HINTS

ON

ARITHMETIC.

BY LADY VERNEY.

Price Sixpence.

HINTS ON ARITHMETIC,

Addressed

TO

A YOUNG GOVERNESS.

BY LADY VERNEY.

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MDCCCLII.

INTRODUCTION.

THE following pages contain, as the title expresses, hints, and only hints. They pretend neither to afford a philosophical treatise on the science of Arithmetic, nor to form a manual of examples for its practice. They simply offer a few plain thoughts on the subject, which the writer has found available in the instruction of her own children, and which she first compiled in their present form for the use of a young person, whom she was assisting in her preparations for going out as governess. She is now induced to publish them from having observed among those engaged in tuition a very general deficiency in the power of explaining, if not in that of themselves understanding, the principles on which arithmetical operations are based.

And glad indeed will she be, should her little work in any measure conduce to render easy and attractive a science which, not only for the uses to which it can be applied, but on its own account, and for the mental training and discipline which it affords, ought always to take a distinguished place among the branches of an intelligent education.

$+$ Plus, the sign of addition.
$-$ Minus, the sign of subtraction.
$\times$ The sign of multiplication.
$\div$ The sign of division.
$=$ Equal to.

HINTS ON ARITHMETIC.

CHAPTER I.—Addition Table.

2+2= 4	4+2= 6	6+2= 8	8+2=10	10+2=12	and so on to 100
1+2= 3	3+2= 5	5+2= 7	7+2= 9	9+2=11	,, ,,

3+3= 6	6+3= 9	9+3=12	12+3=15	15+3=18	and so on to 100
1+3= 4	4+3= 7	7+3=10	10+3=13	13+3=16	,, ,,
2+3= 5	5+3= 8	8+3=11	11+3=14	14+3=17	,, ,,

4+4= 8	8+4=12	12+4=16	16+4=20	20+4=24	and so on to 100
1+4= 5	5+4= 9	9+4=13	13+4=17	17+4=21	,, ,,
2+4= 6	6+4=10	10+4=14	14+4=18	18+4=22	,, ,,
3+4= 7	7+4=11	11+4=15	15+4=19	19+4=23	,, ,,

5+5=10	10+5=15	15+5=20	20+5=25	25+5=30	and so on to 100
1+5= 6	6+5=11	11+5=16	16+5=21	21+5=26	,, ,,
2+5= 7	7+5=12	12+5=17	17+5=22	22+5=27	,, ,,
3+5= 8	8+5=13	13+5=18	18+5=23	23+5=28	,, ,,
4+5= 9	9+5=14	14+5=19	19+5=24	24+5=29	,, ,,

6+6=12	12+6=18	18+6=24	24+6=30	30+6=36	and so on to 100
1+6= 7	7+6=13	13+6=19	19+6=25	25+6=31	,, ,,
2+6= 8	8+6=14	14+6=20	20+6=26	26+6=32	,, ,,
3+6= 9	9+6=15	15+6=21	21+6=27	27+6=33	,, ,,
4+6=10	10+6=16	16+6=22	22+6=28	28+6=34	,, ,,
5+6=11	11+6=17	17+6=23	23+6=29	29+6=35	,, ,,

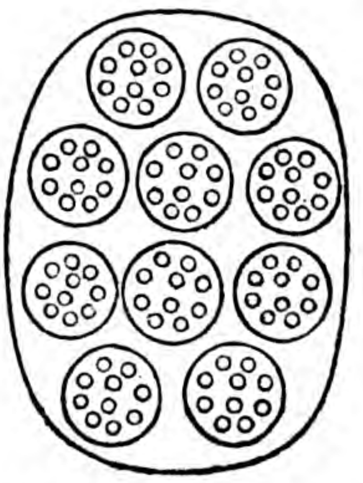
7+7=14	14+7=21	21+7=28	28+7=35	35+7=42	and so on to 100
1+7= 8	8+7=15	15+7=22	22+7=29	29+7=36	,, ,,
2+7= 9	9+7=16	16+7=23	23+7=30	30+7=37	,, ,,
3+7=10	10+7=17	17+7=24	24+7=31	31+7=38	,, ,,
4+7=11	11+7=18	18+7=25	25+7=32	32+7=39	,, ,,
5+7=12	12+7=19	19+7=26	26+7=33	33+7=40	,, ,,
6+7=13	13+7=20	20+7=27	27+7=34	34+7=41	,, ,,

8+8=16	16+8=24	24+8=32	32+8=40	40+8=48	and so on to 100
1+8= 9	9+8=17	17+8=25	25+8=33	33+8=41	,, ,,
2+8=10	10+8=18	18+8=26	26+8=34	34+8=42	,, ,,
3+8=11	11+8=19	19+8=27	27+8=35	35+8=43	,, ,,
4+8=12	12+8=20	20+8=28	28+8=36	36+8=44	,, ,,
5+8=13	13+8=21	21+8=29	29+8=37	37+8=45	,, ,,
6+8=14	14+8=22	22+8=30	30+8=38	38+8=46	,, ,,
7+8=15	15+8=23	23+8=31	31+8=39	39+8=47	,, ,,

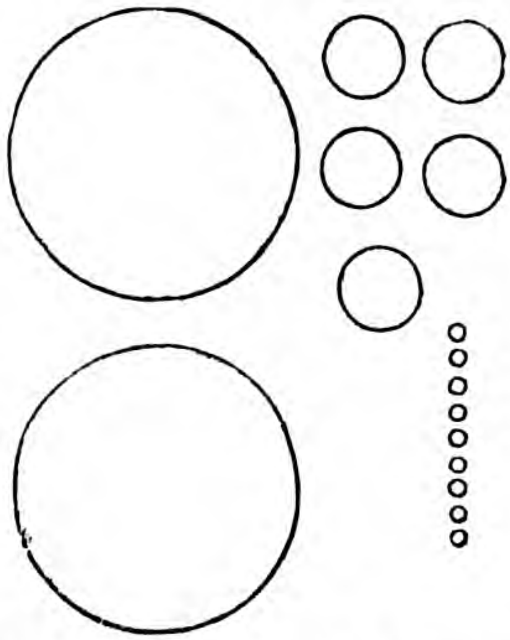
9+9=18	18+9=27	27+9=36	36+9=45	45+9=54	and so on to 100
1+9=10	10+9=19	19+9=28	28+9=37	37+9=46	,, ,,
2+9=11	11+9=20	20+9=29	29+9=38	38+9=47	,, ,,
3+9=12	12+9=21	21+9=30	30+9=39	39+9=48	,, ,,
4+9=13	13+9=22	22+9=31	31+9=40	40+9=49	,, ,,
5+9=14	14+9=23	23+9=32	32+9=41	41+9=50	,, ,,
6+9=15	15+9=24	24+9=33	33+9=42	42+9=51	,, ,,
7+9=16	16+9=25	25+9=34	34+9=43	43+9=52	,, ,,
8+9=17	17+9=26	26+9=35	35+9=44	44+9=53	,, ,,

Before beginning this table, you should teach your pupil how to add up tens, explaining that *twenty* means *twoty*, or two tens; *thirty* means *threety*, or three tens; *forty* means four tens, and so on. In doing the above table, the great point is to make the child perceive the identity in the adding up of the units, whatever number of tens there may be; as, for instance, that if 8 and 5 make 13, 18 and 5 make 23, 28 and 5 make 33, and so on; the 8, the 5, and the 3 reappearing whether you have ten, twenty, or thirty joined to them; exercises on this point should be given till the child is quite familiar with it. Point out that in adding 9 you always have one unit less, and one ten more, than the original number.

It is very desirable, while teaching this addition table, to use one of those frames of balls which are commonly employed in infant schools. By its assistance, you may render evident to the child's sight the truths which you are endeavouring to impress on his memory. When the addition table is pretty well acquired, invert it. Make the child, for instance, begin at 100 and go backwards, always taking away 2 or 3 or whatever the number may be: thus subtraction will be learned. After this, take the first row of each division of this table, and make the child vary the mode of saying it; instead of saying 2 and 2 are 4, let him say 2 twos are 4, 3 twos are 6, 4 twos are 8, then he may gradually get to say twice 2 are 4, three times 2 are six, four times 2 are 8, and so on, and thus the multiplication table is learned. After this you may begin to use a slate, and teach the child notation. Let the child consider units as small balls, thus, o o o o o, tens as larger balls, each containing ten units; hundreds as still larger balls, containing ten tens, or a hundred units. By drawing them as I have done here, the child has ocular demonstration that ten units are one ten, and must therefore be set down as one in the second row of figures; and that ten tens are one hundred, and must be set down as one in the third row of figures. Write



down the number 111, and make the child perceive that each of the three ones there is perfectly different from the other, representing a different object—a different-sized ball. If the size of your slate admit of it, you can go on drawing thousands, tens of thousands, hundreds of thousands, millions.



