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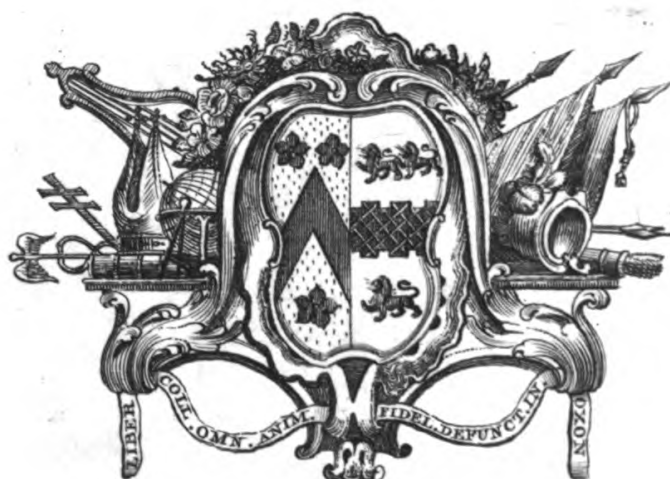


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A ROYAL ROAD
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With Notes interspersed, critical, explanatory, and instructive.

By THOMAS MALTON.

To which is annexed, an Appendix, on the Theory of Mensuration
of Superficies and Solids, as deduced from the Elements.



L O N D O N:

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AUTHOR, in Poland-Street, Oxford-Road.

MDCCLXXIV.



T O

SIR CHARLES FREDERICK;

Knight of the most honourable Order
of the Bath, F. R. S. &c.

S I R,

AS there is an undisputable Propriety in dedicating to a competent Judge of a Science, an Attempt to familiarize the Elements of it, I hope that propriety will plead an excuse for this Presumption; especially, as the Subject is so nearly connected with the business of Your Personal Station, and too highly important to the Publick, in general, to conceive, that any attempt at a clearer Investigation can be beneath the notice of the Surveyor-General of his Majesty's Ordnance, and one of the Representatives of the People.

How far my Method of treating the Elements of Geometry may merit your Approbation, I presume not to imagine; farther than, as I flatter myself, the simplicity with which I have endeavoured to elucidate

D E D I C A T I O N.

the first Principles, to render the knowledge and the use of Geometry more general, and practically useful in common Life, where it is most wanted, may prove an additional inducement to your excusing the liberty I have taken, in thus soliciting Your Patronage and Protection; for a Work, whose Subject, alone, is sufficient to justify the attention You may please to honour it with; whatever may be thought of the manner in which I have treated it; having been, frequently (in order to comprise the whole Elements in less compass) under the necessity of deviating, considerably, from the Path of Euclid.

I am,

S I R,

with due respect,

Your most obedient

humble Servant,

THOMAS MALTON.

A D V E R T I S E M E N T.

IT may appear somewhat strange to call so copious a Work as this, an Abridgment; which, on examination, and comparing with those who have transcribed Euclid, will be found to be a great abridgment of the Elements, both in the number of Theorems and length of the Demonstrations.

But, they will find, in this Work, particularly in the 2nd; the 3rd, the 6th, 7th and 8th Books, several valuable Theorems which are not Euclid's; some of which are elementary, and really necessary to be known. Upon the whole, there are upwards of eighty other Theorems and Problems, exclusive of the Ellipsis.

In respect of the practical Part, with the Introduction, it may be deemed a compleat Work of itself; which, with the Appendix, is more than one third part of the whole.

The Applications, Notes, and Remarks, necessary to a young Student, and the length of some (in the Fifth Book particularly) have swelled the Work considerably. The much greater number of Terms defined; various ways of performing the same Problem, and also of demonstrating some Theorems; the number of Figures, and the Preamble to each Book, &c. also the manner of printing, in order to have the Figure always in View, so that, many Pages are not near full; all which have concurred to extend the Work, greatly beyond my first Design; yet, when compared with the close printing of others, the Elements will be found abridged, almost half; nevertheless, it contains the whole, in Substance. The full and perfect knowledge of all, and more than is contained in Euclid, may be acquired in a third part of the Time, and with infinitely more Ease, Pleasure, and Satisfaction to the Learner (having no *Asses Bridge* * to get over, but the Road smooth and even) consequently, it may, with great Propriety, be called an Abridgment.

* *The 7th (some say the 5th) of the first Book, has, in the Seminaries of Education, been long known by the Appellation of the Asses Bridge; on which, many have flundered and never got over; nor been able to advance one step further.*

A N I N D E X.

Shewing where to find the Propositions of Euclid, in the First, the Third, the Fifth and Sixth, the Eleventh and Twelfth Books; in cases of Reference, to Euclid, by Authors in the Mathematics.

In the second and fourth Books, the Propositions follow in the Order of Euclid.

N. B. The Problems, of Euclid, are collected together, in the first Part; amongst other, select Problems, in Practical Geometry.

Note. The Numbers in the First Column are Euclid's; and, opposite to them is shewn where each Proposition may be found in this Work, whether Problem or Theorem.

I N D E X.

BOOK I.		Prop.	Theo.	Prop.	Theo.	Prop.	Theo.
Prop.	Theo.	5 & 6 Ax.	3.	24—Cor. to	3.	13 C. to Prob.	2.
1 Prob. 11.	7	—	5.	25 —	11	14 —	6.
2 Prob. 2.	8	—	6.	BOOK VI.			
3 Prob. 5.	9—Cor. to	5.		1 —	1.	15 —	7.
4 —	8.	10 Ax. 4.		2 —	2.	16 —	8.
5 —	9.	11 and 12 —	7.	3 —	3.	17 —	10.
6—Cor. 3.	9.	13 Ax. 6.		4 —	4.	18 —	9.
8 —	7.	14 —	3.	5 Cor. 1.	4.	19—Cor. 1.	9.
9 Prob. 9.	15	—	4.	6 —	5.	20 and 22.	13.
10 Prob. 8.	16	—	8.	8 —	7.	21 —	12.
11 Prob. 6.	17 Prob. 42.			9 Prob. 3.		23 fee NB.	2d.
12 Prob. 7.	18—Cor. 3.	8.		10 Prob. 36.		24 —	14.
13 —	19—Cor. 2.	8.		11 Prob. 31.		25 —	15.
14—Cor. 1.	20 —	9.		12 Prob. 32.		26 } fee NB.	
15 —	21 —	10.		13 Prob. 30.		27 } to the	15.
16 and 17 —	22 —	11.		14 and 15.—	8.	28 —	16.
18 —	23 and 24 are			16 —	9.	29, 30 and	
19—Cor. 1.	12	useless.		17—Cor. to	9.	31 are in the	17.
20 —	13	25 Prob. 40.		18 Prob. 16.		32 Cor. 1.	17.
21 —	14	26 and 27—	10.	19 —	12.	33 —	18.
22 Prob. 14.	28 }	Cor. 2. of 3.		20 —	13.	34 —	19.
23 Prob. 4.	29 }			21 Axiom.		35 —	22.
24 and 25 are	30 Prob. 8.			22 —	15.	36 —	23.
in Cor. to	31 —	12.		23 —	11.	37 —	24.
26 —	32 —	13.		24 —	17.	38—Cor. 2.	9.
27 and 28—	4.	33 Prob. 44.		25 Prob. 38.		BOOK VIII.	
29 Cor. 1 & 2.	4.	34 Prob. 43.		26—Cor. to	17.	12 of Euclid.	
30 —	5.	35 —	14.	27 —	18.	1 —	14. 6.
31 Prob. 5.	36 —	16.		28 and 29 are		2 Cor. 1.	14. 6.
32 —	10.	37 Converse.		useless, and pro-		3 and 4 are	
33—Cor. to	15.	BOOK V.		lix Problems.		wholly omit-	
34 —	15.	Those which are		30 Prob. 35.		ted, as useless.	
35, 36, }	—18	not numbered		31 —	16.	5 and 6 —	6.
37 & 38. }		are wholly o-		32 — is useless.		7 —	4.
39 and 40.		mitted, as use-		33 —	19.	8 —	7.
Cor. to	18.	less.		BOOK VII.		9 —	8.
41 —	17.	1 —	1.	11 of Euclid.		10 —	5.
42 Prob. 20.		7 Ax. 4.		1 Ax. 1.		11 Cor. 1 & 2.	6.
43 —	19.	9 Ax. 5.		2 Ax. 4.		12 Cor. 1 & 2.	7.
44 Prob. 23.		10 Ax. 6.		3 —	1.	13 is an Axiom;	
45 Prob. 22.		11 Ax. 13.		4 —	2.	and the same	
24, 20 & 21.		12 —	2.	5—Cor. to	2.	as 15 of B.	7.
46 Prob. 17.		14 Ax. 12.		6 —	3.	14—Cor. 3.	6.
47 —	20.	15 Ax. 7 & 8.		7 Ax. 3.		15—Cor. to	8.
48—Cor. to	20.	16 —	4.	8—Cor. to	3.	16 & 17 are use-	
BOOK III.		17 —	7.	9 —	4.	less, and pro-	
1 Prob. 39.		18 —	6.	10 —	5.	lix Problems.	
2 Ax. 5.		19 —	3.	11 Prob. 1.		18—Cor. to	9.
3 —	1.	22 —	9.	12 Prob. 2.			
4 —	2.	23 —	10.				

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P R E F A C E.

THE mathematical World will, I doubt not, be surprized at a fresh publication of the Elements of Geometry, by one entirely unknown; and, on a plan very different from that of others who have wrote on the Subject. I hope they will suspend their opinion, and not pass a too hasty censure, on account of the obscurity of the Author, till after they have given it a fair and candid perusal, and then proceed to judgment with candour and impartiality.

I do not pretend to much knowledge in the Mathematics, having been brought up in a way of life, very different from my inclination; yet, what time I could spare from business and the demands of my family, I chose to employ in such studies; and have, by dint of study, only, and without any other instruction, made some progress in mathematical Sciences; of which, Geometry is the first, and a sure key to the rest.

Since I have made myself, by self-application, a Proficient in Geometry, and have made some branches of the Mathematics my Study and Profession, I have often been surprized at the negligence and deficiency of our common Schools, for the cultivation of Youth who are intended to fill the middle sphere of Life, in mechanic Trades, &c. They, almost in general, pursue one common Plan or track of Learning. After the first and necessary branches, Reading, Writing, and Arithmetic; which, indeed, might be acquired in half the time it usually is; the next step (if the Pupil has made a progress thro' Arithmetic in any reasonable time) is the Grammar of the Latin Tongue through which, he sweats and labours to little purpose. If the Pupil has three or four years to spare, before he goes out to business, he perhaps gets into the Cordery or Erasmus; or, if he reaches Cornelius Nepos, he is looked on as a prodigy.

Now, it may reasonably be asked, for what purpose all this Time has been spent? which might have been employed to much

A

better

better purpose. For, what has mechanic Trades to do with Latin? any more than a common Porter or Carman with Logic; it may indeed complete him a Pedant or Coxcomb, but can never be of real use in his Profession; even suppose he had made a tolerable proficiency, it could answer no purpose but to set him above his Employment, without being of any service in it.

On the contrary (supposing no particular avocation is intended for the Student) if, instead of Latin, Geometry and Mensuration, &c. were introduced in all public, common Schools, I would ask any person, who has considered these things, and their uses in Life, which is most likely to turn to the Pupil's advantage? Is there a mechanic Profession in which Geometry or Mensuration may not be of some use? in some particular ones it is well known to be of the greatest, the foundation of it; and yet, altho' the Youth was particularly intended for that Profession, it was, perhaps, never once so much as thought on; until, by too late experience, he finds the want of it: I mean all such Trades as relate particularly to Building, in general. Had some Builders, whom I have known, been conversant in the Mathematics, or only in plane Geometry; instead of plodding on in a low sphere of Employment, they would, if their natural, mechanical genius had been properly cultivated, have filled a more elevated station.

For the use of such, I have been at the trouble of composing this Treatise. If only the practical part is well inculcated, it will be of more service, in common Life, than a proficiency in Latin can possibly be. If the young Pupil has a genius, and discovers a relish for mathematical Science, let him go on with the Elements; and if he acquires a competent share of knowledge therein, it will then be time to consider, what particular Profession he either wishes or is destined for. In choosing of which, regard ought particularly to be had to the Boys genius and disposition, which will, ere this, be discernable. But, at all adventures, instead of flogging and driving a useless dead Language into a stupid Boy, which only renders him more so, let the practice of Geometry be introduced in its stead, in every common School; there is something entertaining

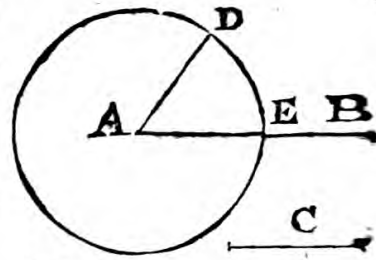
to the Mind, more than in burdening the memory with *As in præfenti*, and other rules of the Latin Grammar.

Accustoming Boys, early, to handle the Compasses and other drawing utensils, in delineating all the Diagrams, as they proceed, will be an entertainment to them, and of great utility, rather than a perplexing study, and gradually enure them to Demonstration; which, under a proper Tutor, they would soon have a relish for, and then they would proceed with pleasure: besides, it is an introduction to Drawing. A foundation being once laid in Geometry, they are then qualified to pursue any other branch of the Mathematics, suitable to the Profession they are intended for; such as Mensuration, Trigonometry, Navigation, Gunnery, Fortification, Architecture, naval or domestic, Surveying, &c. In short, all the useful and necessary Employments, in the mechanic Arts, have their foundation in this most necessary Science; which, being acquired, will, most probably, make its Possessor strike out of the common and vulgar track, and make him eminent and distinguishable, in whatever Profession he is casually fixed in; as he will have laid a solid and permanent foundation, in Theory; whereon, may, very probably, be erected, a lasting monument to his future Fame.

I have perused several Authors on the Subject, and find, that some have treated it in a manner scarce intelligible to a beginner, unless he has some knowledge of Algebra; others would be better understood and approved, if they did not dwell too much on self-evident Propositions, which are, in themselves, perfect Axioms. Perhaps, I shall be blamed for censuring, as useless, several Propositions in that famous Geometrician, Euclid; but must own, I cannot conceive of what use is all that tedious round about method, in the 2nd Problem, Book I. viz. "To put a right Line, at a given point, equal to a given Line," unless some particular direction, in respect of the other Line, was also given.

In Problem 3d. where the 2nd is applied, I ask, for what use? and why, after having taken the line C in the Compasses, as Radius,

we may not as well cut off from the given Line (AB) at once, the measure of C, (at E) without first placing it from A to D? (See the Figure) nor can I perceive that the Demonstration is, at all, the clearer for it.



As the knowledge I have acquired, in such studies, has been entirely from Books, without any other assistance; I may, perhaps, have been able to see deficiencies or superfluities in them, better than those who have studied under a Tutor; and, I dare venture to affirm, that very few, who are not already tolerably versed in Geometry, will be able to form any Idea of what the 7th Proposition, Book I. attempts to prove; who, from inspection only of a proper Figure, might be fully convinced of the truth of it.

I have, in this Treatise, endeavoured to render every Proposition easy and intelligible, to any capacity; and have omitted none that are useful, or necessary to demonstrate the rest; and, I will be bold to affirm, that those who cannot, from it, acquire a knowledge of all that is requisite, in Plane Geometry, will never be able to attain it at all. Where there is not a capacity to understand, they had better desist from the undertaking. An equal talent is not given to all; and, for such as have not a talent and a natural propensity, to expect to attain any tolerable share of knowledge, in Geometry, is aiming at impossibilities.

The 16th and 17th Propositions, Book 1st, are entirely useless; for since, in the 32nd, the external Angle is proved to be equal to the two remote Angles of the Triangle; and, that all the three Angles, of every Triangle, are equal to two right ones; it seems absurd to prove, before hand, that any two of the Angles are less than two right Angles, and, that the external Angle, is greater than either of the remote ones. As if one should undertake, first, to prove that five is greater than either two, or three, and afterwards, that it is equal to them both. I have, therefore, made free, to alter, in some measure, the Elements of the first Book; that, by a different arrangement, in transposing the places of

of some Propositions, others, which depend on them, may with more ease and elegance be demonstrated.

I shall ever be of opinion with Tacquet, and some others, that, to attempt a formal Demonstration of Propositions which are self-evident, is involving a thing, in itself clear and conspicuous, in darkness and obscurity. I have always found more difficulty in demonstrating, to another Person, self-evident Propositions, than the most intricate of others; and, when done, have only confused the Idea which the Pupil had more perfectly from inspection of the Figure. If a knowledge of several properties of Figures be acquired, which is necessary to elucidate other more sublime Propositions, is not the easiest method the most eligible? certainly it is; and, barely to be told several properties of Triangles is sufficient for attaining the rest. Therefore, in this Work, I have reduced some Propositions into Corollaries; as they are but a certain consequence of the preceding Proposition, or of some other.

Dr. Keill, in his Preface to his Translation of Commandine, and also Mr. Cunn seems to think it an unpardonable fault in Tacquet, to omit the Demonstrations of the 5th Book; and asserts, that not one Demonstration, in the 6th, 11th and 12th, can be obtained without it. But, I must beg his pardon, for differing from him in opinion, and am, myself, a living witness against such his assertion; for, I do aver, that, without any other demonstration of the 5th Book, than what Tacquet has delivered, I have been able to go through all the other; and, without vanity, I think I understand them; but, of that, let this Treatise bear witness. I never could, at first, have had patience to go through the dry and tedious Demonstrations, delivered in 25 Propositions, of the 5th Book of his Euclid; and, had it first fallen in my way, I should certainly have lain it aside before I had got through a fourth part of it; yet, I must acknowledge, that Tacquet is as much too brief as the other is tedious.

I cannot think it absolutely necessary, in order to obtain a competent knowledge in Geometry, and of Proportion in particular, to tread exactly in the same steps with Euclid; therefore, I have
made

made free to deviate, where I think it for the better, a readier or clearer way of investigating. I have, likewise, separated the Problems from the Theorems; making, of them, a distinct and separate Book, for the use of Schools and for Mechanics in general; the full Demonstration, of which, may be obtained from the Elements, to which I refer where it is necessary. I have given it the first place, contrary to some others who have made it the last Book, or otherwise disposed of it. By which means, we are frequently at a loss in the References, and are told to form Constructions, before we have learned how.

My reason for separating Practical Geometry from the Theorems is, that there are numbers of People, to whom it may be of great use, who either have not leisure or inclination to go through, or, perhaps, a capacity to understand the Demonstrations of the previous Propositions, which discourages them from proceeding. But, such as are not inclined to take it on credit, will readily obtain the Demonstration from the Theorems.

To make the Work more compleat I have added, after Practical Geometry, a brief Theory of the nature and construction of an Ellipsis, that most useful and valuable Figure. And also, as an Appendix to the whole, I have given a concise Theory of mensuration of Superficies and Solids; shewing their immediate and absolute dependence on Geometry.

I have for some time debated, with myself, whether I should publish a tract of Geometry or not; from which I have been deterred, through the persuasions of some particular Acquaintance; alledging, that there are more learned Treatises already published, than any I could produce: the truth of which, I readily assent to; but, I don't find that they are more intelligible for that, and have, therefore, determined to publish. My chief reason, for which, is to bring the whole Elements into less compass, and to abridge it of that tedious prolixity, which is in many Authors, on that Subject; in dwelling too long on such Propositions as are clear and evident of themselves; and, by that means, to render the study of Geometry more pleasant and entertaining. 'Tis enough to deter any one from the pursuit of a Science, who find, in the Rudiments of it,

so much perplexity; for, I am persuaded, that, unless a Person has a strong inclination to it, and a good natural Capacity, 'tis not a very pleasing Study, at the first, until they begin to feel the sublime Truths it contains; and am therefore of opinion, that, the easier it is made in the begining, the more entertaining to young Students.

Can there need Demonstration, that any two sides of a Triangle, are greater than the third? is there a Person so ignorant as not to know it? it is implanted in us by Nature; every common Porter knows it, or practices it every Day. Who ever saw one of them traverse two sides of a Square, when he could cross the Diagonal? and why is it? but, because he knows it to be shorter than the two Sides. Is it not obvious, that the greatest Angle of every Triangle, must be opposite to the greatest Side? and can there be any need to demonstrate the converse? that the greater Side subtends, or is opposite to the greater Angle; is not the one contained in the other? it is trifling, to no purpose; for, all converse Propositions may be Corollaries to the former. If two Sides of a Triangle, equal to two Sides of another Triangle, contain a greater or less Angle, the Base will be greater or less? Are not all these, and several more, obvious and clear, from a bare inspection of the Figure? nay even without it; 'tis enough to be told they have such properties, and not to lose time in trying the patience of the Student, with a tedious and puzzling Demonstration of what he saw clearer before; for, if the thing is seen or known, what needs there more? is it intended to perplex, only, where it can be of no use? to disgust the beginner, before he is able to see any of the Beauties it contains? Yet, I do not omit these entirely, because the whole Elements depend on them; but have endeavoured to treat them in as simple a manner as the nature of the Subject will admit of; if I have been concise, I doubt not I shall be excused, if I have but said enough.

But, as I think I have said enough, in this place, I shall straight-way proceed to the Subject; through which, if the Reader be inclined to follow, with an intention to learn what it contains, I am
much

much mistaken if he loses his labour ; and, for such as peruse it only with intent to cavil, I hope they will be greatly disappointed, and find but little to cavil at. I am not so vain as to suppose it is without defect, or that it will please all, for that I know is impossible ; but, if I have made it intelligible and useful to common capacities, it is what I aimed at, and shall rest satisfied in the supposition that my labour is not entirely lost.

The greatest fault in Tacquet is, that his Figures are too small and trivial ; and it is a general fault, that they are often incorrect and frequently contradict the description. Correct and well adapted Schemes are certainly of some consequence, in which, I have been very particular ; and have, also, carefully revised the letter press, so that, I hope there are but few errors have escaped my observation. For, Errors, in misplacing the References to the Schemes, and sometimes omitting them entirely (which is better of the two) is unpardonable in mathematical Works ; having frequently experienced, in most Authors that I have perused, the perplexity it occasions ; especially, when the subject is quite new to the Student. But, if any should remain unnoticed, I hope the candid reader will impute it to human fallibility, and not quarrel with it on that account, for I am certain he will not meet with many.

Although I have, in this Treatise, deviated greatly from Euclid, in many particulars, I have endeavoured to make it generally useful ; by putting his Numbers after mine, to each Proposition, and also by means of an Index, I have shewn where to find any Proposition of Euclid ; which, in case of reference, to Euclid, in other Works, may be readily turned to.

I have well considered and digested every Proposition, have carefully revised them over and over with the strictest attention, and I am fully convinced, that there is no omission of any thing that is essential, or necessary to be known. Notwithstanding, I have abridged the whole Elements, particularly the first, the third, the fifth, and the eleventh Books ; yet, I dare venture to affirm, that I have not omitted the substance of any Proposition which will ever be referred to, by Authors in any other Science.

A N
I N T R O D U C T I O N
T O
G E O M E T R Y,

Containing a full Definition of the Terms peculiar to, or made use of in that Science, with explanatory Notes and Remarks, where it is necessary to illustrate or enlarge.

Likewise a short Theory of Plane Angles; in which they are more fully explained than in any other Work of the kind, that I have seen. Indeed, most Authors in Geometry are entirely deficient in that respect; for want of which, the young Geometrician is frequently at a loss, to conceive a clear Idea of Angles. I have therefore, been explicit on that Head.

It also contains an Explanation of all the Abbreviations made use of in this Work; many of which are explained in English Grammars, and other school Books, and ought to be known to every English Reader; yet, as I know that their true significations are not so generally understood, it may not be thought impertinent, or superfluous here.

Nor have I omitted any thing, that is necessary to elucidate the Subject I am about to treat on; at least, I think I have not; for, in the course of my own study of it, and in teaching others, I have been enabled to discover what is needful to most Capacities; and if I have any pretence to merit in this Work, it is chiefly on that account; in rendering an intricate, yet generally useful and most necessary Science, attainable to ordinary Capacities; and, at the same time, I hope, not exceptionable to those of acuter talents.

I flatter myself that it will not be less acceptable to any, for being easy to be attained, to write only for Proficients, is to little purpose. By such I may, in some cases, be thought

B

rather

2 I N T R O D U C T I O N

rather prolix ; yet, I presume, not tedious ; a repetition is sometimes necessary to young Minds, and is more agreeable, in general, than turning back, which they are too frequently obliged to do ; it being impossible to retain, in memory, all that is passed over on the first perusal.

GEOMETRY, according to its original derivation, signifies to measure the Earth. It is a Science which contemplates continued Quantity, Extension or Magnitude, abstractedly considered ; it teaches the nature and properties of Lines, Angles, Figures, Surfaces and Solids.

Geometry is in two parts, speculative and practical ; the first demonstrates the properties of Right Lines, Figures, &c. in speculation ; from which is deduced the practical part for various uses, for the benefit of mankind, in mechanic Arts, &c.

Euclid has judiciously divided the Subject into Books or Sections ; each of which, treats of different Figures, or different properties of Figures, the power of Lines, Proportion, &c. which some Authors have thought proper to deviate from, though without any justifiable reason for so doing.

It treats, in the first, third, fourth, and sixth Books, of Plane Figures, and thence is called Plane Geometry ; and afterwards, in the 11th and 12th of Euclid, the 7th and 8th of these Elements, of Planes and Solids.

Each Book contains sundry Propositions ; from which, are deduced Corollaries, Scolia, &c. the signification of all which I shall first beg leave to explain or define ; and then proceed to the Definitions of the more essential Terms, which are the Subject of Geometry.

A DEFINITION is the defining or explaining the full signification of any Term or particular Word, peculiar to, or made use of, in that Science of which we are about to treat.

A PROPO-

A PROPOSITION is either a Theorem, proposed to be proved or demonstrated, contemplatively ; or, it proposes something to be done, problematically or mechanically.

A CONVERSE PROPOSITION is the contrary of the other; that, which in the foregoing was the Conclusion, drawn from the Premises of it.

e. g. If a Triangle have two equal Sides, the Angles which they subtend are also equal ; the Converse is, that if a Triangle have two equal Angles, the Sides subtending them are also equal. / 1. a. ✓

A THEOREM is a speculative Proposition ; a declaration of certain properties, equality, or other proportion relative to Quantity, or Figure, mathematically considered.

A PROBLEM, is a Proposition which proposes something to be done, practically or mechanically.

An AXIOM is a self evident Proposition, which does not require to be demonstrated. See Axioms, Book 1. El.

A LEMMA is a Proposition, as it were by the bye, or out of the way, which serves, previously, to prepare the way for the more easily comprehending the Demonstration of the following Proposition.

I do not make use of Lemmas in this Work, as some geometrical Authors do ; for if there be a necessity for a Lemma, I see no reason why that Lemma is not as much a Proposition as any other. The 7th and 16th Propositions of the first Book of Euclid, may be called Lemmas, for they are certainly redundant Propositions. In other mathematical Works, Lemmas are frequently necessary, but, in Geometry, they are quite inconsistent.

4 I N T R O D U C T I O N

A COROLLARY is a necessary consequence deducible from some Proposition, already demonstrated.

A SCHOLIUM is a remark, or useful lesson derived from the preceding Proposition.

A POSTULATE is a petition or request which is required to be granted. See Postulates, p. 21.

HYPOTHESIS. Whatever is supposed or premised, in a Proposition, is called the Hypothesis or Premises of it; from which some certain Consequence is deduced, as affirmed, and afterwards demonstrated, called the Thesis or Affirmation.

e. g. If a Right Line, cutting two Right Lines, makes equal Angles with them both, those lines are parallel.

Here, the Hypothesis is, if the Angles are equal; and, the Consequence, that the Lines are parallel.

SUPPOSITION. In demonstrating some Theorems, it is necessary to have recourse, frequently, to suppose such and such things, which are not so in reality; by the absurdity of the consequences, arising from such a supposition, a conclusion is drawn, and the Demonstration is made evidently to appear.

Such kind of Demonstration is called *reductio ad absurdum*, i. e. proving it to be absurd, or impossible to be on that supposition; which, not being direct and positive, is, to many, very unsatisfactory; yet, if rightly considered, is full, though not direct Demonstration.

CONSTRUCTION, is the contriving or disposing, geometrically, Lines and Figures, necessary for making the Demonstration appear, clear and conspicuous; and must
always

always be made of such Figures or Lines as are already well understood; the Properties, of which, being previously demonstrated.

DEMONSTRATION. When any thing is proposed, or affirmed in a Proposition, the Case is first stated and prepared, by drawing such lines, or forming such a Construction as is necessary; and it is afterwards demonstrated; that is, the truth of the Assertion is made to appear, obvious, and without the least doubt remaining; the performance or operation of which is called the Demonstration.

The three last Terms being common words in the English Tongue, may, by some, be thought impertinent; but, notwithstanding the common acceptation of them is almost universal, yet the application of them in Geometry requires to be explained.

DEFINITIONS.

Of the essential and operative TERMS.

The Terms of Art, to be defined in any Science, are Names, arbitrarily given, by the first Authors, or others, to certain Symbols, Figures, Marks or Characters, possessing certain properties or relations, in respect of figure, position, situation, &c.

The operative Terms are, generally, technical Words, peculiar to that Science, though perhaps applicable to others; which are not of common use in Language, or have a different signification.

The following Definitions are frequently referred to, hereafter, for illustration or proof of what is advanced in the Propositions.

When any Figure, &c. which we are contemplating, is found to possess such or such properties, we affirm it to be such a Figure, as answers to them; or, in contemplating any Figure, given in the Premises, we affirm that it has such or such properties, arbitrarily, by the Definition of it; and therefore, it requires no other Demonstration.

DEF.

DEFINITIONS.

DEF. 1. A POINT is, rather to be conceived than understood to be, without dimensions, therefore indivisible.

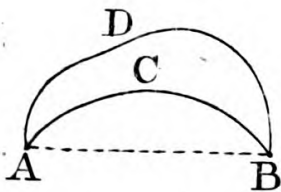
In Plane Geometry, it is a Mark, supposed to be made with a sharp-pointed Needle, or fine drawing Pen. As A.

A

DEF. 2. A LINE has length only, without other dimensions, of breadth or thickness.



DEF. 3. A RIGHT LINE is the shortest that can be drawn between two given Points, (A and B) usually called a straight Line.

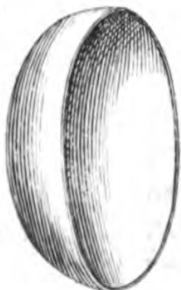


DEF. 4. A CURVE, or CURVED LINE, is any other than a Right Line; either regular, as ACB; or irregular, as ADB.

Curve Lines are of various kinds, as circular, elliptic, parabolic, &c. each of which has particular properties peculiar to them, and some in common.



DEF. 5. A SURFACE, or SUPERFICIES, is the outside or external parts of Bodies, having no regard to thickness or substance; and therefore, has but two dimensions, viz. length and breadth. As AB and CD.



Surfaces are various, as Convex, Concave, and Plane.

Irregular Surfaces are such as are not, uniformly, any of the three, but may be compounded of them all.

A convex Surface is one that is externally round or protuberant. Also, the outside of the curve of a Circle, &c. is convex.

A concave Surface is one that is round internally, or hollow, such as the inside of a round Vessel; the outside is convex. The curve of a Circle towards the Center is concave.

DEF.

D E F I N I T I O N S. 7

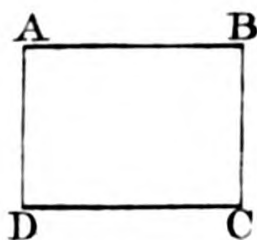
DEF. 6. A PLANE is a perfectly even and regular surface, which is neither convex nor concave, in any part; to which, if a Right Line or streight Ruler be any how applied, it will touch in every Point.

Or, if any two Points, in a Plane, be joined by a Right Line, the whole line is in that Plane.

A Plane may be conceived to be generated by the direct motion of a Right Line, laterally; or whirled around on any Point in it.

If the Right Line AB be moved, directly to CD, there will be generated the Plane ABCD.

N. B. A Plane may be of any Shape or Figure, in respect of its bounds or limits.

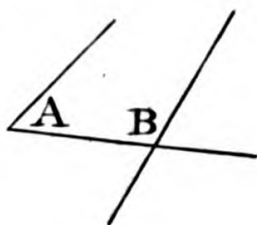
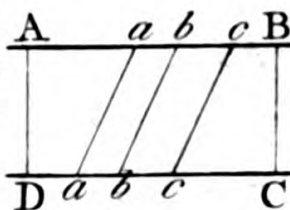


DEF. 7. PARALLEL. Right Lines are Parallel to each other, which, if produced infinitely, either way, and being in the same Plane, would never meet. As AB and CD.

N. B. Parallel Lines are equidistant in every part; between which all Right Lines, as *aa*, *bb*, and *cc*, being also parallel, are equal.

But, the Distance between two parallel Lines is measured by a Perpendicular, AD or BC. See Perpendicular (Def. 10.)

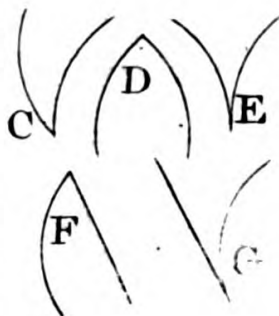
The same holds true in parallel Planes.



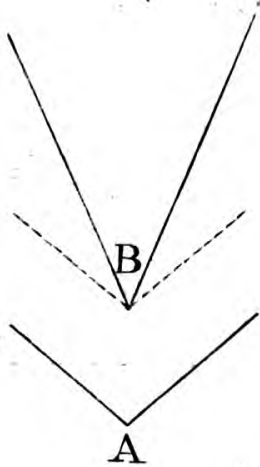
DEF. 8. A PLANE-ANGLE is formed by the meeting, touching, or cutting, of two Lines in the same Plane, so as not to fall into or constitute one Line.

Angles are either right-lined, as A or B; curved, as C, D and E; or mixed, as F and G; which are compounded of a Right Line and a curved or crooked one.

N. B. Angles are neither increased nor diminished by the length of the Lines which form them; for, the Angle A is greater than the Angle B, notwithstanding the Lines which form the Angle B are longer than those of the Angle A; but, the Lines forming the Angle B are more inclined to each other.

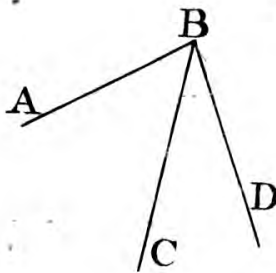


D E F I N I T I O N S.



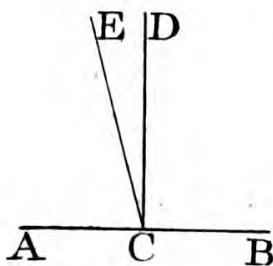
2. Imagine the dotted Lines to represent the Angle A laid upon, or applied to the Angle B; it is evident, that the Angle B is less than the Angle A, having a greater inclination of its Sides, or the aperture between them less; therefore, a less Angle may always be contained in a greater.

3. If the Sides of any Angle (as B) were lengthened infinitely, the Angle is not varied, if the inclination of the Lines remains the same; but, if you suppose them moveable, on the angular point, like a folding Rule, and are parted farther asunder, approaching towards the dotted Lines, their inclination will then be less; and consequently, the aperture or opening between them greater; and therefore, the Angle is said to be greater.



DEF. 9. VERTEX, is the angular Point in which two Lines, forming a Plane Angle, meet and touch each other.

N. B. A single Angle is usually described by one Letter only, or other Character, as A or B; but, if three or more Lines meet in a Point, then three Letters are used, to specify the Angle spoken of; as ABC, CBD or ABD; of which the middle Letter always denotes the Vertex; and means, the Angle made by the lines AB and BC, or CB and BD, &c. wherefore, the middle Letter, denoting the Angle, is understood to be twice named.



DEF. 10. PERPENDICULAR. A Right Line is perpendicular to another, when it does not incline to the other on either Side; but makes the Angle, on each Side, equal to the other.

Thus; if the Angles ACD, DCB, are equal to one another, then CD is perpendicular to AB.

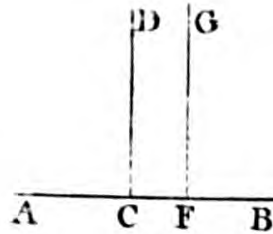
N. B. No more than one Perpendicular can be drawn from the same Point in a Right Line, on the same Side, and in the same Plane.

For, if any other Line, as CE, be drawn from the same Point, C, it cannot be a Perpendicular, but is said to incline to AB; and, the inclination is on the same Side of the Perpendicular CD; i. e. the Angle ACE is the Angle of Inclination of the two Lines AB and CE.

N. B. 2.

D E F I N I T I O N S. 9

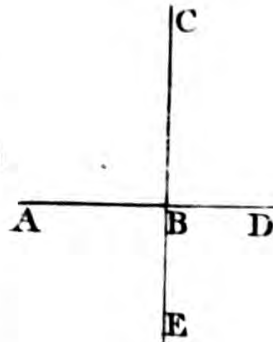
N. B. 2. If another Right Line, as FG, be drawn in the same Plane, from any Point F, perpendicular to AB, it will be parallel to CD.



DEF. 11. A RIGHT ANGLE is that which is formed by the meeting of two Right Lines which do not incline to each other, but either of them is perpendicular to the other.

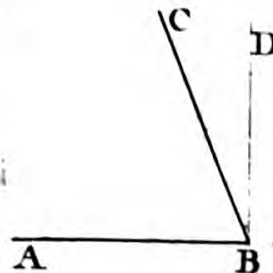
N. B. If either Side of a Right Angle be produced, or drawn out beyond the Vertex, there is necessarily generated another Right Angle. And, consequently, if both Sides are produced, there will be generated four Right Angles.

Thus, ABC is a Right Angle; if, when AB or CB is produced, towards D or E, there is made another, CBD or ABE; and if both are produced, EBD is a fourth Right Angle.



DEF. 12. An ACUTE ANGLE is less than a Right Angle.

If the Line CB, meeting AB in the Point B, falls on this Side of a Perpendicular, BD, at that Point; the Angle ABC, being less than the Right Angle ABD, is therefore called Acute.

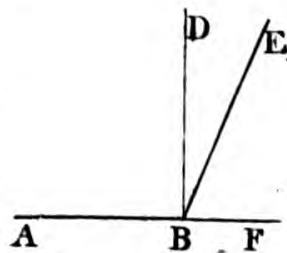


DEF. 13. An OBTUSE ANGLE is greater than a Right one.

If the Line BE falls on the other Side of the Perpendicular, BD; the Angle ABE is Obtuse.

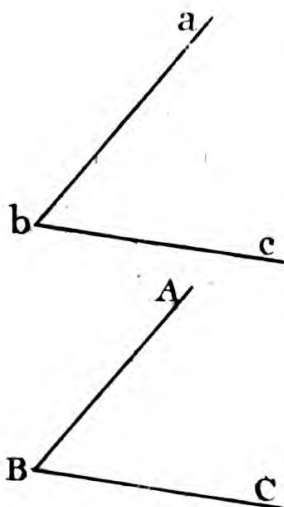
N. B. The difference, CBD, between an Acute Angle, ABC, and a Right one, ABD, is called the COMPLEMENT of the Angle ABC.

And, if either Side of an Obtuse Angle, as AB, be produced, the Angle EBF is the Complement of the Obtuse Angle; or its deficiency to two Right Angles, ABD, DBF.



C

DEF.

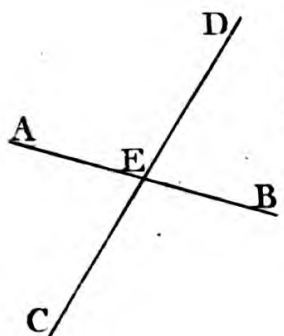


DEF. 14. EQUAL ANGLES. Angles are equal, when the Lines, which form them, have the same Inclination each to the other, respectively.

In the Angles abc , ABC ; if the Vertex, b , of the one, be applied to the Vertex, B , of the other, in such wise, that the Side ab falling on AB , cb also falls on CB ; then, there is the same inclination of cb to ab as of CB to AB , and the Angles abc , ABC are equal.

N. B. The length of the Lines, or Sides, is not considered or regarded in the equality of the Angles, but only their inclination to each other.

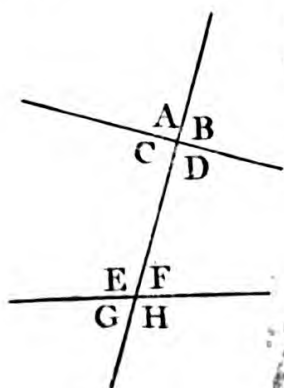
Angles have other Denominations, which are derived to them only from their Situation, in respect of each other; yet still retain the general appellation of Right, Acute, or Obtuse Angles. Such are the following.



DEF. 15. VERTICAL and CONTIGUOUS ANGLES.

If two Lines, AB and CD , cut and cross each other, there are made four Angles, at the Point, E , of their mutual Interfection; either two of which, AED , CEB , or AEC and DEB , touching at their Vertices, only, are called Vertical Angles.

Any other two, as AEC , AED , or AEC and CEB , &c. having one Side, CE or AE , common to both Angles, are called Contiguous or Adjoining Angles.



DEF. 16. ALTERNATE ANGLES, and others.

If a Line crosses or intersects two Lines, there are made eight Angles, A , B , C , D , &c.; of which C and F , also E and D , between the two Lines, one on each Side of the cutting Line, are called Alternate Angles.

C and E , also D and F , are called INTERNAL ANGLES on the same Side.

E and A , F and B , C and G , or D and H are called INTERNAL and OPPOSITE ANGLES, on the same Side.

DEF.

D E F I N I T I O N S. 11

DEF. 17. A **PLANE FIGURE** is a Space bounded on all sides by one or more Lines in the same Plane.

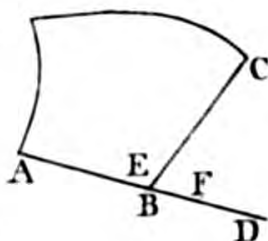
2. Any Line, between two adjacent Angles, forming or bounding a Figure, is called a **SIDE** of that Figure.

N. B. If a Plane Figure be bounded by Right Lines, only, it is called a **Right-lined Figure**; as Z. And if it be formed of Right Lines and curved, it is called a **mixed Figure**. As AC.



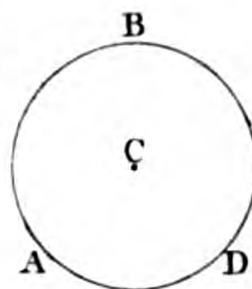
DEF. 18. **INTERNAL** and **EXTERNAL ANGLES**.

If any Side of a Plane Figure be drawn out beyond the Figure, as AB to D, the Angle E, or ABC, within the Figure, is **Internal**; and the Angle F or CBD, without the Figure, is called an **External Angle**.



DEF. 19. A **CIRCLE** is the simplest and most perfect of all Plane Figures, therefore the first; it is bounded by one regular and uniform curved Line, falling again into itself; which is called the **CIRCUMFERENCE** of the Circle.

The curved Line ABD is the Circumference; the Space, included within it, is the Circle.

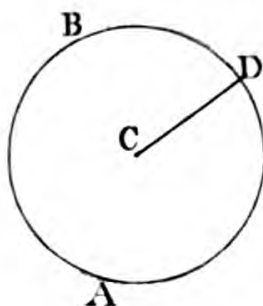


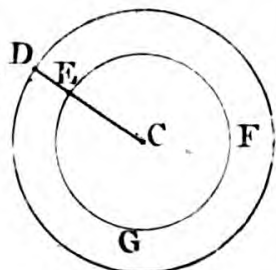
DEF. 20. **CENTER** of a Circle, or **CENTRE**, is a Point in the middle of a Circle, or the middle Point of a Circle; which is equally distant, every way, from the Circumference. As C.

N. B. The genesis of a Circle is thus defined.

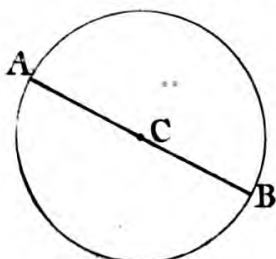
If a Line, CD, be conceived to be revolved quite around, on one extreme, C, fixed to a Pin or Point; the other extreme, D, will, in its revolution, describe the Circumference of a Circle, ABD; and the Line CD, having gone over the whole space, has generated the Circle, bounded by that Circumference.

Hence, it is evident, that all Right Lines, drawn from the Center of a Circle to the Circumference, are equal.





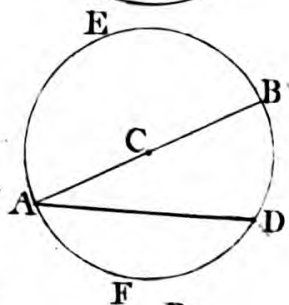
N. B. 2. Equal Right Lines describe equal Circles; but, if another Line, CE, or, if any Point, E, be assumed in the Line CD, another Circle will be described, on the same Center, and in the same space of time; whose Circumference, EFG, is to that of the other Circle, described by the Point D, in proportion to the Lines CE and CD, by which they were generated.



DEF. 21. DIAMETER, of a Circle, is a Right Line drawn through the Center, and terminated on both sides by the Circumference. As AB.

2. Half the Diameter of a Circle is called the RADIUS. As AC or CB.

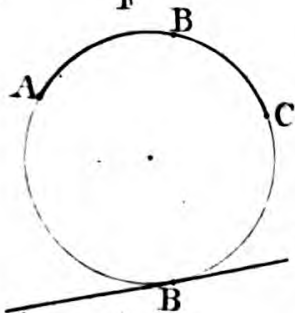
N. B. Every Diameter divides the Circle, and also the Circumference, in two equal Parts.



DEF. 22. A SEGMENT of a Circle, is any portion cut off by a Right Line; which is called a CHORD or SUBTENSE.

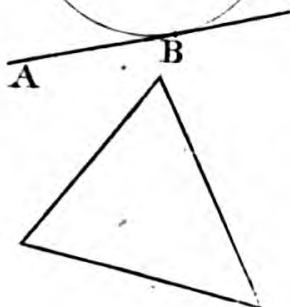
As AD, making two Segments, AED and AFD.

N. B. A Diameter is also a Chord Line.



DEF. 23. A SEMICIRCLE is a Segment, made by a Diameter, AB. As AEB or AFB.

Therefore, the Segment AED, which is greater than a Semicircle, is called a greater Segment; and AFD a lesser Segment.



DEF. 24. An ARK, or ARCH, is any portion of the Circumference of a Circle. As AB, BC, or ABC.

DEF. 25. A TANGENT is a Right Line drawn without a Circle, and touching it in a Point only; which is called the POINT OF CONTACT.

As AB, touching the Circle in B.

DEF. 26. A TRIANGLE is a Plane Figure bounded by three Right Lines, and contains as many Angles.

N. B. Not

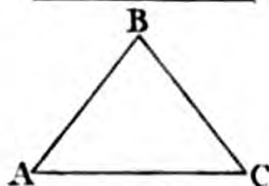
N. B. Not less than three Right Lines can include a Space and form a Figure; wherefore, a Triangle is the first of all Right-lined Figures.

Triangles are of various kinds. As follows.

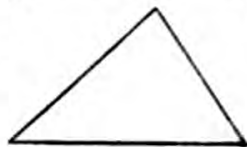
DEF. 27. 1. An **EQUILATERAL TRIANGLE** has all its three Sides equal, to one another.



DEF. 28. 2. An **ISOSCELES TRIANGLE** has only two equal Sides. AB and BC.

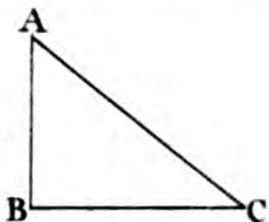


DEF. 29. 3. A **SCALENE TRIANGLE** has all its Sides unequal.



DEF. 30. 4. A **RIGHT-ANGLED TRIANGLE** is one that has a Right Angle. B.

2. The Side AC, opposite the Right Angle, is called the **HYPOTHENUSE**.



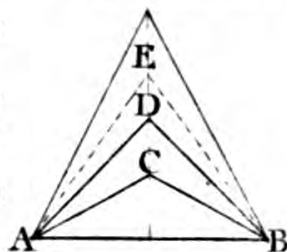
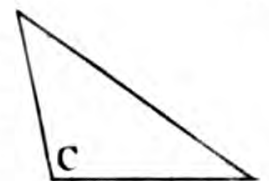
DEF. 31. 5. An **OBTUSE ANGLED TRIANGLE** is one which has an Obtuse Angle. C.

