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HARMONICS,

OR

THE PHILOSOPHY

OF

MUSICAL SOUNDS.

BY

ROBERT SMITH, D.D. F.R.S.
And Master of Trinity College
In the University of Cambridge.

[illegible] ——————
—————— [illegible]
[illegible] lenimen; mihi cumque salve
Rite vocanti. Hor.

CAMBRIDGE,
[illegible]
[illegible]
[illegible]
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O Decus Phœbi ————
———— *ô laborum*
Dulce lenimen, mihi cunque salve
Rite vocanti. Hor.

CAMBRIDGE,
Printed by J. BENTHAM, Printer to the University, and
Sold there by W. THURLBOURN and T. MERRILL, and
by S. AUSTEN and J. WHISTON Booksellers in London.

MDCCXLIX.

Scimus musicen, mathesin, atque adeo veram physicam nostris moribus NON *abesse à Principis persona: Quæ quidem omnia apud Græcos non laude solùm, sed honore et gloria digna ducebantur.*

Epaminondas, Imperator ille insignis, ne dicam summus vir unus omnis Græciæ, philosophiam et musicam egregie didicit. Nam doctus est à Dionysio, qui fuit eximia in musicis gloria. At philosophiæ præceptorem habuit Lysin Pythagoreum, neque prius eum à se dimisit, quam doctrinis tanto antecessit alios, ut facilè intelligi posset, pari modo superaturum omnes in cæteris artibus.

Corn. Nep. vit. Epam. sub initio.

TO

HIS ROYAL HIGHNESS

WILLIAM

DUKE OF CUMBERLAND,

This Philoſophical Treatiſe,

For a laſting Teſtimony of Gratitude,

Is humbly offered and dedicated,

By His ROYAL HIGHNESS'S

moſt devoted and

moſt dutifull ſervant

ROBERT SMITH

THE PREFACE.

THE want of an elementary treatise of harmonics, ſuch as might properly have been quoted in ſupport of my demonſtrations, has obliged me to begin the following work from the firſt principles of the ſcience.

The ancient theoriſts conſidered no other conſonances than ſuch as are perfect, and yet all their muſical ſcales compoſed of theſe conſonances, have in practice been found diſagreeable. The reaſon is, they neceſſarily contain ſome imperfect concords, whoſe imperfections are too groſs for the ear to bear with.

The skill of the moderns has been chiefly employed in the buſineſs of tempering the ancient ſcales, that is, in diſtributing thoſe groſſer imperfections in ſome of the concords, among all the reſt or the greater part of them. By which means, though the number of imperfect concords be greatly increaſed, yet if their ſeveral imperfections be but as much diminiſhed, the ear will be leſs offended than before. Becauſe it is the tranſition from a better harmony to a worſe, which chiefly gives the offence; as is evident to any one that attends to a piece of muſic performed upon an inſtrument badly tuned. It follows then that the inſtrument would be better in tune, if all the conſonances were made as equally harmonious as poſſible, though none of them were perfect.

And if this be the true deſign in tuning an inſtrument, or tempering a ſcale of ſounds, a theoriſt ought to begin with the ſimpleſt caſe; and inquire in the firſt place, whether it be poſſible for two imperfect conſonances to be made equally harmonious; and if ſo, what muſt be the proportion of their temperaments or imperfections; and alſo whether different con-

consonances require different proportions. These and the like questions being rightly settled, we may then determine in what proportion those grosser imperfections in the ancient scales ought to be distributed, so as to make all the concords equally harmonious in their kind, either exactly or as near as possible.

But as none of the writers that I have seen, have attempted to give us the least notion of the nature and constitution of imperfect consonances, nor of any one property or proportion of their effects upon the ear, except a single conjecture whose contrary is true (*a*), it was not possible for them to determine, from the principles of science, what distribution of those grosser imperfections in the ancient systems, would produce the most harmonious scale of musical sounds.

As this is one of the most difficult and important problems in harmonics,

(*a*) Prop. XIII. coroll. 8.

in order to a ſcientific ſolution of it I found it neceſſary to premiſe a Theory of Imperfect Conſonances (*b*), wherein I have demonſtrated as many properties of their Periods, Beats and Harmony as I judged ſufficient for ſolving that problem, and perhaps any other that belongs to harmonics. This theory with its preliminaries and conſequences takes up a large part of the preſent treatiſe. As to the reſt I chuſe to refer the reader to the book it ſelf or the Index, rather than trouble him with a further account of it: a ſhort one would be imperfect or obſcure, and a perfect one, too long for a preface.

Having been asked more than once, whether an ear for muſic be neceſſary to underſtand harmonics, it may not be amiſs to give this anſwer: That a muſical ear is not neceſſary to underſtand the philoſophy of muſical ſounds; no more than the eye, to underſtand that of colours. Our late Profeſſor of Mathematics was an inſtance of the

(*b*) Sect. VI.

latter caſe, and the xx^th^ propoſition of this treatiſe affords an inſtance of the former. For by the ſolution of that propoſition, a perſon of no ear at all for muſic may ſoon learn to tune an organ according to any propoſed temperament of the ſcale; and to any deſired degree of exactneſs, far beyond what the fineſt ear unaſſiſted by theory can poſſibly attain to: and the ſame perſon, if he pleaſes, may alſo learn the reaſon of the practice.

But though an ear for muſic is not neceſſary to underſtand this treatiſe, yet thoſe that are acquainted with muſical ſounds will more readily apprehend many parts of it, and receive more pleaſure from them.

In the firſt ſcholium to prop. xx, I obſerved that the winter ſeaſon had prevented me from tuning an organ by the ſecond table of beats, in order to try what effect the ſyſtem of Equal Harmony might have upon the ear. But upon telling Mr. *Turner*, one of our organiſts

organiſts at Cambridge, how he might approach near enough to that ſyſtem, by flattening the major III^{ds}, till the beats of the vth and vIth major with the ſame baſe, went equally ſlow, by his great dexterity and skill in tuning he preſently put my rule in execution upon a ſtop of his organ; and affirmed to me, he never heard ſo fine harmony before, eſpecially in the flat keys; but he added, that the falſe concords were more intolerable than ever: and no wonder, as their common difference from true concords was then increaſed from one fifth to one fourth of a tone.

This I mention not ſo much to confirm my theory, which I did not doubt of, but chiefly as a motive to baniſh ſuch inſufferable diſſonance from that noble inſtrument, and improve its harmony throughout the ſcale by adding more pipes to the ſtops, in the manner ſuggeſted in this treatiſe (*c*).

Nor

(*c*) Page 200.

Nor will it be improper to mention a like experiment made by the accurate hand of Mr. *Harriſon*, well known to the curious in mechanics by his admirable inventions in watch-work and clock-work, for keeping time exactly both at ſea and land: which if duly encouraged and purſued will undoubtedly prove of excellent uſe in navigation; by correcting the ſea-charts, with reſpect to longitude, as well as the reckonings of a ſhip, to as great exactneſs, in all probability, as need be deſired.

But in regard to the experiment I was going to mention, he told me he took a thin ruler equal in length to the ſmalleſt ſtring of his baſe viol, and divided it as a monochord, by taking the interval of the major III^{d}, to that of the VIIIth, as the diameter of a circle, to its circumference. Then by the diviſions on the ruler applied to that ſtring, he adjuſted the frets upon the neck of the viol, and found the harmony of the conſonances ſo extremely fine, that after

after a very ſmall and gradual lengthening of the other ſtrings, at the nut, by reaſon of their greater ſtiffneſs, he perfectly acquieſced in that manner of placing the frets.

It follows from Mr. *Harriſon*'s aſſumption, that his III^{d} major is tempered flat by a full fifth of a comma. My III^{d} determined by theory, upon the principle of making all the concords within the extent of every three octaves as equally harmonious as poſſible, is tempered flat by one ninth of a comma; or almoſt one eighth, when no more concords are taken into the calculation than what are contained within one octave. That theory is therefore ſupported on one hand by Mr. *Harriſon*'s experiment, and on the other by the common practice of muſicians, who make the major III^{d} either perfect, or generally ſharper than perfect: with a deſign, I ſuppoſe, to improve the falſe concords, though to the manifeſt detriment of all the reſt.

We may gather from the conſtruction of the baſe viol, that Mr. *Harriſon* attended chiefly, if not ſolely to the harmony of the conſonances contained within the octave; in which caſe the differences between his and my temperaments of the major III[d], VI[th] and V[th], and their ſeveral dependents, are reſpectively no greater than 4, 3 and 1 fiftieth parts of a comma. And conſidering that any aſſigned differences in the temperaments of a ſyſtem, will have the leaſt effect in altering the harmony of the whole when at the beſt, I think a nearer agreement of that experiment with the theory could not be reaſonably expected.

Upon asking him why he took the interval of the major III[d] to that of the VIII[th] as the diameter to the circumference of a circle, he anſwered, that a gentleman lately deceaſed had told him, it would bring out the beſt diviſion of a monochord. Whoever was the author of that hypotheſis, for ſo it

muſt

muſt be called, he took the hint, no doubt, from obſerving that as the octave, conſiſting of five mean tones and two limmas, is a little bigger than ſix ſuch tones, or three perfect major III^{ds}, ſo the circumference of a circle is a little bigger than three of its diameters.

When the monochord was divided upon the principle of making the major III^{d} perfect, or but very little ſharper, as in Mr. *Huygens*'s ſyſtem reſulting from the octave divided into 31 equal intervals, Mr. *Harriſon* told me the major VI^{ths} were very bad, and much worſe than the V^{ths}. In which he judged rightly, as I further ſatisfied my ſelf by trying the experiment upon an organ; and being ſolicitous to know the reaſon of that effect, that is, why the V^{ths} and VI^{ths} major when equally tempered ſhould differ ſo in their harmony, after various attempts I ſatisfied my curioſity; and this gave me the firſt inſight into the theory of imperfect conſonances.

With

With a view to ſome other inquiries I will conclude with the following obſervation. That, as almoſt all ſorts of ſubſtances are perpetually ſubject to very minute vibrating motions, and all our ſenſes and faculties ſeem chiefly to depend upon ſuch motions excited in the proper organs, either by outward objects or the power of the will, there is reaſon to expect, that the theory of vibrations here given will not prove uſeleſs in promoting the philoſophy of other things beſides muſical ſounds.

Such readers as can only dip into this treatiſe muſt remember, that by the word Vibration ſo often repeated, I frequently mean the time of a ſingle vibration; which I notified once for all in Sect. I. Art. 8.

Trinity College,
Cambridge Dec. 31. 1748.

A TABLE of the SECTIONS.

HARMONICS.

SECTION I.

Philoſophical Principles of Harmonics.

1. SOUND is cauſed by the vibrations of elaſtic bodies, which communicate the like vibrations to the air, and theſe the like again to our organs of hearing.

Philoſophers are agreed in this, becauſe ſounding bodies communicate tremors to diſtant bodies. For inſtance, the vibrating motion of a muſical ſtring puts others in motion, whoſe tenſion and quantity of matter diſpoſe their vibrations to keep time with the pulſes of air, propagated from the ſtring that was ſtruck. *Galileo* explains this phænomenon by obſerving, that a heavy pendulum may be put in motion by

 the

the leaſt breath of the mouth, provided the blaſts be frequently repeated and keep time exactly with the vibrations of the pendulum; and alſo by the like art in raiſing a large bell; and probably he was the firſt that rightly explained that phænomenon (*a*).

2. If the vibrations be iſochronous the ſound is Muſical, and continues at the ſame Pitch; and is ſaid to be acuter, ſharper or higher than any other ſound whoſe vibrations are ſlower; and graver, flatter or lower (*b*) than any other whoſe vibrations are quicker.

For while a muſical ſtring vibrates, if its tenſion be increaſed or its length be

(*a*) For he ſays, in the perſon of another, il problema poi trito delle due corde teſe all uniſono, che al ſuono dell' una l' altera ſi muova et attualmente riſuona, mi reſta ancora irreſoluto; come anco non ben chiare le forme delle conſonanze et altre particolarità. *Dialogo* 1° *attenente alla Mecanica*, towards the end.

(*b*) As the ideas of acute and high, grave and low, have in nature no neceſſary connection, it has happened accordingly, as Dr. *Gregory* has obſerved in the preface to his edition of *Euclid*'s works, that the more ancient of the Greek Writers looked upon grave ſounds as high, and acute ones as low, and that this conbe diminiſhed, its vibrations will be accelerated; and experience ſhews that its ſound is altered from what is called a graver to an acuter; and on the contrary. And the like alteration of the pitch of the ſound will follow, when the ſame tenſion is given by a weight, firſt to a thicker or a heavier ſtring, and after that to a ſmaller or a lighter of the ſame length, as having leſs matter to be moved by the ſame force of tenſion. And theſe changes in the pitch of the ſound are found to be conſtantly greater or leſſer, according as the length, tenſion, thickneſs or denſity of the ſtring is more or leſs altered (*c*).

3. There-

connection was afterwards changed to the contrary by the leſs ancient Greeks, and has ſince prevailed univerſally. Probably this latter connection took its riſe from the formation of the voice in ſinging, which *Ariſtides Quintilianus* thus deſcribes. Γίνεται δὲ ἡ μὲν βαρύτης, κάτωθεν ἀναφερομένου τοῦ πνεύματος, ἡ δ' ὀξύτης, ἐπιπολῆς προϊεμένου. pag. 8. Et quidem gravitas fit, ſi ex inferiore parte (gutturis) ſpiritus ſurſum feratur, acumen vero, ſi per ſummam partem prorumpat, as *Meibomius* tranſlates it in his notes. pag. 208.

(*c*) The Greek muſicians rightly deſcribe the difference between the manner of ſinging and talking.

They

3. Therefore if ſeveral ſtrings, however different in length, thickneſs, denſity and tenſion, or other ſounding bodies vibrate all together in equal times, their ſounds will all have one and the ſame pitch, however they may differ in loudneſs or other qualities, and are therefore called Uniſons: and on the contrary, the vibrations of uniſons are iſochronous.

This obſervation reduces the theory of all ſorts of muſical ſounds to that of the ſounds of a ſingle ſtring; I mean with reſpect to their gravity and acuteneſs, which is the principal ſubject of Harmonics (*d*).

4. Con-

They conſidered two motions in the voice, κινήσεις δύο; the one continued and uſed in talking, ἡ μὲν συνεχής τε καὶ λογικὴ, the other diſcrete and uſed in ſinging, ἡ δὲ διαστηματική τε καὶ μελῳδική. In the continued motion, the voice never reſts at any certain pitch, but waves up and down by inſenſible degrees; and in the diſcrete motion it does the contrary; frequently reſting or ſtaying at certain places, and leaping from one to another by ſenſible intervals. *Euclid's* Introductio

4. Consequently the wider and narrower vibrations of a musical string, or of any other body sounding musically, are all isochronous very nearly.

Otherwise, while the vibrations decrease in breadth till they cease, the pitch of the sound could not continue the same; as by the judgment of the ear we perceive it does, if the first vibrations be not too large: in which case the sound is a little acuter at the beginning than afterwards.

5. In like manner, since the pitch of the sound of a string or bell or other vibrating body, does not alter sensibly while the hearer varies his distance from it;

ductio Harmonica. pag. 2. I need not observe, that in the former case, the vibrations of the air are continually accelerated and retarded by turns and by very small degrees, and in the latter by large ones.

(*d*) *Ptolemy* says, Ἁρμονικὴ μέν ἐστὶ δύναμις καταληπτικὴ τῶν ἐν τοῖς ψόφοις, περὶ τὸ ὀξὺ καὶ βαρὺ, διαφορῶν. Harmonics is a power apprehending the differences of sounds, with respect to gravity and acuteness.

it; it follows that the larger and lesser vibrations of the particles of air, at smaller and greater distances from the sounding body, are all isochronous: and consequently that the little spaces described by the vibrating particles are every where proportional to the celerity and force of their motions, as in a pendulum (*e*). And this difference of force, at different distances, causes a difference in the loudness of the sound, but not in its pitch.

6. It follows also, that the harmony of two or more sounds, according as it is perfect or imperfect when heard at any one distance, will also be perfect or imperfect at any other distance: which being a known fact, is mentioned here as a confirmation of the principles of Harmonics.

7. If two musical strings have the same thickness, density and tension, and differ in length only, (which for the future I shall always suppose,) mathematicians

(*e*) See Newton's Princip. Lib. II. Prop. 47.

thematicians have demonſtrated, that the times of their ſingle vibrations are proportional to their lengths. (*f*)

8. Hence if a ſtring of a muſical inſtrument be ſtopt in the middle, and the ſound of the half be compared with the ſound of the whole, we may acquire the idea of the interval of two ſounds, whoſe ſingle vibrations (always meaning the times) are in the ratio of 1 to 2; and by comparing the ſounds of $\frac{2}{3}$, $\frac{3}{4}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{8}{9}$, $\frac{9}{10}$, &c, of the ſtring with the ſound of the whole, we may acquire the ideas of the intervals of two ſounds, whoſe ſingle vibrations are in the ratio of 2 to 3, 3 to 4, 3 to 5, 4 to 5, 5 to 6, 8 to 9, 9 to 10, &c.

9. A

(*f*) As a clear and exact demonſtration of this curious Theorem depends upon one or two more, of no ſmall uſe in Harmonics, and requires a little of the finer ſort of geometry, which cannot well be applied in few words, I have therefore reſerved it to the laſt Section of this Treatiſe; which the reader may conſult, or, taking it for granted at preſent, may proceed without interruption; as he likes beſt.

9. A Musical Interval is a quantity of a certain kind (*g*), terminated by a graver and an acuter sound.

In a ring of bells, for example, the sounds of the first and second bells, counting either from the biggest or the least, terminate a certain interval; those of the first and third a greater interval; those of the first and fourth a greater still; &c. So that the interval increases by degrees, either as the graver of the two sounds descends, or as the acuter ascends; and within the interval of the sounds of the biggest and least bells, the intervals between the sounds of all the rest are contained.

10. Musical intervals are Measures of the Ratios of the single vibrations of the

(*g*) See Dr. *Wallis*'s preface to *Porphyrie*'s comment on *Ptolemy*'s Harmonics. Oper. Math. vol. III. *Euclid* says, an Interval is τὸ περιεχόμενον ὑπὸ δύο φθόγγων ἀνωμοίων ὀξύτητι καὶ βαρύτητι, what is contained by two sounds different in gravity and acuteness. Introductio Harmonica. p. 1. *Aristoxenus* defines a musical sound thus, Φωνῆς πτῶσις ἐπὶ μίαν τάσιν ὁ φθόγγος, sonus est vocis casus in unam tensionem; and an interval thus, Διάστημα δέ ἐστι τὸ ὑπὸ δύο φθόγγων ὡρισμένον, μὴ τὴν the terminating ſounds, or, *cæteris paribus*, of the lengths of the ſounding ſtrings (*h*).

For it is obſervable in the experiments laſt mentioned (*i*) and is univerſally allowed by muſicians, that when the lengths of the ſtrings have the ſame ratio, the interval of their ſounds is the ſame, whatever be their pitch; that if the acuter of the two ſounds be raiſed higher, and conſequently the ratio of the lengths of the ſtrings be increaſed, the interval is increaſed; and on the contrary, if the acuter ſound be depreſſed lower, that the interval is diminiſhed, and reduced to nothing when the ſtrings have the ratio of equality, whoſe magnitude is nothing.

FIG.

αὐτὴν τάσιν ἐχόντων; intervallum vero eſt, quod duobus ſonis, non eandem tenſionem habentibus, terminatur. And he adds, that it is τόπος δεκτικὸς φθόγγων, ὀξυτέρων μὲν τῆς βαρυτέρας τῶν ὁριζουσῶν τὸ διάστημα τάσεων, βαρυτέρων δὲ τῆς ὀξυτέρας; a place capable of ſounds, that are acuter than the graver of the two tenſions (tones, or ſounds) that terminate the interval, and graver than the acuter of them. pag. 15.

(*h*) Article 7. (*i*) Art. 8.

FIG. 1. Now let the lengths of ſeveral ſtrings *A*, *B*, *C*, *D*, &c, be continual proportionals in any ratio. Then ſince the interval of the ſounds of *A* and *B* is equal to that of *B* and *C*, or of *C* and *D*, &c, by adding equal intervals together and equal ratios together, it follows, that the interval of the ſounds of *A* and *C*, whoſe ratio is duplicate of *A* to *B* or of *B* to *C*, is double the interval of the ſounds of *A* and *B*, or of *B* and *C*; and that the interval of the ſounds of *A* and *D*, whoſe ratio is triplicate of *A* to *B*, is alſo triple the interval of the ſounds of *A* and *B*, or of *B* and *C* or of *C* and *D*. So that the interval of the ſounds of *A* and *C*, is to that of *A* and *D*, as 2 to 3; and the like is evident of any other equimultiples of the propoſed ratios and intervals, whatever be their number and magnitude.

11. Therefore muſical intervals are proportional to the logarithms of the ratios of the ſingle vibrations of the terminating

minating ſounds, or, *cæteris paribus*, of the lengths of the vibrating ſtrings. Becauſe logarithms are numeral meaſures of ratios; and all ſorts of meaſures of the ſame magnitudes are proportional to one another. (*k*)

SECTION II.

Of the Names and Notation of conſonances and their intervals.

1. FIG. 2. If a muſical ſtring *CO* and its parts *DO*, *EO*, *FO*, *GO*, *AO*, *BO*, *cO*, be in proportion to one another as the numbers 1, $\frac{8}{9}$, $\frac{4}{5}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{8}{15}$, $\frac{1}{2}$, their vibrations will exhibit the ſyſtem of 8 ſounds which muſicians denote by the letters *C*, *D*, *E*, *F*, *G*, *A*, *B*, *c*.

FIG. 3. And ſuppoſing thoſe ſtrings to be ranged like ordinates to a right line *Cc*, and their diſtances *CD*, *DE*, *EF*,

(*k*) See Mr. *Cotes*'s Harmonia Menſurarum. pag. 1.

EF, *FG*, *GA*, *AB*, *Bc*, not to be the differences of their lengths, as in fig. 2, but to be of any magnitudes proportional to the intervals of their sounds, the received names of these intervals are shewn in the following Table; and

C..*D*..*E*.*F*..*G*..*A*..*B*.*c*..

$1 .. \frac{8}{9} .. \frac{4}{5} . \frac{3}{4} .. \frac{2}{3} .. \frac{3}{5} .. \frac{8}{15} . \frac{1}{2} ..$

Musical Ratios,		Interval's Names,	Marks,	Elements.
C : *c* :: 2 : 1	*Cc*	Octave	VIII	3T+2t+2H
B : *c* :: 16 : 15	*Bc*	Hemitone	H	
C : *B* :: 15 : 8	*CB*	VII major	VII	3T+2t+ H
C : *D* :: 9 : 8	*CD*	Tone maj.	T	
D : *c* :: 16 : 9	*Dc*	7th minor	7th	2T+2t+2H
A : *c* :: 6 : 5	*Ac*	3d minor	3d	T+ H
C : *A* :: 5 : 3	*CA*	VI major	VI	2T+2t+ H
C : *E* :: 5 : 4	*CE*	III major	III	T+ t
E : *c* :: 8 : 5	*Ec*	6th minor	6th	2T+ t+2H
G : *c* :: 4 : 3	*Gc*	4th minor	4th	T+ t+ H
C : *G* :: 3 : 2	*CG*	V major	V	2T+ t+ H
F : *B* :: 45 : 32	*FB*	IV major	IV	2T+ t
B : *f* :: 64 : 45	*Bf*	5th minor	5th	T+ t+2H
D : *E* :: 10 : 9	*DE*	Tone min.	t	
81 : 80		Comma	c	T— t

and, except the Tone and Hemitone, are taken from the numbers of the ſtrings or ſounds in each interval incluſively; as a Third, Fourth, Fifth, &c, with the epithet of *major* or *minor*, according as the name or number belongs to a greater or ſmaller total interval; the difference of which reſults chiefly from the difference of the magnitudes of the *tone* and *hemitone*.

2. Hence it is, if the ratio of the ſingle vibrations of any two ſounds, or of the lengths of two vibrating ſtrings, be any of thoſe in the firſt column of the table, that their interval, and the conſonance too, retains the name in the third column, whether the intermediate ſounds be preſent or abſent.

3. FIG. 3. In the line *Cc* produced beyond *c*, if we take the intervals *Dd*, *Ee*, *Ff*, &c, ſeverally equal to the octave *Cc*, and make the length of the ſeveral ſtrings at *d*, *e*, *f*, &c, equal to half the lengths of thoſe at *D*, *E*, *F*, &c, all the intervals within this higher octave *cc′*, will

will alſo conſiſt of major and minor tones and hemitones, ranged in the ſame order as in the lower octave *Cc*.

Now the names of intervals larger than one or more octaves, are alſo taken from the number of the ſtrings in them incluſively. Thus the interval *Cd* is called a Ninth, *Ce* a Tenth, *Cf* an Eleventh, *Cg* a Twelfth, &c, with the epithet of *major* or *minor* as before; and are thus denoted, IX or VIII + T, X or VIII + III, 11th or VIII + 4th, XII or VIII + V, &c, the units in the compound marks being conſtantly one more than thoſe in the ſimple ones, becauſe the intermediate ſtring at the end of the octave is counted twice.

4. A Comma is the interval of two ſounds whoſe ſingle vibrations have the ratio of 81 to 80, and is the difference of the major and minor tones (*l*).

5. When

(*l*) For the ratio of 9 to 8 diminiſhed by the ratio of 10 to 9, is the ratio of 9 × 9 to 8 × 10, or of 81 to 80.

5. When the ſingle vibrations of two ſounds are in any one of the ratios in the table, or in fig. 3 continued, the conſonance and its interval too is called Perfect; and Imperfect or Tempered, when that ratio, and interval belonging to it, is a little increaſed or diminiſhed.

6. Any ſmall increment or decrement of a perfect interval is called its Temperament, and is very conveniently meaſured by the proportion it bears to a comma.

7. As the addition and ſubſtraction of logarithms anſwers to the multiplication and diviſion of their abſolute numbers, that is, to the compoſition and reſolution of the ratios of thoſe numbers to an unit; ſo the addition and ſubſtraction of muſical intervals anſwers to the compoſition and reſolution of the ratios of the ſingle vibrations of the terminating ſounds, or of the lengths of the vibrating ſtrings: and on the contrary.

In the following examples, the composition and resolution of ratios is intimated by the multiplication of their terms, placed, in the form of fractions, upright and inverted, respectively.

$$\begin{cases} \text{As } \frac{1}{2} = \frac{8}{15} \times \frac{15}{16} = \frac{9}{16} \times \frac{8}{9} = \frac{3}{5} \times \frac{5}{6} = \frac{5}{8} \times \frac{4}{5} = \frac{2}{3} \times \frac{3}{4} = \frac{8}{9} \times \frac{8}{9} \times \frac{8}{9} \times \frac{9}{10} \times \frac{9}{10} \times \frac{15}{16} \times \frac{15}{16}. \\ \text{So VIII} = \text{VII} + \text{H} = 7^{\text{th}} + \text{T} = \text{VI} + 3^{\text{d}} = 6^{\text{th}} + \text{III} = \text{V} + 4^{\text{th}} = 3\text{T} + 2\text{t} + 2\text{H}. \end{cases}$$

As $\frac{3}{5} = \frac{3}{4} \times \frac{4}{5}$	$\frac{5}{8} = \frac{3}{4} \times \frac{5}{6}$	$\frac{2}{3} = \frac{4}{5} \times \frac{5}{6}$
So $\text{VI} = 4^{\text{th}} + \text{III}$	$6^{\text{th}} = 4^{\text{th}} + 3^{\text{d}}$	$\text{V} = \text{III} + 3^{\text{d}}$

As $\frac{3}{4} = \frac{4}{5} \times \frac{15}{16}$	$\frac{3}{4} = \frac{5}{6} \times \frac{9}{10}$	$\frac{4}{5} = \frac{8}{9} \times \frac{9}{10}$
So $4^{\text{th}} = \text{III} + \text{H}$	$4^{\text{th}} = 3^{\text{d}} + \text{t}$	$\text{III} = \text{T} + \text{t}$

As $\frac{5}{6} = \frac{8}{9} \times \frac{15}{16}$	$\frac{3}{5} \times \frac{3}{2} = \frac{9}{10}$	$\frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$
So $3^{\text{d}} = \text{T} + \text{H}$	$\text{VI} - \text{V} = \text{t}$	$\text{V} - 4^{\text{th}} = \text{T}$

As $\frac{3}{4} \times \frac{6}{5} = \frac{9}{10}$	$\frac{3}{4} \times \frac{5}{4} = \frac{15}{16}$	$\frac{8}{9} \times \frac{10}{9} = \frac{80}{81}$
So $4^{\text{th}} - 3^{\text{d}} = \text{t}$	$4^{\text{th}} - \text{III} = \text{H}$	$\text{T} - \text{t} = \text{c}.$

8. Hence

8. Hence the hemitones and tones major and minor, being the differences of the intervals, III, 4th, V, VI, and of their complements to the octave, may be considered as the Elements that compose the intervals of all perfect concords (*m*).

SECTION III.

Of perfect consonances and the Order of their simplicity.

1. FIG. 4. When a single sound is heard, the series of equal times between the successive pulses of air, that beat on the ear, may be represented by a series of equal parts contained in a right line; as in o2, o3, o4, &c. Consequently when two sounds are heard, two of those lines, as o2 and o3, will right-

(*m*) The old method of resolving concords into their elements may be seen in Dr. *Wallis*'s division of the monochord, or section of the musical canon, as the ancients called it. Philosoph. Transact. N°. 238. or Abridg. by Lowthorp. vol. 1. p. 698. first edit.

rightly represent the two series of equal times, if the magnitude of the equal parts in one line, be to the magnitude of those in the other, in the ratio of the single vibrations of the sounds: or, the whole lines being supposed equal, if the numbers of aliquot parts in each, as 2 and 3, be severally the same as the least numbers of the vibrations of each sound, made in the same time represented by the line o2 or o3. (*a*)

2. And the sounds being heard together, if we conceive the two equal and parallel lines that rightly represent them, as o2 and o3, to coincide throughout, the points that divide the separate lines, will subdivide the combined lines into smaller portions, as in Fig. 5, representing a third series or cycle of times, in which the pulses of both sounds interchangeably succeed one another in beating upon the ear.

3. Such a mixture of pulses, succeeding one another in a given Cycle of

(*a*) See Art. 12. following.

of Times, terminated at both ends by a perfect coincidence of the pulſes, and ſufficiently repeated, is the phyſical cauſe that excites the idea of a given conſonance: Eſpecially when conſidered as diſtinct from any other conſonance, whoſe ſingle vibrations having a different ratio from that of the former, *will* conſtitute a different cycle, and excite a different idea. But if that ratio be the ſame, though the abſolute times be different, the conſonances may be looked upon as the ſame, in this reſpect, that their cycles have the ſame form; the times in both having the ſame order, and the ſame proportions; and in this other alſo, that the interval of the ſounds is the ſame (*b*).

4. This being premiſed, one conſonance may be conſidered as more or leſs ſimple than another, according as the cycle of times belonging to it, is more or leſs ſimple than the cycle belonging

(*b*) Art. 10. Sect. 1.

longing to the other. And upon this principle all consonances may be ranged in due order of such simplicity, by the help of the following Rule.

5. *One consonance is Simpler than another in the same Order, as the sum of the least terms, expressing the ratio of the single vibrations, is smaller than the like sum in the other consonance; and when several such sums are the same, these consonances are simpler in the same order, as the lesser terms of their ratios are smaller.*

For the simplicity of a consonance or cycle of times, consists partly in the number of times contained in the cycle, and partly in the different proportions they bear to one another.

Fig. 4. When the numbers of times in different cycles are different, and the times in each cycle are equal to one another, as when we combine the sounds o1 and o1, o1 and o2, o1 and o3, o1 and o4, o1 and o5, &c, the cycles of this sort may be ranged in the order

order of their ſimplicity above defined, by the order of the numbers of equal times in the cycles, or of the magnitudes of the numbers 1, 2, 3, 4, 5, 6, &c, or of 2, 3, 4, 5, 6, 7, &c, that is, of the ſums of the terms of the ratios 1 to 1, 1 to 2, 1 to 3, 1 to 4, 1 to 5, &c.

In the other caſe, where the numbers of times in different cycles are the ſame, and the times in each cycle bear different proportions to one another, as when we combine the ſounds o1 and o6, o2 and o5, o3 and o4, that cycle is ſimpler than another, in which the equal times between the pulſes of the acuter ſound, are leſs interrupted and ſubdivided by the pulſes of the graver.

Accordingly in the firſt of theſe cycles compoſed of o1 and o6, not one of the 6 equal times between the pulſes of the acuter ſound o6, is ſubdivided by any pulſe of the graver o1; but in the ſecond cycle compoſed of o2 and o5, one of the 5 equal times, between the pulſes of the acuter ſound o5, is

A table of the Order of the ſimplicity of conſonances of two ſounds.

Ratios of the vibrations.	Order of the ſimplicity.	Intervals of the ſounds.
1 : 1	1	0
1 : 2	2	VIII
1 : 3	3	VIII + V
1 : 4	4	2VIII
2 : 3	$4\frac{1}{2}$	V
1 : 5	5	2VIII + III
1 : 6	6	2VIII + V
2 : 5	$6\frac{1}{3}$	VIII + III
3 : 4	$6\frac{2}{3}$	4^{th}
1 : 7	7	
3 : 5	$7\frac{2}{3}$	VI
1 : 8	8	3VIII
4 : 5	$8\frac{3}{4}$	III
1 : 9	9	3VIII + T
1 : 10	10	3VIII + III
2 : 9	$10\frac{1}{5}$	2VIII + T
3 : 8	$10\frac{2}{5}$	VIII + 4^{th}
5 : 6	$10\frac{4}{5}$	3^{d}
1 : 12	12	3VIII + V
3 : 10	$12\frac{1}{3}$	VIII + VI
4 : 9	$12\frac{1}{2}$	VIII + T
5 : 8	$12\frac{2}{3}$	6^{th}
6 : 7	$12\frac{5}{6}$	

Continuation of the table.

1 : 15	15	3VIII + VII
1 : 16	16	4VIII
2 : 15	$16\frac{1}{8}$	2VIII + VII
5 : 12	$16\frac{1}{2}$	VIII + 3^{d}
8 : 9	$16\frac{7}{8}$	T
1 : 18	18	4VIII + T
3 : 16	$18\frac{2}{9}$	2VIII + 4^{th}
4 : 15	$18\frac{1}{3}$	VIII + VII
9 : 10	$18\frac{8}{9}$	t
1 : 20	20	4VIII + III
5 : 16	$20\frac{2}{5}$	VIII + 6^{th}
1 : 22	22	
3 : 20	$22\frac{2}{11}$	2VIII + VI
5 : 18	$22\frac{4}{11}$	VIII + 7^{th}
8 : 15	$22\frac{7}{11}$	VII
1 : 24	24	4VIII + V
9 : 16	$24\frac{2}{3}$	7^{th}
1 : 28	28	
5 : 24	$28\frac{2}{7}$	2VIII + 3^{d}
9 : 20	$28\frac{4}{7}$	VIII + t
1 : 30	30	4VIII + VII
15 : 16	$30\frac{14}{15}$	H
32 : 45	$76\frac{31}{38}$	IV
45 : 64	$108\frac{22}{27}$	5^{th}

ſubdivided by one pulſe of the graver o2; and in the third cycle compoſed of o3 and o4, two of the 4 equal times in the acuter ſound o4, are ſubdivided by 2 pulſes of the graver o3. By which it appears, that the firſt cycle is ſimpler than the ſecond, and the ſecond ſimpler than the third; and that the order of ſimplicity of this ſort of cycles, anſwers to the order of the magnitudes 1, 2, 3 of the leſſer terms of the ratios.

6. Now by the firſt part of the rule above, the integers in the ſecond column of the table, are the ſeveral ſums of the terms of the oppoſite ratios in the firſt, diminiſhed by 1, which alters not the order of their magnitudes, but only makes the ſeries begin with 1, anſwering to the ſimpleſt conſonance.

By the ſecond part of the rule, the ratios whoſe terms have the ſame ſum, as 1 : 6, 2 : 5, 3 : 4, are ranged in the order of their leſſer terms 1, 2, 3, or, which alters not the order, of thoſe terms ſeverally diminiſhed by 1, as of

0, 1, 2, or of the fractions $\frac{0}{3}$, $\frac{1}{3}$, $\frac{2}{3}$, whose common denominator 3 is the number of the ratios whose terms have the same sum 7. These fractions either by themselves or the mixt numbers 6, $6\frac{1}{3}$, $6\frac{2}{3}$, made by annexing them to the number 6, may therefore denote the order of the Modes of simplicity of such consonances as have the same Degree of simplicity denoted by 6 or 7 — 1. And thus the order of the simplicity of all consonances whatever, is denoted by the order of the magnitudes of the integers and mixt numbers in the second column of the table.

7. This series increases from unity in several arithmetical progressions, except that a term or two is here and there omitted, where ratios occur, which being reducible to simpler terms, have been considered before, or else are not Musical Ratios, which are such only whose terms are 1, 2, 3, 5, with their

their powers and products (*d*), or belong not to the intervals in Fig. 3.

For example, writing down all the ratios in due order, whose terms make a given sum, as 1 to 8, 2 to 7, 3 to 6, 4 to 5, I reject the two middlemost for the reasons just mentioned, and place the rest in the first column of the table; which may thus be continued with certainty and order as far as we please.

8. Hence we may distinguish consonances into two sorts, Pure and Interrupted; pure, where none of the equal times between the pulses of the acuter sound, is subdivided by any intermediate pulse of the graver; and interrupted, when any of those equal times are interrupted by one or more pulses of the graver sound.

In the second column of the table, the least simple or lowest mode of each degree of interrupted consonancy, is every where placed above the next inferior degree of pure consonancy, as $4\frac{1}{2}$ above 5.

(*d*) See Sect. IV.

For ſhould we depreſs the mode $4\frac{1}{2}$ to a place next below the degree 5, why not even to a place next below 6? though not below $6\frac{1}{3}$, as being a more complex mode of a leſs ſimple degree. But if that were allowable, by parity of reaſon we ought to depreſs $4\frac{1}{2}$, $6\frac{1}{3}$, $6\frac{2}{3}$ next below 7, though not below $7\frac{2}{3}$, and likewiſe $4\frac{1}{2}$, $6\frac{1}{3}$, $6\frac{2}{3}$, $7\frac{2}{3}$ below 8, though not below $8\frac{3}{4}$, and alſo $4\frac{1}{2}$, $6\frac{1}{3}$, $6\frac{2}{3}$, $7\frac{2}{3}$, $8\frac{3}{4}$ below 9 and 10, and ſo forth to infinity: which, by depreſſing all the modes of interrupted conſonancy, below all the degrees of pure conſonancy, would render them heterogeneal, and incapable of any order or compariſon with one another. The table is therefore rightly ordered.

9. Hitherto we have only conſidered the number and proportions of the times, in the cycle by which a conſonance

nance is repreſented, without regard to the quality of the pulſes, as to duration, ſtrength, weakneſs or other accidents; whereas the pulſes of graver ſounds are generally ſtronger and obtuſer than thoſe of acuter ſounds, and affect the ear differently. Very true; but ſtill this alters not the rational idea of the conſonance, as above deſcribed, nor does the ear perceive any alteration in the kind of mixture, or in the interval, upon ſoftening or ſwelling either ſound, while the other retains the ſame ſtrength.

10. It is well known in general, that ſimpler conſonances affect the ear with a ſmoother and pleaſanter ſenſation, and the leſs ſimple with a rougher and leſs pleaſant one. And this analogy ſeems to hold true according to the order in the table, as far as the ear can judge with certainty. Thoſe that are willing to try the experiment, may readily do it by the help of the third column of the table, ſhewing the muſical

ſical intervals anſwering to the reſpective conſonances. But the analogy will be plainer perceived by intermitting ſeveral conſonances, and trying it, for example, in this ſeries of all the concords within the octave, VIII, V, 4^{th}, VI, III, 3^{d}, 6^{th}; but then they ſhould not be tempered as uſual, but tuned perfect. And if the experimenter be skilful in melody and compoſition, he muſt endeavour, as much as poſſible, to deveſt himſelf of all habitual prepoſſeſſions in favour of this or that concord, or ſucceſſion of concords, acquired from the rules and practice of his art; in order to an impartial judgment of the ſimple perception of the ſmoothneſs and ſweetneſs of each concord, and a fair compariſon of ſuch perceptions only.