The next exercise should be to write down a sum, as 259, and make the child draw it thus.* A variety of sums should be written until the child can draw them with perfect facility; then he is prepared to understand and learn the general law or theorem, that *according to the decimal system of notation, each place of figures is worth ten times more than that to the right of it, and worth ten times less than than that to the left of it.* So if you want to multiply any figure by ten, you have simply to shove it one place to the left; or by a hundred, two places to the left; or by a thousand, three places to the left; and for dividing in the same way, you must shove the figures to the right; thus: ten times 4 are 40, where, by adding the nought, you shove the 4 to the left hand. The process by which the child has been brought to the rule or law for notation may seem a long one, but the object of education is to gain knowledge not rapidly, but thoroughly, and in such a way as to strengthen the mind; and for this reason it is desirable not to teach a general rule or law to a child, until he is so familiar with all the particular instances of it, and reasons for it, that he could pretty well have found out the rule for himself. Ascend from particular instanees to general rules, do not descend from general rules to particular

* The circles might be drawn on a slate, placed in a line in order of magnitude; the largest circles most to the left, each representing an unit of ten times ten, the next in size each an unit of ten, and the smaller, simple units.

examples. The child is now prepared to do sums of simple addition on the slate, such as the one here.

```
 265
 468
 949
 374
 852
----
2908
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He adds up the units and finds that there are 28 of them; you make him draw them, and ask what he will do with them; he says they must be put into heaps of ten; you therefore make him enclose [0000000000] [0000000000] 00000000, each ten by a line; he then finds that 8 are left which he cannot enclose; he sees that there are 2 tens and 8 units; he puts down the 8 in the row of units, and he carries the 2 tens to the row of tens. The addition of the tens gives him thirty [0000000000] [0000000000] [0000000000] On enclosing these he finds that they make exactly 3 hundreds, all of which must be transferred to the row of hundreds, and that, therefore, he must put a nought into the row of tens, because he has no tens to leave there. A similar process must be repeated for the row of hundreds. When the sum has thus been gone through, let the child recommence it, and this time proceed without drawing. When he has added up the units, let him say thus: "28 units make 2 tens and 8 units. I put the 8 units down in the row of units, and I carry the 2 tens to the row of tens." When he has added up the tens, let him say, "30 tens make 3 hundreds; as there are no tens left, I put down nought in the row of tens, and I carry 3 to the row of hundreds." When he has added up the hundreds, let him say, "29 hundreds make 2 thousands and 9 hundreds. I put the 9 hundreds down in the row of hundreds, and I carry the 2 thousands to the row of thousands." However many rows he may have to add up, he must still use a formula exactly similar. For instance, he might say, "37 tens of thousands are 3 hundreds of thousands and 7 tens of thousands. I put down the 7 tens of thousands in the row of tens of thousands, and I carry the 3 hundreds of thousands to the row of hundreds of thousands." The child will need the practice of many sums before he attains facility in doing them. When he is pretty expert you may point out to him that the mode by which he converts small balls into large ones, is in fact division; that he is finding out how many tens are contained in the number of small balls, and that this is what we call dividing by ten. Familiarity with this idea will prepare him for understanding division at a later period. You may now exercise the child's powers of notation, by dictating numbers which he shall write down upon a slate.

The next step is subtraction. The only difficulty here is in the matter of borrowing. In the sum which I have written down, it is impossible to take 9 units from 2 units, you must therefore take or borrow one of the 6 tens, and break it up into units; you will thus be able, by adding these 10 units to the original 2 units, to get 12 units, from which 9 units can easily be subtracted, leaving 3 units, which you accordingly write down. When you proceed to the next row, you must remember that you have already made use of one of the 6 tens, and as you cannot eat your cake and have your cake, you cannot now have 6 tens, but must call it 5. Therefore, you subtract 3 tens from 5 tens, and finding there are 2 tens left, you write them down accordingly. In the row of hundreds you repeat the same proeess as in the row of units. As you cannot take 8 hundreds from 3 hundreds, you borrow one of the 8 thousands, you break it up into hundreds, you add the ten hundreds thus obtained to the 3 hundreds, from these 13 hundreds you subtract the 8 hundreds, and set down the 5 hundreds which are left. Then in proceeding to the row of thousands, you bear in mind that one has been removed, and that therefore you have only 7 thousands now from which to subtract 4 thousands; you set down the 3 thousands which are left.

$$\begin{array}{r}8362\\4839\\\hline 3523\end{array}$$

There is a little more difficulty where there are noughts in the upper row, as in this sum. How can you borrow from tens when there are no tens?—you must resort to the hundreds. But how will they assist you when there are no hundreds? you must resort to the thousands, and, taking one thence, break it up into 10 hundreds; you must then take one of these and break it up into 10 tens; and lastly, you must take one of these, and breaking it up into 10 units, you must add them to the 3 units which you have already got. Your upper row will now stand thus: 5 thousands, 9 hundreds, 9 tens, and 13 units, and there will be no difficulty in subtracting the lower row from it. You must show your pupil that subtraction is the reverse of addition, and that while in addition small

$$\begin{array}{r}6003\\3249\\\hline 2754\end{array}$$

balls are heaped together to make large ones, in subtraction large balls are broken up to make small ones. It will be well now to give such questions to the pupil as will show the practical use of addition and subtraction. For instance, if I am travelling and the first day I go 49 miles, the next day 65, the third day 37, the fourth day 82, how far have I gone altogether? King William conquered England in the year A.D. 1088, how many years is that ago? Invent as many such questions as you can, and employ also for your pupil's practice any examples in addition and subtraction out of a common arithmetic book. Most arithmetic books are of no use for explanation, though of great use for practice. Colenso's arithmetic is very good.

You now proceed to multiplication. This has, in fact, been learned by the addition tables; but, in order to give facility, a multiplication table might as well be learned also. I have found the rhymes and pictures of the little book called "Marmaduke Multiply" most helpful to the memory of my children.

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  4968
    27
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 34776
 99360
------
134136
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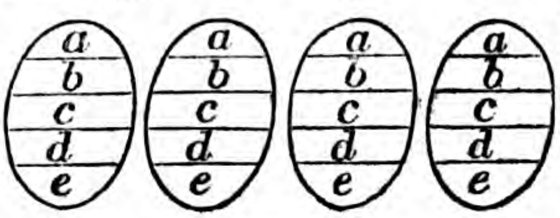
The first row of a multiplication sum is easy enough. The child should use much the same formula as in addition, thus: 7 times 8 are 56; 56 units are 5 tens and 6 units. I put down the 6 units in the row of units, and I carry the 5 tens to the row of tens: 7 times 6 are 42; 42 tens and 5 tens are 47 tens; 47 tens are 4 hundreds and 7 tens. I put down the 7 tens in the row of tens, and I carry the 4 hundreds to the row of hundreds, and so on. The question now arises, how is the second row to be written? You are required to multiply 4968 by 27, that is, by 20 and by 7. You have already multiplied it by 7; you have now to multiply it by 20, that is, by 10 times 2; if you multiply by 2 and this by 10, it comes to the same thing as multiplying at once by 20. Now, twice 4968 are 9936, and the way to multiply this by 10 is to shove every figure one place to the left, which makes the sum 99360; therefore, the way to do the second row of your multiplication sum, or to multiply by tens, is to place every figure one row to the left of the place which it would have occupied had you been multiplying by units. The way to multiply by hundreds is to place every figure two places to the left; by thousands, three places to the left, and so on. You may test the correctness of this rule thus: $20 \times 8 = 160$; 160 units are 16 tens with no units over; therefore put down nought in the row of units; but 16 tens are 1 hundred and 6 tens; therefore put down the 6 in the row of tens, and carry the 1 to the row of hundreds: $20 \times 6 = 120$; 120 tens are 12 hundreds, add to these the 1 hundred you have carried; 13 hundreds are 1 thousand and 3 hundreds; put down the 3 in the row of hundreds, and carry the 1 to the row of thousands, and so on. It is thus evident that the figures must fall into the place assigned to them by the rule. Having multiplied by 7 and then by 20, you have only to add the two products together in order to ascertain how much 27×4968 are. When the theory of multiplication has been mastered, exercise your pupil's skill by numerous examples, and by such practical questions as these: if in a school of 58 boys, each boy possessed 6 marbles, how many marbles would the whole school possess?

The best way to begin division is by such questions as the following: if I had a basket of 35 apples, and desired to give to each of 5 boys an equal number, how many must each boy have? The pupil will perceive that he must classify the apples in heaps much as he did his units and tens, but whereas in the case of the units and tens he knew how many were to go into each heap, and he had to find out how many heaps there would be; in this case of the apples, on the contrary, he knows how many heaps there are, but he has to find out how many apples must constitute each heap. He can find this out thus: as there are 5 boys, and every boy is to have the same number, let him begin by giving each boy one apple—then 5 apples will have been used; then let him give each boy another apple—then 10 apples will have been used; give each boy a third—15 apples will have been used; let him go on so until all the apples are disposed of, then he will find that each boy has just as many apples as there are fives in 35, and that the short way of doing the business would have been to have reversed the multiplication process, and have asked "how many times 5 make 35?" or "what is the fifth part of 35?" If you cut 35 into 5 equal parts, the boys are sure to be justly dealt by. To make the matter plainer, you might call the boys A, B, C, D, E, and write their names over the apples thus,

ABCDE, ABCDE, ABCDE, ABCDE, ABCDE, ABCDE, ABCDE oooo
ooooo ooooo ooooo ooooo ooooo ooooo ooooo

By counting up the number of As or Bs, it becomes plain that each boy will have as many apples as the times that the number of boys is contained in the number of apples. There are 5 boys, $5 \times 7 = 35$; there are therefore 7 fives in 35, and each boy will have 7 apples. Thus division is the reverse of multiplication. Now let there be 39 apples instead of 35 to be divided among the 5 boys. It is quite evident from the drawing above that 4 apples will be left over, and if they are not in some way equally distributed among the boys, either some boys will have more than others, or else all the 39 apples will not have been given to them. The question then is how to deal with this remainder of 4 apples. Cut each apple into 5 equal bits, then you can distribute each



apple among the 5 boys, and each boy will have exactly the same, viz., four bits or four-fifth parts of an apple. Thus you cut your remaining apples into as many parts as there are boys, and you give to each as many of those parts as there are apples remaining unappropriated after the division has been made. Four-fifths is not a whole apple, but only part of one. We call a whole apple or a whole unit of any kind an integer, and we call a part of a unit a fraction. We write the fraction four-fifths thus, $\frac{4}{5}$. The lower number is called the denominator; it tells into how many bits the integer has been cut. If the integer is cut into many bits, the bits are of course much smaller than if it is cut into few bits; thus $\frac{4}{10}$ is just half of $\frac{4}{5}$, because the bits are half the size. A large denominator shows that the bits are small; a small denominator shows that the bits are large. Thus the denominator shows of what denomination, kind, or size the bits are; and hence its name. The upper number of the fraction is called the numerator, because it tells you the number of bits that each boy has got. The reason why the fraction is written thus, $\frac{4}{5}$, is that this is the mode of expressing all division in figures. If I wished in figures to say that 35 divided by 5 was 7, I should write it thus, $\frac{35}{5} = 7$; the fifth part of 35 is 7. So when I write $\frac{4}{5}$, I mean the fifth part of 4 apples, which is exactly what each boy has got; and by looking at the drawing you will perceive that the fifth part of 4 apples is exactly the same as four-fifth parts of 1 apple.

Now in all this transaction with the boys, we have been doing a division sum; the boys have been the divisor, the apples have been the dividend, the portion assigned to each boy has been the quotient, and the sum should be stated and worked as here marked down.