N. B. The two last are not distinct species of Triangles, but only a particular kind; which still come under the general Denomination of Isosceles or Scalene.

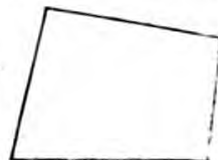
An Isosceles or Scalene Triangle may be either right or obtuse angled, or have all its Angles acute.

The Triangle ADB is right-angled; ACB is obtuse-angled; and AEB has all its Angles acute; yet, they are all Isosceles.

So likewise, the Figures of the two last Definitions are Scalene Triangles.



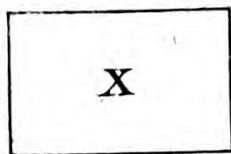
DEF. 32. A **QUADRILATERAL** or **QUADRANGLE** is a Plane Figure which has four Sides, and four Angles.



These are synonymous Terms; the first expressing it by the number of its Sides, the other by its Angles.

DEF. 33.

D E F I N I T I O N S.



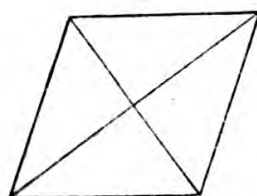
DEF. 33. A PARALLELOGRAM is a Quadri-
lateral, whose opposite Sides are parallel.



DEF. 34. A RECTANGLE is a Parallelogram,
whose Angles are all Right ones. As X.

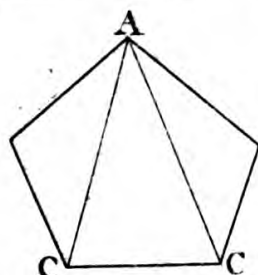
DEF. 35. A SQUARE is a Rectangle, whose
Sides are all equal, to one another. Z.

N.B. All Rectangles and Squares are Parallelograms.

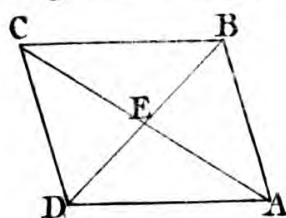


DEF. 36. A RHOMBUS is a Parallelogram,
whose Sides are all equal, and its Angles not
Right ones.

z. If the Sides of a Parallelogram are not all equal,
and the Angles not Right ones, it is called a RHOM-
BOIDES.

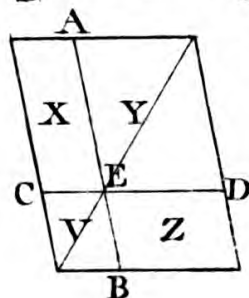


DEF. 37. A DIAGONAL is a Right Line drawn
between any two Angles, that are opposite, in
any right-lined or mixed Figure; i. e. from one
Angle to the other. As AC.



N. B. In Parallelograms, the Diagonal is usually
called a Diameter; because it passes through the Cen-
ter (the middle Point, E, where the two Diagonals,
AC and BD, intersect) and, as in a Circle, it divides
the Parallelogram into two equal Parts.

Any Right Line, cutting a Parallelogram through
its Center, is a Diameter.



DEF. 38. COMPLEMENTS of a Parallelogram.

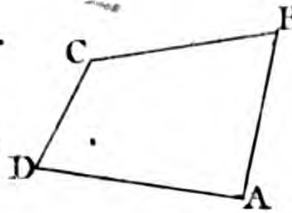
If any Point, as E, be taken in the Diagonal of a
Parallelogram, and, through that Point, two Right
Lines are drawn parallel to the Sides, both ways (AB
and CD) it will be divided into four Parallelograms,
V, X, Y, and Z; the two, X and Z, which touch
the Diameter, in the Point E, only, are called the
COMPLEMENTS; which, with either of the other,
about the Diameter, taken together (as XYZ or XVZ)
is called a GNOMON.

DEF. 39.

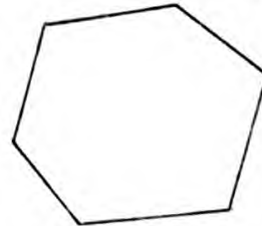
D E F I N I T I O N S. 15

DEF. 39. A **TRAPEZIUM** is an irregular four-sided Figure.

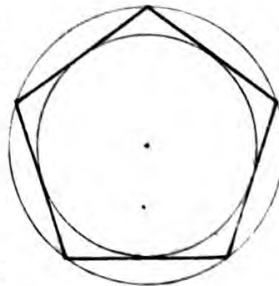
Wherefore, every Quadrilateral which is not a Parallelogram, is, consequently, a Trapezium.
As $ABCD$.



DEF. 40. A **POLYGON**. All right-lined or other Plane Figures, having more than four Sides, have the general appellation of Polygons, signifying many Sides.

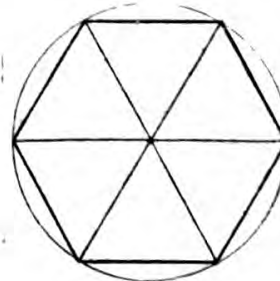


N. B. Ordinate or regular Polygons are such as have all their Sides and Angles equal; about which a Circle may be circumscribed, whose Circumference shall pass through every Angle of the Polygon; and, a Circle may also be inscribed, which shall touch every Side.

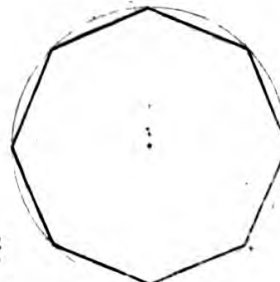


Polygons have various Names, derived to them from the Number of their Sides; as follows.

1. A **PENTAGON** is one that has five Sides.
2. A **HEXAGON** has six Sides.
3. A **HEPTAGON** has seven Sides.
4. An **OCTAGON** has eight Sides.
5. A **NONAGON** has nine Sides.
6. A **DECAGON** has ten Sides.
7. A **DUODECAGON** has twelve Sides.
8. A **QUINDECAGON** has fifteen Sides.



These eight are the most essential, in Geometry, and the most useful amongst Mechanics. To specify every kind of Polygon would be infinite.

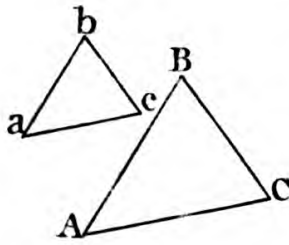


DEF. 41. **PERIMETER** is the sum or measure of all the Sides of a Polygon, or other right-lined or mixed Figure, in one Sum; which is sometimes called its Circumference, or **PERIPHERY**.

DEF. 42.

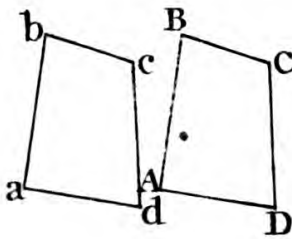
DEF. 42. **EQUIANGULAR FIGURES** are such as have an equal number of Angles ; and the Angles of one also equal, respectively, to the Angles of the other, each to its corresponding Angle.

As, a equal A, b equal B, and c equal C.



DEF. 43. **CONGRUOUS FIGURES** are such as have all their Angles equal, respectively ; also, the Sides, which contain equal Angles, or which are between equal Angles, are equal.

If the Angle a-b-e equal A, b equal B, c equal C, and d equal D ; and, if the Side a-b is equal AB, b-c equal BC, c-d equal CD, and a-d to AD ; then, the Figures a-b-c-d, ABCD, are congruous.

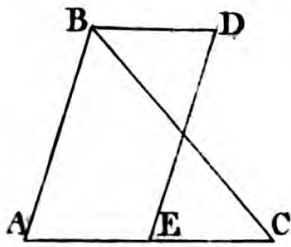


DEF. 44. **EQUAL FIGURES** are such as have an equal Area. See Def. 47.

N. B. If two Figures of different Denominations have an equal Area, they are called equal Figures.

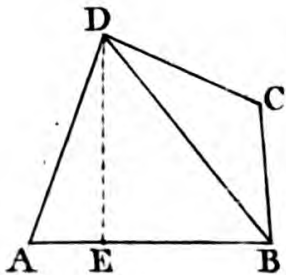
So, the Parallelogram ABDE is equal to the Triangle ABC.

Congruous Figures are equal and similar.



DEF. 45. **BASE**, of a Plane Figure, is the Side on which it is supposed to stand erect. As AB.

N. B. It may be any Side, at discretion ; but is generally applied to the lower Side, or that which is next towards us.



DEF. 46. **ALTITUDE**, of a Figure, is its perpendicular height from the Base.

As ED, perpendicular to the Base AB, is the Altitude of the Trapezium ADCB ; or of the Triangle ADB.

DEF. 47. **AREA** of a Plane Figure, or other Surface, is its superficial Contents ; i. e. the measure, or quantity of Space contained within the bounds of the Figure, expressed in square Feet, Yards, or any other known measure, of length.

DEF. 48.

D E F I N I T I O N S. 17

DEF. 48. SEGMENT of a Line, is any portion of a Line.

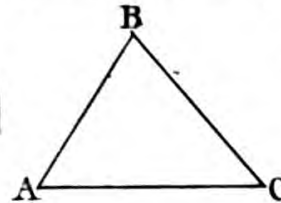
As AC, or CB, of the Line AB.



DEF. 49. SUBTEND. The Side of a Triangle which is opposite to any Angle is said to subtend that Angle.

Thus, AB subtends the Angle C, AC subtends the Angle B, and BC subtends the Angle A.

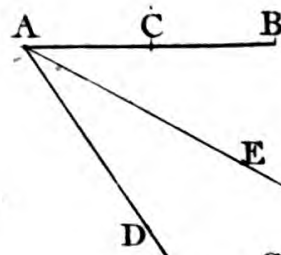
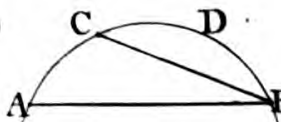
So likewise, the Chord or Subtense AB, subtends the Ark ACDB; and CB subtends the Ark CDB.



DEF. 50. To BISECT, is to cut or divide a Line or Angle, &c. into two equal Parts.

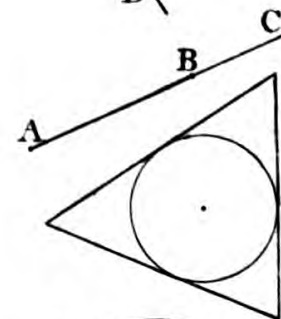
Thus AB is bisected in the Point C; and the Angle BAD, is bisected by the Line AE.

2. To TRISECT, is to cut, equally, into three Parts.



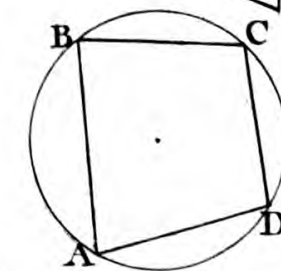
DEF. 51. To PRODUCE, is to draw out a Line or Plane, or to lengthen it at pleasure.

Thus, the Line AB is produced to C.



DEF. 52. To DESCRIBE, is to draw a Line, Circle, or other Figure.

DEF. 53. To INSCRIBE, is to draw a Figure touching every Side of another Figure, internally; or, whose Angles shall all touch the Circumference of a Circle.



DEF. 54. To CIRCUMSCRIBE, is to describe a Circle or other Figure passing through all the Angles of another Figure. As ABCD.

D

DEF. 55.

DEF. 55. EXTREMES or BOUNDS, are the utmost Limits of any thing.

So, the extremes of a Line are Points; the bounds or extremes of a Surface are Lines, (except the surface of a Sphere, which has no Bounds) and, the extremes of a Solid are Surfaces.

A Sphere has but one Surface; also, a Circle is bounded but by one Line, without beginning or end; consequently it has no Extremes.

Some Readers will, perhaps, think I have already deviated from the Plan proposed in the Preface, which was, to abridge the Elements of Euclid; but I hope they will find, that the time spent on the Definitions (by way of Introduction) is not thrown away; having made several of them useful Lessons, as well as defined the Terms; and I am persuaded, that the more perfectly they are understood, the progress, in the Subject they relate to, will be greatly facilitated. But, I must inform the Reader, that it is not the length of the Definition itself, but the number of Terms I have defined, and the Notes to several, which have swelled the bulk of them; having near twice the Number which some Authors have, and, in my opinion, not one superfluous.

However, not to discourage the young Student, I advise him not to burthen his memory too much at once; for there is not the least necessity that he should retain them all, before he proceeds further; 'tis enough, at first, to read them over with attention, and understand them clearly as he reads; they will soon become familiar to him, by frequent use, in the course of his study of the Science.

I also particularly advise him, when he meets with any Term hereafter, of which he has not a clear Idea, to turn immediately back to the Definition of it; he may depend on it, that the time will not be entirely lost.

Several of the Terms which I have here defined, being frequently used in the Subject of Geometry, are absolutely necessary to be defined, but are not, properly speaking, elementary. Such are the 4th, the 9th, the 14th, 15th, 16th, and 18th; the 36th, the 41st, 43d, 44th, 47th, and 48th, and the last seven; which are chiefly operative or practical Terms. Though I must freely own, that I see no reason why the 49th, 50th, 51st, and 52d, are not as necessary to be defined as the 53d and 54th: the 55th is made three separate Definitions, in the 3d, 6th, and 13th of Euclid.

It is, or ought to be, the design of every Author on any Science, to make his Book a perfect Tutor; consequently, no particular Term, made use of in that Science, and which is peculiar
to

to it, should be left undefined: for, we should suppose the Student to be entirely ignorant of all that relates to it. Can any thing be more absurd, than to propose bisecting a Line or Angle, or producing a Line, &c. to talk of Alternate Angles, &c. of Diagonals, Complements, Subtenses, &c. without having first defined what is meant by them? I have often been surprized at the omission of the Definition of a Polygon, and the various kinds of Polygons; the forming, inscribing, and circumscribing of which, is the principal subject of the 4th Book. They are frequently called by the Terms Pentagon, Hexagon, &c. which have never been properly, if at all, defined. In the 23d Definition they are, in general, called many-sided Figures; 'tis a strange ungeometrical Term, and never once made use of after, but other Terms are assumed.

Euclid, himself, has not defined a Parallelogram, that most useful and necessary Figure. I admire Mr. Stone's Apology for that omission, as being unwilling to increase the Number, unnecessarily; and says, that Euclid has sufficiently defined it in the 34th Proposition. Mr. Stone indeed has, but not Euclid; for why else has Keill defined it before the Proposition? But, if it be thought most eligible to define that Term in the Proposition, where it is first made use of, why are not others so defined? why are not all, as well as this?

At the same time, Euclid has, in Mr. Stone's Opinion, given several unnecessary Definitions; viz. the 3d, 6th, and 13th of his Euclid, which I always thought superfluous; the 9th I think so too, also the 20th. He, likewise, thinks the 18th, 26th, the 32d, and 33d, useless. In respect of the 26th (the 29th of this) I cannot say it is essentially necessary; yet, as Scalene Triangle is a general Term, including all that are neither Equilateral nor Isosceles, I cannot think it redundant.

The 29th, of which he says nothing, is really superfluous, viz. "when all the Angles are acute, it is called an acute-angled Triangle." I know of no properties peculiar to such an acute-angled Triangle, that is not common to every Triangle; for every Triangle has, necessarily, two acute Angles. An equilateral Triangle is included in that Definition, and so are the Isosceles and Scalene, frequently.

Why is not a three-sided Figure (Def. 21.) called a Triangle, as it is always called afterwards? we might as well go on, from four to five or fifteen sided Figures. An Oblong (Def. 31.) for a Rectangle, is quite ungeometrical, and is never called so after. The Rhombus and Rhomboides, 32. and 33. (Def. 36. of this) are, in a great measure, useless, being fully signified in a Parallelogram, which also includes Squares and other Rectangles; all which, are only particular species of Parallelograms; the Rhomboides including all unequal sided and acute angled.

I can by no means agree with Mr. Stone, in thinking the 18th Definition, of a Semicircle, needless; I also think, the Radius as necessary to be defined as the Diameter. In speaking of Angles in a Segment of a Circle, Prop. 31. of the 3d Book, the 12. of this, viz. "the Angle in a Semicircle is a Right one," &c.; since there is a necessity for calling it something, I cannot but think a Semicircle more elegant and expressive than half a Circle, which does not confine it to any particular Shape or Figure; provided it has half the Area, it is half a Circle; whereas, the Semicircle (which undoubtedly means half a Circle) is always understood to be contained under a Diameter and half the Circumference.

Mr. Stone is somewhat too dogmatical in his remarks. I cannot be of Opinion, that the manner in which Euclid has compiled his Elements, is the best that can possibly be, because Euclid lived two thousand Years ago. 'Tis not to be imagined that the mathematical Sciences had arrived at their *ne plus ultra* in his Days, for it is notorious they were not; why then, should we suppose, the Elements of Geometry to be in their greatest perfection? I am not so much wedded to antiquity, as to think them infallible; neither do I think that all Euclid's Definitions are necessary, or that he has omitted none that are so: Such, therefore, as are useless I have rejected, and have supplied their places with others, which I think more essential, and absolutely necessary to be defined.

POSTU-

P O S T U L A T E S.

Postulates are fundamental Principles in any Science; which, being plain and simple in themselves, may readily be granted; although, it is not possible to give perfect and indisputable Demonstration.

The following are, therefore, requested to be granted.

1. That there may be a perfect Plane; and that it may be extended at pleasure.
2. That a Right Line may be drawn between any two given Points; i. e. from one Point to the other.
3. That a Right Line may be produced at pleasure; i. e. that a finite Line may be continued or lengthened.
4. That a Circle may be described on any Center, and with any given Radius.
5. That one Figure may be applied to, or laid upon another Figure.

These Postulates must be granted, at least in Idea, or all Geometry falls at once to the Ground; for, if there cannot be a Plane, a Right Line, or a Circle, the whole Elements of Geometry are to no purpose; as it will be impossible to form a Construction, whereby we may demonstrate the most essential Properties of Figures in general, whether Plane Figures or Solids; and consequently, if no Demonstration can be given, there is an end of the Science, having no Data to build on.

In every mathematical or physical Science, there is a necessity for some Data or first Principles to be given, whereon to frame Hypotheses, in order to demonstrate the Theorems which follow; and the more simple those Principles are, the better; because there will be less room to dispute them. But, at the same time, if they are disputed, they are the most difficult to demonstrate, for being the most simple; because, there is no reverting back to any thing more so; and consequently, there can be no Demonstration given. That the thing is so, of itself, is somewhat arbitrary, notwithstanding there is no possibility of denying it; therefore, the more simple the first Principles are, the readier the assent will be given; and the Demonstrations, of the most complex Propositions, which follow after, will be easier obtained and more firmly supported; and consequently, the whole Science, which is built on those Principles, is more solid and permanent, and more securely established.

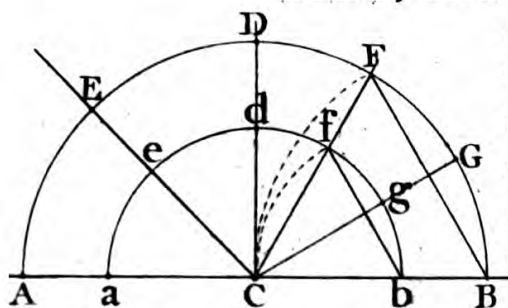
T H E

T H E
T H E O R Y
O F
P L A N E A N G L E S,

BEfore I proceed to practical Geometry, I think it proper, first to explain the Theory of Plane Angles; which I look on as a very necessary Introduction.

In order to have a perfect and clear Idea of Plane Angles, and to determine their Quantity by a certain standard or measure, the Circumference of a Circle is supposed to be divided into 360 equal parts, called Degrees (answering to the number of the Degrees on the Equator) it is evident that those divisions will be less in a small Circle than in a larger,

1. If from two Points in the Circumference of a Circle, as E and F, Lines are drawn to the Center, as EC and FC, there is made an Angle at the Center, C, which is greater or less, according to the number of Degrees on the Ark EDF, but it will be the same in a small Circle as in a large one, i. e. the lines will have the same inclination to each other. *e. g.*



2. ADB is a Semicircle, whose Center is C; the Arch AEDFB contains 180 deg. half 360, the whole Circumference.

From the middle point, D, of the Arch ADB, which is 90 deg. each way, from A and B, if CD be drawn, it will be perpendicular to AB; for ACD and DCB, are
Right

Right Angles, at the Center (Def. 11.) the Arks AD, DB, being each a fourth part of the whole Circumference, or half the Semi-circumference: hence, a Right Angle is said to be of 90 Degrees.

3. If the Ark AD be bisected in E, and EC be drawn, the Angles ACE, ECD, will be each of 45 deg. half ACD a Right Angle, or 90 Degrees.

And, if the Ark DB be trisected, at F & G, (i. e. divided into three equal parts), and FC be drawn, the Ark DF containing 30, and FGB 60 deg. the Angle DCF is said to be an Angle of 30, and FCB an Angle of 60 deg.

By which means, an Angle of any quantity may be obtained, or measured.

4. On the same Center, C, with any other Radius, as Ca, let the Arch aedfb be drawn, which is also a Semicircle.

It is very obvious, that it is also divided into the same number of parts, and in the same proportion, as the Arch AEDFB; for it is bisected in d, and ad is again bisected in e, and db is also trisected at f and g; wherefore, AD, ad are each a fourth; ED, ed an eighth; BF, bf, a sixth; and FD, fd, a twelfth part of their respective Circles; and the Angles ACD, aCd; ECD, eCd, &c. are the same in both.

From all which, it is clear, that, Angles may be formed or measured by an Ark or Circle of any Radius. And also, that equal Arks of the same, or of equal Circles, or an equal number of degrees in a Circle of any Radius, will form equal Angles at the Center.

5. If you would have an Angle of 60 degrees at the point C, of the line BC.

With

With any Radius, at pleasure, describe the Ark BD , cutting the Line BC in B ; with the same Radius, on the Center B , cut the Ark BD at F , and draw CF .

It is very clear, that the Angle BCF would be the same, if a less Radius had been taken, as Cb .

For, draw the Chord Lines FB and fb , each will be equal to the Radius of its respective Circle; and, the Triangles CFB , Cfb are equilateral; whose Angles are of 60 degrees each (Cor. 1. 9. 1.); for the Arks BF and bf are, each a sixth part of the whole Circumference of their respective Circles, of which C is the common Center, (see Prop. 11. 4.); and consequently, each contains 60 degrees on the circumference of that Circle, of which it is a Part; whose Radius is CB or Cb .

A description of the Instrument called a **PROTRACTOR**, with the application of it, in measuring and making Angles, of any known quantity or measure, may not be improper in this place. It is of special use in Surveying, in drawing Plans of any piece of Ground for building on, &c. or of Buildings, already erected, being readier and more exact than a Line of Chords.

The Protractor is a Semicircle, divided on its Limb or Semi-circumference, $AEDFB$, into 180 equal parts; having a small Notch at C , the Center. See the last Figure.

Some Protractors have a Scale, added to the Semicircle, which are the best and readiest in use.

In measuring an Angle, apply the Diameter, i. e. the Edge or Right Line AB , to either Side of the Angle ACE , with the Vertex, C , of the Angle, at the center of the Protractor; and, where-ever the Side CE , cuts the Limb or circular edge of the Instrument, observe how many Degrees there are from A to E , the Ark intercepted between the Sides AC and CE , of the Angle ACE .

If

If it contains 45 or 50, or whatever number of Degrees it happens to be, (as 45 by the Figure) the Angle ACE is of so many Degrees. If it had cut the Arch at D, as ACD, it is a Right Angle; and if beyond D, as ACF, it is obtuse; the Complement FCB, i. e. the Ark FB, being subtracted from the Arch of the Semicircle, ADB, or 180 degrees, gives the quantity of the Angle ACF.

2. If it is required to lie down or make an Angle of some known Quantity (as 45 deg.) at the Point C, of the Line AB.

Apply the edge of the Protractor, as above, with the Center, C, at the Point given; make a Mark or Point at E; take away the Instrument, and draw EC.

Thus, may any Angle whatever be laid down on Paper.

A Scale of equal Parts is nothing more than a Right Line divided into any number of equal Parts, at pleasure.

Each Part may represent any measure you please, as an Inch, a Foot, a Yard, &c.; for, being equal, whatever measure any Object or Figure contains, in length and breadth, a similar Figure may be constructed, on a Plane, having or containing the same number of Divisions, each way, on the Scale, as the real Object contains of Feet, Yards, &c. One of those Parts is generally subdivided into parts of the next inferior denomination, or into tenths and hundredths, denoting the Decimal Parts.

Observe, that the Divisions on the Scale, (whatever measure is represented by them,) must always be adapted to the Proportion you would delineate any Object, or form a Design.

See the Appendix (Page 15 and 16) for the construction of Scales.

N. B. A pair of Compasses or Dividers, a Drawing Pen, and a straight Ruler, are all the Utenils that are requisite, in Plane Geometry.

The Board or Paper, on which we draw any geometrical Figure, is supposed to be a Plane.

ABBREVIATIONS &c. EXPLAINED.

&c. *Et cætera.* And all the rest. When, what is supposed to follow may be readily understood.

i. e. *Id est.* That is. When, what has been said requires to be further explained.

viz. *Videlicet.* To wit; or, that is to say. When any thing advanced is given in Grofs, which is more particularly specified, as follows after.

e. g. *Exempli gratia.* For instance; or, for the sake of example. When an Example is to be given of what is advanced.

N. B. *Nota Bene.* Mark well. That is, take particular notice of that Paragraph.

Q. E. F. *Quod erat faciendum.* Which was required to be done.

Q. E. D. *Quod erat demonstrandum.* Which was to be demonstrated.

DEM.	Demonstration.	Hyp. AB.	Hypothenufe AB.
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COR.	Corollary.	Perp.	Perpendicular.
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SCHOL.	Scholium, or remark.	Dia.	Diameter.
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APPL.	Application, or use.	Diag.	Diagonal.
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Par. AB.	} Parallelogram AB.	Trap.	Trapezium.
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or P. AB.	} Parallelogram AB.	Pent.	Pentagon, &c.
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Rect. AB. Rectangle AB, or ABCD, &c.

R. Ang. ABC. Right Angle ABC, &c.

Line AB. Denotes Right Line AB, &c.

Note. When it is required to join any two Points, it is meant, that a Right Line be drawn between them, i. e. from one Point to the other.

AB-

ABBREVIATIONS, by way of Reference.

Post. 1. or 2. Refers to the first or second Postulate; where it is requested that such or such things may be granted.

Def. 6. or 7. Refers to the sixth or seventh Definition in Or, Def. 6. 3. the general Introduction. But, when there or 6. 5. &c. are two numbers, it refers to the 6th Definitions of the 3d or 5th Book.

Ax. 3. Refers to the third Axiom for Demonstration, arising from self-evident properties of things.

2. Th. P. A. Refers to the second Article in the Theory of Plane Angles, for illustration or Proof.

Pr. 1. or 2. &c. Refers to the first or second Problem, for the constructing of some Figure, &c.

P. 1. or 2. &c. Refers to the first or second Proposition of that Book, for proof of the Assertion.

P. 2. 3. &c. To the second Proposition of the third Book.

C. 2. 4. 1. To the 2d Corollary, of the 4th Proposition of the first Book.

Hyp. That the thing is so by the Hypothesis, or is given in the Premises.

Sup. That it is so by Supposition, only.

Con. That it is so by Construction; i. e. the thing was formed or made so.

Conf. consequently. Th. therefore. Wh. wherefore, it is so.

Note. When there is but one Number within Parenthesis or otherwise, as (15.) &c. it refers to the fifteenth Problem or Theorem of that same Book. But if there be two Numbers, as (10. 1.) or (12. 3.) &c. it refers to the 10th Prop. of the first Book, or to the 12th of the third, &c.

I N S T R U C T I O N S

F O R

Y O U N G S T U D E N T S.

I would advise the young Practitioner to draw every Figure as he proceeds; carefully remarking what things, as Points, Lines, Angles, &c. are given; which are, in general, stronger marked than the operative Lines; they being either dotted or finer drawn; the given Lines, &c. are, by that means, obvious and distinguishable from the other.

In Geometry, as in Arithmetic, there is always some Data or things given; from which, in Theory, other Properties are deduced, as a necessary consequence; and, in Practice, somewhat is required to be done, or performed, from what is given.

Let the Practitioner, therefore, select the given things, and mark them down, first, in the position given in the Premises; but, with as much variation as it will admit of; i. e. he need not put them exactly as in the figure, only observe that they are as required.

e. g. In Prob. 4. an Angle is required to be made at the extremity of a given Line; but the position of that given Line is not absolutely determined; also, the Angle may be made at either extreme, and either above or below the given Line.

Likewise, in the 6th and 7th Problems, the Perpendicular may be drawn, and the Point, in the 7th, given on either Side of the Line; for, let it be observed, and carefully remembered, that, by the Term Perpendicular, nothing more is meant than the Position one Line has to another; which Position, is when they make a Right Angle or Right Angles with each other; no regard being had to the position or situation of either, separately.

These things being premised, and the given Lines, &c. described on Paper; carefully observe the directions given in the operation, and proceed accordingly, step by step, drawing every Line, Angle, Ark, &c. as the Problem directs.

P R A C-

P R A C T I C A L G E O M E T R Y.

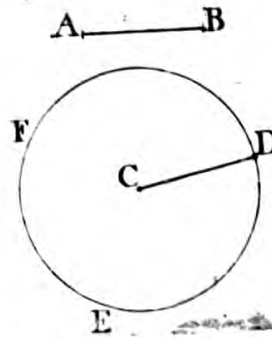
P R O B L E M I.

To describe a Circle of any given Radius, and on
a given Center.

AB is the Radius given, and C the given Center.

FIX one Point of a pair of Compasses in either extreme of the given Line, AB, and extend the other Point to the other extreme, i. e. open the Compasses equal to the given line.

Then, fix one Point of the Compasses in C, the Center given, and revolve the other Point around; which, by its revolution, will describe the Circumference DEF. - - - - - Post. 4.



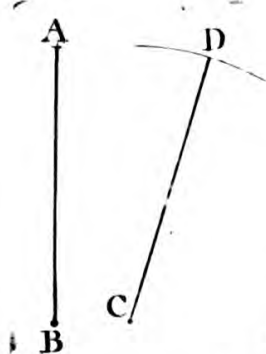
DEF contains the Circle required (Def. 19.)

For, if a Right Line be drawn from the Center to the Circumference, as CD, it is equal to the given Line AB; by Construction.

P R O-

P R O B L E M II. 2nd. I. Euclid.

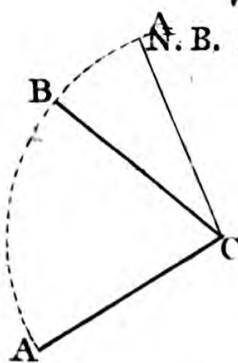
To draw Right Line from a Point given, C, which shall be equal to a given Line; A B.



Extend or open the Compasses equal to the Line given; fix one of their Points in C, the given Point, and, with the other, describe a Circle, or portion of a Circle only, at D. (Pr. 1.)

Apply a straight Ruler close to the Point C, and crossing the Ark at D, (in whatever position you require the Line to be) and, with the point of a Pencil or a Drawing Pen (applied first to the Point C, and drawn, along the edge of the Ruler, to the Ark at D) describe the Right Line CD. (Post. 2.) Which is equal to the given Line AB; by Construction; and by N. B. Def. 20.

SCHOL. Thus, the genesis of a Right Line (Def. 3.) is conceived to be by the direct motion of a Point.



N. B. In the practice of Geometry, it is often required to draw or to make a Line equal to another Line given; which is done by drawing an Ark of a Circle, as A B, from the given Line, A C, till it cuts the other; if the two Lines, A C and C B, touch at the Point given, C, which is made the Center.

But, if they do not touch, the Line given is taken for Radius (as in the Problem) and an Ark drawn where it is required; for equal Circles have equal Radii, as well as all Radii of the same Circle are equal; which needs no other Demonstration than the genesis of a Circle, in N. B. Def. 20.

So that, hereafter, when two Lines are observed to be Radii of the same Circle, it is sufficient Demonstration that the Lines are equal; and also, when they are made Radii of equal Circles.

APPL. The Application of this Problem, in designing, is to delineate or draw, on Paper, &c. a Right Line equal to some known measure, as A B, by a Scale of equal Parts.

P R O-

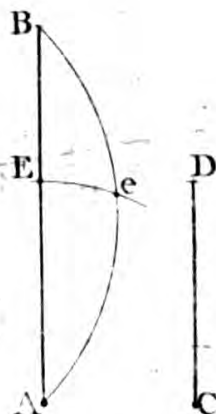
PROBLEM III. 3. I. Euclid.

A Right Line being given, to cut off a portion or segment equal to another given Line, or known measure.

AB is the first given Line, and CD the measure of the Segment required to be cut off.

Extend the Compasses from C to D, i. e. with the Radius CD, setting one Point of the Compasses in either extreme of AB, (from which the given Segment is required to be cut off), as A, with the other Point, draw a small Ark cutting the Line AB, at E. Q. E. F.

2. After the same manner, any portion of an Ark of a Circle may be cut off. As AeB.



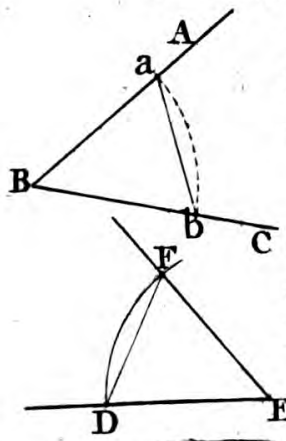
APPL. In delineating, the given Line AB may be supposed to be an indefinite Line already drawn; or, it may represent a certain measure, by some Scale, suppose 5 Feet, being made equal to 5 Divisions on the Scale; and it is required to cut off 3 Feet from the extreme Point A, of the Line AB, or two Feet from the other extreme, B.

By which means, a Right Line may be divided in any Proportion required.

PROBLEM IV. 23. I. Euclid.

To make an Angle, equal to a right-lined Plane Angle given.

ABC is the given Angle, and DE a Line given; it is required to make, at the point E, an Angle, with the Line DE, equal to the given Angle.



With any Radius, at discretion, on the Vertex of the given Angle, B, describe an Ark, $a\ b$, cutting the two Sides AB and BC in the Points a and b . - - - - Post. 4.

With the same Radius, on the Point E, draw the Ark DF; take the measure of the Ark ab in your Compasses; make DF equal ab . - - 2. Pr. 3. Draw EF through the Point F; and it is done; Q. E. F.

The Angle DEF is equal to the given Angle ABC; i. e. FE inclines the same to ED, as AB to BC.

This is evident, from the Theory of Plane Angles, Art. 4, ab and DF being equal portions of equal Circles.

Or, by drawing the Chord Lines ab and DF, the Triangles Bab , EDF, are Congruous. - - - - Con. And the Angle DEF is equal ABC. - - - P. 7. I.

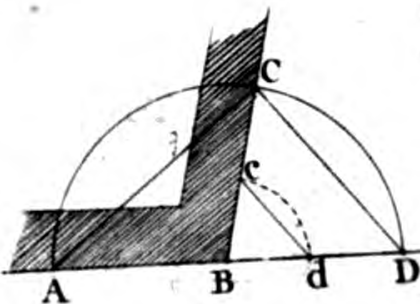
APPL. From this Problem we learn to delineate, i. e. to lay down or draw, on Paper, any right-lined Angle, of a piece of Ground or Building, &c. which we have measured, to form a Plan of it.

The

The Angle may be taken in the manner following.

If it be an internal Angle, as CBD ; set off equal measure, from B , to C and D , and draw, or measure only, the Diagonal, or Chord CD .

Suppose you have made BC and BD each equal 5 feet, and the Chord CD measures 6 feet 6 inches, or as it happens; then, by a Scale of equal Parts, having drawn BD , at discretion, take 5 divisions off the Scale; which may be either inches, half inches, or any other measure, provided they are equal.



With that Radius, on B , draw the Ark CD , cutting BD in D . Take $6\frac{1}{2}$ of the same divisions in your Compasses; and, setting one Point in D , make a small Ark, at C , with the other, and draw BC .

So shall CBD be an Angle laid down, equal to the Angle which was measured.

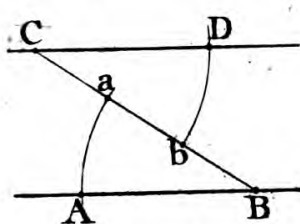
For (by N.B. Def. 8.) the length of the Lines or Sides makes no difference in the Angle; wherefore, if the Lines BC and BD were produced, equal to the Originals; i. e. equal 5 feet each; then would CD measure 6 feet 6 inches, and the Angle CBD would remain the same; which is obvious, if B and Bd be taken a half or a third part, or any other portion of BC or BD ; for then, cd will be the same portion of CD ; viz. a half or third, &c.

But, if ABC be the external Angle of a Building, so that, by reason of the obstruction of the Walls, &c. we cannot obtain the measure of the Chord Line, AC , it must then be got by its Complement of two Right Angles; i. e. by producing one Side, as AB , to D , and proceeding as before.

For, having assumed the Point B in the Right Line AD , and made the Angle CBD equal to the Complement of two Right Angles; the remaining Angle, ABC , is the inaccessible Angle required.

PROBLEM V. 31. I. Euclid.

To draw a Line through a given Point, which shall be parallel to a Right Line given.



AB is the given Line, and C is the Point given.

Through C, draw at pleasure a Right Line, as CB, cutting the given Line, AB, in the point B.

With any Radius, on B, describe the Ark Aa; and, on C, the Ark bD, with the same Radius.

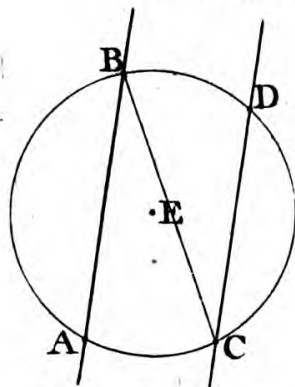
Make bD equal to Aa. - - - Pr. 3.

and through the Points C and D, draw a Right Line.

Which will be parallel to the given Line AB. Q. E. F.

For, the Alternate Angles ABC and BCD are equal. Con.

Therefore, CD is parallel to AB. - - - P. 4. 1.



Otherwise, thus.

With any Radius, at discretion, set one Point of the Compasses in the given Point, C, and fix the other, at pleasure, in E, either in the Line or out of it; on which Center describe a Circle, passing through C; or draw the Arks AC and BD only.

Make BD equal AC, and draw CD;

Then is CD parallel to the given Line AB. Q. E. F.

For, let CB be drawn.

DEM. Then, the Angles ABC, BCD are equal. Cor. 10. 3.

Therefore AB is parallel to CD. - - - P. 4. 1

Take

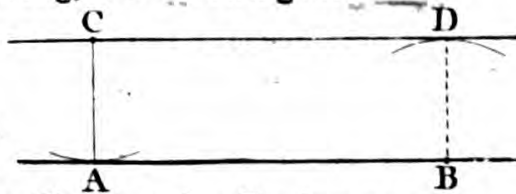
PRACTICAL GEOMETRY. 35

Or, when the Lines are very long, the following method is the most eligible.

Take the distance of the given Line AB , from the given Point C , in the Compasses; as CA ; and, assuming any Point, B , towards the other end of the Line, describe an Ark, at D , with that Radius.

Apply a Ruler to the Point C and the Ark at D , and draw CD ; which will be parallel to AB . - - Def. 7.

For, the Perpendiculars, AC and BD , being drawn, are equal, by Construction.



PROBLEM VI. 11. I. Euclid.

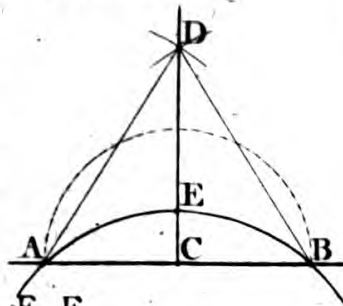
To draw a Perpendicular, from any Point, in a given Right Line,

AB is the given Line, and, let C be the Point given.

With any Radius, on the given Point C , describe a Semicircle, cutting the given Line in two Points, A and B ; i.e. make CB equal CA .

Then, with any Radius, greater than AC , on A and B , describe two Arks intersecting at D , and draw CD .

The Line CD will be perpendicular to AB . Q. E. F.



For, draw DA and DB .

DEM. Then, the Triangles ADC , CDB , are congruous.

Wherefore, the Angles are respectively equal, and ACD is equal DCB . - - - - P. 7. 1.

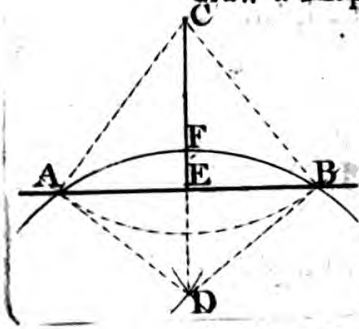
Therefore, CD is perpendicular to AB . - Def. 10.

After the same manner, a Perpendicular may be erected on the Ark of a Circle, AEB , at the point E .

P R O B L E M VII. 12. I. Euclid,

To draw a Perpendicular to a Right Line, from a given Point which is out of the Line.

Let C be the given Point, from which it is required to draw a Perpendicular to AB.



On C, the Point given, describe an Ark, with any Radius greater than CE, the nearest distance to the given Line, cutting it in A and B. - - - - - Post. 4.

With the same or any other Radius, on A and B, describe two Arks, intersecting at D.

Apply a Ruler to the Points C and D, and draw CE; Then will CE be perpendicular to AB. Q. E. F.

For, draw AC and AD, BC and BD.

DEM. The Triangles ACD, DCB are congruous. Con. Wherefore, the Angle ACE is equal to ECB.—P. 7. 1. Conf. the Triangles ACE, ECB are also equal.—8. 1. And the Angles AEC, CEB, being equal, are consequently Right Angles. - - - - - C. 2. 1. 1. Therefore, CE is perpendicular to AB. - - Def. 10.

In the same manner, a Perpendicular may be drawn to the Ark of a Circle, AFB.

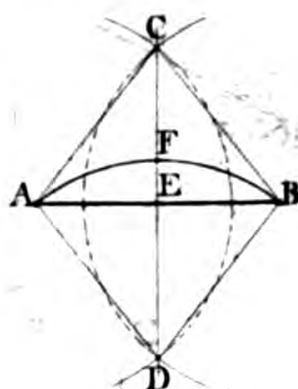
P R O-

PROBLEM VIII. 10. I. Euclid,

To bisect a given Right Line, A B.

With any Radius, at pleasure (provided it be more than half the given Line) on each extreme, A and B, describe two Arks, cutting each other, at C and D.

Draw the Right Line CD; which will bisect, or divide into two equal Parts, the given Line AB, in the Point of their common intersection E. Q. E. F.



Draw AC and AD, BC and BD.

DEM. Then, ACBD is a Parallelogram; by Con. P. 15. 1.
And, the two Diameters AB and CD bisect each other,
in its Center, E. - - - - - P. 16. 1.

This Problem may be deduced from and proved as the foregoing.

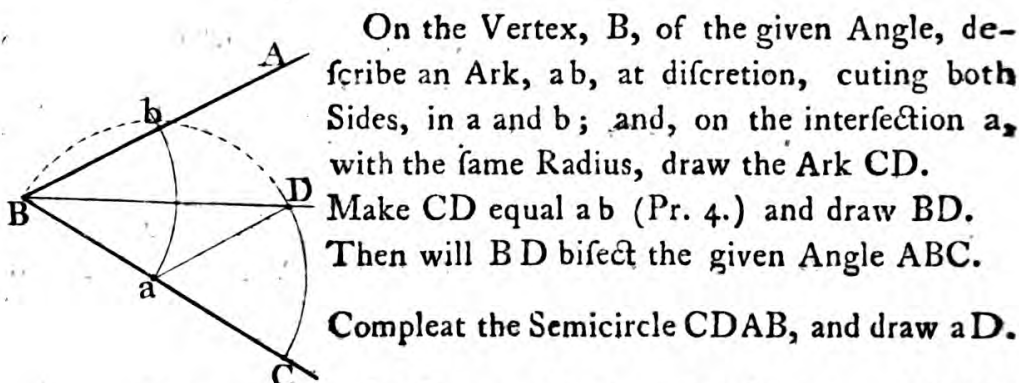
After the same manner an Ark of a Circle may be bisected; as AFB in the Point F.

N. B. It is not necessary to draw the whole Arks from C to D, but only the Intersections at C and D; the rest are useless.

P R O-

PROBLEM IX. 9. I. Euclid.

To bisect a right-lined Plane Angle. ABC.



On the Vertex, B, of the given Angle, describe an Ark, ab , at discretion, cutting both Sides, in a and b ; and, on the intersection a , with the same Radius, draw the Ark CD .

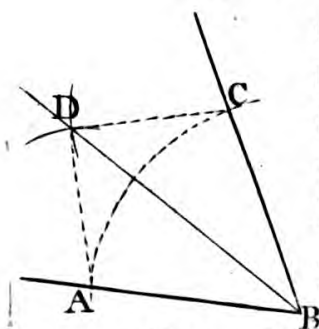
Make CD equal ab (Pr. 4.) and draw BD . Then will BD bisect the given Angle ABC .

Compleat the Semicircle $CDAB$, and draw a D .

DEM. Then DaC is an Angle at the Center, and DBC is at the Circumference of that Circle.

Th. DaC , (eq. ABC , Con.) is equal twice DBC . P. 9. 3. And consequently, ABC is double of DBC . Q. E. D.

Or, an Angle may be readily bisected in this manner.



On the Vertex B describe an Ark, AC , as before. On the two Intersections, A and C , with the same, or any other Radius, describe two Arks, cutting each other, at D .

Draw BD ; which bisects the Angle ABC .

Draw DA and DC .

DEM. Then, the Triangles ADB and DBC are congruous, by Construction.

Therefore, the Angle ABD is equal DBC . - P. 7. 1.

Angles may be thus divided into four, eight, or sixteen equal Parts, by bisecting again and again; but there is no geometrical method, by which, Angles, or curved Lines, may be divided into any equal Parts, at pleasure, as a Right Line may be divided; otherwise than by dividing the Ark with Compasses.

APPL,

APPL. By this Problem, Carpenters and Joiners, &c. find, what they call, the Mitre of any Angle (whether it be Right, Acute or Obtuse) with ease and expedition.

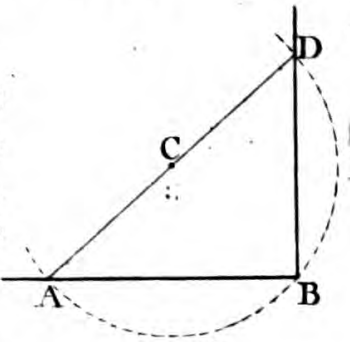
In returning, or breaking Mouldings at an Angle, whether external or internal, the Angle bisected is the Mitre; in which the Mouldings will exactly fit each other.

PROBLEM X.

To make a Right Angle; or, to draw a Perpendicular at the extreme Point of a Right Line.

It is required to make a Right Angle, at the extreme, B, of the Right Line AB.

Set one point of the Compasses in the point B; and, with any Radius, at discretion, fix the other Point, at pleasure, at C (on that side, you require the Perpendicular) and draw the Ark ABD, through the Point B, and intersecting the Line AB in A.



Draw a Right Line through the Intersection A and the Center C, cutting the Ark on the opposite Side, at D. Draw BD, which will be perpendicular to AB. Q. E. F.

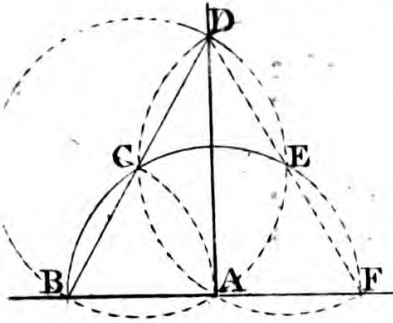
DEM. For the Angle ABD, being in a Semicircle, is a Right Angle. - - - - - P. 12. 3.

Or, it is frequently, and not inelegantly, performed thus.

It is required to draw a Perpendicular, to AB, at the extreme Point A.

Fix

40 PRACTICAL GEOMETRY.



Fix one point of the Compasses at A, and with any Radius, describe the Ark BCE, cutting AB in B.