11. Though nature has appointed no certain limit between concords and diſcords, yet as muſicians diſtinguiſh conſonances by thoſe names for their own uſes, I may do the like for mine; call-

calling unisons, III[ds], V[ths] and VI[ths], and their complements to the VIII[th] and compounds with VIII[ths], Concords, and all other consonances, Discords.

12. If the given times of the single vibrations of any two sounds be V and v, and if $V : v :: m : n$, representing the *least* integers in that ratio; the length of the cycle of times between the successive coincidences of the pulses of V and v, is $nV = mv$. Because these multiples of V and v are the least of any that can be equal.

For the same reason, if $V : x :: r : s$, in the least integers, the cycle $sV = rx$.

13. Hence the length of the cycle of V and v, is to that of V and x, as n to s; that is, the cycles of consonances that have a common sound, string or vibration V, are proportional to the Numerators of the fractions $\frac{n}{m}V = v$, $\frac{s}{r}V = x$, expressing the times of the single vibrations, or the lengths of the strings

ſtrings of the other ſounds, as in Fig. 3, or to the leſſer terms of the ratios in column 1 of the table, of the order of the ſimplicity of conſonances.

14. Conſequently, were the degrees of ſimplicity of conſonances to be eſtimated by the frequency of the coincidences of their pulſes, or the ſhortneſs of their cycles, as is commonly ſuppoſed; the uniſon, VIII, VIII + V, 2 VIII, 2 VIII + III, &c, whoſe cycles are but 1 vibration of the baſe, would be equally ſimple; and the ſame may be ſaid of the Vth, VIII + III, 2 VIII + T, &c, whoſe ſeveral cycles are but 2 vibrations of the baſe; and the ſame alſo of all conſonances having the ſame number for the leſſer term of their ratios; which ſhews that the frequency of coincidences is, of it ſelf, too general a character of ſimplicity and ſmoothneſs, and therefore an imperfect one.

SECTION IV.

Of the ancient Systems of perfect consonances.

1. IF no other primes but 1, 2, 3 were admitted to the composition of musical ratios, a system of sounds thence resulting could have no perfect thirds; nor any perfect consonance whose vibrations are in any ratio having the number 5, or any multiple of it, for either of its terms, as 5 to 4, 10 to 9, 16 to 15, &c: it being impossible for any powers and products of the given primes 1, 2, 3 to compose any other prime or multiple of it.

2. FIG. 3. The minor tones *DE*, *GA* being thus excluded, and major tones being put in their places, every perfect major III[d] will be increased by a comma, as being the difference of the tones (*a*); and every hemitone and per-

(*a*) Sect. II. Art. 4.

perfect minor 3[d] will be as much diminished; because the 4[ths] and v[ths], as *CF* and *Fc*, *cG* and *GC*, are perfect, whether 5 be admitted or not, as depending on the primes 1, 2, 3, only.

3. These diminished hemitones being called Limmas, the octave is now divided into 5 major tones and 2 limmas; as represented to the eye in Fig. 6; where the consonances whose vibrations are expressed by such high terms as the powers of 8 and 9, &c, must needs be disagreeable to the ear, according to the foregoing analogy between the agreeable smoothness of a consonance, and the simplicity of the numbers expressing the ratio of its vibrations (*b*): and that in reality they are so, any one will soon find if he pleases to try the following experiment.

4. Fig. 6. Ascending by a perfect v[th] and descending by a perfect 4[th] alter-

(*b*) Sect. III. Art. 10.

alternately, upon an organ or harpſichord tune the following ſounds, from *F* to *C*, *C* to *G*, *G* to *D*, *D* to *A*, *A* to *E*; and the octave *Ff* will then be divided into 5 major tones and 2 limmas; becauſe the differences of thoſe ſucceſſive v^ths and 4^ths are major tones.

Then having tuned octaves to every one of thoſe notes, try the conſonances that would be perfect if the number 5 were admitted, as thirds major and minor, with their complements to the VIII^th and compounds with VIII^ths; and you will find them extremely diſagreeable, or out of tune as we ſay.

5. But if 5 be admitted among the muſical primes, the ratios 10 to 9 and 16 to 15, belonging to the minor tone and the hemitone, are alſo admitted, and the elements that now compoſe the octave, are 3 major tones, 2 minor and 2 hemitones, as in Fig. 3.

PROPOSITION I.

A system of sounds whose elements or smallest intervals are tones, major and minor, and hemitones, will necessarily contain some imperfect concords.

6. Whatever be the order of those elements in any one octave, it must be the same in every one; to the end that every sound may have an octave to it, as being the best concord. And in order to have as many perfect v^ths as possible, and consequently VIII + v^ths, which concords are the second best (*c*), the elements must be ranged in such order, that the contiguous couples shall make as many perfect thirds as possible, both major and minor; these being the intervals which compose the perfect v^ths. And that order being rightly determined, we shall have the greatest number of perfect concords of all kinds. Because the complements to the octave, of perfect thirds and v^ths, will

(*c*) See the table of the order of concords in Sect. III. Art. 5.

will alſo be perfect, and ſo will their compounds with any number of VIII[ths].

Now it is obſervable of the ſeven elements T, T, T, *t*, *t*, H, H, which compoſe an octave, that T and H, T and *t* are the only couples which make perfect thirds (*d*), all the reſt, T and T, *t* and *t*, *t* and H, H and H, making thirds imperfect by a comma, except H and H, which compoſe an imperfect tone, bigger than the major tone by almoſt a comma (*e*).

Hence either T and H, or T and *t* muſt be the outermoſt elements in the octave, as in the following table.

For

(*d*) Sect. II. Art. 5 and 7.

(*e*) Putting $H = \log. \frac{16}{15} = 0.02803$

Then $2H = 2 \times \log. \frac{16}{15} = 0.05606$

And $T = \log. \frac{9}{8} = 0.05115$

Whence $2H - T = 0.00491$

And the Comma $= \log. \frac{81}{80} = 0.00540$

Difference 0.00049

For if the firſt element in every octave in the ſyſtem be T and the ſeventh be H, the ſeventh in any octave, combined with the firſt in the next octave, will compoſe the interval H+T of a perfect minor 3^{d}, and thus the contiguous octaves will be joined in perfect concord.

Table of the Elements.

	&c 7	1 2	3 4 5	6 7	1&c
Caſ. 1.	H	T + t	H + T + t t + T + H	T + H	T
	H	T + H	t + T + t	T + H	T
Caſ. 2.	t	T + t	H + T + H	T + t	T
	t	T + H	t + T + H H + T + t	T + t	T

Likewiſe if the firſt element in every octave be T and the ſeventh be *t*, here alſo the ſeventh in any octave, together with the firſt in the next octave, will

will compoſe the interval $t+T$ of a perfect major III[d], and thus the contiguous octaves will again be joined in perfect concord; and in no other caſe beſides thoſe two, as appears by the obſervation above.

Caſ. 1. Now if the ſecond element be t, the firſt joined to it compoſes the perfect major III[d], $T+t$. And if the ſixth element be T, the ſeventh joined to it will compoſe the perfect minor third $T+H$.

Two of the ſeven elements being thus diſpoſed of, at each end of the octave, the contiguous couples of the remaining three, cannot compoſe perfect thirds in any order different from this, $H+T+t$, or its reverſe $t+T+H$; both which being transferred into the interval between thoſe extreme couples, ſhew, that the elements in the ſecond and third places, compoſe either the imperfect minor third $t+H$, or the imperfect major third $t+t$.

 If

If H be the ſecond element, as in the third rank of the table, the firſt couple does now compoſe the perfect minor third T+H, and the laſt being alſo T+H, as before, the three remaining elements muſt have this order *t*+T+*t*, to make perfect thirds of their contiguous couples; and being thus transferred into the interval between thoſe extreme couples, they ſhew, that the ſecond and third elements do again compoſe an imperfect minor third H+*t*.

Caſ. 2. Here alſo the ſixth element muſt be T, ſince no other joined to the ſeventh can make a perfect third, as T+*t*.

Now if the ſecond element be *t*, this joined to the firſt makes the perfect major third T+*t*. And two of the ſeven elements being thus ſituated at each end of the octave, the contiguous couples of the remaining three, cannot compoſe the intervals of perfect thirds in any order different from this,

H+T

H+T+H; which being transferred into the interval between the extreme couples, ſhews, that the ſecond and third elements do here alſo compoſe the interval *t*+H of an imperfect minor third.

If H be the ſecond element, as in the next lower ranks, then the firſt couple compoſe the interval T+H of a perfect *minor* 3[d], and the laſt couple being T+*t* as before, the three remaining elements muſt have this order, *t*+T+H, or its reverſe, H+T+*t*, for the reaſon above; and being thus transferred into the middle interval, they ſhew, that the elements in the ſecond and third places do again compoſe an imperfect minor third, H+*t*, or elſe an imperfect tone H+H; which being joined to the major tone on either ſide of it, compoſes an imperfect major third, greater than *t*+T by almoſt two commas, as appears by the preliminary obſervation.

Now any one of those imperfect minor thirds, $t+H$, together with the contiguous perfect major III^d, composes a fifth equally imperfect, and so does the imperfect major third $t+t$ with the perfect minor third next to it. And the complements to the VIII^th of these imperfect concords, as well as their compounds with VIII^ths, are also equally imperfect, which proves the proposition. For having shewn the necessary defects in those six arrangements of the elements, we are freed from the trouble of considering the rest (*f*). Q. E. D.

7. *Coroll.* Of those six arrangements of the elements, the first and fifth in the table are equally good, and better than any of the rest, as producing as many perfect thirds, and a greater number of perfect V^ths.

FIG. 7. In order to enumerate them with certainty and ease, if the circumference

(*f*) Mr. *De Moivre*'s general corollary to the XVI problem of his Doctrine of Chances, gives 210 permutations of these seven things, T, T, T, t, t, H, H, ference of a circle, be divided into ſeven arches, *CD*, *DE*, *EF*, *FG*, *GA*, *AB*, *BC*, proportional to T, *t*, H, T, *t*, T, H, placed in the reſpective angles at the the center; they and their ſums, whether ſmaller or greater than the circumference, here conſidered as a continued ſpiral, will repreſent all the intervals in a ſyſtem compoſed of any number of octaves, and the correſponding intervals in different octaves will be denoted by the ſame arch and letters: as appears by conceiving the baſe of the third Figure coiled round into the circumference of a circle, equal to the line *Cc* or *cc'* &c.

In this ſenſe then we have only three major III[ds], *CE*, *FA*, *GB*, and they all perfect; and four minor thirds, *DF*, *EG*, *AC*, *BD*, the firſt of which being compoſed of $t+$H, inſtead of T$+$H, is too ſmall by a comma; and ſix fifths, *FAC*, *CEG*, *GBD*, *DFA*, *ACE*, *EGB*, all perfect but *DFA*, which being compoſed of the defective minor third *DF* and the

the perfect major III[d] *FA*, is too ſmall by a comma.

Theſe imperfections being cauſed by the contiguity of *t* and H in the cycle of the elements, cannot be avoided while the hemitones are ſeperated; there being but 3 major tones in the cycle; and if they be joined, as in Fig. 12, the conſequences will be worſe.

The reſt will appear by enumerating the thirds and fifths in the 8[th], 9[th], 10[th], 11[th], and 12[th] Figures, made according to the other five arrangements in the Table of Elements (*g*).

8. Now if any one pleaſes to try the following experiment, he will find what effect theſe imperfect fifths and fourths and their compounds with VIII[ths],

(*g*) Sir *Iſaac Newton* happily diſcovered, (Optics Book 1, Part 2, Prop. 3) that the breadths of the ſeven primary colours in the ſun's image, produced by the refraction of his rays through a priſm, are proportional to the ſeven differences of the lengths of the eight muſical ſtrings *D*, *E*, *F*, *G*, *A*, *B*, *C*, *d*, when the intervals of their ſounds are T, H, t, T, t, H, T: which

VIII$^{\text{ths}}$, will have upon the ear; that of the thirds and ſixths having been tryed before (*b*).

In Fig. 3, tune upwards from *C* the two perfect V$^{\text{ths}}$ *CG*, *Gd*, and the perfect XVII$^{\text{th}}$, or 2VIII + III, *Ce'*, then downwards the V$^{\text{th}}$ *e'a*, and the intermediate fifth *ad* will be too little by a comma, as including the imperfect minor third *df*. And by tuning an eighth below *a* we have the imperfect fourth *Ad*, too large by a comma.

9. The diſagreeable effect of this fifth and fourth in every octave, and of their compounds with VIII$^{\text{ths}}$, and alſo of the third and ſixth in every octave and their compounds with VIII$^{\text{ths}}$, and of many more ſuch imperfect concords, when

which order is remarkably regular; but though it agrees beſt with the priſmatic colours, it is not the propereſt for a ſyſtem of concords, as producing one major third, two minor thirds and two fifths ſeverally imperfect by a comma. See Fig. 13. N°. 2.

(*b*) Sect. IV. Art. 4.

when the usual flat and sharp sounds are added to complete the system, has obliged practical musicians, long ago, to distribute that comma, wanting in the fifth *da*, equally among all the four v[ths], *CG*, *Gd*, *da*, *aé*, contained in the xvii[th] *Cé*. Such distribution is therefore called the Participation, or Temperament of the system, and when rightly adjusted is undoubtedly the finest improvement in harmonics.

10. If it be asked why no more primes than 1, 2, 3, 5 are admitted into musical ratios; one reason is, that consonances whose vibrations are in ratios whose terms involve 7, 11, 13, &c, *cæteris paribus* would be less simple and harmonious (*b*) than those whose ratios involve the lesser primes only.

Another reason is this; as perfect fifths and other intervals resulting from the

(*b*) Sect. iii and Table of the order of the simplicity of consonances.

the number 3, make the Schiſm of a comma with the perfect thirds and other intervals reſulting from the number 5, ſo ſuch intervals as reſult from 7, 11, 13, &c would make other ſchiſms with both thoſe kinds of intervals.

11. The Greek muſicians, after dividing an octave into two 4ths, with the diazeuctic or major tone in the middle between them, and admitting many primes to the compoſition of muſical ratios, ſubdivided the 4th into three intervals of various magnitudes, placed in various orders, by which they diſtinguiſhed their Kinds of Tetrachords (*i*). Two of them have occurred in this Section. The firſt, or $\frac{3}{4} = \frac{8}{9} \times \frac{8}{9} \times \frac{243}{256}$, anſwering to the 4th=T+T+L, in Fig. 6, is *Ptolemy*'s *Genus Diatonum ditonicum*, and reſults from that diviſion of a Monochord which bears the name of

(*i*) Dr. *Wallis* has given a table of them in his Appendix to *Ptolemy*'s Harmonics. Oper. Math. vol. III. pag. 166.

of *Euclid*'s Section of the Canon; the ſecond, or $\frac{3}{4} = \frac{8}{9} \times \frac{9}{10} \times \frac{15}{16}$, anſwering to the $4^{th} = T + t + H$, in Fig. 3, is *Ptolemy*'s *Diatonum intenſum*.

12. Since the invention of a temperament, all thoſe ancient ſyſtems have juſtly been laid aſide, as being unfit for the execution of muſical compoſitions in ſeveral parts. But to conclude from thence that the ancients had no muſic in parts, would be a very weak inference. Becauſe it is much eaſier for practical muſicians to follow the judgment of the ear, which naturally leads to an occaſional temperament of any diſagreeable concords, than to learn and put in practice the theories of philoſophers (*k*): And becauſe we are aſſured from hiſtory, that experience and

(*k*) It may not be amiſs to add the opinion of the famous *Salinas*. Sed unum hoc omnes ſcire volo, inſtrumenta quibus antiqui utebantur, conſonantias habuiſſe imperfectas, ut ea, quibus nunc utimur. Neque enim aliter modulatio convenienter exerceri poterat.

and necessity did introduce something of a temperament, before the reason of it was discovered, and the method and measure of it was reduced to a regular theory, as in the following proposition.

SECTION

terat. Quod si de hac consonantiarum imperfectione, neque *Ptolemæus*, neque alius ex antiquis musicis mentionem fecisse reperitur, causam potissimam esse crediderim, quòd ad practicos eam pertinere arbitrarentur; quoniam sensu duce solùm, non arte aut ratione semper fieri solita sit: cujus plenissimum et evidentissimum testimonium reperitur apud *Galenum*, libro primo De Sanitate tuenda, capite quinto; ubi magnam esse latitudinem sanitatis ostendere volens, sic inquit: Καὶ τί θαυμαστὸν εἰ τὴν εὐκρασίαν εἰς ἱκανὸν ἐκτείνουσι πλάτος ἅπαντες, ὅπου γὲ καὶ ἐν αὐταῖς λύραις εὐαρμοστίαν, τὴν μὲν ἀκριβεστάτην δήπου, μίαν καὶ ἄτμητον ὑπάρχειν εἰκός· ἡ μὲν τοι γ' εἰς χρεῖαν ἰοῦσα, πλάτος ἔχει. Πολλάκις γ' οὖν ἡρμόσθαι δοκοῦσαν ἄριστα λύραν, ἕτερος μουσικὸς ἀκριβῶς ἐφηρμόσατο· πανταχοῦ γὰρ ἡ αἴσθησις ἡμῖν ἐστὶ κριτήριον, ὡς πρὸς τὰς ἐν τῷ βίῳ χρείας. hoc est, *Quid mirum, si Eucrasiam in satis amplam latitudinem extendunt universi; quando et in lyris consonantiam ipsam, quæ summa exactissimaque sit, unicam atque insectabilem esse probabile sit, et quæ in usus hominum venit, certe latitudinem habeat. Sæpe namque,* [*quam*] *percommode temperasse lyram videaris, alter superveniens musicus exactius temperavit: siquidem nobis ad omnia vitæ munera sensus ubique judex est.* Ex quibus

Ga-

SECTION V.

Of the temperaments of perfect intervals and their synchronous variations.

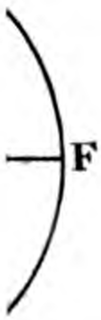
PROPOSITION II.

To reduce the diatonic system of perfect consonances to a tempered system of mean tones.

FIG. 13. When the elements are ranged in this order, T, t, H, T, t, T, H, or this, t, T, H, T, t, T, H, which two we shewed to be the best (*l*), and the arches *CD*, *DE*, *EF*, *FG*, *GA*, *AB*, *BC*, are proportional to them, let the major III[d] *CE*, situated between the two hemitones,

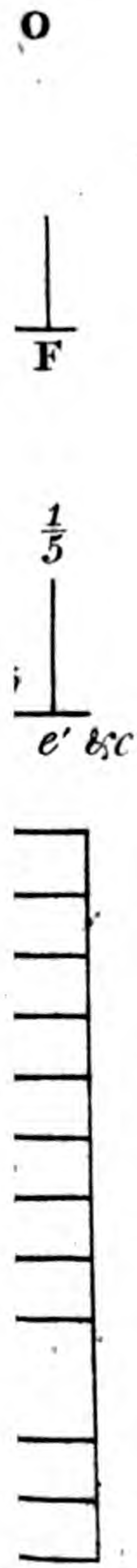
Galeni verbis liquido constat, consonantias, quibus in musicis utebantur instrumentis, jam tunc imperfectas esse, quin potius et fuisse semper et semper esse futuras. De Musicâ lib. III. cap. 14. True; but did they know, that all the concords cannot be tuned perfect, and why they cannot?

(*l*) Sect. IV. Art. 7.


F


F
G

g. 48.



mitones, be biſected in *d*; and let the other two major tones, *FG*, *AB*, be diminiſhed at both ends by the intervals *Ff*, *Gg*, *Aa*, *Bb*, ſeverally equal to half *Dd*; and the octave will then be divided into five mean tones and two limmas, each limma being bigger than the hemitone by a quarter of a comma.

For the interval *Dd* being half the difference between the major and minor tones, *CD*, *DE*, is half a comma (*m*), and therefore the new tone *Cd* or *dE* is an arithmetical mean between them. And each of the temperaments *Ff*, *Gg*, *Aa*, *Bb*, being made equal to half *Dd* or a quarter of a comma, it appears that every major tone is diminiſhed by half a comma, and that every minor tone is as much increaſed, which reduces all the tones to an equality. And the limmas *bC*, *Ef* apparently exceed the hemitones by a quarter of a comma apiece. Q. E. D.

Coroll.

(*m*) Sect. 11. Art. 4.

Coroll. In the ſyſtem of mean tones every perfect v^{th} is diminiſhed by a quarter of a comma: as will appear by going round the 13^{th} figure, and comparing the tempered v^{ths}, *fC*, *Cg*, *g d*, *d a*, *aE*, *E b*, with the perfect ones.

This is uſually called the vulgar temperament and might be proved ſeveral other ways independent of the firſt and ſecond propoſitions (*n*).

PRO-

(*n*) *Salinas* tells us, that when he was at Rome, he found the muſicians uſed a temperament there, though they underſtood not the reaſon and true meaſure of it, till he firſt diſcovered it, and *Zarlino* publiſhed it ſoon after; firſt in his Dimonſtrationi Harmoniche, Ragionamento quinto, propoſta 1^{ma}, and after that, in his Inſtitutioni Harmoniche, part. 2. cap. 43.

After his return into Spain, *Salinas* applied himſelf to the latin and greek languages, and cauſed all the ancient muſicians to be read to him, for he was blind; and in 1577 he publiſhed his learned work upon muſic of all ſorts; where treating of three different temperaments of a ſyſtem, he prefers the diminution of the v^{th} by a quarter of a comma to the other two, which he ſays are peculiar to certain inſtruments. De Muſicâ Lib. III. cap. 22.

Dechales ſays, that *Guido Aretinus*, the inventer of the preſent Scale of muſical Notes, was alſo reputed the inventer of that temperament: Ipſe nulla ha-

PROPOSITION III.

If the five mean tones and the two limmas, that compoſe a perfect octave, be changed into five other equal tones and two equal limmas, of any indeterminate magnitudes; the ſynchronous Variations of the limma L, *the mean tone* M, *and of every interval compoſed of any numbers of them, are all exhibited in the following table, by the numbers and ſigns of any ſmall indeterminate interval v: And are the ſame quantities as the variations of the temperaments of the reſpective perfect intervals.*

For

habita ratione toni majoris et minoris, hunc unius quintæ defectum aliis omnibus quintis communicat, et quaſi dividit, ita ut nulla deficiat, niſi quarta parte commatis. Hoc ſyſtema, quod valde commodum eſt, dicitur eſſe *Aretini*. Curſus Mathem. Tom. I. pag. 62. De Progreſſu Muſicæ, & Tom. IV. pag. 15. But that opinion wants confirmation, eſpecially as *Dechales* makes no mention of the claims of *Zarlino* and *Salinas* to that invention; for it ſeems they had a diſpute about it.

2^d	II^d	3^d	III^d	4^{th}	IV^{th}
L	M	L+M	2M	L+2M	3M
$5v$	$-2v$	$3v$	$-4v$	v	$-6v$
$-5v$	$2v$	$-3v$	$4v$	$-v$	$6v$
L+5M	2L+4M	L+4M	2L+3M	L+3M	2L+2M
VII^{th}	7^{th}	VI^{th}	6^{th}	V^{th}	5^{th}

For ſince the $\text{VIII}^{th} = 2L + 5M$ is ſuppoſed invariable, if the variation of L be put equal to $5v$, as in the table, that of 2L is $10v$, and that of 5M, as being the complement of 2L to the VIII^{th}, is $-10v$; whence the variation of M is $-2v$.

Conſequently the variation of the mean 3^d, L+M, is $5v - 2v = 3v$, and that of the III^d, 2M, is $-4v$, and that of the mean 4^{th}, L+2M, is $5v - 4v = v$, and that of the mean IV^{th}, 3M, is $-6v$.

The variations of the intervals in the lower half of the table, are reſpectively equal to thoſe in the upper half, but have contrary ſigns; the correſponding in-

intervals being complements to the invariable octave.

For which reason the compounds of every one of those intervals with any number of octaves, have respectively the same variations both in quantity and quality.

And if the sign of the variation of any one interval be changed, the signs of *all* the rest will also be changed; because their quantities will vanish all together when *v* or any one multiple of it vanishes.

As to the second part of the proposition, it will appear in Fig. 13, that any variation *v* of the mean interval *CdEf* is the same in quantity as the variation of the temperament *Ff* of the perfect interval *CdEF*: and the like is evident in any other instance. Q. E. D.

Coroll. 1. It is observable in the table, that the variations of all the major mean intervals II^d, III^d, IV^th, V^th, VI^th, VII^th, have the same sign, and those of the minor intervals the contrary sign.

Coroll. 2. Having extended the circumference *CdEfgabC* of Fig. 13 into a right line, as in Fig. 14, at the points *d*, E, *g*, *a*, *b*, that terminate the major mean intervals II^{d}, III^{d}, Vth, VIth, VIIth, meaſured from *C*, (and the minor too meaſured from *c* the other extreme of the octave *Cc*) place the reſpective tabular numbers 2, 4, 1, 3, 5, denoting the proportions of their ſynchronous variations; and in Fig. 15 divide any given line *O*6 into 6 equal parts, at the points 1, 2, 3, 4, 5; then conceive the 14th Fig. transferred to the 15th five ſeveral times, into five parallel poſitions, ſo that the ſeveral points 1, 2, 3, 4, 5 in each Figure may coincide. And it will be evident, by coroll. 1, that any right line *Ovvvv*, drawn from *O*, terminates the ſynchronous variations, 1*v*, 2*v*, 3*v*, 4*v*, 5*v*, of thoſe mean intervals, meaſured from their reſpective origins 1, 2, 3, 4, 5; and that theſe are alſo the ſynchronous variations of the temperaments of the reſpective perfect intervals,

and

and of their complements to the VIII[th] and compounds with VIII[ths], that is, of all the intervals in the ſyſtem.

For as to the mean IV[th] *fb*, Fig. 14, its contemporary variation in Fig. 15, will be the line 6*v* in the ſixth parallel *F*6*B'F'*, when its temperament *B'*6 or *B'b'* is taken equal to twice *Bb* and placed the ſame way from its origin *b'*. Becauſe in Fig. 14 the temperament of the perfect IV[th] *FB* is $Ff + Bb = 2Bb$.

As want of room in Fig. 15 will not permit the ſeveral intervals, even leſs than one octave, to be repreſented in their due proportions to *G*1, the quarter of the comma, which is but the 223[d] part of an octave, we muſt conceive them continued far beyond the margin of the paper.

Coroll. 3. When the III[d] is perfect, the temperaments of the V[th] and VI[th] are ſeverally $\frac{1}{4}$ of a comma, the former in defect, the latter in exceſs: and if either of them be made less, the other will be greater than $\frac{1}{4}$ comma.

FIG. 15, 16. For when the Temperer *Ovvv* falls upon *E*, the III[d] *CE* is perfect, and the tempered V[th] *C1* is less than the perfect V[th] *CG* by *G1*, and the tempered VI[th] *C3* is bigger than the perfect VI[th] *CA* by $A3 = G1 = \frac{1}{4}$ comma.

Hence when *Av*, any other temperament of the VI[th], is less than *A3* or $\frac{1}{4}$ comma, *Gv* the corresponding temperament of the V[th], is greater than *G1* or $\frac{1}{4}$ comma: and on the contrary, when *Gv* is less than *G1*, the respective *Av* is bigger then *A3*. And whatever be the magnitudes of these temperaments of the V[th] and VI[th], those of their complements to the VIII[th] and compounds with VIII[ths] are the same.

Coroll. 4. When the VI[th] is perfect, the temperaments of the V[th] and III[d] are severally $\frac{1}{3}$ comma, and are both negative.

FIG.

FIG. 16. For when the temperer $Ovvv$ falls upon the line $OHAI$, the temperament of the VI^th vanishes, and those of the V^th and III^d are GH and EI, and are equal. For the equal lines $G1$, $A3$ and equal triangles GEO, AOE shew, that the line GE is parallel to AO; whence GH is equal to EI, and the similar triangles IEO, $A3O$ give $IE = \frac{4}{3}A3 = \frac{4}{3} \times \frac{1}{4}$ comma $= \frac{1}{3}$ comma.

Coroll. 5. When the V^th is perfect, the temperaments of the VI^th and III^d are severally equal to a comma in excess.

For when the temperer $Ovvv$ falls upon the line $OGKL$, the temperament of the V^th vanishes, and those of the VI^th and III^d are now AK and EL, which are equal, because of the parallelograms $AEGO$, $AELK$; and EL is $= 4 \times G1$ or four quarters of a comma.

Coroll. 6. When the temperer $Ovvv$ falls within the angle AOE, the temp^t. V^th = temp^t. VI^th + temp^t. III^d, that is, the line $Gv = Av + Ev$, or the lines

G1

$G1+1v=A3-3v+Ev$, that is, putting the letter v for the line $1v$, $\frac{1}{4}c+v=\frac{1}{4}c-3v+4v$, which is evidently true.

Coroll. 7. When the temperer $Ovvv$ falls within the angle EOG, the temp^t. VI^th = temp^t. V^th + temp^t. III^d, that is, the line $Av=Gv+Ev$, or the lines $A3+3v=G1-1v+Ev$, that is, putting $v=$ the line $1v$, $\frac{1}{4}c+3v=\frac{1}{4}c-v+4v$, which is true.

Coroll. 8. When the temperer falls any where out of the angle AOG, the temp^t. III^d = temp^t. V^th + temp^t. VI^th, that is, when it falls beyond the side AO, the temp^t. $EI+Iv=GH+Hv+Av$, or putting the letter v for the line Hv, $\frac{1}{3}c+4v=\frac{1}{3}c+v+3v$, which is true: and when the temperer falls beyond the other side OG, the said temp^t. $EL+Lv=Gv+AK+Kv$, that is, putting v for the line Gv, $c+4v=v+c+3v$, which is true.

Coroll.

Coroll. 9. The ſum of the temperaments of the v[th] and vi[th] is $\frac{1}{2}$ a comma when the iii[d] is perfect; is leſs than $\frac{1}{2}$ a comma by $\frac{1}{2}$ the temperament of the iii[d] when flattened; and greater than $\frac{1}{2}$ a comma by $\frac{1}{2}$ the temperament of the iii[d] when ſharpened.

For in the firſt caſe the ſaid ſum is $G1+A3$; in the ſecond, it is $G1+1v+A3-3v=G1+A3-2v$; and in the third, it is $G1-1v+A3+3v=G1+A3+2v$; in which latter caſes the temperament of the iii[d] is $4v$.

Coroll. 10. Hence the ſum of the temperaments of all the concords is leſs when the iii[ds] are flattened, than the like ſum when the iii[ds] are equally ſharpened; but it is the leaſt of all when the iii[ds] are perfect, as in the ſyſtem of mean tones (*a*).

Scholium.

From the third and tenth corollaries, I think we might juſtly pronounce the ſyſtem

(*a*) Prop. 11.

ſyſtem of mean tones to be the beſt poſſible, were it evident that equal temperaments cauſe different concords to be equally diſagreeable to the ear (*a*).

But if it ſhall appear, that the VI[th] and 3[d] and their compounds with octaves, are more diſagreeable in their kind, than the V[th] and 4[th] and their compounds with octaves, all being equally tempered, as in that ſyſtem; will it not follow, that the temperament of the former Parcel of concords ſhould be ſmaller than that of the latter, to make them all as equally harmonious as poſſible, without ſpoiling the harmony of the III[d] and 6[th] and their compounds with octaves; which third

(*a*) Mr. *Huygens* has pronounced it the beſt, in ſaying that the muſicians in the other planets may know perhaps, cur optimum ſit temperamentum in chordarum ſyſtemate, cum ex diapente quarta pars commatis ubique deciditur; Coſmotheoros pag. 76; but has given us no reaſon for his aſſertion, either in that incomparable book or in his Harmonic Cycle; where he only appeals to the approbation and practice of muſicians and refers to the demonſtrations of *Zarlino* third parcel makes up the ſum of all the concords in the ſyſtem.

For if it be the immediate ſucceſſion of a worſe harmony to a better, as in inſtruments badly tuned, which chiefly offends the ear; it muſt be allowed, that a ſyſtem would be the better, *cæteris paribus*, for having all the concords as equally harmonious in their kinds, as the nature and properties of numbers will permit.

In order to reſolve thoſe queſtions upon philoſophical principles, and to determine the temperament of a given ſyſtem, that ſhall cauſe all the concords, at a medium of one with another, to be equally, and the moſt harmonious in

lino and *Salinas*. But neither of theſe celebrated authors do any thing more, if I rightly remember, (for I have not the books now by me) than reduce the Diatonic ſyſtem of perfect conſonances to that of mean tones, by diſtributing the ſchiſm of a whole comma into quarters; not at all conſidering, whether thoſe equal temperaments have the ſame, or a different effect upon the ſeveral concords.

in their several kinds, I found it necessary to make a thorough search into the abstract nature and properties of tempered consonances; and thence to derive their effects upon our organs of hearing: A large field of harmonics hitherto uncultivated.

But before I enter upon it, it will be convenient to finish this section with a determination of the least sum of any three temperaments in different parcels, when any two of them have any given ratio.

PROPOSITION IV.

To find a set of temperaments of the Vth, VIth *and* III^{d} *upon these conditions; that those of the* Vth *and* VIth *shall have the given ratio of* r *to* s, *and the sum of all three shall be the least possible.*

Part of the 17th and 18th Figures being constructed as before, from *A* towards *K* take *AM* : *G*1 :: *s* : *r*, and through

through the interſection p of the lines AG, $M1$, draw the temperer $Orst$; I ſay Gr, As, Et are the temperaments required.

For by the ſimilar triangles Grp, Asp, and $G1p$, AMp, we have $Gr : As :: (Gp : Ap :: G1 : AM ::)\ r : s$ by conſtruction, as required by the firſt condition.

Again, in the ſame line MAC take $AN = AM$, and through the interſection P of the lines AG, $N1$ produced, draw another temperer $ORST$; and by the ſimilar triangles GRP, ASP, and $G1P$, ANP, we have $GR : AS :: (GP : AP :: G1 : AN$ or $AM ::)\ r : s$ by conſtruction, which likewiſe anſwers the firſt condition; and it is eaſy to underſtand, that no other temperers but thoſe two can anſwer that condition.

Now whatever be the quantity and quality of the given ratio r to s, I ſay the ſum $Gr + As + Et$ is leſs than $GR + AS + ET$.

Caſe

Case 1. Fig. 17. For when r is bigger than s, or the ratio of r to s, or of $G1$ or $A3$ to AM or AN, is a ratio of majority, the temperers Op, OP fall within the angles AOE, AOC respectively; as appears by the construction. Whence, by coroll. 6 and 8. prop. III, $Gr = As + Et$, and $ET = GR + AS$; and therefore $Gr + As + Et : GR + AS + ET :: Gr : ET$, which is a ratio of minority, because Gr is less than GH or EI (*b*) and EI less than ET.

Case 2. Fig. 18. When r is less than s, or the ratio of r to s, or of $G1$ to AM or AN is a ratio of minority, the temperers Op, OP fall within the angles EOG, AOC respectively; as appears by the construction. Whence, by coroll. 7 and 8. prop. III, $As = Gr + Et$, and $ET = GR + AS$, and therefore $Gr + As + Et : GR + AS + ET :: As : ET$, which is a ratio of minority; because $Gr : As :: r : s :: GR : AS$, whence, as Gr is less

(*b*) See coroll. 4. Prop. III.

less than GR, so As is less than AS, which is less than IT, which is less than ET.

Case 3. Fig. 17 and 18. When r to s, or $G1$ to AM or AN, is the ratio of equality, the temperer $Orst$ coincides with the line OE, and $ORST$ is parallel to GA; whence it is plain, that the sum of the temperaments $G1+A3+0$, is less than $GR+AS+ET$, as required. Q. E. D.

Coroll. Putting c for the comma EL or four $G1$, when the temp^t. V : temp^t. VI :: r : s, the required temperaments of the V, VI and III are, $Gr = \frac{r}{3r+s}c$, $As = \frac{s}{3r+s}c$ and $\pm Et = \frac{r-s}{3r+s}c$. And according as r is bigger or less than s, the temperer $Orst$ falls within the angle AOE or EOG.

FIG. 17 and 18. For, $As : Gr :: s : r$, and $As : 3Gr$ or $sK :: s : 3r$, and $As : As+sK$ or c (a) $:: s : s+3r$. Whence $As = \frac{s}{3r+s}c$, and $Gr = \frac{r}{s}As = \frac{r}{3r+s}c$, and

(a) See Dem. coroll. 5. prop. III.

and in the angle AOE, $Et = Gr - As = \frac{r-s}{3r+s}c$, but in EOG, $Et = As - Gr$, by the equations in case 1, 2.

PROPOSITION V.

To find a set of temperaments of the v[th], vi[th] *and* iii[d] *upon these conditions; that those of the* v[th] *and* iii[d] *shall have the given ratio of* r *to* t, *and the sum of all three shall be the least possible.*

Fig. 19, 20. If t to r be a ratio of minority, or of equality, or even of majority less than 1 to $\frac{1+\sqrt{33}}{8}$ or 0. 843070 &c, from E towards I take $EM : G1 :: t : r$, and through the intersection p of the lines $M1$, GE produced, draw the temperer $Orst$, and the required temperaments will be Gr, As, Et.

But if the ratio of t to r be greater than 1 to 0. 843070 &c, in Fig. 20, from E towards L take $EN : G1 :: t : r$, and through the intersection P of the lines

lines $N1$, GE, draw the temperer $ORST$, and the required temperaments will be GR, AS, ET.

And if $t : r :: 1 : 0.843070$ &c, the required temperaments will be Gr, As, Et, or GR, AS, ET, their ſums being equal.

In the firſt caſe, Fig. 19, take $EN = EM$, and in the ſecond, Fig. 20, $EM = EN$; and through the interſections P, p of the lines $N1$, $M1$ with GE, draw two more temperers $ORST$, $Orst$.

Then by the ſimilar triangles Grp, Etp and $G1p$, EMp, we have $Gr : Et :: (Gp : Ep :: G1 : EM ::)\ r : t$ by conſtruction, as required by the firſt condition.

Again, by the ſimilar triangles GRP, ETP and $G1P$, ENP, we have $GR : ET :: (GP : EP :: G1 : EN ::)\ r : t$ by conſtruction, which alſo anſwers the first ~~ſecond~~ condition; and it is eaſy to underſtand that no other temperers but thoſe can anſwer that condition.

Case 1. Fig. 19. Now when t is to r, and therefore EM or EN to $G1$ in a ratio of minority, the temperers Op, OP fall within the angles AOE, EOG respectively by the construction. Whence, by coroll. 6 and 7 prop. 3, $Gr = As + Et$ and $AS = GR + ET$.

But $Gr : Et :: r : t$, and $Gr : \frac{1}{4} Et$ or $r1 :: r : \frac{1}{4} t$, and $Gr : Gr - r1$ or $\frac{1}{4} c :: r : r - \frac{1}{4} t :: 4r : 4r - t$. Whence $Gr = \frac{r}{4r - t} c$, and $Et = \frac{t}{r} Gr = \frac{t}{4r - t} c$, and $As = Gr - Et = \frac{r - t}{4r - t} c$, by the equation in the last paragraph.

Likewise $GR : ET :: r : t$, and $GR : \frac{1}{4} ET$ or $R1 :: r : \frac{1}{4} t$, and $GR : GR + R1$ or $\frac{1}{4} c :: r : r + \frac{1}{4} t :: 4r : 4r + t$. Whence $GR = \frac{r}{4r + t} c$, and $ET = \frac{t}{r} GR = \frac{t}{4r + t} c$ and $AS = GR + ET = \frac{r + t}{4r + t} c$, by the equation above.

