that the BURGERS model is the general linear model.

d) Remarks on the linear models.

In the most general case, the linear relation between stress and strain can be written as

$$P(D)\sigma=Q(D)\varepsilon \quad , \tag{h16}$$

where

$$P(D)=A^0+A^1D+A^2D^2+...+A^nD^n \tag{h17}$$

and

$$Q(D)=B^0+B^1D+B^2D^2+...+B^mD^m \tag{h18}$$

are formal polynomials of the variable $D=\frac{d}{dt}$ representing the derivative with respect to time, applied to stress and strain respectively. Here, $A^0,A^1,...,A^n,B^0,B^1,...,B^m$ are fourth rank tensors. For example, with respect to the MAXWELL body having the constitutive equation (h8), it follows that

$$P(D)=\frac{1}{2\eta_M}+\frac{1}{2\mu_M}D \quad , \quad Q(D)=D \tag{h19}$$

A common way to solve (h16) is by using the LAPLACE transform (e.g. Sokolnikoff and Redheffer, 1958). Consider a certain function of one real variable $f(t)$, providing that

1. $f(t)=0$, for $t<0$;
2. $f(t)$ is piecewise continuous on every finite interval;
3. there are two constants $0<M$, $a\le 0$ in order to have $|f(t)|\le M\exp(at)$, for an arbitrary t.

Under the above conditions, the LAPLACE transform of $f(t)$ is a new function of the variable p, defined by

$$L[f](p)=\int_0^\infty f(t)\exp(-pt)dt \tag{h20}$$

EXERCISE. Show that:

(a) The LAPLACE transform of the first derivative is

$$L\left[\frac{df}{dt}\right](p)=pL[f]-f(0+0) \tag{h21}$$

(b) $L[\exp(-at)](p)=\dfrac{1}{p+a} \quad , \quad 0\le a$ (h22)

(c) **The *convolution theorem*. Consider the functions f, g vanishing for negative values of their argument and a new function (also vanishing for negative values of the argument defined) by the *convolution product***

$$h(t)=(f*g)(t)=\int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau=\int_{0}^{t} f(\tau)g(t-\tau)d\tau \tag{h23}$$

Show that

$$L[f*g]=L[f]L[g] \tag{h24}$$

As an example, consider again the MAXWELL model by applying the LAPLACE transform to both sides of (h8). It follows that

$$pL[\varepsilon]-\varepsilon^0=\frac{1}{2\mu_M}\left(pL[\sigma]-\sigma^0\right)+\frac{1}{2\eta_M}L[\sigma] \quad , \tag{h25}$$

where $\sigma^0=\sigma(0+0), \varepsilon^0=\varepsilon(0+0)$ are the initial stress and strain respectively.
EXERCISE. By using the above properties of the LAPLACE transform, show that

$$\sigma(t)=2\mu_M\varepsilon(t)-\frac{2\mu_M^2}{\eta_M}\int_0^t\varepsilon(t)\exp\left[-\frac{\mu_M}{\eta_M}(t-\tau)\right]d\tau+\left(\sigma^0-2\mu_M\varepsilon^0\right)\exp\left(-\frac{\mu_M}{\eta_M}t\right) \tag{h26}$$

If the initial conditions are elastically coupled, i.e.

$$\sigma^0=2\mu_M\varepsilon^0 \tag{h27}$$

it follows an integral representation of the stress which is independent on the initial conditions

$$\sigma(t)=2\mu_M\varepsilon(t)-\frac{2\mu_M^2}{\eta_M}\int_0^t\varepsilon(t)\exp\left[-\frac{\mu_M}{\eta_M}(t-\tau)\right]d\tau \tag{h28}$$

Hence, the actual value of the stress is related both to the actual value of the deformation and to the previous values of the strain (i.e. the stress depends on the "history" of the deformation). If a constant strain $\varepsilon(t)=\varepsilon^0$, for $t\geq 0$ is applied to the MAXWELL body, it follows from (h28) that the stress decreases as

$$\sigma(t)=2\mu_M\varepsilon^0\exp\left(-\frac{\mu_M}{\eta_M}t\right) \quad , \tag{h29}$$

representing a "relaxation phenomenon" (stress decreases in time if a constant deformation is present). Here, the function

$$G(t)=2\mu_M\exp(-t/\tau_M) \tag{h30}$$

is the "relaxation" kernel and the parameter $\tau_M=\eta_M/\mu_M$ is the relaxation time.