On B, describe the Ark AC, cutting the former in C; on which Center describe a Circle, or an Ark AED only; and on E, where it cuts the Ark BCE, draw ACD, or cut the other Ark, only, at D.

Draw AD, which will be perpendicular to AB. Q. E. F.

Or thus, without the Point E.

Describe the Arks BC and AC, intersecting at C.

Through B and C draw the Right Line BCD, indefinite.

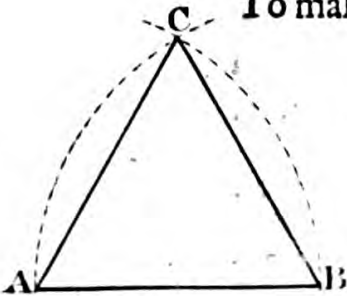
Make CD equal CB, and draw AD; and it is done.

By producing BA to F (making AF equal AB) and FD being drawn, it is demonstrated as the 6th.

The use of Right Angles, perpendicular and parallel Right Lines, are so well known, that it would be impertinent to point them out; particularly to those concerned in building, and various other mechanic Arts. They give beauty, strength, utility, and conveniency to a Building and its several Appurtenances; also the execution, of the several parts thereof, depends on them entirely.

P R O B L E M XI. I. I. Euclid.

To make an Equilateral Triangle, on a Line given, AB.



With the Radius AB, the given Line, and on the extreme Points A and B, describe two Arks AC and BC, intersecting at C.

Draw AC and BC; and it is done. Q. E. F.

This needs no Demonstration; for it is evident, that the three Sides are all equal, seeing, they are all Radii of equal Circles.

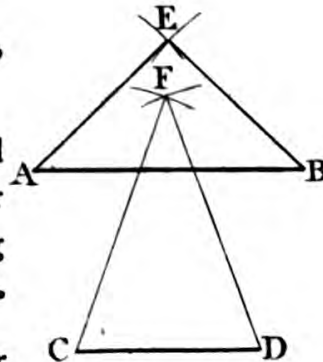
P R O-

P R O B L E M XII.

To make an Iſoſceles Triangle.

AB and CD are two given Lines ; of which, let AB be the Baſe.

Take the Line CD in your Compaſſes, and with that Radius, on the extremes of the other Line, A and B, deſcribe two Arks, interſecting at E ; draw AE and EB ; and it is done. Q.E.F.



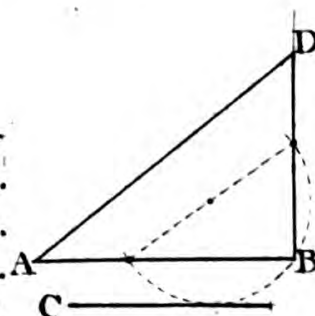
If the other Line, CD, had been required for the Baſe, the operation would be the ſame ; taking AB for Radius, & drawing the Arks, interſecting F ; draw CF & FD.

P R O B L E M XIII.

To make a Right-angled Triangle, having two Lines given for the Sides containing the Right Angle.

Let AB and C be the two given Lines.

On either Line, as AB, and, on either extreme, as B, make a Right Angle ABD.—10. Make BD equal to the other given Line, C,—2. and draw the Hypothenuſe AD ; - - Poſt. 2. ABD is the Triangle required. Q E. F.



The uſe and application of Triangles, in general, are almoſt univerſal, in mathematical Sciences. The Iſoſceles and Right-angled are particularly uſeful in Perſpective.

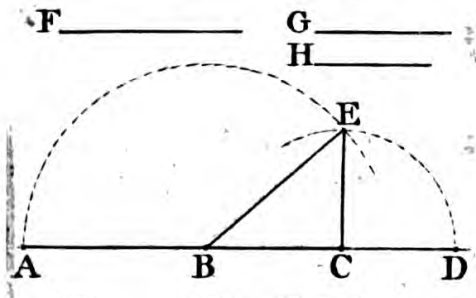
N. B. The Properties of the Right-angled Triangle are fully explained in the ſeventh Propoſition of the ſixth Book of Elements.

P R O-

P R O B L E M XIV. 22. I. Euclid.

To make a Triangle, of three unequal Lines given ;
any two of which must be greater than the other.

F, G, and H, are the three given Lines.



Let them be placed at the ends of each other in one Right Line, in what order you please, i. e. having drawn the Right Line AD indefinite, make the Segments, AB, BC, and CD, respectively equal to the given Lines, F, G, H.

With the Radius AB, on B, describe a Circle, or the Ark AE only; and on C, describe the Ark DE, with the Radius CD, cutting the other Ark in E; draw BE and EC, and it is done. Q. E. F.

This Problem and the two last need no Demonstration; every thing being as required, by Construction; agreeable to Def. 26. 28. & 30.

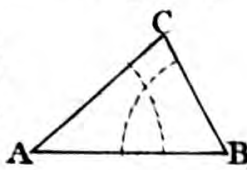
N. B. The Triangle BEC would be the same, if either of the other two Lines; F or A, had been made the Base, by placing it in the middle; only the position of the Triangle would be varied. This follows from Prop. 7. 1. of Elements.

2. There is no necessity for placing them, at all, in this manner; only, with the different Radii of two of the Lines, draw Arks, on the extremes of the other Line, cutting each other; taking any one for the Base.

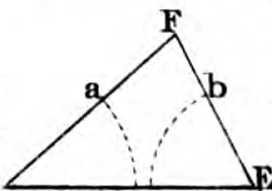
P R O B L E M XV.

To make a Triangle, similar to another Triangle*.

Let ABC be the given Triangle, and DE a Line given, on which to construct a Triangle; similar, and in a similar Position, i. e. alike situated, to ABC .



At the Point D , of the given Line, make an Angle EDa , equal to CAB , of the given Triangle, by - - - - - Pr. 4.



And, at the Point E , make the Angle DEb , equal to ABC .
Produce Da and Eb , meeting in F . - - - Post. 3.
The Triangle DEF will be similar to ABC . Q. E. F.

DEM. For, the three Angles are respectively equal. C. 5. 10. 1.

Therefore, the Sides are proportional. - - - P. 4. 6.

Consequently, the Triangles are similar. - - - Def. 1. 6.

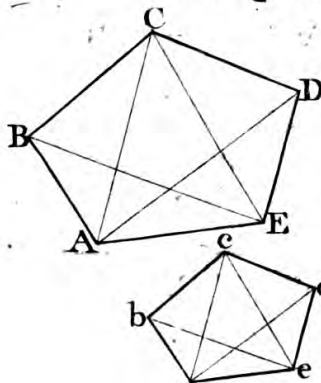
APPL. This Problem is of great use in various Professions. For, by it we learn to take Altitudes and Distances, though ever so inaccessible; or, the Surveyor takes his Bearings, and lays down a Plan of the Ground he surveys; by it, the Mariner plans the Course in which the Ship ploughs the Ocean; and the Mechanic plans the Ground, on which he intends to build, &c.; in short, it is almost of universal Use; which, to enumerate, is not necessary in this place.

* For similar Figures, see Def. 1. of the 6th Book of Elements.

PROBLEM XVI. 18. VI. Euclid.

To make a Figure, similar to any given right-lined Figure.

Let $ABCDE$ be the Figure given, and ae a Line given, on which to construct a Figure, similar, and alike situated to the given Figure.



Draw the Diagonals AC , AD , BE , and CE .

On the given Line, ae , make the Triangle abe , similar to the Triangle ABE , in the original Figure, by the foregoing. - - Pr. 15.

also, make ace similar to ACE , & ade to ADE

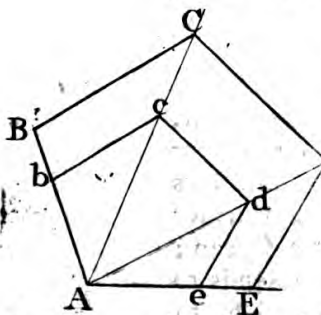
Join bc & cd , which compleats the Figure.

$abcde$ is similar to $ABCDE$.

Q. E. F.

^aThis method is deduced from the foregoing Problem; for, all right-lined Figures are composed of, or may be reduced into right-lined Triangles. It is demonstrable from 4. and 13. of 6. El.

The Application of this Problem is evident.



It may be thus performed, when required bigger.

On AB , the given Line, describe a Pentagon, $Abcde$, congruous to the given one, by reducing the given Figure into Triangles (as above) and making Abc , Acd , and Ade , respectively equal to them. - - - - Pr. 14.

Produce AE , and the Diagonals, Ac , Ad , indefinite.

Draw BC , CD , and DE , parallel to the Sides bc , cd , and de , respectively; cutting the Diagonals and Side AE , in the Points C , D , and E . $ABCDE$ is similar to $Abcde$.

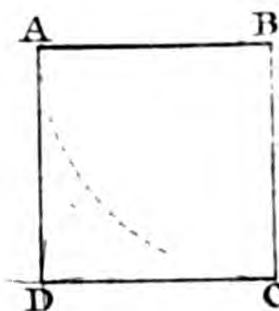
After the same manner, any Polygon may be increased or diminished in any Proportion; demonstrable by 2. 6. El.

P R O-

PROBLEM XVII. 46. 1. Euclid.

To make a Square on a given Line. AB.

On either extreme of AB, make a Right Angle; as ABC. - - - Pr. 10.
Make BC equal AB. - - - Pr. 2.
Then, with the Radius AB, on A and C, describe two Arks, intersecting at D, and draw AD and DC.



The Quadrilateral ABCD is a Square.

DEM. For it is equilateral, by Construction;

consequently it is a Parallelogram. - - - P. 15. 1.

Wh. the Angle D = B; conf. D is a Right Angle. - 10. 1.

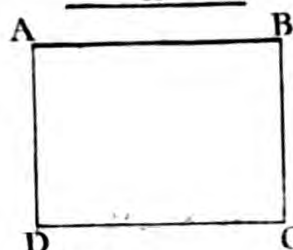
But, the Angles A and C are also equal; conf. Right.

Therefore, ABCD is a Square. - - - Def. 35.

PROBLEM XVIII.

To make a Rectangle; two Lines is given; AB & E.

Make ABC a Right Angle. - - - Pr. 10.
Make BC equal to the Line E; - - - 2.
Draw AD parallel to BC, and CD to AB. - 5.
ABCD is the Rectangle required.



DEM. For, it is a Parallelogram, - - - Con.

Wh. the Angle D is equal B, and A equal C. - P. 15. 1.

But, the Angle B is a Right one. - - - Con.

wherefore they are all Right Angles; - - - Th. 1. 10. 1.

and therefore it is a Rectangle. - - - Def. 34.

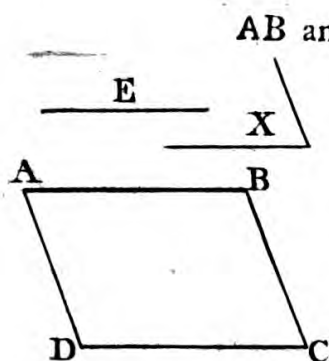
The Square and Rectangle are of great use in the mechanic Arts; as most regular Figures are Rectangles.

In Mensuration, the Area of every Figure is reduced to a standard measure; by the Square, or other Rectangle.

P R O-

P R O B L E M XIX.

To construct a Parallelogram under a given Angle, and having its Sides equal to given Lines, of determinate length.



AB and E are the given Lines, and X the given Angle.

At either extreme, of either Line (as A) make an Angle, BAD equal to the given Angle. Make AD equal to E, the other given Line. Through the Points D and B, draw DC parallel to AB, and BC to AD, meeting in C.—Pr. 5. Then is ABCD the Parallelogram required.

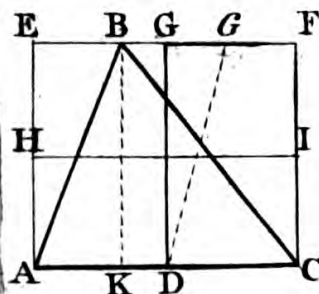
This needs no Demonstration, every thing being as required by Construction.

The opposite Sides and Angles are equal, by - - P. 15. 1.

APPL. By this Problem is delineated Plans, &c. of four sided Objects which are not right-angled, but, whose opposite Sides are equal. Any one Angle, being taken, determines all the rest.

P R O B L E M XX. 42. I. Euclid.

To make a Rectangle, or any other angled Parallelogram, equal to a given Triangle.



ABC is the given Triangle.

Bisect any Side, as AC in the Point D.—8. Draw EF, through the opposite Angle, B, parallel to AC. - - - - Pr. 5. Draw DG, from the point of bisection, perpendicular to the Base AC, cutting EF in G. Lastly, draw CF parallel to DG.

The Rect. DGFC is equal to the Triangle ABC. Q. E. F.

Or,

Or, if the Rectangle AHIC be constructed on the whole Base, AC, and half the perpendicular Altitude, BK, it will be equal to the Triangle ABC.

DEM. For, it is half the Rectangle AEFC, which is double the Area of the Triangle ABC. - - 17. 1.

N. B. If any other angled Parallelogram was required, you must proceed as in Prob. 19. making the Angle CDG or DCF equal to the Angle given; still keeping the same Altitude; i. e. between the same Parallels, AC and EF, and the same Base DC or AD.

SCHOL. From hence, and from the 17th Proposition of the first Book of Elements, the whole Theory of Mensuration is deduced; as all Figures whatever, except Parallelograms, are resolved into Triangles in Mensuration.

From the 17th of the first Book we learn, that every Triangle is equal to half a Parallelogram of the same Base and Height; consequently, the Rectangle DGFC is equal to the Triangle ABC, which is on half its Base, AC, and the same height, GD, equal BK.

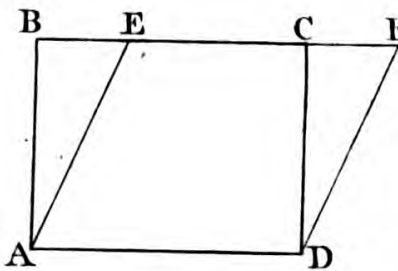
COR. Hence, the Rule for measuring a Triangle is to multiply the Perpendicular, BK (equal GD) by half the Base, AC (equal DC) or the whole Base, AC, by half the Perpendicular BK (equal AH); the first gives the Rectangle DGFC, the other is the Rectangle AHIC; either of which is equal to the Triangle ABC.

For, if the whole Base AC be multiplied by the Perpendicular BK, it gives the Area of the Rectangle AEFC which is double of the Triangle (17. 1.) consequently, half that Sum is the Area of the Triangle ABC; and also of the Rectangle DGFC or AHIC; which is, therefore, equal to the Triangle given.

P R O B L E M XXI.

To make a Parallelogram, under any given Angle,
equal to a Rectangle, and having the same Base.

ABCD is the given Rectangle.



Make the Angle DAE equal to the given Angle, cutting BC in E, by - - - Pr. 4.
Produce BC; and draw DF parallel to AE.-5.
The Parallelogram AEFD is equal to the given Rectangle ABCD. - - - P. 18. 1.
Or, it may be thus demonstrated.

DEM. AB is equal DC (15. 1.) and BE is equal CF. Ax. 7.
for, EF (eq. AD) is equal to BC; and EC is common.
And, the Angle ABE is equal to DCF. - - P. 4. 1.
conf. the Triangle AEB is equal to DFC. - - 8. 1.
Wherefore, if from the Rectangle ABCD there be taken
away the Triangle AEB, and an equal Triangle DFC
be added, the Par. AEFD is equal to the Rect. ABCD.

COR. 1. Hence it is evident, that two Spaces may have the
same Area, yet differ greatly in compass or circuit.

For, the Sides AE and DF, of the Parallelogram AEFD,
are greater than AB and DC of the Rectangle ABCD. 12. 1.
But, EF is equal BC (Ax. 3.) for each is equal AD.—15. 1.
Therefore, the Circuit of the Par. AEFD is greater than that
of ABCD; and they contain equal Areas. - - 18. 1.

COR. 2. From hence is deduced the general Rule for mea-
suring all Parallelograms; which is, to multiply the Base;
i. e. any Side (as AD) by its perpendicular height, (CD)

For it gives the Area of the Rectangle ABCD, and conse-
quently of the Par. AEFD, which is equal to the Rectangle.

P R O-

PROBLEM XXII.

To make a Rectangle equal to a given Trapezium; $ABCD$.

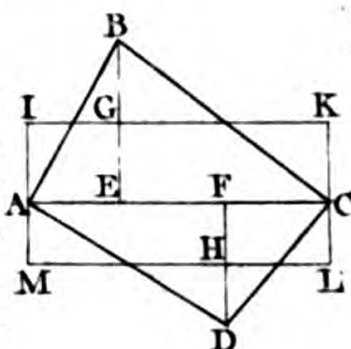
Draw either Diagonal, as AC ; to which, draw the Perpendiculars BE and DF , from the opposite Angles. - - - Pr. 7.

Bisect the Perpendiculars, BE and FD , in G and H . - - - - - Pr. 8.

Through the Points G and H , draw IK and LM parallel to AC . - - - - - Pr. 5.

and through A and C , draw IM and KL , perpendicular to AC . - - - - - Pr. 10.

The Rectangle $IKLM$ is equal to the Trapezium $ABCD$.



DEM. For, the Rect. $AIKC$ is equal to the Triangle ABC , and the Rect. $ACLM$ is equal to the Triangle ACD . 20.

Conf. the Rect. $IKLM$ is equal to the Trap. $ABCD$, Ax. 2.

Otherwise,

Having drawn a Diagonal and Perpendiculars, as before; bisect the Diagonal, in the Point G . - - - - - Pr. 8.

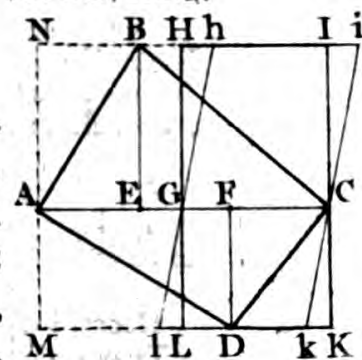
Draw BI and MK through the Angles B and D , parallel to the Diagonal (AC), and, through G and C , draw HL and IK , parallel to BE and FD , cutting BI and MK in the Points H and L , I and K .

Then, the Rect. $HIKL$ is equal to the Trap. $ABCD$.

DEM. For, the Rect. $GHIC$ is equal to the Triangle ABC , and the Rect. $GLKC$ is equal to the Triangle ACD . 20.

H

N B. By



50 PRACTICAL GEOMETRY.

N. B. By this Problem and the last may be performed the 45th of Euclid, without the assistance of the next, which he makes use of, and much readier; making the Parallelogram under a given Angle, as $hikl$, which is equal to $HIKL$ by the last.

COR. Hence we learn, to find the Area of any Quadrilateral whatever; the Rule for which is, to multiply the Diagonal (AC) by half the Sum of the two Perpendiculars (Fig. 1.) or, the Sum of the two Perpendiculars (BE and FD) by half the Diagonal (GC, Fig. 2.) or, if we multiply the whole Diagonal by the Sum of the two Perpendiculars, they will give the Area of the Rectangle MNIK; half the Sum, of which, is the Area of the Trapezium ABCD.

It may appear strange, to those who have not considered it, that the Area of any Figure should be obtained without measuring its Sides; which are of no use in this operation.

But, having well digested what has been advanced, they will find, that the whole business of Mensuration is to find a Rectangle equal to any Figure; for (as I shall make appear hereafter) the multiplication of any two Numbers, applied to measure, denotes a Rectangle of such Dimensions.

Now, since it seldom happens, that a Trapezium has either Right Angles or parallel Sides, it is plain, that they cannot be of any use towards obtaining its Area; whereas, the Diagonals and Perpendiculars are at Right Angles with each other.

A Diagonal divides any Quadrilateral into two Triangles; and every Triangle is equal to half a Parallelogram having the same or an equal Base, and the same Altitude. (Prop. 17. 1.)

Hence, it is easy to account for the Rules given for measuring a Trapezium, as two Triangles having a common Base; which is a Diagonal of the Trapezium.

For, every Trapezium is equal to half a Parallelogram which circumscribes it, having two Sides parallel to either Diagonal; and all Parallelograms having the same Base and Altitude are equal (18. 1.) consequently, they are equal to a Rectangle of those Dimensions.

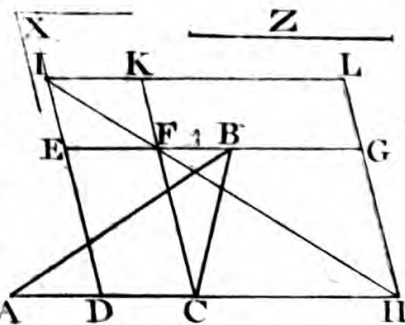
P R O-

PROBLEM XXIII. 44. I. Euclid.

To make a Parallelogram, equal to a Triangle, having an Angle equal to a given one, and a Side equal to a given Right Line.

ABC is the given Triangle; X the Angle, and Z the given Line.

Make the Parallelogram $DEFC$, equal to the Triangle ABC , whose Angle, at D or C , is equal to the given Angle, X .—Pr. 20. Produce DE indefinite; also, produce AC ; and make CH equal to the given Line, Z ;—2. From the Point H , and through the Angle F , of the Parallelogram $DEFC$, draw HI , cutting DE , produced, in I .



EI , is the other Side of the Parallelogram, sought; which may be completed by Prob. 19.

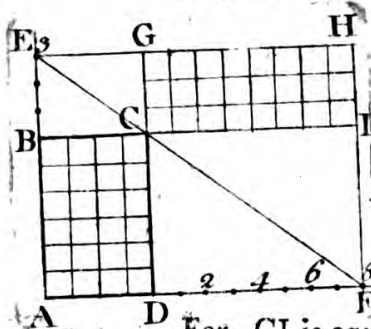
Draw IL parallel to AH , and HL parallel to DI meeting at L ; produce EF to G , and CF to K .

DEM. The Par. $KLGF$ (having its Angles, at K and G , equal to the given Angle X , and a Side, FG , equal to a given Line, Z) is equal to the Par. $DEFC$.—P. 19. 1. which, is equal to the Triangle ABC . - - Pr. 10.

SCHOL. *This is, properly, geometrical Division.*

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Let the Rectangle ABCD be any Quantity given, to be divided; whose Area is 24; the Side AB being 6, and AD 4 equal Parts; it is required to be divided by 3.



Produce any Side, as AB; and make BE equal 3 (half AB) also produce AD indefinite. Through E and the Angle C, draw EC, and produce it, cutting AD produced, in F; DF will be equal 8 of the divisions, or twice AD, which is the Quotient sought.

Compleat the Rect. AEHF; produce BC to I, and DC to G; the Rect. CH = AC.

For, CI is equal DF, i. e. equal 8; and, CG is equal BE equal 3; 8 multiplied by 3, is equal 24, the Area of CH; which is equal to the Dividend, or the Rectangle ABCD.

If the Divisor had been 8, equal DF, the Quotient would have been 3, equal BE.

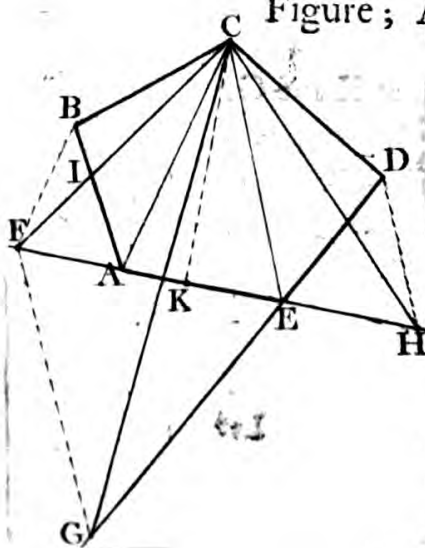
If a fractional Number was given for the Divisor, the Quotient would, most probably, be fractional likewise; for the Rectangle under the Divisor and the Quotient will always be equal to the given Rectangle, which is the Dividend.

As in Division, the Quotient multiplied by the Divisor is (when there is no remainder) equal to the Dividend; which proves the work to be true.

N. B. It is equal, which way the given Rectangle is situated, in this operation; or which Side is produced for the Divisor; the Quotient, will always be the same.

P R O B L E M XXIV.

To make a Triangle equal to any given Right-lined Figure; ABCDE.



Draw the Diagonals AC and CE; and produce EA, indefinite. - - - Post. 3. From the Angle B, and parallel to AC (the adjacent Diagonal) draw BF; cutting EA, produced in F; and draw FC.

Again. Produce the adjoining Side DE, indefinite; and, parallel to the other Diagonal CE, draw FG; cutting DE produced, in G; draw GC, and it is done. Q. E. F.

The

The Triangle GCD, is equal to the given Pentagon ABCDE.

DEM. 1st; the Trap. FCDE is equal to the Pent. ABCDE.

For, since FB is parallel to AC, the Triangle AFC is equal to ABC (18. 1.) and AIC is common to both; wherefore, AFI is equal to IBC. - - Ax. 7. 1.

Then, if from the given Pentagon, ABCDE, there be taken away the Triangle IBC, and its equal FAI be added; the Trapezium FCDE is equal to the given Pentagon. - - - - - Ax. 7. & 6.

But, the Trap. FCDE is equal to the Triangle GCD. For, because FG is parallel to CE, the Triangle EGC is equal to EFC. - - - - - P. 18. 1.

conf. EGC added to ECD = EFC + ECD.—Ax. 6. 1.

i. e. the Triangle GCD is equal to the Trap. FCDE.

But, the Trap. FCDE = the Pent. ABCDE.—proved above

Th. the Triangle GCD is equal to the given Pentagon.

Or, if the Side AE had been produced both ways, and DH drawn parallel to CE, cutting AE in H, and if HC be drawn; the Triangle FCH is also equal to the given Pentagon, equal to the Triangle GCD.

For, the Triangle CHE is equal to CDE. - - 18. 1.

SCHOL. *By which means, a Triangle may be readily constructed, whose Altitude shall be equal to the Altitude of the Polygon; or, to a Perpendicular, from any Angle to any Side; as CI or CK. Or, any one Angle, as D, and an adjoining Side, CD, of the Polygon, may be retained in the Triangle.*

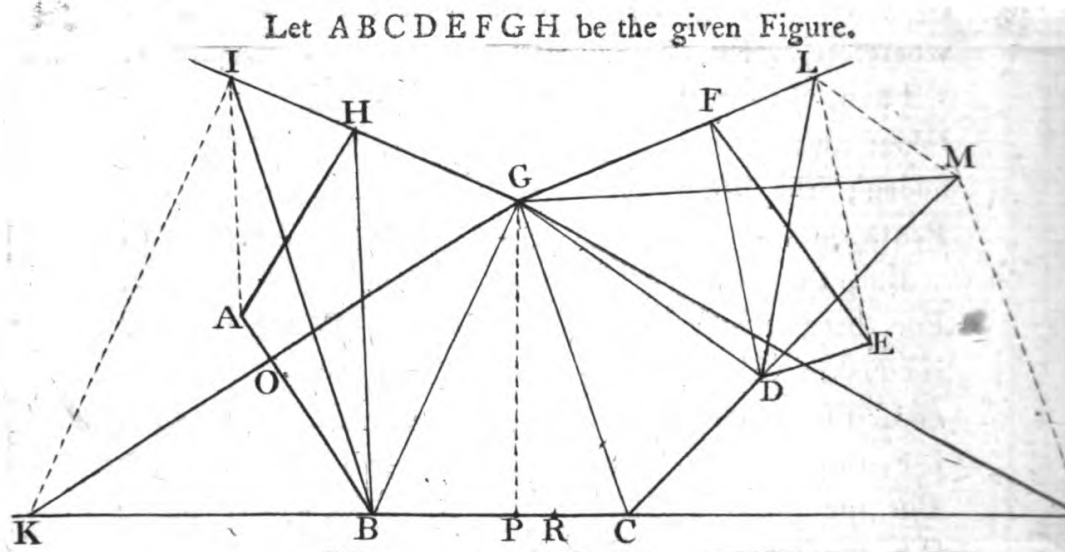
I have been more particular in the Demonstration of this Problem than in any of the former; because, it may, to some, appear, at first sight, rather intricate; which it certainly is not, being properly analyzed.

From hence, and from Prob. 20. a Rectangle may be found or constructed equal to any given right-lined Figure; and, by Prob. 25. the Side of a Square, equal to a given Rectangle, is readily obtained.

But

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But, as the operation, in a more complex Figure, may not be readily performed, without an Example, I shall briefly go through the whole process; and also, shew how much it is preferable to the 14th Prop. of the 2d Book of Euclid; viz. To find the Side of a Square, which is equal to any given right-lined Figure.



First, draw the Diagonals BH, BG, GC, &c.

Produce the Side GH, and, draw AI, parallel to the Diagonal BH; cutting GH produced, at I, and draw BI.

Now, since AI is par. to BH, the Tri. BIH, = BAH. 20. 1. wherefore, the Tri. BIG, is equal to the Trap. BAHG.

Next, produce the Side BC, both ways, indefinite, for the Base of the Triangle sought.

From the new-acquired Angle I, draw IK, par. to the Diagonal BG; cutting CB produced, at K, and draw GK. Then, the Tri. BKG = BIG, having the same Base BG. 18. 1.

But the Tri. BIG is equal to the Trap. BAHG; wh. the Tri. BKG, is equal to the Trap. BAHG.---Ax. 3. Conf. the Triangle BKO, which is added, is equal to the Trap. AHG|O, taken away from the original Figure; for, BOG is common to both.

Again,

Again; produce GF; and, from the Angle E, draw EL, parallel to the Diagonal FD; and draw DL.

The Triangle FLD is equal to FED. - - - 18. 1.

The given Figure is now reduced to a Pentagon, KGLDC.

Then; produce the Side CD; and, from the Angle L, draw LM, parallel to the Diagonal GD; cutting CD produced, in M, and draw GM.

The Triangle GMD is equal to GLD. - - - 18. 1.

Lastly; from the Angle M, and par. to the Diag. GC, draw MN, cutting KC produced, in N, and draw GN. The Triangle GNC is equal GMC; equal to the Trapezium GLDC; equal GFEDC.

But, the Triangle BKG is equal the Trap. BAHG; wherefore, BKG, added to BGC, added to GNC, is equal to the Triangle KGN. - - - Ax. 2.
Conf. the Triangle KGN, is equal to the Trap. KGMC; which is equal to the Pentagon KGLDC;
equal to the Hexagon KGFEDC;
equal to the Heptagon BIGFEDC;
equal to the Octagon AHGFEDCB.
Therefore, the Triangle KGN, is equal to the given Figure.

Having thus reduced the given Figure to a Triangle, it is easily formed into a Rectangle, by Prob. 20.;

For, a Rectangle on KR, half the Base KN, and GP, the perpendicular Altitude of the Triangle, is equal to the Triangle. - - - Cor. 17. 1.

Or, by Prob. 23. it is easily reduced to a Parallelogram under any given Angle and Side.

SCHOL. Let any one compare this, with the 14th Proposition in the 2d Book of Euclid; that is, let him go through the operation, both ways, in a Figure of as many Sides; I am confident to which he will give the preference. To say nothing of the inaccuracy of the other, this may be done in less than a fourth part of the Time. Each Triangle (of which there are six, in this Figure) goes through two operations, viz. 20th and 23d of this; and are added, separately into one Sum, or Rectangle. Whereas, by this method, the Figure is reduced one Side at every operation; from a Hexagon to a Pentagon, from a Pentagon to a Trapezium, from a Trapezium to a Triangle; each, being equal to the original Figure: and, from a Triangle to a Rectangle, or other Parallelogram under any Side or Angle; by the 23d.

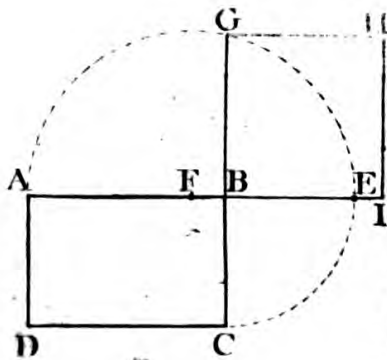
APPL. This Problem, may be applicable in several Cases, in surveying, &c. with the greatest accuracy. e. g.

If you would exchange an irregular piece of Ground in one Place, for the same Quantity in another, suppose to build on; which, by reason of the contiguous Buildings, is confined to a certain Angle, which is the given Angle; the length in Front may be considered as the given Side; the Question is, what Depth of Ground from the Front is required, to be equal in its Area to the other.

Having first reduced the original given Figure to a Triangle, by this Problem, it is then convertible into a Parallelogram under any Side and Angle, by the 23d; or, into a Square by the following.

P R O B L E M XXV.

To make a Square, equal to a Rectangle. ABCD.



Produce any Side, as AB, of the given Rectangle, until it be equal to the adjoining Side, BC; i. e. make AE equal to AB and BC, in a Right Line.

Bisect AE, in F; on which, with the Radius AF, equal FE, describe a Semicircle.

Produce CB, till it cuts the Arch, at G;

or, at the point B, draw BG perpen. to AE.

Describe the Square BGHI on the Line BG.

DEM. It

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DEM. It will be equal to the Rectangle ABCD. - - 14. 3.

Also, BG is a mean Proportional between AB and BE.

Conf. the Rectangle under AB and BE (i. e. DB) is equal to the square of BG (i. e. BH) by—Cor. to 9. 6.

N. B. In the performance of this Problem it is not necessary to construct the Rectangle; but only, to draw a Right Line (AE) in which, take AB and BE equal to the measures of the two Sides, and proceed as above.

PROBLEM XXVI.

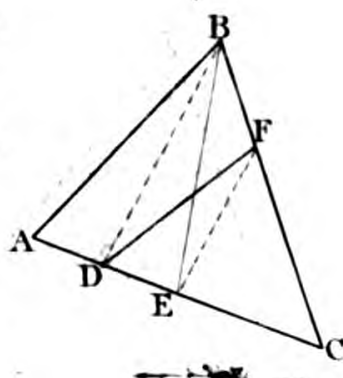
To divide a Triangle into two equal parts, by a Right Line drawn from any Point given, in a Side.

Let D be the given Point, in the Side AC, of the Triangle ABC.

Bisect the Side AC, in E, and draw EB, to the Vertex B.

Draw DB, and EF parallel to it, cutting BC in F, and draw DF.

The Trapezium ABFD, is equal to the Triangle DFC.



DEM. Because $AE = EC$, the Tri. $ABE = EBC$.— 20. 1.

But, EF is par. to DB; wh. the Tri. $BFD = DBE$ —same.

conf. $BFD + ABD$ is equal to $DBE + ABD$. - - Ax. 6.

And, the Triangle ABE is equal to EBC. - - - above.

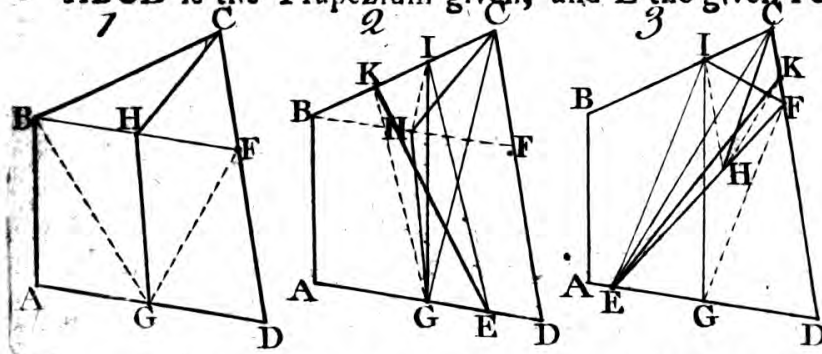
wh. the Trap. ABFD, is equal to the Tri. ABE, eq. EBC; and consequently, to the Triangle DFC.

Therefore, the Right Line DF divides the Triangle ABC into two equal Parts. Q. E. F.

PROBLEM XXVII.

To divide a Trapezium into two equal Parts, by a Right Line, drawn from a Point given in any Side.

ABCD is the Trapezium given, and E the given Point.



Through the Angle B, draw BF, parallel to AD. — 5. Bisection AD and BF, in G and H; and draw GH and HC. Then the Pentagon ABCHG is equal to the Trap. AHCD.

DEM. For (having drawn BG and GF) the Triangle ABG is equal to GFD, the Triangle BGH is equal to HGF, and BCH is equal to HCF. — — — — P. 18. 1.

2dly. In Fig. 2. let ABCHG be equal GHCD, as before. Draw GC, and HI parallel to GC, cutting BC in I, and draw GI.

The Right Line GI bisection the given Trapezium.

DEM. For, because HI is parallel to GC, the Triangle GIC is equal to GHC. — — — — — P. 18. 1.

Wherefore, GIC added to GCD = GHC + GCD. Ax. 6.

Again. Draw EI, and GK parallel to EI, cutting BC in K, and draw EK.

I say, that EK divides the Trapezium ABCD equally.

DEM.

DEM. For first, the Trap. GHCD was proved half ABCD;
secondly; GICD was proved equal to GHCD;
and, EKCD is equal to GICD.
For the Triangle EKI is equal to EGI. - - P. 18. 1.
Therefore, the Trap. ABKE is equal to EKCD.

Now, if E had been given near the Angle A, at the lesser end of the Trapezium, the operation would require more labour.

Let Fig. 3. be supposed the same Figure divided, by the Line GI, into two equal parts, from the middle Point G (as by the second operation) which is necessary to be first done, in all cases.

Join the given Point, E, and I, as before; draw GF, parallel to EI, cutting CD in F, and draw EF and IF.
Then, the Pent. ABIFE is equal to the Tri. EFD + FIC.

For, because GF is parallel to EI, the Tri. IFE = EGI.

But, although the Pentagon ABIFE is equal to half the Trap. ABCD, it is not equally divided by one Right Line. Therefore, draw IH parallel to CD, cutting EF in H, and draw HC.

Then, because IH is parallel to CF, the Triangle ICH is equal to IFH. - - - - - 18. 1.
Wh. the Pent. ABCHE = ABIFE = the Trap. EHCD.

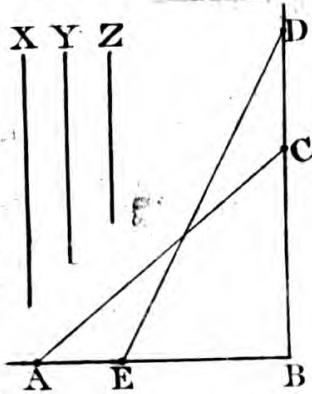
Lastly; join EC, and draw HK parallel to EC, cutting CF in K; and draw EK; which Line divides the Trapezium ABCD into two equal parts, as required.

For, since HK is parallel to EC the Tri. EKC = EHC; wherefore, EKC added to ECBA = EHC added to ECBA. But, ABCHE was equal to ABIFE, equal to ABIG; which, was equal to half the Trapezium ABCD.

PROBLEM XXVIII.

To find the Side of a Square, equal to any Number of Squares.

Let X, Y and Z be the Sides of three given Squares.



It is required to find the proportion of a Line, on which if a Square be constructed, it shall be equal in Area to all the three.

Make a Right Angle ABD. - - Pr. 10.
From the Angle B, take BA and BC, equal to any two of the given Lines, respectively, as X and Y, and join AC.

The Square of AC is equal to the two Squares of AB and BC, or X and Y. - - - P. 20. 1.

Again. Make BD equal AC, and BE equal Z;
Draw ED, which is the Line required.

DEM. For, ED square = the two Squares of EB & BD.
But, BD, equal AC, square = AB + BC square.
Wherefore, ED square = AB + BC + EB square.
i. e. ED square = the three Squares, of X, Y and Z. -- 20. 1.

APPL. By this useful Problem, Quantities may be increased in any Proportion at pleasure.

Also, by means of this Problem, and Prop. 20. 1. Carpenters form a Right Angle, in framing Timber, &c.

For, having made AB equal 3 feet, and BC equal four; then, if AC measures 5 feet ABC is a Right Angle.

N. B. The Numbers 3, 4 and 5 being multiplied, separately, by any one Number, at pleasure, produce the same effect. e. g.

If AB be made 6 or 9 feet, and BC equal 8 or 12 feet, then will AC be equal to 10 or 15 feet, if the Angle ABC be a right one.

For, the Square of 9 is 81; and the Square of 12 is 144, added to 81 is 225; equal to the Square of 15. --- 20. 1. El.

When

PRACTICAL GEOMETRY. 61

When the Timbers are long, it is necessary, in placing them at right angles, to apply greater measures; which, will give the Right Angle with greater accuracy.

COR. Hence, a Perpendicular may be drawn, very readily, at the extremity of a Right Line; by a Scale of equal Parts. As BC , perpendicular to AB . e. g.

Take, by any Scale, 3, 6, 9, or 12 equal Parts, which, set off, from B to A ; at which point, B , a Perpendicular to AB is required.

Then take, for Radius, 4, 8, 12, or 16 Divisions, of the same Scale; and, setting one Point of the Compasses in B , make a small Ark, at C .

By the same Scale, take 5, 10, 15, or 20 equal Parts, and with one point of the Compasses, at A , cross the former Ark, at C , and draw BC ; which will be perpendicular to AB .

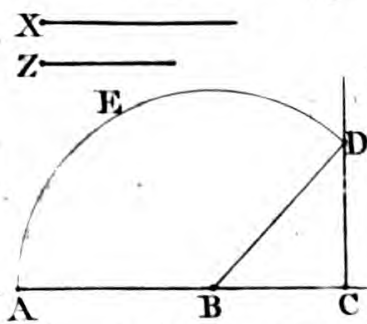
For, the Square of AC is equal to the two Squares of AB and BC , added together; by 20. 1. *El.*

P R O B L E M XXIX.

The Sides of two Squares being given, to find the Side of a Square which is equal to the difference between them; i. e. by how much the greater exceeds the less.

X and Z are the Sides of the given Squares.

Draw AC indefinite; in which, take AB equal X , and BC equal Z . - - - Pr. 3. Make ACD a R. Angle, and draw CD , indef. On B , with the Radius AB , describe the Ark AED , cutting CD at D ; CD is the Side of a Square, equal to the difference between the Squares of AB and BC , or X & Z . Draw BD .



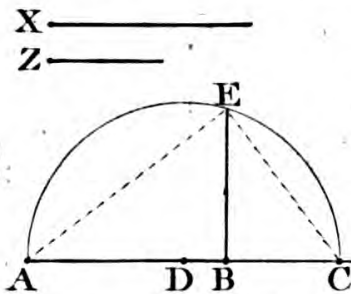
DEM. BD (equal AB , equal X) is the Hypotenuse of the right-angled Triangle DCB ; the square of which, is equal to the square of BC added to the square of CD . Consequently, BD square (equal X) exceeds BC square (equal Z) by the square of CD ; by Prop. 20. 1.

P R O.

PROBLEM XXX. 13. VI. Euclid.

To find a mean Proportional between two given Lines.

X and Z are the two given Lines.



It is required to find a Line, to which, either of the Lines, X or Z, shall have the same Ratio or Proportion, as that Line has to the other. Or, the Square of which, shall be equal to a Rectangle under the two given Lines.

Draw at pleasure A'C; make AB equal to one of the given Lines, as X, and make BC equal to the other, Z.

Bisect A C, in the point D; on which Center, and with the Radius A D, equal D C, describe a Semicircle.

At the Point B, draw a Perpendicular, to A C, cutting the Ark at E; and B E is the Line sought.

DEM. For, draw A E and E C; A E C is a Right Angle--12.1.

And, the Perpendicular, B E, in a right-angled Triangle, A E C, is a mean Proportional, between the Segments of the Base, A B, B C, made by the Perpendicular. C.1.7.6.El.

Consequently, A B is to B E, as B E is to B C. — same.

Therefore, $AB \times BC = \text{the square of } BE$.—Cor. to 9. 6.

SCHOL. This is the very same, in the operation, as the 25th; for the Side of a Square is a mean Proportional between the two Sides of a Rectangle having an equal Area.

P R O-

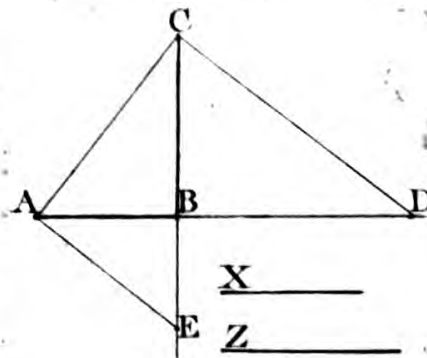
PROBLEM XXXI. 11. VI. Euclid.

To find a third Proportional to two given Lines.

X and Z are the given Lines.

It is required to find a third Line, to which the greater, Z, shall have the same proportion, as the lesser, X, has to Z. Or, which shall have that proportion to the least, as the least to the greatest.

Make a Right-angled Triangle, ABC, whose Catheti or Legs, are equal to the two given Lines, i. e. make AB equal to one (X) and BC equal to the other. --- 13 Produce AB and CB, indefinite, Make ACD and CAE Right Angles; i. e. draw CD and AE perpendicular to AC, cutting AB and CB, produced, in D and E.



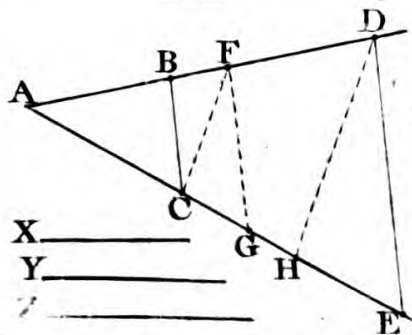
Then will BD be a greater, and BE a less, third Proportional, to the two given Lines, X and Z; Q. E. F.

DEM. For, as in the last, EB is to AB, as AB to BC; and $AB : BC :: BC : BD$. - - - P. 7. 6. Wherefore, EB, AB, BC, and BD are in continual Proportion. - - - Def. 7. 5.

Consequently EB is a less, and BD a greater third Proportional, to the two Lines AB and BC, equal X and Z.

P R O B L E M XXXII. 12. VI. Euclid;

To find a fourth Proportional, to three given, unequal Lines; X, Y and Z.



It is required to find a Line, to which, the third (Z) shall have the same proportion, as the first (X) has to the second (Y). Or, the same proportion to the first Line as the second has to the third.

Draw two Lines, AD & AE, making any Angle at pleasure. Make AB equal X, & AC equal Y; also make BD equal Z, Draw BC, and DE parallel to it; cutting AE in E; and, CE is a fourth Proportional, greater than Z.

i. e. As AB is to AC, so is BD to CE.

Q. E. F.

DEM. For, because BC is parallel to DE, the Sides of the Triangle ADE are cut proportionally; wherefore, $AB :: AC$, as $BD :: CE$. - - P. 2. 6. But, AB is equal to X, AC is equal Y, and BD equal Z; therefore, as X is to Y, so is Z to CE.

If a less Proportional be required; make AG equal to Z, make AF equal Y, and GE equal to X.

Join FG, and draw DE parallel to it.

Then, will FD be the Proportional sought.

DEM. For, as AG (equal Z) is to AF (equal Y) so is GE (equal X) to FD. - - - by the same.

After

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After the same manner, a third Proportional may be found.

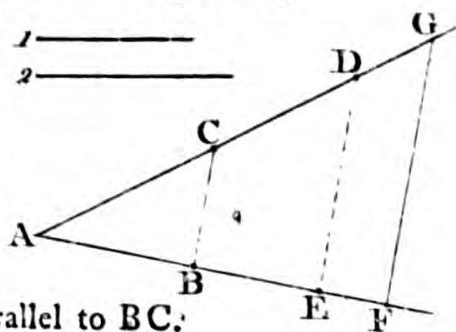
Draw AF and AG, making any Angle, as before.

Let AB & AC be made equal to the given Lines, respectively

If the greater Proportion be required, make BF equal to AC, and draw FG, parallel to BC.

The Segment CG is the Proportional sought.

But if a less Proportional is wanted; make CD equal AB, and draw DE parallel to BC: Then, BE is the Proportional required.



DEM. For, as AB is to AC, so is BF to CG;

and, as AC is to AB, so is CD to BE. - - 2. 6. El.

But, BF was made equal to AC, and CD to AB.

Conf. as $AB : AC :: AC : CG$; and, $AC : CD :: CD : BE$.

N. B. A fourth Proportional to three given Right Lines may be required and found in various orders of the given Lines.

Let X, Y, and Z be three given Lines; of which, let X be the least, and Z the greatest of the three,

By the first, it is, as X is to Y, so is Z to a fourth;

and by the second, as Z is to Y, so is X to a fourth.