There-

Therefore $Gr+As+Et : GR+AS+ET :: Gr : AS :: \frac{r}{4r-t} : \frac{r+t}{4r+t} :: 4rr+rt : 4rr+rt, +2rt-tt$, which is a ratio of minority; becauſe t being leſs than r, tt is leſs than $2rt$.

Caſe 2. Fig. 20. When t to r, and therefore EM or EN to $G1$, is a ratio of majority, the temperers Opt, OPT fall within the angles AOC, EOG reſpectively; as appears by the conſtruction. Whence, by coroll. 8 and 7 prop. 3, $Et=Gr+As$ and $AS=GR+ET$.

In which caſe the theorems for the values of Gr, As, Et, GR, AS, ET are the ſame as before.

Therefore $Gr+As+Et : GR+AS+ET :: Et : AS :: \frac{t}{4r-t} : \frac{r+t}{4r+t} :: 4rt+tt : (4rr+4rt-rt-tt$ or$)$ $4rt+tt, +4rr-rt-2tt$, which is a ratio of minority, except either when $4rr-rt-2tt=0$, or $\frac{4rr-rt-2tt}{4tt}=0$, or

$\frac{rr}{tt} - \frac{r}{4t} - \frac{1}{2} = 0$, which gives $\frac{r}{t} = \frac{1+\sqrt{33}}{8} = 0.843070$ &c (*a*); or when $4rr - rt - 2tt$, or $\frac{rr}{tt} - \frac{r}{4t} - \frac{1}{2}$ is negative, and consequently $\frac{r}{t}$ is less than 0.843070 &c (*b*), or the ratio of t to r is greater than 1 to 0.843070 &c.

In the first case either $Gr+As+Et$ or $GR+AS+ET$, as being equal, are the required temperaments; in the second the latter only, as being less than the former.

Case 3. When $t=r$, we have EM or $EN=G1$; therefore the intersection *p*

(*a*) For supposing $\frac{r}{t} = \frac{x}{1} = x$, we have $\frac{rr}{tt} - \frac{r}{4t} - \frac{1}{2} = xx - \frac{1}{4}x - \frac{1}{2} = 0$. Whence $xx - \frac{1}{4}x = \frac{1}{2}$ and $xx - \frac{1}{4}x + \frac{1}{8} \times \frac{1}{8} = \frac{1}{8} \times \frac{1}{8} + \frac{1}{2} = \frac{1}{8} \times \frac{1}{8} + \frac{32}{8 \times 8} = \frac{33}{8 \times 8}$; whose square roots are $x - \frac{1}{8} = \frac{\pm\sqrt{33}}{8}$; whence x or $\frac{r}{t} = \frac{1 \pm \sqrt{33}}{8} = \frac{1 \pm 5.744562 \text{ \&c}}{8}$.

(*b*) For

p is removed to an infinite diſtance, and the temperer $Orst$ coincides with $OHAI$. Hence $Gr+As+Et$ becomes $=GH+0+EI$, and is to $GR+AS+ET$: : 5 : 6, a ratio of minority, produced by putting $t=r$ in the terms of that ratio in caſe 1 or 2. Q. E. D.

Coroll. When the temp^r. V : temp^t. III : : $r : t$, if $\frac{r}{t}$ be bigger than 0. 843070 &c, the required temperaments of the V, VI and III are, $Gr = \frac{r}{4r-t}c$, $As = \frac{r-t}{4r-t}c$, $Et = \frac{t}{4r-t}c$. And the temperer $Orst$ falls within the angle AOE or AOC, according as r is bigger or leſs than t.

But

(*b*) For ſince the root 0. 843070 &c, when ſubſtituted for x, will make the value of $xx - \frac{1}{4}x - \frac{1}{2}$, or of $x - \frac{1}{4} - \frac{1}{2x} = 0$; a ſmaller number ſubſtituted for x, will produce a negative value of the latter, and conſequently of the former quantity.

But if $\frac{r}{t}$ be less than 0. 84307 &c, they are $GR = \frac{r}{4r+t}c$, $AS = \frac{r+t}{4r+t}c$, $ET = \frac{-t}{4r+t}$; and the temperer $ORST$ falls within the angle EOG.

And if $\frac{r}{t} = 0.843070$ &c, their sums are equal and either of them answers the problem.

PROPOSITION VI.

To find a set of temperaments of the V^th^, VI^th^ *and* III^d^ *upon these conditions, that those of the* VI^th^ *and* III^d^ *shall have the given ratio of* s *to* t, *and the sum of all three shall be the least possible.*

FIG. 21 and 22. From E towards C take $EM : A3 :: t : s$ and through the intersection p of the lines $M3$, AE draw the temperer $Orst$, and the required temperaments will be Gr, As, Et.

For

For by the ſimilar triangles Asp, Etp and $A3p$, EMp, we have $As : Et$ :: ($Ap : Ep$ :: $A3 : EM$ ::) $s : t$ by conſtruction, as required by the firſt condition.

Again, taking $EN = EM$, through the interſection P of the lines $N3$, AE produced, draw the temperer $ORST$, and by the ſimilar triangles ASP, ETP and $A3P$, ENP, we have $AS : ET$ :: ($AP : EP$ :: $A3 : EN$ or EM) :: $s : t$ by conſtruction, which alſo anſwers the firſt condition; and it is plain that thoſe are all the temperers which can anſwer it.

Now whatever be the ratio of s to t, I ſay that $Gr + As + Et$ is leſs than $GR + AS + ET$.

Caſe 1. Fig. 21. When t is to s, or EM to $A3$ in a ratio of minority, the temperers Op, OP fall within the angles AOE, EOG reſpectively, as appears by the conſtruction. Whence by coroll. 6 and 7 prop. III, $Gr = As + Et$ and $AS = GR + ET$.

But

But $Et : As :: t : s$, and $Et : \frac{4}{3}As$ or $It :: t : \frac{4}{3}s$ and $Et : Et + It$ or $\frac{1}{3}c$ (a) $:: t : t + \frac{4}{3}s :: 3t : 3t + 4s$. Whence $Et = \frac{t}{4s+3t}c$. And $As = Et \times \frac{s}{t} = \frac{s}{4s+3t}c$. And $Gr = As + Et = \frac{s+t}{4s+3t}c$, by the equation in the last paragraph.

Again, $ET : AS :: t : s$ and $ET : \frac{4}{3}AS$ or $IT :: t : \frac{4}{3}s$ and $ET : IT - ET$, or IE or $\frac{1}{3}c :: t : \frac{4}{3}s - t :: 3t : 4s - 3t$. Whence $ET = \frac{t}{4s-3t}c$ and $AS = ET \times \frac{s}{t} = \frac{s}{4s-3t}c$.

Therefore $Gr + As + Et : GR + AS + ET :: Gr : AS :: \frac{s+t}{4s+3t} : \frac{s}{4s-3t} :: 4ss + 4st - 3st - 3tt$, or $4ss + 3st - 2st - 3tt : 4ss + 3st$, which is evidently a ratio of minority.

Case

(a) See Dem. coroll. 4. prop. III.

Case 2. Fig. 22. When t is to s, or EM to $A3$ in a ratio of majority, the temperers Op, OP fall within the angles AOE, GOc respectively. Whence, by coroll. 6 and 8 prop. III, $Gr = As + Et$ and $ET = GR + AS$, and $Gr + As + Et : GR + AS + ET :: Gr : ET$, which is plainly a ratio of minority.

Case 3. When $t = s$, or EM or $EN = A3$, the intersection P vanishes, and the temperer $ORST$ coincides with $OGKL$, as appears by the construction. Whence by the conclusion of the second case, $Gr + As + Et : 0 + AK + EL :: Gr : EL$, a ratio of minority, as before. Q. E. D.

Coroll. When the temp^t. VI : temp^t. III :: $s : t$, the required temperaments of the V, VI and III are, $Gr = \frac{s+t}{4s+3t} c$, $As = \frac{s}{4s+3t} c$, $Et = \frac{t}{4s+3t} c$; and the temperer lies within the angle AOE, whatever be the quantity and quality of the ratio of s to t.

Scho-

Scholium.

These three problems comprehend the solution of a more general one, namely, To find the temperament of any given system of sounds upon these conditions; that the octaves be perfect, that the ratio of the temperaments of any two given concords in different parcels be given, and that the sum of the temperaments of all the concords, be the least possible.

The reason is, that the given ratio of the temperaments of any two concords, determines the position of the temperer of the system, and this the three magnitudes of the temperaments of all the concords, whatever be their number. But if both the given concords be contained in any one of the three parcels above mentioned, the given ratio of their temperaments can be no other than that of equality; and this *datum* is plainly insufficient.

SECTION

SECTION VI.

Of the Periods, Beats and Harmony of imperfect conſonances.

DEFINITIONS.

I. Any two ſounds whoſe vibrations have any ſmall given ratio, are called Imperfect Uniſons:

II. And the cycle of their pulſes is called Simple or Complex, according as the difference of the terms of that ratio is an unit or units:

III. And when a complex cycle is divided into as many equal parts, as that difference contains units, each part is called a Period of the pulſes:

IV. And the cycles of perfect conſonances are often called Short cycles, to diſtinguiſh them from the long cycles of imperfect uniſons.

PROPOSITION VII.

In going from either end to the middle of any simple cycle or period of the pulses of imperfect unisons, the alternate lesser intervals between the successive pulses increase uniformly, and are proportional to their distances from that end; and at any distance from it less than half the simple cycle or period, are less than half the lesser of the two vibrations.

Let the vibrations be V and v, and $V : v :: m : n$, the numbers m, n being the least in that ratio; and putting $d = m - n$, we have the complex cycle $nV = mv = nv + dv$ (*a*), and the period $\frac{n}{d}V = \frac{n}{d}v + v$, which, when $d = 1$, is a simple cycle (*b*).

FIG. 23, 24, 25, 26. To assist the imagination, let the successive vibrations V, V, &c, be represented by the equal lines

(*a*) Sect. III. Art. 12. (*b*) Def. II.

lines *AB*, *BC*, *CD*, &c, and the inſtants of their pulſes by the points *A*, *B*, *C*, &c; and the ſucceſſive vibrations v, v, &c. by the equal lines *ab*, *bc*, *cd*, &c, and the inſtants of their pulſes by the points *a*, *b*, *c*, &c.

Then beginning from the firſt coincidence of the pulſes at *A* or *a*, which muſt happen in leſs time than that of a cycle, it is obſervable, that the ſucceſſive intervals of the pulſes are alternately bigger and leſs; and that the alternate leſſer intervals *Bb*, *Cc*, *Dd*, &c, or $V - v$, $2V - 2v$, $3V - 3v$, &c, increaſe uniformly, by the repeated addition of the firſt and leaſt interval $V - v$, at every equal increment V or v of their diſtances from *A*. The alternate leſſer intervals are therefore proportional to their diſtances from the coincident pulſes *A*, *a*.

Now any aſſigned diſtance $3V : nV :: 3v : nv :: 3V - 3v : nV - nv = dv$, by the equation; whence $3V : \frac{n}{d}V ::$

3V

$3V - 3v : v$; consequently if the assigned distance $3V$ or AD be less than half the simple cycle or period $\frac{n}{d}V$, the adjoining interval $3V - 3v$, or Dd is less than half v; but if bigger, bigger than half v.

And the argument is the same in going backwards from the next coincidence of the pulses at U and w, U and x, &c, their larger and lesser alternate intervals being evidently of the same magnitudes as in going forwards.

FIG. 24. Now if the difference $d = 2$, let the length of the complex cycle be the line AU or $ax = nV = nv + 2v$; and having divided it into two equal parts AX, XU, we have the part or period $AX = \frac{n}{d}V$; which because 2 does not measure n (a), consists of a multiple of V, as AK, and a remainder $KX = \frac{1}{2}V = \frac{1}{2}KL$.

We

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We have alſo, by the ſame equation, $AX = \frac{n}{2}v + v$, which becauſe 2 does not meaſure n, conſiſts of a multiple of v (one more than that other of V) as Al, and a like remainder $lX = \frac{1}{2}v = \frac{1}{2}lm$.

Now the diſtances of the ſucceſſive pulſes of V from the point X are XL, XM, XN, &c, or $\frac{1}{2}$ V, $\frac{3}{2}$ V, $\frac{5}{2}$ V, &c, and thoſe of the ſucceſſive pulſes of v are Xm, Xn, Xo, &c, or $\frac{1}{2}v$, $\frac{3}{2}v$, $\frac{5}{2}v$, &c, and the differences of thoſe reſpective diſtances, or the alternate leſſer intervals between the ſucceſſive pulſes of V and v, are Lm, Mn, No, &c, or $\frac{1}{2}V - \frac{1}{2}v$, $\frac{3}{2}V - \frac{3}{2}v$, $\frac{5}{2}V - \frac{5}{2}v$, &c; which increaſe uniformly by the repeated addition of $V - v$ or $\frac{2V - 2v}{2}$ to the firſt and ſucceeding intervals.

Assign any distances *XL*, *XN*, or $\frac{1}{2}$V and $\frac{5}{2}$V; then $\frac{1}{2}V : \frac{5}{2}V :: \frac{1}{2}v : \frac{5}{2}v :: \frac{1}{2}V - \frac{1}{2}v : \frac{5}{2}V - \frac{5}{2}v$, that is, $XL : XN :: Xm : Xo :: Lm : No$, or the alternate lesser intervals are proportional to their distances from the periodical point *X*.

Now any assigned distance $\frac{5}{2}V : \frac{n}{2}V :: \frac{5}{2}v : \frac{n}{2}v :: \frac{5}{2}V - \frac{5}{2}v : \frac{n}{2}V - \frac{n}{2}v = v$, by the given equation, that is, $\frac{5}{2}V : \frac{n}{2}V :: \frac{5}{2}V - \frac{5}{2}v : v$; consequently if the assigned distance $\frac{5}{2}$V or *XN*, be less than half the period $\frac{n}{2}$V or half *XU*, the adjoining interval $\frac{5}{2}V - \frac{5}{2}v$ or *No*, is less than half *v*; but if bigger, bigger than half *v*.

And in going backwards from *X*, the alternate lesser intervals *Kl*, *Ik*, *Hi*, &c, are respectively equal to *Lm*, *Mn*, *No*, &c, at equal distances on each side of *X*.

Fig. 25. In like manner if $d = 3$, or $nV = nv + 3v$, having divided this cycle *AU* or *ay* into three equal periods *AX*, *XY*, *YU*, that equation gives $AX = \frac{n}{3}V$, which conſiſts of a multiple of V, as *AG*, and a remainder $GX = \frac{1}{3}V$ (or $\frac{2}{3}V$ hereafter to be conſidered) $= \frac{1}{3}GH$, whoſe complement $XH = \frac{2}{3}V$.

The ſame period *AX* is alſo $= \frac{n}{3}v + v$ by the ſame equation, and therefore conſiſts of a multiple of v (one more than that other of V) as *Ab*, and a like remainder $bX = \frac{1}{3}v = \frac{1}{3}bi$, whoſe complement $Xi = \frac{2}{3}v$.

Hence the diſtances from *X* of the ſucceſſive pulſes of V are *XH*, *XI*, *XK*, &c, or $\frac{2}{3}V$, $\frac{5}{3}V$, $\frac{8}{3}V$, &c, and thoſe of the ſucceſſive pulſes of v are *Xi*, *Xk*, *Xl*, &c, or $\frac{2}{3}v$, $\frac{5}{3}v$, $\frac{8}{3}v$, &c, and the alternate lesser intervals between the successive pulses of V and v, are Hi, Ik, Kl, &c, or $\frac{2}{3}V - \frac{2}{3}v$, $\frac{5}{3}V - \frac{5}{3}v$, $\frac{8}{3}V - \frac{8}{3}v$, &c; which increase uniformly by the repeated addition of $V - v$ or $\frac{3V - 3v}{3}$ to the first and succeeding intervals.

Assign any distances XH, XK or $\frac{2}{3}V$ and $\frac{8}{3}V$; then $\frac{2}{3}V : \frac{8}{3}V :: \frac{2}{3}v : \frac{8}{3}v, :: \frac{2}{3}V - \frac{2}{3}v : \frac{8}{3}V - \frac{8}{3}v$, that is, $XH : XK :: Xi : Xl :: Hi : Kl$, or the alternate lesser intervals are proportional to their distances from the periodical point X.

Now any assigned distance $\frac{8}{3}V : \frac{n}{3}V :: \frac{8}{3}v : \frac{n}{3}v :: \frac{8}{3}V - \frac{8}{3}v : \frac{n}{3}V - \frac{n}{3}v = v$ by the equation, that is, $\frac{8}{3}V : \frac{n}{3}V :: \frac{8}{3}V - \frac{8}{3}v : v$; so that if the assigned distance $\frac{8}{3}V$ or XK be less than half the period $\frac{n}{3}V$ or half XY, the adjoining interval

$\frac{8}{3}V - \frac{8}{3}v$ or KL is leſs than half v; but if bigger, bigger than half v.

By doubling the period $AX = AG + \frac{1}{3}V$, we have $AY = 2AG + \frac{2}{3}V = AN + \frac{2}{3}V$, ſo that NY is $= \frac{2}{3}V$ and its complement $YO = \frac{1}{3}V$. Again by doubling $AX = Ah + \frac{1}{3}v$, we have $AY = 2Ah + \frac{2}{3}v = Ap + \frac{2}{3}v$, ſo that $pY = \frac{2}{3}v$ and its complement $Yq = \frac{1}{3}v$.

Hence the alternate leſſer intervals of the pulſes of V and v, in going oppoſite ways to equal diſtances from X and Y, are equal. And in going contrary ways from X towards A, or from Y towards U, the alternate leſſer intervals are $\frac{1}{3}V - \frac{1}{3}v$, $\frac{4}{3}V - \frac{4}{3}v$, $\frac{7}{3}V - \frac{7}{3}v$, &c, which increaſe uniformly as before; and $\frac{7}{3}V$ being an aſſigned diſtance from *X* or *Y*, we have $\frac{7}{3}$V : $\frac{n}{3}$V :: $\frac{7}{3}v$: $\frac{n}{3}v$ :: $\frac{7}{3}$V $-$ $\frac{7}{3}v$: $\frac{n}{3}$V $-$ $\frac{n}{3}v = v$ as before. So that if the aſſigned diſtance $\frac{7}{3}$V be leſs than half the period $\frac{n}{3}$V, the adjoining interval $\frac{7}{3}$V $-$ $\frac{7}{3}v$ is leſs than half v; but if bigger, bigger than half v.

Fig. 26. Laſtly when the period *AX*, $= \frac{n}{3}$V, conſiſts of a multiple of V as *AG* and a remainder *GX* $= \frac{2}{3}$V, which remained to be conſidered, its complement *XH* is $= \frac{1}{3}$V, and the demonſtration would proceed in the ſame method as before.

Whoever deſires a general proof of the propoſition for any value of the difference *d*, need only read the laſt example over again, with a deſign to make the proof general; and he will perceive that what has been ſaid of the number 3 as a value of *d*, *mutatis mutandis*,

tandis, is plainly applicable to any other value. Q. E. D.

Coroll. 1. The periodical points *X*, *Y*, &c, always fall within certain values of v that are ſeverally contained within as many values of V, and the number of thoſe points in each complex cycle is $d-1$.

Coroll. 2. The leſſer intervals that lye neareſt to the periodical points and the points of coincidence, are leſs than the reſt and are $\frac{V-v}{d}$ and all its multiples, whereof the greateſt multiplier is d; as $\frac{V-v}{3}$, $\frac{2V-2v}{3}$, $\frac{3V-3v}{3}$ when $d=3$; $\frac{V-v}{4}$, $\frac{2V-2v}{4}$, $\frac{3V-3v}{4}$, $\frac{4V-4v}{4}$, when $d=4$; &c.

Coroll. 3. Some of the alternate leſſer intervals of the pulſes of imperfect uniſons, are the differences of equal numbers of their vibrations, counted from the neareſt coincident pulſes; and others are the differences of equal numbers

bers

bers of the ſame parts of their ſingle vibrations, counted from the neareſt periodical point.

Coroll. 4. If the vibrations of two couples of imperfect uniſons (or of any two conſonances) be proportional, the periods and cycles of their pulſes, whether ſimple or complex, will be in the ratio of the homologous vibrations.

Let T and t be the vibrations of one couple, and V and v thoſe of the other; and ſince $T : t :: V : v :: n+d : n$, the cycles of their pulſes are $nT = \overline{n+d} \times t$ and $nV = \overline{n+d} \times v$, and the periods are $\frac{n}{d}T = \frac{n+d}{d}t$ and $\frac{n}{d}V = \frac{n+d}{d}v$; and are in the ratio of T to V, or of t to v.

Coroll. 5. If two couples of imperfect uniſons have one vibration common to both, the periods and ſimple cycles of their pulſes, are in the inverſe ratio of the intervals of their ſounds very nearly.

Let

Let the vibrations of one couple be V and v, and of the other V and x, and let $V : v :: n+d : n$, and $V : x :: N+D : N$. Then the periods of their pulſes are $\frac{n}{d}V = \frac{n+d}{d}v$ and $\frac{N}{D}V = \frac{N+D}{D}x$, and are in the ratio of $\frac{n}{d}$ to $\frac{N}{D}$, or of Dn to dN, and are ſimple cycles when d and D are units.

Now the intervals of the ſounds of V and v, V and x, being meaſures of the magnitudes of the ſmall ratios $n+d$ to n, $N+D$ to N (*a*), are very nearly in the ratio of $\frac{d}{n}$ to $\frac{D}{N}$ (*b*), or of dN to Dn, the inverſe ratio of the periods.

Coroll.

(*a*) Art. 10. Sect. 1.

(*b*) See coroll. 1. prop. 1. of Mr. *Cotes*'s Harmonia Menſurarum, or take the following proof of it.

1. *The magnitudes of very ſmall ratios, which have a term common to both, are very nearly proportional to the differences of their terms.*

Fig. 29. Let the lines ao, bo, co, do, &c be continual proportionals in any very ſmall ratio. Then ſince $ao : bo :: bo : co :: co : do$ &c, and disjointly $ab : ao :: bc : bo :: cd : co$ &c; the equal quanti-

ties

Coroll. 6. The periods and ſimple cycles of the pulſes of any two couples of imperfect uniſons, are in the ratio compoſed of the direct ratio of their ſingle vibra-

ties $\frac{ab}{ao}$, $\frac{bc}{bo}$, $\frac{cd}{co}$, &c, may expound the equal magnitudes of the ratios *ao* to *bo*, *bo* to *co*, *co* to *do*, &c.

Conſequently the magnitude of the ratio *ao* to *co*, compoſed of *ao* to *bo*, *bo* to *co*, is to the magnitude of the ratio *ao* to *do*, compoſed of *ao* to *bo*, *bo* to *co*, *co* to *do*, as $\frac{ab}{ao}+\frac{bc}{bo}$ to $\frac{ab}{ao}+\frac{bc}{bo}+\frac{cd}{co}$ exactly; or, ſuppoſing *mo* and *no* to be arithmetical means among the denominators of each ſum, as $\frac{ab}{mo}+\frac{bc}{mo}$ to $\frac{ab}{no}+\frac{bc}{no}+\frac{cd}{no}$ very nearly; (becauſe ſome of theſe fractional quantities in each ſum, are a little bigger, and others about as much ſmaller, than they were before) that is, as $\frac{ac}{mo}$ to $\frac{ad}{no}$, or as $ac \times no$ to $ad \times mo$, or as *ac* to *ad* very nearly; becauſe the ratio *no* to *mo* is extreamly ſmall in compariſon to the ratio *ac* to *ad* by hypotheſis.

The magnitudes of any other multiple ratios, as well as duplicate and triplicate, are therefore in the ratio of the differences of their terms very nearly, and the nearer, as the ratios are ſmaller.

Fig. 30. Now let *ao* to *eo* and *ao* to *io* be any two very ſmall contiguous ratios; and their magnitudes, if

vibrations, and of the inverſe ratio of their intervals very nearly.

Fig. 31, 32. Let the perpendiculars T and *t*, V and *v* to the line *T t V v x* repre-

if commenſurable, are multiples of the magnitude of ſome ſmaller ratio; in which caſe the propoſition is already proved; becauſe a term of a geometrical progreſſion interpoſed between the extremes *ao*, *io*, will coincide with *eo*, the third given term; and if their magnitudes be incommenſurable, the error in that coincidence will be infinitely diminiſhed by diminiſhing the ratio of the progreſſion and increaſing the number of its terms to infinity. *Q. E. D.*

2. Fig. 30. *Hence the magnitudes of any very ſmall ratios* ao *to* eo, po *to* qo, *are very nearly in the ratio of* $\frac{ae}{ao}$ *to* $\frac{pq}{po}$, *or of* $\frac{ae}{eo}$ *to* $\frac{pq}{qo}$, *compoſed of the direct ratio of the differences of their terms, and the inverſe ratio of the terms themſelves.*

For ſuppoſing $ao : io :: po : qo$, the magnitudes of theſe two ratios are equal. Whence the magnitudes of the propoſed ratios *ao* to *eo*, *po* to *qo*, or *ao* to *io*, are very nearly as *ae* to *ai* (by N°. 1) or as $\frac{ae}{ao}$ to $\frac{ai}{ao} = \frac{pq}{po}$, becauſe $ai : ao :: pq : po$ by conſtruction: Or, taking $ko : eo :: po : qo$, the magnitudes of the ſaid ratios *ao* to *eo*, *po* to *qo*, or *ko* to *eo* are very nearly as *ae* to *ke* (by N°. 1) or $\frac{ae}{eo}$ to $\frac{ke}{eo} = \frac{pq}{qo}$, becauſe $ke : eo :: pq : qo$. *Q. E. D.*

represent the two couples of vibrations, and the intercepted lines *Tt*, *Vv* the intervals of their sounds; and taking a vibration *x* : V : : *t* : T, the intervals *Tt*, *Vx* of the sounds of T and *t*, V and *x*, are equal (*c*).

Hence, by coroll. 4, the period of the pulses of T and *t* : per. V and *x* : : T : V; and by coroll. 5, the per. V and *x* : per. V and *v* : : interval *Vv* : *Vx* or *Tt* very nearly. Therefore the per. *T* and *t*, is to the per. V and *v*, in the ratio composed of T to V and of the interval *Vv* to the interval *Tt* very nearly.

Coroll. 7. Hence the periods of the pulses of imperfect unisons are very nearly equal, when their single vibrations and intervals are proportional, that is, when T : V : : interval *Tt* : *Vv*.

By single vibrations in this and the former corollary, we may understand, either those of the graver sounds or of the

(*c*) Art. 10. Sect. 1.

the acuter or promiſcuouſly; becauſe the ratio T to t and of V to v is ſuppoſed very ſmall in compariſon with T to V, or t to v.

Coroll. 8. The length of the period of the Leaſt Imperfections in any conſonance of imperfect uniſons, is the ſame as that of the period of its pulſes.

Fig. 23, 24, 25, 26. For the conſonance grows gradually more imperfect in going from the beginning to the middle of any period or ſimple cycle of its pulſes, and gradually leſs imperfect in proceeding from the middle to the end: the pulſes which would be conſtantly coincident, if the uniſons were perfect, being gradually ſeperated by their alternate leſſer intervals. And thus the periods of the leaſt degrees of imperfection in the conſonance, are the very ſame as the periods of its pulſes.

PROPOSITION VIII.

If either of the vibrations of imperfect unisons and any multiple of the other, or any different multiples of both, be considered as the single vibrations of an imperfect consonance, the length of the period of its least imperfections, will be the same as that of the pulses of the imperfect unisons.

Fig. 23. 27. For instance, if *AB* and *ab* be the vibrations of imperfect unisons, 2*AB* or *AC* and *ab* will be the vibrations of an imperfect Octave; whose treble is one of the unisons, and whose base is derived from the other by intermitting every other pulse of the series *A*, *B*, *C*, *D*, *E*, &c.

Now if this octave were perfect, every pulse of its base would coincide with every other pulse of its treble; but here they are gradually seperated by some of the alternate lesser intervals *Cc*, *Ee*, &c, of the imperfect unisons.

The

The intermediate pulſes of the treble, which in a perfect octave would biſect the intervals of the pulſes of the baſe, are alſo gradually ſeperated from the round points which biſect them, by the reſt of the alternate leſſer intervals of the ſaid uniſons. And thus the imperfections of the tempered octave, or the diſlocations of the pulſes in its ſucceſſive ſhort cycles (*a*), are every where the ſame as thoſe of the imperfect uniſons, and conſequently have the ſame periods.

The argument is the ſame if $2ab$ or ac and AB be the vibrations of the tempered octave, as in Fig. 28; and alſo, if any other multiple of AB or ab, as mAB or mab, be one of the vibrations of the tempered conſonance; as appears by ſuppoſing $m-1$ pulſes of AB or ab, to be ſo intermitted, as to leave only ſingle equidiſtant pulſes in Fig. 23, 24, 25, 26.

Fig. 34. Now let any different multiples of AB and ab, as $3AB$ and $2ab$, or

(*a*) Defin. IV.

or AD and ac be the vibrations of the imperfect consonance; and if AB were $=ab$, then would $3AB$ or $AD : 2ab$ or $ac :: 3 : 2$, and all the short cycles of the vibrations AD, ac would be perfect, or their exterior pulses G and g, N and n, &c, would be coincident, as in Fig. 33: because $2 \times 3AB$ or $2AD$ or $AD + DG$ would then $= 3 \times 2ab$ or $3ac$ or $ac + ce + eg$.

But AB in Fig. 34 being bigger than ab, the multiple $6AB$ or AG is also bigger than the equimultiple $6ab$ or ag; and so the exterior pulses G and g, N and n, &c, which before were coincident, are now separated by some of the alternate lesser intervals Gg, Nn, &c, of the pulses of AB and ab (a): the distances of the pulses G and g, N and n, &c, from A and a, being equimultiples of AB and ab.

For the like reason the interior pulses of the imperfect short cycles are also sepe-

(a) Prop. VII. coroll. 3.

$\frac{c}{4}$th

$\frac{c}{3}$d

$\frac{c}{6}$th

ſeperated from the pulſes of *AB* and *ab* (denoted by round points when different from thoſe of *AD* and *ac*) by ſome of their alternate leſſer intervals.

Hence the diſlocations of the pulſes in all the ſhort cycles, are ſome of the alternate leſſer intervals of the pulſes of *AB* and *ab*.

For though we began the firſt cycle from two coincident pulſes *A*, *a*, yet the argument is the ſame if we ſuppoſe them ſeperated by any one of the alternate leſſer intervals; or begin to count the vibrations of the conſonance from any two pulſes of *AB* and *ab*, as Q and *r*, whoſe diſtances from the next periodical point or coincident pulſes Z, η, are equimultiples of the vibrations *AB*, *ab*, that is, whoſe interval Q*r* is an alternate leſſer interval of their pulſes.

For ſince the ſeveral lengths Q*X* and *ry*, *X*Δ and *y*ε, Δ*K* and ελ, &c, of the ſubſequent ſhort cycles, are equimultiples of *AB* and *ab*, the remaining diſtances *XZ* and *y*η, Δ *Z* and εη, *KZ* and λη, &c, are alſo equimultiples of *AB* and *ab*; which ſhews, that the diſlocations of the exterior pulſes *X* and *y*, Δ and ε, *K* and λ, &c, and of the interior too, are conſtantly ſome of the alternate leſſer intervals of the pulſes of *AB* and *ab*. And thus the period of the leaſt diſlocations of the pulſes of the tempered conſonance, or of the leaſt imperfections in its ſhort cycles, is conſtantly the ſame as that of the pulſes of *AB* and *ab*.

FIG. 35 Shews the ſame thing, when 2*AB* and 3*ab*, or *AC* and *ad*, are the different multiples of *AB* and *ab*, ~~and~~ whose ~~their~~ pulſes make long ſimple cycles; and when they make periods, the like is evident by inſpection of the pulſes about the periodical points *X*, *Y*, in Fig. 24, 25, 26, ſuppoſing the proper numbers of pulſes to be intermitted. And the argument is the ſame for any other different

ferent multiples of *AB* and *ab*, that is for any imperfect consonance whatever. Q. E. D.

Coroll. The same multiples or submultiples of the vibrations of imperfect unisons, will always be the vibrations of other imperfect unisons, whose period is the same multiple or submultiple of the period of the given unisons (*a*), and whose interval is the same too at a different pitch; because the ratio of the vibrations is the same (*b*).

PROPOSITION IX.

If either of the vibrations of imperfect unisons, and any aliquot part or parts of the other, be the vibrations of an imperfect consonance, the length of the period of its least imperfections, will be the same part of the period of the imperfect unisons.

Fig.

(*a*) Prop. VII. coroll. 4. (*b*) Sect. 1. Art. 10.

FIG. 36. For if the vibrations of imperfect unisons be T and t, and $\frac{n}{m}$ be any irreducible fraction, proper or improper, and the vibrations of a perfect consonance be T and $\frac{n}{m}$T, or t and $\frac{n}{m}t$; those of the imperfect consonance will be T and $\frac{n}{m}t$, or $\frac{n}{m}$T and t; which being the same as $m \times \frac{1}{m}$T and $n \times \frac{1}{m}t$, or as $n \times \frac{1}{m}$T and $m \times \frac{1}{m}t$, are different multiples of $\frac{1}{m}$T and $\frac{1}{m}t$; whose period is therefore the same as that of the least imperfections of either of either of the tempered consonances (*a*), and is equal to $\frac{1}{m}$ of the period of T and t (*b*). Q. E. D.

Coroll. 1. Fig. 37. If T and $\frac{n}{m}$T, V and $\frac{s}{r}$V be the vibrations of any two perfect consonances, the periods of their least im-

(*a*) Prop. VIII. (*b*) Coroll. Prop. VIII.

imperfections when tempered, will be in the ratio composed of the direct ratio of T to V, the inverse ratio of the denominators of the fractions $\frac{n}{m}$, $\frac{s}{r}$, and the inverse ratio of the temperaments.

For if the temperaments be the small intervals of the sounds of the vibrations T and t, V and v; by this proposition, the periods of the tempered consonances, whose vibrations are T and $\frac{n}{m}t$, V and $\frac{s}{r}v$, or t and $\frac{n}{m}$T, v and $\frac{s}{r}$V, are respectively equal to $\frac{1}{m}$ × Per. of T and t and $\frac{1}{r}$ × Per. of V and v; whose ratio is composed of $\frac{1}{m}$ to $\frac{1}{r}$, or r to m and of the other two ratios abovementioned (*c*).

Coroll. 2. Hence, when the temperaments are equal, and the fractions $\frac{n}{m}$, $\frac{s}{r}$ are the same, or their denominators

(*c*) Prop. VII. coroll. 6.

tors only, the periods of the consonances are in the ratio of their homologous vibrations, T, V.

Coroll. 3. If T=V, that is, if the consonances have a common sound, the ratio of their periods is composed of the inverse ratio of the denominators of the fractions $\frac{n}{m}$, $\frac{s}{r}$, and the inverse ratio of the temperaments, by coroll. 1.

Coroll. 4. Consequently when the denominators are the same, the periods are in the inverse ratio of the temperaments.

Coroll. 5. And when the temperaments are the same or equal, the periods have the inverse ratio of the denominators, or the direct ratio of their reciprocals $\frac{1}{m}$, $\frac{1}{r}$.

Coroll. 6. Therefore the periods of simpler consonances are generally longer than those of the less simple when equally tempered; the denominators of the fractions belonging to the simpler

pler consonances being generally smaller (*a*).

PROPOSITION X.

When either of the sounds of a perfect consonance is tempered flat, the length of the period of its least imperfections is the same as it would have been, if the other sound had been equally tempered flat; and when either of the given sounds is equally tempered sharp, the period will be of another given length; and if the flat and sharp temperaments be equal, the former equal periods will be longer than the latter, in the ratio of the vibrations which terminate the given temperament.

Fig. 38. For let t and $\frac{n}{m}t$ be the vibrations of the given perfect consonance, and let its flat temperament be the

(*a*) Sect. III. Art. 5 and the Table of the order of the simplicity of consonances, also Fig. 3.

the interval of the sounds of T and t, or of $\frac{n}{m}$T and $\frac{n}{m}t$. Then is the period of T and $\frac{n}{m}t$ = Per. t and $\frac{n}{m}$T = $\frac{1}{m}$ × Per. T and t (*a*).

Again if the interval of the sounds of t and v, or of $\frac{n}{m}t$ and $\frac{n}{m}v$, be the sharp temperament of the same consonance of t and $\frac{n}{m}t$; then the period of t and $\frac{n}{m}v$ = Per. v and $\frac{n}{m}t$ = $\frac{1}{m}$ × Per. t and v (*a*).

And the flat and sharp temperaments being supposed equal, we have $T : t :: t : v$, whence $\frac{1}{m}$ × Per. T and t : $\frac{1}{m}$ × Per. t and v :: T : t :: t : v (*b*). Q.E.D.

Coroll. 1. The given vibrations of any perfect consonance being t and $\frac{n}{m}t$, the period of its least imperfections, when tempered flat or sharp by any part

(*a*) Prop. IX. (*b*) Prop. VII. coroll. 4.

part or parts of a comma, denoted by $\frac{q}{p}$, will be respectively $\frac{161p+q}{2q} \times \frac{t}{m}$ or $\frac{161p-q}{2q} \times \frac{t}{m}$ very nearly.

For suppoſing thoſe equal temperaments to be terminated by the vibrations *T* and *t*, *t* and *v*, we have $T : t :: t : v :: 161p+q : 161p-q$ (*c*). Whence the

(*c*) Fig. 38. N°. 2. *If any given part, parts or multiple* ef *of the difference* ad *of the terms* ao, do *of any ſmall given ratio, be placed in the middle of it, the magnitude of the ratio* eo *to* fo *will be the ſame part, parts or multiple of the magnitude of the ratio* ao *to* do, *as* ef *is of* ad *very nearly.*

For it would be ſo very nearly, if *ef* were placed contiguous to either end of the difference *ad*, as at *ab* or *cd*, (by N°. 1. note (*b*) pag. 89.) in which caſe the magnitude of the ratio *ao* to *bo* is ſomething leſs, and that of *co* to *do* is ſomething greater, than the magnitude of *eo* to *fo*; which therefore approaches ſtill nearer to the truth, and gives the following algebraical theorem.

If s *be the half ſum and* d *the half difference of the terms of any ſmall given ratio, and any fraction, proper or improper, be* $\frac{q}{p}$, *the magnitude of the ratio* $s+\frac{q}{p}d$ *to* $s-\frac{q}{p}d$, *is to that of the given ratio, as* $\frac{q}{p}$ *to* 1 *very nearly.*

Hence,

the cycle of the pulses of T and t is $\overline{161p+q}\times t=\overline{161p-q}\times T$, and their period is $\frac{161p+q}{2q}t$. Likewise the cycle of the pulses of t and v is $\overline{161p-q}\times t = \overline{161p+q}\times v$, and their period is $\frac{161p-q}{2q}t$. And these periods severally divided by the denominator m, are the periods of the least imperfections of the given consonance, tempered flat or sharp respectively, by prop. IX.

Coroll. 2. Whatever be the situation of a small given temperament, the period of the tempered consonance will be exactly, or else very nearly of the same length. For instance, if the temperament be a comma in dissimilar situations, the ratio of the periods will be

Hence, the ratio of the vibrations of any two sounds, whose interval is any given part or parts of a comma, denoted by $\frac{q}{p}$, is $161p+q$ to $161p-q$ very nearly.

For the terms of the given ratio belonging to the comma are 81 and 80, whence $s=\frac{161}{2}$ and $d=\frac{1}{2}$ and be 81 to 80, if $\frac{1}{4}$ comma, 645 to 643 very nearly.

PHÆNOMENON.

In tuning musical instruments, especially organs, it is a known thing that while a consonance is imperfect, it is not smooth and uniform as when perfect, but interrupted with very sensible Undulations or Beats; which while the two sounds continue at the same pitch, succeed one another in equal times, and in longer and longer times while either of the sounds approaches gradually to a perfect consonance with the other; till at last the Undulations vanish and leave a smooth, uniform consonance.