```
5)39(7 4/5
  35
  --
   4
```

Try how many times the divisor will go in the dividend, the number of times is the quotient; perform the multiplication, and subtract its product from the dividend; this will show how much is still left undivided, corresponding to the remaining 4 apples, which had to be cut up among the boys. This division is represented by a fraction, of which the remainder forms the numerator, and the divisor the denominator; and the quotient is thus obtained, consisting of 7 integers and the fraction $\frac{4}{5}$. A little reflection will show that this arithmetical process is precisely that explained at length in the previous page, and there supposed to be done without the aid of figures. You should give many such familiar examples to your pupil, requiring him first to work them in his head or by drawing, and then afterwards requiring him to express the process in figures.

The next step is the case when the dividend consists of several places of figures; but the familiarity the pupil has by this time acquired with tens, hundreds, etc., will obviate much difficulty here.

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5)392036(78407 1/5
  35
  --
   42
   40
   --
    20
    20
    --
     036
      35
      --
       1
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Division is the reverse of multiplication: in multiplication you begin with the lowest place of figures, and carry on to the higher places; in division you begin with the highest place, and divide or reduce or break up what you cannot manage there, so as to use it in the lower places. You first ask what is the fifth part of 3 hundreds of thousands, and as 5 is greater than 3, and you cannot therefore express the answer in integers of hundreds of thousands, you break them up into tens of thousands, and adding to them the 9 tens of thousands already there, you find you have now 39 tens of thousands to deal with; 5 goes in this 7 times; the 7 is of course 7 tens of thousands. By multiplying and subtracting, you find you have still 4 tens of thousands left undivided, which, in order the more conveniently to deal with, you break up into thousands, and adding to them the 2 thousands which you were originally told to divide, you now inquire what is the fifth part of 42 thousands; 8 is the nearest answer. After multiplying and subtracting as before, you find you have 2 thousands left, which you break up into hundreds, and having no hundreds to add from your original dividend, you inquire what is the fifth part of 20 hundreds; the answer is 4 hundreds, and as this is exactly the fifth part, after multiplying and subtracting, there is no remainder; therefore you proceed at once to divide the 3 tens, but as 3 is less than 5, you cannot express the answer in integers; therefore you state in the quotient by writing nought that there are no tens, and you proceed to break up your 3 tens into units, and to add the units of the original dividend. You then inquire what is the fifth part of 36 units; the answer is 7 units, and 1 unit is still left undivided. That must now be cut into 5 parts, and one-fifth part added to the quotient, which is now complete.

In short division all the processes above mentioned are gone through, but to save trouble they are not expressed, and only the quotient is written down; but the pupil should never do short division until he has obtained perfect facility in long division. The next step is to proceed to division in which there are several places of figures in the divisor as well as in the dividend.

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5)392036
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 78407 1/5
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You inquire, for instance, what is the fifty-eighth part of 392 thousands, that is, how many times 58 will go in 392? Now, if the question were how many times 50, the answer would be easy, because we need merely say, how many times are 5 tens contained in 39 tens, or how many times are 5 contained in 39; the answer would be 7; but 58 is much nearer to 60 than to 50, therefore it cannot be contained so often in 392 as 50 would; it will only go 6 times. The way, therefore, to find out how many times a divisor with several places of figures will go in the dividend, is to try how many times its highest place of