But it may be, as Y is to either X or Z, so is the other to a fourth.

Also, as X is to Z, or Z to X, so is Y to a fourth.

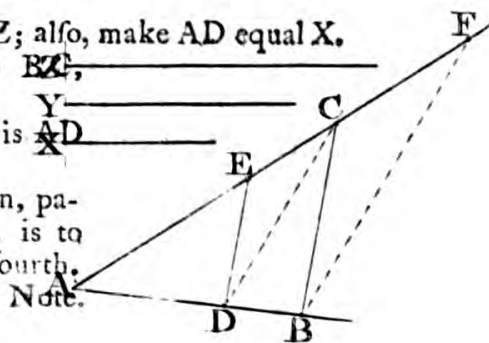
It may also be observed, that it is not necessary to draw the Lines, making an Angle, for the operation, longer than the greatest given Line; except when a Proportional is required greater; for the measures may all be set off from the Vertex. e. g. Let AB and AC make any Angle (BAC) at discretion.

Make AB equal to Y, and AC equal Z; also, make AD equal X. Join BC; and draw DE, parallel to BC, cutting AC in E.

Then, as AB or Y, is to AC or Z; so is AD or X, to AE, a fourth.

But, if DC be joined, and BF drawn, parallel to DC; it is then, as AD or X, is to AC or Z, so is AB or Y, to AF, a fourth.

Note.



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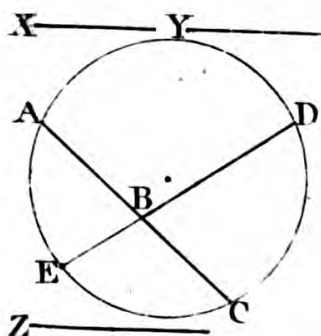
Note. This method, of setting off all the measures from the Angle, is the most eligible, when we are clear in the manner of placing them; being less liable to error than when they are set forward on a longer Line; on account of the Parallels being nearer together. The Demonstration is in the 4th of 6th El.

A third Proportional may also be found after the same manner. For, if AD be the first; and, if AB and AC be each made equal to the second; DC being joined, and BF drawn parallel to DC gives AF, a greater third Proportional; or, if AF be the first, AD is a lesser third.

A third or a fourth Proportional may be very elegantly found after this manner.

Let X, Y and Z be three given Lines.

If you require a less Proportional, X is the first Term taken; if a greater be required, Z must be the first, contrary to the order, after the former method; nor can the Terms be taken alternately, as in the other.



Draw a Right Line, AC, at pleasure.

Make AB and BC equal, respectively, to the first and second Terms, X and Y.

Through the point B, draw, at pleasure, DE and make BD equal to the third Term (Z)

Describe a Circle through the three Points A, C, and D, cutting DE, at E.—Pr. 40.

Then, BE is a fourth Proportional; in the order Z to Y, as X to BE.

DEM. For, the Rectangles under the segments of Chord Lines, cutting each other, are equal, i. e. the Rectangle, under AB and BC, is equal to that under DB and BE. Conf. as DB : AB, or BC :: BC, or AB : BE.— 9. 6. El. Therefore, BE is a fourth Proportional.

If a third Proportional be required, AB and BC must be each equal to that Term, of two given Lines, which you require to be the middle Term of the three.

For, if three Quantities are Proportionals, the middle Term is a Mean between the other two.

In

In analogous or equal Proportion of four Quantities, since the first has the same proportion to the second, as the third has to the fourth; and consequently, the first is to the third as the second to fourth; a Rectangle under the two extreme Terms (in equal Ratios of Right Lines) is equal to a Rectangle under the two middle Terms (9. 6. El.); which, in Numbers, is easily proved.

Take any four proportional Numbers, either continual; e. g. as 3, 6, 12, and 24; or in equal Ratios, as 3, 5; 9, and 15.

Now, it is plain that the first Number, 3, has the same proportion to the second, 5, as 9, the third, has to the fourth, 15.

For, if 3 be multiplied three times, it is equal 9; and 5 multiplied three times, is equal 15; consequently, 3 has the same proportion to 9, the third Number, as 5, the second, has to 15, the fourth; and so it will ever be, when any two Numbers are multiplied equally.

As 4 is to 6, so is 10 to 15; consequently, $4 : 6 :: 10 : 15$.

Now, in both these Cases, the two extreme Terms, i. e. the first and the last, viz. 4 and 15, and the two middle Terms, 6 and 10, remain the same; only, the middle Terms have changed places; but, the second multiplied by the third or the third by the second is the same thing; and is always equal to the fourth multiplied by the first, or the first by the fourth.

Wherefore, if four Lines are proportional, as above; a Rectangle under the first and the fourth, the two Extremes, is equal to a Rectangle under the two mean or middle Terms; that is, the Rectangles have equal Areas, seeing, the Area, of a Rectangle, is produced by the multiplication of one Side by the other.

Hence, a fourth Proportional may very readily and accurately be found; as follows.

Let X, Y and Z be three given Lines.

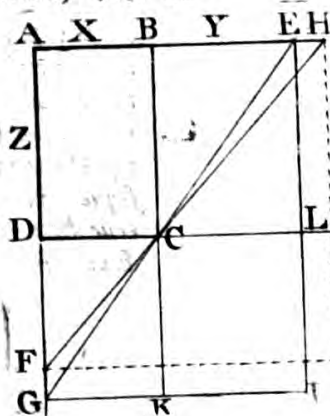
It is required to find a fourth Proportional, which shall have the same Ratio or Proportion to Z, as X has to Y.

In this Case, X and Z will be the two Extremes; for since X is less than Y; consequently, the Proportional required, will be less than Z, and is, properly, a Mean.

Make a Rectangle, ABCD, under the two given Lines X, and Z.

Produce any two Sides, from the same Angle, as AB and AD; on either of which, make BE or DF, equal Y.

Draw EG or FH, through the Angle C, cutting the other Side, produced, in G or H; then is DG, or BH, the Proportional sought.



K 2

For,

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For, complet the Rectangle AEIG.

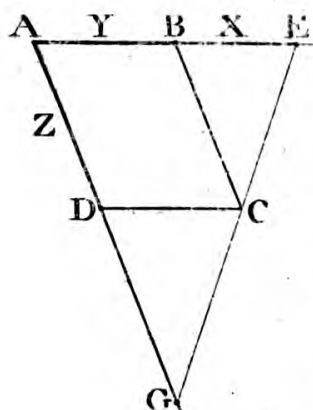
Produce BC to K, and DC to L.

DEM. The Rectangle CI, is under BE (equal Y) and DG, the fourth Proportional required.

AC and CI are Complements of the Par. AEIG. Def. 38.

But, the Comp. in every Parallelogram are equal. - 19.1.

Th. the Par. or Rect. CI is equal to the Rectangle ABCD.



But, if the Proportional was required to be to Z, as Y to X; then the Rectangle, or any Parallelogram, ABCD, must be under the two Lines Y and Z; which will, in this Case, be the two Means; and DG the Proportional sought, is now one of the Extremes; being the greatest of the four; which, in the former Case, was one of the middle Terms.

Thus may a fourth Proportional be found, either greater or less than either of the three given Lines, X and Z.

For, if a less Proportional was required, which should be an Extreme of the four; the Rectangle or Parallelogram, must be made under X and Y.

SCHOL. When three Lines are given; a fourth Proportional is generally understood, to be either greater than the greatest, or less than the least of the three; but if another mean Proportional is required, to three Lines given, as it must be between the two Extremes of the three; so it will be, either greater or less than the middle Line, as that is either greater or less than a true Mean, between the other two. And if it be already a true Mean, there can no other be found, on either Side; for, the Square of a mean Proportional, being equal to the Rectangle under the two Extremes (C. 2. 6.) consequently, no other Rectangle but a Square, which shall have the true Mean for its Side, can be equal to a Rectangle under the two Extremes.



PROBLEM XXXIII.

To find two mean Proportionals, to two given Lines.

Let X and Z be the two given Lines.

It is required, to find two other Lines; which not only contain an equal Rectangle, but, are in continual Proportion.

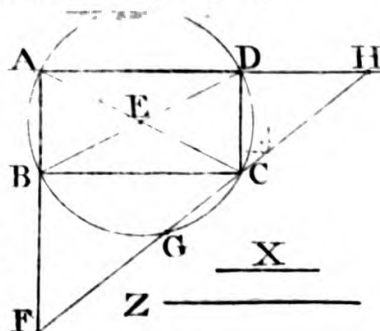
Construct the Rectangle $ABCD$, on the two given Lines; and on its Center, E , describe a Circle, circumscribing it.

Produce the two Sides AB & AD , indef.

Apply a Ruler to the Angle C , and move it on that Point, till it makes FC & GH equal;

and draw FH through G and C .

Then will BF and DH be the two Proportionals sought.



For, DC , DH , BF , and BC are in geometrical Progreſſion. And, a Rectangle under the two Means, DH and BF , is equal to the Rect. $ABCD$, under the two given Lines, X & Z .

Or, briefly thus.

Make a Right Angle FAH ; in which, from the Angle A , make AB & AD reſpectively equal to the given Lines, X & Z . Join BD ; which biſect in E ; and, on E , with the Radius EB , deſcribe a Semicircle, BCD .

Make DC equal AB (eq. X) and apply a ſtreight Ruler to the Point C , making EF equal to EH , and it is done. $Q.E.F.$

N. B. This method, though very ingenious, is not perfectly geometrical; ſeeing, the Points F and H cannot be aſcertained, but by trial; yet, it is the beſt I have met with; for thoſe methods which are performed by an Inſtrument, are not practical otherwiſe, and conſequently cannot be called geometrical.

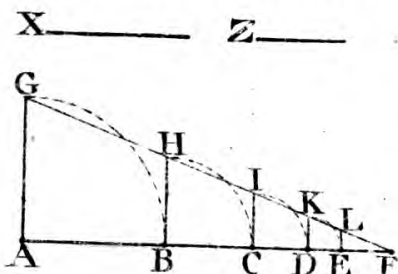
The method for finding a third Proportional (Pr. 31.) exhibits the reverſe of this; for AB and BC , the two given Lines, in that Problem, are two Means, between the two Extremes BE and BD . For, $BD : BC :: BC : BA :: BA : BE$. - - - P. 7. 6.

P R Q.

PROBLEM XXXIV.

To continue a progressive Proportion, between two given Lines, infinitely; and to represent the sum of them all.

X and Z are the two given Lines.



Draw an indefinite Right Line, AF; in which, take AB equal to X, and BC equal Z.

Make BAG a Right Angle ; and draw
BH parallel to AG.

Make AG equal AB, and BH eq. BC.

Through the Points G and H, draw a Right Line, cutting AC, produced, at F ; and AF is the whole Sum of the infinite Proportionals.

Draw CI par. to BH ; make CD equal CI , and draw DK .
Make DE equal DK , and draw EL , perp. to AH .

Then is CI a third, DK a fourth, and EL a fifth Proportional; and, after the same manner, it may be continued, *ad infinitum*.

For, since AG is less than AF ; so BH is less than BF ; and consequently, CI , DK and EL , will still have the same Ratio to CF , DF and EF ; wherefore, the last may always be taken from the remainder, and therefore, AF , is equal to the whole sum of the infinite Proportionals; and AB , BC , CD , &c. or, AG , BH , CI , &c. are in a progressive, geometrical Proportion.

The Demonstration of this Problem and the last are given at the end of the sixth Book.

P R O.

PROBLEM XXXV. 11. II. Euclid.

To divide a Line in extreme and mean Proportion.

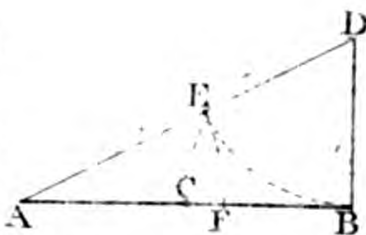
AB is the given Line.

It is required to cut it so, that the lesser Segment shall have that Proportion to the greater, as the greater Segment has to the whole Line.

Or, the Rectangle under the whole Line and the least Segment shall be equal to the Square of the greater.

Bisect AB, in C; draw BD perpendicular to AB and equal to BC, half the given Line, and draw AD.

On D, with the Radius DB, describe an Ark BE, i. e. make DE equal DB; and, on the point A, describe the Ark EF, cutting AB in F, the Point sought, Q. E. F.



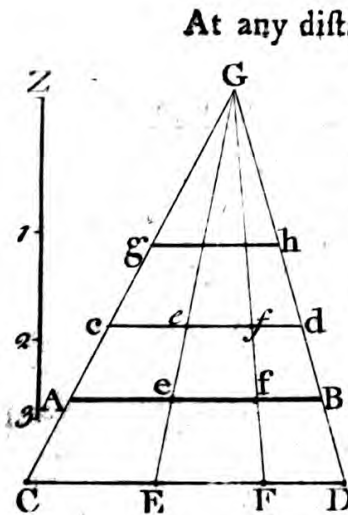
N. B. AF, the greater Segment, is the difference between the least Side, DB, and the Hypothenufe, AE, of the Right-angled Triangle ABD; constructed on the whole given Line, and half the Line, for the Base and Perpendicular. - - - Pr. 13.

This Problem is otherwise performed and demonstrated in Prop. 11. of the second Book of Elements.

PROBLEM XXXVI. 10. VI. Euclid.

To divide a given Line, in any known Proportion.

AB is the given Line; and Z, is a Line divided in the Proportion required, at 1, 2, and 3.



At any distance from AB, at discretion, draw CD parallel to AB; and make CE, EF, and FD equal, respectively, to the Divisions on Z.

Through the extreme Points of the two Lines, AB and CD, draw CA and DB, produced till they intersect, at G.

Draw EG and FG, cutting AB in e and f.

Then, is the given Line, AB, divided (in e and f) in the same Ratio as CD (in E and F) or as the given Line, Z, is divided, in the Points 1, 2, 3. - - - P. 2. 6. El.

If the Measure given had been less than the Line given to be divided (as in this Example it is greater) AB would be divided the same.

Let cd, less than AB, be divided in the known Ratio, e & f.

Draw, as before, the Lines Ac, and Bd, through their Extremes, meeting at G; and, from G, through the Divisions e and f, draw the Lines Ge, Gf, cutting the given Line, AB, in the same Points e and f. Q. E. F.

The difference, it is evident, is only in the operation, for the effect is the same.

In the first Case, when the Measure is greater than the Line to be divided; then, the Vertex G will fall on the opposite Side, from CD; but, when it is less, as cd, the Vertex, G, will fall on the same Side with cd; as is obvious from the Figure.

N. B. Any

N. B. Any other Line drawn between the two Lines GC and GD, as gh, parallel to AB or CD, will be divided in the same Ratio, by the Lines GE and GF.

2. The same thing may be very readily and accurately done after this manner.

AB is the given Line, to be divided; and HI is a Line divided in the given or known Ratio, in 1 and 2.

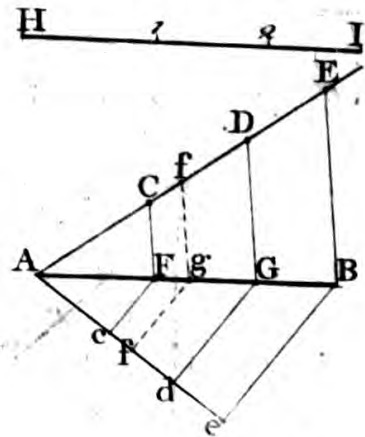
At either extreme of the given Line, draw AE, in any Angle at pleasure.

An acute Angle, not too small, is best.

Make the divisions at C, D, and E, equal, respectively, to the divisions on HI; according as you require them on AB, begin at either end, H or I.

Join the Point E, and the other extreme, B, of AB; draw DG and CF, parallel to EB, cutting AB in G and F.

So shall AB be divided in the same Ratio, as HI, in F and G, - - - P. 2. 6.



If the Measure given had been less than AB, as Ae, divided in c and d, the Divisions, on AB, would be the same,

N. B. If it was required to divide AB into any number of equal Parts; make so many equal Divisions on CD, in the first, or on AE, in the last method, at pleasure; and proceed as directed.

AB may be readily bisected by either method; by the last, making Af, fE, or fe, two equal divisions, and drawing fg, as before, parallel to EB or eB.

The first method is demonstrable, from Prop. 6. 6. & Corol.

For the Triangles CGE and AGE, also EGF and eGf, &c, are similar. Conf. Ae : ef :: CE : EF, and as fB to FD.

The last method is easily deduced from Prop. 32, and the Demonstration of it, from Prop. 2. 6. El.

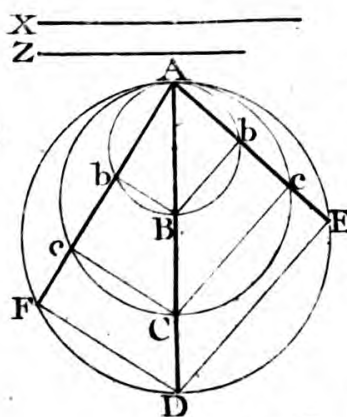
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APPL. It is almost needless to give an Application of this Problem, as it speaks its use sufficiently. It is extremely useful in the practice of Perspective, as well as in all geometrical Drawings; as Plans, Elevations, &c.

AB is considered as a finite Line, already drawn, in some Plan, &c. and the Ratio or Proportion given, is certain known measures, or divisions, to be represented on AB, from either a greater or less Scale of Proportion.

3. There is another method, by which a Line may be divided from a greater Measure given; which, not so much for its utility as the singular elegance of it, I shall give, as follows.

Let AD be divided in the given Ratio, at B and C; and let X or Z be a Line given, to be divided, in equal Ratio.



Describe three Circles, on the three Diameters AB, AC, and AD.

Take the Line X or Z in your Compasses; and, setting one Point in A, cut the greatest Circumference at E or F, with the other Point, and join AE or AF; which, will be divided in the same Ratio as AD, in b & c.

Draw Bb and Cc.

DEM. The Angles A b B, A c C, also A E D or A F D, are Right. - - - - - P. 12. 3. El. wherefore, A b B, A c C, &c. are similar Triangles, the Angle at A being common to them all. - - C. 3. 2. 6. Therefore, AE or AF is divided in the same Ratio as AD, the Lines B b, C c, and ED, or FD, being parallel.—2. 6.

4. A Right Line may be accurately divided into any number of equal Parts, by the following Method.

Let AB be the given Line, to be divided into five equal Parts.

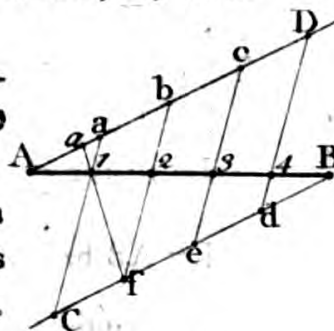
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From either extreme, as A, draw AD, making any Angle, BAD, at pleasure; and, at the other extreme, B, make the Angle ABC equal BAD.

On AD and BC take, at pleasure, four equal Divisions, from A and B, at a, b, c, D and, d, e, f, C.

Join, a C, b f, &c. as in the Figure, which will divide AB into five equal Parts, as required; at 1, 2, 3, and 4. Q. E. F.



This is very evident from the second method; for, the Divisions on AD and BC being equal, they are consequently equal on AB; the Lines AC, bf, &c. being parallel (C. 15. 1.) for, AD is parallel to BC; by P. 4. 1.

N. B. If the Divisions on AD and BC were either greater or less, AB would be divided the same; which is obvious by taking Aa 3 fourths of Aa, and joining af; for, Bf is also 3 fourths of BC.

I have never seen this method used for dividing a Right Line in any given Ratio; which may be applied with success.

Let AB be a Line given to be divided; and Z, a Measure known, i.e. a Right Line divided in the Ratio required, in a & b.

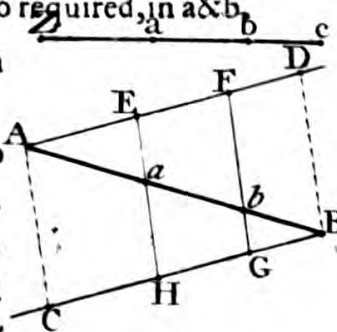
Make AD and BC in equal Angles with AB, as before.

Transfer the measures Za, ab, and bc, to AD and BC, in the order required; viz.

make AE equal Za, and EF equal ab.

also, make BG equal bc, and GH equal ab, &c. and join the Points EH, FG;

which will divide AB, in the Ratio of Z, in the Points a & b.



Note. The two extreme Divisions FD and CH are of no use in the operation; but AC and DB, being joined, will be parallel to EH & FG; and exhibits the former Method, applied on both Sides.

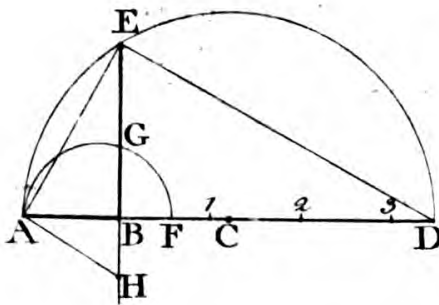
The only difference is, that, in this Method, there is no regard had to the parallelism of the Lines EH and FG, but only to join the Points, and they are necessarily parallel.

P R O B L E M XXXVII.

To find the Side of a Square, or any other Right-lined Figure (similar to a given one) which shall be to the given Figure in any proportion required.

Let AB be the given Line, or Side of the given Figure.

It is required to find the length of a Line, for a corresponding Side; on which, if a similar Figure be constructed, its Area shall be to the given one in the Ratio of $3\frac{1}{2}$ to 1.



Produce the given Line, AB, indef.
Make BD equal to $3\frac{1}{2}$ times AB.
Bisect AD; and, on the point of bisection, C, describe a Semicircle, or the Ark AE only.
At the Point B, draw BE perpendicular to AB, cutting the Ark at E;
BE is the Line sought. Q. E. F.

DEM. For, BE is a mean Proportional between AB & BD.
Wherefore, the square of AB, to the square of BE, is duplicate of AB to BE. - - - P. 10. & 12. 6.
i. e. their Ratio or Proportion, to each other, is as AB to BD; and all similar Figures are in the same Ratio, as the Squares of their corresponding Sides. - - C. 2. 13. 6.

By this Problem, any right-lined Figure, whatever, may be increased or diminished in any Proportion. e. g.

If you would decrease it a fifth, a fourth, a third, or a half, &c. make BF to AB in that Ratio; bisect AF, and describe the Semicircle AGF, cutting the Perpendicular BE (at the Point B) in G; and BG is the Line sought.

Hence the Ratio between any two Figures, may be known.

For,

For, having, by Prob. 24. and 25. reduced the given Figures to Squares; i. e. having found the side of a Square, equal to each Figure, respectively, make AB and BE respectively equal to them, and forming a Right Angle, ABE.

Produce AB or EB, indefinite; join AE, and make AED, or EAH, a Right Angle; i. e. draw ED, or AH, perpendicular to AE, cutting AB, or EB, produced, in D or H.

Then, as AB is to BD, or, as HB to BE, so is one Figure to the other;—by Cor. 1. P. 13. 6. El.

For, the Squares of AB and BE are, respectively, equal to those Figures.

N. B. If the Figures, whose Ratio is required, are similar, make AB and BE, respectively, equal to any two corresponding Sides, and proceed as above.

PROBLEM XXXVIII. 25. VI. Euclid.

To construct a Polygon similar to a given one, and equal to any given Right-lined Figure.

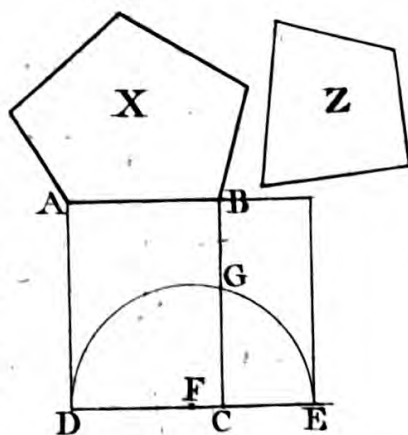
X and Z are the two Figures given.

It is required to make a Pentagon similar to X, and equal to the Trapezium Z.

On any Side of the Pentagon, as AB, make a Rectangle ABCD equal to the Pentagon; by Prob. 24, 20, and 23.

Then, on the Side BC, of the Rectangle BD, make another Rectangle, BE, equal to the Trapezium; by the same Problems. Bisect DE, in F; and, on that Center, describe a Semicircle, with the Radius FD, cutting BC, at G.

On CG, if a Pentagon be constructed, similar to X, making CG the corresponding Side to AB, it will be equal to the given Trapezium Z. Q. E. F.



DEM. For

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DEM. For, CG is a mean Proportional between DC & CE.

Wherefore, since the Ratio of any two similar Figures, is duplicate of their corresponding Sides (13. 6.) and the Ratio of DB to BE is as their Bases, DC to CE; - 1. 6.

Conf. the Ratio of the two Pentagons, will be as DC to CE; for, they are respectively equal to the Rectangles DB & BE.

Therefore, a Pentagon constructed on CG, similar to X, will be equal to the Trapezium Z. Q. E. D.

By the 24th any Figure is readily reduced to a Triangle, by the 20th a Rectangle is found equal to a Triangle; and by the 23d, another Rectangle may be found equal to that, having one Side equal to AB, or any given Line.

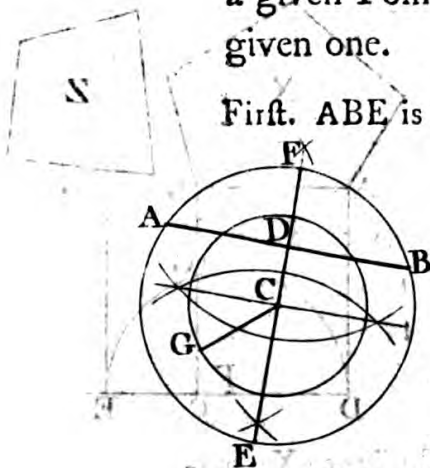
For, it will produce AD, for the other Side of the Rectangle AC, which is a fourth Proportional to AB and the two Sides of that Rectangle, found by the 20th, equal to the Pentagon; as by the last method of Prob. 32.

Conf. the Rectangle BD is equal to the Pentagon X, and BE to Z.

PROBLEM XXXIX. 1. III. Euclid.

To find the Center of a given Circle; and through a given Point, to describe a Circle parallel to the given one.

First. ABE is the Circle given; and G the given Point.



Draw a Chord Line, AB, at pleasure. Bisection AB (Pr. 8.) and through D, the point of bisection, draw EF perpendicular to AB (6.) cutting the Circumference in E & F. Bisection EF, in C, which is the Center of the Circle ABE. Q. E. F.

For EF is a Diameter, and consequently passes through the Center of the Circle. - - - P. 1. 3.

2nd. Join

2nd. Join CG. With the Radius CG, and on the Center C, describe a Circle GH, and you have done. Q. E. F.

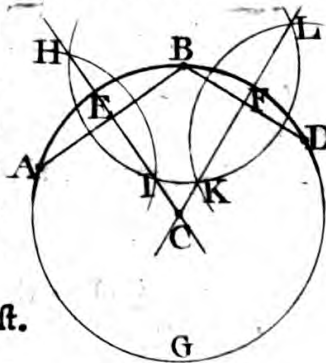
SCHOL. All parallel Circles, in the same Plane, have the same Center, and are called concentric; but, if they are in parallel Planes, as on a Cylinder, Cone or Sphere; a Right Line perpendicular to those Planes, if it passes through the Center of one Circle, it will pass through the Centers of them all; which line is the Axis of the Cylinder, Cone or Sphere.

PROBLEM XL. 25. III. Euclid.

To perfect or compleat a Circle, from a given Ark or Segment of that Circle.

ABD is the Ark given.

Draw, at pleasure, two Chord Lines; AB and BD. Bisect the two Chords, at E and F; from which, draw the Perpendiculars EC and FC, intersecting at C. - - Pr. 8. & 6. Then will C be the Center of the Circle; on which, with the Radius CA, CB, or CD, the Circle ABDG may be compleated



This Problem is demonstrated in the last.

N. B. By this method, may be found the Center of a perfect Circle, as readily as in the foregoing; nor is it necessary to draw the Chords; only, assuming, at pleasure, three points, A, B, & D, and proceed after the manner following.

With any Radius, at discretion, on B, as a Center, describe the Ark HIKL; and, with the same Radius, on A and D, describe two Arks, HI and KL, cutting the former in H, I, K and L; through which Points, draw Right Lines, HI and LK, intersecting at C, the Center sought.

COR.

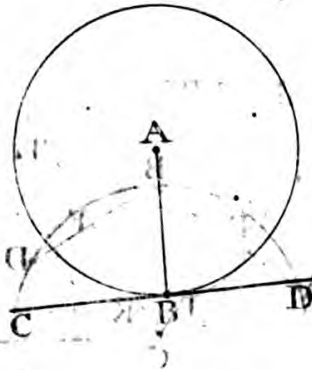
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COR. Hence, through three given Points (not lying in a Right Line) as A, B, and D, the Circumference of a Circle may be described. whose Center (being found, as above) is C.

P R O B L E M XLI.

To draw a Tangent to a Circle, through a given Point in the Circumference. And, to find the Point of Contact of a Tangent to a Circle.

First; B is the Point given; through which a Tangent is required to be drawn.



Having found A, the center of the Circle (by the foregoing) join the point B, and the center of the Circle by a Right Line, AB.

At the Point B, make a Right Angle, ABC, and produce CB, towards D.

The Right Line CD will touch the Circle, in the Point B. - - P. 8. 3. El.

2nd. CD is the Tangent given; to find the Point of Contact.

Draw a Perpendicular AB to the Tangent, from the center of the Circle; cutting CD in B; - - - Pr. 7.

B is the Point of Contact, in which the Tangent, AD, touches the Circle. - - - C. 3. 8. 3.

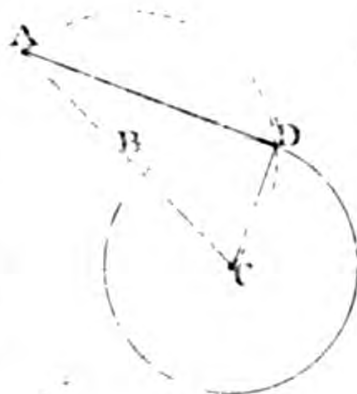
P R O.

PROBLEM XLII. 17. III. Euclid.

To draw a Tangent to a given Circle, from a Point given without the Circle; i. e. to determine the Point, in which a Right Line drawn from the given Point shall touch the Circle.

From the given Point, A, draw AC; to the Center of the Circle.

Bisect AC; and, on the point of bisection, B, with the Radius AB, describe a Semicircle; cutting the circumference of the given Circle in D, the Point sought. Q.E.F.



DEM. Having drawn AD and CD, the Right Line AD will touch the Circle at D.

For, the Angle ADC is a Right one. - - P. 12. 3.

Wherefore, AD touches the Circle in D. - - C. 2. 8. 3.

Otherwise.

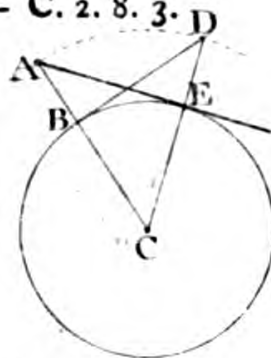
Join the Point A, and the Center C, as before, cutting the Circumference in B.

With the Radius CA describe the Ark AD.

Draw the Perpend. BD, cutting AD at D.

Lastly, draw CD, cutting the given Circle at E, the Point sought. Draw AE.

The Right Line AE will touch the Circle, at E.



DEM. The Triangles AEC, BDC are congruous.

For, the sides AC, CE are equal to DC and CB, respectively; and the Angle C is common to both; therefore,

AE is equal to BD; the Angle at A equal D, and the Angle AEC equal DBC. - - P. 8. 1. El.

But CBD is a Right Angle (Con.) wh. AEC is a R. Angle.

Therefore, AE touches the Circle at E. - C. 2. 8. 3.

DEM. From any Point, as F, draw FA and FB.

The Ang. $AFB = BAD$, eq. to the given Ang. X.-13.3.

Or, without drawing AD, on the given Line AB, make an Isosceles Triangle, whose Angles at the Base, BAC, ABC, are each equal to the Complement of the given Angle, to a Right Angle.

The Vertex C will be the Center, and CA, or CB, the Radius of the Circle required.

P R O B L E M XLV.

The Angles being given, under which three Objects, situate in a Right Line, are seen, and their Distances from each other known; to determine the Point from which they are seen.

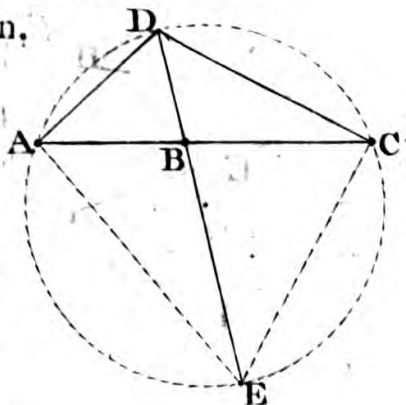
A, B, and C are the three Objects.

Make the Angles ACD, CAD, alternately, equal to the given Angles, i. e. make ACD equal to the Angle under which AB is seen, and CAD equal to the other Angle given.

Produce CD and AD, intersecting at D.

Describe a Circle through the two extreme Objects, A & C, and the Angle D; - - by Pr. 40.

Draw DB, and produce it, till it cuts the opposite Circumference, at E, the Point sought.



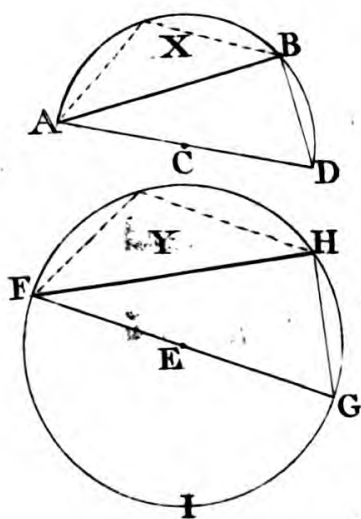
DEM. For (having drawn AE and CE) the Angle $AEB = ACD$, and the Angle $CEB = CAD$. - P. 10. 3.

P R O B L E M XLVI.

A Circle being given, to cut off a Segment, similar to a given Segment; and, on a given Line, to construct a Segment, similar to a given one.

First. It is required to cut off, from the given Circle FHI, a Segment similar to X.

Find the Center, C, of the given Segment, X, - Pr. 40. and complete the Semicircle ABD.



Through the Center, E, draw FG, and make the Angle GFH equal DAB; - Pr. 4. The Segment, Y, cut off by the Chord FH, will be similar to the given Segment.

DEM. For, the Angle GFH = BAD. - Con. And (having joined BD and HG) the Angle FHG = ABD - - - Ax. 9. (for they are Right Angles; 12. 3.) conf. FGH = ADB - - C. 5. 10. 1. wherefore, the Triangles ABD, FHG are similar. - - - C. 2. 4. 6.

Therefore, as FG : AD :: FH : AB. - 4. 6.

And, by taking away the Triangles ABD, FHG, from the Semicircles, there is left the Segment Y similar to X.

N. B. If the Segment given had been greater than a Semicircle, the operation is the same; the Triangle GFH being added to the Semicircle FIG, instead of taking it away from FHG.

2nd. FH is the given Line, on which to construct a Segment as required.

Make the Angle HFG equal to BAD; which, AB, the Chord of the given Segment, makes with the Diameter AD; found as above.

Draw HG perpendicular to FH, cutting FG in G.

Bisect FG, and on E, the Center, with the radius EF or EG, describe the Ark FCH, forming the Segment required.

APPL. This Problem is very useful to Builders, to form Arches, and scheme Heads for Windows, Doors, &c. similar to others, either greater or less, of any proportion required.

The following is an universal Rule, to find, arithmetically, the Diameter of a Circle, from any given Segment, by the measure of the Chord Line or Subtense; i. e. the Base of the segment, and the height of the Ark.

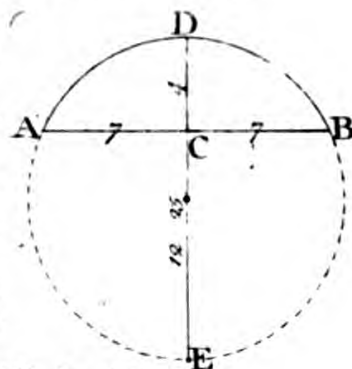
ADB is the Segment given.

Divide the Square of half the Base by the perpendicular height and add the Quotient to the Divisor. See Case 2. 14. 3.

e. g. Let the Base AB be 14 and the Perpendicular CD, 4.

The Square of AC or CB, 7 multiplied by 7 is equal 49; which, divided by 4, gives the Quotient 12, 25; equal CE; to which add the divisor, 4, the height of the Ark, CD, it gives 16, 25, for the Diameter.

For a Rectangle under CE and CD is equal to the Square of AC. - - - P. 14. 3.

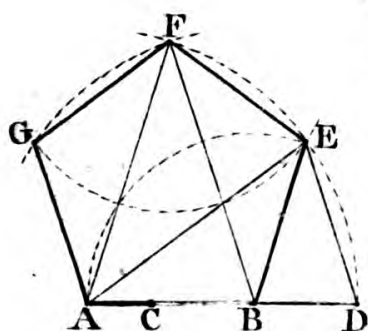


Supposing the Integers Feet, the Diameter, DE, is 16 Feet 3 In.

The Rule is the same, if the Segment be greater than a Semi-circle. But, having found the Center (by Pr. 39 or 40) a Right Line drawn through the Center, cutting the Ark in two Points, is a Diameter.

P R O B L E M XLVII.

To make a regular Pentagon, on a given Line, AB.



Divide AB in extreme and mean Proportion, in the Point C. - - Pr. 35. Produce AB; and make BD equal to BC, the greater Segment; then, AD is to AB, as AB to BC, and as BC to AC.*

On A, with the Radius AD, describe the Ark DEF; and, on B, describe the Ark GF, intersecting at F.

On F, with the Radius AB, describe the Ark EG; i. e. make FE and FG each equal AB, and join the Points A and G, B and E; also E, F and F, G, which compleats the Pentagon. Q. E. F.

The Demonstration of this construction of a Pentagon may be obtained, from the 7th and 8th Prop. of the 4th of Elements.

For, AED is an Isosceles Triangle, having its Angles at the Base, ED, each double the Angle EAD, at the Vertex; AFB is the same, which may be considered as inscribed.

And, by 34. of the 6th. the Diagonal of a Pentagon, BF or AE (equal AD) has that ratio to the Side, AB, BE, &c. as the greater Segment to the less, of a Line divided in extreme and mean Proportion.

Or it may be constructed thus.

Having found the Point D, as above, and drawn the Ark DEF, on the Center A.

On B, with the Radius AB, describe the Ark AE, cutting the other at E, and draw AE and DE.

Draw BF parallel, to DE; and also AG, indefinite.

Draw FG parallel to AE, cutting AG in G, and join FE and BE, which compleats the Figure.

* See N. B. 2. Prop. 35. 6th of Elements.

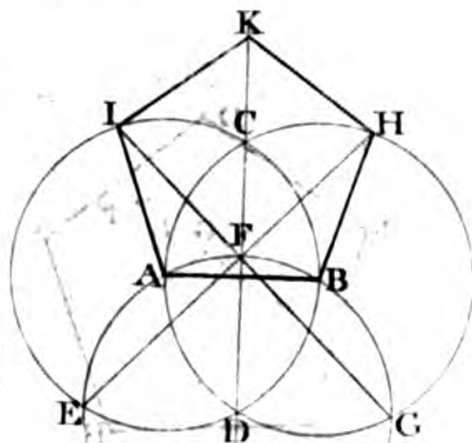
Otherwise,

Otherwise, mechanically; AB being the given Side.

On A and B, with the Radius AB, describe two Circles, cutting each other in C and D, and draw CD.

On D, with the same Radius, describe the Ark EABG, cutting the two Circles and the Right Line CD, in E, F, and G. Draw the Right Lines EF and GF, and produce them till they cut the Circumferences, in H and I.

Then, on H and I, with the Radius AB, describe two Arks, intersecting at K, and join the Points AI, IK, KH, and HB, which completes the Pentagon, AIKHB. Q. E. F.



Of this construction of a Pentagon, there is no Demonstration has yet been given.

I shall, here, give a general Method for constructing every kind of regular Polygon, from a Pentagon to a Duodecagon, by means of the following Table; in which, the Angle of the Polygon is determined by the proportion it has to a Right Angle, and the difference there is between them.

A Right Angle is to the Angle of a Polygon as follows,

	Ratio	Diff.		Ratio	Diff.
Pentagon	as 5 to 6	— 1	Nonagon	as 9 to 14	— 5
Hexagon	as 3 to 4	— 1	Decagon	as 5 to 8	— 3
Heptagon	as 7 to 10	— 3	Undecagon	as 11 to 18	— 7
Octagon	as 2 to 3	— 1	Duodecagon	as 3 to 5	— 2

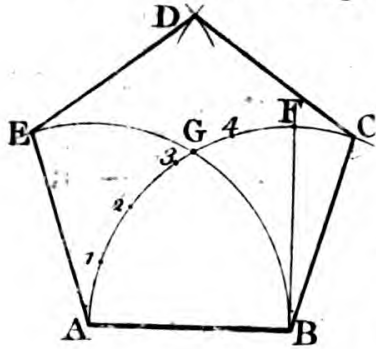
Note; the Hexagon, Octagon, &c. having an equal number of Sides, are reduced to the lowest Denomination; otherwise, the Right Angle is always supposed to be divided into the same number of Parts as the Polygon has Sides; which will then be, for a Hexagon, as 6 to 8, difference 2; in an Octagon, as 8 to 12, difference 4; and so of the others.

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To construct a Pentagon, by the Table, on the Line AB.

Make a Right Angle, ABF, on the extreme Point B.



By the Table, the Angle of a Pentagon is to a Right Angle in the ratio of 6 to 5, difference 1.

With the Radius AB (or any other) on B describe an Ark from A to C.

Divide the Ark (of the Right Angle) AGF into five equal Parts, as in the Figure; the Angle of the Pentagon is six of those Parts.

Add the difference, 1, from F to C, and draw BC.

The Angle ABC is the Angle of a Pentagon, containing 6 fifts of a Right Angle, on the Ark AGC.

On A, with the same Radius, AB, describe the Ark BGE, cutting the other at G; make GE equal GC, and draw AE.

With the same Radius, on E and C, describe two Arks, intersecting at D, and join CD and DE; which compleats the Pentagon.

DEM. The Sides AB, BC, &c. are equal, by Construction.

And the Angles are also equal, being subtended by equal Arks, AGC, BGE, &c. of equal Circles. - C. 2. 9 3.

Therefore, ABCDE is a regular Pentagon.

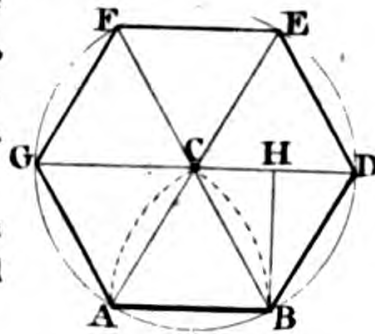
P R O B L E M XLVIII.

To make a Hexagon, on a given Line, AB.

In the construction of a Hexagon you need not regard the Table, nor proceed after that method; seeing that, the Side of a Hexagon is always equal to the Radius of a circumscribing Circle. - - - - - 11. 4. El.

Therefore, with the Radius AB, describe two Arks, on A and B, intersecting at C, and compleat the equilateral Triangle ACB. On the Center C, with the same Radius, describe a Circle.

Produce AC to E, and BC to F; and, through the Center C, draw GD parallel to AB, and join the Points A, G, F, E, D, and B.



Or, having found the Center C, and described a Circle, as above.

Apply the given Side, AB (equal to the Radius, AC) fix times round the Circumference, from B to D, E, F, G, and join the Points as before.

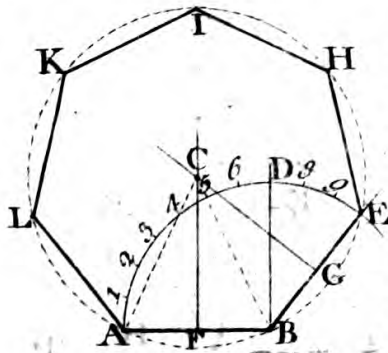
AGFEDB is a regular Hexagon. Q. E. F.

The circumference of a Circle contains the Radius exactly six times inscribed (Th. P. Ang. Art. 5.) consequently, it is equal to the Side of a Hexagon inscribed. - P. 11. 4.

This needs no other Demonstration; for, the Sides are all equal by Construction; and the Triangles ACB, BCD, DCE, &c. are Equilateral, whose Angles are also equal (C. 1. 9. 1.) Each Angle of the Hexagon, contains two Angles of a Triangle, ABD, equal ABC added to CBD, &c. which are therefore equal; and consequently have that proportion to a Right Angle as 4 to 3; as the Perpendicular BH, bisecting the Angle CBD, indicates.

P R O B L E M XLIX.

To make a Heptagon, on a given Line, AB.



Make a Right Angle ABD ; and divide the Ark of it into seven equal parts, as many as the Figure has Sides.

Observe, by the Table, that the angle of a Heptagon is ten of those Parts, the difference is three.

Set off three Parts from the Perpendicular BD to E, and draw BE. Make BE equal AB. Bisect the two Sides AB and BE; - Pr. 8.

and draw the Perpendiculars, FC and GC, intersecting at C. On C, the Center, and Radius CA, or CB, describe a Circle; which will contain the given Line, AB, seven times in its Circumference, at A, B, E, H, I, K, L; which Points, joined by Right Lines, will compleat the Heptagon required. Q E. F.

DEM. It is equilateral by Construction; and it is also equiangular, because inscribed in a Circle; for, equal Segments contain equal Angles. - - - P. 10. 3.

N. B. It is plain that the Angle ABE, which is an Angle of the Heptagon, is equal to the Right Angle ABD added to the Angle DBE; the Right Angle containing seven parts, the Acute Angle, DBE, three; and consequently, the Angle ABE contains ten; therefore, the Ratio is as 10 to 7; as by the Table.

After the same manner a Circle may be found, which shall contain a given Line any number of times, to twelve, applied to the Circumference, by the Table, given above.

P R O B L E M L.

To describe an Octagon, on a given Line, mechanically, without the Table.

On the extremes of the given Line, AB, draw the Perpendiculars AG and BH, indefinite.

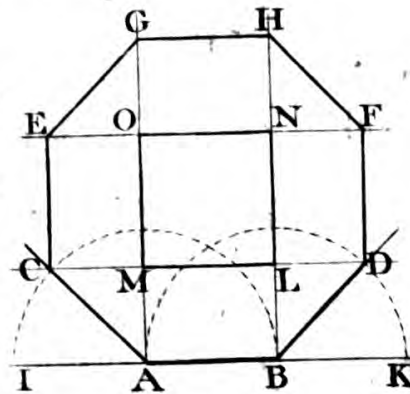
Produce AB, both ways, to I and K.

Bisect the external Angles, IAG & HBK, by the lines AC and BD; which, make equal to AB, and draw CD; which, will be parallel to AB.

Make LN equal LM; and, through N, draw EF parallel to CD.

Draw CE & DF, parallel to AG & BH.

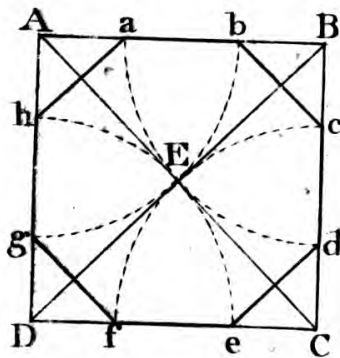
Make NH equal NF, and draw GH parallel to EF; and join EG and FH; which compleats the Octagon. Q. E. F.



N. B. It is obvious, in this Figure, that its Angles, as ABD, has the proportion to a Right Angle, ABL, of 3 to 2, as in the Table; for, the Angle LBD is half the Right Angle LBK; and LBD added to ABL, equal ABD, is the Angle of the Octagon.

PROBLEM LI.

To find the Side of an Octagon, in a given Square, ABCD.



Draw the two Diagonals AC and BD, which gives the Center E. - P. 16. 1. El. With the Radius of half the Diagonal, AE, on every Angle of the Square, describe an Ark or Quadrant, bEg, aEd, &c. i. e. make Ab, Ag, Ba, Bd, &c. equal half the Diagonal; and, joining the Points a h, bc, &c. you will have a regular Octagon abcd efgh. Q. E. F.