Quick-

and $s+\frac{q}{p}d : s-\frac{q}{p}d :: \frac{161}{2}+\frac{q}{p}\times\frac{1}{2} : \frac{161}{2}-\frac{q}{p}\times\frac{1}{2}$ $:: 161p+q : 161p-q$.

For example, when $\frac{q}{p}=\frac{1}{4}$, that ratio is 645 to 643, and the following logarithms shew how very near that theorem approaches to exactness.

The

Quicker undulations are beats, and are remarkably disagreeable in a concert of strong, treble voices, when some of them are out of tune; or in a ring of bells ill tuned, the hearer being near the steeple; or in a full organ badly tuned: Nor can the best tuning wholly prevent that disagreeable battering of the ears with a constant rattling noise of beats, quite different from all musical sounds, and destructive of them, and chiefly caused by the compound stops called the Cornet and Sesquialter, and by all other loud stops of a high pitch, when mixed with the rest. But if we be content with compositions of unisons and octaves to the Diapason, whatever be the quality of their sounds, the best manner of tuning will render the noise of their

The log. $\frac{81}{80} = 0.\ 00539.\ 503$

$\frac{1}{4}$ of it ...... $= 0.\ 00134.\ 876$

The log. $\frac{645}{643} = 0.\ 00134.\ 874$

their beats inoffenſive if not imperceptible.

PROPOSITION XI.

The time between any two ſucceſſive beats of any imperfect conſonance, is equal to the periodical time of its leaſt imperfections.

Fig. 23 to 27. 34. 35. Any ſimple cycle or any period of the pulſes of imperfect uniſons, contains one more of the ſhorter than of the longer vibrations, as appears by the equation to it (*a*): and the ſhort cycle of any imperfect conſonance conſiſts of equal numbers of the longer and ſhorter vibrations of the imperfect uniſons (*b*). Conſequently after taking away the greateſt equal numbers of ſhort cycles, that can be taken from both ends of the ſimple cycle or period of the imperfect uniſons, ſome part of another ſhort cycle or two, as

(*a*) See the demonſtration of Prop. VII.
(*b*) See pag. 96.

as consisting of unequal numbers of the longer and shorter vibrations of the imperfect unisons, will always remain in the middle. And this part, by interrupting the succession of the short cycles, wherein the harmony of the consonance consists, interrupts its harmony and causes the noise which is called a beat: especially as the interruption is made where the short cycles on each side of it are the most imperfect or confused. Therefore the time between two successive beats, made in the middle of each period, or simple cycle of the pulses of the imperfect unisons, or of the least imperfections of the consonances(*a*), is equal to the time of this period.

And the cause of the beats of imperfect unisons is a like interruption of the succession of their short cycles, in the middle of every period or simple cycle of their pulses, where they are most confused (*b*). Q. E. D.

Scho-

(*a*) Prop. VIII.
(*b*) See the 2[d] Schol. following, Note (*c*).

Scholium 1.

The ratio of the numbers of beats made by any two conſonances in any equal times, being the inverſe ratio of the times between their ſucceſſive beats, is alſo the inverſe ratio of the periods of their leaſt imperfections; and is therefore determined in all caſes by the corollaries to prop. IX, which however are tranſlated hither with that alteration, to prevent the frequent repetition of this obſervation.

Coroll. 1. If T and $\frac{n}{m}$ T, V and $\frac{s}{r}$ V be the vibrations of any two perfect conſonances, when they are tempered the numbers of their beats, made in equal times, will be in the ratio compoſed of the inverſe ratio of T to V, the direct ratio of the denominators of the fractions $\frac{n}{m}$, $\frac{s}{r}$, and the direct ratio of the temperaments.

Coroll. 2. Hence when the temperaments are equal and the fractions $\frac{n}{m}$, $\frac{s}{r}$ are the ſame, or their denominators only, the numbers of beats, made in equal times, are in the inverſe ratio of the homologous vibrations T, V: or in the direct ratio of the numbers of theſe vibrations made in equal times.

Coroll. 3. If T = V, that is, if the conſonances have a common ſound, the numbers of their beats, made in equal times, are in the ratio compoſed of the direct ratio of the denominators of the fractions $\frac{n}{m}$, $\frac{s}{r}$, and the direct ratio of the temperaments, by coroll. 1.

Coroll. 4. Conſequently when the denominators are the ſame, the numbers of beats, made in equal times, are in the ratio of the temperaments.

Coroll. 5. And when the temperaments are the ſame or equal, the numbers of the beats, made in equal times, are

are in the direct ratio of the denominators, m, r.

Coroll. 6. Therefore ſimpler conſonances generally beat ſlower than the leſs ſimple do when equally tempered, the denominators belonging to the ſimpler being generally ſmaller (*a*).

Coroll. 7. Let the ſingle vibrations of any perfect conſonances be t and $\frac{n}{m}t$, and the numbers of them made in one ſecond of time, be N and M reſpectively; then the number of the beats of that conſonance, when tempered flat by any part or parts of a comma denoted by $\frac{q}{p}$, will be $\frac{2q}{161p+q} \times m\text{N}$, or $\frac{2q}{161p+q} \times \text{M}n$ in one ſecond of time very nearly; and when tempered ſharp by the ſame quantity, the number will be $\frac{2q}{161p-q} \times m\text{N}$, or $\frac{2q}{161p-q} \times \text{M}n$ in one ſecond very nearly.

For

(*a*) Sect. III. Art. 5 and the Table of the order of conſonances: alſo Fig. 3.

For the time between the successive beats, being equal to the period of the least imperfections of the tempered consonance (*a*), is $\frac{161p+q}{2q} \times \frac{t}{m}$ in the first case, and $\frac{161p-q}{2q} \times \frac{t}{m}$ in the second (*b*); that is (because $t = \frac{1''}{N}$ and $\frac{n}{m}t = \frac{1''}{M}$ by hypothesis) $\frac{161p+q}{2q} \times \frac{1''}{mN}$ or $\frac{161p+q}{2q} \times \frac{1''}{Mn}$ in the first case, and $\frac{161p-q}{2q} \times \frac{1''}{mN}$ or $\frac{161p-q}{2q} \times \frac{1''}{Mn}$ in the second: and the number of beats in one second, being the reciprocal of the time between the successive beats, is the reciprocal of that quantity in each case, as above.

Coroll. 8. All the foregoing corollaries will also follow from this last, by considering

(*a*) Prop. XI. (*b*) Prop. X. Coroll. 1.

(*c*) M. *Sauveur* ayant cherché la cause de ce Phenomene, a imaginé avec une extrême vraisemblance, que le son des deux tuyaux ensemble devoit avoir plus de force, quand leurs vibrations, après avoir été quelque temps separées, venoient à se réunir et s'accordoient à frapper l' oreille d'un même coup. Histoire de l'Acad.

ſidering every quantity in thoſe theorems as indeterminate, and by obſerving that the quantity $\frac{2q}{161p+q}$ or $\frac{2q}{161p-q}$ is as the temperament of the conſonance very nearly, by the notes in pag. 91 and 106.

Scholium 2.

Mr. *Sauveur* is the only writer I know of that takes any notice of the beats of conſonances. He imagined that they beat at every coincidence of their pulſes (*c*), and obſerving that he could diſtinguiſh the beats pretty well when they went no quicker than 6 in one ſecond, and ſtill plainer when they went ſlower, he concluded that we could not perceive them at all when they went faſter than at that rate (*d*); and

l'Acad. Royale des Sciences, année 1700, pag. 171. 8vo.

(*d*) Donc dans tous les accords où les vibrations ſe rencontreront plus de 6 fois par ſeconde, on ne ſentira point de battemens, et on les ſentira au contraire avec d'autant plus de facilité que les vibrations ſe rencontreront moins de 6 fois par ſeconde. ibid. pag. 176.

and thence he inferred that octaves and other ſimple concords, whoſe vibrations coincide very often, are agreeable and pleaſant becauſe their beats are too quick to be diſtinguiſhed, be the pitch of the ſounds ever ſo low; and on the contrary, that the more complex conſonances whoſe vibrations coincide ſeldomer, are diſagreeable becauſe we can diſtinguiſh their ſlow beats; which diſpleaſe the ear, ſays he, by reaſon of the inequality of the ſound (*a*). And in purſuing this thought he found, that thoſe conſonances which beat faſter than 6 times in a ſecond, are the very ſame that muſicians treat as concords; and that others which beat ſlower are the diſcords; and he adds, that

(*a*) Les battemens ne plaiſent pas à l'oreille, à cauſe de l'inégalité du ſon, et l'on peut croire avec beaucoup d'apparence, que ce qui rend les octaves ſi agréables, c'eſt qu'on n'y entend jamais de battemens. ibid. pag. 177.

(*b*) En ſuivant cette idée, on trouve que les accords dont on ne peut entendre les battemens, ſont juſtement that when a conſonance is a diſcord at a low pitch and a concord at a high one, it beats ſenſibly at the former pitch but not at the latter (*b*).

As Mr. *Sauveur* appeals to numbers, let us ſee what evidence they produce. The tones and ſevenths major and minor being diſcords, muſt beat ſlower than 6 times in one ſecond by his own hypotheſis. Then let them beat but 4 or 5 times, and it will follow that the major IVth and minor 5th cannot beat above once in a ſecond.

For the lengths of the cycles of perfect conſonances to a common baſe, are proportional to the leſſer terms of the ratios of their vibrations (*c*), which being but 8 and 9 in the former diſcords and

ment ceux que les muſiciens traitent de conſonances, et que ceux dont les battemens ſe font ſentir, ſont les diſſonances; et que quand un accord eſt diſſonance dans une certaine octave, & conſonance dans une autre, c'eſt qu'il bat dans l'une, et qu'il ne bat pas dans l'autre. ibid. pag. 177.

(*c*) Sect. III. Art. 13.

and 32 and 45 in the latter (*a*), ſhew, that the latter muſt beat 4 or 5 times ſlower than the former, that is, as ſlow at leaſt as a clock that beats ſeconds.

But in ſounding the latter diſcords upon an Organ, Harpſichord or Violoncello, even at a low pitch, I find their beats are ſo quick that I cannot count them; or rather they do not beat at all, like imperfect conſonances, but only flutter, at a ſlower or quicker rate according to the pitch of the ſounds.

The truth is, this gentleman confounds the diſtinction between perfect and imperfect conſonances, by comparing imperfect conſonances (*b*) which beat becauſe the ſucceſſion of their ſhort cycles is periodically confuſed and interrupted (*c*), with perfect ones which cannot beat, becauſe the ſucceſſion of their

(*a*) Table of muſical ratios pag. 12.
(*b*) Memoires de l' Acad. 1701, Syſtême general, Sect. XII, maniere de trouver le ſon fixe. pag. 473. 8vo.
(*c*) Dem. of Prop. XI.

their ſhort cycles is never confuſed nor interrupted.

The fluttering roughneſs abovementioned is perceivable in all other perfect conſonances, in a ſmaller degree in proportion as their cycles are ſhorter and ſimpler, and their pitch is higher; and is of a different kind from the ſmoother beats and undulations of tempered conſonances; becauſe we can alter the rate of the latter by altering the temperament, but not of the former, the conſonance being perfect at a given pitch: And becauſe a judicious ear can often hear, at the ſame time, both the flutterings and the beats of a tempered conſonance, ſufficiently diſtinct from each other.

Scholium 3.

In all tempered ſyſtems the times of the vibrations of moſt of the conſonances are incommenſurable quantities.

In the system of mean tones, for instance, the vibrations of the tone are in the ratio of $\sqrt{5}$ to 2, the subduplicate of 5 to 4, as the mean tone is half the III^d. Likewise the vibrations of the V^th tempered by a quarter of a comma, are in the ratio of $\sqrt[4]{5}$ to 1, the subquadruplicate of 5 to 1, as that V^th is a quarter of the XVII^th or 2VIII+III. For as $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{81}{80} = \frac{1}{5}$, so V+V+V+V$-c=$XVII; whence V$-\frac{1}{4}c+V-\frac{1}{4}c+V-\frac{1}{4}c+V-\frac{1}{4}c=$XVII. Lastly the ratio of the vibrations of two sounds whose interval is a quarter of a comma, is $\sqrt[4]{81}$ to $\sqrt[4]{80}$, or $\sqrt[4]{3 \times 3 \times 3 \times 3}$ to $\sqrt[4]{2 \times 2 \times 2 \times 2 \times 5}$, or 3 to $2\sqrt[4]{5}$; and consequently the ratio of the vibrations of any consonance tempered by a quarter of a comma, is also incommensurable, as being composed of the ratio of the vibrations of the perfect consonance

nance, and the ratio of the vibrations belonging to its temperament.

The ſame may be ſaid of any perfect conſonance tempered by any aliquot part or parts of a comma; whoſe vibrations are always incommenſurable, becauſe 81 and 80 are not equal powers of any two numbers whatever (*a*) We may conclude then, that in tempered ſyſtems the vibrations of moſt of the conſonances are incommenſurable, as reſulting from force and motion, which are continued quantities.

Now if the agreeableneſs of conſonances, according to the received principle in Harmonics, be the reſult of the frequent coincidences of their pulſes, and conſequently be greater or leſſer according as the coincidences are more or leſs frequent, all the conſonances in tempered ſyſtems, whoſe vibrations are incommenſurable, ought to be the greateſt diſcords in nature: it being im-

(*a*) See coroll. Prop. 2. VIII. Elem.

impossible for their pulses to coincide more than once in an infinite time. For as no two numbers how large soever, can express the ratio of such vibrations, so no multiple of one vibration can ever be equal to any multiple of the other. And yet experience shews that such consonances are much more agreeable than perfect discords whose pulses coincide very often.

We may approach indeed as near as we please, and certainly much nearer than our senses can distinguish, towards the exact magnitude of an incommensurable ratio, by the ratios of whole numbers; but as these will grow larger and larger without bounds, so will the time between the successive coincidences, or the length of the approximating cycle of the pulses: by which I mean the time of either of the incommensurable vibrations multiplied by the heterologous term of the approximating ratio.

Let

Let any man tell us then where we may ſtop, and which of thoſe cycles it is, whoſe repetition excites the determinate idea of the conſonance.

The like difficulty occurrs in approaching gradually even to a commenſurable ratio of the vibrations of any perfect conſonance. For if either of its vibrations be pretty much altered at once, and then be made to approach by degrees to its former length, the terms of the ſeveral approximating ratios will grow larger and larger without bounds and in regular order, except when ratios occur whoſe terms are reducible; and the cycles of their pulſes will accordingly be longer and longer and their coincidences fewer and fewer without limit, thoſe interruptions excepted; and yet the conſonance will grow better and better by regular degrees till it arrives at perfection, as is certain by experience. For inſtance the ratios 30 to 21, 300 to 201, 3000 to

2001,

2001, &c, approach nearer and nearer to 3 to 2, and the v[ths] whose vibrations are in those ratios grow more and more harmonious, though the cycles of their pulses grow longer and longer to infinity.

It is therefore impossible to account for the phænomena of imperfect consonances upon the principle of coincidences, which indeed is applicable to none but perfect ones. Accordingly Dr. *Wallis* (*a*), Mr. *Euler* (*b*) and others disapprove incommensurable vibrations as impracticable and inharmonious.

But supposing the vibrations V, v of imperfect unisons to be incommensurable

(*a*) It hath been long since demonstrated, that there is no such thing as a just hemitone practicable in music, and the like for the division of a tone into any number of equal parts, three, four or more. For supposing the proportion of a tone or full note to be as 9 to 8, that of the half note must be as $\sqrt{9}$ to $\sqrt{8}$, that is as 3 to $\sqrt{8}$, or as 3 to $2\sqrt{2}$, which are incommensurable quantities; and that of a quarter note as $\sqrt[4]{9}$ to $\sqrt[4]{8}$, which is yet more incommensurate;

rable, or $V : v :: \sqrt{p} : \sqrt{q}$, and x to be an indeterminate vibration, and $V : x :: m : n$, and the ratios of the indeterminate numbers m, n to approach gradually to the given ratio of $\sqrt{p}$ to $\sqrt{q}$; though the length $nV, = mx$, of the indeterminate cycle of the pulſes of V and x, increaſes without bounds, nevertheleſs the length $\frac{n}{m-n}V, = \frac{m}{m-n}x$, of the indeterminate period of their pulſes tends gradually to a determinate limit $\frac{\sqrt{q}}{\sqrt{p}-\sqrt{q}}V = \frac{\sqrt{p}}{\sqrt{p}-\sqrt{q}}v$. And this is the period of the pulſes of the incommenſurable vibrations V, v, which excites the determinate idea of the imperfect

menſurate; and the like for any other number of equal parts: which will therefore never fall in with the proportions of number to number. *Upon the imperfection of an Organ.* Phil. Tranſ. N°. 242, or Abridgm. vol. 1. pag. 705. edit. 1.

(*b*) Denique ob nullam ſonorum rationem rationalem præter octavas, hoc genus [muſicum] harmoniæ maxime contrarium eſt cenſendum; etiamſi hebetiores aures diſcrepantiam vix percipiant. *Tentamen novæ Theoriæ muſicæ*, cap. IX. ſect. 17. Petropoli. 1739.

perfect unisons, be the complex cycle of their pulses ever so long, infinite or impossible.

I say determinate idea. For though the alternate lesser intervals of the pulses in the several successive periods of V and v, even when commensurate, are not precisely equal (*a*), yet it is highly probable that the ear could not distinguish a repetition of any one period from the succession of them all, and seems agreeable to experience in observing the identity of the tone of imperfect unisons held out upon an organ.

For further illustration I will add an example or two. We shewed above that the vibrations V, v of the mean tone are as $\surd 5 : 2 :: 2.23606796 \&c : 2 :: m : n$. Whence the length of the period of the pulses of V and v, is $\frac{n}{m-n}V = \frac{2V}{0.23606796 \&c} = 8.47213 \&c \times V$; which is a medium between 8V and 9V,

(*a*) Coroll. 2. Prop. VII.

9V, the cycles of the pulſes of the major and minor tones, ſomething leſs than the arithmetical, or even the geometrical mean, but not quite ſo little as the harmonical mean between them (*a*).

Again, when V and v are the vibrations of two ſounds whoſe interval is a quarter of a comma, we found $V : v :: 3 : 2\sqrt[4]{5}$ or 2.99069756 &c $:: m : n$; whence the period of the pulſes of V and v is $\frac{n}{m-n}V = \frac{2.99069756 \text{ \&c}}{0.00930244 \text{ \&c}} \times V =$ 321.4960 &c $\times$ V.

Or thus. In approximating towards the ratio of V to v, or 3 to $2\sqrt[4]{5}$, or 3 to 2.990697, or 3000000 to 2990697 by ſmall numbers (*b*), the ratios greater than V to v are 322 to 321, 967 to 964, 1612 to 1607, &c. Whence the cycle 321V and the periods $321\frac{1}{3}$V, $321\frac{2}{5}$V, &c, are all too ſhort.

And

(*a*) See Sect. VII. Def. 11.

(*b*) See Mr. *Cotes*'s Harmonia Menſurarum, prop. 1. Schol. 3.

And the ratios leſs than V to v being 323 to 322, 645 to 643, &c, the cycle 322V and periods $321\frac{1}{2}$V, &c, are all too long. Therefore the true period falls between the laſt mentioned limits, agreeably to the former computation.

From what has been ſaid of imperfect uniſons the difficulty vaniſhes in other imperfect conſonances, by obſerving the reduction of the periods of their imperfections to thoſe of imperfect uniſons, as in prop. VIII and IX.

Scholium 4.

Having obſerved a very ſtrict analogy between the undulations of audible and viſible objects, I will here deſcribe it, as an illuſtration of the foregoing theory of imperfect conſonances.

Fig. 39. Let the points a, b, c, &c and α, β, γ, &c repreſent the places of two parallel rows of equidiſtant and parallel objects, ſuch as pales, palliſadoes, &c, and

and let them be viewed from any large diſtance by an eye at any point z. In a plane paſſing through the eye and cutting the axes of the parallel objects at right angles in the points a, b, c, &c, α, β, γ, &c, let lines drawn from z through α, β, γ, &c, cut the line of the other row in A, B, C, &c. Then by the ſimilar triangles ABz and $\alpha\beta z$, BCz and $\beta\gamma z$, CDz and $\gamma\delta z$, &c, we have $AB : \alpha\beta :: (Bz : \beta z ::) \; BC : \beta\gamma :: (Cz : \gamma z) :: CD : \gamma\delta ::$ &c. Therefore the antecedents AB, BC, CD, &c, which are to the equal conſequents $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, &c in the ſame ratio, are alſo equal to one another, and are the apparent projections of the conſequents upon the line abc of the other row.

Hence ſuppoſing m and n to repreſent the leaſt whole numbers in the given ratio of AB to ab, we have a line $m \times ab = n \times AB$, equal to the length of the cycle between the apparent coincidences of ſome of the objects in one row with ſome in the other; as of α

and a at A, of n and m at K, &c: and if $m-n$ be not an unit we have a shorter line $\frac{m}{m-n}ab = \frac{n}{m-n}AB = AX$ or XK, equal to the length of the apparent period of their nearest approaches towards coincidences; as on each side of the point X, according to the demonstration of the VII[th] proposition.

But if the point z be so situated, that the lines AB and ab or $\alpha\beta$, or Bz and βz, or Cz and γz, &c, which are all in the same ratio, happen to be incommensurable, it will be impossible, mathematically speaking, for more than one couple of objects to appear coincident (a), and yet the periods of their apparent approaches will subsist in this case as well as in the other.

Now if the objects be white, or of any colour that reflects more light to the eye than what comes to it from the spaces between them, and their breadth be considerable as usual, the rows will appear

(a) See pag. 122. line 3.

appear the leaſt luminous about the coincident objects and the periodical points, *A*, *X*, *K*, &c, where the objects of the nearer row hide the whole or ſome part of thoſe behind them in the remoter row; and the rows will appear gradually more luminous towards the middle of the periods, where the objects will be ſeen diſtinct from one another if they be not too broad. And the contrary will happen if the objects in the rows be leſs luminous than the ſpaces between them.

Conſequently if the ſpectator ſtands ſtill and moves his eye from one end of the rows to the other, he will ſee an alternate ſucceſſion of light and ſhade; and while he moves forwards in any tranſverſe direction $z\omega$, and fixes his eye upon a given place of the rows, he will then ſee an undulation of light and ſhade, moving forwards quicker or ſlower according to the celerity of his own motion.

For then the apparent coincidences which were at *A*, *K*, &c, and consequently the intermediate periodical points *X*, *Y*, &c, will gradually shift from *A* to *B*, &c, and from *K* to *L*, &c, as is evident from the angular motion of the visual rays about the fixt points or objects α, β, γ, &c, κ, λ, μ, &c: And this is a known phænomenon.

If the spectator recedes from the rows, the period $\frac{m}{m-n}ab$ will grow longer, and then upon moving transversely, the visible undulations will be broader and slower than before, and at a very great distance from the rows, will become imperceptible; as being changed into an uniform appearance of both rows in the place of one: Quite analogous to the audible undulations of imperfect unisons, as they grow slower and less perceptible while the unisons are approaching to perfection.

The like phænomenon results from two rows of pales that meet in any angle.

PRO-

PROPOSITION XII.

Imperfect consonances of the same sort or name are equally harmonious when their short cycles are equally numerous in the periods of their imperfections.

If all the tempered intervals of any one sort or name be equal to one another in any system, those of any other sort will be every where equal and equally tempered. For instance, if the intervals between any two octaves be equal, the interval between their bases will be equal to the interval between their trebles; which is plain from the ratios of the times C, c, G, g of their single vibrations. For if $C : c :: G : g$, then $C : G :: c : g$. But if the former ratios and intervals be unequal, the latter will be so too: and conversely.

Therefore as it is universally agreed, and with good reason too (*a*), to have all the octaves equal by making them perfect, it appears that all tempered intervals of the same name must have equal temperaments; these being the diffe-

(*a*) See Prop. XVI. Schol. 2, Art. 12.

differences of the tempered and perfect intervals.

Consequently the lengths of the periods of the least imperfections of all consonances of the same name, are in the ratio of the times of the single vibrations of their bases or trebles (*a*), or of the lengths of their short cycles, as being equimultiples of the times of the vibrations. The short cycles are therefore equally numerous in the periods of their imperfections.

And whatever be the quantity of the temperament of any consonances of the same name, as of v[ths] for instance, I believe no musician ever doubted their being equally harmonious, or agreeable to his ear, in every place of the scale, when well tuned. Q. E. D.

Coroll. Consonances of the same name are equally harmonious when equally tempered.

Scholium.

After an organ had been well tuned by making all the tempered v[ths] as equally

(*a*) Prop. IX. coroll. 2.

qually harmonious as the ear could determine, I found that the numbers of their beats, made in equal times, were in the inverſe ratio of the times of the ſingle vibrations of their baſes or trebles, as nearly as could be expected: or that the times between their ſucceſſive beats, which are equal to the periods of their leaſt imperfections (*a*), were in the direct ratio of thoſe homologous vibrations, or of equimultiples of them, or of the lengths of the ſhort cycles; which therefore were equally numerous in thoſe periods.

PROPOSITION XIII.

Imperfect conſonances of different ſorts are equally harmonious, in their kind, when their ſhort cycles are equally numerous in the periods of their imperfections.

Fig. 34. The times of the ſingle vibrations of imperfect uniſons being repreſented by *AB* and *ab*, let *AD* and *ac*,

(*a*) Prop. XI.

ac, that is 3*AB* and 2*ab* be those of imperfect v[ths]. And one length of their imperfect short cycle being 2*AD*=*AG*, and the other being 3*ac*=*ag*, their difference *Gg* is the dislocation of the pulses *G*, *g* at the end of the first short cycle *AagG*, measured from the coincident pulses *Aa*. And the greater of the two dislocations which terminate the several succeeding cycles, is double, triple, &c of *Gg* (*a*).

Again, conceiving the pulses *c*, *g*, *l*, &c, to be now intermitted, let *AD* and *ae*, that is 3*AB* and 4*ab* be the single vibrations of imperfect 4[ths]. And the two lengths of their first short cycle *ANna* being 4*AD*=*AN* and 3*ae*=*an*, their difference *Nn* is the dislocation of the pulses *N*, *n* at the end of that cycle; and in the several succeeding cycles the greater of the two dislocations is double, triple, &c of *Nn*.

And the common period *AZ* or *an* of those dislocations or imperfections in the short cycles of the v[ths] and 4[ths], is

(*a*) Prop. VII.

is the ſame as the period or ſimple cycle of the pulſes of the vibrations *AB*, *ab* of the imperfect uniſons (*b*).

Now the two diſlocations *Gg*, *Nn*, in the firſt imperfect cycles of the v^{ths} and 4^{ths} in that period, are in the ratio of *AG* to *AN* (*c*), the lengths of the cycles, that is of 2*AD* to 4*AD*, or 1 to 2: and the two greater diſlocations *Xy*, *Qr*, in the laſt imperfect cycles *Xyε*Δ, *Qrε*Δ, in the ſame period *AZ*, are in the ratio of their diſtances *ZX*, *ZQ*, from this end of it: and this ratio is leſs than that of Δ*X* to Δ*Q*, or 1 to 2. But the two greater diſlocations *K*λ, Πσ in the ſubſequent cycles *K*λεΔ, ΠσεΔ, of the next period, are in the ratio of *ZK* to *Z*Π, which, on the contrary, is greater than that of Δ*K* to ΔΠ, or 1 to 2.

The periods muſt be conceived to contain a much greater number of ſhort cycles than can be well repreſented in a ſcheme. And then, as the correſponding diſlocations in the v^{ths} and 4^{ths} lye farther and farther from *Z*, the ratio of

(*b*) Prop. VIII. (*c*) Prop. VII.

their

their distances and magnitudes will approach nearer and nearer to 1 to 2.

Therefore 1 to 2, or the ratio of the lengths of the short cycles of the v[ths] and 4[ths], is either the exact or the mean ratio both of the greater and the lesser dislocations in all their corresponding short cycles: because the lesser of the increasing dislocations in any subsequent cycle, is the same as the greater in the antecedent one.

Now while the length *AG* or *ag* remains unaltered, imagine the dislocation *Gg* of the v[ths] to be increased in that ratio of 1 to 2, and then it will be equal to the former magnitude of the dislocation *Nn* of the 4[ths], or to *Nn* in Fig. 35, supposing the pulses *C*, *G*, *L*, &c to be absent. And the first dislocation *Bb* of the pulses *B*, *b*, of the imperfect unisons, being at the same time increased in the same ratio, their period *AZ*, which is also that of the dislocations in the v[ths] (*a*), will be diminished in that ratio inverted (*b*). And thus the present period

(*a*) Prop. VIII.
(*b*) Prop. VII. coroll. 5 and the note to it.

period of the imperfect v^{ths} and the former period of the 4^{ths}, are in the ratio of the lengths of their ſhort cycles; which therefore are equally numerous in their reſpective periods.

And ſince the greater and leſſer diſlocations at the ends of the correſponding ſubſequent ſhort cycles of the v^{ths} and 4^{ths}, are now reſpectively equal, either exactly or at a medium of one with another, and equally numerous too, the whole periods compoſed of theſe ſhort cycles, will be equally harmonious. Becauſe thoſe equal diſlocations of the pulſes in the correſponding ſhort cycles, are the cauſes that ſpoil their harmony: and cauſes conſtantly equal will have equal effects.

The concluſion will be the ſame if the diſlocation *Nn*, in the firſt cycle of the 4^{ths} in either figure, be contracted to the magnitude of the diſlocation *Gg* belonging to the v^{ths} in the other. For then the new period of the 4^{ths}, being double of the old one (*a*), will be

(*a*) Prop. VII. coroll. 5 and the note to it.

to

to the old one, or that of the v[ths], as *AN* to *AG*, that is, in the ratio of the lengths of their ſhort cycles, which therefore are equally numerous in theſe periods: and the diſlocations at the ends of the ſeveral ſubſequent ſhort cycles of the 4[ths], being likewiſe contracted to the reſpective magnitudes of thoſe of the v[ths], the conſonances are again made equally harmonious.

And laſtly, ſince either of thoſe conſonances is equally harmonious to another of the ſame name, at any other pitch, when their ſhort cycles are equally numerous in their periods (*a*), it appears that 4[ths] and v[ths] are equally harmonious at any pitches, when their ſhort cycles are equally numerous in their periods. And the like proof is plainly applicable to any other caſe of theſe or any other conſonances: I mean when their common period, or that of the imperfect uniſons, is terminated at firſt either by coincident pulſes or periodical points, as will plainly appear by conceiving a ſhort cycle or two to re-

(*a*) Prop. XII.

refult from a proper intermiffion of the pulfes of imperfect unifons, on each fide of fuch points in fig. 24. 25. Q. E. D.

Coroll. 1. Imperfect confonances are more harmonious in the fame order as their fhort cycles are more numerous in the periods of their imperfections.

For let any two imperfect confonances be fuppofed equally harmonious; and their fhort cycles will be equally numerous in their periods, by the propofition. Then if either of the given periods be lengthened, the fhort cycles will be more numerous in it, and the diflocations of their pulfes will be fmaller than they were before, and confequently will firft increafe and then decreafe by fmaller degrees from one end of the period to the other. And thus the confonance will be more harmonious than it was at firft, or than the other given confonance.

And on the contrary, if the period of either of them be fhortened, the number of its ſhort cycles will be diminiſhed, and the diſlocations of their pulſes will be increaſed. And thus the conſonance will be leſs harmonious than it was before, or than the other given conſonance.

Coroll. 2. Imperfect conſonances are more harmonious in the ſame order, as their temperaments multiplied by both the terms of the ratios of the vibrations of the perfect conſonances, are ſmaller; and are equally harmonious when thoſe products are equal.

Fig. 34. 35. The vibrations of imperfect uniſons being AB and ab, and r and s being the terms of any muſical ratio, rAB and sab will be the ſingle vibrations of the tempered conſonance; and the period of its imperfections will be the ſame as that of the pulſes of AB and ab (*a*); which period, ſuppoſing $AB : ab :: m : n$ in the leaſt integers, and $d = m - n$, is $\frac{n}{d} AB$. Call it p.

And

(*a*) Prop. VIII.

And ſince $rAB : sAB :: r : s$, the ſhort cycle of the vibrations rAB, sAB of the perfect conſonance, is $rsAB$. Call it c.

Then $\frac{p}{c} = \frac{nAB}{d} \times \frac{1}{rsAB} = \frac{1}{rs} \times \frac{n}{d} = \frac{1}{rs} \times \frac{1}{t}$ by taking $t = \frac{d}{n}$, which being the difference of the terms of the ſmall ratio of AB to ab divided by one of them, is very nearly the meaſure of that ratio (*a*), or of the ratio of the vibrations rAB to rab, or of sAB to sab, or of the interval of their ſounds (*b*), which is the temperament of the imperfect conſonance.

Therefore in the ſame order in which the values of $\frac{p}{c}$ or $\frac{1}{rst}$ are greater, or the values of rst are ſmaller, the correſponding conſonances are more harmonious, by corol. 1; and are equally harmonious when the values of rst are equal, by the preſent propoſition.

Coroll.

(*a*) Note in page 91. n°. 2. (*b*) Sect. 1. Art. 10.

Coroll. 3. Consequently imperfect consonances are equally harmonious when their temperaments have the inverse ratio of the products of the terms of the ratios of the vibrations of the perfect consonances.

For when the values of the product $rs \times t$ are equal, the values of t have the inverse ratio of the values of rs.

Coroll. 4. When the products rs of the terms of those musical ratios are equal, the tempered consonances are more harmonious in the same order as their temperaments are smaller; and are equally harmonious if their temperaments be equal.

For if the values of rs or $\frac{1}{rs}$ be equal, the values of $\frac{p}{c} = \frac{1}{rs} \times \frac{1}{t}$ are greater in the same order as those of $\frac{1}{t}$ are greater, or as those of t are smaller; and are equal when the values of t are equal.

Coroll. 5. Therefore consonances of the same name are more harmonious in in the ſame order as their temperaments are ſmaller; and are equally harmonious when they are equal.

For the vibrations of thoſe conſonances are in the ſame ratio, and the terms of it, being the leaſt poſſible, are the ſame, and their product the ſame alſo, as in the laſt corollary.

Coroll. 6. Conſonances equally tempered are more harmonious in the ſame order as the products of the terms of their muſical ratios are ſmaller; and are equally harmonious when thoſe products are equal.

For the values of t being ſuppoſed equal, thoſe of $\frac{p}{c} = \frac{1}{rs} \times \frac{1}{t}$ are greater in the ſame order as the values of $\frac{1}{rs}$ are greater, or as thoſe of rs are ſmaller; and the former values are equal when the latter are ſo.

Coroll. 7. Conſonances equally tempered are generally more harmonious in the ſame order as they are ſimpler, the pure ones chiefly excepted (*a*), which are more harmonious than fome others that are fimpler; though feperately confidered they follow that order exactly.

This will appear from the laft corollary by a feries of the products of the terms of the ratios in the firft column, compared with the feries of numbers in the fecond column of the table in Sect. III. Art. 5, fhewing the order of the fimplicity of confonances.

Coroll. 8. Confequently fimpler confonances will generally bear greater temperaments than the lefs fimple will; or the lefs fimple ones generally fpeaking will not bear fo great temperaments as the fimpler will: contrary to the common opinion (*b*).

Coroll.

(*a*) Sect. III. Art. 8.

(*b*) Octavarum autem omnium unica eft fpecies, eaque perfecta ratione 1 ad 2 contenta. Hoc enim intervallum, propter perfectionem, vix aberrationem à ratione 1 ad 2 pati poffet, quin fimul auditus ingenti moleftia afficeretur. Namque quo perfectius per-

Coroll. 9. The tempered concords in the ſyſtem of mean tones (*c*) are not equally harmonious in their kinds.

For by coroll. 6, and by inſpection of the terms of the muſical ratios annexed to the characters of the concords in the firſt of the tables in the next ſection, it will appear, that the V[th] and 4[th] and their compounds with VIII[ths], are more harmonious than the VI[th] and 3[d] and their compounds with equal numbers of VIII[ths], as being all equally tempered in that ſyſtem (*d*).

Coroll. 10. The harmony of thoſe concords is ſtill more unequal in the *Hugenian* ſyſtem, reſulting from a diviſion of the octave into 31 equal intervals (*e*).

Be-

perceptuque facilius eſt intervallum, eo magis ſenſibilis fit error minimus; minus autem ſentitur exigua aberratio in intervallis minus perfectis. *Tentamen novæ Theoriæ muſicæ*, cap. IX. ſect. 10. *Petropoli* 1739.

(*c*) Prop. II. (*d*) Prop. III. coroll. 3.

(*e*) See Prop. XVII, ſcholium.

Becauſe the common temperament of the VIth and 3^{d} and their compounds with VIII^{ths}, which by coroll. 4 and 9, ſhould be ſmaller than that of the Vth and 4th and their compounds with VIII^{ths}, to render them equally harmonious, is on the contrary ſomething greater.

Coroll. 11. Imperfect conſonances are more harmonious both as they beat ſlower, and as the cycles of the perfect conſonances are ſhorter.

For the quantities $\frac{t}{c}$ will be greater on both accounts (*a*) and the harmony better (*b*).

Coroll. 12. Conſonances to a common baſe whoſe vibration is denoted by 1, as in Fig. 3, are more harmonious in the ſame order as the numbers of their beats, made in equal times and multiplied by the numerators of the fractions $\frac{s}{r}$, denoting the vibrations of the trebles, are ſmaller.

For

(*a*) Prop. XI. (*b*) Prop. XII and XIII.

For the vibrations of the baſe and treble being 1 and $\frac{s}{r}$, and 1 being to $\frac{s}{r}$ as r to s, the ſhort cycle of their vibrations is $s \times 1 = c$: and putting β for the number of beats made in any given time, we have the period p as $\frac{1}{\beta}$: becauſe p is equal to the time between the ſucceſſive beats (*a*), which is inverſely as β or directly as $\frac{1}{\beta}$. Therefore the indeterminate quantity $\frac{p}{c}$ is as $\frac{1}{\beta c}$ or as $\frac{1}{\beta s}$; that is, the values of the quantity $\frac{p}{c}$ have conſtantly the ſame ratio as thoſe of $\frac{1}{\beta s}$ have.

Coroll. 13. And univerſally, if the vibrations of the baſes and trebles of ſeveral perfect conſonances be indeterminately repreſented by V and $\frac{s}{r}$ V, when they are tempered their harmony

(*a*) Prop. XI.

ny will be as $\frac{1}{\beta s V}$; that is, it will be better in the same order as the values of $\frac{1}{\beta s V}$ are greater, or as those of $\beta s V$ are smaller; and equally good when they are equal.

For now $sV = c$, because $V : \frac{s}{r} V :: r : s$.

SECTION VII.

Of a system of sounds wherein as many concords as possible, at a medium of one with another, shall be equally and the most harmonious.

DEFINITION I.

The Arithmetical Mean among any number of quantities, is to the sum of them under their given signs, as an unit is to their number; and has the same sign as their sum has: Or if they be expressed by numbers, it is the quotient of their sum divided by their number.

Thus

Thus the arithmetical mean among the quantities $a, b, c, -d$ is $\frac{a+b+c-d}{4}$.

Coroll. 1. Hence the ſum of the exceſſes of all the greater quantities above their arithmetical mean, is equal to the ſum of the defects of all the ſmaller from the ſame.

For let the arithmetical mean $\frac{a+b+c-d}{4} = r$, then $a+b+c-d=4r=r+r+r+r$. Whence if a and b be ſeverally greater than r, we have $a-r+b-r=r-c+r+d$.

Fig. 40. Accordingly if the lines ao, bo, co, $-do$ and ro, be proportional to $a, b, c, -d$ and r, the ſum of the parts ra, rb on one ſide of the point r, is equal to the ſum of the parts rc, rd on the other ſide of it.

Coroll. 2. If any quantity q be added to, or taken from every one of the quantities $a, b, c, -d$, their arithmetical mean will be augmented or diminiſhed by that quantity q.