A very similar approach (e.g. Wahr 1996) is based on the use of FOURIER transform (see Chapter F.13). Formally, the results derived by using FOURIER transform are derived from the same results obtained with LAPLACE transform by performing the substitution $p=iu$.

The above 1-D models can be generalised for the 3-D case. For example, consider again the MAXWELL body. There is a strong experimental evidence that the MAXWELL Rheology applies only to the dissipation of the shear energy, i.e. the stress and strain tensors in (h8) are the deviatoric tensors

$$\sigma\leftrightarrow\sigma-\frac{1}{3}\mathrm{tr}\,\sigma\,1 \quad , \quad \varepsilon\leftrightarrow\varepsilon-\frac{1}{3}\mathrm{tr}\,\varepsilon\,1 \tag{h31}$$

It will be further assumed that there is no dissipation of the compressional energy, i.e. a proportionality like

$$\mathrm{tr}\,\sigma=(3\lambda_M+2\mu_M)\mathrm{tr}\,\varepsilon \quad , \tag{h32}$$

is valid. By differentiating with respect to time

$$\mathrm{tr}\,\dot{\sigma}=(3\lambda_M+2\mu_M)\mathrm{tr}\,\dot{\varepsilon} \tag{h33}$$

Substituting (h31) and (h33) in (h8) it follows

$$\dot{\varepsilon}-\frac{1}{3}\mathrm{tr}\,\dot{\varepsilon}1=\frac{1}{2\mu_M}\left(\dot{\sigma}-\frac{3\lambda_M+2\mu_M}{3}\mathrm{tr}\,\dot{\varepsilon}1\right)+\frac{1}{2\eta_M}\left(\sigma-\frac{3\lambda_M+2\mu_M}{3}\mathrm{tr}\,\varepsilon1\right) \tag{h34}$$

Hence

$$\dot{\sigma}+\frac{\mu_M}{\eta_M}\sigma=2\mu_M\dot{\varepsilon}+\lambda_M \operatorname{tr}\dot{\varepsilon}\,1+\frac{\mu_M}{\eta_M}\frac{3\lambda_M+2\mu_M}{3}\operatorname{tr}\varepsilon\,1 \tag{h35}$$

By applying the FOURIER transform to both sides of (h35), it follows

$$\Phi\left[\sigma\right]=\tilde{\lambda}\operatorname{tr}\Phi\left[\varepsilon\right]1+2\tilde{\mu}\Phi\left[\varepsilon\right] \quad , \tag{h36}$$

i.e. a relation similar to HOOKE's law is valid between the FOURIER (or LAPLACE) transforms of stress and strain. Here, the coefficients similar to LAMÉ parameters are

$$\tilde{\lambda}=\lambda_M\frac{iu+\frac{\mu_M}{\eta_M}\left(1+\frac{2}{3}\frac{\mu_M}{\lambda_M}\right)}{iu+\frac{\mu_M}{\eta_M}} \quad , \quad \tilde{\mu}=\mu_M\frac{iu}{iu+\frac{\mu_M}{\eta_M}} \tag{h37}$$

At short periods T correspond high values of the pulsation $u=2\pi/T$. From (h37) it follows

$$\tilde{\lambda}=\lambda_M \quad , \quad \tilde{\mu}=\mu_M \tag{h38}$$

and the behaviour of the MAXWELL body is an elastic one.
At long periods T correspond low values of the pulsation and

$$\tilde{\lambda}=\frac{3\lambda_M+2\mu_M}{3}=K_M \quad , \quad \tilde{\mu}=0 \tag{h39}$$

and the MAXWELL body is a fluid having the compressional coefficient denoted by K_M.