APPL. The uses of these Problems, to various Mechanics, are very obvious. By the 50th we learn how to form an Octagon Building; as a Temple, Library, &c. an Alcove, or Bow Window, &c. (which are frequently half a regular Octagon or Hexagon) of a given measure, for a Side of the Building, &c. By this Problem, we find the measure of a Side, when the width or Diameter is first determined.

To enumerate all the uses of Poligons, would be impertinent and foreign to the purpose; my design being to shew how to construct them, in the easiest and readiest manner, the application of them will readily occur, as occasions require.

These are the most necessary Poligons for mechanical uses; and since, by the Table, and Rules already given, any Polygon, to twelve Sides, may be readily constructed, I shall defer treating more fully on them to the fourth Book; which teaches how to inscribe and circumscribe all kinds of regular Figures.

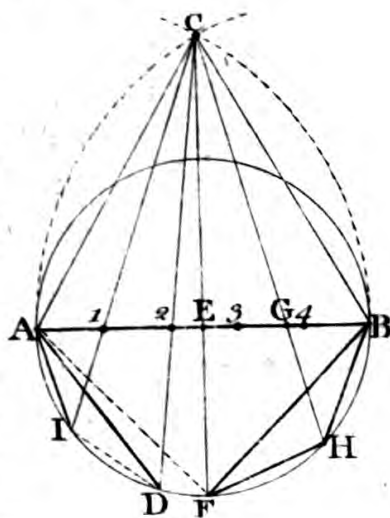
Although it is entirely Problematical, yet I think it best to follow the order of Euclid, in that; seeing, it cannot be properly treated on without the Elements of the first three Books. The practical Part is, nevertheless, easy and intelligible to any Capacity.

PROBLEM LII.

To find the Side of any regular Polygon in a given Circle.

Draw a Diameter, AB , on which, construct an equilateral Triangle, ACB ; or draw the two Arks only, intersecting at C . (This preparation is the same for any Polygon whatever.)

Then, divide the Diameter into as many equal Parts as the Polygon, required, has Sides; (Pr. 36.) and, through the second division, from either extreme, draw a Right Line, from C to the opposite Side, of the concave Circumference. e. g.



If a Pentagon be required, the Diameter (AB) must be divided into five equal Parts; through the second Division, from A or B , draw CD ; then, AD is the side of a Pentagon; i. e. a fifth part of the whole Circumference, or two fifths of the Semicircumference AFB .

The Side of a Hexagon is equal to the Radius; and the Side of an equilateral Triangle is the Diagonal of two sides of a Hexagon; yet the same Rule holds equally true in all.

If a Right Line, be drawn, from C , through the Center, E , to F , it divides the Circle equally into four, and AF or FB , is the Side of a Square.

For, if the Diameter be divided into four equal Parts, the Center being equally distant from each extreme of the Diameter, AE is, consequently, two of those Parts.

One

One of those Parts, GB, equal half the Radius, EB, is two eighths of the Diameter; wherefore, if CG be drawn and produced, it will cut the concave Circumference in the Point H; and BH, or FH, being joined, is the Side of an Octagon; for it will bisect the Ark, FHB, of the Side of a Square.

Again. AI, being a fifth part of the Diameter, is equal to two tenths; wherefore, if CI be drawn, to I, it will bisect the Ark AID; and AI or ID, being joined, is the side of a Decagon, inscribed; for, it is equal to two tenths of the concave Circumference AFB.

Thus may the Side of any Polygon whatever, contained in a Circle, be obtained; by observing the Rules given above. And it is truly worthy of notice; that, any Right Line, drawn from C, cutting the Diameter and the concave Circumference, will cut them both in the same Proportion; or, in whatever Ratio one of them is divided, a Right Line being drawn, from C, through the point of division, will also cut the other in the same Ratio.

Of this Construction, or equal division of the Diameter and the Circumference, no Demonstration can be given, having consulted several able Geometricians concerning it; who say, that it is only an approximation and not mathematically true. Yet, I must own, that I do believe it to be perfectly true, or it could never answer so very accurately, as it does, in all Divisions, whatever.

O F T H E
E L L I P S I S.

HAVING now, gone through all the most useful and valuable Problems in Geometry; I shall, next, before I proceed to the Elements, shew how to describe that useful and elegant Figure called an Ellipsis; which is so very necessary to Mechanics, and particularly to Architects and Builders, in general; insomuch, that I should reckon a compleat System of practical Geometry deficient without it.

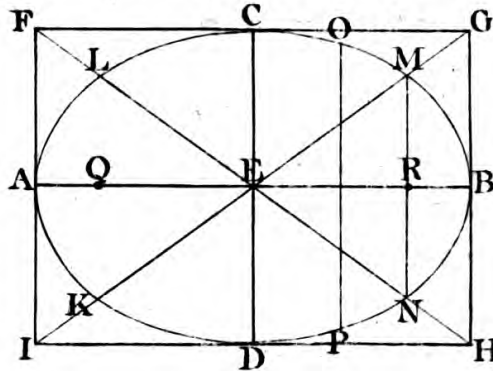
The Ellipsis, is a Figure which is not admitted into Plane Geometry; as the nature of its Curve and the properties peculiar to it, are of no use in the Elements of Euclid. Nevertheless, as it is a very useful Figure, I shall define all its parts, particularly, and shew how, by various ways, it may be described of any Proportion required; a thing much wanted and but little known, to many who have a constant occasion for it; how many lame and imperfect Ovals may be seen amongst the Works of our most eminent Artists, at an Exhibition, is but too obvious to a judicious Eye.

I shall also explain some of its peculiar Properties; shewing the affinity between the Circle and Ellipsis (as between a Square and Rhombus) the Ellipsis being considered as a Circle, pressed gently, or drawn out, at the two extremes of any Diameter.

D E F I-

D E F I N I T I O N S.

DEF. I. An ELLIPSIS, or OVAL, is a Plane Figure bounded by a regular curved Line, falling into itself, which, is not circular in any part, being described on two Centers.



DEF. II. PERIPHERY, or CIRCUMFERENCE, is the curved Line which bounds the Ellipsis; ACBD.

DEF. III. CENTER, of an ELLIPSIS, is the Point E, where any two Diameters intersect, and, consequently, bisect each other.

DEF. IV. DIAMETER, of an ELLIPSIS, is any right Line, as AB, MN, &c. passing through its Center, E, and terminated by the Periphery.

For, every such Line divides the Ellipsis equally in two; and is also bisected in the Center, E. Therefore, any two Diameters bisect each other; as in a Circle.

DEF. V. TRANSVERSE DIAMETER is the longest which can be drawn in an Ellipsis. As AB.

DEF. VI. CONJUGATE DIAMETER. This Term is generally confined to the shortest Diameter, CD; but, that is conjugate only in respect of the Transverse AB; which is also conjugate to CD.

If Tangents to the Ellipsis be drawn, through the extremes, A and B, of the Transverse, and CD its conjugate Diameter, cutting each other in F, G, H and I, they will

will form a Rectangle; and if the Diagonals of the Rectangle, FH and IG, be drawn, they will pass through the Center of the Ellipsis; the parts, KM and LN, which are terminated by the Curve, are Diameters of the Ellipsis; each of which, is Conjugate, in respect of the other.

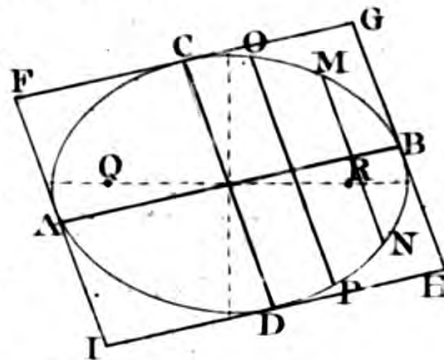
These are the only two Conjugate Diameters which are equal.

The Transverse Diameter, and its Conjugate, are the only two which are perpendicular to each other, or cut at right Angles.

The Transverse Diameter, AB, bisects the acute Angles FEI, GEH; and the obtuse Angles, FEG, IEH, are bisected by the Conjugate, CD.

Every Diameter, drawn within the acute Angles, is greater than its Conjugate, which falls within the obtuse Angles.

DEF. VII. ORDINATES are Right Lines drawn parallel to the Conjugate of any Diameter, as MN, OP, &c. and they are bisected by the Diameter, AB. The whole Lines MN, OP, are, therefore, double Ordinates to the Diameter AB.



DEF. VIII. A TANGENT is a Right Line touching the Periphery, at the extreme of a Diameter, parallel to its Ordinates. As FI, FG, &c.

DEF. IX. AXES, of an Ellipsis, are the Transverse and its Conjugate Diameter; AB and CD, Fig. 1st.

DEF. X. FOCI, of an Ellipsis, are the Points Q and R, on which it is described; as a Circle on its Center.

DEF. XI. LATUS RECTUM, or RIGHT PARAMETER, is that Ordinate to the Transverse Axe which passes through either Focus; as MN; Fig. 1st.

The Latus Rectum, and every Parameter, is a third Proportional to the two conjugate Diameters, to which it is the Parameter.

O

Note.

Note. The Parameter to the Diameters KM and LN, Fig. 1st. coincides with them; for, being equal, there cannot be a third Proportional to them.

DEF. XII. ABSCISSA. If the Transverse Diameter be cut, in any Point at pleasure, the two Segments, made by that Section, are called Abscissas.

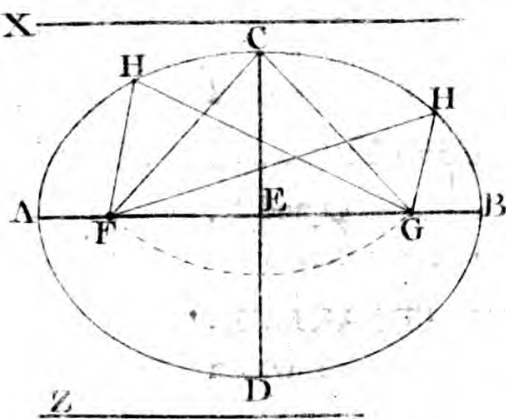
Note. An Ellipsis may be generated, by the Shadow of a Circle, or Ring, projected on a Plane, not parallel to the Ring; provided, the luminous Point is not in the Plane of the Circle.

Or it is an oblique Section, made by a Plane, of a Cylinder or Cone. (See Def. 15 and 18, 7th. El.)

PROBLEM I.

The Transverse and its Conjugate Diameter being given, how to determine the Foci, on which an Ellipsis may be described, of the Proportion required.

Let X and Z be the measures given.



Draw, at right Angles, two Right Lines, AB and CD, equal, respectively, to X and Z, and bisecting each other, in the Point E.

On either extreme, C or D, of the Conjugate Diameter, describe an Ark, with the radius AE, or EB, half the Transverse, cutting it in F and G; which are the Foci, or Centers, on which the Ellipsis may be described.

Then, proceed in the manner following.

Take

Take a fine, smooth Cord or string, and, having fixed two Pins in F and G, carry the string round both Pins, and draw it, on both Sides, to the Point C, or D; where, fix a Pencil, and revolve it around, on the two Pins, keeping the String at full stretch; the point C will describe a Curve, which will pass through the four Points A, C, B and D.

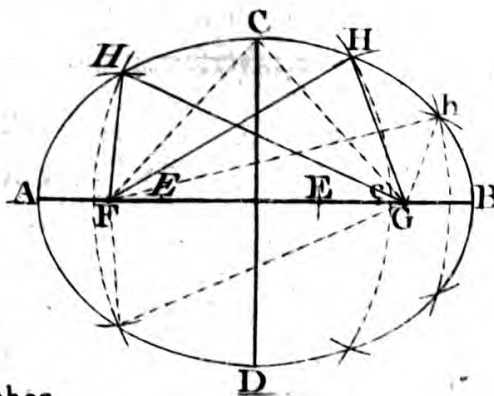
From which construction it is evident, that any two Right Lines, FH and GH, drawn from the Foci to any Point in the Circumference, are equal to FC and CG; that is, to the Transverse Diameter, AB.

For, FG being common to both the Triangles FCG and FHG, the remaining Sides, added together, are equal; i. e. FC added to CG is equal to FH added to HG.

As it is difficult to keep the Point or Pencil true, in a string, this Method is not so eligible, for small Ovals, or for any, which require the Curve to be exact; but, from what has been advanced, (being well considered and understood) it will be found practicable to describe small Ovals with tolerable exactness.

Having determined the two Foci, F and G, in the given Transverse Diameter, AB, whose Conjugate is CD (as above) as many Points, H, may be determined, in the Periphery, as are necessary; through which a Curve may be described, by a steady Hand, which will be a true Ellipsis; after the following manner.

With any Radius, at pleasure, setting one Point of the Compasses in either Focus, as at F; with the other Point, make an Ark, at H. Then, with the same Radius, FH, setting one Point in either extreme of the transverse Diameter, as at A, cut the Transverse, at E with the other.



Take the remaining Segment, EB, as Radius, and, on the Center, G, describe another Ark, cutting the former in H; which will be a Point in the Periphery.

Thus, as many Points, H or h, may be obtained as you please; which are all in the Periphery.

That the Point *H* is in the Periphery is manifest. For, seeing that the Foci are determined from the extremes, *C* and *D*, of the Conjugate Axe, making *CF* and *CG* each equal to *AE*, half the Transverse; (by Prob. 1.) consequently, *FH* and *GH* being made also equal to *AB*, the Point *H* is in the Periphery.

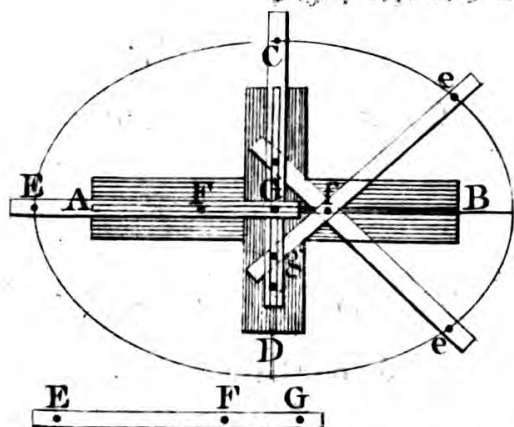
2nd. To describe an Ellipsis, by means of an Instrument called a Trammel.

The Trammel is made of two pieces of wood, fixed together at right Angles, in the form of a Cross; or, for small Ovals, it may be of Metal, with two streight Grooves, truly perpendicular to each other, as *AB* and *CD*.

Then, having provided a streight Ruler, of wood or metal, fix a Point or Pencil at one end, as at *E*; and, let two short, round Pins (equal, in Diameter, to the width of the Grooves in the Trammel, which will slide in them, freely) be fixed at *F* and *G*, in the manner, or order, as follows.

Make *EG* equal to half the Transverse, and *EF* half the Conjugate Diameter.

The Trammel and the Ruler being thus prepared, let the Trammel be placed, as in the Figure; exactly on the Transverse and Conjugate Diameters.



Then, apply the Ruler, with the Pins in the Groove, along the Transverse Diameter, *G* being in the Center, and the point *E* in the Periphery, at the extremity of the Transverse.

When the point *E* is moved towards *C*, the Pin at *G*, in the Center, falls into the Groove towards *D*, whilst the Pin at *F*, moves towards the Center; which is continued till it falls into the Center, at *G*, and the point *E* being arrived at *C*, has described, by its motion, the

the Curve AC, a fourth part of the Periphery; which has all the variety of the whole; for every fourth part is the same; from C to B, and from A to D, it is inverted.

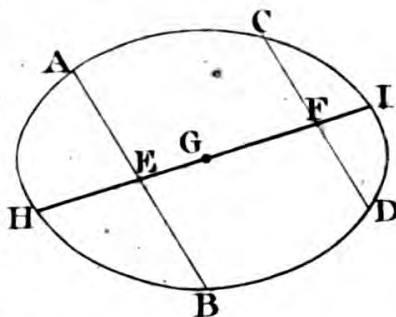
The Ruler being now in the position of the Conjugate Diameter; the Pin, at F, in the Center, and that at G fallen down to g; the motion is continued to B; the Pin at F, being now supposed to be in the Center, moves towards B, and the Pin at g goes back again to the Center; when the Point has described the Curve CB, another 4th part, the converse of the former; thus continuing the motion until the Curve is compleated, and the point E arrives again at A; the Pin, at F, still moving, to and again, in the Transverse, from F to f and back again to F; and the Pin at G, to and again, in the Conjugate Groove; by which means a true Ellipsis is formed; which, it is evident, has no part of the Curve of a Circle in its composition, seeing it is described on two Centers.

The nearer the Foci are together, that is, the less the difference is between the Transverse and Conjugate Diameters, the nearer it approaches to a Circle, and at last ends in a Circle, when the Centers unite.

PROBLEM II.

To find the Center of an Ellipsis.

Draw, at pleasure, two parallel lines, AB and CD, in the Ellipsis. Bisection them, in E and F; through which Points draw HI, which is a Diameter, of the Ellipsis. Bisection the Diameter, HI, in G, which is the Center of the Ellipsis.



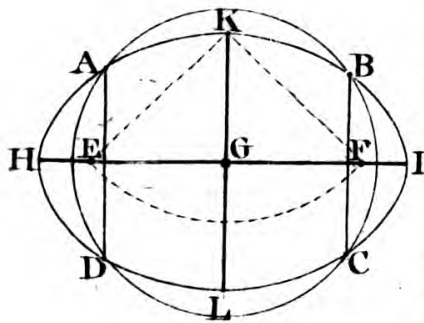
For, every Right Line drawn through the Center is a Diameter; and every Diameter is bisected in the Center.

P R O-



P R O B L E M I I I .

To find the Axes of an Ellipsis.



Having found the Center, by the last, with any radius, less than the Transverse Diameter and greater than the Conjugate, describe a Circle, cutting the Ellipsis in four points, A, B, C, and D.

Draw the Ordinates AD and BC and bisect them, in E and F; through E and F draw HI, which is the Transverse Axe.

Through the Center, G, draw KL, at right Angles with the Transverse, or parallel to the Ordinates, AD and BC, and that is the Conjugate Axe.

P R O B L E M I V .

To find the Foci of an Ellipsis.

Having found the Center, G, and drawn the Transverse and Conjugate Diameters, or Axes, take GH, half the Transverse, for radius, and on either extreme, K or L, of the Conjugate, as a Center, describe an Ark, cutting the Transverse in two points, E and F, which are the Foci, or Centers, on which the Ellipsis is described.

PROBLEM V.

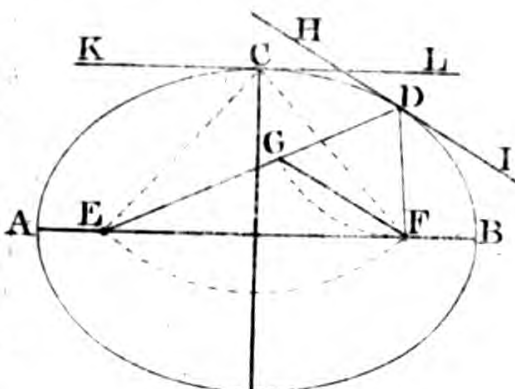
To draw a Tangent to an Ellipsis at any Point given;
in the Periphery.

If the Tangent was required at either extreme of the transverse or conjugate Axes, as B or C, it will be perpendicular to them, as in the Circle, and consequently makes equal Angles.

Let D be the given Point, through which a Tangent is required to be drawn.

Find E and F, the Foci of the Ellipsis, by the foregoing; and draw ED and DE.

Make DG equal DF, and draw GF; to which, if HI be drawn parallel, through the given Point D, it will touch or be a Tangent to the Ellipsis in that Point.



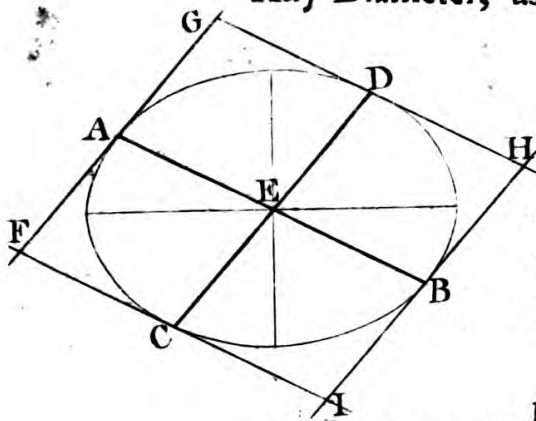
For, let KL be drawn, through C, parallel to the Transverse, AB; it will touch the Periphery in C.

Draw EC & CF; they are equal, by Construction - Pr. 4. and, consequently, the Angle ECK, is equal LCF - 4. 1. El. for the Angles CEF, CFE are equal - - - 9. 1. El.

But, the Tri. GDF is Isosceles, and HI is par. to GE. conf. the Angle HDG or E is equal to FDI. - 4. 1. El. Therefore, HI touches the Ellipsis, in the Point D.

PROBLEM VI.

Any Diameter, as AB, being given, to find its Conjugate, CD.



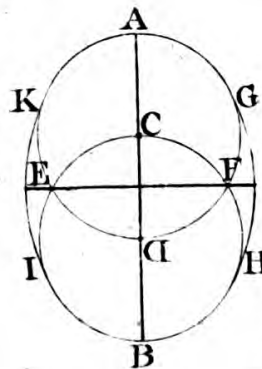
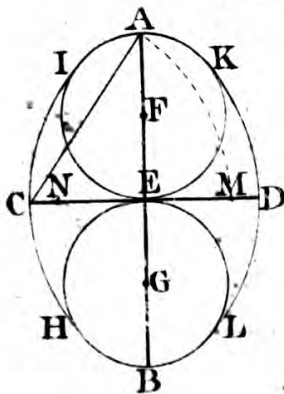
Through either extreme of the Diameter, as A, draw the Tangent FG; through the Center E, draw CD, parallel to the Tangent; which is Conjugate to the Diameter AB.

If Tangents are drawn, through the four points A, B, C and D, of any two Conjugate Diameters, meeting in F, G, H and I, the Parallelogram FGHI, is equal to a Rectangle under the two Axes.

PROBLEM VII.

To make a Representation of an Ellipsis, with Compasses.

Let AB be the given Transverse Axe, and CD the Conjugate.



Bisect AE and EB, in F and G, and on the Centers F and G, describe the Circles AIEH, & EKBL, touching at E.

Draw CA; and, with that radius, on C, describe the Ark AM, cutting the Conjugate Axe in M.

Make EN equal EM; on which Centers, M and N, with the radius MC and ND, describe the Arks HI and KL, falling into the Circles AE and EB, at H, I, K, and L.

Or

Or thus, when the difference between the Axes is less.

Divide the Transverse, AB, into three equal Parts, at C and D; on which, with the radius AC, describe two Circles, cutting each other in E and F; on which Centers, with the radius AD, describe the Arks GH and IK.

This last, is nearer to an Ellipsis than the other; and they may be lengthened or shortened at pleasure. It is very obvious that they are not Ellipses, but compounded of circular Curves; whereas, a true Ellipsis has no part of the curve of a Circle, in its composition, the Curve, being every where described on two Points, is continually varying; wherefore, no Oval, formed by Circles, can partake of the properties of an Ellipsis.

Having explained or defined all the Terms peculiar to the Ellipsis, and shewn how to construct it of any given Dimensions, and to fix the Points on which it is described, there remains nothing more to be done in respect of the utility of it, to Mechanics, &c.

To treat at large of its Properties would neither be proper nor necessary, in this Work, for the reason mentioned in the Prelude or Introduction.

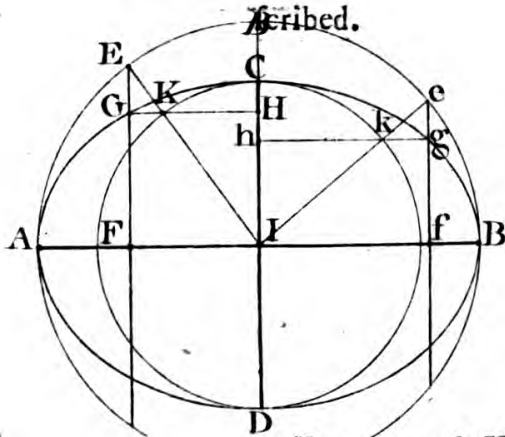
The Properties of the Ellipsis are really very extraordinary and surprizing; many of which have so near Affinity to those of a Circle, that they are almost necessarily deduced from them. But, as it is requisite to have some acquaintance with the Conic Sections, in order to a right and clear understanding of the Properties of the Ellipsis, which would not be proper to enter upon in this place; I shall content myself with giving one or two general Theorems, from which, the preceding Problems are deducible; and just mention two or three particular Properties, and then proceed to the Elements of Geometry, from which this may properly be called a Digression, though a very useful one; and which, I am persuaded, will not be unacceptable to many.

T H E O R E M I.

The Rectangle, under any two Abscissas, has the same Proportion to the Square of the Ordinate which divides them, as the Rectangle under any other two Abscissas, has to the Square of the Ordinate dividing them.

In a Circle, the Rectangle, under the two Segments of a Diameter is equal to the Square of an Ordinate, at the point of bisection; consequently they have all the same Ratio (Pr. 14. 3. El.)

Let ACBD be an Ellipsis; and let there be described two Circles, the one circumscribing it, the other, cCFD , inscribed.



From any Point, E, in the outer Circles Periphery, draw the Right Line EF, parallel to the Conjugate Axe, CD, cutting the Periphery, of the Ellipsis, in the Point G; from which Point, draw GH, parallel to the Transverse, AB; also draw the Radius, EI, which will cut the inscribed Circle in the same Point K.

For, because KG is par. to IF, $FE:FG::IE:IK$, - 2. 6. i. e. as $IB:IC$. And, EF par. to CD will be cut, in G, by the Periphery of the Ellipsis, as IB to IC . - 19. Em. C. Sec.

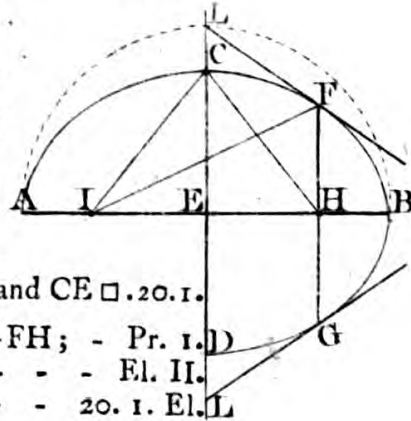
Now, GH is parallel to IB, and EF to CI - - - Con. wh. the Angle $HKI=EIF$, and $HIK=IEF$ - - - 4. 1. El. consequently, the Triangles, IHK and IEF , are similar; wh. as $IE:EF::IK:IH$. - - - - - 4. 6. But, $AI=IE$, $IC=IK$; and $FG=IH$; - - - - - 15. 1. wh. $AI:EF::IC:FG$; conf. $AI:IC::EF:FG$, alternately; and consequently, $AI \square : IC \square :: EF \square : FG \square$. - - - 15. 6. But, $AF \times FB$ is equal to $EF \square$, - - - - - 14. 3. and, $AI \times IB$ is the same as $AI \square$; for $AI=IB$. Therefore, as $AI \times IB : IC \square :: AF \times FB : FG \square$.

The same holds true of any Abscissas. whatever, and their Ordinate. For, $Af \times fB : fg \square :: AI \times IB : IC \square$ conf. $Af \times fB : fg \square :: AF \times FB : FG \square$.

2. The

2. The Latus Rectum (FG) is a third Proportional to the Transverse and Conjugate Axes.

The common method of proving it, is to make FG to AB, as CE $\square$ to AE $\square$; i. e. as CD $\square$ to AB $\square$ (4. 2. El.) which is making FG a third Proportional to AB and CD (12 and 13. 6.) and then, proving it to be what it is by Construction.

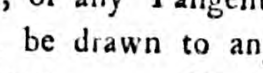


The distance of the Focus from the Center =
the Sq. Root of the diff. between AE (eq. CH) $\square$ and CE $\square$. 20. I.

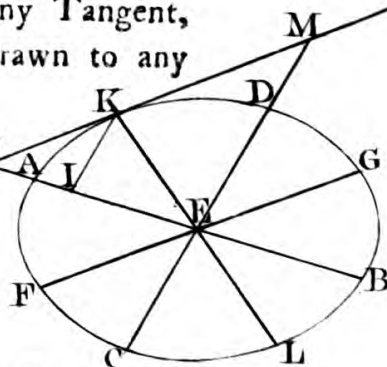
Now, $IF + FH = AB = 2EB$, i. e. $IF = 2EB - FH$; - Pr. 1. D
wh. $IF \square = 4EB \square + FH \square - 4EB \times FH$. - - - El. II. E
But, $IF \square = IH \square + FH \square$; - - - - - 20. 1. El. L
wh. $4EB \square + FH \square - 4EB \times FH = IH \square + FH \square$.
conf. $4EB \square - 4EB \times FH = IH \square = 4EH \square$, - - - Ax. 7.
and, $4EB \square = 4EB \times FH + 4EH \square$. - - - - - El. II.
i. e. $4EB \square = 4EB \times FH + 4CH \square - 4CE \square$.
wh. $4EB \square + 4CE \square = 4EB \times FH + 4CH \square$.
But, $CH = EB$ (Pr. 1.) wh. $4CE \square = 4EB \times FH$;
i. e. $4CE \square = 2EB \times 2FH$; i. e. $CD \square = AB \times FG$.
conf. FG is a third Proportional to AB and CD . - - 9. 6. El.

3. If from the Center, E, of an Ellipsis, EL (on the conjugate Axe) be made equal to the Semitransverse, AE, the Tangents LF and LG, drawn from that Point, will pass through the Extremes of the Latus Rectum.

4. If from the Point of Contact, K, of any Tangent, KN, to an Ellipsis, an Ordinate, KI, be drawn to any Diameter, cutting it in I; half that Diameter, AE, is a mean Proportional between the Segment EI, made by the Ordinate, and the Distance of the Point N, from the Center, where the Tangent cuts the Diameter, AB, produced; i. e. $EI:EA::EA:EN$.



The diagram shows an ellipse with a horizontal major axis AB and a vertical minor axis CD intersecting at center E. A tangent line KN is drawn from point K on the upper right quadrant of the ellipse to point N on the extension of the major axis AB to the right. An ordinate KI is drawn from point K perpendicular to the major axis AB, meeting it at point I. The segment EI is the distance from the center E to the foot of the ordinate I. The segment EN is the distance from the center E to the point where the tangent intersects the major axis extension, N.

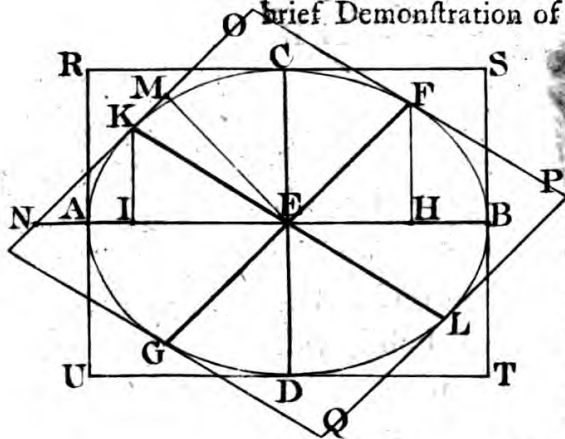


5. If a Tangent MN be drawn through the Extreme of any Diameter, KL, cutting any other two conjugate Diameters, AB and CD, in M and N; the Rectangle under KM and KN, the Segments of that Tangent, will be equal to the square of EF, half the Diameter FG, conjugate to the Diameter KL.

6. If from the Extremes, F and K, of any two conjugate Diameters, Ordinates are drawn to the transverse Axe, cutting it in H and I; then is the Ordinate FH to the semi-conjugate Axe, as the Segment EI, made by the other Ordinate, to the semi-transverse Axe,
i. e. $FH:CE::EI:EA$; and $IK:CE::EH:EB$.

7. Every Parallelogram formed by Tangents at the Extreme of any two conjugate Diameters, is equal to a Rectangle, under the transverse and conjugate Axes.

As this is a curious and extraordinary Property, I shall give a brief Demonstration of it, as follows.



Let NOPQ be a Parallelogram, made by the Tangents, NO, OP, &c. at the extremes of the Diameters GF and its Conjugate KL.

Draw the Ordinates, FH and KI, to the transverse Axe, cutting it in H and I; and draw EM perpendicular to NO; and consequently,

the Triangles EFH, ENM are similar. - - - - 4. 6. El.
for, the Angles NME, EHF are Right; and MNE = FEH. 4. 1.
Wherefore, - $EF:FH::EN:EM$. - - - - by 4. 6.

Now, - $FH:CE::EI:EA$ (6.) and $EI:EA::EA:EN$ (4.)
wherefore, - $FH:CE::EA:EN$; by equality of Ratios.

Then, since, $EF:FH::EN:EM$; and, $FH:CE::EA:EN$;
consequently, $EF:CE::EA:EM$; by inordinate equality;
wherefore, - $EF \times EM = CE \times EA$ - - - - by 9. 6. El.

But, - $CE \times EA = \frac{1}{4}$ of the Rectangle under AB and CD.
And, - $EF \times EM = \frac{1}{4}$ of the Parallelogram NOPQ. - 18. 1.
Therefore, the Par. NOPQ is equal to the Rect. RSTU. Ax. 5.

E L E M E N T S
O F
G E O M E T R Y.

B O O K I.

THIS first Book of Elements treats of particular properties of Right Lines and right lined Figures, respecting themselves, and also their relation to, or connection with one another; of Angles, formed by the intersections of Right Lines, &c. which, being entirely unconnected with Figures, in this Work, are first treated on, before we begin with Figures.

Next we proceed with Triangles, the first, yet most useful and extensive, of all right lined Figures. Then is shewn the affinity between Triangles and Parallelograms, and other four sided Figures; all which, is necessary, in some degree, to prepare the way for the last grand Proposition, the 47th of Euclid, which concludes this Book.

The Order in which these things are treated, according to Euclid, seems to me not very regular; the chain of connection is frequently broken or interrupted; beginning first with Figures, then Lines and Angles; again Figures and Lines, alternately. The fourth Proposition begins the properties of Triangles, in general; the 5th introduces the Isosceles, a particular species of Triangles, merely (in this place) for the Demonstration of the 7th; and that, is of no other use but to demonstrate the 8th, which may be clearly demonstrated (according to Proclus) without it. In the 24th
and

25th is again introduced the same general properties of Triangles, which are evidently deducible from the fourth. For, having, in the fourth, proved, that, in two Triangles, if there are found two Sides, in each, respectively equal, and that the Angles, contained between the equal Sides, are also equal, the remaining Sides are equal; can any Person of common sense, who knows what a Plane Angle means, want Demonstration, that if, in the Case of equality of the Angles, the Sides opposite those Angles are equal; if one of the Angles be greater or less, the side opposite will also be greater or less? and, *vice versa* (in the 25th) if the Side or Base, lying between the equal Sides, be greater or less, the Angle it subtends must necessarily be so too: these things seem, to me, such a necessary consequence, that I would not hesitate to take them for granted.

This Book is much encumbered with useless Demonstrations, of several Propositions, already fully demonstrated or need none; for, what is evident in itself, needs no other Demonstration; and such Propositions, as may easily and clearly be deduced from some preceding ones, are as well made Corollaries to them; such are all converse Propositions; for I cannot conceive it useful to perplex and embarrass the minds of Youths, with demonstrating what is not essential and absolutely necessary.

I have, therefore, greatly abridged, and altered the Elements of Euclid's first Book; yet, I am greatly mistaken, if I have not retained all that is essential in it; and which, I think, may be much easier and more regularly acquired. After abstracting the 14 Problems, I have but 20 other Propositions in this first Book; three of which number (the 3d the 6th and 16th) are not in Euclid; whereas, exclusive of the Problems, Euclid has 34 Theorems, twice the number, of the remaining seventeen.

The

ELEMENTS OF GEOMETRY. III

The 7th of Euclid is not only useless, in itself, but the Demonstration is intricate, to a beginner; and since the 8th. which is made to depend on it, is clearly deducible from the 4th, I see no use for it all. In respect of the 4th (my 8th) I cannot think the Demonstration of it perfectly geometrical; but, such as it is, has been generally accepted; and although it has been thought somewhat deficient, yet, I have not seen even an attempt at any other kind of Demonstration; which is sufficient conviction that no other can be given. As it is the first Proposition on the properties of Figures, which might well pass for an Axiom, the Demonstration, being purely mental, is satisfactory; though it supposes a manual, or mechanical application of one to the other; the proof arising, from which, would be, merely, ocular.

Having received the first rudiments of Geometry from Pardie, I may perhaps have imbibed a prejudice, in respect of his manner of treating the first Elements. Upon the whole, I look on that Tract as very imperfect and irregular; but I have always thought it more introductory to treat, first, of Lines and Angles, before Figures; keeping them distinct and unconnected, as much as possible. The 13th. of Euclid I have, therefore, made the first; for, on the foundation of it, depends, in a great measure, the whole first Book. I am the more confirmed in it, having seen (since my Plan was formed) two or three Authors, of some Fame, who have pursued the same Plan, nearly; which at first (I freely own) hurt me, not a little, but, which, I am now reconciled to, and think it does me credit.

According to Euclid, the properties of parallel Lines &c. cannot be obtained without having recourse to Triangles, and is the reason why the 27th, and the following, are not introduced sooner, which are previously necessary to demonstrate the 32nd; in which, the 16th and 17th are more fully and perfectly demonstrated: wherefore, Euclid

was

was obliged to have recourse to the 16th, as a Lemma, (which is much more complex and difficult to conceive than the 32nd) in demonstrating several Propositions previous to them. But, for what use the 17th is introduced, I am at a loss to devise, it being never once referred to, I believe, in the whole Elements; it might (if it be necessary) be a Corollary after the 32nd, and it is never used before. We can never be at a loss to know, that any two Angles of a Triangle are less than two Right Angles, having full Demonstration (in the 32nd) that, the three Angles of every Triangle are, together, equal to two Right ones; consequently it is of no use at all.

In the 18th of this, is contained the 35, 36, 37 and 38th of Euclid including also, in a Corollary, the 39th and 40th.

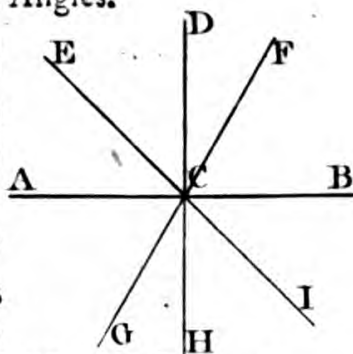
This property of Parallelograms and Triangles is, undoubtedly, very extensive; but it is all included in the first part of the 18th (i. e. in the 35th of Euclid) having previously demonstrated, in the 17th, that every Triangle is equal to half a Parallelogram, having the same Base and Altitude; which, Euclid deduces and demonstrates from the other. To dwell so long on one Property, in six Propositions, becomes tedious, if not trifling; for where is the difference, whether Parallelograms, or Triangles, have the same or an equal Base?

The alterations, which I have made in the first Book, are very considerable; and the manner of demonstrating is, in general, different, and more concise; as to the clearness of it, it must be left for others to determine. I do not doubt but the judicious, candid, and impartial Reader will concur with me, or not disapprove, entirely, the liberty I have taken. I thought it incumbent on me to advertise him of the alterations, and, by pointing them out, he may, with more ease, form a judgment of them.

THE first six Propositions, and the Corollaries deduced from them, contain all the properties of Right Lines, and Angles formed by their Intersections; which are essentially necessary to be known; before we can obtain a knowledge of the properties of Plane Figures. But, I think it scarce worth the loss of time to demonstrate them, they being in a manner self evident; insomuch, that the greatest part of them may be, and are, by some, given as first Principles, or self evident Propositions.

1. For, since (by Def. 10.) one Right Line standing perpendicularly on another makes the Angles on each side equal to the other, they are, therefore, Right Angles.

The Perpendicular, CD, being common to both Angles, ACD and DCB; the two other Sides AC, CB are, consequently, in one Right Line, ACB. And, it is clearly evident, that any other Line, or number of Lines, as CE, CD, CF, at the same Point C, must make all the Angles, ACE + ECD + DCF + FCB, equal to the two Right Angles (ACD + DCB.)



Hence, the first Proposition and its Corollaries are evidently clear and manifest.

2. Again; (by Def. 11.) if the Perpendicular, CD, be produced, towards H, there is generated equal Angles, to the opposite, on either Side, for, they are all Right Angles, consequently equal, by the 9th Axiom.

So likewise, if any other, inclined, Line be produced, as EC, or FC, it will, necessarily, produce equal Angles, to the opposite, on the other side of AB; as ACG equal FCB, and GCB = ACF; and which, together, are also equal to two Right Angles.

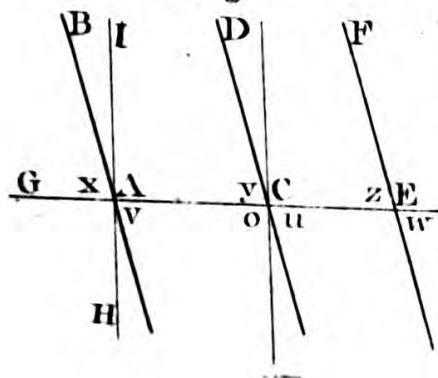
Q

For,

For, if EC bisects the Right Angle ACD , CI will also bisect the opposite, HCB ; and, if CF trisect the Right Angle DCB ; the opposite Angle, ACH , will also be trisected, if FC be produced, towards G ; and so, in whatever proportion DCB is divided, by FC , the continuation of FC will, necessarily, divide the opposite, ACH , in the same Proportion; wherefore, the Ang. $GCH = DCF$, and $ACG = FCB$; also, $GCH + HCB = DCF + ACD$; and $ACG + GCB = FCB + ACF$; i. e. they are each equal to two Right Angles; and consequently, all the Angles about the Point C are equal to four Right Angles.

Hence, it is plain, that the first and second Propositions, with their Corollaries, are clearly deducible from the 10th and 11th Definitions; after what has been said in the Theory of Plane Angles is well understood.

3. It is also, I presume, easy to be conceived, and which I think no Person, at least, who has any Talent for Geometry) can be at a loss to conceive; that, two or more parallel Lines must necessarily have the same Inclination to any Right Line which cuts them all; i. e. the Angle of Inclination, with each Line, is equal to one another; the bare Idea of Parallelism seems to indicate it; which being known, or understood, all the rest follow of course, viz. that the Alternate Angles are equal to one another, and the two internal Angles, on each Side, are equal to two Right Angles.



For, suppose the Right Lines AB , CD , and EF to be parallel amongst themselves, and all cut by any Right Line, GE ; making, with them, the Angles x , y , z ; and, suppose AB to be moved, directly, to CD , still keeping parallel to CD , it must necessarily coincide with CD ; and, consequently, the

the Angle GAB will always remain the same, and, at last, coincide with ACD .

And if AB be moved directly forward, to EF , still keeping parallel to its first position, it will also coincide with EF , and the Angle GAB , or ACD , with CEF .

Hence, it is plain that the Angles x , y and z , made with GE , are all equal to one another; and, being equal to one another, the Right Lines AB , CD and EF are consequently parallel between themselves.

4. Again; since the Angles x , y and z are equal to one another; and the opposite Angles, v , u , w , are also equal between themselves, and to the other (as it was made evident in the foregoing) consequently, the Alternate Angles, v and y , u and z , &c. are equal; for, they are all equal to one another, as above (the Lines AB , CD and EF being parallel) and, conversely, if the Alternate Angles are found to be equal, the Lines, forming them, are consequently parallel.

5. Also; since a Right Line cutting another Right Line makes the Angles, on both Sides, equal to two Right Angles (by the first) it is evident, that, if the Angles x , y and z are equal to one another, and the adjoining Angles $x+A$ and $y+C$, &c. are each equal to two Right Angles; consequently, A , C , and E are also equal between themselves, and the internal Angles (between each two Lines) A added to y , and C added to z , are also, each, equal to two Right Angles; by the 6th. Axiom.

6. And, because $A+v$ and $y+o$ are each equal to two Right Angles; and, $A+y$, on one Side GE =two Right Angles; conf. $v+o$, on the other Side GE , are, also, equal to two Right Angles. Therefore, if two parallel Right Lines, AB & CD , &c. are cut by any other Right Line, GF , the internal Angles, on each Side, are equal to two Right

Angles; and, consequently, if a Right Line cuts two Right Lines, which are not parallel (as CD and HI) they will meet on that Side where the two internal Angles ($DCE + CEI$) on the same Side, are less than two Right Angles; which, is Euclid's 12th. Axiom.

Now since all these properties of Right Lines, and of Angles formed by Right Lines, are manifest, from what has been advanced already, there is, I think, but little occasion for Demonstration; nevertheless, in conformity to the Antients, I have formed them into Propositions, for the satisfaction of the scrupulous, and for the sake of reference, in other Works, as well as in this; yet, I must own, I am clearly of opinion, that this brief extract is sufficient, towards attaining all the necessary knowledge inculcated by the first six Propositions. It will, at least, be of use to the young Student, in giving him a clear Idea of the Properties therein demonstrated; which, by being more familiarized to them, will appear more evident and conspicuous; the necessary knowledge, contained in them, will be deeper rooted and more securely established, and the Pupil better prepared to proceed with the Demonstrations of the properties of Right lined Figures.

An explanation of the Notes and Abbreviations made use of in this and in other mathematical Treatises.

$=$ Note of Equality.

Thus ; $A=B$, signifies that A and B are equal ; and is thus read, A is equal to B ; frequently, A equal B.

$+$ Note of Addition, Plus, or more.

Thus $A+B$, signifies the Sum of A added to B, A and B representing any Quantity, whatever. Let A be 5, and B, 3 ; then $A+B=8$.

$-$ Note of Subtraction ; Minus, or less.

Thus $A-B$, signifies the sum of B taken from A, $A-B=2$, must be read, A, less B, is equal to 2.

$\times$ Note of Multiplication.

Thus $A \times B$, signifies, A multiplied into B. $A \times B=15$. A multiplied into B is equal 15.

$\div$ Note of Division.

Thus, $A \div B$ signifies, A divided by B. Let A be 6 and B 2 ; then, $A \div B=3$.

$:: ::$ Note of Equality of Proportion.

Thus, $A:B::C:D$, signifies, that A bears the same Proportion to B, as C to D. $6:9::2:3$.

$\div \div$ Note of continued Proportion, or geometrical Progression.

Thus, $A:B:C:D \div \div$ signifies, that A has the same Proportion to B, as B has to C, and, as C to D. In Numbers, thus, $1:3:9:27 \div \div$

$\square$ Square. Thus, $AB \square$, signifies, the Square of the Line AB. $A \text{---} B$.

$\begin{matrix} A & B \\ \square & \\ D & C \end{matrix}$ Rectangle. Thus, $AC \square$; or ABC , signifies the Rectangle under the Lines AB and BC ; or, $AB \times BC$. And, $ABC \square = BD \square$, must be read thus ; the Rect. ABC is equal to BD Square, or to the Square of BD.

Rect-

Rectangles and other Parallelograms, are frequently denoted by two Letters, at opposite Angles. As AC or DB.

2 ABC; or 2 AB□; must be read, two Rectangles ABC, or twice the Rect. AB; and, 2 or 3 AB□ Square must be understood, twice or thrice the Square of AB.

△ Triangle. Thus, △ABC, signifies the Triangle ABC.

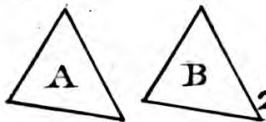
A X I O M S.

An Axiom (as already defined) is a manifest Truth, clear in itself, and therefore, does not require to be demonstrated: Nevertheless, an Illustration will not, I presume, be thought superfluous or unnecessary.

1. Things which correspond, coincide, or mutually agree with each other, in every Part, are equal.

And if the wholes be equal, the halves are also equal.

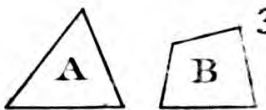
Thus, A is equal B; if every part of the Figure A coincides with B.



2. The whole is greater than a Part of the same Thing. For it is equal to all its Parts.

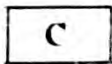


Thus, $AD = AB + BC + CD$.