For let $a+b+c-d=4r$, then r is their arithmetical mean. But $a+q+b+q+c+q-d+q=4r+4q=4\times\overline{r+q}$, and therefore $r+q$ is the aritmetical mean among thoſe augmented quantities $a+q, b+q, c+q, -d+q$: and by changing the ſign of every q, it appears that $r-q$ is the like mean among the diminiſhed quantities $a-q, b-q, c-q, -d-q$.

Coroll. 3. If every one of the quantities $a, b, c, -d$, be increaſed or diminiſhed in any given ratio of 1 to n, their arithmetical mean will alſo be increaſed or diminiſhed in the ſame given ratio.

For let $a+b+c-d=4r$, then r is their arithmetical mean. But $na+nb+nc-nd=4nr$, and therefore nr is the arithmetical mean among the quantities $na, nb, nc, -nd$.

DE-

DEFINITION. II.

The Harmonical Mean among any number of quantities, is the reciprocal of the arithmetical mean among their reciprocals.

Fig. 41. In an hyperbola, for inſtance, where the ordinates parallel to an aſymptote are the reciprocals of their abſciſſes, meaſured from the center upon the other aſymptote; if an abſciſs *ro* be the arithmetical mean among the abſciſſes *ao*, *bo*, *co*, *do*, its ordinate $r\varrho$ is the harmonical mean among the ordinates $a\alpha$, $b\beta$, $c\gamma$, $d\delta$ by the definition, $r\varrho$ being the reciprocal of the arithmetical mean *ro* among their reciprocals, *ao*, *bo*, *co*, *do*.

Likewiſe if an ordinate $m\mu$ be the arithmetical mean among the ordinates $a\alpha$, $b\beta$, $c\gamma$, $d\delta$, its abſciſs *mo* is the harmonical mean among their abſciſſes, *ao*, *bo*, *co*, *do*.

Coroll.

Coroll. 1. The arithmetical mean is greater than the harmonical mean among the same quantities, if they all have the same sign.

For let the line $\beta\varrho$, produced through the top of either of the ordinates next to $r\varrho$, cut the rest in f, g, h, and the asymptote ro in e. Then because ro is the arithmetical mean among ao, bo, co, do, the line re is the arithmetical mean among the lines ae, be, ce, de (*a*); and $r\varrho$ the arithmetical mean among the proportional lines af, $b\beta$, cg, dh (*b*), which, excepting the common ordinate $b\beta$, are severally smaller than the hyperbolical ordinates $a\alpha$, $b\beta$, $c\gamma$, $d\delta$; whose arithmetical mean $m\mu$ is therefore greater than $r\varrho$ (*c*), the harmonical mean among the same ordinates.

Coroll. 2. The difference between the arithmetical and the harmonical means among the same quantities, will be very

(*a*) Defin. 1. coroll. 1 or 2. (*b*) Defin. 1. coroll. 3.
(*c*) Defin. 1.

very ſmall when the differences of the quantities themſelves are ſo.

This will appear by conceiving the ordinates $a\alpha$, $b\beta$, $c\gamma$, $d\delta$ to approach gradually towards one another till they coincide. For then the differences between the hyperbolical ordinates $a\alpha$, $b\beta$, $c\gamma$, $d\delta$, and the lines af, $b\beta$, cg, dh, and conſequently between their arithmetical means $m\mu$, $r\varrho$, will gradually decreaſe to nothing. But $r\varrho$ is alſo the harmonical mean among thoſe ordinates.

Coroll. 3. By increaſing every quantity in any given ratio, the harmonical mean among them will be increaſed in the ſame ratio.

For the reciprocals of the increaſed quantities, and the arithmetical mean among them (*a*), will ſeverally be diminiſhed in that ratio; and the reciprocal of this mean, which is the harmonical mean among the increaſed quan-

(*a*) Defin. 1. coroll. 3.

quantities, will of consequence be increased in the same ratio.

Coroll. 4. Fig. 42, 43. Whatever be the signs of the proposed quantities *aα*, *bβ*, *cγ*, *dδ*, their harmonical mean *rρ* has always the sign of the sum of their reciprocals *ao*, *bo*, *co*, *do*, or of, *ro*, the arithmetical mean among them.

For the reciprocal of each quantity has the sign of the quantity it self, and according as their sum is affirmative or negative, so is their arithmetical mean (*b*), and so is its reciprocal, or the harmonical mean among the proposed quantities.

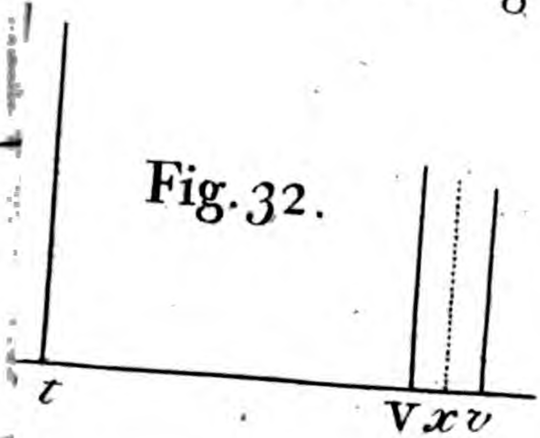
PROPOSITION XIV.

Instead of several imperfect concords resulting from the same perfect one differently tempered, if it be necessary to use but one, let the period of its imperfections be the arithmetical mean among all the periods of those concords, and it will best answer the several purposes of every one.

(*a*) Defin. 1.

Be-

Fig. 32.


t
V x v

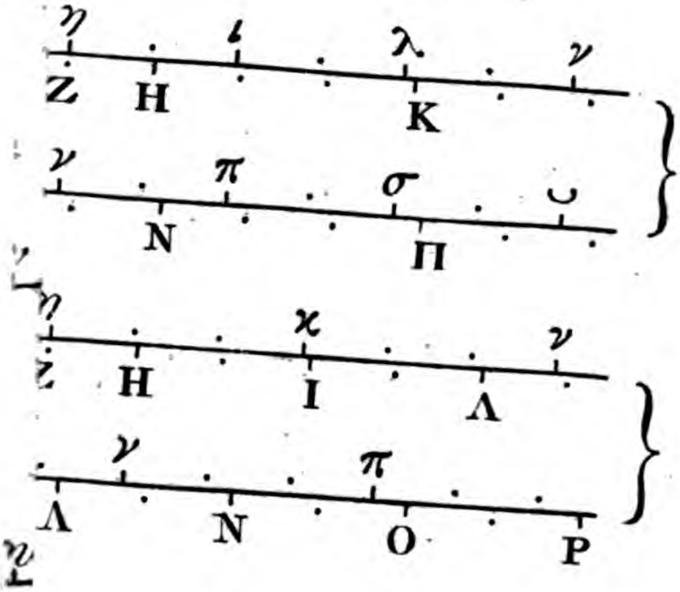
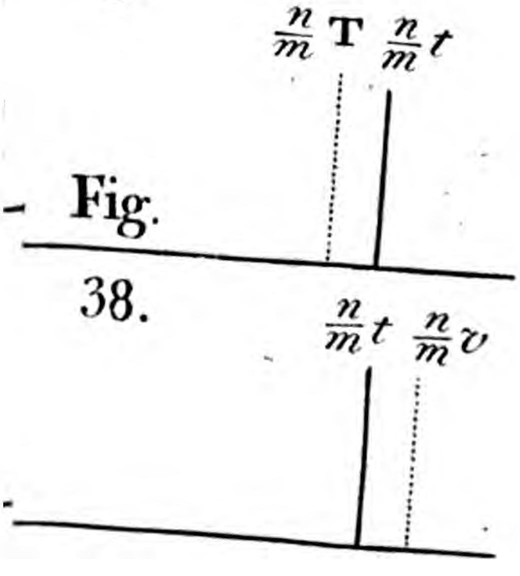

η ι λ ν
Z H K
ν π σ
N Π
η κ ν
H I Λ
γ π
Λ N O P

Fig. 38.


$\frac{n}{m}$T $\frac{n}{m}t$
$\frac{n}{m}t$ $\frac{n}{m}v$

Becauſe the exceſſes of the longer periods above the arithmetical mean are equal, one with another, to the defects of the ſhorter from the ſame, and becauſe the arithmetical mean period is longer (*a*) and therefore more harmonious (*b*) than the harmonical mean among the ſame. Q. E. D.

PROPOSITION XV.

The tempered concords in any one of the parcels derived from the V[th], VI[th] *or* III[d] (*c*) *whatever be their common temperament, are conſtantly more harmonious in the ſame immutable order as the products of the terms of the ratios of the vibrations of the perfect concords are ſmaller; and thoſe only are equally harmonious which have equal products belonging to them; and no others can be made ſo, becauſe they cannot have different temperaments while the octaves are perfect.* The

(*a*) Def. 2. coroll. 1. Sect. VII. (*b*) Prop. XIII. coroll. 1. (*c*) In the ſcholium to prop. III, the concords were diſtributed into three parcels, which may be ſeen at one view in Table I pag. 173.

The truth of the propoſition appears from prop. XIII. coroll. 6. Q. E. D.

PROPOSITION XVI.

To find the temperament of a ſyſtem of ſounds of a given extent, wherein as many concords as poſſible, at a medium of one with another, ſhall be equally and the moſt harmonious.

Into the ſecond and third columns of the II[d] Table following, transfer every couple of concords in the given ſyſtem, whoſe characters can be taken from the different parcels in the I[st] Table; omitting all other couples whoſe characters are both ſituated in any one of the parcels. And after each couple place the ratio of the temperaments which would make the two concords equally harmonious in their kind (*a*).

Then will the corollaries to the IV[th], V[th] and VI[th] propoſitions give the temperaments themſelves, or the poſitions of the temperers *Orst*, *Orst*, &c, in Fig. 44, belonging to every one of thoſe ratios.

(*a*) Prop. XIII. coroll. 3.

Among

Among the temperaments in the three ſeveral parcels *Gr*, *Gr*, &c, *As*, *As*, &c, *Et*, *Et*, &c, taking three harmonical means *GD*, *AH*, *EM*, and transferring them to Fig. 45, draw three temperers *ODef*, *OgHi*, *OklM*, and taking *Eq* the arithmetical mean among the three temperaments *EM*, *Ef*, *Ei*, the temperer *Onpq* will approach very near to the poſition required in the propoſition.

If greater exactneſs be deſired, among the three temperaments in the three ſeveral parcels *GD*, *Gg*, *Gk*; *AH*, *Ae*, *Al*; *EM*, *Ef*, *Ei*, taking three harmonical means *GD′*, *AH′*, *EM′*, and transferring them to Fig. 46, draw three new temperers *OD′e′f′*, *Og′H′i′*, *Ok′l′M′*; and taking *Eq′* the arithmetical mean among the three temperaments *EM′*, *Ef′*, *Ei′*, the temperer *Gn′p′q′* will approach ſtill nearer towards the required poſition.

And by repeating the like conſtruction we may approach as near as we pleaſe (*a*). Q. E. I.

THE DEMONSTRATION.

In combining the concords all the couples whose characters are both in any one of the parcels in Tab. I are omitted; their harmony with respect to one another, or the proportions of their periods being immutable, by reason of their common temperament (*b*).

Fig. 44. Now supposing *Ga* to be the arithmetical mean among all the temperaments *Gr*, *Gr*, &c, of the first parcel of concords, and drawing the temperer *Oabc*, the temperaments *Ab* and *Ec* will be the like Means among *As*, *As*, &c, and *Et*, *Et*, &c (*c*). Whence if the periods of concords were proportional to their temperaments, the best temperer would be *Oabc* (*d*):

Because the period of the v[th], for instance, belonging to the temperament *Ga*,

(*a*) Want of room in such small plates made it necessary to alter the true proportions of the lines in the figures; otherwise some parts of them would have appeared confused. (*b*) Prop. xv.

(*c*) Def. 1. coroll. 1. 2. 3. Sect. vii. (*d*) Prop. xiv.

Ga, would then be the arithmetical mean among all the other periods of the v[ths] anſwering to the ſeveral temperaments *Gr*, *Gr*, &c. And the like may be ſaid of the periods of the 4[th] and of every other concord in this firſt parcel, as having the temperaments *Gr*, *Gr*, &c, common to them all, and likewiſe of the periods of the ſeveral concords in the other two parcels, with reſpect to their temperaments *Ab*, *As*, &c, and *Ec*, *Et*, &c.

But ſince the periods of given concords to a given baſe are not directly but inverſely proportional to their temperaments (*a*), the period of the v[th], or any other concord, belonging to the arithmetical mean temperament *Ga*, is not the arithmetical, but the harmonical mean among the other periods of that concord, anſwering to the temperaments *Gr*, *Gr*, &c; and conſequently is ſhorter (*b*) and therefore leſs harmonious than

(*a*) Prop. IX. coroll. 4. (*b*) Def. 2. coroll. 1. Sect. VII.

than the arithmetical mean period among them (*a*) anſwering to the harmonical mean temperament *GD*. And what has been ſaid of the periods in that parcel is applicable to thoſe of the other two, with reſpeĉt to the arithmetical and harmonical mean temperaments *Ab* and *AH*, *Ec* and *EM*.

Hence the arithmetical mean periods belonging to the harmonical mean temperaments *GD*, *AH*, *EM*, would beſt anſwer the deſign of the propoſition, if the points *D*, *H*, *M* were all ſituated in one temperer.

For ſince the ſums of the temperaments terminated at the ſeveral temperers *Orst*, *Orst*, &c, are the leaſt that can render the concords in each couple equally harmonious in their kind (*b*), it follows that the ſums of all the temperaments *Gr*, *Gr*, &c, in the firſt parcel, of all the temperaments *As*, *As*, &c, in the ſecond, and of all the temperaments *Et*, *Et*, &c, in the third, taking one ſum

(*a*) Prop. XIII. coroll. 1. (*b*) Prop. IV. V. VI.

ſum with another, are alſo the leaſt poſſible: the ſum total being the ſame in both diſtributions of the particulars.

The ſum of the harmonical temperaments *GD*, *AH*, *EM* being therefore the leaſt poſſible (*a*), and that of all the correſponding arithmetical mean periods being the greateſt (*b*), would render the ſyſtem of periods, at a medium of one with another, the moſt harmonious.

But in reality the three harmonical points *D*, *H*, *M* can never fall into any one temperer. Becauſe the harmonical means *GD*, *AH*, *EM* are leſs than the reſpective arithmetical means *Ga*, *Ab*, *Ec* (*c*), which are all terminated by one temperer *Oabc*.

Fig. 45. In the ſolution of the problem we reduced the three temperers *ODef*, *OgHi*, *OklM* to one, by ſo drawing the temperer *Onpq*, as to make *Eq* the arithmetical mean among *EM*, *Ef*, *Ei*, and conſequently *Ap* the like mean among

(*a*) Def. 1 and cor. 1. Def. 2. Sect. VII.
(*b*) Prop. IX. cor. 4. (*c*) Def. 2. cor. 1. Sect. VII.

among *AH*, *Ae*, *Al* and *Gn* the like among *GD*, *Gg*, *Gk* (*a*).

Now the differences of the three temperaments in each of those parcels being but small, as will appear by the following calculation (*b*), the arithmetical means among them will differ but little from the respective harmonical means among the same (*c*), which would be fitter for the purpose if their extremities *D′*, *H′*, *M′* could be situated in any one temperer (*d*). Consequently as the temperer *Onpq* falls in the middle among the three temperers conceived to pass through the harmonical points *D′*, *H′*, *M′*, it will nearly answer the several purposes of those three, and approach very near to the situation of the required temperer.

Fig. 46. Hence and by prop. XIV, it appears that a repetition of this last construction, as described in the solution, will give a temperer *On′p′q′* approaching

(*a*) Def. 1. coroll. 1. 2. 3. (*b*) Tab. VI column 2.
(*c*) Def. 2. coroll. 2. (*d*) Prop. XIV.

proaching ſtill nearer to the required ſituation. Becauſe the latter temperaments EM', Ef', Ei' differ leſs from one another (*a*) and conſequently from their arithmetical mean Eq', than the former, EM, Ef, Ei did from one another and from their arithmetical mean Eq.

And as the ſame may be ſaid of the temperaments of the other two parcels, it appears that by a further repetition of the ſame conſtruction, we may find a temperer approaching as near as we pleaſe towards the poſition required in the propoſition. Q. E. D.

Coroll. 1. Fig. 45. The comma, or four times the line $G1$, being the unit, and ſuppoſing any three temperaments of different parcels to be given, as $GD=d$, $AH=h$ and $EM=m$, it will be eaſy to collect, (from the ſimilar triangles under the line $O13E$, the three temperers $ODef$, $OgHi$, $OklM$, and the three parallels GD, AH, EM,) that $Gg = \frac{1-h}{3}$ and $Gk = \frac{1+m}{4}$; $Ae = 1-3d$ and

(*a*) Tab. VII at the end of it.

and $Al = \frac{1-3m}{4}$; $Ef = 4d - 1$ and $Ei = \frac{1-4b}{3}$; provided the three temperers be all situated within the angle EOA; but if OH or OM lyes out of it beyond A or E respectively, the sign of b or m will accordingly be changed in those theorems.

Coroll. 2. Hence we have the three arithmetical mean temperaments, $Gn = \frac{1}{3} \times \overline{d + \frac{1-b}{3} + \frac{1+m}{4}}$, $Ap = \frac{1}{3} \times \overline{1 - 3d + b + \frac{1-3m}{4}}$, and $Eq = \frac{1}{3} \times \overline{4d - 1 + \frac{1-4b}{3} + m}$.

Scholium. 1.

Fig. 47 serves to illustrate part of the demonstration of the proposition, by representing to the eye the proportions of the periods of the concords. It is thus constructed. The line AI being parallel to EO, the middlemost hyperbolas $v\frac{1}{5}$, $x\frac{1}{6}$, $y\frac{1}{10}$, $z\frac{1}{12}$, &c, are drawn

to

to the aſymptotes *AI*, *A*3; and their ordinates to their common abſciſs *A*3 are made proportional to the fractions $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{10}$, $\frac{1}{12}$, &c.

Hence the periods of the VI, 3[d], VI + VIII, 3[d] + VIII, &c, when tempered by a quarter of a comma, repreſented by the common abſciſs *A*3, are proportional to thoſe ordinates (*a*); and when they have any other temperament repreſented by the abſciſs *As*, their periods are then proportional to the ordinates *sv*, *sx*, *sy*, *sz*, &c.

And the like conſtruction being made for the other two parcels of concords, the ordinates erected from the interſections *r*, *s*, *t* of any temperer *Orst*, ſhew the proportions of the periods in the whole ſyſtem: and theſe proportions are the ſame whatever be the unit of the fractional ordinates.

Scho-

(*a*) Prop. IX. coroll. 5, and Tab. I. pag. 173.

Scholium 2.

In order to calculate the required temperaments of a ſyſtem of any given extent, it will not be amiſs to explain the following tables.

1. According to the ſolution of the problem, ſee whether every two characters, each of which lye in the different parcels of concords in Tab. I, be placed over againſt one another in the ſecond and third columns of Tab. II Part I.

2. Then examine whether the ratios placed after thoſe characters in the 4th column of that table, be rightly deduced from the fractions annexed to the ſame characters in Tab. I, according to the rule in prop. XIII. coroll. 3.

3. See whether all the temperaments in Tab. IV be rightly deduced from thoſe ratios in Tab. II, by the corollaries to prop. IV, V, VI; and whether the numbers in the firſt column of each table correſpond to the ſame ratios and concords.

4.

4. Examine whether the reciprocals in Tab. V, of the temperaments in Tab. IV be right, that is, whether the product of the quotient by the divisor, differs from the devidend by less than half the divisor. When a reciprocal is negative, as coming from a negative temperament, I subtract it from 0 and place the remainder in the table instead of the reciprocal it self. Thus at N°. 10, $-6)26(=-4.33333$ &c, which subtracted from 0 gives $\overline{5}.66667$ to be transferred to Tab. II Part II and there added to the positive reciprocals, for the sake of uniformity in the work; the integer $\overline{5}$ being only negative and the decimals .66667 affirmative. For n being any given integer, the number $n-4-\frac{1}{3}=n-5+\frac{2}{3}$.

5. See whether the reciprocals in Tab. V be rightly transferred into the respective columns of Tab. II Part II, which is readily done by means of the corresponding numbers in the first column of each table.

6. Cast up the several dozens of reciprocals in Tab. II Part II, and transfer the sums to Tab. III and there cast them up.

7. Tab. VI is thus deduced from Tab. III. By the solution of the problem the fraction $\frac{12}{42.72013}$, $=GD$ in Fig. 44, is the harmonical mean among the temperaments *Gr*, *Gr*, &c; because its reciprocal $\frac{42.72013}{12}$ is the arithmetical mean among their reciprocals, as being their sum divided by their number. The same is to be understood of all the other fractions: and as the value of the temperament *Eq*, computed from coroll. 2. prop. XVI, comes out affirmative; by the coroll. to prop. IV, V, VI it is part of the interval *EC* of the perfect III^{d}, and therefore is a negative temperament of that concord, or an affirmative one of its complement to the octave. This is the first approximation towards the required temperament.

8. Tab. VII contains the calculation of Eq', the ſecond approximation towards the true temperament of the III^d, in a ſyſtem whoſe extent is but one octave, and is ſufficiently evident from cor. 1, 2, prop. XVI, and Tab. VI. And by a like calculation the values of Eq', in a ſyſtem of two and of three octaves, will be found as put down under thoſe of Eq in Tab. VI; care being taken in the operations to continue the quotients in decimals as far as they are juſt.

9. Therefore the reſult of the whole is this. As all the parts of muſical compoſitions, ſetting aſide double baſes, are generally contained within three octaves, and as their harmony is ſtronger and better within that compaſs than it would be in a larger; I chooſe to make all the concords within every three octaves equally harmonious and no more, be the extent of the ſyſtem ever ſo great; and conſequently to diminiſh the III^d by $\frac{1}{9}$ comma, this being

ing very nearly the value of the laſt Eq' $=0.11024$ in Tab. VI.

10. Hence in the ſyſtem of equal harmony the temperaments of the V[th], VI[th] and III[d] are $-\frac{5}{18}$, $+\frac{3}{18}$ and $-\frac{2}{18}$ of a comma reſpectively (*a*) and are proportional to the muſical primes 5, 3 and 2. (*b*)

11. In determining theſe temperaments of the diatonic ſyſtem, I have regarded no more conſonances than the concords. 1. Becauſe the diſcords are ſeldomer uſed than the concords. 2. Becauſe the ear is generally leſs critical in the diſcords than in the concords. 3. Becauſe a mean temperament among thoſe of the concords and diſcords too, would differ

(*a*) Prop. III. coroll. 2, 3.

(*b*) But if any one chooſes to have all the concords in 4 octaves made equally harmonious, he will find by continuing the tables, that the III[d] muſt be diminiſhed by $\frac{987}{10000}$ of a comma, which being nearly $\frac{1}{10}$ comma, the temperaments of the V[th], VI[th] and III[d] will then be $-\frac{11}{40}$, $+\frac{7}{40}$ and $-\frac{4}{40}$ of a comma reſpectively.

differ from that of the concords alone, and therefore be leſs ſuitable to them.

12. Laſtly I have kept the octave untempered. 1. Becauſe it is the ſimpleſt and moſt harmonious of all the concords, both in itſelf and its multiples. 2. Becauſe ſome one interval muſt be kept perfect, in order to determine the variations of the temperaments of the reſt (*c*). 3. Becauſe upon ſeveral trials of keeping other intervals untempered inſtead of the octave, many reaſons have occurred to me for rejecting every one of them.

13. Does it not follow then, that the ſyſtem of equal harmony, as above derived from the beſt ſyſtem of perfect intervals (*d*), is the beſt tempered ſyſtem that the nature of ſounds is capable of? (*e*)

(*c*) Prop. III. (*d*) Sect. IV. Art. 7.
(*e*) See Scholium. Prop. III.

TABLE I

Contains the characters and te
times of the vibrations o

1st Parcel.	2d Parc
V. $\frac{2}{3}$ 4th. $\frac{3}{4}$	VI. $\frac{3}{5}$ 3d. $\frac{5}{6}$
V + VIII. $\frac{1}{3}$ 4th + VIII. $\frac{3}{8}$	VI + VII 3d + VII
V + 2VIII. $\frac{1}{6}$ 4th + 2VIII. $\frac{3}{16}$	VI + 2VII 3d + 2VII
V + 3VIII. $\frac{1}{12}$ 4th + 3VIII. $\frac{3}{32}$	VI + 3VII 3d + 3VII
&c. &c.	&c. &c.

RT I.

mperaments for
ny of the

as	5 : 2 5 : 1 5 : 4 5 : 2
	10 : 3 20 : 3 5 : 3 10 : 3
	4 : 3 8 : 3 2 : 3 4 : 3
tave	1 dozen
+ VIII + VIII + VIII + VIII	5 : 1 10 : 1 5 : 2 5 : 1
+ VIII + VIII + VIII + VIII.	5 : 3 40 : 3 5 : 6 20 : 3
+ VIII + VIII + VIII + VIII	2 : 3 16 : 3 1 : 3 8 : 3
	2 dozen

TAB. II. PART II.

Nº	Reciprocals of the temperaments of the v, 4th & Comp.	vi, 3d & Comp.	iii, 6th & Comp.
1	3.40000	8.50000	5.66667
2	3.20000	16.00000	4.00000
3	3.80000	4.75000	19.00000
1	3.40000	8.50000	5.66667
4	3.70000	5.28571	12.33333
5	3.85000	4.52941	25.66667
1	3.40000	8.50000	5.66667
4	3.70000	5.28571	12.33333
6	3.57143	6.25000	8.33333
7	3.72727	5.12500	13.66667
1	3.40000	8.50000	5.66667
6	3.57143	6.25000	8.33333
Sums	42.72013	87.47583	126.33334
2	3.20000	16.00000	4.00000
8	3.10000	31.00000	3.44444
1	3.40000	8.50000	5.66667
2	3.20000	16.00000	4.00000
1	3.40000	8.50000	5.66667
9	3.92500	4.24324	52.33333
10	5.20000	2.36364	5.66667
5	3.85000	4.52941	25.66667
1	3.40000	8.50000	5.66667
11	3.84211	4.56250	24.33333
12	3.25000	13.00000	4.33333
7	3.72727	5.12500	13.66667
Sums	43.49438	122.32379	144.44445

TAB. II. PART I.

N°	Ratios of the temperaments for equal harmony of the		
2	V + VIII	VI	5 : 1
8	V + VIII	3^{d}	10 : 1
13	4th + VIII	VI	5 : 8
3	4th + VIII	3^{d}	5 : 4
5	V + VIII	III	20 : 3
9	V + VIII	6th	40 : 3
10	4th + VIII	III	5 : 6
1	4th + VIII	6th	5 : 3
1	VI + VIII	III	2 : 3
6	VI + VIII	6th	4 : 3
12	3^{d} + VIII	III	1 : 3
1	3^{d} + VIII	6th	2 : 3
			3 dozen
8	V + VIII	VI + VIII	10 : 1
14	V + VIII	3^{d} + VIII	20 : 1
3	4th + VIII	VI + VIII	5 : 4
1	4th + VIII	3^{d} + VIII	5 : 2
4	V + VIII	III + VIII	10 : 3
15	V + VIII	6th + VIII	80 : 3
16	4th + VIII	III + VIII	5 : 12
4	4th + VIII	6th + VIII	10 : 3
12	VI + VIII	III + VIII	1 : 3
7	VI + VIII	6th + VIII	8 : 3
17	3^{d} + VIII	III + VIII	1 : 6
6	3^{d} + VIII	6th + VIII	4 : 3
	In two	Octaves	4 dozen

TAB. II. PART II.

N°	Reciprocals of the temperaments of the v, 4th & Comp.	vi, 3d & Comp.	iii, 6th & Comp.
2	3. 20000	16. 00000	4. 00000
8	3. 10000	31. 00000	3. 44444
13	4. 60000	2. 87500	8. 33333
3	3. 80000	4. 75000	19. 00000
5	3. 85000	4. 52941	25. 66667
9	3. 92500	4. 24324	52. 33333
10	5. 20000	2. 36364	5. 66667
1	3. 40000	8. 50000	5. 66667
1	3. 40000	8. 50000	5. 66667
6	3. 57143	6. 25000	8. 33333
12	3. 25000	13. 00000	4. 33333
1	3. 40000	8. 50000	5. 66667
Sums	44. 69643	110. 51129	122. 11111
8	3. 10000	31. 00000	3. 44444
14	3. 05000	61. 00000	3. 21053
3	3. 80000	4. 75000	19. 00000
1	3. 40000	8. 50000	5. 66667
4	3. 70000	5. 28571	12. 33333
15	3. 96250	4. 11688	105. 66667
16	6. 40000	1. 88235	3. 33333
4	3. 70000	5. 28571	12. 33333
12	3. 25000	13. 00000	4. 33333
7	3. 72727	5. 12500	13. 66667
17	3. 14286	22. 00000	3. 66667
6	3. 57143	6. 25000	8. 33333
Sums	44. 80406	168. 19565	188. 98830

TAB. II. PART I.

N°	Ratios of the temperaments for equal harmony of the		
8	V	VI + 2VIII	10 : 1
14	V	3^{d} + 2VIII	20 : 1
2	4^{th}	VI + 2VIII	5 : 1
8	4^{th}	3^{d} + 2VIII	10 : 1
10	V	III + 2VIII	5 : 6
15	V	6^{th} + 2VIII	80 : 3
16	4^{th}	III + 2VIII	5 : 12
9	4^{th}	6^{th} + 2VIII	40 : 3
12	VI	III + 2VIII	1 : 3
18	VI	6^{th} + 2VIII	32 : 3
17	3^{d}	III + 2VIII	1 : 6
11	3^{d}	6^{th} + 2VIII	16 : 3
			5 dozen
14	V + VIII	VI + 2VIII	20 : 1
19	V + VIII	3^{d} + 2VIII	40 : 1
1	4^{th} + VIII	VI + 2VIII	5 : 2
2	4^{th} + VIII	3^{d} + 2VIII	5 : 1
1	V + VIII	III + 2VIII	5 : 3
20	V + VIII	6^{th} + 2VIII	160 : 3
21	4^{th} + VIII	III + 2VIII	5 : 24
5	4^{th} + VIII	6^{th} + 2VIII	20 : 3
17	VI + VIII	III + 2VIII	1 : 6
11	VI + VIII	6^{th} + 2VIII	16 : 3
22	3^{d} + VIII	III + 2VIII	1 : 12
7	3^{d} + VIII	6^{th} + 2VIII	8 : 3
			6 dozen

TAB. II. PART II.

N°	Reciprocals of the temperaments of the		
	V, 4th & Comp.	VI, 3d & Comp.	III, 6th & Comp.
8	3. 10000	31. 00000	3. 44444
14	3. 05000	61. 00000	3. 21053
2	3. 20000	16. 00000	4. 00000
8	3. 10000	31. 00000	3. 44444
10	5. 20000	2. 36364	$\overline{5}$. 66667
15	3. 96250	4. 11688	105. 66667
16	6. 40000	1. 88235	$\overline{3}$. 33333
9	3. 92500	4. 24324	52. 33333
12	3. 25000	13. 00000	4. 33333
18	3. 91429	4. 28125	45. 66667
17	3. 14286	22. 00000	3. 66667
11	3. 84211	4. 56250	24. 33333
Sums	46. 08676	195. 44986	243. 09941
14	3. 05000	61. 00000	3. 21053
19	3. 02500	121. 00000	3. 10256
1	3. 40000	8. 50000	5. 66667
2	3. 20000	16. 00000	4. 00000
1	3. 40000	8. 50000	5. 66667
20	3. 98125	4. 05732	212. 33333
21	8. 80000	1. 51724	$\overline{2}$. 16667
5	3. 85000	4. 52941	25. 66667
17	3. 14286	22. 00000	3. 66667
11	3. 84211	4. 56250	24. 33333
22	3. 07692	40. 00000	3. 33333
7	3. 72727	5. 12500	13. 66667
Sums	46. 49541	296. 79147	302. 81310

TAB. II. PART I.

N°	Ratios of the temperaments for equal harmony of the		
1	V +2VIII	VI	5 : 2
2	V +2VIII	3^{d}	5 : 1
23	4^{th} +2VIII	VI	5 : 16
13	4^{th} +2VIII	3^{d}	5 : 8
4	V +2VIII	III	10 : 3
5	V +2VIII	6^{th}	20 : 3
16	4^{th} +2VIII	III	5 : 12
10	4^{th} +2VIII	6^{th}	5 : 6
12	VI +2VIII	III	1 : 3
1	VI +2VIII	6^{th}	2 : 3
17	3^{d} +2VIII	III	1 : 6
12	3^{d} +2VIII	6^{th}	1 : 3
			7 dozen
2	V +2VIII	VI +VIII	5 : 1
8	V +2VIII	3^{d} +VIII	10 : 1
13	4^{th} +2VIII	VI +VIII	5 : 8
3	4^{th} +2VIII	3^{d} +VIII	5 : 4
1	V +2VIII	III+VIII	5 : 3
9	V +2VIII	6^{th} +VIII	40 : 3
21	4^{th} +2VIII	III+VIII	5 : 24
1	4^{th} +2VIII	6^{th} +VIII	5 : 3
17	VI +2VIII	III+VIII	1 : 6
6	VI +2VIII	6^{th} +VIII	4 : 3
22	3^{d} +2VIII	III+VIII	1 : 12
1	3^{d} +2VIII	6^{th} +VIII	2 : 3
			8 dozen

TAB. II. PART II.

N°	Reciprocals of the temperaments of the v, 4th & Comp.	VI, 3d & Comp.	III, 6th & Comp.
1	3. 40000	8. 50000	5. 66667
2	3. 20000	16. 00000	4. 00000
23	6. 20000	1. 93750	$\bar{3}$. 18182
13	4. 60000	2. 87500	$\bar{8}$. 33333
4	3. 70000	5. 28571	12. 33333
5	3. 85000	4. 52941	25. 66667
16	6. 40000	1. 88235	$\bar{3}$. 33333
10	5. 20000	2. 36364	$\bar{5}$. 66667
12	3. 25000	13. 00000	4. 33333
1	3. 40000	8. 50000	5. 66667
17	3. 14286	22. 00000	3. 66667
12	3. 25000	13. 00000	4. 33333
Sums	49. 59286	99. 87361	48. 18182
2	3. 20000	16. 00000	4. 00000
8	3. 10000	31. 00000	3. 44444
13	4. 60000	2. 87500	$\bar{8}$. 33333
3	3. 80000	4. 75000	19. 00000
1	3. 40000	8. 50000	5. 66667
9	3. 92500	4. 24324	52. 33333
21	8. 80000	1. 51724	$\bar{2}$. 16667
1	3. 40000	8. 50000	5. 66667
17	3, 14286	22. 00000	3. 66667
6	3. 57143	6. 25000	8. 33333
22	3. 07692	40. 00000	3. 33333
1	3. 40000	8. 50000	5. 66667
Sums	47. 41621	154. 13548	101. 61111

TAB. II. PART I.

N°	Ratios of the temperaments for equal harmony of the		
8	v +2VIII	VI +2VIII	10 : 1
14	v +2VIII	3^{d} +2VIII	20 : 1
3	4^{th} +2VIII	VI +2VIII	5 : 4
1	4^{th} +2VIII	3^{d} +2VIII	5 : 2
10	v +2VIII	III+2VIII	5 : 6
15	v +2VIII	6^{th} +2VIII	80 : 3
24	4^{th} +2VIII	III+2VIII	5 : 48
4	4^{th} +2VIII	6^{th} +2VIII	10 : 3
22	VI +2VIII	III+2VIII	1 : 12
7	VI +2VIII	6^{th} +2VIII	8 : 3
25	3^{d} +2VIII	III+2VIII	1 : 24
6	3^{d} +2VIII	6^{th} +2VIII	4 : 3
	In three	Octaves	9 dozen

TAB. III.

The numbers and sums of the - - - - -		
Dozen	N°	v, 4^{th} & Comp.
In 1 VIII^{ve}. 1^{st}	12	42.72013
2^{d}	12	43.49438
3^{d}	12	44.69643
4^{th}	12	44.80406
In 2 VIII^{ves}.	48	175.71500
5^{th}	12	46.08676
6^{th}	12	46.49541
7^{th}	12	49.59286
8^{th}	12	47.41621
9^{th}	12	53.22812
In 3 VIII^{ves}.	108	418.53436

N°
8
14
3
1
10
15
24
4
22
7
25
6
Sums

v 4th &c	vi 3d &c
20	1
	61
80	77
	317
5	17
	32
7	1
	22
In two Octa	
35	32
	137
40	1
	121
160	157
	637
5	29
	44
13	1
	40
5	16
	31
5	53
	68
25	1
	76
In three O	

different temperaments

iii, 6th & Comp.	
3) 17	5. 66667
4) 16	4. 00000
1) 19	19. 00000
3) 37	12. 33333
3) 77	25. 66667
3) 25	8. 33333
3) 41	13. 66667
9) 31	3. 44444
3) 157	52. 33333
— 6) 26	5. 66667
3) 73	24. 33333
3) 13	4. 33333
— 3) 23	8. 33333
19) 61	3. 21053
3) 317	105. 66667
— 12) 32	3. 33333
6) 22	3. 66667
3) 137	45. 66667
39) 121	3. 10256
3) 637	212. 33333
— 24) 44	2. 16667
12) 40	3. 33333
— 11) 31	3. 18182
— 48) 68	2. 58333
24) 76	3. 16667

vi, 3
87
122
110
168
488
194
296
95
154
245
148

TAB. VI.

The values of Eq *and* Eq' *in Fig.* 45 *and* 46, - - - *wards the temperament of the* III[d], *for* - - - *equally and the most harmonious.*

In 1 Octave.

$$\frac{12}{42.72013} = 0.2808980 = GD = d$$

$$\frac{12}{87.47583} = 0.1371808 = AH = h$$

$$\frac{12}{126.33333} = 0.0949868 = EM = m$$

In 2 Octaves.

$$\frac{48}{175.71500} = 0.2731696 = GD = d$$

$$\frac{48}{488.50656} = 0.0982587 = AH = h$$

$$\frac{48}{581.87720} = 0.0824916 = EM = m$$

In 3 Octaves.

$$\frac{108}{418.53436} = 0.2580433 = GD = d$$

$$\frac{108}{1480.43123} = 0.0729517 = AH = h$$

$$\frac{108}{1449.65428} = 0.0745005 = EM = m$$

TAB. VI.

--- *being the first and second approximations to-*
--- *making all the concords in* 1, 2 *or* 3 *octaves*

Hence

$Ef = 4d - 1 = 0.1235920$

$Ei = \frac{1-4b}{3} = 0.1504256$

$EM = m = 0.0949868$

$3)\ 0.3690044$

$Eq = 0.1230015$

$Eq' = 0.122233.$ in 1 Octave,

See Tab. VII.

$Ef = 4d - 1 = 0.0926784$

$Ei = \frac{1-4b}{3} = 0.2023217$

$EM = m = 0.0824916$

$3)\ 0.3774917$

$Eq = 0.1258306$

$Eq' = 0.124719.$ in 2 Octaves.

$Ef = 4d - 1 = 0.0321732$

$Ei = \frac{1-4b}{3} = 0.2360634$

$EM = m = 0.0745005$

$3)\ 0.3427371$

$Eq = 0.1142457$

$Eq' = 0.11024..$ in 3 Octaves.

TAB. VII.

The computation of E q′ *in Fig.* 46, *being the* - - - *ment of the* III[d], *for making all the concords* - - -

$$\frac{1}{GD} = \frac{1}{d} = \frac{42.72013}{12} = 3.560011$$

$$\frac{1}{Gg} = \frac{3}{1-b} = \frac{3}{0.8628192} = 3.476974$$

$$\frac{1}{Gk} = \frac{4}{1+m} = \frac{4}{1.0949868} = \underline{3.652913}$$

3) 10.689898

Arith. mean 3.563299

$$GD' = d' = \frac{1}{3.563299} = 0.280631$$

$$\frac{1}{AH} = \frac{1}{b} = \frac{87.47583}{12} = 7.289653$$

$$\frac{1}{Ae} = \frac{1}{1-3d} = \frac{1}{0.1573060} = 6.357031$$

$$\frac{1}{Al} = \frac{4}{1-3m} = \frac{4}{0.7150396} = \underline{5.494096}$$

3) 19.240780

Arith. mean 6.413593

$$AH' = b' = \frac{1}{6.413593} = 0.155919$$

TAB. VII.