H.3) Non-linear models.

For tectonic applications, large stresses and periods of thousands to millions of years are appropriate. Here, the non-elastic behaviour is probably related to the diffusion or dislocation creep of the molecules, a major factor being the high temperatures.

There are great difficulties to consider constitutive equations with non-linear relations between stress and strain, but some attempts have been made. A very common non-linear model is the work-hardening plasticity (e.g. Ranalli, 1994)

$$\varepsilon_{ij}=\frac{3}{2}\left(\frac{\overset{*}{\sigma}_E}{\sigma_0}\right)^{n-1}\frac{\overset{*}{\sigma}_{ij}}{\sigma_0} \quad , \tag{h40}$$

where $\overset{*}{\sigma}_{ij}$ is the component of the deviatoric stress, $\overset{*}{\sigma}_E=\sqrt{\frac{3}{2}\overset{*}{\sigma}_{ij}\overset{*}{\sigma}_{ij}}$ is related to the second invariant of the deviatoric stress, and σ_0, n are material parameters.

H.4) Brittle. Creep. Empirical criteria.

The usual materials are reacting in an elastic manner only for small values of the (deviatoric) stress, i.e. for stress values smaller than a limiting value representing the yield strength (or the yield stress), denoted by σ_Y. The yield stress is a function of the nature of the material, of the temperature, pressure, the chemical composition of the adjacent rocks and, finally, of the history of the deformation (i.e. the intermediate steps followed to attend the yield value). When the yield stress is attended, there are two possibilities of behaviour of the material:

- a rupture deformation of the rock, when the continuity of the deformation is lost, usually along a fault surface; this is the case of the **brittle materials**. The process is illustrated in Fig.H2a and b.
- a plastic, irreversibly flow of the material (**creep**), when, apparently, the continuity holds. The phenomenon is quite similar to the usual viscous flow, but it can be observed only when the yield stress is attended. This is the case of the **ductile materials**. The process is illustrated in Fig.H2c. An usual non-linear constitutive equation is the BYERLEE power-law creep

$$\dot{\varepsilon}=A\sigma^n\exp(-H/RT) \quad , \tag{h41}$$

where A, n are material parameters, H is the activation enthalpy, R is the gas constant and T is the absolute temperature. It should be noted that the same material can act as a brittle or a ductile one according to the external conditions.

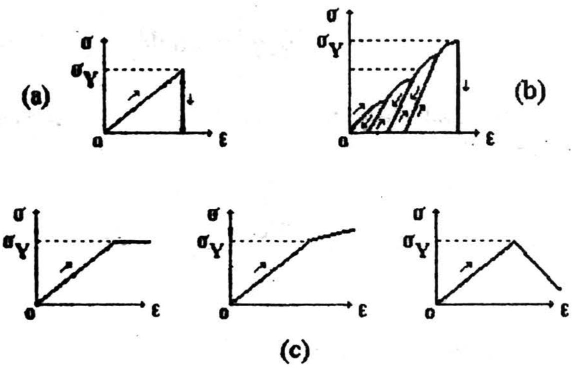

Fig.H2. (a) Faulting of a brittle material; (b) Increasing of the yield stress due to the history of deformation; (c) Creep of a ductile material.

H.5) Empirical criteria for shear-faulting. TRESCA criterion. COULOMB-NAVIER criterion.