3. If two, or more, Things, or Quantities, are each equal to the same third Thing or Quantity, they are equal between themselves.

Thus, if $A = C$, and if $B = C$, then $A = B$.



4. Things, or Quantities, which are each equal to half (or any other equal portion) of the same third Thing, are equal to one another.

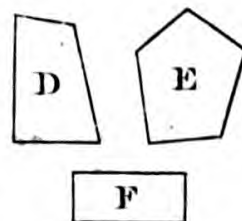
Thus, if A and B are each equal to half of C, A and B are equal.



5. Things,

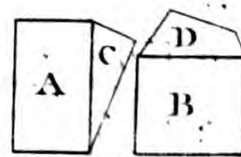
5. Things, or Quantities, which are each double, or triple, &c. of the same third Thing, or Quantity, are equal to one another.

Thus, if D and E are each equal twice or three times F, D and E are equal.



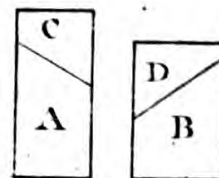
6. If to equal Things be added equals, the Sums are equal.

If to the equal Rectangles A and B, be added equals, C and D; then, $C + A = D + B$.



7. If from equal Things be taken away equals, the remainders will be equal.

If from the equal Rectangles, A and B, be taken away equals, C and D, the remainders of A and B are equal.



8. If to or from unequal Things be added or taken away equals, the Sums or Remainders will be unequal.

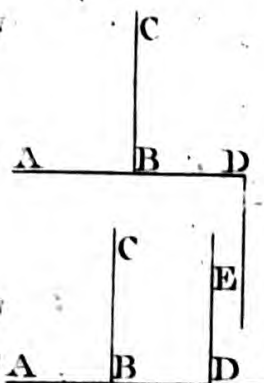
This necessarily follows from the two last.

9. All Right Angles are equal to one another.

The Right Angle $ABC =$ the Right Angle BDE .

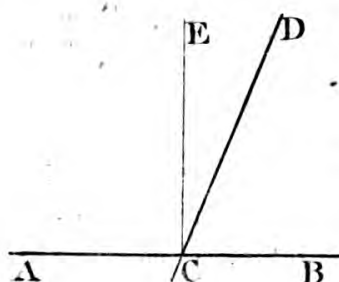
10. Two or more Right Lines, being perpendicular to the same Right Line, and being all in the same Plane, are parallel. BC is parallel to DE.

The Angles ABC , BDE , being Right Angles, are equal; by the last.



THEOREM I.

If one Right Line touch or cut another Right Line, at any Point apart from its Extremes, it will make two Angles; which are either both Right Angles, or, both together, equal to two Right Angles.



Let DC be a Right Line, touching AB in the Point C, making two Angles, ACD, DCB; I say, they are either Right Angles, or both together equal to two Right Angles.

If CD be perpendicular to AB the thing is manifest; by Def. 10.

But if CD does not make Right Angles with AB; draw CE perpendicular to AB.

DEM. Now, ACE, ECB are Rt. Angles (Con.) th. equal.

But, $ACD = ACE + ECD$; also $DCB = ECB - ECD$.

Therefore, ECD, the excess of the Obtuse Angle, ACD, to the Right Angle ACE, is also the deficiency of the acute Angle DCB to the Right Angle ECB.

Conf. $ACD + DCB$ is equal to $ACE + ECB$. - Ax. 2.

COR. 1. Hence, if from the same Point in a Right Line (as C in the Line CD) two Right Lines AC, CB are drawn, on contrary Sides and in the same Plane, making Right Angles, or Angles equal to two Right Angles, with the given Line, they will constitute one Right Line. ACB.

2. Two Angles at the same Point, and on the same Side of a Right Line, being equal, they are consequently Right ones.

Wherefore, when two Angles are contiguous or adjoining, if one of them ACE, be a Right Angle, the other ECB is also a Right one; and if one of them ACD be Obtuse, the other, ECB, is consequently Acute.

3. Two

3. Two or more Right Lines meeting in the same Point of another Right Line, make all the Angles, together, equal to two Rt. Angles. $ACE + ECD + DCB = ACE + ECB$.
4. When one Right Line makes Angles with another, if one of the Angles be known the other is also known.

For, it is the difference between the Angle known and two Right ones. As, $ACD = ACE + ECB - DCB$.

T H E O R E M II. 15. Euclid.

Vertical or opposite Angles, made by the Intersection of two Right Lines, are equal to one another.

Let AB and CD intersect in E; then, the Angles AEC, BED, are equal; also, AED is eq. to CEB.

DEM. The Right Line AE, touching the Right Line CD, makes the Angles AEC, AED, together, equal to two Right Angles. - Th. 1.

And, because DE touches the Right Line AB, the Angles AED, DEB, are also equal to two Right ones. - - - - - same.

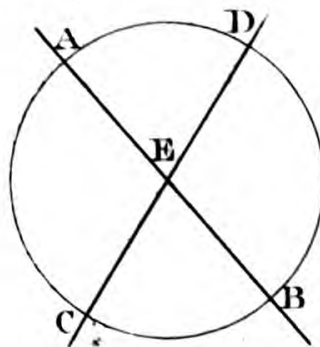
Therefore, the Ang. $AEC + AED = AED + DEB$. - Ax. 9.
and if there be taken away the common Angle AED,
there will be left, the Ang. AEC equal to DEB. - Ax. 7.

In the same manner, AED may be proved equal to CEB,

Or, it may be demonstrated after another manner, thus.

About E, the point of Intersection, as a Center, describe a Circle (with any Radius at discretion) cutting the two Lines in A, C, B, and D. Then, the Ark $AD + DB = CB + BD$.

For, they are, each two, equal to the Semi-circumference of a Circle. - - - - - See N. B. Def. 21.
Wh. taking away the Ark DB, there remains $AD = CB$;
and, the Angles AED, CEB, having equal Arks, are equal,



R

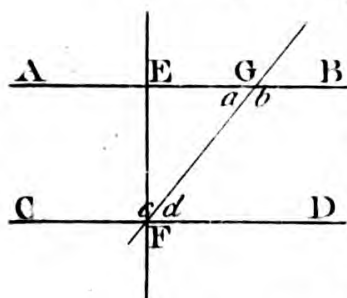
COR.

COR. Hence, if two Right Lines intersect each other, the Angles they make are equal to four Right Angles.

2. All the Angles which can be made about a Point are equal to four Right Angles.

T H E O R E M III.

If a Right Line, cutting two Right Lines, makes all the Angles, on both Sides, equal between themselves, the two Lines are parallel.



Let the Right Line EF cut two Right Lines, AB and CD, making the Angles AEF, EFC, BEF, and EFD equal to one another; then AB is parall. to CD.

DEM. Because the Angle $\angle AEF = \angle BEF$, EF is perpendicular to AB. - - - - - Def. 10.
 And, because $\angle EFC = \angle EFD$, EF is, also, perpend. to CD.
 Wherefore, since EF cuts both Lines, AB and CD, perpendicularly, the Angles AEF, EFC, &c. (being equal) are Right Angles; and, AE, EB, CF, and FD are perpendicular to EF. - - - - - Def. 11.
 But, AE, EB is the same with AB, & CF, FD, with CD.
 Therefore, AB and CD, being both perpendicular to EF, are parallel. - - - - - Ax. 10.

COR. From hence it is manifest, that, any Right Line, cutting two parallel Lines, makes the internal Angles, on each Side, equal to two Right Angles.

For,

For, if they are cut perpendicularly, as EF, it is clear.

And, since any other Line (FG) cutting them both, makes the adjoining Angles (a and b , c and d) with each Line (AB and CD) equal to two Right Angles. - - - Th. 1. Conf. all the internal Angles (a , b , c , and d) which it makes with them both, are equal to four Right Angles. - Ax. 2.

Now, since AB is parallel CD (Theo.) and the two internal Angles, on one Side EF, are equal to those on the other Side (Hyp.) it is manifest, that the internal Angles, on one Side, are, also, equal to those on the other Side, of any Right Line (FG) cutting them both; wh. $a + c = b + d$.

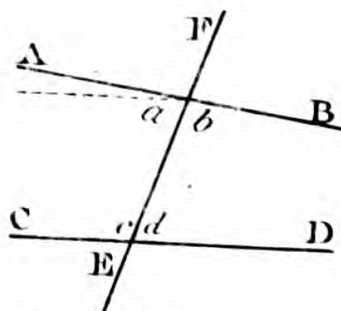
But, all the four internal Angles are equal to four Right Angles; consequently, the internal Angles, $a + c$, and $b + d$, on each Side, are, each two, equal to two Right Angles.

COR. 2. If two Right Lines, which are not parallel, be cut by another Right Line, they will meet on that Side where the internal Angles are less than two Right ones.

For, if $b + d$ be less than two R. Angles $a + c$ is greater; because, $a + b + c + d =$ four Right Angles. - - - Th. 1.

Wherefore, AB will incline to CD, and they will, if produced, meet on that side EF, towards B and D, on which the Angles $b + d$ are less than two Right ones.

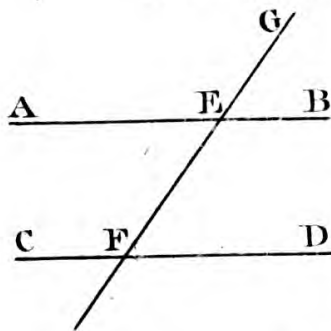
Because, the Angles, b and d , on that Side EF, are, together, less than $a + c$, on the other Side,



If this Theorem cannot be admitted, as perfect Demonstration, it may be supposed a Lemma to the next; being an illustration, at least, of Euclid's 12th Axiom; which, in my opinion cannot be received as such; by any Person as yet unacquainted with Geometry. From that consideration, it may pass with the most scrupulous. My design is to make Geometry understood, and intelligible to common Capacities, in the easiest manner.

THEOREM IV. 27. Euclid;

If a Right Line, cutting two Right Lines, makes the Alternate Angles equal, to one another; or, shall make the external Angles equal to the internal and opposite; the two Lines are parallel.



Let FG be a Right Line, cutting the two Right Lines AB and CD, and making the Angles AEF, EFD, equal to one another; then, the two Lines, AB and CD, are parallel.

For, if the Angles AEF, EFC, be not equal to two Right Angles, they are either greater or less; suppose them greater; then, the Lines, AB and CD, will meet on the other Side, towards B and D. - - - - - C. 2. 3.

Now, the Angles BEF, EFD, are less than AEF + EFC. And the Ang. AEF + FEB = EFC + EFD, - - Ax. 9. for, they are, each two, equal to two Right Angles. - Th. 1. But, the Ang. AEF = EFD (Hyp.) wh. BEF = EFC. - Ax. 7. Consequently, BEF + EFD = EFC + EFD; and, BEF + EFD = AEF + EFC, - - - - - Ax. 6 i. e. they are, on each side, equal to two Right Angles. Therefore, AB is parallel to CD. - - - Th. 3. and Cor.

2. Let the Angle GEB be equal to EFD.

Now, the Ang. EFD = GEB (Hyp.) and, AEF = GEB. 2. Wherefore, AEF is equal to EFD. - - - - - Ax. 3. But, the Angles AEF, EFD, are Alternate; and, it is already proved, that, if Alternate Angles be equal the Lines are parallel. Therefore AB is parallel to CD.

COR.

COR. 1. Conv. If a Right Line cuts two Right Lines, which are parallel, the Alternate Angles are equal to one another.

For, $\angle AEF + \angle BEF$ are equal to two Right Angles. Th. 1.
Also, $\angle BEF + \angle EFD$ are equal to two Rt. Angles. C. 1. 3.
Wherefore, $\angle AEF = \angle EFD$ ($\angle BEF$ being common) Ax. 3.
and consequently, $\angle BEF = \angle EFC$; by the same.

COR. 2. If a Right Line cuts two parallel Lines, it makes equal Angles with them both. viz. $\angle GEB$ equal to $\angle GFD$.

N. B. From this Theorem, is deduced the fifth Problem; according to the first Method, there prescribed.

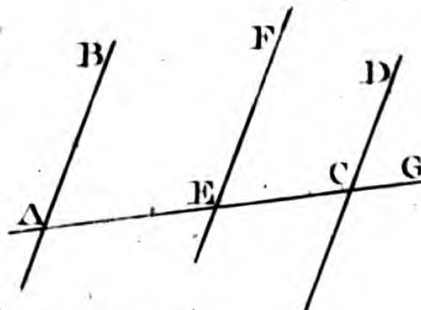
THEOREM V. 30. Euclid.

If two, or more, Right Lines are parallel to any other Right Line, they are parallel between themselves.

Let the two Right Lines, AB and CD , be both parallel to the same Right Line, EF .

I say, that AB is parallel to CD .

Let any Right Line, (AG) cut them all; being produced, if necessary



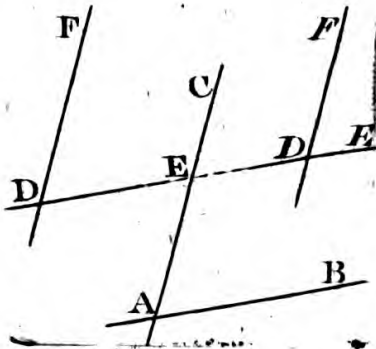
DEM. Because AB is parallel to EF , and they are both cut by the Rt. Line AG , the Angle $\angle BAE = \angle FEC$. - - C. 2. 4.
And, for the same reason, the Angle $\angle DCG = \angle FEC$; wherefore, the Angle $\angle BAE$ is equal to $\angle DCG$. - Ax. 3.
But, those Angles are internal and opposite. - Def. 16.
therefore, AB is parallel to CD . - - - 2. Th. 4.

SCHOL. This Theorem might well pass for an Axiom; for, by adopting parallelism for Things, or Quantity, 'tis the same as Axiom the third.

THEO-

THEOREM VI.

If two Right Lines cut each other, and, if there be two other Right Lines (in the same Plane with the former) parallel to them, respectively, and also cut each other, they contain equal Angles.



Let the Right Lines DE and DF (or DF) be respectively parallel to AB and AC.

I say, the Angle FDE is equal to CAB.

Produce either Side (if it be necessary) as DE (or ED) cutting AC, at E.

DEM. Now, because DE is parallel to AB, and they are both cut by AC, the Angle $CAB = CED$. - - C. 2. 4. And, because DF (or DF) is parallel to AC, and are both cut by DE, the Angle $FDE = CED$. - - - same. But, the Angle $CAB = CED$; and $FDE = CED$. proved. Therefore, the Angle FDE is equal to CAB. - - Ax. 3.

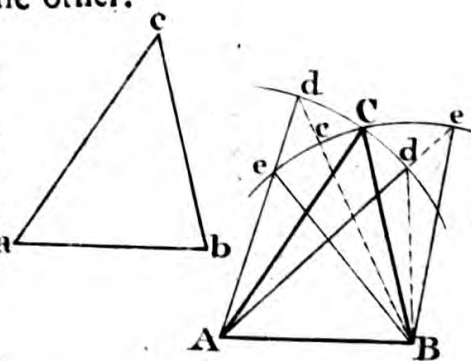
This valuable Theorem is not in Euclid, except the 10th of the eleventh Book, may be compared with it; in which, the two Lines (cutting each other) are, respectively, in different Planes; but they are, necessarily, either in the same Plane, or in parallel Planes, and holds equally true.

THEOREM VII. 8. Euclid,

If the three Sides of one Triangle are equal, respectively, to the three Sides of another Triangle, they are also equiangular; i. e. the Angles opposite to equal Sides, are also equal, and the whole Triangles are congruous, or equal, each to the other.

Let the Triangles abc , ABC , have all their three Sides equal; viz. ab equal to AB , bc equal BC , and ac equal AC .

I say, the Angle a is also equal to A , b equal B , and c equal C ; and the Triangle, abc , equal to ABC .



DEM. Suppose the Triangle abc applied

to ABC ; any Side, ab , to its equal AB . - Post. 5.

Then, if the Triangles are alike situated, ac will fall on AC , and bc on BC ; the Angle a will coincide with A , b with B , and c with C .

For, because the Side ab is equal to AB , the Extremes, a and b , will fall on A and B ; and, because ac is equal to AC , and bc to BC , the Point c must necessarily fall on C .

If not, let c , if possible, fall at d , in the Ark Cd , described on A , with the radius ac ; wh. $Ad = AC$, equal ac .

But, Bd is not equal to bc ; for, if an Ark of a Circle be described, on B , with the Radius bc , being equal to BC , it will cut the Ark, dd , at C , and the Line Ad , at e . Wherefore, the Point c cannot fall at d , nor elsewhere but on C , in the intersecting Point of the two Arks.

Th. the Triangles, agreeing in every part, are congruous.

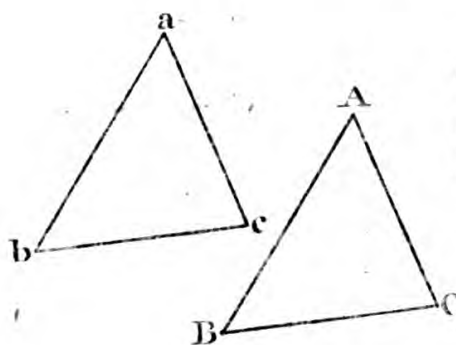
N. B. This Property does not hold good in any other Figure.

Hence, an Angle is made equal to a given Angle, as in Pr. 4; and hence, an Angle is bisected according to the 2nd method, Prob. 9th; viz by making two Triangles, congruous with each other,

THEO-

THEOREM VIII. 4. Euclid,

Triangles having two Sides, in each, respectively, equal, and if the Angles, contained by those Sides, are also equal; then, the remaining Side, of the one, shall be equal to the remaining Side of the other; the remaining Angles, of the one, equal to the remaining Angles of the other, viz. those that are opposite equal Sides; and the whole Triangles are equal, and congruous,



Let the Triangle, abc , have two Sides, ab , bc , equal, respectively, to AB , and BC , of the Triangle ABC ; and, let the Angle abc be equal to ABC .

Then, the Side ac is equal to AC ; the Angle a is equal A , and c equal C ; and the whole Triangle, abc , equal to ABC .

DEM. For, suppose the Triangle, abc , applied to, or laid upon, the Triangle ABC , in such wise, that the Side ab , of the one, agrees with AB of the other Triangle; the Extreme a , with A , and b with B . - - Post. 5.

Then, the Side bc will fall upon BC ; - by Def. 14; for, they contain equal Angles, by the Hypothesis.

The Extreme c , of the one, will coincide with C , of the other; because, bc is equal to BC .

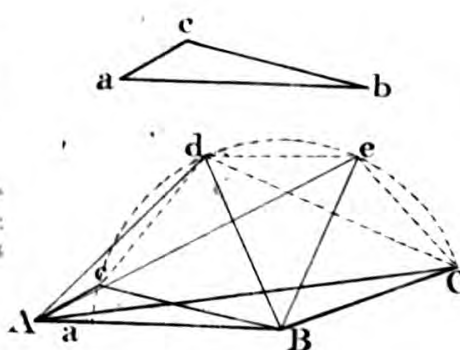
Therefore, the remaining Sides, ac and AC , must necessarily agree, and be equal to each other; the Angle a will coincide with A , and c with C ; and consequently, the whole Triangles, abc , ABC , are congruous; therefore the Triangle abc is equal to ABC .

COR.

COR. Hence, if two Sides of one Triangle be equal, respectively, to two Sides of another Triangle, and the contained Angle, of one, be greater or less than the contained Angle of the other, the remaining Side will also be greater or less; and conversely.

For, suppose two Triangles, abc , ABC , having their Sides, ab and bc , respectively equal, to AB and BC .

It is evident, that, if the Angle abc was the least possible; i. e. if the inclination of the Sides, ab , bc , was such, that it is not possible to conceive a less Angle, the two Sides, ab , bc , would nearly coincide; and consequently, the length of the remaining Side, ac , would be somewhat more than the difference between ab and bc ; viz. $ac = ab - bc$, nearly.



Again. Suppose the Angle made by the Sides AB , BC , equal ab , bc , the largest possible, so as, nearly to fall into one Right Line; it is manifest, that the remaining Side, AC , would be somewhat less than the length of both, AB and BC ; i. e. $AC = AB + BC$, nearly.

Hence it is plain, that, whether the Angle, abc or ABC , contained between ab and bc , or AB and BC , be greater or less, the remaining Side, ac or AC , will also be greater or less; and conversely, the greater or less the Side, ac or AC , the greater or less is the Angle, abc or ABC .

For, suppose the Triangle abc applied to ABC , the Side ab to AB , the Angle a to A , and b to B ; and let the Ark of a Circle be described, from c to C , on the Center B .

Then, suppose the Angle aBc opened or enlarged, to the several Apertures, aBd , aBe , aBC ; it is evident, that as the Angle aBc is increased, at aBd , &c. so is the remaining Side, Ad , Ae , &c. continually longer; from Ac , the shortest possible, to AC , the longest.

Hence the 24th and 25th of Euclid are clear, and manifest.

S

What

What I have here advanced is, I presume, sufficient conviction, if not Demonstration, of the properties therein contained.

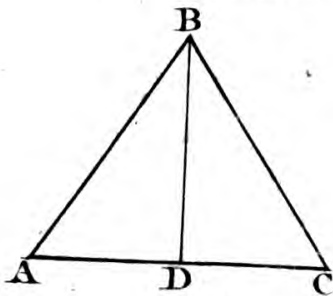
The shortest way to knowledge is the best; where perfect Demonstration is necessary, I shall be as brief as is consistent with perspicuity, and where it is not, I shall briefly illustrate.

Those who are not satisfied with this Demonstration of the two last, self evident, Propositions (of Euclid) may refer back, after the 12th (and this is not of use between) and, by drawing the Right Lines de , eC , &c. they will find, that the Side Ae , or AC opposite to an obtuse Angle, Ade , or AeC , is greater than the Side Ad , or Ae , which is opposite to an acute Angle, Aed , or ACe .

The 7th might be a Corollary to the 8th. For since, in two Triangles, if there be found two Sides, in one, equal respectively to two Sides in the other, and containing equal Angles, the third Sides are equal; it certainly follows, that if all the three Sides are respectively equal, the Triangles are equal and congruous; seeing it is not possible for an Angle to be greater or less, but the Side opposite is also greater or less.

THEOREM IX. 5. Euclid.

In Isosceles Triangles, the Angles at the Base are equal to one another, i. e. if a Triangle have two equal Sides, the Angles, opposite to them, are also equal.



Let ABC be an Isosceles Triangle; the Side AB equal to BC .

Then, the Angle A is equal to C .

Bisect the Angle ABC (Pr. 9.) made by the equal Legs, AB , BC , and draw BD ; it will also bisect the Base, AC .

DEM. In the Triangles ABD , DBC , the Sides AB , BC are equal (Hyp.) and, BD is common to them both.

Now, the Sides AB , BD are $= CB$, BD , respectively.

And the Angle ABD is equal to DBC . - - - Con.

Th. $AD=DC$, and the Triangle $ABD=DBC$. - 8.

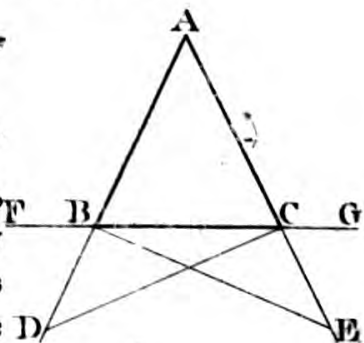
But, in congruous Triangles, opposite equal Sides are equal Angles. - - - - - Th. 7 and 8.

Th. the Ang. $A=C$, being opposite the common Side BD .

Or

Or it may be demonstrated thus, after Euclid.

Let the equal Sides, AB, AC , be produced below the Base, to D and E .
Make AD eq. AE , and join DC and BE .



Then, in the Triangles ABE, ACD , there are two Sides, AB, AE , respectively equal to AC, AD ; and the Angle A is common; wherefore, $DC = BE$; and the Angle ABE is equal to ACD . - - 8.

But, $BD = CE$ (Ax. 7.) and BC being common; the Triangle, and consequently, the Angle, $DBC = BCE$. - 7.
Also, the Angle $BCD = CBE$; conf. $ABC = ACB$. - Ax. 3.

COR. 1. The external Angles, made by producing equal Sides of a Triangle, or the Side which lies between them, are also equal, amongst themselves.

For, the Ang. $DBC = BCE$; conf. $ABF = ACG$. - 2.

2. An Equilateral Triangle, is also equiangular.

For, opposite to equal Sides are equal Angles.

3. If a Triangle have two equal Angles, the Sides opposite them are also equal. And, if all the three Angles of a Triangle are equal, it is equilateral.

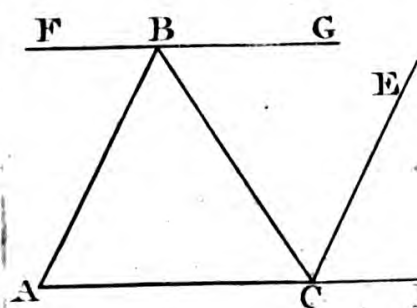
4. A Right Line, bisecting the Angle, and the Base, contained between equal Sides, is perpendicular to the Base; and consequently, two Right Lines drawn from the same Point in a Perpendicular to a Right Line, to Points, equally distant from the Perpendicular, in that Line, are equal.

For, the Angles ADB, BDC , being equal, are consequently Right Angles. (See first Figure.) C. 2. Th. 1.

N. B. Hence are performed the 6th, 7th and 8th Problems;
viz. by constructing Isosceles Triangles.

THEOREM X. 32. Euclid.

If any Side of a Triangle be produced, the external Angle is equal to the two remote Angles of the Triangle. And, the three Angles of every Triangle are, together, equal to two Right Angles.



In the Triangle ABC, let any Side, as AC, be produced. Then, the Ang. BCD is equal to the two Angles, A and B.

From C, draw CE parallel to AB.

Because CE is parallel to AB - Con. and they are both cut by the Right Line BC, the Angle $BCE = ABC$. - Th. 4.

And, because AB and CE, being parallel, are cut by the Right Line AD, the external Angle, $ECD = BAC$, the internal and opposite; - - - - - 2. Th. 4.

But, the Ang. BCE (eq. B) + ECD (eq. A) = BCD. Ax. - 2.

Therefore, the external Angle, BCD, = A + B, the two remote Angles of the Triangles. Q. E. D. - Ax. 3.

2. The three Angles, A + B + BCA, are equal to two Right Angles.

Having produced AC, and drawn CE, parallel to AB.

DEM. The Angle $BCE = ABC$, and $ECD = BAC$ - 4.

Wh. BCD (eq. BCE + ECD) is equal to ABC + BAC.

But, the Angle ACB, + BCD = two Rt. Angles. - - 1.

Therefore, the three Angles, A + B + BCA, are equal to two Right Angles.

This 2nd part is also elegantly proved, by drawing a Right Line, FG, through any Angle, B, parallel to the opposite Side, AC.

DEM.

DEM. For, the Angle $FBA = A$, and $GBC = BCA$ - 4.
 conf. the Ang. $A + ABC + BCA$ (eq. $FBA + ABC + CBG$)
 are equal to two Right Angles. - - - C. 3. Th. 1.

COR. 1. Hence, if one Angle of a Triangle be equal to the other two Angles, it is a Right one.

For, if either Side, containing the Right Angle, be produced; the external Angle, being equal to the two remote ones, is equal to the Right Angle to which it is contiguous. (Defin. 11.)

2. There can be but one Right Angle, or one Obtuse Angle, in a Triangle.

Because, if there be one Right Angle, the other two are, together, equal to a Right Angle.

3. If a Triangle have one Right Angle, the other two are acute, and equal to a right one; and, if the Triangle be Ifofceles, the two Angles, at the Hypothenufe, are each equal half a Right Angle. For, they are equal. - Th. 9.

4. If the fum of two Angles of a Triangle be known, the other is alfo known.

For, it is the Complement of two Right Angles.

Alfo, if one Angle be known, the Sum of the other two is known.

5. If the Sum of two Angles of a Triangle, either together or feparately, be equal to two Angles of another Triangle, the two remaining Angles are equal.

For, the Sum of all the Angles of every Triangle is the fame; i. e. they are equal to two Right Angles.

6. The equal Angles of every Ifofceles Triangle are acute.

For, however fmall the Angle at the Vertex, the Angles at the Bafe are, each, equal to half the difference between that Angle and two Right Angles.

7. The

134 ELEMENTS OF GEOMETRY.

7. The Angles of an Equilateral Triangle are each 60 Degs.

For, it is two thirds of a Right Angle, or one third of two Right Angles.

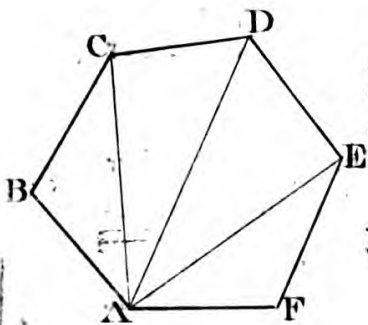
8. All the Angles of every Quadrilateral are, together, equal to four Right Angles.

For, by drawing a Diagonal, it is divided into two Triangles; and all the Angles, of each Triangle, are equal to two Right Angles; consequently, the sum of all the Angles, of both, are equal to four Right Angles; and they are equal to all the Angles of the Quadrilateral.

Note. The two following Theorems are also deducible from the last Proposition; which, not being elementary, are not numbered amongst the other Theorems.

THEO. I. All the internal Angles, of every right lined Figure, are equal to twice as many Right Angles, wanting four, as the Figure has Sides.

Every right lined Figure, may be resolved into as many Triangles, as the Figure has Sides, wanting two; by drawing Diagonals, from any Angle, to all the opposite Angles.



In the irregular Hexagon ABCDEF, let there be drawn the Diagonals AC, AD, and AE; which divides it into four Triangles, ABC, ACD, &c.

DEM. Because, by the Theorem, the three Angles of every Triangle are equal to two Right Angles, the Sum of all the Angles of the four Triangles, ABC, ACD, &c, are equal to eight Right Angles.

And, they are equal to all the internal Angles of the Hexagon. Therefore, all the Angles of the Hexagon are equal to eight Right Angles.

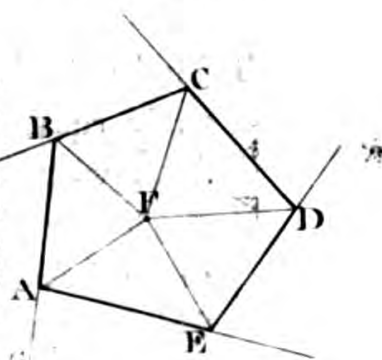
But, a Hexagon has six Sides, and twice six is twelve; by taking away four, there will remain eight Right Angles; which are equal to all the Angles of the Figure.

Or;

Or; by assuming any Point, as F, within the Figure, ABCDE, and drawing Lines to every Angle, from that Point, FA, FB, &c. it will be divided into as many Triangles, as the Figure has Sides.

Now, the Angles of all the five Triangles are equal to ten Right Angles. - - Theo. 10.

But, one Angle, of each Triangle, is at the point F, within the Figure; and all the Angles about a Point are equal to four Right Angles (Cor. 2. 2.) consequently, the remaining Angles, of the Triangles, are equal to all the Angles of the Figure; which, in a Pentagon, are equal to six Right Angles.



COR. Hence, the Sums of all the Angles, of every right lined Figure, having an equal number of Sides, are equal.

THEO. 2. All the external Angles, made by producing every Side, of any right lined Figure, are equal to four Right Angles.

For, since the internal and contiguous external Angle, BAE + BAG, &c. are equal to two Right Angles (Th. 1.) the Sum of all the internal and external Angles, together, are equal to twice as many Right Angles, as the Figure has Sides.

But, all the internal Angles are equal to as many, wanting four; consequently, the external Angles are equal to four.

COR. The sums of all the external Angles (taken together) of every Right lined Figure, are equal.

SCHOL. *This property of right lined Figures is, at first, really surprizing; for, the external Angles of a Triangle, are equal to all the external Angles of any Polygon, whatever, and of any number of Sides.*

The general utility of the knowledge of this property of Triangles, in the last Theorem, is very extensive. It is, perhaps, the most perfect Demonstration that can be given; and we may conceive an Idea of its extensiveness, from the Corollaries and Theorems already deduced from it.

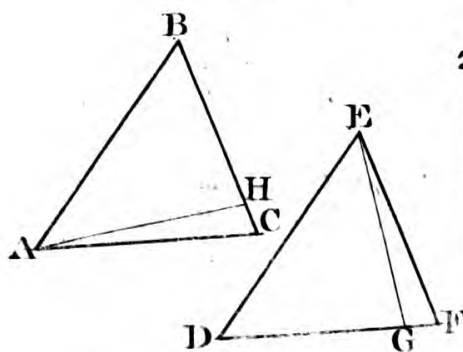
THEO-

THEOREM XI. 26. Euclid.

Triangles having two Angles, and one Side, respectively equal to two Angles, and a corresponding Side; whether it be the Side, between the equal Angles, or either of the other, which are opposite to equal Angles; the remaining Sides and Angles shall be equal, and the Triangles are congruous.

The first part of this Proposition, viz. when the equal Sides lie between the equal Angles, is so very obvious, that it might have been made a Corollary to the 8th, from which it is easily deduced.

DEM. For, if AB, equal DE, be applied to DE (Post. 5.) if the Angle A be equal D, and B equal E (the equal Sides, AB, DE, lying between) then will AC fall on DF, and BC on EF. - - - - - Def. 14. Conf. the Angle C, will fall on F, and the Triangle ABC, coincide, or agree in every respect, with DEF.



2. Let ABC and DEF be two Triangles, whose Angles A and C, of the one, are, separately, equal to D and F, of the other; and let the Side AB, of the one, be equal to DE, of the other.

Then, the Triangles, ABC, DEF, are equal, and congruous.

DEM. First, if the Side AC be not equal to DF, it is either greater or less; suppose it less, and take DG equal AC, and draw EG.

Then,

Then, because $AB=DE$ (Hyp.) and $DG=AC$, by Supposition; the Sides, AB , AC , are respectively equal to DE , DG ; and the Angle $BAC=EDG$. - - Hyp. Wh. $EG=BC$ and the Triangle $DEG=ABC$ - - 8. consequently, the remaining Angles, of the one, are equal to the remaining Angles of the other, which are opposite equal Sides; wherefore, the Angle DGE is equal ACB . And, $DFE=ACB$ (Hyp.) wh. $DGE=DFE$. - Ax. 3.

But, DGE is greater than DFE ; - - - Th. 10. for it is an external Angle of the Triangle FEG ; it is, therefore, both equal and greater, which is absurd. Therefore, AC is not less than DF .

After the same manner, it may be proved not greater; consequently, AC is equal to DF .

Th. the Triangle ABC is equal to DEF ; and congruous.

Again; if it be affirmed, that BC may be either greater or less than EF ; suppose BC greater than EF .

Take BH equal EF , and join AH .

Then, the Angle BAC , being equal to EDF , is greater than the Angle BAH ; which destroys the Hypothesis.

Wherefore, BC cannot be greater or less than EF .

This Theorem concludes the general properties of Triangles, mentioned in the preamble to this Book; which I have kept as near together as possible; by which properties, peculiar to Triangles, only, we can, with certainty, affirm them to be congruous, consequently equal.

First, when all the three Sides, of one Triangle, are equal, respectively, to the three Sides of another Triangle. (Th. 7.)

2nd. When there are found only two Sides respectively equal to two Sides, and containing equal Angles between those Sides. (8.)

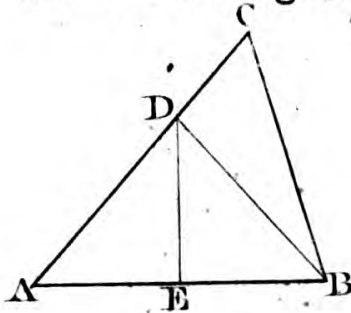
3dly. When two Angles, of the one, are respectively equal to two Angles of the other; and one Side of each Triangle, either lying between the equal Angles, or opposite to equal Angles, also equal; by this Theorem.

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THEOREM XII. 18. Euclid.

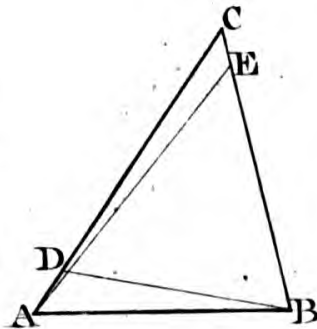
In every Triangle, the greatest Side subtends the greatest Angle; and the least Side subtends the least Angle.



Let ABC be a Triangle, whose Side AC, is greater than CB; I say, the Angle B, is greater than the Angle A.

Bisect the Side AB, in E; draw ED perpendicular to AB, and join DB.

DEM. In the Triangle ADB, the Side $AD = DB$; - C. 2. 9. wherefore, the Angle BAD is equal ABD. - Th. 9. But, the Ang. ABD (eq. BAD) is less than ABC. Def. 14. Therefore, the Angle ABC (opposite to AC) is greater than BAC, which is opposite to a less Side, BC.



Or thus, after Euclid.

By Hypothesis, AC is greater than CB. Make CD equal to CB, and join DB.

Then, the Ang. $CBD = CDB$. - Th. 9. But the Ang. CDB is greater than CAB. 10. And, CBD (eq. CDB) is less than ABC. Th. the Angle ABC is greater than CAB.

2nd. The least Side subtends the least Angle.

AB is less than BC; then, the Angle C is less than A. Make BE equal BA, and draw AE.

DEM. Now, the Angle $AEB = BAE$. - Th. 9. But, AEB is greater than ACB. - 19. Th. BAC, which is greater than BAE, is greater than ACB, which is less than AEB (equal EAB) Q.E.D.

COR.

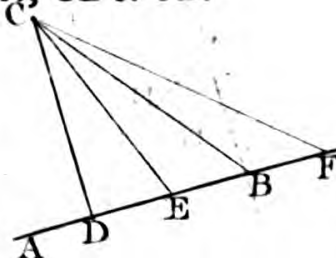
COR. 1. In Triangles, opposite the greatest Angle is the greatest Side; and the least Side is opposite the least Angle.

COR. 2. A Perpendicular is the shortest Line, which can be drawn from any Point to a Right Line. And that Line, which falls nearest to the Perpendicular, is shorter than the more remote ones.

Let CD be perpend. to AB. Draw, at pleasure, CE & CB.

Now, since CDE is a Right Angle, CED is acute. - - - C. 3. 10. Therefore, CE, which subtends the Right Angle, is greater than CD. - - Theo.

2ndly. Because the Angle CED is acute, CEB is obtuse. Consequently, CB is greater than CE. - - - Theo. Therefore, CE, which is nearest to the Perpendicular, CD, is less than CB, which is more remote from it.



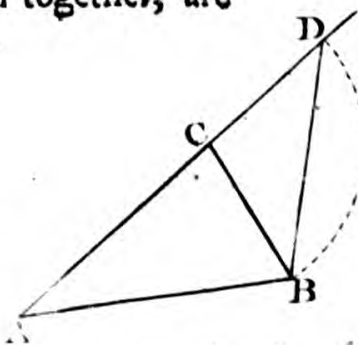
THEOREM XIII. 20. Euclid.

In every Triangle, any two Sides, taken together, are greater than the remaining Side.

Let AB be the greatest Side of the Triangle ABC.

I say, the Sides AC, CB, are, together, greater than AB.

Produce AC to D; make CD equal CB, and join BD.



DEM. Then, because $CD = CB$, the Ang. $CBD = CDB$ - 9.

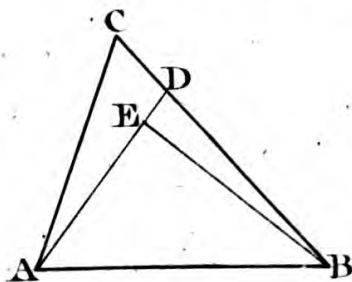
But, the Angle ABD is greater than CBD, equal CDB.

Then, since in the Triangle ADB, the Angle ABD is greater than ADB; the Side AD is greater than AB - 12.

But $AD = AC + CB$ (Con.) therefore, $AC + CB$, is greater than AB. Q. E. D.

THEOREM XIV. 21. Euclid.

If, from any Point within a Triangle, two Right Lines are drawn to the extremes of any Side; those Lines, taken together, are less than the other two Sides of the Triangle, but they contain a greater Angle.



In the Triangle ABC, assume any Point, E, and draw EA, EB.

I say, that AE added to EB, is less than AC added to CB.

And, the Angle AEB is greater than ACB.

Produce AE to D.

DEM. In the Triangle ACD, the Sides, AC, CD, are greater than AD. - - - - - 13.

Add DB to both; and $AC + CD + DB$, (eq, $AC + CB$) is greater than $AD + DB$. - - - - - Ax. 8.

Again. In the Triangle EDB, the Sides, ED, DB, are greater than EB.

Add AE to both; and $DB + AD$ is greater than $EB + AE$.

But, $AC + CB$ is greater than $AD + DB$, - - proved.

Therefore, $AC + CB$ is still greater than $AE + EB$.

2ndly. The Angle ADB is greater than ACB, and AEB is greater than ADB; - - - - - 1. Th. 10, consequently, AEB is still greater than ACB.

For, the Angle ADB is external, in respect of the Triangle ACD; and AEB, in respect of BED.

COR. Hence; if two Triangles have one Side, in each, equal to one another, or have one Side common; and, if the remaining Sides, of the one, be less, respectively, than the remaining Sides of the other, they will contain a greater Angle.

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THEOREM XV. 34. Euclid.

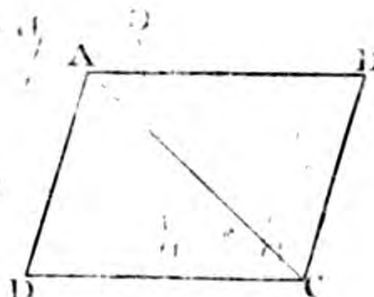
In every Parallelogram, the opposite Sides and Angles are equal; and, it is cut into two equal Triangles by a Diagonal.

Let ABCD be a Parallelogram.

The Side AB is equal to DC, and AD is equal to BC.

Also, the Angle D is equal to B, and A equal to C.

Draw either Diagonal; as AC,



DEM. By Def. 33d. AB is parallel to DC; and they are both cut by AC; wh. the Angle $BAC = ACD$. - Th. 4. And, because AD is parallel to BC, and are both cut by AC; the Angle $DAC = ACB$; and AC is common; wherefore, the Triangles, ABC, ADC, are congruous-11. Therefore, the Side $AB = DC$, and $AD = BC$. - Def. 43. Also, the Angle $B = D$ (Th. 11.) & $DAB = BCD$ -Ax.6. (For, $BAC = ACD$, and $DAC = ACB$ (Th. 4.) as above; and consequently, $BAC + CAD = ACD + ACB$.)

COR. Hence, it is manifest, if Right Lines be drawn between the extreme Points of two Right Lines, which are equal and parallel (so as not to cross each other) they are also equal and parallel.

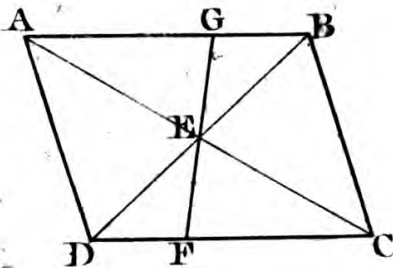
As AD and BC, joining the Points A & D, B & C.

Hence, the 33d of Euclid is evident.

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THEOREM XVI.

The two Diagonals of a Parallelogram bisect each other. And, every Diameter divides a Parallelogram into two equal and congruous Figures.



In the Parallelogram ABCD, draw the Diagonals, AC and BD.

I say the Diagonal AC is bisected by DB; and, DB is also bisected by AC, in the Point E, of their mutual Intersection.

DEM. Because, AB is parallel to CD, and are both cut by AC and BD; the Ang. $BAC = ACD$, & $ABD = BDC$. 4.
But, $AB = DC$ (15.); wh. the Triangle $AEB = DEC$; and also congruous. - - - - - Th. 11.
wh. $DE = EB$ & $AE = EC$, being opposite equal Angles.
Therefore, the Diagonals, AC, & BD, are bisected, in E.

2ndly. The Diameter, FG, divides the Parallelogram ABCD, into equal and congruous Figures.
AGFD is equal to FGBC.

DEM. Now, the Triangle ABC is equal to ACD. - 15.
The Angle $AEG = FEC$ (2.) and $GAE = ECF$, - 4.
and $AE = EC$ (above) wh. the Tri. $AGE = EFC$. - 11.
Let them be taken away; there remains $Aefd = CEGB$.
and conf. $\triangle AGE + Aefd = \triangle EFC + CEGB$. - Ax. 6.
Th. the Trapezium AGFD is equal to FGBC. Q. E. D.

COR. Every Right Line, which divides a Parallelogram, and passes through the Center, is bisected at the Center.

As, FG is bisected in E,

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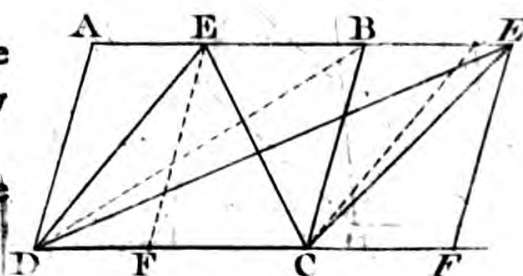
THEOREM XVII. 41. Euclid.

If a Triangle have the same Base as a Parallelogram and lie between the same Parallels, i. e. having the same Altitude, the Triangle is equal to half that Parallelogram.

In the Parallelogram ABCD, assume any Point, E, in the Side AB, and draw EC and ED:

The Triangle DEC is equal half the Parallelogram, ABCD.

Draw EF, parallel to AD and BC.



DEM. The Parallelograms, AEFD and EBCF, are bisected, by the Diags DE & EC; $ADE = DEF$, & $FEC = EBC$ - 15. Wh. the Triangle $DEF + FEC = ADE + ECB$. - Ax. 6. But, the Triangle $DEC = DEF + FEC$. - - Ax. 2. Therefore, $\triangle DEC = ADE + ECB$. Q. E. D.

Or; produce AB, make BE equal to AE, and draw CE.

Then, $\triangle BEC$ (equal AED. 8.) compleats the Par. DEEC (15. Cor.) which is equally divided by the Diag. EC. The Triangle $DEC = EEC$ (equal AED + ECB) - - 15.

Case the Second.

When E falls without the Par. ABCD, in AB produced. The Triangle DEC is equal half the Par. ABCD.

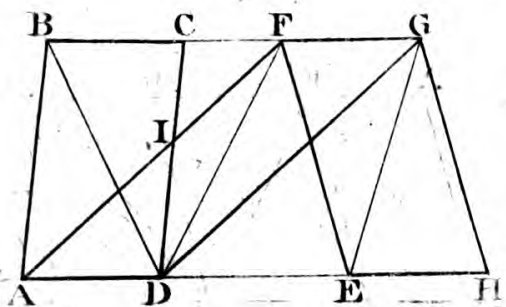
Draw EF, as before, parallel to BC; meeting the Base, DC, produced, in F

DEM. DE bisects the Par. AEFD; wh. $AED = DEF$ - 15. and, the Diag. CE, cuts the Par. CBEF, in $BCE = ECF$. Wherefore, the Triangle $DEF - CEF$ (equal DEC) is equal half the Parallelogram AF - BF (equal ABCD.) i. e. the Tri. DEC is equal half the Par. ABCD. Q. E. D.

THEOREM XVIII.

Containing the 35th, 36th, 37th, and 38th of Euclid.

Parallelograms, or Triangles, having the same or equal Bases, and equal Altitudes, are equal.



The Parallelogram $ABCD$ is equal to the Parallelogram $AFGD$; standing on the same Base AD , and between the same Parallels AH, BG ; i. e. they have the same or equal Altitudes.

Also, the Par. $ABCD$ is equal to $EFGH$, having equal Bases, $AD=EH$.

DEM. Now, $AB=CD$, and $AF=DG$. - - Th. 15.

And, the Angle $BAF=CDG$. - - - - 6.

Wherefore, the Triangle $ABF=DCG$. - - - 8.

But, the Triangle ICF is common to both;

wherefore, the Trapezium $ABCI=IFGD$. - Ax. 7.

Add, to both, the Triangle AID ; which is common;

the Parallelogram $ABCD$ is equal to $AFGD$ - - Ax. 6.