---second approximation towards the tempera-
---in 1 octave equally and the most harmonious.

$$\frac{1}{EM} = \frac{1}{m} = \frac{126.33333}{12} = 10.527777$$

$$\frac{1}{Ef} = \frac{1}{4d-1} = \frac{1}{0.1235920} = 8.091138$$

$$\frac{1}{Ei} = \frac{3}{1-4b} = \frac{3}{0.4512768} = \underline{6.647805}$$

3) 25.266720

Arith. mean 8.422240

$$EM' = m' = \frac{1}{8.422240} = 0.118733$$

Hence $Ef' = 4d' - 1 = 0.122524$

$$Ei' = \frac{1-4b'}{3} = 0.125441$$

$$Em' = m' = \underline{0.118733}$$

3) 0.366698

$$Eq' = 0.122233$$

PROPOSITION XVII.

A ſyſtem of rational intervals deduced from dividing the octave into 50 *equal parts, and taking the limma* L = 5 *of them, the tone* T = 8 *and conſequently the leſſer* 3[d] L+T = 13, *the greater* III[d] 2T = 16, *the* 4[th] L + 2T = 21, *the* V[th] L + 3T = 29, *&c, according to the table of elements*(a), *will differ inſenſibly from the ſyſtem of equal harmony: I mean with regard to the harmony of the reſpective conſonances in both.*

For ſince L = 5, T = 8 and the III[d] 2T = 16 and the VIII = 5T + 2L = 50, we have the III[d] 2T : VIII :: 8 : 25; whence the III[d] $2T = \frac{8}{25}$ VIII $= \frac{8}{25}$ log. $2 = \frac{8}{25} \times$ 0. 30102. 99957 = 0. 09632. 95962, which ſubtracted

(a) Prop. III. pag. 52.

ed from the perfect III^d = log. $\frac{5}{4}$ = 0.09691.00130, leaves the temperament 0.00058.04168, which is to the comma c = log. $\frac{81}{80}$ = 0.00539.50319 as 4 to 37 very nearly (*b*). Hence the temperament of the III^d is $-\frac{4}{37}c$, and those of the Vth and VIth as in Tab. I, by prop. III. coroll. 1. 2. 3.

TABLE I.

T : L :: 8 : 5 VIII = 50	The system of equal harmony.	Ratios of the temperaments and of the beats made in the same time (*c*).
$\text{V} - \frac{1}{4}c - \frac{1}{37}c$	$\text{V} - \frac{1}{4}c - \frac{1}{36}c$	$\frac{1}{4} + \frac{1}{37} : \frac{1}{4} + \frac{1}{36} :: 369 : 370$
$\text{VI} + \frac{1}{4}c - \frac{3}{37}c$	$\text{VI} + \frac{1}{4}c - \frac{3}{36}c$	$\frac{1}{4} - \frac{3}{37} : \frac{1}{4} - \frac{3}{36} :: 75 : 74$
$\text{III} \quad - \frac{4}{37}c$	$\text{III} \quad - \frac{4}{36}c$	$\frac{4}{37} : \frac{4}{36} :: 36 : 37$

Now though the concords of the same name in this system and that of equal

(*b*) See an example of the like reduction in the next Scholium. (*c*) Prop. XI. coroll. 4.

qual harmony are not exactly equally harmonious (*d*), and though the difference of an unit in the largest numbers of beats made in a given time may be distinguished by counting them; yet when the numbers are no larger than those in the table, the difference in the harmony of the concords will be deemed insensible by proper judges; which are those only that have carefully attended to the beats of concords in tuning instruments. But any one else may be satisfied experimentally, by causing two concords to the same base to beat as in the table. Q. E. D.

Scholium.

In like manner if T = 5 and L = 3, then the octave 5 T + 2 L is = 31 and the temperament of this system, which *Hugenius* has adopted (*e*), will be found as in the third column of the next table.

TAB.

(*d*) Prop. XIII. coroll. 5.

(*e*) Cyclus harmonicus, at the end of his Works, or Histoire des Ouvrages des Sçavans, Octob. 1691, pag. 78.

TABLE II.

T : L :: 2 : 1 VIII = 12	T : L :: 3 : 2 VIII = 19	T : L :: 5 : 3 VIII = 31
$\text{V} - \frac{1}{4}c + \frac{3}{19}c$	$\text{V} - \frac{1}{4}c - \frac{3}{35}c$	$\text{V} - \frac{1}{4}c + \frac{1}{110}c$
$\text{VI} + \frac{1}{4}c + \frac{9}{19}c$	$\text{VI} + \frac{1}{4}c - \frac{9}{35}c$	$\text{VI} + \frac{1}{4}c + \frac{3}{110}c$
$\text{III} \quad + \frac{12}{19}c$	$\text{III} \quad - \frac{12}{35}c$	$\text{III} \quad + \frac{4}{110}c$

On the contrary, if from the given temperament of a ſyſtem it be requir-ed to find the ratio of T to L, we may proceed as follows. Let it be propoſed to approximate to the ſyſtem of equal harmony, where $2\text{T} = \text{III} - \frac{1}{9}c$ (*f*); then ſince $5\text{T} + 2\text{L} = \text{VIII}$, we have $2\text{L} = (\text{VIII} - 5\text{T} =)\ \text{VIII} - \frac{5}{2} \times \overline{\text{III} - \frac{1}{9}c}$, whence $\text{T} : \text{L} :: \text{III} - \frac{1}{9}c : \text{VIII} - \frac{5}{2} \times \overline{\text{III} - \frac{1}{9}c}$.

To

(*f*) Prop. XVI, Scholium 2. Art. 9.

To find this ratio, we have the III $= \log. \frac{5}{4} = 0.09691.00130$ and the comma $c = \log. \frac{81}{80} = 0.00539.50319$ and $\frac{1}{9} c = 0.00059.94480$. Whence $2T = \text{III} - \frac{1}{9} c = 0.09631.05650$ and $\frac{5}{2} \times \overline{\text{III} - \frac{1}{9} c} = 0.24077.64125$ and the VIII $= \log. 2 = 0.30102.99957$ and $2L = \text{VIII} - \frac{5}{2} \times \overline{\text{III} - \frac{1}{9} c} =$ $0.06025.35832$, and lastly $T : L ::$ $963105650 : 602535832$.

Now the quotients of the greater term of this ratio divided by the lesser and of the lesser divided by the remainder and of the former remainder by the latter &c, are 1, 1, 1, 2, 24, &c. Whence the ratios greater than the true one are 2 to 1, 5 to 3, 8 to 5, &c, and the lesser are 3 to 2, 11 to 7, &c (*g*).

Hence taking T to L successively in those ratios, by the method used in the de-

(*g*) See Mr. *Cotes*'s Harmonia Mensurarum, Schol. 3. prop. 1.

demonſtration of the propoſition the temperaments of the approximating rational ſyſtems will be found as in the tables. By which we ſee how much and which way they differ from that of mean tones, as well as from that of equal harmony in Table I.

SECTION VIII.

Of complete and defective ſcales of muſical ſounds.

1. Fig. 48. Conceiving the circumference of a circle to repreſent an octave and to be divided into 31 equal parts, of which the tones *AB*, *CD*, &c, are 5 apiece and the limmas *BC*, *EF* are 3, take the interval AA^{b} towards the graver ſounds, and AA^{*} towards the acuter, each equal to the leſſer limma $AB - BC = 2$ of thoſe parts; and do the like at all the other primary ſounds, *B*, *C*, *D*, *E*, *F*, *G*. Then will the addition of theſe 7 couples of ſecondary ſounds

ſounds A^b and A※, B^b and B※, C^b and C※, &c, produce 2 ſecondary ſyſtems, A^b, B^b, C^b, D^b, E^b, F^b, G^b, and A※, B※, C※, D※, E※, F※, G※, each like the firſt; every ſound of each being equally depreſſed below, or raiſed above the correſponding ſound in the primary ſyſtem *A, B, C, D, E, F, G*.

2. Theſe three diatonic ſyſtems compoſe a ſcale of 21 ſounds in the octave, which is ſufficiently complete for muſical compoſitions. But if it happen now and then that others are wanted, they are always ſituated at the points that biſect the leſſer limmas AA※, BB^b CC※, DD^b, &c, and by analogy may be thus denoted A※※, B^{bb}, &c, ſignifying twice the diſtance of a ſingle flat or ſharp ſound from the primary ſounds *A, B*, &c.

3. The 7 Dieſes or ſmalleſt intervals A※B^b, $C^b B$※, C※D^b, D※E^b, &c, ſituated in the middle of the primary tones and limmas, are ſeverally equal to $\frac{1}{5}$ of the tone

tone or $\frac{1}{31}$ of the octave. This is the *Hugenian* syſtem.

4. Fig. 49. In like manner conceiving the circumference of a circle to be divided in 50 equal parts, as in the approximation to the ſyſtem of equal harmony (*a*), the tone *AB* is 8 of them and the greater limma *BC* is 5, and the leſſer $AA^{*} = AB - BC$ is $= 3$; and each of the 5 greater dieſes $A^{*}B^{b}$, $C^{*}D^{b}$, &c, within the primary tones is 2, and each of the leſſer $C^{b}B^{*}$, $F^{b}E^{*}$, within the two primary limmas is 1 of thoſe 50th parts. And the ſyſtem compoſed of theſe 21 ſounds in the octave is likewiſe complete enough for muſical compoſitions.

5. Hence the *Hugenian* dieſis is to the leſſer of theſe dieſes as $\frac{1}{31}$ to $\frac{1}{50}$, or as 50 to 31, and is to the greater as 50 to 62, or as 5 to $6\frac{1}{5}$.

6. The ſeveral columns in the firſt table of concords (*b*) contain the notes that

(*a*) Prop. XVII. (*b*) Plate 19.

that make the ſix concords to the ſeven primary baſe notes *A*, *B*, *C*, *D*, *E*, *F*, *G*, placed in the loweſt tranſverſe row, and alſo to the 5 ſecondary baſe notes *B*ᵇ, *C*※, *E*ᵇ, *F*※, *G*※ in the ſame row continued; an organ or harpſichord having generally no more than thoſe 5.

Conſequently when a compoſer uſes any of theſe other ſecondary notes *A*ᵇ, *A*※, *C*ᵇ, *B*※, *D*ᵇ, *D*※, *F*ᵇ, *E*※, *G*ᵇ, which in the table and Fig. 48 or 49 are incloſed in circles, the performer is obliged to take the note next to it, as *G*※ inſtead of *A*ᵇ, *B*ᵇ inſtead of *A*※, *B* inſtead of *C*ᵇ, &c, whoſe interval is a dieſis.

7. Hence each concord in the table between any baſe note and the note uſed inſtead of the true one in the circle, is either too flat or too ſharp by a dieſis, according as the mark ᵇ or ※ is annexed to the true but abſent note incloſed in the circle.

8. Theſe falſe concords render ſome of the Keys, or Modes of harmony, more imperfect than others with reſpect to the

$\frac{4^{th}\ \&c}{G\quad c}$

$\frac{3^{d}\ \&c}{c}$

$\frac{{}^{th}\ \&c}{c}$

the concords, in the ſame order as the numbers 1, 2, 3, placed under the baſe notes, increaſe in magnitude. For one falſe concord makes a mode leſs imperfect than two, and two leſs than three, and I do not here conſider the falſe diſcords.

9. Hence it is obſervable, that the modes both flat and ſharp, whoſe baſes *A*, *D*, *E*, *G*, are the notes of the four open ſtrings of the violin, are all free from falſe concords, though not all from falſe diſcords.

As to the reſt, perhaps I need not obſerve, that the mode whoſe baſe is *B*, for inſtance, with a flat 3^{d} above it, is as perfect with reſpect to the concords as the like mode whoſe baſe is *A*; or that the degrees of imperfection 1, 2, 3, relate only to the imperfect concords in the columns above them.

10. The deſign of placing but one ſecondary ſound inſtead of two within each primary tone, is to facilitate the

practice of music and the structure of instruments. And the reason for choosing these five B^b, $C^{※}$, E^b, $F^{※}$, $G^{※}$, rather than any other, was, I suppose, to make as many perfect modes as possible with that number of sounds.

Now the numbers in Tab. 3, placed below the five couples of secondary notes, shew how often each note is found among the concords to the seven primary bases *A*, *B*, *C*, *D*, *E*, *F*, *G*, whose modes should first be regarded; and that note of each couple, placed within the primary tones in Fig. 48 or 49, which occurs seldomer than the other, is justly excluded by the circle, as being generally less used.

But A^b and $G^{※}$ occur equally often, and musicians have excluded A^b and admitted $G^{※}$, as in the first table of concords.

On the contrary, if $G^{※}$ were excluded and A^b admitted, as in the second table of concords, two of the modes, whose

whoſe common baſe is E^b, would be freed from the falſe 4^{th}, and the whole number of falſe concords, reckoned up in the laſt column, would be but 15 inſtead of 16 in Tab. 1.

11. The numbers in the 4^{th} table placed under the notes excluded in the firſt, ſhew how often each of theſe notes is found in this whole table of concords to the 12 baſes; by which it appears that A^b and D※ are ofteneſt wanted; and that an addition of theſe two to an organ or harpſichord, would reduce the 16 falſe concords in the firſt table to 10, by making 6 of them as good as any in the ſcale.

For the like reaſon the 5^{th} table ſhews, that an addition of a couple of notes would alſo reduce the 15 falſe concords in the ſecond table to 10.

12. We may obſerve of thoſe two ſyſtems of rational intervals, that the *Hugenian* is ſimpler than the other, and has this advantage too in a defec-

tive ſcale, that the dieſis being ſomething ſmaller (*c*), the falſe concords are ſomething better than in the other, though ſtill extremely bad: but in thoſe reſpects the ſyſtem of 12 equal parts in the VIII far excels them both.

13. So that conſidering the extreme diſagreeableneſs of the imperfect modes in any defective ſcale, in compariſon to the perfect ones when tuned in the beſt manner, and what a charming inſtrument an organ would be if thoſe defects were ſupplied, I propoſe to do it as follows, without adding a ſingle key to the uſual number, or ſplitting any one of them.

Let every octave in an organ be furniſhed with ſeven couples of pipes, anſwering to the ſeven couples of ſecondary notes *A*※*B*♭, *C*♭*B*※, *C*※*D*♭, *D*※*E*♭, &c, which terminate the 7 dieſes, and by means of ſeven ſtops, let either of the pipes in each couple be made to ſound

(*c*) Art. 5. Sect. VIII.

ſound by the ſame key while the other is ſilent; and ſince both the pipes in any couple are very ſeldom uſed in the ſame piece of muſic, the organiſt before he begins to play, may preſently adjuſt the ſeven ſtops for ſounding ſuch ſecondary notes as he foreſees he has moſt occaſion for.

If ſuch an addition of pipes to the preſent number be thought too expenſive, or too cumberſome in the inſtrument, ſome of the leaſt uſeful pipes may be omitted in each octave, and the expence and incumbrance of the reſt may well be balanced by rejecting ſome entire ſtops of little or no uſe, eſpecially the Cornet and Seſquialter, which only add a noiſe to an organ and deſtroy its harmony.

The like contrivance is eaſily applicable to a harpſichord by adding more ſtrings, and poſſibly without ſuch addition.

14. Another great convenience in ſuch complete ſcales would be the tranſpoſition of muſic into different keys, but little higher or lower than the given one, and yet equally perfect; the frequent want of which in the preſent inſtruments is a great diſadvantage to the voices of the ſingers.

SECTION IX.

Methods of tuning an organ and other inſtruments.

PROPOSITION XVIII.

To find the pitch of an organ.

By the following experiment made upon our organ at Trinity College, I found that the particles of air in the cylindrical pipe called *d*, or *d-la-ſol-re* in the middle of the open diapaſon (*a*) made 262 vibrations, or returns to the places

(*a*) See the notation, Tab. IV. Plate XX.

places they went from, in one ſecond of time. And this number of vibrations is what I call the pitch of the organ.

Having ſuſpended a braſs weight of ſeven pounds averdupois at one end of a copper wire, commonly uſed for ſome of the loweſt notes of a harpſichord, I lapped the other end round a peg, taken from a violin, that turned ſtiffly in a hole made in the wainſcot near the organ. Then by turning the peg to and fro I lengthened or ſhortened the vibrating part of the wire, till it ſounded a double octave below the ſound of the pipe *d* abovementioned. Then having meaſured the length of the vibrating part of the wire, while ſtretched by the weight, from the loop below to the under ſide of the peg, and added to it the ſemidiameter of the peg, I cut the wire, firſt at the point of contact with the ſide of the peg, and then at the loop below; for I found no need of a bridge either above or below. And having repeated the experiment with another piece of wire taken from the same bunch, I found no sensible difference either in the length or weight of the vibrating part; the length being 35.55 inches and the weight 31 grains troy; and the seven pounds averdupois that stretched it, was equal to 49000 grains troy, allowing 7000 for each pound.

Hence by a Theorem hereafter demonstrated (*b*), the number of semivibrations, forwards and backwards together, made by the wire in one second was 131, and the number of such vibrations made by the air in the pipe *d*, two octaves higher, was 4 × 131 (*c*), and the number of complete vibrations was 2 × 131 or 262. Q. E. I.

Scholium.

I made that experiment in the month of September, at a time when the thermometer

(*b*) Prop. XXIV. coroll. 1, 2. (*c*) Sect. 1. Art. 3 and 7.

mometer ſtood at temperate or thereabouts.

But upon a cold day in November I found by a like experiment, that the ſame pipe gave but 254 complete vibrations in one ſecond; ſo that the pitch of its ſound was lower than in September by ſomething more than $\frac{1}{4}$ of a mean tone.

And upon a pretty hot day in Auguſt I collected from another experiment, that the ſame pipe gave 268 complete vibrations in a ſecond of time; which ſhews that its pitch was higher than in November by almoſt half a mean tone.

By ſome obſervations made upon the contraction and expanſion of air, from its greateſt degree of cold in our climate to its greateſt degree of heat (*d*), compared with Sir *Iſaac Newton*'s theory of the velocity of ſounds, I find alſo

(*d*) See Mr. *Cotes*'s xv[th] Lecture upon Hydroſtat. and Pneumat. towards the end.

ſo that the air in an organ-pipe may vary the number of its vibrations made in a given time, in the ratio of 15 to 16; which anſwers to the major hemitone or about $\frac{7}{12}$ of the mean tone and agrees very well with the foregoing experiments.

Coroll. In order to know when the pitch of an organ varies, and when it returns to the ſame again, it is convenient to keep a thermometer conſtantly in the organ-caſe.

PROPOSITION XIX.

The pitch of an organ and the temperament of the Vth *being given, to find the numbers of beats that every* Vth *will make in a given time.*

The muſical notes in Tab. 1, 2, Plate 20, ſhew all the V^{ths} of different names in a complete ſcale of ſounds.

Let v to 1 be the given ratio of the times of the ſingle vibrations of the baſe and

and treble of the given tempered v[th], and N be the given pitch of the organ, that is, the number of complete vibrations made in 1″ by the ſound d in the middle of the Tables, and β the number of beats made in a given time by the v[th] above d, to be found by coroll. 7. prop. XI.

Then if a ſeries of v[ths] equally tempered, aſcended continually from d, as in Tab. 3, the numbers of their beats would continually increaſe in the ratio of 1 to v (a) and therefore would be $\beta, \beta v, \beta v^2, \beta v^3, \beta v^4, \beta v^5, \beta v^6, \beta v^7, \beta v^8, \beta v^9$.

Now as often as theſe v[ths] are depreſſed by an octave, as in Tab. 1 and 2, ſo often muſt theſe beats be divided by 2 (a); which changes that ſeries of beats into this, $\beta, \frac{\beta v}{2}, \frac{\beta v^2}{2}, \frac{\beta v^3}{4}, \frac{\beta v^4}{8}, \frac{\beta v^5}{8}, \frac{\beta v^6}{16}, \frac{\beta v^7}{16}, \frac{\beta v^8}{32}, \frac{\beta v^9}{32}$.

In

(a) Prop. XI. coroll. 2.

In like manner, if the former series of beats were continued downwards from *d*, as in Tab. 3, the numbers of their beats would continually decrease in the ratio of v to 1 or of 1 to $\frac{1}{v}$, and therefore would be, $\frac{\beta}{v}$, $\frac{\beta}{v^2}$, $\frac{\beta}{v^3}$, $\frac{\beta}{v^4}$, $\frac{\beta}{v^5}$, $\frac{\beta}{v^6}$, $\frac{\beta}{v^7}$, $\frac{\beta}{v^8}$, $\frac{\beta}{v^9}$, $\frac{\beta}{v^{10}}$.

Now as often as these v^{ths} are raised by an octave, as in Tab. 1 and 2, so often must these beats be multiplied by 2; which produces this series of beats, $\frac{\beta}{v}$, $\frac{2\beta}{v^2}$, $\frac{2\beta}{v^3}$, $\frac{4\beta}{v^4}$, $\frac{8\beta}{v^5}$, $\frac{8\beta}{v^6}$, $\frac{16\beta}{v^7}$, $\frac{16\beta}{v^8}$, $\frac{32\beta}{v^9}$, $\frac{32\beta}{v^{10}}$, Q. E. I.

Scholium

For example, be it proposed to calculate the number of beats, in Tab. 1, Plate xx, which every vth in the system of mean tones, will make in 15 seconds of time.

Here

Here the temperament of the v^th^ is $\frac{1}{4}$ of a comma, and ſuppoſing the interval of the perfect v^th^ $=$ log. $\frac{3}{2} =$ 0. 17609. 13, we have the comma $c =$ log. $\frac{81}{80} =$ 0. 00539. 50, and $\frac{1}{4}\ c =$ 0. 00134. 88, and the v^th^ $- \frac{1}{4}\ c =$ 0. 17474. 25; which is the logarithm of the number 1. 4953 or $\frac{1.4953}{1}$, that is, of the ratio of 1. 4953 to 1, which in the ſolution of the problem we repreſented by *v* to 1.

Now when the thermometer ſtood at temperate, the pitch of our organ at Trinity College, or the number of complete vibrations made in 1 ſecond by the air in the pipe denoted by *d*, in the middle of our table, was 262 (*c*).

Hence to find the number of beats of the v^th^ above *d*, when tempered flat by $\frac{1}{4}$ comma, in coroll. 7. prop. XI, we have

(*c*) Prop. XVIII.

have $\frac{n}{m}=\frac{2}{3}$, N=262 and $\frac{q}{p}=\frac{1}{4}$, and the required number in $15''=15\times\frac{2q}{161p+q}\times mN=\frac{15.2.1.3.262}{161\times4+1}=\frac{1572}{43}=\beta$, whose logarithm is 1.56298.41; to which adding continually the log. of v, we

Abacus 1.

0.17474.25	v	N° of beats.
1.56298.41	$\beta=36,558$	$\beta=37$
1.73772.66	$\beta v=54,667$	$\frac{1}{2}\beta v=27$
1.91246.91	$\beta v^2=81,747$	$\frac{1}{2}\beta v^2=41$
2.08721.16	$\beta v^3=122,24$	$\frac{1}{4}\beta v^3=31$
2.26195.41	$\beta v^4=182,79$	$\frac{1}{8}\beta v^4=23$
2.43669.66	$\beta v^5=273,34$	$\frac{1}{8}\beta v^5=34$
2.61143.91	$\beta v^6=408,73$	$\frac{1}{16}\beta v^6=26$
2.78618.16	$\beta v^7=611,20$	$\frac{1}{16}\beta v^7=38$
2.96092.41	$\beta v^8=913,95$	$\frac{1}{32}\beta v^8=29$
3.13566.66	$\beta v^9=1366,7$	$\frac{1}{32}\beta v^9=43$

we get the logarithms of βv, βv^2, βv^3 &c, as in the first Abacus, and thence the corresponding numbers, which divided

Abacus 2.

$\bar{1}$. 82525. 75	$\frac{1}{v}$	N° of beats
1. 56298. 41	$\beta = 36, 558$	$\beta = 37$
1. 38824. 16	$\frac{\beta}{v} = 24, 448$	$\frac{\beta}{v} = 24$
1. 21349. 91	$\frac{\beta}{v^2} = 16, 349$	$\frac{2\beta}{v^2} = 33$
1. 03875. 66	$\frac{\beta}{v^3} = 10, 933$	$\frac{2\beta}{v^3} = 22$
0. 86401. 41	$\frac{\beta}{v^4} = 7, 3116$	$\frac{4\beta}{v^4} = 29$
0. 68927. 16	$\frac{\beta}{v^5} = 4, 8896$	$\frac{8\beta}{v^5} = 39$
0. 51452. 91	$\frac{\beta}{v^6} = 3, 2699$	$\frac{8\beta}{v^6} = 26$
0. 33978. 66	$\frac{\beta}{v^7} = 2, 1867$	$\frac{16\beta}{v^7} = 35$
0. 16504. 41	$\frac{\beta}{v^8} = 1, 4623$	$\frac{16\beta}{v^8} = 23$
$\bar{1}$. 99030. 16	$\frac{\beta}{v^9} = 0, 9779$	$\frac{32\beta}{v^9} = 31$
$\bar{1}$. 81555. 91	$\frac{\beta}{v^{10}} = 0, 6540$	$\frac{32\beta}{v^{10}} = 21$

vided by the proper powers of 2, as directed in the solution of the problem, give the latter part of the set of beats opposite to the pitch 262 in Tab. 1.

The log. of v subtracted from 0 gives the log. of $\frac{1}{v}$, which log. continually added to the log. of β, gives the logarithms of $\frac{\beta}{v}$, $\frac{\beta}{v^2}$, $\frac{\beta}{v^3}$, as in the 2[d] Abacus. And these logarithms give the numbers themselves, which multiplied by the proper powers of 2, as above directed, give the former part of the same set of beats opposite to the pitch 262 in Tab. 1.

The superior sets of beats are designed for tuning the same or different organs, when their pitch is higher than this by 1, 2, 3 or 4 quarter-tones, as noted at the beginning of the table, and may be found by the continual addition of the logarithm of a quarter-tone to the logarithms in the Abacus; and the first inferior set of beats may be found

found by the ſubtraction of the log. of a quarter-tone from the ſaid ſet of logarithms, or by the addition of its arithmetical complement: remembering to divide and multiply the correſponding numbers by the ſame powers of 2 as before in the Abacus.

And as the $\frac{1}{4}$ tone is $\frac{1}{8}$ of the III[d], its logarithm is $\frac{1}{8}$ log. $\frac{5}{4} = 0.01211.38$, which continually added to the logarithm of 262, gives the ſucceſſive logarithms of the higher pitches 269, 277, &c, in the firſt column of the table, over againſt the correſponding ſuperior ſets of beats; and ſubtracted from the ſame logarithm of 262, gives the log. of the loweſt pitch 255, over-againſt the loweſt ſet of beats.

From the given temperament of the ſyſtem of equal harmony (*d*) the beats of all the V[ths] may be calculated by the ſame

(*d*) Prop. XVI, Schol. 2, Art. 10.

ſame method; and will be found as in Tab. II. Plate XX.

Coroll. 1. Suppoſing the middlemoſt notes *d*, in the firſt and ſecond tables, to be uniſons, the numbers of the beats in a given time, of any two correſponding v^{ths} are very nearly in the given ratio of their temperaments $\frac{9}{36}c$ and $\frac{10}{36}c$ (*d*), or as 9 to 10. For they would be exactly in that ratio if their ſeveral baſe notes were exactly uniſons (*e*); and the difference of their pitches, at the diſtance of the tenth vth from the middle note *d*, is but ten times the difference of the temperaments, or $\frac{5}{18}c$; which produces the difference of but 1 beat in 290 in the extreme v^{ths} in the tables, and leſs in the reſt in proportion to their diſtances from the note *d* (*f*).

Coroll. 2. In the ſyſtem of equal harmony the ratio of the numbers of beats of

(*d*) Tab. I. prop. XVII. (*e*) Coroll. 4. Prop. XI.
(*f*) Coroll. 2. Prop. XI. and the note in pag. 106.

of the III^{d} and Vth above the ſame baſe note, is 2 to 3 in a given time; as being compounded of the ratio of their temperaments $\frac{2}{18}c$ and $\frac{5}{18}c$, and the ratio of the denominators of the fractions $\frac{4}{5}$ and $\frac{2}{3}$ denoting the ratio of the ſingle vibrations of the trebles, that of the common baſe being 1 (*g*).

Coroll. 3. For the ſame reaſon the beats of the Vth and VIth above the ſame baſe note are iſochronous in the ſyſtem of equal harmony, whereas in that of mean tones they are in the ratio of 3 to 5 in a given time.

PROPOSITION XX.

To tune an organ by a given table of beats.

Caſe 1. If the Pitch of the organ be given (*a*), look for it, or the neareſt to it, in the firſt column of the given table

(*g*) Coroll. 3. Prop. XI. (*a*) Prop. XVII.

ble, Plate XX, and over againſt it you have the proper ſet of beats for tuning the organ, at a time when the thermometer has the ſame height as it had when the pitch was found by experiment.

At that time flatten the treble of the vth above *d* (*b*) more or leſs, till the number of its beats, made in 15 ſeconds, agrees with the tabular number placed over that vth in the proper ſet.

From the treble of that vth *dà* tune downwards the octave *àa*, ſo as to be quite free from beats, and repeat the like operation upon the next aſcending vth *ae*, and the like again upon the next till you have tuned all the ſharp notes in your organ.

Then going backwards from *d*, ſharpen the baſe of the vth below it, more or leſs, till the number of its beats, made in 15 ſeconds, agrees with the tabular number, placed over this vth in the proper ſet.

From

(*b*) See the notation Tab. IV. Plate XX.

From the baſe of the v^{th} *dG*, tune upwards the octave *Gg* and repeat the like operation upon the next deſcending v^{th} *gc*, and the like again upon the next, till you have tuned all the flat notes in your organ.

This being done, let all the other ſounds be made octaves to theſe, and the ſtop will be exactly tuned according to the temperament in the given table; that is, all the v^{ths} will be equally tempered, and conſequently equally harmonious (*c*), and ſo will all the VI^{ths}, and every other ſet of concords of the ſame name, which anſwers the deſign of tuning by a table of beats.

The time may be meaſured either by a watch that ſhews ſeconds, or a pendulum-clock, or a ſimple pendulum that vibrates ſeconds. And the perſon that counts them muſt give a ſtamp with his foot at the beginning and end of the 15 ſeconds; and another perſon that

(*c*) Prop. XIII. coroll. 4.

that counts the beats, at the firſt ſtamp ſhould ſay, None, one, two, three, &c; otherwiſe if he begins with ſaying one, he muſt count one beat more between the ſtamps than the tabular number, as properly ſignifying the number of intervals between the beats.

Caſe 2. If the pitch of the organ be not given, you may try to tune by that ſet of beats in Tab. I, which by any kind of eſtimation you think is the neareſt to the proper ſet; and having tuned upwards any four of the ſucceſſive v^ths^, you muſt attend to the III^d^ or X^th^ between the baſe of the firſt and the treble of the fourth.

If that III^d^ or X^th^ be perfect, you have hit upon the proper ſet in this table, and conſequently have found the pitch of the organ, placed oppoſite to it, which directs you to the proper ſet in the other table in caſe you chooſe to tune by it.

But if that concord be ſharper than a perfect III^d^ or X^th^, the four v^ths^ are not

not flat enough, and muft be flatten-ed a little more, till their beats agree with the numbers of the next fuperior fet, which if the III^{d} or Xth comes out perfect, is the fet required; if not, you muft repeat the like trial upon the fet next fuperior to that, till you find one that produces a perfect III^{d} or Xth.

On the contrary, if at the firft trial the III^{d} or Xth comes out flatter than a perfect one, the four V^{ths} are too flat, and their trebles muft be fharpened, till their beats agree with the next inferior fet and bring out a perfect III^{d} or Xth, either at this or fome other trial upon the fucceffive inferior fets.

Or the required pitch of the organ may be found fomewhat quicker by Tab. II. Tune upwards any three of the fucceffive V^{ths}, except where two octaves intervene, by that fet of beats which you think moft likely to belong to the unknown pitch; then if the beats of the firft V and of the VI above the affumed bafe be ifochronous, you have

 hit

hit upon the proper ſet and the required pitch over againſt it (*d*).

But if that v^{th} beats ſlower than the vi^{th}, make each of the three v^{ths} beat quicker according to the next ſuperior ſet of numbers; and on the contrary, if the v^{th} beats faſter than the vi^{th}, make each of the three v^{ths} beat ſlower according to the next inferior ſet, till you find one that makes the v^{th} and vi^{th} beat equally faſt.

And as the production of perfect iii^{ds}, in tuning by the firſt table, is a ſure teſt of the exactneſs of the performance, ſo that of iſochronous beats of the v^{th} and vi^{th} above the ſame baſe, is as ſure a teſt of exact tuning by the ſecond table; and each of them muſt be conſtantly attended to in the progreſs of the reſpective methods of tuning.

If you chooſe to tune the organ according to the *Hugenian* ſyſtem, the ſet of beats in Tab. I, next below that which anſwers to the pitch, found by any

(*d*) Prop. xix. Schol. coroll. 3.

any of the foregoing methods, will ſerve your purpoſe.

For the *Hugenian* v^{th}, having its temperament $-\frac{1}{4}c + \frac{1}{110}c$ ſmaller than $-\frac{1}{4}c$, in the ratio of 53 to 55 (*e*), beats ſlower than the tabular v^{th} in that proportion, which is but very little ſlower than it would do, if its pitch were depreſſed by $\frac{1}{4}$ of a mean tone. Q. E. F.

Scholium I.

Since our organ at Trinity College was new voiced, and by altering the diſpoſition of the keys was depreſſed a tone lower, and thereby reduced to the Roman pitch, as I judge by its agreement with that of the pitch-pipes made above 30 years ago; by the help of ſuch a pipe one may know by how many quarter-tones the pitch of any other organ is higher than that of ours, and thus determine the proper ſet of beats

(*e*) Prop. XVII. Schol. Tab. 2.

beats for tuning it (*e*); allowance being made for the ſeaſon of the year (*f*).

At a time when the thermometer ſtood at Temperate, as it did alſo when the pitch of our organ was found to be 262, I aſſiſted at the tuning of the v[ths] of the open Diapaſon by the ſet of beats oppoſite to that pitch in Tab. I; and upon examining the III[ds] and X[ths] I found them all perfect: a manifeſt proof of the theory of beats and of the certainty of ſucceſs in tuning by it.

At that time the whole organ was tuned to the open diapaſon, and is now univerſally allowed to be much more harmonious than before, when the major thirds were ſharper than perfect ones; and its harmony, I doubt not, is ſtill improveable by flattening thoſe thirds ſtill more according to the ſyſtem of equal harmony. But at that time I had not finiſhed the calculation of

(*e*) See Column 2, Tab. 1, 2, Plate XX.
(*f*) Scholium Prop. XVIII.

of it, and to repeat the tuning of the organ over again, would be troublesome and improper at the present season, when cold and damp weather is coming on very faſt.

For the propereſt times for tuning an organ ſeem to be from the latter end of Auguſt to the middle of October, when the air being dry, temperate and quiet, will keep nearer to the ſame degree of elaſticity for a given time. Becauſe a very ſmall alteration in the warmth of moiſt air will ſuddenly and ſenſibly alter its elaſtic force and thereby the pitch of ſome of the pipes before the whole ſtop can be accurately tuned.

For that reaſon conſtant care muſt be taken not to heat the pipes by touching them oftener than is needful; not to ſtay too long at a time in the organ-caſe; not to tune early in the morning, but rather towards the evening, when the air is drier and its declining warmth is kept at a ſtay by the warmth of of the perſons about the organ; in a word not to proceed with the tuning whenever the temperature of the air is ſenſibly altered, which is diſcoverable either by a thermometer in the organ-caſe, or by the preſent number of beats made by a given pipe in a given time, compared with a former: and this pipe ſhould be of a ſmaller ſort, as being ſubject to alter its pitch ſooner than the larger ones do; an alteration of the elaſticity and denſity of air in a ſlender column, being ſooner effected by heat, cold or moiſture than in a groſſer.

But theſe and the like cautions may ſooner be learned by a little practice than by any deſcription, and if not altogether neceſſary, will however contribute to the accuracy of tuning by ſo nice a method; which is plainly capable of any deſired degree of exactneſs, provided the blaſt of the bellows be uniform: becauſe the numbers of the beats of each concord may be taken larger, in proportion to any larger time de-

deſigned for counting them, if the tabular numbers be thought too ſmall.

I will add but one obſervation more. After tuning an organ according to any new ſyſtem whatever, we muſt be cautious of judging too haſtily of it. Some muſicians here, who had conſtantly been uſed to major thirds and conſequently major ſixths tuned very ſharp, could not well reliſh the ſmoother harmony of perfect thirds and better ſixths in the organ newly tuned. At firſt they thought it dull and lifeleſs, till after a little uſe they became better ſatisfied with it, and after a longer uſe they could not bear the harſhneſs of other organs tuned in the former manner.

It is therefore neceſſary to have equal experience in different objects of ſenſe, in order to judge impartially, which of the two is more grateful than the other: as is evident in many inſtances, beſides muſic, which any thinking perſon cannot miſs of.

Scho-

Scholium 2.

The following table shews the numbers of beats, made in 15 seconds, by the several concords to the base note D, or D-sol-re

TAB. I.

The beats in 15″ *of all the* ----

VI +2VIII. $\frac{3}{20}$	80	120	$133\frac{1}{11}$
V +2VIII. $\frac{1}{6}$	40	36	$34\frac{8}{11}$
III+2VIII. $\frac{1}{5}$	13	0	$4\frac{1}{4}$
VI + VIII. $\frac{3}{10}$	40	60	$66\frac{6}{11}$
V + VIII. $\frac{1}{3}$	20	18	$17\frac{4}{11}$
III+ VIII. $\frac{2}{5}$	13	0	$4\frac{1}{4}$
VI. $\frac{3}{5}$	20	30	$33\frac{3}{11}$
V. $\frac{2}{3}$	20	18	$17\frac{4}{11}$
III. $\frac{4}{5}$	13	0	$4\frac{1}{4}$
Syftem of	equal harm.	mean tones	M. *Huygens.*
Column	1	2	3

D-ſol-re, at the Roman Pitch, and likewiſe the proportions of the beats of the ſame concords to any other baſe note; with this deſign, that perſons wanting leiſure or proper qualifications for examining

TAB. I.

---- *concords to the baſe* D-ſol-re.

$3^{d}+2$ VIII. $\frac{5}{24}$	96	144	160
$4^{th}+2$ VIII. $\frac{3}{16}$	107	96	93
$6^{th}+2$ VIII. $\frac{5}{32}$	83	0	27
$3^{d}+$ VIII. $\frac{5}{12}$	48	72	80
$4^{th}+$ VIII. $\frac{3}{8}$	53	48	46
$6^{th}+$ VIII. $\frac{5}{16}$	42	0	14
3^{d}. $\frac{5}{6}$	24	36	40
4^{th}. $\frac{3}{4}$	$26\frac{2}{3}$	24	23
6^{th}. $\frac{5}{8}$	$20\frac{4}{5}$	0	7
Syſtem of	equal harm.	mean tones	M. *Huygens.*
Column	1	2	3

amining the principles and conclusions requisite to determine the system of equal harmony, may yet form some judgment of its advantages and disadvantages, when compared with the system of mean tones and that of Mr. *Huygens*; upon this allowed principle, that, *cæteris paribus*, concords to the same base are more or less disagreeable in their kind, for beating faster or slower respectively.

To assist the reader's judgment I have added the following observations resulting from inspection of the table.