A first criterion also (valid for plasticity) is due to TRESCA. It assumes that faulting (for brittle materials) or creep (for ductile ones) is attended at that points of the material where the maximum value of the shear (tangential) stress is equal to a yield value denoted by σ_Y. Consider now a homogeneous (constant) stress field inside the material. Such a case can be obtained either by considering an infinitesimal volume of material or by taking into account a prismatic body with very large (infinite) sides. Consider the eigen-values $\sigma_1, \sigma_2, \sigma_3$ of the stress tensor. It will be assumed that they are denoted in order to have $\sigma_3 < \sigma_2 < \sigma_1$, with the remark that ,in real life, stress is assumed to be positive for compression. Hence, with respect to Fig.H3, let $\sigma_{11} = -\sigma_1$, $\sigma_{22} = -\sigma_3$ and $\sigma_{12} = 0$ in eqs. (d11) and (d13). It follows that

$$\sigma = -\sigma_{rr} = \frac{\sigma_1 + \sigma_3}{2} - \frac{\sigma_1 - \sigma_3}{2}\cos 2\psi \quad , \qquad \tau = -\sigma_{r\theta} = \frac{\sigma_1 - \sigma_3}{2}\sin 2\psi \tag{h42}$$

where σ , τ are so-called "normal stress" and "tangential (shear) stress" respectively, acting on a plane inside the material. The plane is at an angle $\psi = \theta - \frac{\pi}{2}$ with the direction of the maximum compressive stress. The outer-pointing normal at that plane makes an angle θ with the direction of the maximum compressive stress. With respect to a $\sigma - \tau$ reference system, eqs.(h42) are the parametric equations of the MOHR circle (see Section D4), plotted in Fig. H3.

In the most general case, stress field is varying from point to point inside the material. Hence both the eigen-vectors of the stress tensor (i.e. the local directions of the maximum / minimum compressive stress) and the eigen-values of that tensor (i.e. the magnitudes of the maximum / minimum compressive stress) are also changing from point to point. For a fixed point inside the material, both normal stress and shear (tangential) stress are varying with the angle between the plane (with respect to normal and tangential stresses are defined) and the direction of the local maximum compressive stress. Consider a certain point inside the material and imagine various planes passing through that point. Hence, according to TRESCA empirical criterion, failure of the material is produced here if

$$\max_{\psi} |\tau| = \sigma_Y \tag{h43}$$

Using eq.(h42b), it follows

$$\max_{\psi}\left|\frac{\sigma_1 - \sigma_3}{2}\sin 2\psi\right| = \frac{\sigma_1 - \sigma_3}{2}\max_{\psi}|\sin 2\psi| = \sigma_Y \tag{h44}$$

Consider now a homogeneous stressed material subject to progressive increasing values of the difference $\sigma_1 - \sigma_3$. The material is characterised by a yield value denoted by σ_Y. Eq.(h44) shows that:

- if $\left(\sigma_1 - \sigma_3\right)/2 < \sigma_Y$, there is no failure inside the material;
- when the equality

$$\sigma_1 - \sigma_3 = 2\sigma_Y \tag{h45}$$

is attended, a failure is produced along the planes at angles $\psi = \pm\frac{\pi}{4}$ with the direction of the maximum compressive stress.