Again. The Par. $ABCD$, is proved equal to $AFGD$.

For the same reason, $EFGH=AFGD$, FG being common.

Therefore, the Par. $ABCD=EFGH$. - - - Ax. 3.

But, $AD=FG$ and $EH=FG$ (15); wh. $AD=EH$. -Ax. 3.

Th. the Parallelograms, AC and FH , have equal Bases.

Secondly. Draw the Diagonals BD, DF and EG .

Now, since every Triangle is equal to half a Parallelogram having the same Base and Altitude, - - - Th. 17.

the Tri. ABD =half the Par. $ABCD$; &, AFD =half $AFGD$

Therefore, the Triangle ABD is equal to AFD . - - Ax. 4.

And, for the same reason, the Triangle $ABD=EGH$.

(For, the Parallelogram $ABCD$ is equal to $EFGH$.)

And, their Bases, AD, EH , are also equal. Q. E. D.

COR.

COR. Equal Parallelograms or Triangles, having the same or equal Bases, have the same Altitude ; or, are contained between the same Parallels.

This Corollary is the converse of the Theorem, and contains the 39th and 40th Propositions of Euclid.

N. B. From this Theorem, the 24th, 26th, and 27th Problems are deduced ; which may be found particularly useful to Surveyors of Land, &c. ; seeing, by its means, any Plot or Survey, though ever so irregular, in Figure, may be reduced, with the greatest facility, and accuracy, to a Trapezium or Triangle ; and, its Area obtained at one Operation. (See Prob. 24.)

Also, by its means, almost any right-lined Figure, as well as a Trapezium, may be divided into two equal Parts, by a Right Line from any determined Point, in any Side.

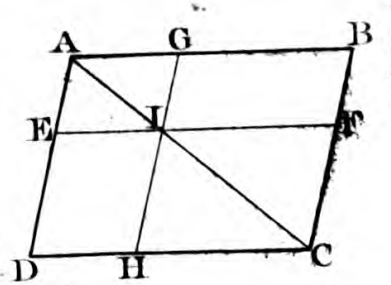
An Example, of which, will be found in the Appendix.

SCHOL. Hence it is evident, that two Spaces may differ greatly in Circumference, yet contain the same Space ; and also, that two Figures or Spaces, of equal Circuit, may contain very different Areas.

THEOREM XIX. 43. Euclid.

In every Parallelogram, the Complements are equal.

In the Parallelogram ABCD ; assume any point, as I, in the Diameter AC ; through which, draw EF and GH, parallel to AB and AD.



I say, the Complements, DI and IB, are equal to each other.

DEM. AC divides the Par. DB into two Tri^s. $ABC = ACD$.

And the Parallelograms AGIE and IFCH, are divided into equal Triangles $AGI = AIE$, and $IFC = ICH$. - Th. 15.

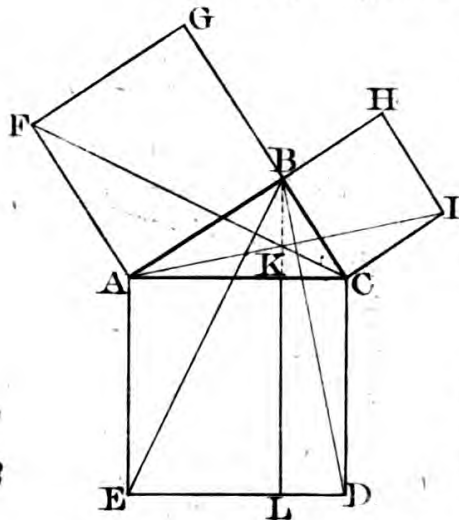
Now, if from the equal Triangles ABC, ACD, there be taken away the equal Triangles AGI, IFC = AIE, ICH, there will remain, the Par. DEIH equal to IGBF. - Ax. 7.

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THEOREM XX. 47. Euclid.

In every Right-angled Triangle, the Square of the Hypothenufe, is equal to the Squares, of the two Sides, containing the Right Angle.



Let ABC be a Right-angled Triangle.

On the Hypothenufe AC, describe the Square ACDE ; and, on the Sides AB, CD, describe the Squares AFGB and BHIC.

I say, the Square AD, of the Hyp. AC is equal to the two Squares, AG and BI, of the Sides, AB and BC.

From the Right Angle, B, draw BK perpendicular to AC ; which, produce to L, parallel to CD.

The Square AG is equal to the Rectangle AL ; and, the Square BI is equal to the Rectangle KD. Draw FC & BE.

DEM. In the Triangles FCA, ABE, AF, AC are respectively equal to AB, AE ; and they contain equal Angles. For, $\angle FAB = \angle EAC$ (Ax. 9.) and, if $\angle BAC$ be added to both, the Angle $\angle FAC = \angle BAE$. - - - - - Ax. 6. Wh. $FC = BE$, and the Triangle $FCA = ABE$. - Th. 8.

But, the Tri. FCA is equal to half the Square FB - 7. (for, they have the same Base, FA ; and, they are between the same Parallels, GC and FA.)

And, the Tri. ABE (eq. FCA) = half the Rect. AKLE ; (on the same Base, AE, & between the Parallels BL & AE) Therefore, since the Tri. $FCA = ABE$ (and they are each equal to half the Square AG, or the Rectangle AL) the Square AFGB is equal to the Rectangle AKLE - Ax. 5.

Again ;

Again; the Square BI is equal to the Rectangle CKLD;

Draw AI and BD.

Then, in the Triangles AIC, CBD, the Sides IC, CA, are equal to BC, CD, respectively.

And, the Angle ICA = BCD (ACB being common) - Ax. 6.

wherefore, the Triangle AIC = CBD. - - - Th. 8.

But, the Triangle IAC, = half the Square CH; - 17.

(having the same Base, CI, and between the Par^s. AH, CI)

And the Tri. CBD (eq. IAC) = half the Rect. KD - 17.

(being on the same Base, DC, & between the Par^s. LB, DC)

Wh. the Square BI is equal to the Rectangle KD - Ax. 5.

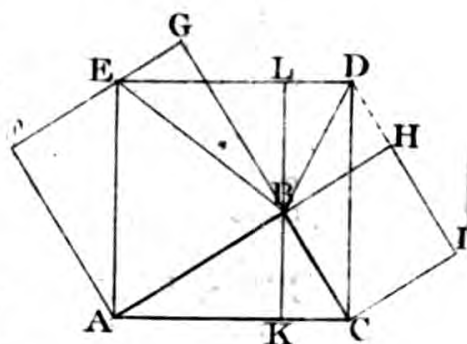
But, the Square AG was proved equal to the Rect. AL.

and $AL + KD = AD$ (the Square of the Hyp. AB.) - Ax. 2.

Therefore, the Square AFGB + BHIC = ACDE. Q. E. D.

Or, it may be, as elegantly and more briefly, demonstrated after the following manner.

Having described the Squares FB and BI, of the two Sides AB, BC, let the Square of AC, the Hypothenuse, be inverted; as AEDC.



Through B, draw KL parallel to CD. Join BD & BE; and produce IH to D.

Then, the Square AG is equal to the Rectangle AL; and the Square CH is equal to the Rectangle CL.

DEM. For, the Triangle AEB = half the Square AFGB - 17. having the same Base, AB, and Altitude, BG.

The Tri. AEB is also equal half the Rect. AELK - same having the same Base, AE, and Altitude, AK.

Th. the Square AG is equal to the Rect. AL. - - Ax. 5.

Again ; the Tri. BDC = half the Square BHIC ; - 17.
 (on the same Base, BC, & between the Parallels DI, BC).
 And the Tri. BDC is also equal half the Rect. KLDC ;
 (having the same Base, CD, and Altitude, LD.)
 Wh. the Square BHIC = the Rect. KLDC. - Ax. 5.
 But, the Rect. AL + LC = the Square AEDC. - Ax. 2.
 Th. the Square AEDC (of the Hyp. AC) is equal to the
 Squares, AFGB and BHIC, of the two Sides, AB & BC.

COR. Hence ; if, in a Triangle, the Square of one Side be
 equal to the Squares of the other two Sides, the Angle,
 contained by those two, is a Right Angle.

I have been more particular in the Demonstration of this famous Theorem, than any of the preceding ones ; not because it is the last of this Book, but, because it is so extraordinary in itself, and of such singular use throughout the Mathematics. It is also of great use in the following Books of the Elements ; inasmuch, that, without it, we should not be able to advance much further. Pythagoras is said to be the inventor of it ; for joy of which, it is said, that he offered a hundred Oxen to the Deity, who inspired him with so noble, and so useful, a Theorem.

That there may not remain the least doubt, in the mind of any, concerning the truth it contains, I have added another Figure ; the Demonstration of which is both ocular and mathematical.

EX. 3. Let the Square AEDC, of the Hypothenufe AC, be inverted, as before ; and, having drawn the Squares of the two Sides, AB, and BC ; produce FA and FG, IC and IH, meeting in K and L ; forming the Square FLIK.

The two Rectangles ABCK, and BGLH, are equal ; being under equal Lines, AB = BG, and BC = BH.
 And the four Triangles ACK, AFE, ELD, and DIC, are equal between themselves, and also to the Rectangles, BL, BK ; each being equal to half a Rectangle ; as ACK, equal half ABCK. - - - - Th. 15.

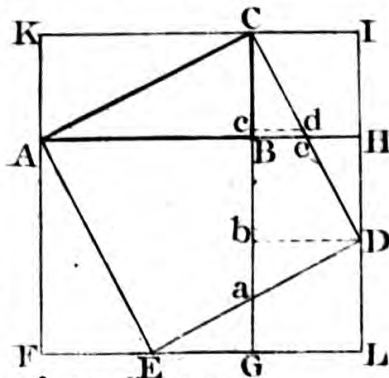
But,

But, the large Square FLIK, is equal to the two Squares FB, and BI, added to the two Rectangles, BL and BK. - Ax. 2.

It is also equal to the Square AEDC, added to the four Triangles ACK, AFE, ELD, and DIC.

And, the four Triangles are equal to the two Rectangles, BL and BK.

Consequently, the Square AEDC is equal to the two Squares, FB and BI, of the Sides, AB and BC.



Note. If the Squares, AFGB and BHIC, are cut according to this Figure, by inverting the Square AEDC; the Parts, so divided, will exactly cover the Square AEDC.

For, the Perpendicular Db being drawn, and bc made equal to bD (equal CI) draw cd parallel to bD.

Then, the Trapezium, AEaB, is common to both; the Tri. AFE is congruous with ABC, & EaG with Cdc; also, the Trap. CIHe will agree with Dbcd; and, CeB will coincide with Dab, which completes the Figure.

Therefore, the Square AFGB + BHIC = AEDC. - Ax. 2.

Having, in these twenty Theorems, gone through the first Book of the Elements of Geometry, (seventeen of which, contain the whole thirty-four, according to Euclid) it will not, I presume, be impertinent, before we proceed further, to say something of its utility. It is reasonable for a person, who has gone so far, to ask, of what use are all those properties of Triangles and Parallelograms, &c. ? which question, though pertinent enough, it may not be so easy to give an answer to, which shall be satisfactory; unless we are acquainted with almost every branch of the Mathematics.

If the study of Geometry was of no other use, than for the enuring us to Demonstration; by which we may have conviction in the quest of Truth, it would be great. Geometry is Truth itself; the manner of reasoning geometrically, is solid and convincing. It is necessary for a Person who would be a great Orator, at the Bar, &c. to study Geometry; it teaches how to range our Ideas, to state the Premises fairly, and to draw Conclusions from them with clearness and certainty. But, when we consider that it is

an

an Introduction to the Mathematics, a Key to let in the first Ideas, to further and more extensive knowledge, we shall pursue it with avidity.

The knowledge of the properties of Triangles and Parallelograms is very extensive, and of great use in almost every mathematical science; as Surveying, Navigation, Perspective, Astronomy, and numerous others, depend on them. The Demonstrations of the most simple general properties of Triangles, in Theo. 7, 8, 9, 11, 12, 13, and 14th, are necessary for the Demonstration of others, as well as for reference, for the proof of what is asserted in other Works. The 10th, a general property, and the 20th, a particular one, are more sublime properties of Triangles; and are, perhaps, the most perfect Demonstrations that can be given; which are frequently quoted and referred to hereafter.

The affinity between Triangles and Parallelograms, in Theorems 15 and 17th, is the foundation of Mensuration of Superficies: The properties of Parallelograms and Triangles in the 18th, are very extensive; the equality of the Complements of Parallelograms, in the 19th, is also of great utility; both, which, carry with them the strongest conviction imaginable.

In respect of the properties of Right Lines, and Angles formed by their Intersections, in the first six Theorems, the knowledge of which, though in a manner self-evident, is absolutely necessary for attaining the properties of Figures. I shall just give one instance, as a specimen, of what great things may be effected, from the knowledge inculcated by the fourth Theorem; viz. that the alternate Angles, formed by a Right Line cutting two parallel Lines, are equal.

It would hardly gain credit, with some, to assert, that it is possible from the measure of a small portion of the Earth's Circumference to determine the whole, by means of that simple property; yet nothing is more certain, and that, without much more previous knowledge, than, that the same portion or ark of every Circle, gives or subtends equal Angles at the Center; and consequently, equal Angles being formed at the Centers of Circles of any Radius, will cut off equal portions of the Circumference: (See the Theory of Plane Angles.)

It is very practicable to measure the distance between two places, on a level Plane; and, by means of an Instrument, contrived for that purpose, any space or distance between two places, very remote, may be ascertained, let the ground, between, be ever so irregular, interspersed with Hills and Vallies; in which case, it must be observed, that the measure, so obtained, is not the real Distance between those places, which can only be measured by a Right Line (see Def. 3, and 4.) and which is impossible to be had by an Instrument. And, let the Ground between two remote places be ever so apparently level, it is not a Plane, but a portion of a Sphere;

Sphere; and consequently, the measure, by an Instrument, over the Surface, is the measure of an Ark or a Circle, of which the Chord Line is the true Distance. (See Def. 22.)

This convexity of the Earth would be obvious, to ocular conviction, if its Surface was regular. In a perfect Calm, when the Surface of the Sea is at rest, it is discernable, that a Ship at a distance, in going from Land, will gradually appear to descend; the Bottom or Hull of the Vessel will soon be lost to sight; and, by means of Telescopes, the Sails and Rigging of the Ship may be seen, when the Hull is apparently sunk in the Water; which alone, is sufficient conviction, to any thinking Person, of the Earth's rotundity.

Now, although it is possible to measure a small portion of the Earth's Circumference, with tolerable accuracy, it would be impossible to measure the whole; because it is, in no part of it, entirely circumscribed with Land; and there is no dependence on the measures taken at Sea; therefore it is impracticable, for several reasons, to obtain its true Circumference by actual measurement.

It is easy to conceive, that two Right Lines or Poles erected perpendicular to the Earth's Surface, at any tolerable distance from each other, are not parallel between themselves; because, the Earth, being a Globe, or globular, every Right Line or Plane, which is perpendicular to its Surface, would, if produced, pass through its Center. Consequently any two Right Lines, which are perpendicular, would, if produced, form an Angle at the Center; which Angle, is greater or less in proportion to the distance of the Perpendiculars from each other (see Art. 1. Theory of Plane Angles.) Hence it is plain, from what has been said, that nothing more is required, than to determine the Angle which two Perpendiculars would form at the Earth's Center, according to the Distance between them.

Suppose a small portion of the Circumference (which may be represented by the Circle F G H) be measured, from B to D, where the Earth is as nearly on a level as possible. (By a small portion, I would not be understood to mean less than two or three Degrees, or about 200 miles.)

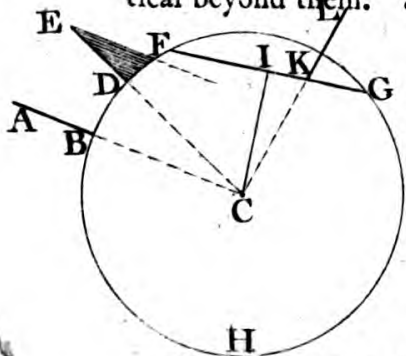
Let AB represent a Pole, or Stile, erected perpendicular to the Horizon, at B; it will consequently, tend to the Center, at C. Now the Ark BD being measured, due North or South, and another Pole, DE, erected perpendicular to the Horizon, at D, it will also tend to the Center, C, making an Angle, ACE, with AB.

The whole mystery is to determine that Angle.

From the immense distance and magnitude of the Sun, in respect of the Earth, its Rays, and consequently the Shadows of Objects, on the Earth, are projected in Right Lines, parallel amongst themselves.

It

It must also be observed, that this operation can only be performed at or between the Tropics; because, the Sun is never vertical beyond them. Suppose the Perpendicular AB to be erected.



When it is in the Meridian of that place, the Shadow of AB will be projected towards C, the Center of the Earth, for no Shadow of it can be projected on the Surface, the Sun being perpendicular over it. At the same Time, any other perpendicular Line, as ED, at a great distance from AB, will cast a Shadow, as DF, in proportion to the length of the Perpendicular DE.

Now, because ED is perpendicular to FD the Angle EDF is a Right one. (Def. 11.) Wherefore, the measure of ED being known, and the length of the Shadow DF taken, DF being considered as a Tangent, and DE the Radius, the Angle DEF is determined, by Trigonometry; or, by a Scale of equal parts, making a right-angled Triangle of the two given Lines, ED and DF (Prob. 13.) which will give the Angle DEF.

But, EF is parallel to AB; and EDC is a Right Line cutting them both; wherefore, the Angle ACE is equal to CEF (Th. 4.). Therefore, the Angle, at C is determined.

Now the Angle BCD being known, which, suppose equal two and a half or three Degrees; then, as so many Degrees is to 360, so is the Ark BD in Miles, &c. to the whole Circumference; by the Rule of Three, or Proportion.

Or, having described a Circle of any Radius, and made an Angle BCD, equal to DEF, which a Ray of Light makes with the Perpendicular DE (Prob. 4.) it will intercept a portion of the Circumference, BD, which being taken in your Compasses (beginning at B or D) see how oft it is contained in the Circumference FGH; which, multiplied into the number of Miles, measured on the Ark BD, will give the measure of the Earth's Circumference, in Miles.

Hence it is also evident, that, if a large portion of the Earth's Surface was a Plane, instead of being spherical (of which let FG be a Section) we should, at any distance from I, where a Right Line, CI, from the Center, is perpendicular to FG, appear as if on a declining Plane; and the farther from the point I, the greater will be the inclination; insomuch, that in going from I, towards F or G, we should seem as if climbing an ascent; making, at K, the acute Angle LKG, with the Plane FG. For, a Person, standing upright at K, would tend towards the Center C, in the direction LC; and consequently, would be more inclined to FG, the farther the point K is from the Perpendicular CI.

E L E M E N T S
O F
G E O M E T R Y.

B O O K II.

THE second Book of Elements treats of the properties or powers of Right Lines, divided at pleasure, and otherwise. It generally seems, at first, somewhat strange and obscure to beginners, who are unacquainted with Mensuration; but, if they carefully observe the instructions, I have given, concerning Rectangles, all difficulty will soon vanish. The method or manner of Demonstration, is, in the first eight Theorems, both ocular and mathematical; it is also, in the ten first, numerical, and does not admit of the least doubt to any, who are tolerably versed in the fundamental rules of Arithmetic.

It is of especial use in attaining a knowledge of the properties of the Circle, as contained in the third Book. For which reason, I advise the young Student, not to pass it over slightly and with indifference, but to digest it carefully, and be sure he understands it perfectly; the ten first particularly, before he advances further; else, he will frequently be obliged to turn back to them. Therefore, I advise him to get them by Heart, so, as to be clearly certain of the power of a Line; so and so divided, whenever it occurs in the course of this Treatise; it will greatly facilitate his pursuit of the remainder, and save him some trouble.

This second Book treats also, more fully, of the Theory of Triangles, in general; shewing, as in the right-angled, the proportion which the Square of any Side of a Triangle, of any species, has to the Squares of the other two; by which

X

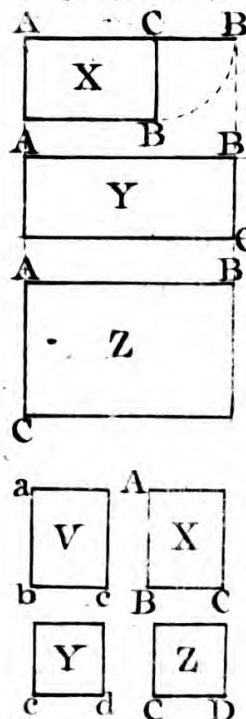
means,

means, the Perpendicular, and, consequently, the Area of any Triangle may be obtained by calculation, in Numbers, by the measures of the Sides only, the Perpendiculars being otherwise unattainable. Likewise some other properties of Triangles and Parallelograms, which are not in Euclid.

I have made no alterations in this Book, nor abridgements; but have added some very elegant Theorems. The Demonstrations, in general, as well in the other Books as this, I have made as brief as is consistent with perspicuity, and the nature of Demonstration.

RECTANGLES and SQUARES are already defined.
(See Def. 34, & 35, in the general Introduction to Geometry.)

A Rectangle, or right-angled Parallelogram, is said to be under, or contained under two Lines, which are the measures of its Sides.



If a Right Line be any how divided, as AB, in C, the Rectangle ACB or BCA is the same; and signifies a Rectangle under the two Segments, AC & CB; i. e. $AC \times CB$; as X.

But the Rects. ABC and BAC, are different from that, and from each other, if the Line be divided unequally.

$ABC \square$ signifies a Rectangle under the whole line, AB, and the Segment, BC, or $AB \times BC$; as Y.

And, BAC means one under the whole line, AB, and the other Segment, AC, or $BA \times AC$; as Z.

The middle letter is always twice meant.

AXIOM. Rectangles or Squares, contained under equal Right Lines, are equal.

As V and X, under the Line ab, eq. AB, and bc, eq. BC.
And the Squares Y and Z, under cd, equal CD.

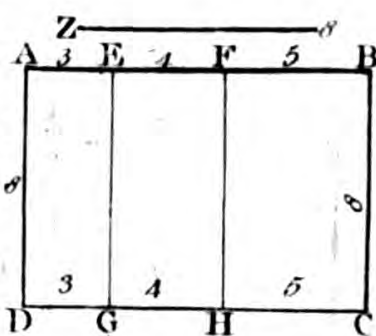
THEOREM I.

The Rectangle contained under any two Right Lines, is equal to all the Rectangles, under either Line, and the several Segments of the other, divided into any number of parts, at pleasure.

Let AB and Z be the two given Lines; and let AB be divided, at pleasure, in E and F.

I say, that $Z \times AE + Z \times EF + Z \times FB = AB$ multiplied into Z.

On the Line AB, construct the Rectangle ABCD, whose Sides AD and BC are each equal to Z. - - - Pr. 18. Through E and F, draw EG and FH, parallel to AD and BC; dividing the Rect. ABCD into three Rect. AG, EH & FC.



DEM. Now, because AD and BC are both equal to Z; - Con EG and FH are also equal Z. - - - Th. 15. 1. & Ax. 3. For, ABCD is a Rectangle, by Construction; wherefore, AG, EH and FC are Rectangles. But, the Rect. AC = all the Rects. AG, EH & FC - Ax. 2. and the Sides AD, EG, FH and BC, are each equal Z. Therefore, $Z \times AE + Z \times EF + Z \times FB = Z \times AB$.

In Numbers thus.

Let AB represent, by some Seale, 12 Feet, or any other measure; and, let Z be equal 8 of those Parts.

Let the Segment, AE be 3, EF 4, and FB 5.

Then, the Rect. AG, i. e. $Z \times AE$, or $8 \times 3, = 24$.

+ the Rect. EH, i. e. $Z \times EF$, or $8 \times 4, = 32$.

+ the Rect. FC, i. e. $Z \times FB$, or $8 \times 5, = 40$.

= the Rect. AC, i. e. $Z \times AB$, or $8 \times 12 = 96$.

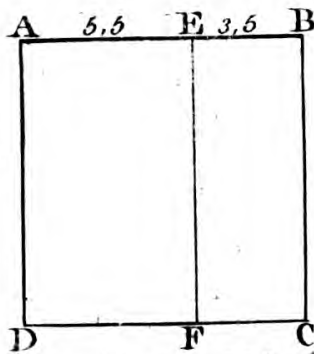
X 2

COR.

COR. If the two Lines are divided into Parts, at pleasure ;
the Rectangle under the two whole Lines, is equal to all
the Rectangles under the several segments of both Lines.

THEOREM II.

If a Right Line be divided, any how, into two parts ;
the Rectangles under the whole Line and each
Segment, are equal to the Square of the whole Line.



Let AB be divided, at pleasure, in E.
I say, that $AB \times AE + AB \times EB = AB \times AB$;
i. e. the Square of AB.

On the given Line, AB, describe the Square
ABCD ; and, through E, draw EF, paral-
lel to AD.

DEM. Now, ABCD is a Square ; by Construction ;
wherefore, AD, and BC are, each, equal to AB - Def. 35.
And AEFD, EBCF, are Rectangles. (Con.)
wherefore EF is equal to AD, or BC ; - - - - 15. 1.
But, the Square ABCD = the Rectangles AF, FB - Ax. 3.
Therefore, $AB \times AE + AB \times EB = AB \times AD$, or $AB \square$.

In Numbers, let the whole Line, AB, be 9 ;
and let it be divided in AE 5,5, and EB 3,5.

Now, the Rect. AF, i. e. $AB \times AE$, or $9 \times 5,5 = 49,5$.
+ the Rect. FB, i. e. $AB \times EB$, or $9 \times 3,5 = 31,5$.
= the Square AC, i. e. $AB \times AB$, or $9 \times 9 = 81,0$.

N. B. This Theorem is only a particular case of the former,
viz. when the two given Lines are equal ; and might have been
a Corollary to the first ; but I was not willing to break the order
of Euclid unnecessarily.

THEO-

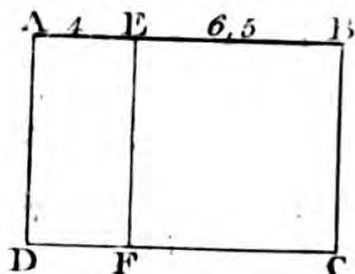
THEOREM III.

If a Right Line be divided at pleasure; a Rectangle under the whole Line and either of the Segments, is equal to a Rectangle under the two Segments, added to the Square of the Segment, first taken.

Let AB be divided in E.

I say, that $AB \times BE = AE \times EB + EB \square$.

Construct the Rectangle ABCD, under the whole Line AB and the Segment EB; and draw EF parallel to AD.



DEM. The Rectangle AEFB added to the Square EBCF is equal to the Rectangle ABCD. - - - Ax. 2.

But, the Rect. AF is under the Segments AE, EB, (eq. EF) and, FB is the Square of EB, the Segment first taken.

Th. $AB \times EB$ (or $ABE \square$) = $AE \times EB$ ($AEB \square$) + $EB \square$

Also, $AB \times AE = AEB \square + AE \square$, of the lesser Segment.

Let the whole Line, AB be 10,5; let AE be 4, & EB, 6,5:

Then, the Rect. AF, i. e. $AE \times EB$, or $4 \times 6,5 = 26$

+ the Square FB, i. e. $EB \times EB$, or $6,5 \times 6,5 = 42,25$

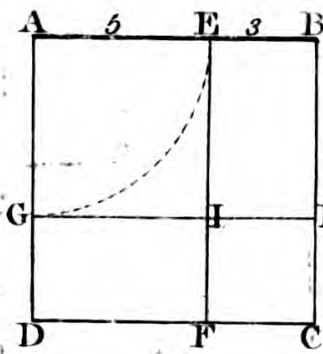
= the Rect. ABCD, i. e. $AB \times EB$, or $10,5 \times 6,5 = 68,25$

N. B. I have given this Line, in Numbers, fractional; and also divided the former, fractional-wise, in order to shew, that it will hold true in Numbers (as in Lines) any how divided; which, I advise the young practitioner to try, by various fractional divisions, if he be conversant in Decimals; it will be of some service, and make him more perfect in the whole Theory of this second Book.

THE O-

THEOREM IV.

If a Right Line be divided at pleasure; the Squares of the two Segments, added to two Rectangles under the Segments, are equal to the Square of the whole Line.



Let AB be the given Line; divided, at pleasure, in E.

Then will the two Squares of AE & EB added to two Rectangles under AE & EB, be equal to the Square of AB.

On AB, construct the Square ABCD. and draw EF parallel to AD.

Make AG equal AE, and draw GH parallel to AB.

DEM. Then, because ABCD is a Square (Con.) the Sides AD, DC, & BC, are each equal, to AB; and the Angles A, B, C, and D are all Right ones. - - - Def. 35. Wh. since AG was made equal to AE, GD is equal to EB. But, GH is parallel to AB, and EF is parallel to AD; wh. HC, and FI, IH, and FC, are each equal to EB. EI and GI are, for the same reason, each equal to AE. And, the Angles AEI, IHC; AGI, and IFC, are each equal to ABC, and ADC. - - - 2. Th. 4. 1. Wherefore, AEIG and IHCF are Squares of AE and EB; and GF, EH are Rectangles under the Segments AE & EB. And, they are all equal to ABCD, the Square of AB-Ax.2 Therefore $AE^2 + EB^2 + 2 AE \times EB = AB^2$.

Let the Line AB be 8, divided at E, in 5 and 3.

Then, the Square GE, i. e. AE^2 , or $5 \times 5 = 25$
 + the Square FH, i. e. EB^2 , or $3 \times 3 = 9$
 + two Rect. GF & EA. $2 AE \times EB$, $2. 5 \times 3 = 30$
 = the Square ABCD, i. e. AB^2 , or $8 \times 8 = 64$

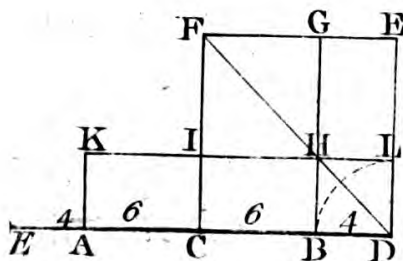
160 ELEMENTS OF GEOMETRY.

Let the whole Line AB be 16; AC=CB each =8;
let CD be 3, and DB 5; then AC+CD, =AD, =11.

Then, the Rect. AH, i. e. ADB $\square$, or $11 \times 5 = 55$
+ the Square FH, i. e. CD $\square$, or $3 \times 3 = 9$
= the Square CFEB, i. e. CB $\square$, or $8 \times 8 = 64$

THEOREM VI.

If a Right Line be divided into two equal Parts, and then produced at pleasure, or another Line be added; the Rectangle contained under the whole compounded Line and the Part added, together with the Square of half the given Line, is equal to the Square of the half Line and the Part added, in one Line.



Let the given Line AB be bisected in C,
and let BD be added to it, at pleasure.

I say, the Rectangle ADB added to CB
Square is equal to the Square of CD.

On CD, describe the Square CDEF;
and on AD the Rectangle AKLD; by drawing AK pa-
rallel to CE, and KL to AD, making DL equal to BD.
Draw BG parallel to DE, and join FD.

DEM. The Rect. AI=IB (Ax.) and IB=GL (19. 1.)
therefore, AI is equal to GL. - - - - - Ax. 3.
Conf. the Rect. AI+IB+BL=IB+BL+GL - Ax. 6.
And, if to both, be added the Square IFGH, of CB.
The Rect. AKLD+IFGH=CFED. - - - Ax. 2.
i. e. ADB $\square$, or $AD \times DB + CB \square = CD \square$.

Q.E.D.

Or it may be demonstrated as the foregoing.

Produce BA, and make AE equal to BD.

Then is ED bisected, in C, and unequally cut, in B.

Wherefore, the Rect. EBD, under the unequal parts,
(equal ADB) i. e. $EB \times BD, + CB \square = CD \square$.

Let AB be 12, bisected in C, AC, 6, and CB, 6,
and let BD, the part added, be 4.

Then, the Rect. AKLD, i. e. $AD \times BD$, or $16 \times 4 = 64$
+ the Square IFGH, i. e. $CB \square$, or $6 \times 6 = 36$
 $=$ the Square CFED, i. e. $CD \square$, or $10 \times 10 = 100$

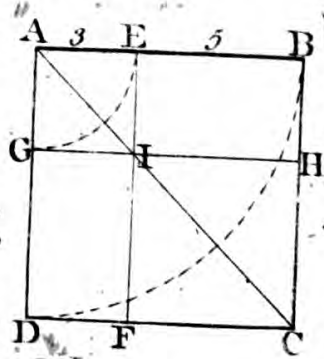
THEOREM VII.

If a Line be divided, equally or unequally, at pleasure; the Square of the whole Line, added to the Square of either Segment, is equal to two Rectangles, under the whole Line and that Segment, together with the Square of the other Segment.

Let AB be divided, any how, in the point E.

Then, the Square of AB, added to the Square of AE, is equal to two Rectangles under AB and that Segment, added to the Square of EB, the other Segment.

Describe the Square ABCD; through E, draw EF parallel to AD; make AG equal AE, and draw GH, parallel to AB; and join AC.



DEM. The Rect. $DI = IB$ (19. 1.) add GE to both; & $DE = GB$

But, the Rectangle $DE + GB = DI + IB + 2GE$; - Th. 3.

and, if FH, i. e. the Square of EB, be added;

they are equal to the Square ADCB + GE. - - Ax. 2.

But, the Rectangles DE, GB, are under the whole Line, AB, and the Segment AE; for, $AD = AB$ and $AG = AE$ - Con.

Therefore $2BA \times AE + EB \square = AB \square + AE$.

Also, $DH + FB + GE = DB + FH$;

or $2ABE \square + AE \square = AB \square + EB \square$.

Y

Let

162 ELEMENTS OF GEOMETRY.

Let the whole Line, AB, be 8; let AE be 3, and EB 5.

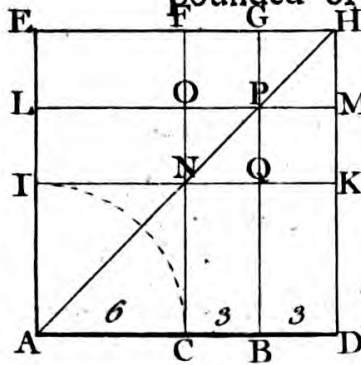
Then the Square ABCD, of AB, i. e. $8 \times 8 = 64$ }
 + the Square AEIG, of AE, i. e. $3 \times 3 = 9$ } = 73

But, the two Rects. DE, GB, i. e. $2 AB \times AE$, $8 \times 3 = 48$

+ the Square IFCH, of EB, ————— $5 \times 5 = 25$
 is also = 73

THEOREM VIII.

If a Right Line be divided, any how, in two Parts;
 four Rectangles under the whole Line and either
 Segment, added to the Square of the other Seg-
 ment, is equal to the Square of a Line, com-
 pounded of the whole Line and the Segment first
 taken.



Let AB be divided, at pleasure, in C;
 and, if CB be the Segment taken,
 make BD equal to BC.

Then, four Rectangles under AB and CB
 added to the Square of the Segment AC,
 will be equal to the Square of AD.

On the whole Line, AD, construct the Square AEHD.
 Draw CF and BG, parallel to AE; make AI equal AC,
 and AL equal to AB, and draw IK, LM, parallel to AD.

DEM. Now CF & BG are parallel to AE; & IK, LM, to AD;
 wherefore, FG, GH, HM, & MK, &c. are each equal CB;
 consequently, FP, GM, OQ, and PK are equal Squares.
 And, EO, LN, NB, and QD are equal Rectangles.
 But the Rect. $EO + FP = ABC \square$, or $AB \times BC$.

Conf. the four Rectangles EO, LN, &c. added to the four
 Squares FP, GM, &c. are equal to four times $AB \times BC$;
 and, if the Square IC (of AC) be added, they are all
 equal to the Square AEHD, of the Line, AD, com-
 pounded of AB and the Segment CB, equal BD. - Ax. 2.

i. e. $4 ABC \square$ or 4 times $AB \times BC + AC \square = AD \square$.

Let AB be 9, divided at C, in 6 and 3.
and if BD be equal CB, then AD will be equal 12.

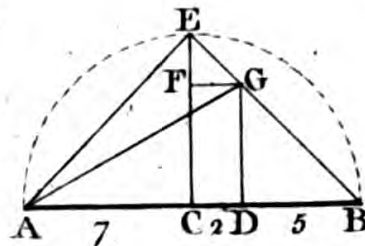
Then, the Rect. ABC, or $AB \times BC$, i. e. $9 \times 3 = 27$.
Conf. 4 Rectangles, $AB \times BC$ is equal to 108
+ the Square of AC, — $6 \times 6 = 36$
= the Square of AD, — $12 \times 12 = 144$

THEOREM IX.

If a Right Line be divided, into two equal and two unequal Parts; the two Squares of the unequal Parts are, together, double the Square of half the Line, together with the Square of the intermediate Part.

Let AB be bisected in C, and cut unequally in D.

Then, the Square of AD, added to the Square of DB; is equal to twice the Square of AC, added to twice the Square of CD, the difference between AC and AD.



Draw CE perpendicular to AB, and equal to AC or CB.
Join AE and EB, and draw DG parallel to CE.
Through G, where DG cuts EB, draw FG parallel to AB;
and, lastly, draw AG.

DEM. Now since ACE is a right Angle, & CE is equal AC,
the Angles CAE and AEC are half right. - C. 3. 10. 1.
And $AE^2 = AC^2 + CE^2$; i. e. equal $2 AC^2$. - 20. 1.
for, ACE is a Right Angle, and $AC = CE$. - - Con.
And, because CG is a Rectangle, $FG = CD$;
and FE is also equal CD; for EFG is a Right angle;
FEG (eq. FEA) is half right; th. the Angle $FEG = EGF$
consequently, FE is equal to FG (equal CD) C. 3. 9. 1.
and $EG^2 = EF^2 + FG^2 = 2 CD^2$. - - - 20. 1.

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Now,

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Now, $AE^2 = 2 AC^2$; and $EG^2 = 2 CD^2$ - above
But AEG is a right Angle, and $AG^2 = AE^2 + EG^2$;
i. e. equal $2 AC^2 + 2 CD^2$.

Also, $AG^2 = AD^2 + DG^2$; for ADG is a right Angle.

And $DG = DB$; for, GDB is a right Angle;

and $DBG = DGB$, equal CEB, half a Right one. - 4. 1.

Th. AG^2 , eq. $AE^2 + EG^2$, i. e. eq. $2 AC^2 + 2 CD^2$;
is also equal to $AD^2 + DB^2$, equal DG^2 .

Therefore, $AD^2 + DB^2 = 2 AC^2 + 2 CD^2$. Q. E. D.

Let the whole Line AB be 14, divided equally, in C,
and cut unequally, in D, in 9 & 5; AC 7, CD 2, & DB 5.

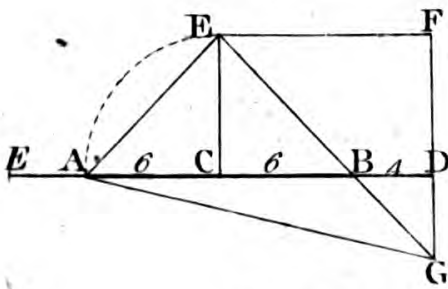
Then, AD^2 , i. e. $9 \times 9 = 81$ }
+ DB^2 $5 \times 5 = 25$ } = 106

And AC^2 , i. e. $7 \times 7 = 49$ }
+ CD^2 $2 \times 2 = 4$ } = 53, $\times 2 = 106$.

THEOREM X.

If a Right Line be bisected, and then produced at pleasure; the Square of the whole compounded Line, together with the Square of the additional Part, will be double the Square of the half Line and the Part added, together with twice the Square of the half Line.

Let AB be bisected in C, and let BD be added.



I say, that the Square of AD added to the Square of BD, is equal to twice the Square of CD, added to twice AC Square.

Make ACE a Right angle, & CE eq. AC.

Compleat the Rectangle CEFD, and produce the Side FD.

Draw EB, cutting FD in G; and lastly, draw AE & AG.

DEM,

DEM. Then, because ACE is a R. Angle, (con.) & $AC=CE$ the Angles CAE and AEC are half Right. - C. 3. 10. 1.
And, for the same reason, CEB and EBC are half right.
And, because CEFD is a Rectangle, $EF=CD$;
and since EFG is a Right Angle, and FEG (eq. DBG, eq. EBC) is half right, FGE is half right. - - Ax. 3.
wherefore, $FG=FE$, eq. CD ; and $DG=DB$ - C. 3. 9. 1.

Now, since ACE is a right Angle, and $CE=AC$
 $AE^2 = AC^2 + CE^2$; i. e. equal 2 AC Square. - 20. 1.
And, $EG^2 = EF^2 + FG^2$; i. e. equal 2 CD².
But, AEG is a R. Angle; for AEC, CEG are half right.
wh. $AG^2 = AE^2 + EG^2$; i. e. equal 2 AC² + 2 CD²
But, ADG is a R. Angle; conf. $AG^2 = AD^2 + DG^2$;
and, since $BD=DG$, $BD^2 = DG^2$; - - - Ax.
Therefore, $AD^2 + BD^2 = 2 AC^2 + 2 CD^2$.

This Proposition may also be demonstrated as the former.

Let the given Line AB be bisected in C and let BD be added;
also make AE equal BD in BA produced.

Then, because ED, is bisected in C and cut unequally in B,
 $EB^2 + BD^2 = 2 CD^2 + 2 CB^2$.

But $EB=AD$; for AE was made equal to BD.

Wh. $EB^2 + BD^2$ (eq. $AD + BD^2$) $= 2 CD^2 + 2 CB^2$.

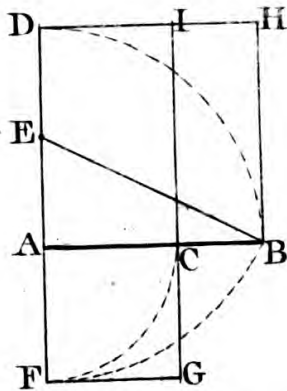
Let AB be 12; divided, in C, equally; and let BD be 4.

Then, the Square of AD,	i. e. $16 \times 16 = 256$	} = 272
+ the Square of BD,	$4 \times 4 = 16$	
But, the Square of CD,	$10 \times 10 = 100$	
+ the Square of AC,	$6 \times 6 = 36$	
	$136 \times 2 = 272$	

T H E O-

THEOREM XI.

To divide a Right Line, in such wise, that the Rectangle under the whole Line and one Segment, shall be equal to the Square of the other; consequently, the greater Segment.



AB is the Line given, to be divided.

It is required so to cut AB, in C, that the Rectangle, ABC, under the whole Line AB and the Segment BC, shall be equal to the Square of AC.

At the extreme A, draw AD perpendicular to AB.

Make AD equal AB, and bisect it in E; join EB.

Produce DA. Make EF equal EB, and AC equal to AF.

Then is AB so divided, in C, that $AB \times BC = AC^2$.

Compleat the Squares ADHB and AFGC, and produce GC to I.

DEM. Then, because AD is bisected in E, and AF is added, the Rect. DFA, i. e. $DF \times AF + AE^2 = EF^2$. - Th. 6. But, $EF = EB$; wh. $EF^2 = EB^2$; i. e. eq. $AB^2 + AE^2$. Conf. $DF \times AF + AE^2 = AB^2 + AE^2$.

Take AE^2 from both, and there remains $DF \times AF = AB^2$. But, the Rect. DIGF. is under the whole Line DF, and the Segment AF; and ADHB is the Square of AB; wherefore, the Rect. FI = the Square AH; and if the Rect. ADIC, which is common to both, be taken away, there is left the Rect. CIHB equal to the Square ACGF. i. e. the Rect. ABC, or $AB \times BC = AC^2$.

This

This Proposition cannot be exemplified in Numbers; i. e. the point C. cannot be found, nor the proof ascertained arithmetically; for there is no Number, whatever, can be so divided, that the product of the whole Line multiplied by one part, shall be equal to the Square of the other.

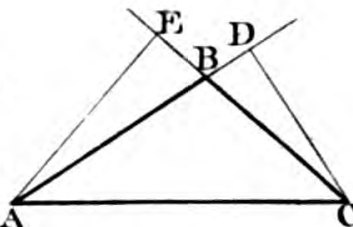
Therefore, a Line so divided is incommensurable.

THEOREM XII.

In an obtuse angled Triangle, the square of the Side which subtends the obtuse angle, exceeds the squares of the other two Sides, by two Rectangles, under either Side containing the obtuse Angle, and the part intercepted between the obtuse Angle, and a Perpendicular, let fall from the adjacent Angle, to the same Side, produced.

ABC is an obtuse angled Triangle.
Produce the Sides AB and CB; and to them,
draw the Perpendiculars AE and CD.

I say, the Square of AC exceeds the Squares of AB and BC, by two Rectangles ABD, or CBE.



DEM. First, AC^2 is equal to $AD^2 + DC^2$ 20. 1.

and $AD^2 = AB^2 + BD^2 + 2 ABD$ 4. 2.

wherefore, $AC^2 = AB^2 + BD^2 + CD^2 + 2 ABD$.

But BC^2 is equal to $CD^2 + BD^2$.

Therefore, $AC^2 = AB^2 + BC^2 + 2ABD$.

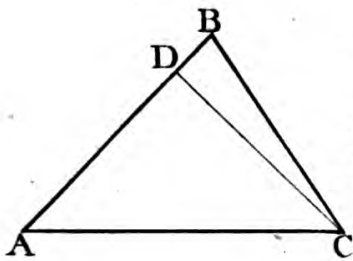
In the same manner, AC^2 may be proved equal to $AB^2 + BC^2 + 2CBE$ Rectangles.

Consequently, the Rectangles ABD and CBE are equal.

THE O-

T H E O R E M XIII.

In every Triangle, the Square of the Side subtending an acute Angle, is exceeded by the Squares of the other two Sides, which contains the acute Angle, by two Rectangles under either of those Sides, and a part of it, intercepted between the acute Angle and a Perpendicular, let fall from the opposite Angle.



In the Triangle ABC, draw a Perpendicular, CD, to the Side AB.

Then, the Square of AC, is exceeded by the two Squares of AB and BC, by two Rectangles, ABD, under the whole Side, AB, and the Segment BD.

And, $AB^2 + AC^2 = BC^2 + 2BA \times AD$.

DEM. First, $AB^2 = AD^2 + DB^2 + 2ADB^2$. - Th. 4.

And, $BC^2 = CD^2 + DB^2$. - - - - - 20. 1.

Wh. $AB^2 + BC^2 = AD^2 + DC^2 + 2DB^2 + 2ADB^2$

But, $ADB^2 + DB^2 = ABD^2$. - - - - - Th. 3.

consequently, $2ADB^2 + 2DB^2 = 2ABD^2$.

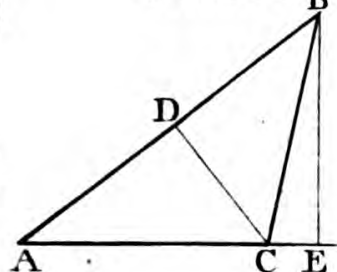
wherefore, substituting these for the other;

then, $AB^2 + BC^2 = AD^2 + DC^2 + 2ABD^2$.

But, $AC^2 = AD^2 + DC^2$. - - - - - 20. 1.

Therefore $AB^2 + BC^2 = AC^2 + 2ABD^2$.

Secondly, thus, more briefly, when an Angle is obtuse. Produce either Side, AC, and from B draw BE perpendicular to it.



Now, $AB^2 = BC^2 + AC^2 + 2ACE^2$

Add, on both sides, the Square of AC.

Then $AB^2 + AC^2 = BC^2 + 2AC^2 + 2ACE^2$. - - - - - Ax. 6.

But, $2ACE^2 + 2AC^2 = 2EAC^2$;

or $2AE \times AC$. - - - - - Th. 3.

Therefore, $AB^2 + AC^2 = BC^2 + 2AE \times AC$.

From which it is obvious, that the Rect. BAD = EAC.

By the first method it may be demonstrated; that the square of BC, is exceeded by the squares of AB and AC; by two Rectangles BAD; and by the second, the squares of AB and AC, exceeds the squares of BC, by two Rectangles EAC; therefore the Rectangle BAD is equal to EAC.