```
58)392036(6759 14/58
   348
   ---
    440
    406
    ---
     343
     290
     ---
      536
      522
      ---
       14
```

figures will go, and then, making allowance for the remaining figures, to put down something less in the quotient than you would otherwise have done.

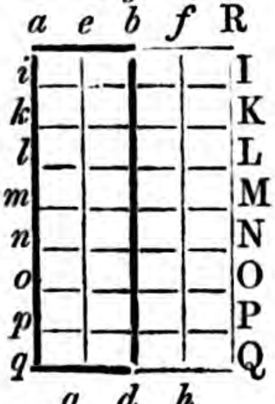
In doing sums, it is very important that you should thoroughly ascertain of what size and kind your units are, because on them depend your tens and hundreds, and all your other figures. You cannot add or subtract sums whose units are not of the same kind; for instance, you cannot say that 4 sheep and 3 oxen make 7 anything; you may reduce the sheep and the oxen to the common term of animals, and then you may say 4 animals and 3 animals are 7 animals. Therefore, before working a sum, it is frequently necessary to reduce all its terms to some one common denomination, so that all the units may have the same size and mean the same thing. You must make up your mind what unit you will take as your standard, and then you must multiply or divide all your terms according to the proportion which they bear to the standard unit. If you take a penny as your standard, you must multiply all your shillings by 12, and all your pounds by 240, because a shilling is 12 times, and a pound 240 times the value of a penny. If a pound is your standard unit, then you must divide your pennies by 240, and your shillings by 20. If an inch is your standard unit, you must multiply your feet by 12, and your yards by 36. If a yard is your standard unit, you must divide your inches by 36, and your feet by 3. All such processes come under the general term of reduction.

CHAPTER II.—Vulgar Fractions.

When the numerator and denominator of a fraction are alike, the value is that of an integer. For instance, $\frac{6}{6}=1$, because if a thing is cut into six equal bits, and you have all the six bits, you have the whole thing. If the numerator be less than the denominator, the fraction is less in value than an integer; if the numerator be greater than the denominator, the value of the fraction is more than an integer: and in this last case the fraction is called an improper fraction. If you have the improper fraction $\frac{39}{5}$, and you desire to know how many integers it contains, you have merely to divide it by five, because $\frac{5}{5}$=one integer; and therefore you will have as many integers as there are times 5-fifths in 39-fifths. Your answer will be $7\frac{4}{5}$, which is called a mixed number, because it contains integers and a fraction mixed. Hence *the rule for reducing an improper fraction to a mixed number is to divide the numerator by the denominator.* If, on the contrary, you desire *to convert a mixed number into an improper fraction, multiply the number of integers by the denominator of the fraction, and add the numerator to the product:* for instance, in the case of $7\frac{4}{5}$, as each integer contains $\frac{5}{5}$, or is a fraction having the same numerator as denominator, 7 integers must contain 7 × 5-fifths, or $\frac{35}{5}$, to which, of course, the $\frac{4}{5}$ must be added, $\frac{39}{5}$. Make your pupil practise a number of examples of converting improper fractions into mixed numbers, and *vice versâ.*

As it is often possible to express the same value in many different ways, for instance, 240 pennies, or 20 shillings, have the same value as one pound, so it is possible to find a great many different fractions all of which have the same value.

Fig. 1.



Let *a b q d* be half of the figure here drawn: if, by drawing the lines *e g* and *f h*, I divide the figure into four parts, the half will evidently contain two of those parts; therefore, $\frac{1}{2}=\frac{2}{4}$. By dividing the figure further by the line *m* M into eight parts, we find that the half contains four of those; therefore, $\frac{1}{2}=\frac{2}{4}=\frac{4}{8}$; and so by further division we find that $\frac{1}{2}=\frac{2}{4}=\frac{4}{8}=\frac{8}{16}=\frac{16}{32}$. In the next figure, by drawing the lines *k* K and *m* M, we find that $\frac{1}{2}=\frac{3}{6}$; by further drawing the lines *e g* and *f h*, we find that $\frac{1}{2}=\frac{6}{12}$; and by further division that $\frac{1}{2}=\frac{12}{24}$. By a different mode of division, Figure 3 shows that $\frac{1}{2}=\frac{5}{10}=\frac{15}{30}$. In Figure 4, we start with a fresh value altogether; *a b q d* is not one-half, but two-thirds of the whole figure. By drawing the lines *eg ik lm*, we find that $\frac{2}{3}=\frac{4}{6}$, for those lines divide the whole figure into six parts, and *a b q d* contains four of them. Draw the additional line B D, and

Fig. 2.

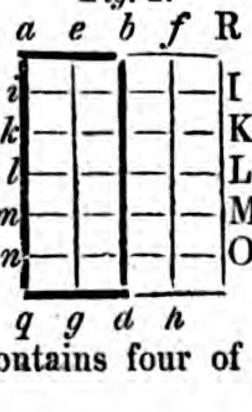


Fig. 3.

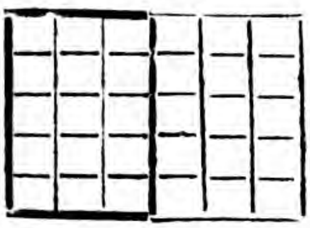
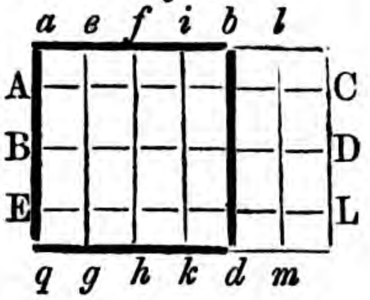


Fig. 4.



you find that $\frac{2}{3} = \frac{8}{12}$. Draw the lines A C and E L, and you find that $\frac{2}{3} = \frac{16}{24}$. Figure 5 shows that $\frac{2}{3} = \frac{6}{9} = \frac{12}{18} = \frac{24}{36}$. Figure 6 shows that $\frac{2}{3} = \frac{10}{15} = \frac{20}{30}$. If we now put together the result of our researches, we have as follows:— $\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{8}{16} = \frac{16}{32} = \frac{3}{6} = \frac{6}{12} = \frac{12}{24} = \frac{5}{10} = \frac{15}{30}$; $\frac{2}{3} = \frac{4}{6} = \frac{8}{12} = \frac{16}{24} = \frac{6}{9} = \frac{12}{18} = \frac{24}{36} = \frac{10}{15} = \frac{20}{30}$. On examination, we perceive that the numerators in each of these sets of equal fractions are exactly the same part of the denominators, and the denominators exactly the same multiple of the numerators. In the first set, every numerator is half of its denominator, and every denominator is twice its numerator. In the second set, every numerator is two-thirds of its denominator, and every denominator is three times the half of its numerator. In short, the numerators and denominators are in the same ratio to each other. Hence the general rule that *any fractions, whose numerators and denominators are in the same ratio, have the same value;* and hence, further, *if you multiply the numerator and denominator of any fraction by the same number, or divide them by the same number, you leave the value of the fraction unchanged.* If you have the fraction $\frac{1}{5}$, and multiply both its numerator and denominator by 3, you obtain $\frac{3}{15}$, a fraction of exactly the same value as $\frac{1}{5}$; because 1 is the fifth part of 5, and 3 is the fifth part of 15. Indeed, that this stands to reason may be shown thus: in the first instance, having cut your figure into 5 equal parts, you take only one of them; in the next instance, you cut your figure into 3 times the number of equal parts, and as the parts are therefore

Fig. 5.

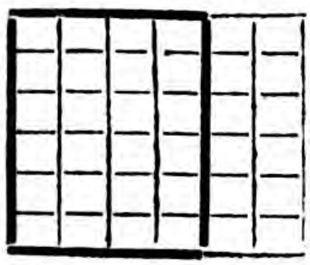
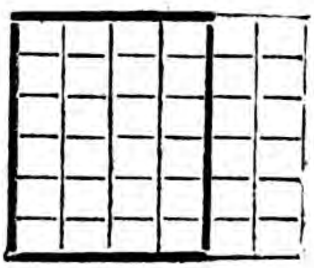


Fig. 6.



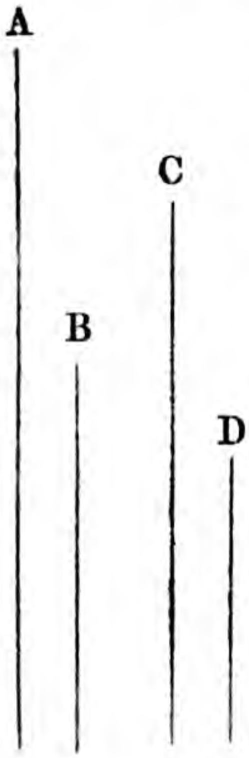
3 times smaller, you take 3 times as many, or $\frac{3}{15}$, which comes to the same thing. On the other hand, if you start with the fraction $\frac{4}{16}$, and divide both numerator and denominator by 4, you obtain $\frac{1}{4}$, which has exactly the same value; for you have made the parts 4 times larger, and taken only one-fourth of their number. You must now exercise your pupil in finding as many fractions as he can of equivalent values. Give him one fraction to start with, and let him write a row of fractions equal to it, and let him as he goes along cut a square by lines into corresponding parts. You may either give him a fraction with a large numerator and denominator, and let him, by dividing them by different numbers, find equivalent fractions, or you may give him a fraction with a small numerator and denominator, and let him, by multiplying them by different numbers, find equivalent fractions. In the course of all this, the pupil observes the ratios between the numerators and denominators, and his mind is directed to the idea of ratios in general; a difficult idea for a child to take in, and one needing much illustration.

The ratio between two numbers is that number by which one must be multiplied or divided, in order to make it equal to the other. Thus the ratio which 6 has to 3 is 2, because 3 must be multiplied by 2 in order to make it equal 6. The ratio which 3 has to 6 is $\frac{1}{2}$, because 6 must be divided by 2 in order to make it equal to 3. The question, what ratio has one number to another? is equivalent to the question, what part or what multiple is one number of another? If the one number be a part of the other, the ratio will be expressed by a fraction: if the one number be a multiple of the other, the ratio will be expressed by an integer. Exercise your pupils in finding the ratios between various numbers, and when he is familiar with the idea, give him such series of questions as the following:—

What ratio has		3 to 9	What ratio has		9 to 3
	—	4 to 12	—	—	12 to 4
	—	5 to 15	—	—	15 to 5
	—	6 to 18	—	—	18 to 6
	—	7 to 21	—	—	21 to 7
	—	8 to 24	—	—	24 to 8
	—	9 to 37	—	—	27 to 9
	—	10 to 30	—	—	30 to 10
	—	11 to 33	—	—	33 to 11
	—	12 to 36	—	—	36 to 12

To every question of the first series, the answer is $\frac{1}{3}$; to every question of the second series, the answer is 3. Therefore the ratios are equal throughout each series, and a number of equal fractions can be arranged out of the number composing them, thus:—$\frac{3}{9}=\frac{4}{12}=\frac{5}{15}=\frac{6}{18}=\frac{7}{21}=\frac{8}{24}=\frac{9}{27}$, etc. A series of proportions can also be deduced from them, for proportion is the equality of ratios: 3 has the same ratio to 9 that 4 has to 12; therefore 3 is to 9 as 4 is to 12, or, expressing this by signs, 3 : 9 :: 4 : 12; in the same way, 9 : 3 :: 12 : 4. The pupil should be exercised in arranging the number in each of the above series as proportions.

Whenever there are four numbers, and the first is the same part or multiple of the second, as the third is of the fourth, these four numbers are said to be proportionals, and the first two have to each other the same ratio as the last two. Thus 2 is the third part of 6, and 3 is the third part of 9; therefore 2, 6, 3, 9 are proportionals, and the fact of their proportion is stated thus, 2 : 6 :: 3 : 9, which means, 2 is to 6 as 3 is to 9. Again, 8 is four times 2, and 12 is four times 3; therefore, 8, 2, 12, 3 are proportionals, and we say, 8 : 2 :: 12 : 3. The line A is twice the line B, and the line C is twice the line D; therefore, A : B :: C : D. Again, the line B is half the line A, and the line D is half the line C; therefore, B : A :: D : C. Thus it appears that things of very different sizes may yet have the same ratio to each other. The France in a map has to the England in a map the same ratio as the real France has to the real England; that is, map France : map England :: real France : real England. From what has been said, it is plain that if we have two equal fractions, we have four proportional numbers, thus, $\frac{3}{5}=\frac{6}{10}$; therefore, 3 : 5 :: 6 : 10; consequently, whenever we have equal fractions, we can always separate them into proportions. On the other hand, if we have a proportion, we can reverse the process and



arrange it into equal fractions, thus, 3 : 7 :: 9 : 21; therefore, $\frac{3}{7} = \frac{9}{21}$.

It is often convenient, if you have a fraction with a very large numerator and denominator, to change it into an equivalent fraction with a small numerator and denominator. This is called reducing a fraction to its lowest terms; and the way to do it is to *divide the numerator and denominator by the same number;* for instance, if I divide both numerator and denominator of $\frac{12}{24}$ by 12, I get $\frac{1}{2}$. The difficulty is to guess what number will divide both numerator and denominator, without leaving a remainder. Thus in the fraction $\frac{6}{10}$, 3 would divide 6, but it would not divide 10; 5 would divide 10, but it would not divide 6: on the contrary, 2 will divide both 6 and 10, and give the answer $\frac{3}{5}$. A little practice will make you expert in guessing at a right divisor. You will observe that when the last digit is an even number, as in 34, for instance, you can divide by 2; when it is an odd number, as in 73, you cannot. When the last digit is either 5 or 0, as in 160 or 85, you can divide by 5, otherwise you cannot. Some fractions can be divided by various divisors, thus, $\frac{12}{24}$ can be divided by 2, or by 3, or by 4, or by 6, or by 12. If you divide by several of the smaller divisors successively, you obtain the same result, as if you had divided at once by the larger divisor, thus, $\frac{12}{24}$ divided by $4 = \frac{3}{6}$; divide by $3 = \frac{1}{2}$; the same as if you had divided at once by 12. Thus you may often employ several successive divisions in order to reduce a fraction to its lowest terms. This depends on a property of numbers, which it is well to consider for a moment. $2+3=5$, $4+7=11$. If you wish to add 11 and 5, it comes to the same thing whether you say $5+11=16$, or $2+3+4+7=16$. It comes to the same thing whether you add the various parts of which numbers are composed, or whether you add the numbers themselves. It is an excellent exercise for a child to make him separate numbers in various ways, and then by adding them prove that their sum is still the same: thus, $1+4+1+6+4=16$; $2+3+2+5+4=16$; $3+2+3+4+4=16$; $1+4+4+3+4=16$; $5+5+2+4=16$; $5+6+1+4=16$.

The same thing can be applied to subtraction. In multiplication we have something of the same kind, but here the numbers must not be divided into parts which, when added, will make up the original numbers, but into parts which, when multiplied, will make up the original numbers: thus, $2 \times 3 = 6$; $4 \times 7 = 28$. It is the same thing, whether we say $6 \times 28 = 168$, or $2 \times 3 \times 4 \times 7 = 168$. Numbers which multiplied into each other produce another number, are called the factors of that number. You can produce the number 168 either by the two factors 6 and 28, or by the four factors 2 and 3 and 4 and 7. Some numbers are composed of various factors: thus, $24 = 2 \times 12$; $24 = 3 \times 8$; $24 = 4 \times 6$; $24 = 2 \times 4 \times 3$; $24 = 2 \times 2 \times 2 \times 3$. If any number is to be multiplied by 24, it comes to the same thing whether you use 24 itself, or any one of these sets of factors which compose it: thus, $24 \times 9 = 216$; or else $2 \times 9 = 18$, $12 \times 18 = 216$; or else $3 \times 9 = 27$, $8 \times 27 = 216$; or else $4 \times 9 = 36$, $6 \times 36 = 216$; or else $2 \times 9 = 18$, $4 \times 18 = 72$, $3 \times 72 = 216$; or else $2 \times 9 = 18$, $2 \times 18 = 36$, $2 \times 36 = 72$, $3 \times 72 = 216$. If any number is to be divided by 24, it comes to the same thing whether you use 24 itself, or any one of the sets of factors which compose it: $72 \div 24 = 3$; $72 \div 2 = 36$, $36 \div 12 = 3$; $72 \div 3 = 24$, $24 \div 8 = 3$; $72 \div 4 = 18$, $18 \div 6 = 3$; $72 \div 2 = 36$, $36 \div 4 = 9$, $9 \div 3 = 3$; $72 \div 2 = 36$, $36 \div 2 = 18$, $18 \div 2 = 9$, $9 \div 3 = 3$. Hence the reason why, in reducing fractions to their lowest terms, you find a succession of small divisors produce the same effect as one large one, because in multiplying or dividing it comes to the same thing, whether you use the number itself or the factors which compose it. Exercise your pupil in dividing various numbers into all the factors of which they are composed. It is often convenient, in reducing a fraction to its lowest terms, to know at once what is the largest divisor that will do so, or, as it is called, the greatest common measure, and the following plan will enable you to discover it. *Divide the denominator by the numerator, divide the last divisor by the remainder, and so on till nothing remain; the last divisor is the greatest common measure.* Let $\frac{28}{388}$ be the fraction, you wish to discover what is the largest number that will divide both 28 and 388 without leaving any remainder, that is, the greatest common measure. Divide 388 by 28, it goes 13 times, and 24 is the