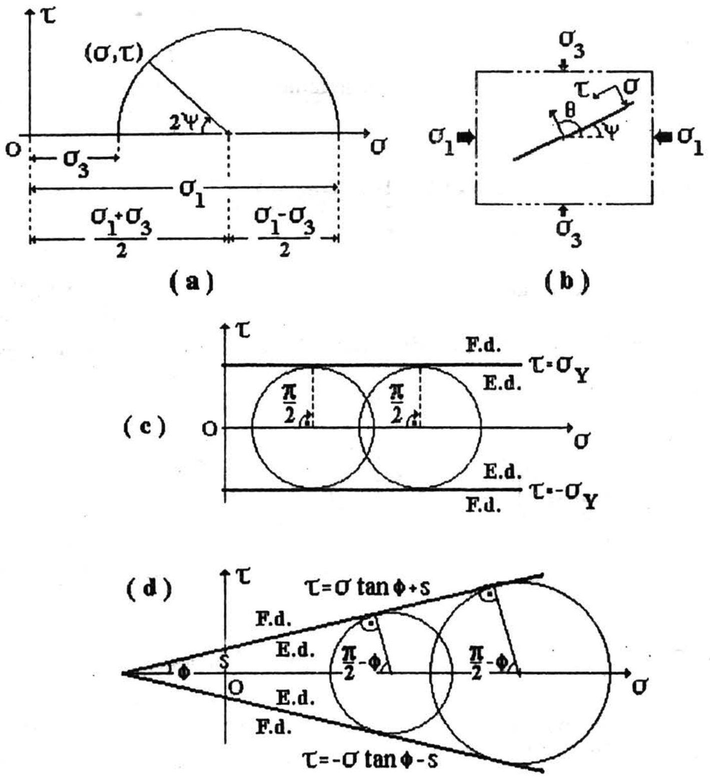

Fig.H3.(a) The MOHR circle;
(b) A plane inside the material, at an angle ψ with the direction of the maximum compressive stress;
(c) TRESCA criterion. E.d and F.D. denote the elastic domain and the failure domain respectively;
(d) COULOMB - NAVIER criterion.

Eq.(h45) represents the TRESCA criterion in terms of the eigenvalues of the stress tensor. In the case of a material subject to a non-homogeneous stress field, eq.(h45) is a local condition. Here, the eigen-values are obtained (see Section A). For a compressive stress, that values are expected to be negative ones. They have to be denoted by $-\sigma_1, -\sigma_2, -\sigma_3$, where $\sigma_3 < \sigma_2 < \sigma_1$.

A second criterion is due to COULOMB and NAVIER. It can be used to describe only the shear fracture. According to it, a material is characterised by the cohesive strength denoted by S and by the coefficient of friction, denoted by $\mu^* = \tan\phi$. Here, ϕ is the angle of internal friction ($\phi = 30^\circ$ in most rocks). According to COULOMB - NAVIER empirical criterion, shear failure of the material is produced at its points where

$$\max_{\psi}\left(|\ \tau\ | - \mu^* \sigma \right) = S \tag{h46}$$

Using eqs.(h42), it follows

$$\max_{\psi}\left[\frac{\sigma_1-\sigma_3}{2}|\sin 2\psi| - \tan\phi\left(\frac{\sigma_1+\sigma_3}{2}-\frac{\sigma_1-\sigma_3}{2}\cos 2\psi\right)\right] = S \qquad \text{(h47)}$$

If ψ is a solution of (h47), $\pi-\psi$ (or just $-\psi$) is a solution too. Hence, without loss of generality, the values of the angle ψ will be limited to the first quadrant, where eq.(h47) is

$$\frac{\sigma_1-\sigma_3}{2}\frac{1}{\cos\phi}\max_{\psi}\left[\sin(2\psi+\phi)\right] = S + \tan\phi\frac{\sigma_1+\sigma_3}{2} \qquad \text{(h48)}$$

Eq.(h48) shows that:

- if $\frac{\sigma_1-\sigma_3}{2}\frac{1}{\cos\phi} - \tan\phi\frac{\sigma_1+\sigma_3}{2} < S$, there is no failure of the material;
- when the equality

$$\frac{\sigma_1-\sigma_3}{2}\frac{1}{\cos\phi} - \tan\phi\frac{\sigma_1+\sigma_3}{2} = S \qquad \text{(h49)}$$

is attended, a share fracture is produced along the planes at angles $\psi = \pm\left(\frac{\pi}{4}-\frac{\phi}{2}\right)$ with the direction of the maximum compressive stress. Eq.(h49) represents the COULOMB - NAVIER criterion in terms of the eigenvalues of the stress tensor. Taking into account that $\mu^* = \tan\phi$, eq.(h49) can be written as

$$\sigma_1\left[\sqrt{\left(\mu^*\right)^2+1}-\mu^*\right] - \sigma_3\left[\sqrt{\left(\mu^*\right)^2+1}+\mu^*\right] = 2S \quad , \qquad \text{(h50)}$$

outlining that COULOMB - NAVIER criterion is a generalisation of TRESCA criterion for a non-zero internal friction.