1. The beats in column 1 differ less from one another than those in column 2 and 3 do, agreeably to the Name of the system.

2. The beats of the V, V+VIII, V+2VIII in col. 1, are a little quicker than those in col. 2, in the ratio of 10 to 9, and quicker than those in col. 3, in a ratio something greater. But the beats of the VI, VI+VIII, VI+2VIII in col.

col. 1 are much ſlower than thoſe in col. 2 and 3, in the ratio of 2 to 3 and more.

3. Theſe quick beats of the VIth in col. 2 and 3 have the diſadvantage to be doubled in every ſuperior octave, whereas the quick beats of the Vth in col. 1 remain the ſame in the ſecond octave and are only double in the third &c.

4. The beats of the III^{d} and compounds are quicker indeed in col. 1 than thoſe in col. 2 and 3; but being ſlower than the beats of the other concords in all the ſyſtems, they can ſcarce be ſo offenſive as theſe will be.

5. Likewiſe the beats of the 6th and compounds in col. 1, being ſlower than thoſe of the 4th and 3^{d} and compounds with the ſame number of VIII^{ths} in all the ſyſtems, can hardly offend the ear ſo much as the quicker beats of theſe other concords will.

6. The ſums of the beats of all the concords to the note D-ſol-re in both

the col. 1, 2, 3, are respectively 759, 702, 805, whose proportions with respect to the same or any other base note are 38, 35, 40 very nearly. The small excess of the first sum above the second arises chiefly from the beats of the III[d] and 6[th] with their compounds, which

TAB. II.

The order of the harmony ----

VI + 2VIII	240	360	$399\frac{3}{11}$
V + 2VIII	40	36	$34\frac{8}{11}$
III + 2VIII	13	0	$4\frac{1}{4}$
VI + VIII	120	180	$199\frac{7}{11}$
V + VIII	20	18	$17\frac{4}{11}$
III + VIII	26	0	$8\frac{1}{2}$
VI	60	90	$99\frac{9}{11}$
V	40	36	$34\frac{8}{11}$
III	52	0	17
System of	equal harm.	mean tones	M. *Huygens.*
Column	1	2	3

which in all probability are inoffenſive, as we ſaid above.

But a completer rule for comparing the harmony of imperfect concords to a given baſe, appears in the ſecond table; *That concord to which a ſmaller number correſponds, being more harmonious*

TAB. II.

---- *of all the concords.*

3^d +2VIII	480	720	800
4^th+2VIII	321	288	279
6^th+2VIII	415	0	135
3^d + VIII	240	360	400
4^th+ VIII	159	144	138
6^th+ VIII	210	0	70
3^d	120	180	200
4^th	80	72	69
6^th	104	0	35
Syſtem of	equal harm.	mean tones	M. *Huygens.*
Column	1	2	3

nious, in its kind, than any other to which a larger number corresponds; which affords two or three more observations.

7. In col. 1 and 3, the III[ds] become more harmonious by the addition of VIII[ths].

8. In all the systems, the V + VIII is more harmonious than the V and V + 2VIII.

9. In col. 1, the V[th] is more harmonious than the III[d] and VI[th], the V + VIII than the III + VIII and VI + VIII, and likewise the 4[th] and compounds than the 6[th] and 3[d] and compounds with equal numbers of VIII[ths].

10. It may be objected to the system of equal harmony that the beats of the V[ths] are not only a little quicker, but something stronger and distincter than those of the other concords; which deserves to be considered. On the other hand it should be considered too, whether those very quick and less distinct beats of the VI[th] and compounds, have not

not a worſe effect in deſtroying the clearneſs of their harmony.

Theſe are the principal advantages and diſadvantages that occur in comparing theſe ſyſtems. For as to the falſe concords being ſomething worſe in the ſyſtem of equal harmony, than in the other two (*d*), this is no objection to the ſyſtem, but only to the application of it to defective inſtruments; and I have ſhewn above how to ſupply their defects, without the leaſt inconvenience to the performer (*e*).

I ſhall only obſerve, that the firſt table was calculated from the temperaments of the ſyſtems in prop. XVII and ſcholium, by the corollaries to prop. XI; and that the numbers in that table multiplied by the numerators of the known fractions, annexed to the characters of the correſponding concords, produce the correſponding numbers in the ſecond table, according to coroll. 12, prop. XIII.

(*d*) Sect. VIII. Art. 12. (*e*) Sect. VIII. Art. 13.

Scholium 3.

In transferring the foregoing method of tuning from an organ to a harpsichord, as the scale of it is defective, the readiest way is to begin at *Elami* flat in the base (*e*), and thence to ascend by two successive v[ths] and descend by a perfect VIII[th] alternately, till you arrive at *g-sol-re-ut* sharp in the treble; and as you go on, to compare the treble of every third and fourth v[th] with the base of the first, as above directed in tuning an organ. But since the beats of the concords die away too soon for counting a sufficient number of them in a given time, we must be content with estimating and correcting their degrees of celerity by the judgment of the ear; in doing which those that are conversant in tuning organs will find less difficulty than others unacquainted with that instrument.

And

(*e*) See Tab. v. Plate xx.

And herein this obſervation is of uſe, that the higher concords of the ſame name muſt be made to beat quicker than the lower, in the ſame ratio as the vibrations of their ſounds are quicker; that is, if a tone higher in the ratio of 10 to 9 or 9 to 8 nearly, if a III^{d} higher in the ratio of 5 to 4, if a 4th higher, of 4 to 3, if a Vth higher, of 3 to 2, &c, which the ear muſt determine by eſtimation.

But if a machine were contrived, as it eaſily might, to beat like a clock or watch, any given number of times in 15 ſeconds, between 20 and 53 or thereabouts; by ſetting it to beat according to any given number in the table for tuning an organ, and by comparing its beats with thoſe of the correſponding Vth, the ear would determine immediately and exactly enough, whether they were iſochronous or not: and thus a harpſichord might be tuned almoſt to the ſame exactneſs as an organ; and the tuning of an organ might be performed much quicker, by the help of ſuch a machine, than by counting the beats as above.

As the known method of tuning an inſtrument by the help of a monochord is eaſier than any other to leſs skilful ears, and pretty exact too, if the *apparatus* to the monochord be well contrived, it may not be amiſs to ſhew the manner of dividing it, according to any propoſed temperament of the ſcale.

PROPOSITION XXI.

To find the parts of a given monochord, whoſe vibrations ſhall give all the ſounds in an octave of any propoſed tempered ſyſtem.

Let the ſyſtem of equal harmony be propoſed, and let the ſeveral parts of the monochord be meaſured from either end of it, and be to the whole, in the ratios of the ſeveral numbers in the 3[d] column of the table, to 100000; I ſay the vibrations of the parts ſo found, and of the whole, will give all the ſounds

sounds in an octave of the proposed system, as denoted in the first column of the table. Q. E. I.

For in the scholium to prop. XVII we had $2T = 0.09631.05650$ and $2L = 0.06025.35832$; whence we have

$$\begin{aligned} T &= 0.04815.52825 \\ L &= 0.03012.67916 \\ \hline T - L = l &= 0.01802.84909 \\ L - l = d &= 0.01209.83007 \end{aligned}$$

From these logarithms of the tone, limma major and minor and the diesis, and from the logarithm 4.69897.00043 of the number 50000, the uppermost in the table, all the logarithms below it will be found by the following additions: where the musical notes in column 1, are supposed also to represent the logarithms over against them:

$c + d = B$※, $c + l = c^b$, $c + L = B$
$B + l = B^b$, $B + L = A$※, $B + T = A$
$A + l = A^b$, $A + L = G$※, $A + T = G$
$G + l = G^b$, $G + L = F$※, $G + T = F$
$F + d = E$※, $F + l = F^b$, $F + L = E$
&c &c &c

till

till you come down to C, which comes out 5.00000.00000 and ſhews that the logarithms of the principal notes B, A, G, F, E, D are right; and thoſe of the ſecondary notes will be right too, if the operations in the addition be right.

The correſponding numbers in column 3, which may be found by the tables of logarithms, ſhew the required parts of the monochord; as a very little reflection will ſatisfy any one that underſtands the common properties of logarithms, and attends to the intervals of an octave in Fig. 49 deſcribed in Sect. VIII, but not divided as there into 50 equal parts, which is only an approximation to the ſyſtem propoſed. Q. E. D.

Scholium.

The numbers in the 4th and 5th columns of the table ſhew the parts of a monochord, whoſe vibrations will give the

the ſounds of the oppoſite notes in the ſyſtem of mean tones and that of Mr. *Huygens*, who has ſhewn how to find the laſt column of numbers in his *Harmonic Cycle*. And as all the meaſures in the 3 ſyſtems may be taken and marked upon the ſounding-board of the ſame monochord, the different effects of thoſe ſyſtems upon the ear, may be eaſily tried and compared together, provided the tone of the monochord be good and the diviſions accurate, and the moveable bridge does not ſtrain it in one place more than in another.

SECTION X.

Of occaſional temperaments uſed in concerts well performed upon perfect inſtruments.

By a perfect inſtrument I mean a voice, violin or violoncello, &c, with which a good performer can perfectly expreſs any ſound which his ear requires.

PROPOSITION XXII.

The ſeveral parts of a concert well performed upon perfect inſtruments, do not move exactly by the given intervals of any one ſyſtem whatever, but only pretty nearly, and ſo as to make perfect harmony as near as poſſible.

For inſtance, if the baſe be ſuppoſed to move by the beſt ſcale of perfect intervals (*a*), the other part or parts cannot conſtantly move by it too, without making ſome of the concords imperfect by a comma (*b*), which would grievouſly offend the performers (*c*). Conſequently if they are pleaſed, thoſe intervals are occaſionally tempered by the upper part or parts, which therefore do not move by the ſame ſcale which the baſe is ſuppoſed to move by.

Likewiſe if the baſe be ſuppoſed to move by the ſcale of mean tones and lim-

(*a*) See Sect. II. Art. 1. (*b*) Prop. 1. and coroll.
(*c*) Sect. IV. Art. 9.

limmas (*d*), the other part or parts cannot conſtantly do ſo too, without making about two thirds of all the concords imperfect by a quarter of a comma (*e*). But whenever concords are held out by good performers, they ſeem to me to be always perfect. And if ſo, the upper part or parts cannot move by the ſcale of mean tones, which the baſe is ſuppoſed to move by. And the argument is the ſame if the baſe be ſuppoſed to move by any other ſcale of tempered intervals: and that it cannot conſtantly move by perfect ones, I ſhall ſhew in the next ſcholium.

What has been ſaid of perfect and imperfect concords, is applicable to diſcords too, a good ear being critical in both. Now the reaſon why the beſt performers acquire a habit of making perfect harmony, as near as poſſible, is plainly this. When the harmony is made

(*d*) Prop. II. (*e*) Prop. III. coroll. 3.

made perfect they are pleaſed and ſatisfied, though the ſeveral parts do not move by perfect intervals. For the paſſing from one ſound to the next, whether by a perfect or an imperfect interval, being nearly inſtantaneous, is almoſt, if not altogether imperceptible, and therefore is indifferent to the performer. But the ſucceeding conſonance is long enough held out to give him pleaſure or pain, according as he makes it perfect or imperfect. Q. E. D.

Coroll. 1. *Cæteris paribus* the harmony will be the ſame whatever be the ſcale which the baſe moves by; but the ſum of the occaſional temperaments will be the leaſt poſſible, if it moves by the ſcale of mean tones and limmas (*g*), and but very little bigger, if it moves by the ſcale of equal harmony (*h*).

Coroll. 2. The propoſition holds true though ſome of the inſtruments be imperfect; as when the thorough-baſe is play-

(*g*) Prop. III. cor. 10. (*h*) Prop. XX. Schol. Art. 6 and Prop. XI. coroll. 4.

played upon an organ or harpſichord: becauſe the performers of the upper parts are more attentive to make perfect harmony with the baſe notes, than with the chords to them. Conſequently thoſe parts do not move by the tempered ſcale of the thorough-baſe.

Coroll. 3. *Cæteris paribus* the ſame piece of muſic well performed upon perfect inſtruments, is more agreeable than it would be if it were as well performed upon imperfect ones, as an organ, &c.

For nothing gives greater offence to the hearer, though ignorant of the cauſe of it, than thoſe rapid, piercing beats of high and loud ſounds, which make imperfect conſonances with one another. And yet a few ſlow beats, like the ſlow undulations of a cloſe ſhake now and then introduced, are far from being diſagreeable.

Coroll. 4. Therefore the harmony of a concert will be ſmoother and diſtincter, and generally more pleaſing, for taking

taking the chords of the thorough-base as near as can be to the base notes, and no more of them than are necessary, and these few upon the softer and simpler stops of an organ.

Because the beats will then be fewer, slower and softer, and so the voices and other instruments will appear to greater advantage.

Coroll. 5. It appears also from the reasons above, that no voice-part ought ever to be played on the organ, unless to assist an imperfect singér, and keep him from making worse concords with the base and other parts than the organ it self does.

Scho-

(*a*) Aio itaque, si quis canat deinceps sonos, quos Musici notant literis C, F, D, G, C, per intervalla consona, omninò perfecta, alternis voce ascendens descendensque; jam posteriorem hunc sonum C, toto commate, quod vocant, inferiorem fore C priore, unde cani cœpit. Quia nempe ex rationibus intervallorum istorum perfectis, quæ sunt 4 ad 3, 5 ad 6, 4 ad 3, 2 ad 3, componitur ratio 160 ad 162, hoc est 80 ad 81, quæ est commatis. Ut proinde, si novies idem cantus repetatur, jam propemodum tono majore

Scholium.

Mr. *Huygens* obſerved long ago, that no ſingle voice, or perfect inſtrument, can always proceed by perfect intervals, without erring from the pitch at firſt aſſumed (*a*). But as this would offend the performer, he naturally avoids it by his memory of the pitch, and by tempering the intervals of the intermediate ſounds, ſo as to return to it again (*b*).

This is alſo confirmed by what we are told of a monk (*c*), who found, by ſub-

jore, cujus ratio 8 ad 9, deſcendiſſe vocem, tonoque excidiſſe oportet. *Coſmotheoros* lib. 1. pag. 77.

(*b*) Hoc verò nequaquam patitur aurium ſenſus, ſed toni ab initio ſumpti meminit, eodemque revertitur. Itaque cogimur occulto quodam temperamento uti, intervallaque illa canere imperfecta; ex quo multo minor oritur offenſio. Atque hujuſmodi moderamine ferè ubique cantus indiget; uti colligendis rationibus, quemadmodum hic fecimus, facile cognoſcitur. ibid.

(*c*) Methode generale pour former les Syſtême temperés de muſique. Mem. de l' Acad. des Scienç. Ann. 1707. pag. 263. 8vo.

R

ſubtracting all the aſcents of the voice in a certain chant from all its deſcents, that the latter exceeded the former by two commas: ſo that if the aſcents and deſcents were conſtantly made by perfect intervals, and the chant were repeated but four or five times, the final ſound, which in that chant ſhould be the ſame as the initial, would fall about a whole tone below it. But finding that the voices in his quire did not vary from the pitch aſſumed, he concluded that the muſical ratios, whereby he meaſured thoſe ſucceſſive aſcents and deſcents, were erroneous. But had he but known Mr. *Huygens*'s remark, it would have ſolved his difficulty.

This

(*d*) Stetti lungo tempo perpleſſo intorno à queſte forme delle conſonanze, non mi parendo che la ragione, che communemente ſe n' adduce da gli autori, che ſin quì hanno ſcritto dottamente della muſica, foſſe concludente à baſtanza. Dicono eſſi &c. *Diſcorſi attenenti alla Mecanica, Dialogo* 1°, towards the end.

(*e*)

This was not the firſt time that the truth of thoſe muſical ratios had been called in queſtion. For *Galileo* obſerves that the reaſon commonly alledged for it, appeared to him inſufficient (*d*). At laſt indeed he hit upon a couple of experiments which gave him ſatisfaction (*e*), but a ſcientific proof was ſtill wanting till Dr. *Taylor* publiſhed his theory of the vibrating motion of a muſical chord (*f*), which has ſince been cultivated by ſeveral able mathematicians (*g*), and being the principal foundation of Harmonics, deſerves to be further conſidered in the next ſection.

SECT.

(*e*) Ibidem.

(*f*) Methodus incrementorum, prop. 22, 23, and Philoſ. Tranſ. N° 337, IV. or Abridg. by Jones Vol. 4. p. 391.

(*g*) Commentarii Acad. Petropol. Tom. III.
Comment on the Principia Vol. 2. pag. 347.
Mr. Maclaurin's Fluxions, Art. 929.

SECTION XI.

Of the vibrating motion of a musical chord.

PROPOSITION XXIII.

When a musical chord vibrates freely, the force which urges any small arch of it towards the center of its curvature, is to the tension of the chord, in the ultimate ratio of the length of that arch when infinitely diminished, to the radius of its curvature.

I suppose the chord to be uniform and very slender, or rather to be a mathematical line, flexible by the least force and elastic; and its tension or quantity of elastic force to be measured by a weight, which if hung to one end of it, would distend it to the same length which it has when it vibrates freely, by the force of its elasticity.

Fig.

Fig. 50. Let *ACB* represent such a chord, fixed at the points *A* and *B*, *CD* any arch of it, *CE* and *DE* tangents at *C* and *D*; in either of which as *EC* produced if need be, take *EF* equal to *ED*, and draw *FG* perpendicular to *FE*, and *DG* to *DE*, and joining *DF* produce *GE* towards *H*.

Then imagine the chord to keep its curvature while a force applied at *E* or *H* draws the tangents *EC*, *ED* and these the points of contact *C*, *D*, so as to keep them in *æquilibrio*. And since the elastic forces at *C* and *D* are each equal to the force of tension, the direction of the third force at *E* will bisect the angle *CED* under the other two directions, and consequently will coincide with the line *GEH*, agreeably to the construction of the equal triangles *EDG*, *EFG*.

Hence the three forces at *H*, *C* and *D*, which would keep the point *E* at rest, are proportional to the sides *DF*, *FG*, *GD* of the isoscelar triangle *DFG*, to which their several directions *EH*, *EC*, *ED* are perpendicular; becauſe that triangle is ſimilar to any other, as *EDI*, whoſe ſides are either parallel to, or in part coincident with thoſe directions, and therefore proportional to the forces acting in them, by the known theorem in Mechanics (*a*).

Now ſuppoſe *CL* and *DM* to be the *radii* of the curvatures of the chord at the points *C*, *D*, and the curve *LM* to be the *locus* of all the centers of the curvatures at every point of the arch *CD*. Then conceiving the point *D* to move up to *C*, and conſequently *M* and *G* up to *L*, the limit of the variable ratio *DF* to *FG*, of the ſaid forces, will be that of the evaneſcent arch *CD* to *CL* the *radius* of its curvature. And a force conſtantly equal and oppoſite to the former of the two, is that which urges the vaniſhing arch *CD* in the direction *EG*, which ultimately coincides with *CL*; and the latter was the force of tenſion. Q. E. D.

Coroll.

(*a*) See Theorem 33 of Keill's Phyſics.

Coroll. When a musical chord vibrates freely, the forces which accelerate its smallest equal arches, are constantly proportional to their curvatures very nearly, provided the latitude of the vibrations be very small in proportion to the length of the chord.

For the force of its tension being then very nearly invariable, the forces which accelerate its smallest equal arches are very nearly in the inverse ratio of the *radii* of their curvatures (*b*), which is the same as the direct ratio of the curvatures themselves.

DEFINITION OF THE HARMONICAL CURVE.

Fig. 51. *Let* C *be the common center of any two circles* DF, EG, *and* CDE, CFG *any two semidiameters, and of either of the included arches as* DF, *let* FH *be the sine, in which produced both ways, let the lines* HI *and* HK *be severally equal to the other arch* EG; *then*

(*b*) By the present Proposition.

then while the ſemidiameter CFG moves round the center C and carries with it the line IFHK, parallel to it ſelf and conſtantly equal to twice the arch EG, the extremities I, K will deſcribe a curve whoſe vertex is D and axis DC, and whoſe baſe ACB is equal to the ſemicircumference of the circle EG.

Coroll. 1. Fig. 52. Drawing *FL* perpendicular to the baſe *ACL*, a line *KP*, perpendicular to the curve at *K*, will be parallel to *EL*.

For drawing *KN* perpendicular to the baſe, let the radius *CFG* go forwards a little into the place *Cfg*, and carry the line *KHFI* into the place *khfi*, cutting *KN* in *O* and *FL* in *r*. Then ſince $HK = EG$ by the definition, and alſo $hk = Eg$, their difference Ok is $= Gg$. Now by the ſimilar triangles *CLF* and *Frf*, *CFf* and *CGg*, *OKk* and *NPK*, we have $CL : CF :: Fr : Ff$, and $CF : CG :: Ff : Gg$, and *ex æquo* $CL : CG$ or *CE*

::

:: ($Fr : Gg :: OK : Ok ::$) $NP : NK$. Consequently the right angled triangles CLE, NPK are equiangular, and the perpendicular KPM is parallel to the line EL.

Coroll. 2. At any point K the radius of curvature $KM : LE :: LE$ *quad.* : $KN \times CE$.

For drawing fl parallel to FL; another line kM, perpendicular to the curve at k, will be parallel to El, by coroll. 1; consequently if the arch Kk be infinitely diminished, either of the coinciding perpendiculars KM, kM will be the radius of the curvature at K.

In the line El take $Es = EL$ and joining Ls, the triangles LEs and KMk, CEL or CEl and sLl, are ultimately equiangular.

Now KN or $FL : LC :: fr : rF$ or KO, and $LC : LE :: PN : PK :: KO : Kk$, and *ex æquo*, $KN : LE :: fr : Kk$;

But $CE : LE :: sL : Ll$ or fr, therefore *componendo*, $KN \times CE : LE$ *quad.* :: ($sL : Kk ::$) $LE : KM$.

Coroll.

Coroll. 3. Hence if the ratio of the circles *CEG*, *CDF* be vaſtly great, the curvature at any point *K* will be extremely ſmall, and its radius *KM* : *CE* :: *CE* : *KN* very nearly; becauſe the lines *LE* and *CE* will be very nearly equal.

Coroll. 4. Upon the ſame ſuppoſition, the very ſmall curvatures at any points *D*, *K* are very nearly in the ratio of their diſtances *DC*, *KN* from the baſe *AB*.

For when *CE* and conſequently *AB* is given, the curvature at *K*, being reciprocally as its radius *KM*, is directly as *KN* by coroll. 3.

Coroll. 5. Fig. 53. While the greater circle remains let the leſſer be diminiſhed, and the curve *AKDB* will be changed into another *Aϰδ B* of the ſame *ſpecies*, and every ordinate to the common baſe will be diminiſhed in the ſame ratio, that is, *NK* : *Nϰ* :: *CD* : *Cδ*.

Fig. 52. For while any arch *EG* equal to *HK* or *CN* is given in magnitude, let the other radius *CD* or *CF* be diminiſhed,

ed, and becauſe the triangle *CFH* retains its *ſpecies*, the line *CH* or *NK* is diminiſhed in the ſame ratio with *CF* or *CD*.

Coroll. 6. Fig. 53. When the axes *CD*, *Cδ* of two curves are very ſmall in compariſon to their common baſe *AB*, the curvatures at the tops of any two coincident ordinates *NK*, *Nϰ*, are in the ratio of the ordinates.

For if $\varkappa\mu$ be the radius of curvature at $\varkappa$, by coroll. 3 we have $KM \times KN = CE\,quad. = \varkappa\mu \times \varkappa N$; whence $\varkappa N : KN :: KM : \varkappa\mu$, that is, as the curvature at $\varkappa$ to the curvature at *K*.

Coroll. 7. Hence, ſuppoſing the curve *AKDB* to have the elaſticity and tenſion of a muſical chord, it will vibrate to and fro in curves very nearly of the ſame ſpecies with the given curve *AKDB*, provided none of the vibrations be too large.

For let the firſt effort of the tenſion reduce that curve into ſome other, as *Aϰδ B*, in the firſt moment of time; and

and ſince the ordinates *DC*, *KN* are in proportion as the curvatures at *D* and *K* by coroll. 4, and theſe curvatures as the accelerating forces at *D* and *K* (*c*), acting in the directions *DC*, *KM* or *KN* very nearly, and theſe forces as the velocities generated by them in that time, and the velocities as the naſcent ſpaces *Dδ*, *Kκ*; *alternando*, we have *DC* : *Dδ* :: *KN* : *Kκ* and *dividendo*, *DC* : *δC* :: *KN* : *κN*. Conſequently by coroll. 5 the curve *AκδB* is very nearly of the ſame ſpecies with *AKDB*. And in the next moment it will be changed into another of the ſame ſpecies, and ſo on, till every point of the chord be reduced to the baſe *AB* at the ſame inſtant. And by the motion here acquired it will be carried towards the oppoſite ſide of the baſe, till by the oppoſition of the tenſion, it ſhall loſe all its motion by the ſame degrees, and in the ſame curves, by which it was acquired; and thus

(*c*) Coroll. prop. XXIII.

thus the chord will continually vibrate in curves of the ſame ſpecies as the firſt, neglecting the ſmall difference in the directions *KN*, *KM*, and the reſiſtance of the air.

Coroll. 8. The ſmall vibrations of a given muſical chord are iſochronous.

For if the chord at the limit of its vibration aſſumes the form of the harmonical curve, it will vibrate to and fro in curves of that ſpecies by coroll. 7, and its ſeveral particles, being accelerated by forces conſtantly proportional to their diſtances from the baſe *AB* (*d*), will deſcribe thoſe unequal diſtances in equal times, like a pendulum moving in a cycloid.

If the chord at the limit of its vibration aſſumes any other form, it will cut an harmonical curve, equal in length to it, in one or more points, as *A*, *K*, *L*, *B* in Fig. 54; and the intercepted parts of the chord will be more or leſs incurvated towards *AB* than the correſponding parts of the curve, according as they fall

(*d*) Cor. prop. XXIII and cor. 6. Defin. curve.

fall without or within them; and will accordingly be accelerated by greater or ſmaller forces than thoſe of the correſponding parts of the curve (e). Therefore, ſuppoſing the chord and curve to differ in nothing but their curvatures, the difference of the curvatures of the correſponding parts will be continually diminiſhed by the difference of their forces, till the parts coincide either before, or when they arrive at the baſe *AB*. And thus the times of the ſeveral vibrations of the chord will be the ſame as thoſe of the curve, and therefore equal to one another.

Coroll. 9. The Figure contained under the harmonical curve and its baſe, is of the ſame ſpecies as the Figure of Sines.

Fig. 52. For ſuppoſing the circle *DFQ* to grow bigger till it becomes equal to *EGR*, the figure *AKDB* will become a figure of ſines. Becauſe any ordinate *KN* to the abſciſs *AN* or arch *GR*, being conſtantly equal to *FL*, will then

(e) Cor. prop. XXIII.

then be equal to the ſine of the arch *GR*; and thus every ordinate as *KN* is increaſed in the given ratio of *CF* to *CG*, or *CD* to *CE*. And on the contrary the ſeveral ordinates in the ſaid figure of ſines diminiſhed in that conſtant ratio of *CE* to *CD*, are the ordinates in the figure *AKDB* of a muſical chord in motion.

PROPOSITION XXIV.

The vibrations of a muſical chord ſtretched by a weight, are ſynchronous to thoſe of a pendulum, whoſe length is to the length of the chord, in a compound ratio of the weight of the chord to the weight that ſtretches it, and of the duplicate ratio of the diameter of a circle to its circumference.

Fig. 54. If P be the weight that ſtretches the chord *ADB*, and *DM* be the radius of its curvature at the vertex *D*, the force that urges any ſmall particle *Dd* towards *C* is $= \frac{Dd}{DM} \times$ P by prop. XXIII.

And

And ſince *Dd* vibrates like a pendulum (*d*), if it were ſuſpended by a ſtring $OP = DC$ in a cycloid $QPR = 2DC$ or *DCF*, and were urged at the higheſt points *Q*, *R* by a force acting downwards like that of gravity, but equal to the ſaid force $\frac{Dd}{DM} \times P$, which urges *Dd* at the limits *D*, *F* of its vibrations; the times of thoſe oſcillations and of theſe vibrations would be equal to one another. Becauſe the forces being alſo equal at all other equal diſtances of the particle from *P* and from *C*, would impel it through equal parts of the equal lines *QP*, *DC* in equal times.

Again putting *p* for the weight of the chord *ADB* or *ACB*, the weight of its particle *Dd* is $= \frac{Dd}{AB} \times p$.

Hence if another ſtring L be to the ſtring *OP* or *DC*, as this latter weight $\frac{Dd}{AB} \times p$ is to the former $\frac{Dd}{DM} \times P$, equivalent

(*d*) Coroll. 9. Def. of the curve.

õo.

B

valent to the force at D; and the particle Dd be again ſuſpended by the ſtring L in another cycloid of the length 2L; ſince at the higheſt points of this cycloid the particle is urged downwards by the whole force $\frac{Dd}{AB} \times p$ of its own gravity, its oſcillations will be ſynchronous to thoſe of the former pendulum (*e*). Becauſe we took their lengths in the ratio of the forces that act upon them at the higheſt points of the cycloids, that is, $L : DC :: p \times DM : P \times AB$; which two ratios compounded with DC to AB, give $L : AB :: p \times DM \times DC$ or $p \times CEq$ (*f*) : $P \times ABq$, which was to be proved. For CEq is to ABq in the duplicate ratio of the diameter to the circumference, by the definition of the curve; and we ſhewed above, that every particle of the curve vibrates in the ſame time with the middlemoſt. Q. E. D.

Coroll.

(*e*) Coroll. Theor. 4 De Motu Pend. in Mr. *Cotes*'s Harmon. Menſurarum. (*f*) Coroll. 3. Defin.

Coroll. 1. The time of a ſemivibration, forwards or backwards, of the chord AB meaſured by inches and decimals, is $\frac{113}{355}\sqrt{\frac{p}{P}} \times \frac{AB}{39.126}$, and its reciprocal is the number of ſuch vibrations made in one ſecond.

For the length of a pendulum that vibrates forwards or backwards in one ſecond, is 39.126 inches in the latitude of London, and the diameter to the circumference of a circle as 113 to 355 very nearly, and the times of the vibrations of pendulums are in the ſubduplicate ratio of their lengths. Whence putting t for that of the pendulum L, we have $t'' : 1'' :: \sqrt{L} = \frac{113}{355}\sqrt{\frac{p}{P}} \times AB : \sqrt{39.126}$, and $t'' = \frac{113}{355}\sqrt{\frac{p}{P}} \times \frac{AB}{39.126}$, and $\frac{1}{t} = \frac{355}{113}\sqrt{\frac{P}{p}} \times \frac{39.126}{AB}$, the number of ſemivibrations made in one ſecond.

Coroll. 2. Suppoſing the laſt number to be n, we have the logarithm of $n^2 =$ log. $\frac{P}{p \times AB} + 2,58676.52698$, which gives n very expeditiouſly.

For

For the logarithm of $\overline{\frac{355}{113}}|^2 \times 39.126$ $= 2,58676.52698$.

Coroll. 3. If the lengths and tensions of two chords be equal, the times of their vibrations are in the subduplicate ratio of their weights, by coroll. 1.

Coroll. 4. If their lengths and weights be equal, the times of their vibrations are reciprocally in the subduplicate ratio of their tensions, by coroll. 1.

Coroll. 5. If their tensions be in the ratio of their weights, the times of their vibrations are in the subduplicate ratio of their lengths, by coroll. 1.

Coroll. 6. The weights of cylindrical chords are in a compound ratio of their specific gravities, lengths and squares of their diameters, that is, p is as $s \times AB \times d^2$; whence t is as $AB \times d\sqrt{\frac{s}{p}}$, by coroll. 1.

Coroll. 7. Hence, if the tensions and diameters of homogeneal chords be equal, the times of their vibrations are in the ratio of their lengths.

Coroll. 8. If the tenſions and lengths of homogeneal chords be equal, the times of their vibrations are in the ratio of their diameters.

Coroll. 9. If the tenſions of ſimilar chords be as their ſpecific gravities, the times of their vibrations are in the duplicate ratio of their lengths or of their diameters (*g*).

Scholium.

Hence we may find the number of vibrations made in a given time by any muſical ſound, if we compare it with the ſound of a given chord ſtretched by a given weight.

For example in the experiment abovementioned (*h*) I found the length of the vibrating chord $AB = 35.55$ inches

(*g*) See *Galileo*'s experiments on chords. Dialogo 1°. attenente alla Mecanica, towards the end.

(*h*) Prop. XVIII.

inches and its weight $p = 31$ grains troy; and the ſound of it, when ſtretched by the weight $P = 7$ pounds averdupois $= 49000$ grains troy, was two octaves below the ſound of the pipe *d* there mentioned. Hence, by coroll. 2, we have $n = 131.04$, the number of ſemivibrations made in one ſecond by the wire *AB*, and $4n = 524.16$, the number of ſemivibrations made by $\frac{1}{4}AB$, by coroll. 7, or by the pipe *d*; which is double the number 262 of its whole vibrations.

The length of the cylindrical part of that pipe was exactly 21.6 inches, and its diameter 1.9, which I mention becauſe the experiment, being accurately made, is of uſe upon other occaſions.

ADVERTISEMENT.

THOUGH the theory of imperfect consonances has been demonstrated pretty clearly, I hope, in the sixth Section, yet as I had considered some parts of it in different lights and searched a little further into some others for my own diversion, I thought it not amiss to print my papers in the form of the following Additions; that if the reader should desire any further information, he may have recourse to them whenever he pleases.

A scholium to prop. viii, explaining an uncommon case of it.

An illustration of prop. xi, with a scholium confirming the theory of the beats of imperfect consonances.

Another demonstration of scholium 4. prop. xi, (concerning the analogy between audible and visible undulations) and of prop. vii.

Another demonstration of prop. xiii and its third corollary, with an illustration of the first, and a scholium or two confirming the theory of the harmony of imperfect consonances, and shewing the absolute times and numbers of their vibrations, short cycles and dislocations of their pulses, contained in the periods between their beats.

A scholium to prop. xx, shewing the methods of altering the pitch of an organ-pipe in order to tune it,

Scho-

Scholium to Prop. viii.

Fig. 63. WHEN different multiples as $3AB$ and $2ab$ of the vibrations AB, ab of imperfect unisons, are the single vibrations AD, ac of an imperfect consonance, the multipliers 3 and 2 are in the ratio of the single vibrations $3AB$ and $2AB$, or $3ab$ and $2ab$ of the perfect consonance, and therefore should be irreducible to smaller numbers. The different multiples of the vibrations of imperfect unisons are therefore supposed in the proposition to be the least in the same ratio.

Fig. 63, 64. But if different multiples of the vibrations AB, ab, as $6AB$ and $4ab$, whose multipliers 6 and 4 are reducible by a common divisor, be the single vibrations of an imperfect consonance, (as they may by intermitting 6—1 pulses of AB and 4—1 of ab, so as to leave single pulses at first and between every intermission,) the period of the imperfections of this consonance will not be equal to that of the imperfect unisons AB, ab, but multiple of it by 2, the greatest common divisor of the multipliers 6 and 4.

For those multiple vibrations $6AB$ and $4ab$ are the same as $3\times 2AB$ and $2\times 2ab$, or $3AC$ and $2ac$, in which the equimultiples $2AB$ and $2ab$, or AC and ac in fig. 64, are the single vibrations of other imperfect unisons, resulting from an intermission of every second pulse of AB and ab in fig. 63; and the period of their imperfections is equal to that of $3AC$ and $2ac$, or AG and ae by this VIII[th] proposition, and is the same multiple of the period of AB and ab, as AC is of AB, or ac of ab by the coroll. to the proposition, that is by 2, the greatest common divisor of the multipliers 6 and 4.

An

An illustration of Prop. xi.

THUS in Fig. 23, after taking away 9 ſhort cycles from each end of the cycle *AU* of imperfect uniſons, there remains *kKLm*, part of two more; and in Fig. 25, after taking away 3 ſhort cycles from each end of the period *AX*, there remains *dDe*, part of another; and in Fig. 27, after taking away 4 ſhort cycles of imperfect octaves from each end of the period *Aw*, there remains *iILm*, part of two more; and laſtly in Fig. 34, after taking away 2 ſhort cycles of imperfect v[ths] from each end of the cycle *AZ* of the imperfect uniſons, there remains *nNQr*, part of another: which though not ſituated exactly in the middle of *AZ*, by reaſon of the part *Δ ε Z*, of another ſhort cycle, containing equimultiples of *AB* and *ab*, is very near it, when the number of ſhort cycles in the period is very large as uſual; in which caſe the beats will be made very nearly in the middle of every period.

Scholium.

It is very unreaſonable to ſuppoſe with Mr. *Sauveur* that the beats are made by the united force of the coincident pulſes of imperfect uniſons (*a*).

For while the imperfect uniſons are made to approach gradually to perfection, experience ſhews that they always beat ſlower and ſlower (*b*) and by theory (*c*) the periods of their pulſes grow longer and longer. Therefore in conſequence of this gentleman's hypotheſis, the uniſons ſhould alſo beat at the ends of the periods where the pulſes do not coincide: Becauſe

(*a*) See Note pag. 114. (*b*) See Phænomenon pag. 107.
(*c*) Prop. VII. coroll. 5.

caufe it is very improbable that the cycle of the firft unifons fuppofing it fimple, while it lengthens gradually, will not fometimes be changed into periods as well as into other fimple cycles.

Nor can it be allowed that the unifons will beat only at the ends of their complex cycles. For according as the numeral terms expreffing the ratios of the fingle vibrations of the feveral fucceffive unifons, happen to be reducible or not, or to be irrational, the cycles of the pulfes will fometimes be fhortened, fometimes lengthened again, fometimes invariable and fometimes impoffible, as fhall be explained by and by; which accidents difagree with the gradual and conftant retardation of the beats in the prefent cafe.

If it be faid that the pulfes next to the periodical points, fall fo clofe to one another, as to affect the ear in the fame manner as if they were quite coincident; it may be fo, and moft probably is fo. And then it will follow that the harmony of the fhort cycles terminated by fuch clofe pulfes, will there be the fame as that of perfect unifons; at leaft it will certainly be better about the periodical points and coincident pulfes than any where elfe in the periods. But the found of a beat has no harmony in it; on the contrary it rather refembles the common found of a beat or ftroke upon any grofs, irregular body. And this found refults from pulfes of air which rebounding from different parts of the body, difpofed to vibrate in different times, will ftrike the ear one after another at irregular intervals, like thofe of the pulfes in the middle between the periodical points of imperfect unifons. Therefore thefe are the only pulfes in each period which can excite the idea of the beats of imperfect unifons. And the like argument is applicable to any other imperfect confonance by prop. VIII.

Fig.

Fig. 55. As to thoſe uncertain lengths abovementioned of the ſimple and complex cycles of the pulſes of imperfect uniſons, while their interval is continually diminiſhed or increaſed; let one of the ſounds be fixt and the time of its ſingle vibration be repreſented by any given line V and thoſe of the variable ſound by the ſucceſſive lines A, B, C, &c, all which lines may conſtitute any increaſing or decreaſing progreſſion; and ſuppoſing n to repreſent any large given number, let $A : V :: n : a$, $B : V :: n : b$, $C : V :: n : c$, &c.

Then will the cycles of the pulſes of V and A, V and B, V and C, &c, be $nV = aA$, $nV = bB$, $nV = cC$, &c, provided every one of the numbers a, b, c, &c, be integers and primes with reſpect to the aſſumed number n. In which caſe the ſeveral cycles are equal to one another and to nV.

But if the terms of all or any of thoſe ratios have a common diviſor, the correſponding cycles will be ſhortened in proportion as the greateſt common diviſors are larger; and therefore their lengths cannot increaſe or decreaſe ſucceſſively in regular order while the ſucceſſive intervals of the uniſons continually decreaſe or increaſe, unleſs the greateſt common diviſors decreaſe or increaſe in regular order too; which can happen but very rarely.