```
28)388(13
   28
   ---
   108
    84
   ---
    24)28(1
       24
       --
        4)24(6
          24
          --
```

remainder; now make 28 your dividend and divide by 24; it goes once, and 4 is the remainder: now make 24 your dividend, and divide by 4; it goes 6 times, and there is no remainder. Therefore 4 is the greatest common measure; it will divide 28 and 388, and give you the fraction $\frac{7}{97} = \frac{28}{388}$. The reason of the process is this: if 4 is contained exactly in 24, and if 24 is contained in 28 with a remainder of 4, then 4 must be contained in 28. If 28 is contained in 388 with a remainder of 24, as 4 is contained both in the 24 and the 28, it must be contained in the 388 too. Thus we have proved it to be contained in both 28 and 388, and therefore to be the divisor we want. If the fraction be $\frac{366}{388}$, the process is exactly the same, and 2 is the greatest common measure. As 2 is exactly contained in 6, and as 6 is contained in 8 with a remainder of 2, therefore 2 is contained in 8; so that we have 2 contained in the fifth divisor and dividend. As 2 is contained in 8, and as 8 is contained in 14 with a remainder of 6, in which 2 is also contained, therefore 2 is contained in 14; thus we find 2 contained both in 8 and 14, the fourth divisor and dividend. As 2 is contained in 14, and as 14 is contained in 22 with a remainder of 8, in which 2 is also contained, therefore 2 is contained in 22; thus we have 2 contained both in 14 and 22, the third divisor and dividend. As 2 is contained in 22, and as 22 is contained in 366 with a remainder of 14, in which 2 is also contained, therefore 2 is contained in 366; thus we have proved that 2 is contained both in 22 and in 366, the second divisor and dividend. As 2 is contained in 22, and 22 is contained in 388 with a remainder of 366, in which 2 is also contained, therefore 2 is contained in 388; thus we have proved that 2 is contained both in 366 and 388, which are the original numbers of our fraction.

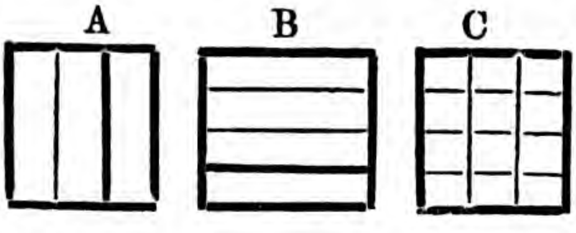
```
366)388(1
    366
    ---------
2nd—22)366(16
        22
        ---
        146
        132
        ---
    3rd—14)22(1
           14
           ---
      4th—8)14(1
             8
            ---
        5th—6)8(1
              6
             ---
          6th—2)6(3
                6
                -
```

This is rather a long piece of reasoning, and needs attention to follow. The only way to do so is to fix the mind upon each separate thing that is proved, and having clearly understood it, keep it in mind so as to use it in proving the next thing; as in building a wall, you lay one brick firmly, and can then use it to rest the next brick upon. You first prove that 2 is contained in the sixth divisor and dividend; then you use this knowledge to prove that 2 is contained in the fifth divisor and dividend; and so you travel backwards, making a kind of chain, fastening each new link to the last which you have made. It is thus that in arithmetic, as in many other things, by making a good use of what we do know we find out a great many things that we did not know.