H.6) Von MISES-HENCKY criterion for ductile flow (plasticity).

Because the ductile (plastic) flow is independent of the co-ordinate system used, it depends only on the invariants of the stress tensor (Section A, eq.(a22)). Hence an equation like

$$f(I_1, I_2, I_3) = 0 \quad , \qquad \text{(h51)}$$

will be valid. There are strong experimental evidence that the plastic flow does not depend on the hydrostatic pressure, being also similar for compressive and tensile states of stress. It follows the function f in (h51) depends only on the second invariant of the stress deviator (see eq.(a23)). Hence, ductile flow occurs only at those points of material where the second invariant of the deviator stress reaches a certain value, depending on the nature of the material. Using eq.(a24), the criterion of Von MISES - HENCKY assumes that ductile flow occurs at those points of the material where

$$(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6\left(\sigma_{12}^2+\sigma_{23}^2+\sigma_{13}^2\right) = 6k^2 \qquad \text{(h52)}$$

In terms of the principal stresses, eq.(h52) is

$$(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 6k^2 \qquad \text{(h53)}$$

Hence the criterion of Von MISES - HENCKY can be regarded too as a generalisation of TRESCA criterion, by taking into account the presence of the intermediate stress.

H.7) Rheological models.

a) SAINT-VENANT body (elastic-plastic material).

The behaviour of that material is characterised by linear elasticity for stress values below the yield strength. When the yield stress is attended, the body exhibits a pure plasticity. Its constitutive equation (using deviator tensors) has the symbolic form

$$\begin{cases} \overset{*}{\sigma} = 2\mu_S \overset{*}{\varepsilon} & , \quad \sigma < \sigma_Y \\ f(\overset{*}{I_2}, \overset{*}{I_3}) = \sigma_Y & , \quad \sigma = \sigma_Y \end{cases} \qquad \text{(h54)}$$

The above material has the mechanical analogue presented in Fig. (h4a), being referred as a SAINT-VENANT body.

b) BINGHAM body (visco-plastic material).

Similar to the SAINT-VENANT body, that material exhibits linear elasticity for stress values lower than the yield strength., but flows linearly above that value. The strain rate is proportional to the difference between the deviatoric stress and the yield strength. Its constitutive equation is

$$\begin{cases} \overset{*}{\sigma} = 2\mu_B \overset{*}{\varepsilon} & , \quad \sigma < \sigma_Y \\ \overset{*}{\sigma} = \sigma_Y + 2\eta_B \dfrac{d\overset{*}{\varepsilon}}{dt} & , \quad \sigma \geq \sigma_Y \end{cases} \qquad \text{(h55)}$$

The above material has the mechanical analogue presented in Fig. (h4b), being referred as a BINGHAM body.

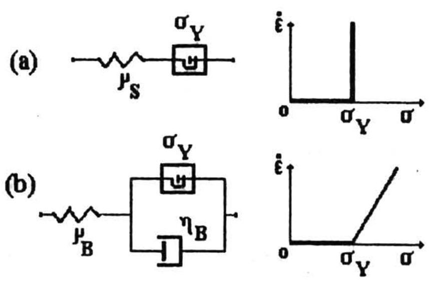

Fig.H4. (a) The SAINT-VENANT body; (b) The BINGHAM body.

I) THE ACCRETION WEDGE.

I1) The model.

Consider a 2-D prismatic body having a triangular vertical section (Fig.I1), in the presence of gravity. The wedge rests on a rigid basement having the slope equal to θ_0. Both the compressional force acting on the left side of the wedge and the friction to the basement cause thickening of the incompressible material and the development of a topographical slope equal to α. It is assumed that the material is into a state of plastic yielding according to the VON MISES-HENCKY criterion. It follows to obtain a condition relating the slopes of the topography and that of the basement to the geometry of the wedge, its yield strength and the friction coefficient to the basement.