And when the terms of the ratios of any of the vibrations happen to be incommenſurable, a ſecond coincidence of their pulſes will be impoſſible: becauſe no multiple of one vibration can be equal to any multiple of the other.

But in all caſes whatever, the periods of the pulſes of V and A, V and B, V and C, &c, which are $\frac{nV}{n-a}$, $\frac{nV}{n-b}$, $\frac{nV}{n-c}$, &c (*a*), will decreaſe continually

(*a*) Def. III. Sect. VI.

ly in the ſame proportions with the fractions $\frac{n}{n-a}$, $\frac{n}{n-b}$, $\frac{n}{n-c}$, whoſe magnitudes can never be altered by any common diviſors of their terms, whether integers, fractions or ſurds.

Coroll. 1. Hence in all caſes, the periods of V and A, V and B, V and C, &c, are inverſely in the ſame proportions as $\frac{n-a}{n}$, $\frac{n-b}{n}$, $\frac{n-c}{n}$, &c, or as $\frac{A-V}{A}$, $\frac{B-V}{B}$, $\frac{C-V}{C}$, &c, or as the magnitudes of the ratios of the vibrations, A to V, B to V, C to V, &c (*a*), or as the intervals of their ſounds (*b*); as we proved before in coroll. 5. prop. VII.

Coroll. 2. If the vibrations V, A, B, C, &c, be a geometrical progreſſion, and conſequently the ſucceſſive intervals of their ſounds be equal, the periods of the pulſes of V and A, V and B, V and C, &c, will be alſo a geometrical progreſſion.

For ſince $nV = aA = bB = cC =$ &c, the numbers n, a, b, c, &c, are the reciprocals of V, A, B, C, &c, and conſequently are alſo a geometrical progreſſion. Whence $n-a$, $n-b$, $n-c$, &c, and $\frac{nV}{n-a}$, $\frac{nV}{n-b}$, $\frac{nV}{n-c}$, &c, are alſo geometrical progreſſions.

Coroll. 3. Suppoſing n, a, b, c, &c, to be an arithmetical progreſſion, the vibrations V, A, B, C, &c, and the periods of V and A, V and B, V and C, &c, will be muſical progreſſions. Becauſe $n-a$, $n-b$, $n-c$, &c, is alſo an arithmetical progreſſion, and the pro-

(*a*) See pag. 89. Note (*b*) Sect. 1. Art. 10.

progressions of the reciprocals V, A, B, C, &c, and $\frac{nV}{n-a}, \frac{nV}{n-b}, \frac{nV}{n-c}$, &c, of arithmetical progressions are called musical progressions.

For further illustration I will add an example or two of this last corollary.

$n = 319$	Simp.&comp.cycles	Simple cycle & periods.
$a = 320$	$319V = 320A$	$\frac{1}{1} \times 319V = \frac{1}{1} \times 320A$
$b = 321$	$319V = 321B$	$\frac{1}{2} \times 319V = \frac{1}{2} \times 321B$
$c = 322$	$319V = 322C$	$\frac{1}{3} \times 319V = \frac{1}{3} \times 322C$
$d = 323$	$319V = 323D$	$\frac{1}{4} \times 319V = \frac{1}{4} \times 323D$

$n = 320$	Simp.&comp.cycles	Simple cycles & periods.
$a = 321$	$320V = 321A$	$320V = 321A$
$b = 322$	$320V = 322B$	$160V = 161B$
$c = 323$	$320V = 323C$	$106\frac{2}{3}V = 107\frac{2}{3}C$
$d = 324$	$320V = 324D$	$80V = 81D$

It is well known that the terms of an arithmetical progression as n, a, b, c, &c, whose common difference is but a small part of any of them, are very nearly in geometrical progression. Consequently since $80V = 81D$, these 4 simple cycles and periods are equal respectively to the times between the successive beats of imperfect unisons when their interval is about

1,

1, 2, 3, 4 quarters of a comma (*a*), or to the times between the ſucceſſive beats of any imperfect conſonance whoſe vibrations are equal to any different multiples of thoſe of the uniſons (*b*) and whoſe temperament is reſpectively about 1, 2, 3, 4 quarters of a comma.

I have lengthened this ſcholium not ſo much to confute Mr. *Sauveur*'s conjecture, which I had done before, as to illuſtrate the theory of the periods and beats of imperfect conſonances: and with the ſame deſign I will add

Another demonſtration of ſcholium 4. *Prop. xi, and of Prop. vii.*

THE breadth of the apparent Undulations of the lights and ſhades ſeen at a diſtance upon two rows of parallel objects, may be alſo found by the following conſtruction.

Fig. 56. Let a plane paſſing through a diſtant eye at *z*, cut the axes of the parallel objects at right angles in the points *a*, *b*, *c*, &c, *α*, *β*, *γ*, &c, which are ſuppoſed equidiſtant in both the parallel lines *abc*, *αβγ*. From any object in one of theſe lines to any ſucceſſive objects in the other, draw the lines *αa*, *αb*, *αc*, &c, and the lines *zvV*, *zxX*, *zyY*, &c, drawn parallel to them, will intercept the equal breadths of the apparent undulations.

Becauſe while the eye is gradually directed from the middle of any of the breadths *VX*, *XY*, &c towards either of its extremities, the objects will appear cloſer together in couples, in proportion to their ſmaller diſtances

(*a*) Sect. II. Art. 4. (*a*) Prop. VIII.

diſtances from the next extremity, (which was ſhewn in the ſcholium to be the cauſe of the undulations).

For the lines $d\delta$, $e\varepsilon$, $f\zeta$, $g\eta$, &c, being parallel to $a\alpha$, are parallel to Vv by conſtruction; and the lines $i\theta$, $k\iota$, $l\kappa$, $m\lambda$, &c, being parallel to $b\alpha$, are parallel to Xx; and ſo on. Let lines drawn from z through the objects δ, ε, ζ, &c, of one row, cut the line of the other in D, E, F, &c. Then becauſe the rows are parallel, the ratio of $D\delta$ to δz, $E\varepsilon$ to εz, $F\zeta$ to ζz, &c, is the ſame as Vv to vz, or Xx to xz, &c (*a*). Whence alſo, becauſe of the parallels between the rows, we have

$$\left.\begin{array}{l} Dd : dV :: D\delta : \delta z \\ Ee : eV :: E\varepsilon : \varepsilon z \\ Ff : fV :: F\zeta : \zeta z \\ Gg : gV :: G\eta : \eta z \\ Hi : iX :: H\theta : \theta z \\ Ik : kX :: I\iota : \iota z \\ Kl : lX :: K\kappa : \kappa z \\ Lm : mX :: L\lambda : \lambda z \\ \text{\&c.} \qquad\quad \text{\&c.} \end{array}\right\} :: Vv : vz;$$

That is, all thoſe ratios are equal, and, alternately, the leſſer apparent intervals Dd, Ee, Ff, Gg, are proportional to their diſtance dV, eV, fV, gV, from the next extremity V of the breadth VX; and alſo Hi, Ik, Kl, Lm, proportional to iX, kX, lX, mX, their diſtances from the next extremity X of the ſame breadth VX. And the breadths VX, XY, &c, are equal, becauſe ab, bc, &c are ſo, and the triangles VzX and $a\alpha b$, XzY and $b\alpha c$, &c are ſimilar by conſtruction. *Q. E. D.*

Coroll.

(*a*) 2. VI. Euclid.

Coroll. 1. The projections DE, EF, &c of the equal intervals $\delta\epsilon$, $\epsilon\zeta$, &c, are to these intervals in the constant ratio of Dz to δz, or $E\epsilon$ to ϵz, or Vv to vz, and consequently are equal to one another. Therefore supposing the lines DE and $\delta\epsilon$ or de to represent the times of the single vibrations of imperfect unisons, the periods of the nearest approaches of their pulses D, E, &c, d, e, &c, are VX, XY, &c; And in going from their extremities V, X to the middle, the alternate lesser intervals between the successive pulses, are proportional to their distances from the next extremity, as we shewed just now: which is another proof of prop. VII.

Coroll. 2. If the eye be moved in a line parallel to the rows, the breadths of the apparent undulations will be constantly the same, and if it be moved uniformly in any other right line, their breadths will vary uniformly, and be constantly proportional to the distance of the eye from the rows. Because the triangles VzX, VzY, &c, are constantly similar to $a\alpha b$, $a\alpha c$, &c. And this conclusion seems to agree with what I have transiently observed of these undulations.

But it is easy to collect from the construction of the figure, and the different ratios of zV to zv expressed by numbers, that the intervals between the apparent conjunctions of the objects will increase and decrease very irregularly; and that no conjunctions can happen except when the eye arrives at certain points of its course, and none at all, mathematically speaking, when its distances from the two rows, measured upon any right line, happen to be incommensurable. Which conclusions being contrary to the continual appearance of the undulations to the eye in all places, and to the regular increase or decrease of their breadth, shew, that the breadth is not equal

equal to the interval between the apparent conjunctions, no more than the interval between the beats of imperfect unisons is equal to the interval between their coincident pulses.

LEMMA.

In any period between the successive beats of any imperfect consonance, any given number of short cycles next to one side of the least dislocation of the pulses, is more harmonious, and the same number of them next to the other side is less harmonious than the same number of them next to either side of the coincident pulses: and these degrees of harmony differ more in those periods where the two least dislocations differ less, and most of all in the periods where these dislocations are equal when possible.

Fig. 59. Let AB and ab represent the times of the single vibrations of imperfect unisons, A and a their coincident pulses, B, C, D, &c, b, c, d, &c, their successive pulses on each side of A, a; Rr their least dislocation in any given period, and consequently the nearest to the periodical point z, which is here placed under A, for the convenience of seeing at one view, the short cycles next to both sides of Rr and Aa.

First I say, the short cycles RS, ST, &c, which include z, are more harmonious, and RQ, QP, &c, less harmonious than AB, BC, &c, the numbers of them being the same: and that the degrees of their harmony differ more in the periods where the two least dislocations Rr, sS differ less, and most of all where $Rr = sS$, when possible (*a*).

For $bB = (AB - Ab = RS - rs =)\ Rr + sS$ (*b*).
And $cC = (AC - Ac = RT - rt =)\ Rr + tT$.
&c. &c.

Hence

(*a*) See prop. VII. coroll. 2. (*b*) Prop. VII. coroll. 1.

6.

Hence the succeſſive diſlocations *sS*, *tT*, &c, are reſpectively ſmaller than *bB*, *cC*, &c, by *Rr*, as appears alſo by their ſmaller diſtances from *z* (*a*). But on the other ſide of *Rr*, the diſlocations *Qq*, *Pp*, &c, are reſpectively greater than *bB*, *cC*, &c, by the ſame *Rr*, for the like reaſons.

Now the ſhort cycle *RS* which includes *z*, is more harmonious than *AB* next to *A*. For though the diſlocations $Rr + sS$ are $= bB$, yet thoſe parts of *bB*, as being ſmaller than *bB*, will give leſs offence to the ear than the whole: the whole may be perceived and give ſome offence even when one or both its parts are imperceptible. And for the ſame reaſons the ſhort cycle *RS* will be ſtill more harmonious than *AB* in other periods where *Rr*, *sS* are leſs unequal, and the moſt harmonious where they are equal when poſſible (*b*): their ſum being every where the ſame.

The next ſhort cycle *ST* is alſo more harmonious than *BC*; the diſlocations *sS*, *tT* being reſpectively ſmaller than *bB*, *cC*. Therefore the ſhort cycles *RS*, *ST*, taken together, are more harmonious than *AB*, *BC* taken together; and ſtill more harmonious in other periods where *sS*, *tT* are ſmaller, till *sS* be equal to *Rr*.

But on the other ſide of *Rr* and *Aa*, the ſhort cycle *RQ* is leſs harmonious than *AB*, the diſlocations *Qq*, *Rr* being larger than *Bb* and *o*. The next ſhort cycle *QP* is alſo leſs harmonious than *BC*; the diſlocations *Pp*, *Qq* being reſpectively larger than *Cc*, *Bb*. Therefore *RQ*, *QP* together are leſs harmonious than *AB*, *BC* together; and ſtill leſs harmonious in other periods where *Rr*, *Qq*, *Pp* are larger, till *Rr* be equal to *sS*. And the ſame is evident in any larger

(*a*) Prop. VII. (*b*) Prop. VII. coroll. 2.

larger equal numbers of ſhort cycles, throughout the period between the ſucceſſive pulſes.

Secondly, any imperfect uniſons will be changed into imperfect octaves whoſe ſingle vibrations are *AC* and *Ab*, or *Ac* and *AB*, by conceiving every ſecond pulſe of the ſeries *A*, *B*, *C*, &c, or *a*, *b*, *c*, &c, to be intermitted, which would depreſs one of the uniſons an octave lower.

Now if that intermiſſion ſhould take away the alternate pulſes *S*, *U*, &c, or *s*, *u*, &c, the ſhort cycles of the octaves, next to one ſide of *Rr*, will be *RT*, *TW*, &c, and on the other, *RP*, *PN*, &c: I ſay the former as including *z* are more, and the latter leſs harmonious than *AC*, *CE*, &c, the numbers of them being equal.

For we had $Rr + tT = cC$, conſequently the ſhort cycle *RT* is more harmonious than *AC*, for the ſame reaſon as in uniſons, and becauſe the intermediate diſlocations *sS*, *bB* are vaniſhed, one of their conſtituent pulſes in each being taken away. And *RT* is ſtill more harmonious than *AC* in other periods where *Rr* and *tT* are leſs unequal.

The next ſhort cycle *TW* is alſo more harmonious than *CE*, the diſlocations *tT*, *wW* being reſpectively ſmaller than *cC*, *eE*, as in uniſons; and is ſtill more harmonious in other periods where *tT*, *wW* are ſmaller, that is where *tT* and *Rr* are leſs unequal.

But on the other ſide of *Rr* and *Aa*, the ſhort cycle *RP* is leſs harmonious than *AC*, and *PN* than *CE*, the diſlocations *Rr*, *Pp* being reſpectively bigger than *o* and *Cc*; and *Pp*, *Nn* bigger than *Cc*, *Ee*, reſpectively: and is ſtill leſs harmonious in other periods where *Rr*, *Pp*, *Nn* are larger, that is where *Rr*, *tT* are leſs unequal.

There-

Therefore the ſhort cycles *RT*, *TW*, &c are more, and *RP*, *PN*, &c are leſs harmonious than *AC*, *CE*, &c.

Likewiſe if that alternate intermiſſion ſhould take away the pulſes *R*, *T*, *W*, &c, or *r*, *t*, *w*, &c, then the leaſt diſlocation is *sS*, and the ſhort cycles *SQ*, *QO*, &c, as including *z*, will be more, and *SU*, *UX*, &c, leſs harmonious than *AC*, *CE*, &c, for the very ſame reaſons as before.

Thirdly, any imperfect uniſons will be changed into imperfect v^{ths}, whoſe vibrations are *Ac* and *AD*, (or *AC* and *Ad*) that is, 2*ab* and 3*AB*, by intermitting 2—1 pulſes of the ſeries *a*, *b*, *c*, *d*, &c, which depreſſes the acuter uniſon an VIII[th] lower, and 3—1 pulſes of the ſeries *A*, *B*, *C*, *D*, &c leaving ſingle ones between, which depreſſes the graver uniſon a XII[th] or VIII+V[th] lower; and thus the interval of the new ſounds is an imperfect V[th], as repreſented in the uppermoſt parallel in the figure.

Now in the period where thoſe intermiſſions leave the pulſes *r*, *t*, *w*, *y*, &c, *R*, *U*, *Y*, &c, (as in the 4[th] parallel) the intermediate ones will be taken away, and then *Rr* being the leſſer of the two diſlocations in the ſhort cycle *RY* which includes *z*, is the leaſt of all in this period. And the ſhort cycles *RY*, &c, on this ſide of *Rr*, will be more harmonious than *AG*, &c (in the firſt parallel); and on the other ſide, the ſhort cycles *RL*, &c, will be leſs harmonious than *AG*, &c: For the ſame reaſons as above.

Likewiſe in the period where the pulſes *q*, *s*, *u*, *x*, &c, *Q*, *T*, *X*, &c are left (in the 5[th] parallel), the intermediate ones will be abſent, and then *Qq* is the leaſt diſlocation in this period, and a greater difference than before will be found in the harmony of the ſhort cycles on each ſide of *Qq* and *Aa*; the difference

Xx — *Qq* being leſs than *Yy* — *Rr* in the former caſe.

Laſtly in the period where *p*, *r*, *t*, *w*, &c, *P*, *S*, *W*, &c, are left (in the loweſt parallel), the intermediate ones are intermitted, and then *Pp* is the leaſt diſlocation in this period, and a difference ſtill greater will be found in the harmony of the ſhort cycles on each ſide of *Pp* and *Aa*, for the like reaſon. And the greateſt difference will be found where theſe diſlocations are equal when poſſible; that is, when a periodical point *z* biſects a ſhort cycle of any conſonance, which conſiſts of any odd number of thoſe of the uniſons; and alſo when either of the coincident pulſes at the ends of the complex or ſimple cycles of the uniſons, biſects a ſhort cycle of any conſonance, conſiſting of any even number of thoſe of the uniſons as in Fig. 35. Plate XII. The like proof is plainly applicable to the vibrations *AC*, *Ad*, or to thoſe of any other conſonance. *Q. E. D.*

Coroll. Hence any two imperfect conſonances will be as equally harmonious as they poſſibly can be, when the periods (between their ſucceſſive beats) which are biſected by their coincident pulſes, are made equally harmonious; theſe periods having a mean degree of harmony among thoſe of all the other periods in each conſonance.

All thoſe degrees of harmony occur in practical muſic, and whether ſenſibly different or not (*a*), muſt be uſed as if they were equal, and in theory we muſt take the medium among them.

As the proof of this concluſion has been pretty long, I avoided it in the Book by a paragraph in the demonſtration of prop. XIII, which may now be proved ſomewhat differently.

Another

(*a*) See pag. 126.

Another demonstration of Prop. xiii and its third corollary.

Fig. 60, 61. Let *op* and *OP* represent the times of the single vibrations of imperfect unisons; *ab* and *AB* those of other imperfect unisons; *o* and *O*, *a* and *A* their coincident pulses; and if $ab = op$, the period of the pulses of the former unisons, will be to that of the latter, as *bB* to *pP* (*a*). Because the inverse ratio of the intervals of the imperfect unisons *op* and *OP*, *ab* and *AB*, or of the magnitudes of the ratios between *op* and *OP*, *ab* and *AB* (*b*), is *bB* to *pP* (*c*).

Taking *bB* to *pP* as 1 to 2, this is now the ratio of the lengths of the periods of the unisons *op* and *OP*, *ab* and *AB*; and the latter is of the same length as the period of the dislocations of imperfect octaves, whose single vibrations are *ab* and *AC* or 2*AB*, by intermitting 2—1 pulses of the series *A*, *B*, *C*, *D*, &c, by prop. VIII.

Now the shorter length of the short cycles of the unisons *op*, *OP*, is $op = ab$, and that of the short cycles of the imperfect octaves is *ac* or 2*ab*, and the ratio of their lengths is 1 to 2, which being the same as that of the periods of the unisons and octaves, shews that their short cycles are equally numerous in them.

The longer length of the short cycle of the octaves is *AC* or 2*AB*, and the difference of the lengths is $2AB - 2ab = 2bB = cC$, the dislocation of the pulses at the end of the first short cycle, and is equal to *pP*, because we took $bB : pP :: 1 : 2$; therefore the several dislocations *eE*, &c, *qQ*, &c, at the ends of the subsequent short cycles of the octaves and unisons, are equal respectively throughout their half periods, which are therefore equally harmonious.

Be-

(*a*) Prop. VII. coroll. 5. (*b*) Sect. I. 10. (*c*) Note in pag. 89.

Becauſe thoſe diſlocations are the cauſes that ſpoil the harmony, more or leſs according as they are greater or ſmaller; and cauſes conſtantly equal muſt have equal effects: And becauſe the harmony of theſe half periods is the medium among the degrees of harmony of all the reſt, by the coroll. to the lemma.

Fig. 60, 62. Again, taking *bB* to *pP* as 1 to 3, this is now the ratio of the periods of the imperfect uniſons *op* and *OP*, *ab* and *AB*; and the latter period is equal to that of imperfect XII^ths^, whoſe vibrations will be *ab* and *AD* or 3*AB* by intermitting 3 — 1 pulſes of the ſeries *A*, *B*, *C*, *D*, &c, ſo as to leave ſingle pulſes between every intermiſſion. And ſince *Pp* = (3*bB* =) *dD*, it appears that the ſeveral ſubſequent diſlocations *qQ*, &c, *gG*, &c, of the uniſons and XII^ths^, are equally numerous and equal reſpectively throughout the half periods on each ſide of *oO* and *aA*; which render the conſonances as equally harmonious as they poſſibly can be, for the reaſons above.

Fig. 60, 63. Laſtly, taking *bB* to *pP* as 1 to 2×3, this is now the ratio of the lengths of the periods of the diſlocations of the imperfect uniſons *op* and *OP*, *ab* and *AB*, for the reaſon above. And the latter period is of the ſame length as that of imperfect V^ths^, whoſe ſingle vibrations 2*ab* and 3*AB* (or 2*AB* and 3*ab*) reſult from intermitting 2—1 pulſes of the ſeries *a*, *b*, *c*, *d*, *e*, *f*, *g*, &c, and 3—1 pulſes of the ſeries *A*, *B*, *C*, *D*, *E*, *F*, *G*, &c, ſo as to leave ſingle pulſes at the beginning and between every intermiſſion, by prop. VIII.

Now the ſhorter length of the ſhort cycle of the uniſons *op*, *OP* is *op* = *ab*, and that of the ſhort cycle of the imperfect V^ths^ is 2×3*ab* (becauſe 2*ab* : 3*ab* :: 2 : 3) and the ratio of theſe lengths is 1 to 2×3, the ſame as that of the periods of the imperfect uniſons fons op, OP and the v[ths], whofe fhort cycles op and ag are therefore equally numerous in them.

The longer length of the imperfect fhort cycle of the v[ths] is $2\times3AB$ (becaufe $2AB : 3AB :: 2 : 3$) and the difference of the longer and fhorter lengths is $2\times3AB-2\times3ab=2\times3,\ \overline{AB-Ab}=2\times3bB=gG$, the diflocation of the pulfes at the end of that fhort cycle, and is equal to pP, becaufe we took $bB : pP :: 1 : 2\times3$. Therefore the feveral diflocations nN, &c, qQ, &c, at the ends of all the fubfequent fhort cycles of the v[ths] and unifons, are refpectively equal in magnitude and number too, throughout the half periods on each fide of the coincident pulfes aA, oO; which equalities make thefe confonances as equally harmonious as they poffibly can be, for the reafons above.

Inftead of the terms 2 and 3 of the ratio of the vibrations of perfect v[ths], if we fubftitute thofe of any other perfect confonance, or r and s indeterminately for them, the method of demonftration will be evidently the fame as in the laft example.

Now thofe imperfect confonances of VIII[ths], XII[ths], v[ths], &c are not only equally harmonious with the fame imperfect unifons op, OP, but alfo with one another; the diflocations pP, cC, dD, gG, at the ends of their firft and fubfequent fhort cycles, being equal and equally numerous in their periods. And fince any one of them is equally harmonious to another of the fame name, at any other pitch, when their fhort cycles are equally numerous in their periods (a), it appears that all forts of imperfect confonances are as equally harmonious as poffible, when their fhort cycles are equally numerous in the periods of their imperfections. *Q. E. D.*

Coroll.

(a) Prop. XII.

Coroll. Hence imperfect consonances are equally harmonious, when their temperaments have the inverse ratio of the products of the terms expressing the ratios of the single vibrations of the perfect consonances.

This is the third corollary to prop. XIII and may be demonstrated in this other manner.

The interval of the sounds of imperfect unisons is the temperament of the interval of any consonance, whose single vibrations are different multiples of the vibrations of those unisons. Thus in Fig. 63 the interval of the imperfect unisons *ab* and *AB* is the temperament of the interval of the V[ths] whose perfect vibrations are 2*ab* and 3*ab*, and whose tempered ones are 2*ab* and 3*AB*, or *ac* and *AD*; because the difference of their two ratios is 3*ab* to 3*AB*, or *ab* to *AB*, the ratio of the vibrations which terminate the temperament.

Now in all the examples of tempered consonances we made the term *ab* constant and *AB* variable. Consequently the several magnitudes of the ratios of *ab* to *AB* were proportional to the corresponding magnitudes of the differences *bB* (*a*), which in the VIII[ths] and V[ths] were made equal to $\frac{1}{1\times 2}pP$ and $\frac{1}{2\times 3}pP$ respectively. Therefore when these consonances are equally harmonious, the ratio of their temperaments is $\frac{1}{1\times 2}$ to $\frac{1}{2\times 3}$.

Now when either of them is equally harmonious to another of the same name at a different pitch, their temperaments are equal (*b*), and the terms of the ratio of the vibrations of the perfect consonances of that name are the same.

Con-

(*a*) Note in pag. 89. (*b*) Prop. XII. coroll.

Consequently the direct ratio of the temperaments and the inverse ratio of the products of those terms, are the same in all equally harmonious consonances.

An illustration of coroll. 1. *Prop. xiii.*

Fig. 57. In the line AP let the intervals of the equidistant points A, I, II, III, &c be the longer or the shorter lengths of the imperfect short cycles of any given consonance; A and a its coincident pulses; ab the lesser of the vibrations of the imperfect unisons whose half period is also AP. Make the perpendicular $PQ = \frac{1}{2} ab$, and draw AQ cutting the perpendiculars at I, II, III, &c, in D, D, D, &c. Then are these perpendiculars equal to the dislocations of the pulses between the successive short cycles of the imperfect consonance, by prop. VII and VIII.

Fig. 58. Make the like construction denoted by the greek letters, for any other imperfect consonance of the same or a different name. And if it be equally harmonious to the former, its half period $\alpha\pi$ will contain the same number of short cycles as AP does (*a*); suppose 6 in each. By lessening its temperament, let its half period be lengthened to αp, where erecting the perpendicular $pq = \pi\varkappa$ join αq cutting all the intermediate perpendiculars in e, e, &c. Then the several new dislocations $1e$, $2e$, $3e$, &c will be smaller than 1δ, 2δ, 3δ, &c respectively. Therefore the short cycles $\alpha 6 e$, contained in part of the new half period αp, are not only more harmonious than the short cycles $\alpha 6 \delta$, contained in the old half period $\alpha\pi\varkappa$, or than AVID, but those in the remaining part $e678e$ continue the harmony in the new half period

(*a*) Prop. XII, XIII.

period αp, when that of the former half periods is quite extinguiſhed by the beats at the ends of them.

Coroll. Since only the correſponding ſhort cycles of imperfect conſonances can admit of a juſt compariſon, one by one, in the order of their ſucceſſion, beginning from the coincident pulſes, or from their leaſt diſlocations next to the periodical points; (as explained in the demonſtration of the lemma;) if the periods of two conſonances contain unequal numbers of their ſhort cycles, the compariſon will be imperfect: which is another argument *à priori* for the truth of the XII and XIII propoſitions.

Scholium 1.

In any pure conſonance (*a*) the ſhort cycle contains but one vibration of the baſe, as in Fig. 61, 62, and the equal times between the pulſes of the treble are never ſubdivided by any pulſes of the baſe, except at the ends of the ſhort cycles; and here the diſlocations *cC*, *dD* are conſidered and adjuſted with the analogous ones in other pure conſonances, by the XIII propoſition.

But in any other conſonance whoſe ſhort cycle contains ſeveral vibrations of the baſe, the equal times between the pulſes of the treble are ſubdivided by the pulſes of the baſe, not only at the ends of the ſhort cycles, but between them, as at *D*, *K*, &c, Fig. 63; where the conſideration of the inequalities of the intervals *cD* and *De*, *iK* and *Kl*, &c, ſeems to have been neglected in the ſaid propoſition, but in reality is implied in it.

Fig. 63. For ſuppoſing the alternate pulſes *D*, *K*, &c to be intermitted or taken away, thoſe V[ths] will be changed into XII[ths] an octave lower than the XII[ths] in Fig. 62; but will not be equally harmonious with them,

(*a*) Sect. III. Art. 8.

them, as the v[ths] were ſuppoſed to be before that intermiſſion, till the diſlocations *gG*, *nN*, &c, in Fig. 63 be doubled; that the temperaments of both the XII[ths] may be equal and their periods proportional to their vibrations and ſhort cycles (*a*).

While the diſlocations *gG*, *nN*, &c, remain doubled, reſtore the pulſes *D*, *K*, &c, to their places, and now the intermediate inequalities *Dc* — *De*, *Ki* — *Kl*, &c are alſo the doubles of their former magnitudes and the new v[ths] are leſs harmonious than the XII[ths] in Fig. 62, and will not be equally harmonious with them till the diſlocation, *gG*, *nN*, &c, and conſequently the inequalities *Dc*—*De*, *Ki*—*Kl*, &c be contracted to their former magnitudes.

Therefore theſe interrupted conſonances are not conſidered as pure ones in the XII and XIII propoſitions, but allowance is made on courſe for the effect of the intermediate pulſes of the baſe.

Scholium 2.

Fig. 63. Suppoſing the letters *d*, *k* to be reſtored to the places of the abſent pulſes of the imperfect uniſons, that fall next to *D* and *K*, I call the lines or times *dD*, *kK* the Aberrations of the interior pulſes *D*, *K*, from the places *d*, *k* which they have in the perfect ſhort cycles. Likewiſe in the upper part of the ſame figure, if *AE* and *ad* be the ſingle vibrations of an imperfect 4[th], then *Ee* is an aberration of one of the interior pulſes of the baſe in the firſt ſhort cycle.

Now if the ratio of the times of the ſingle vibrations of any perfect conſonance be *m* to *n* in the leaſt integers, and when it is tempered, if 2*D* be the ſum of the exterior diſlocations in any given ſhort cycle, the aberration or ſum of the aberrations of the interior

(*a*) Prop. XII. coroll.

rior pulſes of the baſe, from the places they have when the conſonance is perfect, will be $\overline{n-1} \times D$.

The reaſon of the theorem will ſoon appear by drawing a ſhort cycle or two of a 4[th], III[d], &c, and by obſerving, that as n is the number of the vibrations of the baſe contained in any ſhort cycle, ſo $n-1$ is the number of its pulſes excluſive of the extreams; and that the ſum of the exterior diſlocations is equal to the ſum of any two interior aberrations equidiſtant from them, or to double the aberration in the middle; as is plain from the arithmetical progreſſion of the alternate leſſer intervals of the imperfect uniſons, from which the given conſonance is derived.

Therefore in two equally harmonious conſonances, as the ſum of the exterior diſlocations in any ſhort cycle of the one, is to the ſum of them in the correſponding ſhort cycle of the other, in a certain conſtant ratio (*a*), ſo the interior aberration or the ſum of the interior aberrations in the former ſhort cycle, is to the ſum of them in the latter in another conſtant ratio; and *componendo*, the totals of the exterior diſlocations and interior aberrations are alſo in another conſtant ratio.

But the temperaments and periods of the two conſonances muſt be adjuſted by the firſt given ratio alone, without any regard to the ſecond or third.

1. Becauſe the exterior diſlocations are of a different kind from the interior aberrations. For as in ſeeing ſo in hearing, it is more difficult for the ſenſe to perceive the quantity of a ſmall inequality in the larger ſucceſſive intervals of the points or pulſes *c*, *D*, *e*, Fig. 63, than to perceive the ſame or a different ſmall quantity when bounded by two viſible points or pulſes *g*, *G*. And the difficulty is greater in

(*a*) By the foregoing illuſtration.

in more complex ſhort cycles of imperfect 4ths, IIIds, VIths, &c, where the ſucceſſive intervals between the points analogous to *c*, *D*, *e*, do not err from the ſimpleſt ratio of 1 to 1, but from the more complex ones of 1 to 2, 1 to 3, &c; as will eaſily appear from the diſpoſition of the pulſes in ſuch cycles in Fig. 5, Plate I. For which reaſon the ratio of the ſum of the interior aberrations ought not to be compared and compounded with that of the exterior diſlocations.

2. Becauſe it appears from the laſt corollary, that no other regard can be had to the interior aberrations, than what follows on courſe from the given ratio of the exterior diſlocations, determined by the equality of their numbers in the periods of the two conſonances, as in prop. XII and XIII.

Scholium 3.

To give the reader more determinate ideas of the numbers of vibrations, ſhort cycles and diſlocations contained in the periods of imperfect conſonances, and of their abſolute duration in practical muſic, I will conclude with a computation of them in a conſonance of vths tempered by $\frac{1}{4}$ comma, as it uſually is, more or leſs, in organs and harpſichords.

Fig. 63. If $AB : ab :: 322 : 321$, the interval of the ſounds of theſe vibrations is $\frac{1}{4}$ comma very nearly *(a)*. Whence $321AB, = 322ab = AZ$ in Fig. 34 or 35 Plate XII, is the length of the ſimple cycle of the diſlocations of the pulſes of the vibrations *AB*, *ab*, or of the period of the imperfections of any conſonances whoſe vibrations are different multiples of *AB* and *ab (b)* and whoſe temperament is the interval of the ſounds of *AB* and *ab (c)*.

Now

(*a*) Pag. 127. (*b*) Prop. VIII. (*c*) See dem. coroll. pag. 284.

Now the vibrations of imperfect v^{ths} are AD and ac, or $3AB$ and $2ab$, and the two constant lengths of their short cycles are $AG=2AD=2\times 3AB$ and $ag=3ac=3\times 2ab$.

Hence $AZ=321AB=\frac{321}{3}\times 3AB=107AD=\frac{321}{6}\times 6AB=53\frac{3}{6}AG$;

Likewise $AZ=322ab=\frac{322}{2}\times 2ab=161ac=\frac{322}{6}\times 6ab=53\frac{4}{6}ag$.

And after the coincidence of the pulses, their first dislocation is $gG=\frac{1}{161}AD$: and the limit of the greatest dislocation is $\frac{1}{2}ab=\frac{107}{644}AD$.

For $AB:AB-Ab$, or $bB::322:1$, whence $bB=\frac{1}{322}AB$, and $gG=6bB=\frac{6AB}{322}=\frac{3AB}{161}=\frac{1}{161}AD$: and $\frac{1}{2}ab=\frac{1}{2}\times\frac{321}{322}AB=\frac{1}{2}\times\frac{321}{322}\times\frac{1}{3}AD=\frac{107}{644}AD$, and is less than the greatest dislocation or alternate lesser interval of the imperfect unisons AB, ab in any half period by prop. VII.

Now by an experiment mentioned in prop. XVIII, I found that the particles of air in an organ-pipe called *d* or *d-la-sol-re*, in the middle of the scale of the open diapason, made 262 complete vibrations or returns to the places they went from, and consequently propagated 262 pulses of air to the ear (*a*), in one second of time, though the pitch of the organ was above

(*a*) Newt. Princip. prop. 43. lib. 2.

above half a tone lower than the preſent pitch at the Opera. And taking that ſound for the baſe of our v'ths, whoſe vibration AD repreſents a certain quantity of time, we have $262AD = 1$ ſecond, and hence the abſolute times AZ, AG, Gg and $\frac{1}{2}ab$.

For, $262AD : AZ$ or $107AD :: 1'' : \frac{107}{262} \times 1''$.

and $262AD : AG$ or $2AD :: 1'' : \frac{1}{131} \times 1''$.

and $262AD : Gg$ or $\frac{1}{161}AD :: 1'' : \frac{1}{262 \times 161} \times 1''$
$= \frac{1}{42182} \times 1''$.

and $262AD : \frac{1}{2}ab$ or $\frac{107}{644}AD :: 1'' : \frac{107}{262 \times 644} \times 1''$
$= \frac{1}{1568} \times 1''$.

And the reciprocal of the periodical time $AZ = \frac{107}{262} \times 1''$ is the number of periods and alſo of beats in $1''$ (*a*), namely $\frac{262}{107} = 2.45$ nearly in $1''$, or 245 in $100''$ nearly.

And the leaſt diſlocations in the ſhort cycles which include the ſucceſſive periodical points Z, are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ of the diſlocation Gg next to the coincident pulſes.

And theſe meaſures are to thoſe in any other v'ths in the ſcale of that organ, in the given ratio of the times of the ſingle vibrations of their baſes. And the like

(*a*) Prop. XI.

like meaſures in any other given conſonance, whoſe temperament is given, may be computed in the like manner, or derived from theſe by the corollaries to prop. IX.

In this example the v[ths] were tempered ſharp, and when they are equally tempered flat, by taking *Ad* and *AC* for the ſingle vibrations, the computation and the meaſures will be but very little different (*a*).

Scholium 4. to prop. xx.

The ſound of an open metalline pipe will be flattened or ſharpened a little, by bending a ſmall part of the metal at the open end, a little inwards or outwards, reſpectively.

The ſound of a ſtopt pipe, made of metal, will be flattened or ſharpened a little, by bending the ears, at the ſides of the mouth, a little inwards or outwards, reſpectively.

The ſound of an open wooden pipe will be flattened or ſharpened a little, by depreſſing or raiſing the leaden plate that hangs over the open end, reſpectively.

The ſound of a ſtopt wooden pipe will be flattened or ſharpened by drawing the plugg outwards or forcing it inwards, reſpectively.

The ſound of a reed-pipe will be flattened or ſharpened by cauſing the brazen tongue to be lengthened or ſhortened, reſpectively.

There is ſomething curious in the reaſons of theſe effects, but as they cannot be well explained in few words, I muſt omit them.

(*a*) Prop. x.

AN

ALPHABETICAL INDEX,

referring to the Pages by the common numbers, to the Propositions by the Roman numbers and to the Notes by the letters in books.

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CORRECTIONS & ADDITIONS.

PAG. 15. line 3. *read* in the table, except 81 to 80, or in any one of them compounded once or oftener with the ratio of 2 to 1, the consonance &c. P. 33. l. 4. *r.* to *E*, *E* to *B*; and &c. P. 41. l. 17. *r.* In this notation then &c. P. 43. l. 16. *r.* the third *df* and sixth *fd'* &c. P. 48. *note* l. 4. *for* True *read* Be it so. P. 64. l. 15. *r.* or *AN* is a ratio &c. P. 65. l. 1. *r.* is less than *AS*, *and* l. 2. *r.* less than *ET*. P. 67. l. 22. *r.* first condition. P. 72. l. 3. *r.* $= \frac{-t}{4r+t}$. P. 77. Def. II. l. 3. *r.* the least terms &c. P. 79. l. 8. *r.* which physically speaking must &c. P. 80. l. 18. *r.* $\frac{n}{2}$V: *To note* (*a*) *add* For if 2 measured *n*, it would measure $m = n + 2$, and thus the terms *m*, *n* of the ratio of V to *v* would not be the least. P. 85. l. 1. *r.* *Kl*. P. 97. l. 3. *r.* of *AD* and *ac*. P. 98. l. 17. *r.* *AB* and *ab* whose pulses &c. P. 103. prop. x. l. 3. *r.* period of the least imperfections of the tempered consonance is &c. P. 115. l. 9. *r.* notice of the cause of the beats &c. P. 116. *note* (*a*) l. 1. *r.* Les battements &c. P. 171. l. 14. *r.* the parts in any assigned places of &c. P. 189. Tab. I. l. 4. *r.* 369 : 370. P. 199. l. 4. *r.* would be 16 as before, *and* l. 19. *r.* the 16 false &c. P. 216. l. 8 & 20. *r.* the perfect v[ths] &c. P. 217. l. 15. *r.* answers one part of the design &c. P. 222. l. 9. *r.* III[ds] and x[ths]. P. 224. l. 7. *r.* made by two given pipes &c.—l. 8. *r.* and these pipes &c.—l. 10. *r.* alter their pitch. P. 242. l. *penult.* *r.* though one of &c.— *note* (*b*) *r.* prop. xx. Schol. 2. Art. 6. P. 259. l. 8. *r.* of the harmonical curve. P. 260. *note* (*d*) *r.* corol. 8. Def. P. 268. l. 16. *r.* is comparatively very near it.

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