```
365)388(1
    365
    ———
2nd—23)365(15
       23
       ——
       135
       115
       ———
   3rd—20)23(1
          20
          ——
     4th—3)20(6
           18
           ——
      5th—2)3(1
            2
            —
       6th—1)2(2
             2
             —
```

If the fraction be $\frac{365}{388}$, we find that our last divisor is 1, so that 1 is the greatest common measure; but dividing by 1 will not render a number any smaller; so this proves that the fraction $\frac{365}{388}$ cannot be reduced to any lower terms.

It is impossible to add together fractions with different denominators; you cannot express $\frac{2}{3}$ and $\frac{3}{4}$ in one sum, any more than you can express two pounds and three shillings, or two sheep and three oxen in one sum. You must reduce the two quantities to some common denomination; that is, to some numbers of which the units shall all have the same value. The first fraction supposes the integer cut into three equal parts, as A; the second fraction supposes the integer cut into four, as B. If we could cut the integer into such a number of parts, as that the thirds in A should exactly contain a certain number of those parts, and also the fourths in B should exactly contain another certain number of those parts, then these small pieces would be a common denomination, to which we could reduce both the fraction of A and the fraction of B. If you had to add a pound and a half crown, you would find some smaller kind of money which would be



exactly contained in the pound, and also exactly contained in the half crown; this kind of money would of course be a penny—a penny is contained exactly 240 times in a pound, and exactly thirty times in a half crown, and thus it is a common denomination to which both can be reduced. To return to our fractions, place A on the top of B, the result will be as in C; the integer is now divided both by 4 and by 3; that is, it is divided by $4 \times 3 = 12$, and of course each fourth part will contain an exact number of twelfths, viz., $\frac{3}{12}$, and each third part will contain an exact number of twelfths, viz., $\frac{4}{12}$, so that by multiplying the two denominators of our fractions together, we have found a common denominator that will express both; $\frac{2}{3}$ must be reduced to twelfths, and $\frac{3}{4}$ must be reduced to twelfths, and then they can be added together; in order to reduce $\frac{2}{3}$ to twelfths, its denominator must be multiplied by 4, and consequently its numerator also, because, if the value of a fraction is to remain unchanged, its numerator and denominator must both be multiplied by the same number; therefore, $\frac{4 \times 2}{4 \times 3} = \frac{8}{12}$. In order to reduce $\frac{3}{4}$ to twelfths, its denominator must be multiplied by 3, and consequently its numerator also; therefore, $\frac{3 \times 3}{3 \times 4} = \frac{9}{12}$. We have now $\frac{8}{12}$ and $\frac{9}{12}$ and can add them together, $\frac{8}{12} + \frac{9}{12} = \frac{17}{12} = 1\frac{5}{12}$. If you retrace the process which we have performed, you will find it to have been this: we first multiplied the two denominators together to ascertain what the common denominator should be, and then to ascertain what the numerators of the new fractions should be, we multiplied each numerator of an old fraction by the denominator of the other. Suppose that having obtained the $\frac{17}{12}$, it were still required to add to them $\frac{4}{5}$, we should repeat the same process, $5 \times 12 = 60$, which is the common denominator, $\frac{17}{12} = \frac{5 \times 17}{5 \times 12} = \frac{85}{60}$ $\frac{4}{5} = \frac{12 \times 4}{12 \times 5} = \frac{48}{60}$ $\frac{85}{60} + \frac{48}{60} = \frac{133}{60} = 2\frac{13}{60}$. Had the fractions, instead of being given to us separately, been given to us all at once; that is, had we at once had $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$ to add together, we should have performed exactly the same operations, only in one process instead of in two. Thus to ascertain the common denominator, we should have said

$3 \times 4 \times 5 = 60$, then we should have taken care to multiply the numerator of each fraction by the same number as that which had multiplied its denominator. Since 3 was multiplied by 4 and by 5 to bring it to 60, we should multiply 2 also by 4 and by 5. Since 4 was multiplied by 3 and by 5 to bring it to 60, we should multiply it by 3 and by 5. Since 5 was multiplied by 3 and by 4 to bring it to 60, we should multiply 4 also by 3 and by 4. In short, we should *multiply the numerator of each fraction by all the denominators except its own, and place beneath each new numerator thus obtained the new denominator obtained by multiplying all the denominators together.* And this is the rule usually given in arithmetic books for reducing fractions to a common denominator. Subtraction of fractions is performed on exactly the same principles as addition.

Difficulty occasionally arises from what are called compound fractions, for instance, $\frac{2}{3}$ of $\frac{3}{4}$; such a fraction must be reduced to a simple one. You can always divide a fraction by multiplying its denominator, the third part of $\frac{3}{4}$ is $\frac{3}{12}$, because twelfths are just one-third the size of fourths; if one-third of $\frac{3}{4}$ is $\frac{3}{12}$, two-thirds of $\frac{3}{4}$ must be $\frac{2 \times 3}{12} = \frac{6}{12}$, which is a simple fraction, exactly equivalent to the compound fraction $\frac{2}{3}$ of $\frac{3}{4}$, and this simple fraction has been obtained by multiplying the numerators together and the denominators together, which formed the compound fraction. Whenever in a sum you find a compound fraction, reduce it by this means to a simple one; and in so doing, you are in fact performing the multiplication of fractions.

When I tell you to multiply $\frac{3}{4}$ by $\frac{2}{3}$, I mean that you are to take it not 2 times, but the third part of 2 times. You have, therefore, first to ascertain what the sum would be if you took it 2 times, and then to find out the third part of that. Twice three-fourths are six-fourths; that is to say, $2 \times \frac{3}{4} = \frac{6}{4}$. You may find the third part of $\frac{6}{4}$, either by dividing the numerator by 3 or by multiplying the denominator by 3. In the former case you have $\frac{2}{4}$, in the other case you have $\frac{6}{12}$, both which fractions have the same value. The latter method is usually preferred, and the result stated thus, $\frac{3}{4} \times \frac{2}{3} = \frac{6}{12}$. Hence the rule for the multiplication of fractions is to *multiply the numerators together and the denominators together.*

We now proceed to division.

If you are to divide $\frac{3}{4}$ by $\frac{2}{3}$, you are not to divide it by 2, but by the third part of 2. First divide it by 2; that is, multiply the denominator by 2, $\frac{3}{4} \div 2 = \frac{3}{8}$. As you were desired to divide by the third part of 2, your divisor has now been three times too great, and your quotient is therefore 3 times smaller than it ought to be; therefore, multiply it by 3, and the result will be the answer sought, $\frac{3}{8} \times 3 = \frac{9}{8}$. You will observe that what we have in fact been doing in this process, has been to multiply the numerator of the dividend by the denominator of the divisor for the numerator of the quotient, and the denominator of the dividend by the numerator of the divisor for the denominator of the quotient. A more simple way of expressing the rule is this, *invert the divisor and proceed as in multiplication:* that is, $\frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \times \frac{3}{2} = \frac{9}{8}$. A learner is generally much puzzled to find that the result of the multiplication of fractions is a product less than the multiplicand, and that the result of the division of fractions is a quotient greater than the dividend. For instance, $6 \times 3 = 18$, but $\frac{6}{7} \times \frac{3}{7} = \frac{18}{49}$, a quantity much smaller than $\frac{6}{7}$. On the other hand, $6 \div 3 = 2$, but $\frac{6}{7} \div \frac{3}{7} = \frac{42}{21}$, a quantity more than twice as great as $\frac{6}{7}$. The fact is that, in this case, the words multiplication and division are somewhat misleading. When you multiply by a fraction, you are in truth performing a kind of division. When you multiply 8 by $\frac{1}{4}$, you take 8 one-fourth number of times; that is, you take the fourth part of 8, or divide 8 by 4. The same is the case if you multiply $\frac{8}{9}$ by $\frac{1}{4}$, you take the fourth part of $\frac{8}{9} = \frac{8}{36} = \frac{2}{9}$. If the numerator of your multiplier be greater than one, you have to multiply as well as divide; but as in every proper fraction the numerator is smaller than the denominator, the effect of the multiplication is less powerful than that of the division, and so the total result is still a diminution of the multiplicand; thus in the case of $\frac{8}{9} \times \frac{3}{4}$, after dividing by 4 you must multiply by 3, because you are to take $\frac{8}{9}$ three-fourth times; but the multiplication by 3 will not be sufficient to counterbalance the effect of the division by 4, and so the result $\frac{24}{36} = \frac{6}{9}$, will still be less than $\frac{8}{9}$. There is also an important distinction between the division of fractions and the division of integers. In division of integers, two separate questions are answered by the same quotient.

For instance, $18 \div 6 = 3$. This 3 answers the question, What is the sixth part of 18? And it also answers the question how many times is 3 contained in 18. Now, it is only this last question which division of fractions will solve. The former question is answered by the multiplication of fractions. When you divide $\frac{8}{9}$ by $\frac{1}{4}$, you do not inquire what is the one-fourth part of $\frac{8}{9}$, that would be discovered by multiplying $\frac{8}{9}$ by $\frac{1}{4}$. But you inquire how many times one-fourth is contained in $\frac{8}{9}$. Reduce both fractions to a common denominator, $\frac{32}{36}$ and $\frac{9}{36}$. Then it is plain that $\frac{9}{36}$ will be contained in $\frac{32}{36}$, as often as 9 is contained in 32; that is, $3\frac{5}{9}$ times, or, by reducing this to an improper fraction, $\frac{32}{9}$ times, which is the very same result as would have been obtained by the ordinary rule for the division of fractions. That rule is in fact only a short expression for the simpler but longer process of reducing both fractions to a common denominator, and then ascertaining how often the one is contained in the other. A superficial glance will satisfy of the correctness of the result, for as $\frac{1}{4}$ is something more in value than $\frac{2}{9}$, it will naturally be contained rather less than 4 times in $\frac{8}{9}$, and this is just what is expressed by $\frac{32}{9}$ times, or $3\frac{5}{9}$ times. The application of division of fractions to money helps to explain it. How many shillings are contained in three-fourths of a pound? that is, how many twentieths of a pound are contained in three-fourths of a pound? $\frac{3}{4} \div \frac{1}{20} = \frac{3}{4} \times \frac{20}{1} = \frac{60}{4} = 15$. Fifteen shillings are three-fourths of a pound.

CHAPTER III.—Proportion.

If you refer to my remarks in introducing the subject of fractions (pages 16, 17), you will find an explanation of proportion, and I would ask you to read that explanation over again as a preparation for the study of the Rule of Three on which we now enter. It is called the Rule of Three, because the object of it is by means of three given numbers to find a fourth, which fourth number is to bear to the third the same ratio as the second bears to the first. For instance, if the numbers 2, 6, and 3 are given, a number must be found which shall bear to 3 the same ratio as 6 does to 2; in this case, as 6 is three times 2, the required number, which for shortness sake I will call x, must evidently be three times 3, that is 9, and we can then state the proportion, 2 : 6 :: 3 : 9. If the given numbers are 15, 5, and 12, the required number, or x, must bear to 12 the same ratio as 5 does to 15; in this case, as 5 is the third part of 15, the required number, or x, must be the third part of 12, that is to say 4, and we may state the proportion thus: 15 : 5 :: 12 : 4. The great point, in order to find a fourth proportional to three given numbers, is to ascertain exactly what the ratio is which the second of these numbers bears to the first, and when the second is either an exact part or an exact multiple of the first, there is no difficulty. But sometimes this is not so; for instance, let the three numbers be 4, 7, and 12, x must bear to 12 the same ratio that 7 does to 4. But what ratio does 7 bear to 4? It is neither the half of 4, nor the third part of 4, nor twice 4, nor three times 4. We must find its ratio to 4 in a more roundabout way. Every unit is the fourth part of 4, and 7 contains 7 units, so that 7 is seven times the fourth part of 4; this is the ratio which 7 bears to 4, and which x must bear to 12, x must be 7 times the fourth part of 12. In order then to find x, we must divide 12 by 4, and then multiply the quotient by 7, or, which is more convenient, do the multiplication first and the division afterwards. Multiply 12 by 7 and divide the product by 4, $12 \times 7 = 84$, $84 \div 4 = 21$, 4 : 7 :: 12 : 21. 7 is seven times the fourth part of 4. 21 is seven times the fourth part of 12. In order to obtain 21, *we have multiplied the third of our three given numbers by the second, and we have divided the product by the first.* This is the Rule of Three, and by this rule you can always with the utmost ease, find a fourth proportional to any three given numbers. But the application of this to practical purposes is sometimes puzzling. When a question about yards of silk or pounds of mutton is given, the difficulty lies in putting it into the shape of a proportion sum, not in working the proportion when stated. For instance, if 5 yards of silk cost 7 shillings, how much will 8 yards cost? In order to state this sum, take care to compare together things of the same kind; you cannot compare silk and shillings, but you can compare 5 yards of silk with 8 yards of silk. Common sense will tell you that whatever ratio the 8 yards bear to the 5 yards, that ratio must the cost of the 8 yards bear to the 7 shillings which the 5 yards cost, or as much greater as 8 is than 5 yards, so much greater is x the cost of 8 than 7 the cost of 5 yards. Express this by signs thus: $5 \angle 8 :: 7 \angle x$, then repeat the statement in the ordinary manner of proportion,

Y. Y. S. S.

$5 : 8 :: 7 : x$, and ascertain the value of x by the Rule of Three, $x = 7 \times 8 \div 5 = \frac{56}{5} = 11\frac{1}{5}$. Therefore, 8 yards of silk will cost 11 shillings and one-fifth of a shilling. If 12 pounds of mutton cost 5 shillings, how much will 9 pounds cost? Compare mutton with mutton; whatever ratio 9 pounds of mutton bear to 12 pounds, that ratio must the cost of the 9 pounds bear to 5 shillings the cost of 12 pounds, or as much less as 9 is than 12 lbs., so much less is x than 5 shillings. Express this by signs thus: $12 \urcorner 9 :: 5 \urcorner x$, then repeat the statement in the ordinary manner of propor-

LBS. LBS. S. S.