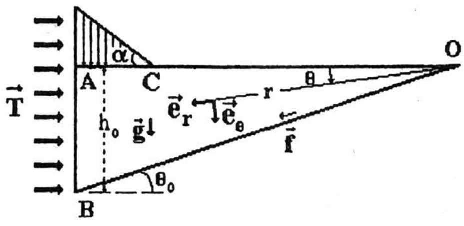

Fig.I1. The 2-D accretion wedge.

I2) Equations of equilibrium. Yield condition. Stress field.

Taking into account that the 2-D case is discussed, the stress in polar co-ordinates is

$$\sigma = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & 0 \\ \sigma_{r\theta} & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \nu(\sigma_{rr}+\sigma_{\theta\theta}) \end{pmatrix} \quad . \tag{i1}$$

Because the material is assumed to be incompressible, the POISSON coefficient is $\nu = 1/2$.
Using polar co-ordinates, the equilibrium equation (d40)-(d41) in the presence of gravity are

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{r\theta}}{r\partial\theta} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + \rho g \sin\theta = 0 \tag{i2}$$

$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{\partial\sigma_{\theta\theta}}{r\partial\theta} + 2\frac{\sigma_{r\theta}}{r} + \rho g \cos\theta = 0 \tag{i3}$$

Using (i1), the yield condition (r54) is

$$\sigma_1 - \sigma_3 = 2k \tag{i4}$$

Taking into account again that the compressive stress is assumed to have positive sign, eqs.(d11)-(d13) give

$$\begin{cases} \sigma_{rr} = -f(r,\theta) - k\cos 2\psi \\ \sigma_{\theta\theta} = -f(r,\theta) + k\cos 2\psi \\ \sigma_{r\theta} = k\sin 2\psi \end{cases} \quad , \tag{i5}$$

where the trace of the stress has been denoted by $f(r,\theta) = \frac{1}{2}\mathrm{tr}(\sigma)$ and $\psi = \psi(r,\theta)$ is the angle between the radius and the local direction of the maximum compressive stress (variable inside the wedge). However, it will be assumed that $\psi = \psi(\theta)$ only. After some manipulations, substituting (i5) into (i2)-(i3) gives

$$\begin{cases} \frac{\partial f}{\partial r} = \frac{2k}{r}\cos 2\psi \frac{d\psi}{d\theta} - \frac{2k}{r}\cos 2\psi + \rho g \sin\theta \\ \frac{\partial f}{\partial\theta} = -2k\sin 2\psi \frac{d\psi}{d\theta} + 2k\sin 2\psi + \rho g r\cos\theta \end{cases} \tag{i6}$$

Eqs.(i6a) and (i6b) are differentiated with respect to θ, r respectively, the results of the differentiation being equal each other. After some elementary manipulations, it follows that

$$\frac{d}{d\theta}\left(\cos 2\psi \frac{d\psi}{d\theta} - \cos 2\psi\right) = 0 \quad , \tag{i7}$$

i.e.

$$\frac{d\psi}{d\theta} = 1 + \frac{C}{\cos 2\psi} \quad , \tag{i8}$$

where C is a constant of integration. Substituting (i8) into (i6a) gives

$$\frac{\partial f}{\partial r} = \frac{2Ck}{r} + \rho g \sin\theta \quad , \tag{i9}$$

i.e.

$$f = 2Ck \ln r + \rho g r \sin\theta + g(\theta) \quad , \tag{i10}$$

To find the unknown function $g = g(\theta)$, eq.(i10) is substituted into (i6b) to obtain

$$\frac{dg}{d\theta} = -2Ck \tan 2\psi \quad . \tag{i11}$$

Using (i8), it follows that

$$\frac{dg}{d\psi} = -2Ck \frac{\sin 2\psi}{\cos 2\psi + C} \quad , \tag{i12}$$

i.e.

$$g = Ck \ln(C + \cos 2\psi) + A \quad , \tag{i13}$$

where A is another constant of integration. Hence the final stress inside the wedge is

$$\begin{cases} \sigma_{rr} = -2Ck \ln r - Ck \ln(C + co2\psi) - \rho g r \sin\theta - A - k\cos 2\psi \\ \sigma_{\theta\theta} = -2Ck \ln r - Ck \ln(C + co2\psi) - \rho g r \sin\theta - A + k\cos 2\psi \\ \sigma_{r\theta} = k \sin 2\psi \end{cases} \quad . \tag{i14}$$

I3) Boundary conditions. Final results.

Consider the segment AC placed on the side OA of the wedge, having $\theta = 0$ and $OC \le r \le OA$, where the point C is very closed to the point A. The outward pointing normal vector is $\vec{n} = -\vec{e}_\theta$. Here, is acting the lithostatic pressure due to the topography. Hence

$$\sigma\left(-\vec{e}_\theta\right) = \rho g r \tan\alpha\, \vec{e}_\theta \quad , \tag{i15}$$

i.e.

$$\begin{cases} \sigma_{r\theta}(\theta = 0) = 0 \\ \sigma_{\theta\theta}(\theta = 0) = -\rho g r \tan\alpha \end{cases} \quad . \tag{i16}$$

Because the angle θ_0 has very small values, it will be assumed for all angles θ that

$$\sigma_{\theta\theta} = -\rho g r \tan\alpha \quad , \tag{i17}$$

or

$$\frac{\partial \sigma_{\theta\theta}}{\partial r} = -\rho g \tan\alpha \quad . \tag{i18}$$

However, all the next derivations are supposed to be valid at the rear of the wedge, where the topography is generated due to the horizontal compression, i.e. the radius r is a mean value of the lengths OA and OC, the point C being closed to A. By differentiating (i14b), eq (i18) leads to

$$\frac{2Ck}{r} = \rho g \tan\alpha \quad , \tag{i19}$$

where

$$r \cong \frac{h_0}{\theta_0} \qquad \text{(i20)}$$

Consider now the side OB of the wedge, having $\theta = \theta_0$ and the outward pointing normal vector $\vec{n} = \vec{e}_\theta$. Here, is acting the friction force due to the basement, assumed to have the magnitude equal to λk, where λ is a friction coefficient. Hence

$$\sigma \vec{e}_\theta = \lambda k \vec{e}_r \quad , \qquad \text{(i21)}$$

i.e.

$$\sigma_{r\theta}(\theta = \theta_0) = \lambda k \quad . \qquad \text{(i22)}$$

The next partial derivative follows to be evaluated in two ways. In the first approach, eqs.(i22), (i16a) and (i20) are used to give

$$\frac{\partial \sigma_{r\theta}}{r\partial\theta} \approx \frac{1}{r}\frac{\sigma_{r\theta}(\theta = \theta_0) - \sigma_{r\theta}(\theta = 0)}{\theta_0 - 0} = \frac{\lambda k}{h_0} \quad . \qquad \text{(i23)}$$

The dominant stress is into the wedge is the horizontal compression. Hence $\psi \cong \theta$, both angles having small values. It follows $\cos 2\psi \cong 1$. Eqs.(i5c), (i8), (i19) and (i20) give

$$\frac{\partial \sigma_{r\theta}}{r\partial\theta} = \frac{2k}{r}\cos 2\psi \frac{d\psi}{d\theta} = \frac{2k}{r}(C + \cos 2\psi) \approx \frac{2Ck}{r} + \frac{2k}{r} = \rho g \tan\alpha + \frac{2k\theta_0}{h_0} \qquad \text{(i24)}$$

From (i23) and (i24) it follows

$$\rho g h_0 \tan\alpha + 2k\theta_0 \cong \lambda k \quad , \qquad \text{(i25)}$$

showing that the friction force (resistance to sliding of the wedge onto the basement) is balanced by tqo forces. The first one is due to the topography and the second force is related to the compressive stress and to the slope of the basement. Further details related to the application of eq.(i25) in real cases are presented by Ranalli (1987).

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