tion, $12 : 9 :: 5 : x$, and ascertain the value of x by the Rule of Three, $x = 5 \times 9 \div 12 = \frac{45}{12} = 3\frac{9}{12}$. Therefore, 9 lbs. of mutton cost three shillings and ninepence. The principle of all these sums is, by comparing things which you know, to find out something which you do not know. Take care to put the thing which you do not know last in the row of proportion, and the thing of the same kind next to it; then in placing your two first terms, take care to put them in an order corresponding to the two last, so that the first of the first set may have some relation to the first of the last set, and the second of the first set to the second of the last set. For instance, if a man can walk 8 miles in 3 hours, how many miles can he walk in 11 hours. Here the unknown quantity is miles, therefore make the 8 miles your third term, then your two first terms will be hours; but as the 3 hours correspond to the 8 miles, take care to make them the first term, so that the second term may be the 11 hours which correspond to the fourth unknown quantity of miles. As much greater as 11 hours are than 3 hours, so much greater is x, the number

H. H. M. M.

sought than 8 miles, $3 \angle 11 :: 8 \angle x$, $x = 11 \times 8 \div 3 = \frac{88}{3} = 29\frac{1}{3}$. The man could walk $29\frac{1}{3}$ miles in 11 hours.

If I can buy 9 yards of ribbon for 19 shillings, how many can I buy for 13 shillings? Here the unknown quantity is yards, therefore make the 9 yards your third term, then your two first terms will be shillings; but as the 19 shillings correspond to the 9 yards, take care to make them the first term, so that the second term may be the 13 shillings which correspond to the fourth unknown quantity of yards. As much less as 13 shillings are than 19 shillings, so much less will x, the number sought, be than 9 yards, $\overset{\text{S.}}{19} \urcorner \overset{\text{S.}}{13} :: \overset{\text{Y.}}{9} \urcorner \overset{\text{Y.}}{x}$, $\overset{\text{S.}}{19} : \overset{\text{S.}}{13} :: \overset{\text{Y.}}{9} : \overset{\text{Y.}}{x}$, $x = \frac{13 \times 9}{19} = \frac{117}{19} = 6\frac{3}{19}$. I can buy $6\frac{3}{19}$ yards for 13 shillings.

In the above examples you will observe that though in some cases the angular signs of ratio point to the right, and in other cases to the left, yet they never point different ways in the same sum; if the first angle points to the left, so does the second; if the first angle points to the right, so does the second; the reason is because, in the one case, *as much greater* as the second term is than the first, *so much greater* is the fourth term than the third; and, in the other case, *as much less* as the second term is than the first, *so much less* is the fourth term than the third. But there are other cases in which *as much greater* as the second term is than the first, *so much less* is the fourth term than the third; or in which *as much less* as the second term is than the first, *so much greater* is the fourth term than the third, in such cases the two angles in the same sum point contrary ways. Thus, if 7 men can build a wall in 11 days, how many men will be required to build it in 9 days. The unknown quantity here is men, so the 7 men will be the third term of our proportion; the first and second terms will be days, and as the 11 days is the time occupied by the 7 men, it will be the first term, leaving the 9 days which correspond to the x number of men as the second term. Now, it is evident that the fewer the days given to build the wall, the greater must be the number of men employed—to build the wall in half the time would need twice the number of men. Therefore, *as much less* as 9 days are than 11 days, *so much greater* must x number of men be than 7 men. Express this by signs $\overset{\text{D.}}{11} \urcorner \overset{\text{D.}}{9} :: \overset{\text{M.}}{7} \angle \overset{\text{M.}}{x}$. To get this sum into working order, you must invert the two first terms, and then state the whole as a common proportion, $\overset{\text{D.}}{9} : \overset{\text{D.}}{11} :: \overset{\text{M.}}{7} : \overset{\text{M.}}{x}$, you then obtain the value of x by the Rule of

Three, $x = 11 \times 7 \div 9 = 77 \div 9 = 8\frac{5}{9}$. The wall can be built in 9 days by $8\frac{5}{9}$ men.

If 5 men will drink a barrel of beer in 7 days, in how many days will 8 men consume it? Here the unknown quantity is days, so your third term must be 7 days, and your two first terms will be men. As the 5 men consume the beer in the seven days, they must occupy the first term, leaving the 8 men who correspond to the x number of days for the second term. Now it is evident that the more men there are the fewer will be the number of days in which the beer will be consumed, twice the number of men would consume the beer in half the time. Therefore, *as much greater* as 8 men are than 5 men, *so much less* will x days be than 7 days.

M. M. D. D.

Express this by signs thus: $5 \angle 8 :: 7 \urcorner x$. To bring this sum into working order, you must invert the two first terms, and then state the whole as a common proportion,

M. M. D. D.

$8 : 5 :: 7 : x$, you then obtain the value of x by the Rule of Three, $x = 5 \times 7 \div 8 = 35 \div 8 = 4\frac{3}{8}$. The barrel of beer will be consumed by the 8 men in $4\frac{3}{8}$ days. When both the angular signs in a sum point the same way, the proportion is called *Direct Proportion;* when they point contrary ways it is called *Inverse Proportion.* By using your common sense to judge of the nature and meaning of the question which you have to solve, you will ascertain without difficulty whether it is a case of direct or of inverse proportion, and if you always state your proportion with the angular signs, before you arrange it in the common form for working, you will not be likely to get into any confusion.

The best way to make your pupil expert in proportion sums is, to make him fully state the explanation of *every* sum that he works, somewhat in the following form:—

If 25 men will consume 129 lbs. of bread in 6 days, in how many days will 31 men consume as much?

M. M. D. D.

$$25 \angle 31 :: 6 \urcorner x$$

M. M. D. D.

$$31 : 25 :: 6 : x$$

25 is 25 times the $\frac{1}{31}$ part of 31

x is 25 times the $\frac{1}{31}$ part of 6

M. M. D. D.

$$31 : 25 :: 6 : 4\tfrac{26}{31}$$

$$\begin{array}{r} 6 \\ \hline 31)150(4\tfrac{26}{31} \\ 124 \\ \hline 26 \end{array}$$

If you refer to my early remarks upon fractions, you will find the following fact stated: if we have two fractions of equal value, we have four proportional numbers, thus, $\frac{3}{5} = \frac{6}{10}$, therefore 3 : 5 :: 6 : 10; therefore, whenever we have a proportion sum, we can convert it into two equal fractions—thus, 4 : 7 :: 12 : 21, therefore $\frac{4}{7} = \frac{12}{21}$ Reduce these equal fractions to fractions having a common denominator, $\frac{4 \times 21}{7 \times 21} = \frac{12 \times 7}{21 \times 7}$ Now, as these fractions are equal, and as they have the same denominators, it is plain that their numerators must be the same also; therefore, $4 \times 21 = 12 \times 7$, which is evidently true, as $4 \times 21 = 84$, and $7 \times 12 = 84$. Observe, that 4 and 21 are the first and last terms, or what we call the extremes of the proportion sum, and that 7 and 12 are the middle terms, or what we call the means, and then from the fact that $4 \times 21 = 12 \times 7$, we deduce the general law, that *when any four numbers are proportionals, the product of the extremes is always equal to the product of the means.* By this law you may prove whether you have worked any rule of three correctly. For instance, in the question of the yards of silk (page 27), you found that 5 : 8 :: 7 : $11\frac{1}{5}$, therefore $5 \times 11\frac{1}{5} = 8 \times 7$, both $= 56$; had your last term not been correctly found, these products would not so have agreed.

In the question of the barrel of beer, you found that 8 : 5 :: 7 : $4\frac{3}{8}$, $8 \times 4\frac{3}{8} = 5 \times 7$, both $= 35$. From this fact that the product of the extremes equals the product of the means, we may learn how the terms in a proportion sum may be transposed. Thus, 4 : 7 :: 12 : 21, $4 \times 21 = 12 \times 7$; suppose we make the 7 and the 12 change places, it will make no difference; for still $4 \times 21 = 7 \times 12$, and then we shall have 4 : 12 :: 7 : 21. So in every proportion not only is the first term to the second as the third term is to the fourth, but also the first term is to the third as the second term is to the fourth. Again, 4 : 7 : : 12 : 21, $4 \times 21 = 7 \times 12$; suppose we change the places of all the four terms, and make 7 and 12 the extremes, and 4 and 21 the means, still there will be no difference, for $7 \times 12 = 4 \times 21$, and then we have 7 : 4 : : 21 : 12, so that we have found in this proportion a third property, that the second term is to the first as the fourth term is to the third. If we call the four terms *a*, *b*, *c*, and *d*, you will find that the above reasoning has taught us that we may arrange these terms in three different ways, in all of which they will be proportionals—

$$a : b :: c : d$$
$$a : c :: b : d$$
$$b : a :: d : c$$

CHAPTER IV.—Decimal Fractions.

Decimal Fractions are fractions whose denominators are 10, or multiples of 10 by 10, such as $\frac{1}{100}$, $\frac{1}{1000}$, $\frac{1}{10000}$, etc. You will remember that, in explaining the system of arithmetical notation, I observed that each place of figures has the tenth part of the value of that to the left of it: thus, a hundred is the tenth part of a thousand, a ten is the tenth part of a hundred, a unit is the tenth part of a ten; suppose that instead of stopping here, we just put a point to mark where the place of units is, and then put another figure to the right hand of it thus 1111.1, this last figure will be the tenth part of a unit, and in fact be equivalent to $\frac{1}{10}$. Add another figure thus 1111.11, this will be the tenth part of the previous figure, or the tenth part of $\frac{1}{10}$ which is $\frac{1}{100}$. Add another figure thus 1111.111, this will be the tenth part of the previous figure, or the tenth part of $\frac{1}{100}$ which equals $\frac{1}{1000}$. We may thus have a row of figures of indefinite length, each of which indicates a fraction having a value ten times smaller, and therefore a denominator ten times greater than the preceding one. Thus, as by adding figures to the left hand there is no conceivable limit to the size of the number which we may express, so by adding places of figures to the right hand there is no conceivable limit to the minuteness of the fraction which we may express. The value of 1111.111 equals 1111. $+ \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} = 1111. + \frac{100}{1000} + \frac{10}{1000} + \frac{1}{1000} = 1111. +$

$\frac{111}{1000}$. The value of $29.3806 = 29. + \frac{3}{10} + \frac{8}{100} + \frac{0}{1000} + \frac{6}{10000} = 29. + \frac{3000}{10000} + \frac{800}{10000} + \frac{0}{10000} + \frac{6}{10000} = 29. + \frac{3806}{10000}$. The value of $29.087 = 29. + \frac{0}{10} + \frac{8}{100} + \frac{7}{1000} = 29. + \frac{0}{1000} + \frac{80}{1000} + \frac{7}{1000} = 29. + \frac{87}{1000}$. From these examples you may gather the rule for reading off decimals as vulgar fractions. Reckon from the decimal point to the right hand by tens, hundreds, thousands, and, as in common numbers, take the number thus indicated by the last place of figures on the right as your denominator, and for the numerator read the figures like a common sum. Thus, in the case of 29.3806, the 6 is in the place of tens of thousands, reckoning from the decimal point, which marks off the 9 as in the row of units; therefore the denominator is 10000, and the fraction is $\frac{3806}{10000}$. The fact is, that each place of figures corresponds to a cipher in the denominator. In the case of 29.087, the nought, which has no effect upon the numerator of the fraction, which is 87, just as though it had been written 29.87, has great effect upon the denominator, by placing the last figure 7 in the row of thousands, and so making the denominator 1000 instead of 100 which otherwise it would have been, the fraction therefore is $\frac{87}{1000}$. A cipher placed on the left of the figures of an ordinary sum does not change its value, and indeed is perfectly useless, because it does not alter the position of any of the figures in reference to the row of units; but if added on the right hand of the figures, by itself usurping the place of units, it increases the value of each figure tenfold, and makes the whole sum ten times greater than it was before. The reverse of this is the case in decimals. A cipher added on the left hand, by removing the figures from the row of units, reduces the value of each to one-tenth of what it was before, and therefore increases the denominator of the whole fraction tenfold; while a cipher added on the right hand leaves the relation of the figures to the row of units undisturbed, and though the rule for reading decimals will cause the reckoning of it to make the denominator ten times greater than it should have been, yet by the same rule the numerator will also be ten times greater, and so the fraction will have the same value as if it were not there. For instance, $.8 = \frac{8}{10}$, $.80 = \frac{80}{100}$, which is exactly equivalent to $\frac{8}{10}$, so the cipher has made no difference; but $.08 = \frac{8}{100}$, which is only the tenth part of $\frac{8}{10}$. Here the cipher has reduced the value of the fraction to one-tenth of that which it had before. You may work sums in decimals exactly in the same manner as in ordinary numbers, all you have to mind is to keep the figures in their proper places. For instance, in common addition you take care to place units under units, tens under tens, hundreds under hundreds; and in decimals take care to place tenths under tenths, hundredths under hundredths, and so on as in the annexed example, where this is simply effected by placing the decimal points vertically under each other. You carry exactly as in common addition. For example, the addition in the second row gives the number 17; these are $\frac{17}{1000}$, or $\frac{10}{1000}+\frac{7}{1000}$, but $\frac{10}{1000}=\frac{1}{100}$, so the 7 thousandths must be put down in the row of thousandths, and the one hundredth must be carried to the row of hundredths. In the fourth row the addition gives the number 13. These are $\frac{13}{10}$, or $\frac{10}{10}+\frac{3}{10}$, but $\frac{10}{10}=$ one unit, so the three tenths are put down in the row of tenths, and the one unit is carried to the row of units. Exactly similar rules apply to subtraction, where you may borrow as in ordinary sums. In the annexed sum you have to take 8 ten thousandths from no ten thousandths, you therefore convert one of the 8 thousandths in the next row into 10 ten thousandths, and then taking 8 from 10 leaves you 2 ten thousandths, which you write down. You have now only 7 thousandths left in the upper row of thousandths, which, as you have nothing to subtract, you just write down. You have no hundredths at all, so you state the fact by writing a nought. You have then 6 tenths to take from no tenths, so you convert one of the 7 units in the next row into 10 tenths, and taking the 6 tenths from them, you have 4 tenths left. You have then only 6 units remaining, which, having nothing to subtract from them, you just write down. In multiplication you proceed as in ordinary sums, but you must ascertain where to place the point in the product, by considering to what the denominator of the fraction must amount. In the annexed sum it is plain that, as two fractions are multiplied whose denominators are each 100, the denominator of the product must

```
 3.15
 2.081
  .4085
30.67
  .0084
-------
36.3179
```

```
7.008
 .6008
------
6.4072
```

```
  3.07
   .75
------
  1535
 2 149
------
2.3025
```

be 10000, and as each place of figures in a decimal fraction corresponds to a cipher in the denominator of a vulgar fraction, in order to secure to our fraction in the product the requisite denominator, we must make our decimal point cut off four places of figures on the right hand. We shall thus have 2 integer units, and the decimal fraction .3025 equal $2 + \frac{3025}{10000}$. As multiplying such numbers as 10, 100, 1000, 10000, etc., together, is just equivalent to adding so many ciphers, and as these ciphers in the denominator correspond to the places of figures in the decimal fraction, the rule for multiplying decimal fractions together is to *count up how many places of figures there are in the multiplier and multiplicand together, and mark off by the point that number in the product.* Sometimes the multiplication of the figures composing the decimal fractions does not produce places of figures sufficient to enable us to comply with this rule, in this case ciphers must be added on the left hand of the product, $2 \times 45 = 90$, and $.90 = \frac{90}{100}$, whereas $\frac{45}{1000} \times \frac{2}{100} = \frac{90}{100000}$. In order to express this, 3 ciphers must be prefixed to the 9, then there will be five places of decimals in the product, equal to the sum of the places in the multiplier and the multiplicand. Of course, after you have got the answer .00090, you may strike out the cipher on the right hand, as it is of no use, and you will then have .0009 for $\frac{90}{100000} = \frac{9}{10000}$.

$$\begin{array}{r} .045 \\ .02 \\ \hline .00090 \end{array}$$

Division being the reverse of multiplication, you *subtract the number of places of decimals in the divisor from those in the dividend, in order to ascertain the number of places in the quotient,* $\frac{1}{1000} \div \frac{1}{10} = \frac{1}{1000} \times \frac{10}{1} = \frac{10}{1000} = \frac{1}{100}$. You perceive here, that the result of dividing any fraction, whose denominator is tenths, hundredths, etc., by another fraction with a denominator of the same kind, is to produce a fraction whose denominator shall have a number of ciphers equal to the difference between the number of ciphers in the denominators of the dividend and divisor; hence the reason of the rule. If, after the division, there is a remainder, add ciphers to the right hand of the dividend, and go on dividing until there shall be no remainder. These ciphers do not, as explained before, alter the value of the dividend, but simply convert it into a fraction with a larger denominator. Thus, $7.5 \div 8 = 9$, and 3 would be the remainder. Add 3 ciphers, by which means $\frac{5}{10}$ are converted into $\frac{5000}{10000}$, changing their denominator, but having the same value as before. Proceed with the division as below. The process here is very similar to that employed in division of integers. Were there no decimal point in this sum, after dividing 75 thousands by 8, we should find 3 thousands remain, which we should convert into 30 hundreds, and then divide. The 6 hundreds which would then remain, we should convert into 60 tens and divide, and so on. In the sum with the decimal point, we first consider $7\frac{5}{10}$ as $\frac{75}{10}$, and dividing it by 8 get the answer $\frac{9}{10}$ with 3 tenths over; these tenths we convert into 30 hundredths, and dividing again get the answer $\frac{3}{100}$ and 6 hundredths over, which we convert into thousandths, and so on. If there are not sufficient figures in the quotient to supply a number of places equivalent to the difference between the places of decimals in the dividend and divisor, one or more ciphers must be prefixed. Thus, .8040 ÷ 10 = .804, but $\frac{8040}{10000} \div 10 = \frac{804}{100000}$, which in decimals can only be expressed by .0804, because the denominator has two more places of figures than the numerator.

```
8)7.5000(.9375
  72
  --
   30
   24
   --
    60
    56
    --
     40
     40
     --
```

Common division is never carried beyond the place of units, because no fractional places of figures are supposed to exist. When we divide 794 by 23, we first divide the 79 tens and then the 104 units that are left, and then, instead of breaking the 12 units that remain into tenths or hundredths, we cut each of them into as many parts as there are units in the divisor; that is, into twenty thirds, and thus discover the twenty-third part of the whole twelve, namely, $\frac{1}{23}$. But decimals enable us to proceed with the remainder on the same plan as that on which we had commenced the sum. The 12 units are converted into 120 tenths, and divided by 23; the 5 tenths that remain are converted into 50 hundredths, and divided by 23. The 4 hundredths that remain are converted into 40 thousandths, and so on. It is manifest that the decimal fraction in the second example corresponds to the vulgar fraction in the first—not exactly indeed, on account of the remainder of 1 thousand millionth. The fraction $\frac{12}{23}$ expresses fully and entirely the twenty-third part of the

```
23)794(34 12/23
   69
   ---
   104
    92
   ---
    12

23)794.00000000(34.52173913
   69
   ---
   104
    92
   ---
    120
    115
    ---
      50
      46
      ---
       40
       23
       ---
       170
       161
       ---
         90
         69
         ---
         210
         207
         ---
           30
           23
           ---
            70
            69
            ---
             1
```

12 units; whereas the fraction .52173913 expresses it with a remainder of one thousand milllionth, which ought still further to be reduced and divided. The decimal fraction can never perfectly represent the vulgar fraction, unless the decimal division be carried on till it leave no remainder. From this example, the rule for converting a vulgar into a decimal fraction is easily deduced; *divide the numerator by the denominator, adding as many ciphers to the numerator as shall enable you to carry on the division until there be no remainder.*

THOMAS HARRILD, PRINTER, SILVER STREET, FALCON SQUARE, LONDON.