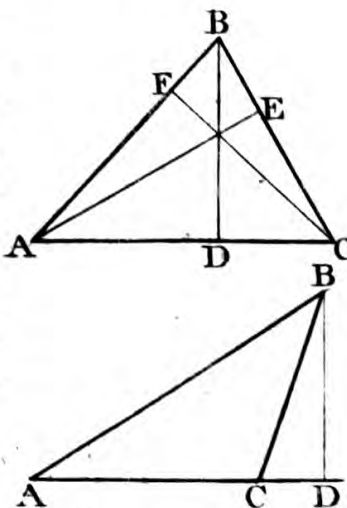
SCHOL. From these two Theorems it is evident, that if the square of one Side of a Triangle exceeds the squares of the other two, the Angle subtended by that Side, and contained by the other two, is obtuse; and, if the square of one Side, be exceeded by the squares of the other two; it subtends an acute Angle; consequently, when the square of one Side is equal to the squares of the other two, the Angle it subtends is a Right one.

COR. Hence, the Perpendicular, and consequently the Area of any Triangle, may be found or obtained, by the measure of the Sides only.

In the Tri. ABC, $AC^2 + BC^2 = AB^2 + 2AC \times CD$.
Wherefore, $AC^2 + BC^2 - AB^2 = 2AC \times CD$; by Th.
i. e. if AB^2 be subtracted from the sum of $AC^2 + BC^2$,
the remainder will be two Rectangles, under AC & DC.
Wherefore, if half that Product be divided by the Side
AC, the Quotient will be the Segment DC.

But $BC^2 = BD^2 + DC^2$. Or, $AB^2 = AD^2 + BD^2$.
wherefore, $BC^2 - DC^2 = BD^2$. - - - 20. 1.
Conf. having subtracted the square of DC from BC^2 ,
or of AD from AB^2 , the remainder will be the square
of the Perpendicular BD, the square Root of which,
gives the Perpendicular required.

By the same means, either of the other Perpendiculars, AE or CF, may be found.



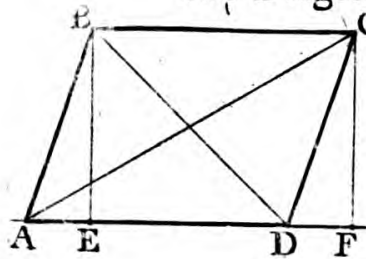
2ndly. When the Perpendicular falls without the Triangle, from the acute Angles of an obtuse angled Triangle, it is thus found.

In the obtuse-angled Triangle ABC; let fall the Perpendicular BD to the Side AC, produced.

Then, $AB^2 = AC^2 + CB^2 + 2AC \times CD$. - - - Th. 12.
Wherefore $AB^2 - AC^2 + CB^2 = 2AC \times CD$;
half of which Sum being divided by the Side AC gives CD.
But, $CB^2 = BD^2 + CD^2$ (20. 1.) wh. $CB^2 - CD^2 = BD^2$,
the square Root of which gives the Perpendicular BD.

THEOREM XIV.

In every Parallelogram, the sum of the Squares of the two Diagonals is equal to the Squares of all the Sides, together.



Let ABCD be a Parallelogram.

Draw the Diagonals, AC and BD; produce the Base AD, and draw the Perpendiculars BE and CF.

DEM. In the obtuse angled Triangle ACD;

$$AC^2 = AD^2 + DC^2 + 2ADF. \quad \text{Th. 12.}$$

$$\text{And, in the Tri. ABD; } BD^2 = AB^2 + AD^2 - 2DAE.$$

$$\text{But, EBCF is a Parallelogram, wh. EF} = BC. \quad \text{15. 1.}$$

$$\text{and, AD} = BC; \text{ wherefore AD} = EF \quad \text{Ax. 3.}$$

$$\text{conf. AE} = DF, \text{ ED being common; wh. ADF} = \text{DAE}$$

$$\text{Therefore, as much as AC}^2 \text{ exceeds AD}^2 + \text{DC}^2, \text{ viz.}$$

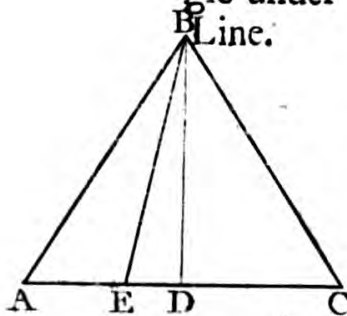
$$\text{by the Rect. ADF, twice; by so much is BD}^2 \text{ exceeded}$$

$$\text{by AB}^2 + \text{AD}^2, \text{ eq. BC}^2; \text{ viz. by } 2\text{DAE} = 2\text{ADF.}$$

$$\text{Th. AC}^2 + \text{BD}^2 = \text{AB}^2 + \text{BC}^2 + \text{AD}^2 + \text{DC}^2.$$

THEOREM XV.

In Isosceles Triangles, the Square of one of the equal Sides is equal to the Square of any Line, drawn from the Vertex to the Base, added to a Rectangle under the Segments of the Base made by that Line.



In the Isosceles Triangle ABC, if BD be perpendicular, it bisects the Base, and is evident; for $AB^2 = BD^2 + AD^2$; i. e. ADC^2 , or $AD \times DC$. - 20. 1.

Let BE be drawn, at pleasure.

I say, the Square of AB is equal to BE Square, added to the Rectangle AEC.

DEM. AB^2 is equal to $BD^2 + AD^2$. - - - 20. 1.
 and AD^2 is equal to $AEC^2 + DE^2$. - - - 5. 2.
 (for AC , is cut equally in D , and unequally in E).
 wherefore, $AB^2 = BD^2 + DE^2 + AEC^2$.
 But, $BE^2 = BD^2 + DE^2$. - - - - - 20. 1.
 Therefore, $AB^2 = BE^2 + AE \times EB$, or AEC^2 .

THEOREM XVI.

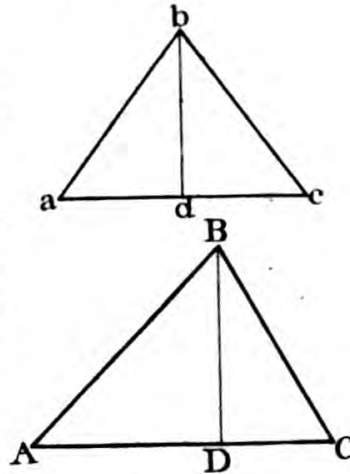
If a Perpendicular be drawn from any Angle of a Triangle to the opposite Side; the Squares of the Sides, containing that Angle, added to the Squares of the alternate Segments of the Base, are equal; and the difference between the Squares of the Sides, is equal to the difference of the Squares of the Segments.

In Isosceles Triangles the thing is manifest.

In the Scalene Triangle ABC , draw the Perpendicular BD .

Then, the Square of AB added to DC Square is equal to BC Square added to AD Square.

And the difference, between the Squares of AB and BC , is equal to the difference, between the Squares of AD and DC .



DEM. For, $AB^2 = AD^2 + BD^2$.

And, $BC^2 = BD^2$ added to DC Square. - - 20. 1.

for the Angles, ADB and BDC , are Right ones. - Con.

Wh. $AB^2 + BD^2 + DC^2 = BC^2 + BD^2 + AD^2$.

that is, $AB^2 + DC^2$, is equal to $BC^2 + AD^2$. - Ax. 3.

Let there be taken, from both, BD^2 , which is common; there is left $AB^2 + DC^2$ equal to $BC^2 + AD^2$.

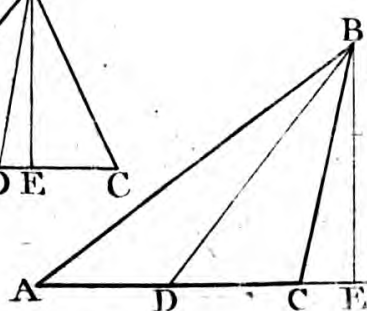
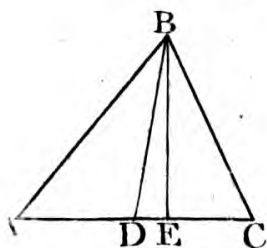
2nd. $AD^2 = AB^2 - BD^2$; & $DC^2 = BC^2 - BD^2$.

Therefore, $AD^2 - DC^2 = AB^2 - BC^2$.

THEOREM XVII.

If any Side of a Triangle is bisected, and a Right Line be drawn from the opposite Angle to the bisecting Point; the Squares of the other two Sides of the Triangle, will be equal to twice the Square of the bisecting Line, added to half the Square of the Side bisected.

If the Triangle be Isosceles, the thing is clear; for the Squares of the equal Sides, are equal to twice the Square of the Perpendicular, added to twice the Square of half the Base.



Let the Triangle ABC be Scalene; let the Side AC be bisected in D; and let BD be drawn.

Then the Squares of AB and BC, together, are equal to twice BD Square, added to twice AD or DC Square.

Draw the Perpendicular BE.

DEM. $AB^2 = AE^2 + BE^2$; and $BC^2 = BE^2 + EC^2$.

Wherefore, $AB^2 + BC^2 = AE^2 + EC^2 + 2 BE^2$.

But $AE^2 + EC^2 = 2 AD^2 + 2 DE^2$. - - 9. & 10.

Wh. $AB^2 + BC^2 = 2 AD^2 + 2 DE^2 + 2 BE^2$.

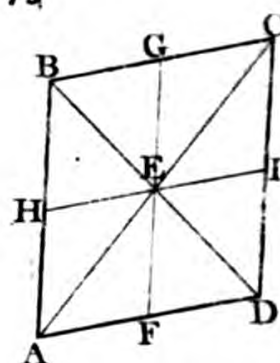
But, $BD^2 = BE^2 + DE^2$; conf. $2 BD^2 = 2 BE^2 + 2 DE^2$

Th. $AB^2 + BC^2 = 2 BD^2 + 2 AD^2$, equal ACD^2 , or half AC^2 . Q. E. D.

By this Theorem, may also be demonstrated the 13th; viz. The Squares of the two Diagonals of every Parallelogram, are equal to the Squares of all the four Sides.

In the Par. ABCD (having drawn the Diagonals) through the Center, E, draw FG and HI, parallel to AB and AD.

In the Tri. AED, $AE^2 + ED^2 = 2EF^2 + AF^2 + FD^2$
 conf. $AE^2 + ED^2 = AH^2 + AF^2 + FD^2 + DI^2$.
 And, $BE^2 + EC^2 = HB^2 + BG^2 + GC^2 + CI^2$.
 But, $AE^2 + EC^2 = \frac{1}{2}AC^2$; & $BE^2 + ED^2 = \frac{1}{2}BD^2$.
 Also, $AF^2 + FD^2 = \frac{1}{2}AD^2$; & $AH^2 + HB^2 = \frac{1}{2}AB^2$.
 and, the $\square$ s of BG, GC, CI, & ID $= \frac{1}{2}BC^2 + \frac{1}{2}CD^2$.
 Th. $AC^2 + BD^2 = AD^2 + AB^2 + BC^2 + CD^2$.



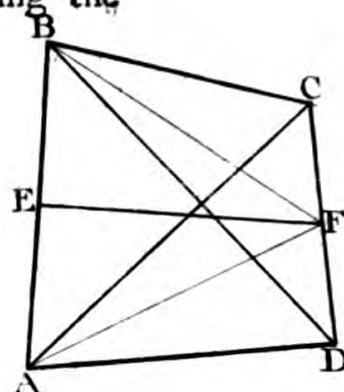
THEOREM XVIII.

In any Trapezium, if the middle Points of two opposite Sides be joined by a Right Line; the sum of the Squares of the two other Sides, together with the Squares of the Diagonals, is equal to the sum of the Squares of the bisected Sides, added to four times the Square of the Line joining the middle Points.

Let the Sides AB and CD of the Trapez. ABCD be bisected in E & F; and draw EF.

I say, the Squares of the two Sides, AB and CD, added to four times the Square of EF, is equal to the Squares of BC and AD, together with the Squares of AC and BD.

Draw AF and BF.



DEM. Now, $AF^2 + BF^2 = 2AE^2 + 2EF^2$. - Th. 17.

and, AD^2 added to $AC^2 = 2AF^2$ added to $2FD^2$;

also, BC^2 added to $BD^2 = 2BF^2$ added to $2FD^2$.

conf. AD^2 added to AC^2 added to BC^2 added to BD^2 is equal to $2AF^2$ added to $2BF^2$ added to $4FD^2$.

But, AF^2 added to $BF^2 = 2AE^2$ added to $2EF^2$.

conf. $2AF^2$ added to $2BF^2 = 4AE^2$ added to $4EF^2$ Square.

But, $4AE^2$ Square is equal to AB^2 Square;

and, $4FD^2$ Square is equal to CD^2 Square. - Cor. to 4.

Therefore, $AD^2 + AC^2 + BC^2 + BD^2 = AB^2 + CD^2 + 4EF^2$ Square. Q E. D.

E L E M E N T S
O F
G E O M E T R Y.
B O O K III.

THE knowledge of the properties of a Circle is the subject of the third Book of Elements. As a Circle is the most perfect of Plane Figures, the properties peculiar to it, are most extraordinary: some of which are really surprising, and are very extensive in their application.

The general Definitions of a Circle, and its attributes, are so well known, and, as they are given in the General Introduction, in Def. 19, and the four following, it would be impertinent to repeat them here; save only what are particularly necessary. The most self-evident Propositions I have reduced into Axioms, for the more expeditiously attaining the knowledge of other, more essential, Properties.

My chief aim, throughout the whole, being to render the Study of Geometry easy, and agreeable to young Students; for which end, I consider, brevity, if clear, most conducive thereto; and am persuaded, that if the knowledge of the most simple properties of Figures can be attained, by any other means, Demonstration is useless.

If a perfect knowledge of all that is contained in the Axioms of this third Book, be not acquired, by a bare recital, and inspection of the Figure, I shall pronounce that person's capacity insufficient, to pursue the Study of the more sublime properties of the Circle; and, if they are clearly evident without, of what use is Demonstration?

Where

Where the truth or evidence of an Assertion cannot be perceived, otherwise, Demonstration is absolutely necessary; but, where the evidence of sight is sufficient, the other is, in a great measure, useless.

There are some critical Geometricians who will not admit of any thing without Demonstration; although they cannot but acknowledge many Propositions to be self-evident truths. It would, perhaps, be no easy matter to give geometrical Demonstration, that all Right Lines drawn from the Center of a Circle to the Circumference are equal; at the same time, no Person of common sense can deny it, who knows any thing of the Figure, and genesis of a Circle. If, therefore something must be given or granted, why not others, which are as clear and self-evident? for my part, I freely own, I can see no reason against it; and have, therefore, pursued the readiest method for attaining the end aimed at; viz. to acquire a knowledge of Geometry in the most easy and familiar manner possible, by divesting it of all that is superfluous and unnecessary; I mean unnecessary Demonstration, of what is clear without it.

I think it, however, necessary to apologize for the liberty I have taken in abridging it. Instead of 31 Theorems, in this third Book, according to Euclid, I have no more than 15, from Euclid; the rest are disposed of in the manner following.

The 2nd, the 5th and 6th, the 10th and 13th, are made Axioms.

The 9th is a Corollary to the 5th.

The 18th and 19th are Corollaries to the 8th; and the 23d and 24th may be deduced from the 3rd.

The 11th and 12th are both included in the 7th.

The 26 and 27th are the 2nd Corollary to the 9th; the 28th and 29th, Corollary 2nd of the 3rd; and the 37th is the 3rd of the 16th.

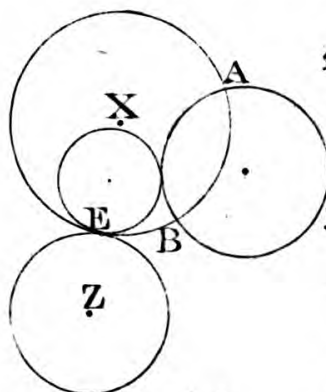
The six Problems are in Practical Geometry.

D E F I-

DEFINITIONS.

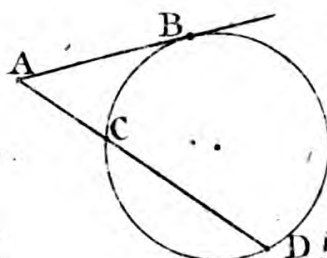
For the definition and genesis of a Circle, see Def. 19, and 20, in the General Introduction.

DEF. 1. EQUAL CIRCLES are such as have equal Diameters or equal Radii.



2. Circles are said to cut one another, when their Circumferences cross or intersect each other in two Points. A and B.

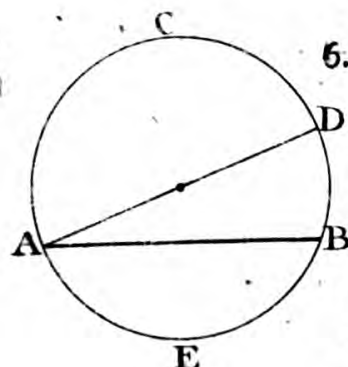
3. Circles are said to touch, when their Circumferences meeting, either internally or externally, in any part, they do not cut each other. As X and Z touching in the Point E.



4. A TANGENT. A Right Line is called a Tangent, when, being in the same Plane with a Circle, it touches the Circumference, only, without cutting it. As AB in B.

And, the Point B, in which it touches the Circle is called the POINT OF CONTACT.

5. A SECANT is a Right Line, drawn from any Point without a Circle cutting the Circumference in two Points. As AD, in C and D.



6. A SEGMENT, of a Circle, is any portion of a Circle, cut off by a Right Line; which is called a CHORD LINE, or SUBTENSE.

As AB making two Segments.

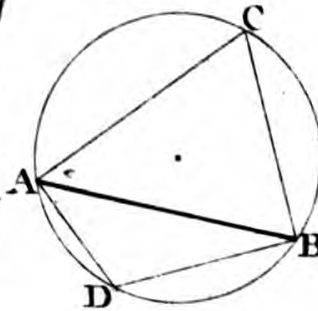
If the Segment be greater or less than a Semi-circle (ACD or AED) it is called a GREATER or LESSER SEGMENT.

7. The Angle of a Segment, is the mixed Angle which is contained under the Chord Line and a portion of the Circumference. As ABC or ABE.

The

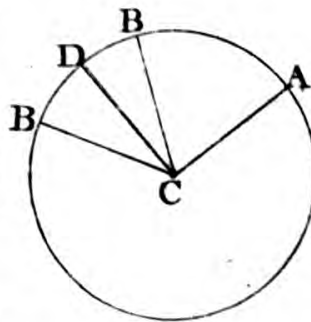
The Angle of a greater Segment is obtuse, of a lesser Segment it is acute.

8. An Angle in a Segment is that which is contained by two Right Lines, drawn from each extreme of its Base, or Subtense, to any Point in the ark of the Segment. As ACB or ADB .



9. Similar Segments are such as contain equal Angles, or whose Angles are equal.

10. A SECTOR of a Circle, is comprehended between two Radii or Semi-diameters, and an Ark, or portion of the Circumference, intercepted between them. As AC , CB and the Ark AB .



If the Radii AC , CD contain a Right Angle, it is called a QUADRANT; as ADC .

11. An Angle in a Circle is said to insit or stand upon an Ark of the Circumference, which is opposite to the Angle; or that portion of the Circumference below the Chord Line.

The Circumference, or curved Line which bounds a Circle, is concave towards the Center; and, externally it is convex.

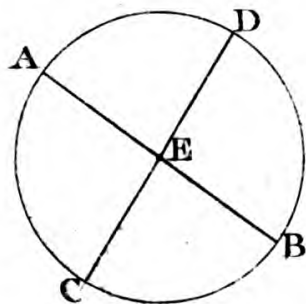
A X I O M S.

The Six Axioms, or self-evident Propositions, which follow, contain properties of a Circle which are necessary to be known, previous to the Demonstration of the Theorems which follow after. I call them Axioms, because they are self-evident; but since that will not, by some, be allowed, arbitrarily, I have endeavoured to illustrate them, and to make the truth appear clear and manifest.

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AXIOM 1st. All Lines drawn from the Center of a Circle to the Circumference are equal. As EA, EC, &c.

For, they are all Radii or Semidiameters.

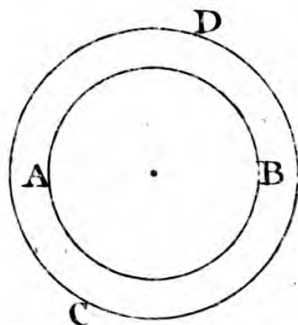


2nd. Two or more Diameters, of a Circle, mutually bisect each other.

For they all pass through the Center; consequently, they are bisected, in the Center.

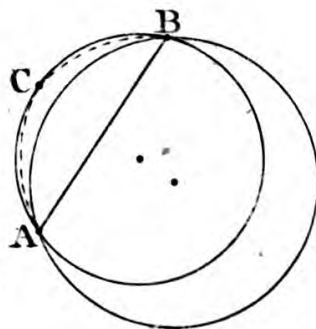
As AB and CD, in the Point E.

3rd. Circles, in the same Plane, which cut, or touch each other, inwardly (outwardly it is manifest) have not the same Center.



For, if they have the same Center, and equal Radii, they must agree in every part. And, if they have the same Center, and unequal Radii, they are parallel Circles, and can neither cut nor touch each other, in any part; for they are, every where, equi-distant. As AB, and CD.

4th. Circumferences of Circles cut each other in two points only.



For, if it were possible to cut, or touch only, in three Points, A, B, & C, each of the points of Section, will be equally distant from the Center of each Circle; and consequently they will have the same Center, which is contrary to the 3rd.

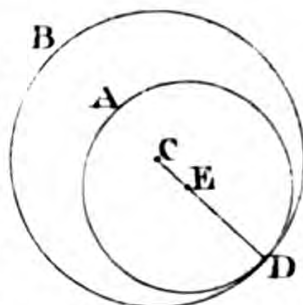
5th. A Right Line joining two Points (A and B) in the Circumference of a Circle, falls entirely within the Circle. This is manifest.

For if not, the Curve of the Circle, ACB, must coincide in some entire part, with the Line AB; which is contrary to the 19th Definition, in the General Introduction, and the Genesis of a Circle, in N. B. Def. 20.; from which it is manifest, that the Curve of a Circle cannot fall in with a Right Line, in any part, it being uniform in every part.

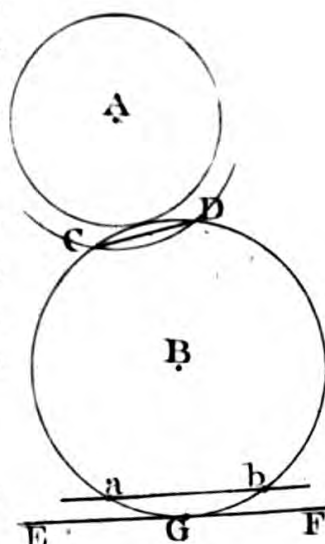
6th. Cir-

6th. Circles touch each other, or a Right Line, in one Point only.

1. For, if the Circles, AD and BD, touched inwardly, in more than a Point, as at D, the Curve of the lesser Circle, AD, must coincide in some part, entirely, with the curve of the larger Circle, BD, which, from the genesis of a Circle, cannot be; seeing, they have unequal Radii, CD and ED; which produce different Curves, according to the Radius, which cannot, in any Part, fall into each other (for if they did they must coincide entirely in some part) therefore, they touch each other but in one Point.



2. If the Circles, A and B, touched outwardly, in more than a Point, they must cut each other; or their Circumferences will be a Right Line, in some part, as CD; which cannot be, for, it falls within both Circles; by the 5th.



3. The Circle, B, touches the Right Line, EF, but in one Point, at G.

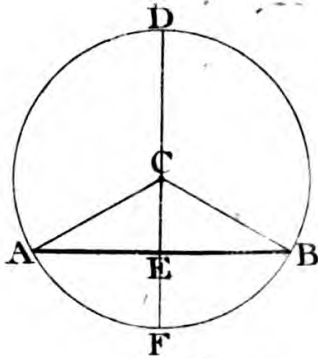
For, if it touched in more than a Point, it must coincide, in some entire part, with the Right Line EF, which cannot be (Def. 19. and 20.)

If any other Line, ab, be drawn between the Line EF and the Center, it will cut the Circle in two points of its Circumference, and the part, ab, of that Line falls within the Circle; by the last.

As this Property of Circles can only be known speculatively, I presume, that, what I have said is as convictive, and satisfactory, as any Demonstration whatever,

THEOREM I. 3. Euclid.

If a Right Line, drawn through the Center of a Circle, bisects a Chord Line, not drawn through the Center, it will cut it perpendicularly.



In the Circle ADB; let the Right Line DE pass through C, the Center, dividing the Chord Line AB into two equal parts, at E.

I say, the Line DE will be perpendicular to AB.

From the Center, C, draw CA, CB.

DEM. The Triangles ACE, ECB are equilateral and equiangular to each other.

For, CE is common to both; $AE = EB$, - - - Hyp.
and AC is equal to CB. - - - - - Ax. 1.

Wh. because $AC = CB$ the Angle $CAE = CBE$. - Th. 9. 1.

Also, the Angle $AEC = CEB$. - - - - - 8. 1.

consequently, they are Right ones. - - - C. 2. 1. 1.

Therefore, DCE is perpendicular to AB - Def. 10. & 11. 1.

COR. 1. A Chord (AB) being bisected by another Right Line, at Right Angles with it (in E) the perpendicular Line, (DE) will pass through the Center of the Circle.

COR. 2. A Perpendicular drawn from the Center of a Circle to a Chord Line, bisects the Chord; and also the Ark of that Chord. As AFB, in F.

Hence, a Right Line, or an Ark of a Circle, is bisected.
Prob. 8th.

THEO-

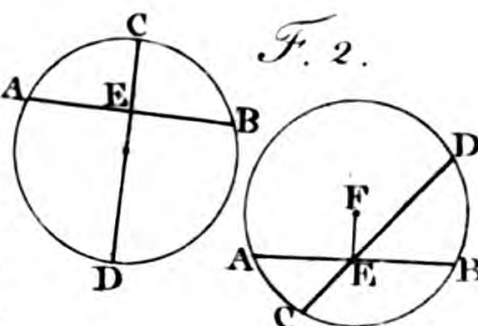
THEOREM II. 4. Euclid.

If, in a Circle, two Chord Lines cut each other, and are not both drawn through the Center, they cannot bisect each other.

Let AB and CD be two Chord Lines cutting each other, in E.

If one of them, CD (Fig. 1.) passeth through the Center, it is evident, that it cannot be bisected by the other, AB, which does not pass through the Center.

If neither of them passes through the Center, one of them may be bisected, but both cannot.

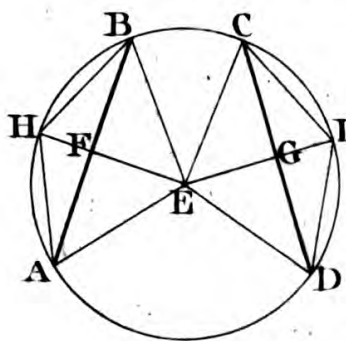


DEM. For, if AB (Fig. 2.) is bisected, in E, a Right Line, FE, joining the Center, F, and the point of Section, E, will be perpendicular to AB. - - - Th. 1. consequently, AEF & FEB are Right Angles. - Def. 10. 1. And, if CD be also bisected, in E; FE is, also, perpendicular to CD. - - - Th. 1. Wherefore, CEF and FED are Right Angles. - - Sup. But, AEF, FEB are Right Angles. - - - proved. consequently, CEF, FED are not Right Angles. - Ax. 2. 1. for it is evident, that one is greater and the other less. Therefore, CD is not bisected in E. Q. E. D.

THEO.

THEOREM III. 14. Euclid.

Equal Chord Lines, in a Circle, are equally distant from the Center; and Chord Lines which are equi-distant, are equal.



Let AB and CD be equal Chord Lines in the Circle ADB.

I say, they are equally distant from the Center, E.

From E, the Center, draw EF and EG perpendicular to AB and CD; and join AE, EB, EC, and ED.

DEM. Now, AB is equal to CD, by the Hypothesis.

and AE, EB, EC, and ED are all equal. - - Ax. 1.

wh. the Triangles AEB, CED are congruous. - Def. 44.

(for, they are equilateral, to each other, by Construction) consequently, they are equiangular, - - - 7. 1.

And, being also Isosceles, the Angles at A and B, C and D, are all equal amongst themselves. - - - 9. 1.

Now, since AB & CD are bisected in F & G, $AF = DG$.

Wherefore, in the Triangles AFE, EGD, the Sides, AE, AF, are equal to ED, DG, respectively;

and they contain equal Angles, $\angle FAE = \angle EDG$. - proved.

Therefore, EF is equal to EG. - - - - 8. 1.

2nd. AB and CD are two Chord Lines, equally distant from the Center of the Circle ADB. I say, $AB = CD$.

Draw the equal Lines, EF, EG, perpendicular to AB and CD; which will bisect them, in F and G. - Cor. 2. 1.

Draw AE and ED.

DEM. Because $AE = ED$, $AE^2 = ED^2$; & $EF^2 = EG^2$.

But, $AE^2 = AF^2 + EF^2$; & $ED^2 = EG^2 + GD^2$.

conf. $AF^2 + EF^2 = EG^2 + GD^2$; wh. $AF^2 = GD^2$

Therefore, $AF = GD$ (Ax. B. 2.) & conf. $AB = CD$ - Ax. 5. 1.

The second Part of this is the converse of the former; which, I should have made a Corollary to it; but on account of the different manner of Demonstration; by the last, of which, Euclid demonstrates both.

COR. 1. Equal Chord Lines, in the same or equal Circles, subtend equal Angles at the Center.

For, since $AB=CD$, and AE , EB , EC , and ED are all equal, the Triangles AEB , CED are congruous; conf. the Angle $AEB=CED$. - - - - 7. 1.

And, since equal Circles have equal Radii, the Angles which equal Chords subtend, in equal Circles, are also equal.

COR. 2. Equal Chords, in the same or equal Circles, subtend equal Arks, and cut off equal Segments. And, equal Arks have equal Chords.

$AB=CD$; and, since $EF=EG$, if EF and EG be produced, to H and I , $FH=GI$. - - - - Ax. 7. 1.

And, since EF and EG are perpendicular to AB and CD , the Arks AHB , CID , are bisected, in A and I . - C. 2. 1. wherefore, the Triangle $AHB=CID$;

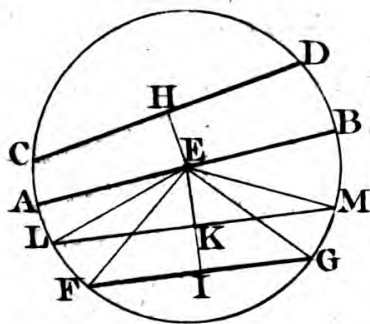
the Arks, AH , HB , CI , and ID , are all equal, and the whole Ark AHB is equal to CID . - - - Ax. 5. 1.

For, if AB was applied to CD , being equal, they would perfectly agree; the Point A with D , and B with C ; also, the Point H would agree with the Point I , and every other Point, in the Ark AHB , with a corresponding Point, in the Ark CID ; consequently, the whole Ark, AHB , would coincide with the Ark CID ; therefore the Segment $AHB=CD$; and, conf. $DAHC=ADIB$.

The same may be said of equal Circles, having equal Radii,

THEOREM IV. 15. Euclid.

A Diameter is the greatest Right Line, which can be drawn in a Circle; and, of all other Chord Lines, that is the greatest which is nearest to the Center.



Let AB be a Diameter, and E the Center of the Circle CDG.

CD and FG are Chord Lines, at different distances from the Center.

I say, that AB is the greatest; and FG, the farthest off, is the least of the three.

Draw the Perpendiculars EH and EI, to CD and FG, from the Center.

DEM. Because FG is farther from the Center than CD, EI is greater than EH. (3.) Make EK equal to EH. Draw KL, perpendicular to EK, and produce LK to M; and, lastly, draw EL, EM, EF, and EG.

Now, because EH is equal to EK, and are perpendicular to CD and LM; LM is equal to CD. - Th. 3. But AE + EB (eq. LE + EM) is greater than LM. - 13. 1. Therefore, AB is greater than LM; i. e. than CD, eq. LM.

2nd. In the Triangles LEM, FEG, the two Sides LE, EM, are equal to the two Sides FE, EG, and the Angle LEM is greater than FEG.

Therefore, FG is less than LM; i. e. than CD, which is equal to LM. - - - - Cor. to 8. 1.

THE Q.

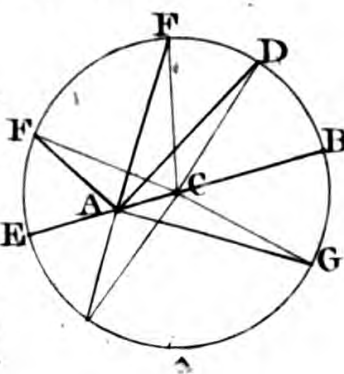
THEOREM V. 7. Euclid.

If any Point, except the Center, be taken within a Circle, and, from that point, divers Right Lines be drawn to the Circumference; the greatest of those Lines is that which passes through the Center; and, the least is the remainder of that Line, produced to the Circumference.

Of all others, drawn from that Point, that which falls nearest to the Line, passing through the Center, is greater than the more remote ones; and, but two equal Right Lines can be drawn, from that Point, to the Circumference.

Let A be the Point assumed, in the Circle EFG; from which draw AB, AD, AE, &c.

First; AB, which passes through the Center; C, is greater than AD, or AF, or any other Line, which can be drawn from the point A. Draw DC.



DEM. In the Triangle ADC, the two Sides, AC, CD, are greater than the remaining Side, AD. - - 13. 1.
But, CD is equal to CB, - - - Ax. 1.
wherefore, $AC + CD = AC + CB$, i. e. AB. π Ax. 6. 1.
Therefore, AB is greater than AD.

2nd. AE is the shortest Line, which can be drawn from the Point A. Draw any Line, AF, and join FC.

Then, in the Tri. AFC, $AF + AC$ is greater than CF.
But, CF is equal to CE, - - - Ax. 1.
Wh. (taking away AC) AF is greater than AE - Ax. 8. 1.

B b

3rd. That

3rd. That Line is the longest, which falls nearest to AB.

In the Triangles ADC, AFC, AC, CD are equal to AC, CF, respectively; for AC is common, & $CD = CF$ - Ax. 1. But, the Angle ACD is greater than ACF. - Ax. 2. 1. Therefore, the Side AD is greater than AF. - Cor. to 8. 1.

4th. No more than two equal Lines can be drawn, from the Point A, to the Circumference.

For, since AF is proved less than AD, and AB greater, every Line which can be drawn, from A, between D and F, will be less than AD, and between B and D greater; wherefore, no other equal Line can be drawn on that Side EB.

Make the Angle $BCG = BCD$; (Pr. 4.) and draw AG. Then, because the Angle $BCG = BCD$, $GCA = ACD$. For, $BCG + GCA = BCD + DCA$. - Th. 1.1. & Ax. 9. And, because $CG = CD$, and AC is common; we have AC, CG, respectively equal to AC, CD; and the Angle $ACG = ACD$; therefore $AG = AD$ - 8. 1.

And, as no other Line, equal to AG, can be drawn on that Side BE; consequently, no more than two equal Lines can be drawn from the same Point, A, to the Circumference.

COR. Hence it is evident, that, if from any Point in a Circle, more than two equal Lines can be drawn to the Circumference, that Point is the Center.

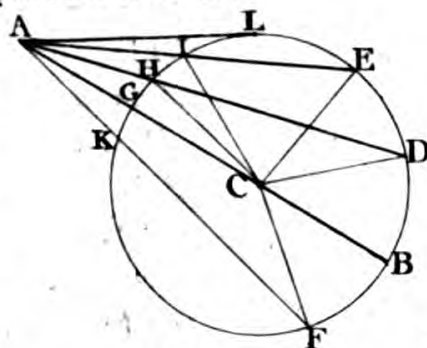
THEOREM VI. 8. Euclid.

If from any Point without a Circle, Right Lines are drawn to the concave part of the Circumference; that which passes through the Center, is the greatest; and that Line which falls nearest to it, is greater than that which is more remote.

Of those, which fall upon the convex part of the Circumference; that is the least, which, if produced would pass through the Center; and that Line, which falls nearest to it, is less than any other, further off. And, no more than two equal Lines, can be drawn from any Point, without the Circle, either to the convex or concave part of the Circumference.

Assume the Point, A, at pleasure, and draw AB, AD and AE.

First; AB which passes through the Center, C, is greater than AD or AE, not passing through the Center. Draw CD.



DEM. Then, in the Triangle ACD, the two Sides AC, CD are greater than AD. - - - - - Th. 13. 1.
But, $CD = CB$; wherefore, $AC + CB$, i. e. AB , $= AC + CD$;
consequently, AB is greater than AD.

2nd. AD is greater than AE. Draw CE.

Then, in the Triangles AEC, ADC, the Sides AC, CE, are respectively equal to AC, CD. - - - Ax. 1.
and the Angle ACD is greater than ACE. - - Ax. 2. 1.
therefore, AD is greater than AE. - - Cor. to 8. 1.

B b 2

3rd.

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3rd. AG is less than AH or AI. Draw CH and CI.

Then, in the Tri. ACH; $AH + HC$ is greater than AC .

But, GC is equal to HC ; - - - - Ax. 1.

wherefore, $AC - GC$, i. e. AG , is less than AH - Ax. 8. 1.

4th. AH is less than AI, which falls further from AC.

For, in the Tri. AIC, the two Sides, AI, IC, are greater than AH, HC, to any Point, H, within the Triangle - 14. 1.

But $IC = HC$; wherefore, AH is less than AI. - Ax. 8. 1.

5th. No more than two equal Lines can be drawn from any point (as A) without the Circle, either to the convex or concave part of the Circumference.

For, since all Lines drawn from A, between G and H, are less than AH, and between H and I, greater; no other Line, drawn on that Side AB, can be equal to AH.

But an equal Line, AK, may be drawn on the other Side.

So likewise; all Lines, drawn from A to the concave Periphery, between AB and AD, are greater than AD; and between AD and AE, less; consequently, no other Line can be drawn, on that Side AB, equal to AD.

But, if BF be made equal BD, and AF be drawn, AF is equal to AD. Draw CF; which is equal to CD.

For, the Angle $FCB = BCD$, by Construction, wherefore, $FCA = ACD$ - - - Th. 1, 1. & Ax. 7. 1.

Wh. in the Triangles ADC, ACF, the Sides AC, CD are respectively equal to AC, CF; for CF, CD are Radii; and, the Angle $ACD = ACF$, therefore $AF = AD$ - 8. 1.

No other Line, equal to AF (eq. AD) can be drawn on that Side of AB; wherefore, but two equal Lines can be drawn from any Point, A, without the Circle, either to the convex or concave part of the Circumference.

COR. The greatest Right Line that can be drawn, from a Point without a Circle, to the convexity of the Circumference, is equal to the least drawn to the concave part.

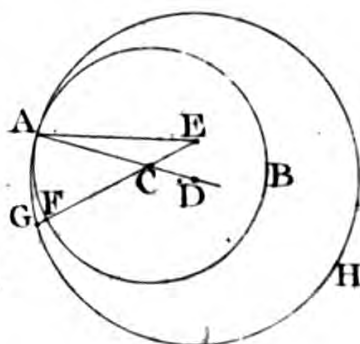
For they unite in a Tangent to the Circle, at L.

THEOREM VII. 11. 12. Euclid.

If two Circles touch each other, a Right Line, joining their Centers, will pass through the Point of contact of the two Circles.

First; let AB and AH be two Circles, touching each other, inwardly, in A. C is the Center of the lesser Circle, AB.

Draw the Right Line AC; it will, if produced, pass through D, the Center of the other Circle.



If not; let any other Point, as E, be the Center of AH, and draw EC, cutting the two Circles in F & G, and join AE.

DEM. Then, because C is the Center of the Circle AB, CA is equal to CF. - - - - - Ax. 1.

If CE be added to both, then $CE + CF = CE + CA$ - 6. 1.

But, $CE + CA$ is greater than AE. - - - Th. 13. 1.

and, EA is equal EG (by the Supposition). Ax. 1.

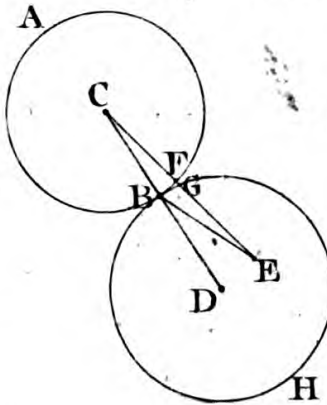
wherefore, EG is less than EF (the greater than the less) which is absurd, and cannot be.

Therefore, E is not the Center of the Circle AH; consequently D is the Center; and, the Right Line DC passes through the Point of contact, A.

2ndly. AB and BH are two Circles, touching each other, outwardly, at B; C is the Center of AB.

Draw the Right Line CB; which, if produced, will pass through the Center, D, of the other Circle.

If it be denied, let E be supposed to be the Center of BH, and draw CE and BE.



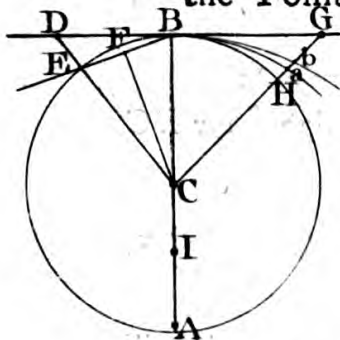
Then, because C is the Center of the Circle AB, CF is equal to CB ;
and, if E be the Center of BH, $EG = EB$.
Wherefore, $CF + GE = CB + BE$. - Ax. 6. 1.
And, if FG be added to the former ; CE, i. e. $CF + FG + GE$, is greater than $CB + BE$.
But, $CB + BE$ is greater than CE - Th. 13. 1
wherefore, $BC + BE$ is both greater and less than CE, which is absurd, and cannot be.

Therefore, E is not the Center of the Circle BH ;
consequently D is the Center : and, the Right Line, CD,
joining the two Centers, C and D, passes through B, the
Point of Contact.

COR. Circles touch each other, either inwardly or outwardly, in a Point only.

THEOREM VIII. 16. Euclid.

If a Right Line be drawn, through the extreme point of a Diameter of a Circle, at right angles with the Diameter, it will fall wholly without the Circle ; and no other Right Line can be drawn, from the Point of contact, between the Tangent and the Circle.



Let AB be a Diameter, and DG a Right Line drawn at right angles with AB, through the extreme B.

Draw CD cutting D in DG.

I say, the Point D is without the Circle.

DEM. In the Tri. CDB, the Angle DBC is a right one ;
conf. BDC is acute ; wh. DC is greater than BC. - 12. 1.
But, the Point B, is in the Circumference.
wherefore, the point D is beyond the Circumference - Ax. 1.
consequently BD falls wholly without the Circle.

Or thus after Euclid.

If BD does not fall without, it will cut the Circle;
suppose in B and E. Draw EC.

Then, in the Triangle EBC, $EC = CB$; - Ax. 1.
wherefore, the Angle $CEB = CBE$.

But, CBE is a right angle (Hyp.) conf. CEB is a right one.
And the two angles, CEB, EBC, of the Triangle CEB,
are equal to two Right Angles.

But all the three Angles of every Triangle = two R. angles.
Therefore, BE, i. e. BD, does not cut the Circle, but
must necessarily fall without it.

2nd. Let BE be drawn (if possible) between DB and the
Circumference of the Circle.

Now, since CBD is a Right Angle, CBE is acute.

Let CF be drawn perpendicular to BE.

Then, CF, subtending an acute Angle, is less than CB.

But, the Point B is in the Circumference.

conf. CF does not reach the Circumference. - Ax. 1.

wherefore, the Point F is within the Circle.

Therefore, the Right Line BE cannot be drawn between
the Tangent, BD, and the Circumference.

COR. 1. A Tangent touches a Circle in one Point only:

For if it touched in two Points it would cut the Circle.

COR. 2. A Right Line perpendicular to the Tangent, at
the Point of Contact, passes through the Center.

COR. 3. A Right Line drawn from the center of the Circle
to the Point of contact, of a Right Line touching the
Circle, is perpendicular to that Tangent.

COR. 4. The Angle of a Semicircle is a right one.

For, it is greater than any right-lined acute angle.

N. B.

N. B. The external Angle, made by a Tangent and the adjoining Ark of the Circle, is less than any right-lined Angle whatever.

For it has been proved, that a Right Line cannot be drawn, from the Point of contact, between the Tangent and the Circumference; wherefore, the Angle GBH cannot be made less by a Right Line, seeing it will cut the Circle, and consequently make a larger Angle, at some other Point in the Circumference.

Yet may the Angle GBH be lessened infinitely, by curved Lines; which is, apparently, a Paradox.

For since the Angle of a Semicircle is Right, and the Tangent BG is perpendicular to the Diameter, AB, there cannot be left any remainder of the Angle as a Complement to it; because, the Complement of an Angle is its deficiency of a Right Angle; or (being obtuse) to two Right Angles.

Now, since it has been proved that a Right Line cannot touch a Circle but in one Point only; it must hold equally true, in respect of a large Circle as of a small one.

Wherefore, if any other Radius, greater than BC, be taken, as Bl, and an Ark, Ba, be drawn, it is evident that the Angle GBa is less than GBH.

If a larger Radius, BA, be taken, and, on the Center A, an Ark, Bb, be drawn, the Angle GBb is less than GBa; notwithstanding, the Angles, ABH, ABa, and ABb, are the same (by the 4th Cor.) and it is evident, that if a still larger Radius be taken, the Angle GBb may still be lessened infinitely.

SCHOL. Hence it is manifest, that any Right Line, GH, is infinitely dividible, by enlarging the Radius BA infinitely; seeing that, the Circumference of a Circle cannot coincide with two Points, B and G, of the Right Line BG.

T H E O-

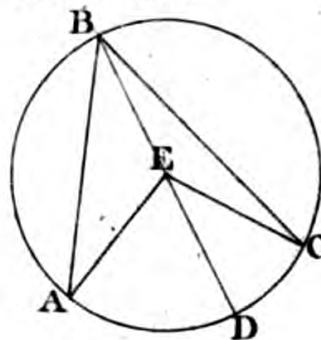
THEOREM IX. 20. Euclid.

An Angle, at the Center of a Circle, is double of an Angle touching the Circumference, when both stand on the same Ark.

In the Circle ABC, let AEC be an Angle at its Center, and ABC is an Angle touching the Circumference; both standing on the same Ark AC.

I say, the Angle AEC is double the Angle ABC.

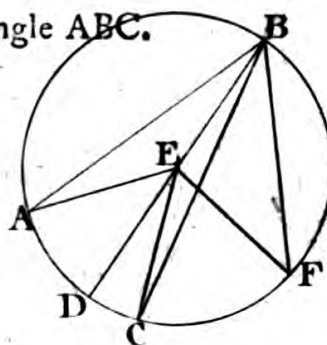
Draw the Right Line BE, and produce it to D.



DEM. In the Triangle ABE, because the side $AE = EB$,
the Angle A is equal to ABE - - - - - 9.1.
But, the external Angle AED = the Angle A + ABE - 10.1
Therefore, the Angle ABD = half the Angle AED
Also $DBC = \frac{1}{2}DEC$; for the Triangle EBC is Isosceles.
Conf. $AED + DEC$ (i. e. AEC) = twice ABD + DBC;
i. e. equal to twice ABC.

Therefore, the Angle AEC is double the Angle ABC.

Case 2nd. When the Angle CEF, at the Center, falls without the Angle CBF, at the Circumference. Draw BED, as before.



Now, the external Angle, $DEC = ECB + EBC$; (10. 1.) and $ECB = EBC$ - 9.1
wherefore, DEC is equal to twice DBC.

And, the Angle $DEF = EFB + EBF$;
therefore. equal twice DBF.

Consequently, $DEF - DEC$, i. e. CEF, = $2DBF - 2DEC$,
i. e. $2CBF$. Therefore, the Angle CEF is double CBF.

C c

Coq.

COR. 1. The Angle at the Circumference, standing on any Ark, is equal to an Angle at the Center, on half that Ark.

For, if the Ark AC be $= CF$, the Angle $AEC = CEF$.
But, the Angle AEC is double ABC , &, CEF double CBF consequently the Angle AEC (eq. CEF) $= ABC + CBF$;
i. e. the Angle ABF , on the Ark ACF , is equal to AEC , (or CEF) on AC , or CF , half that Ark.

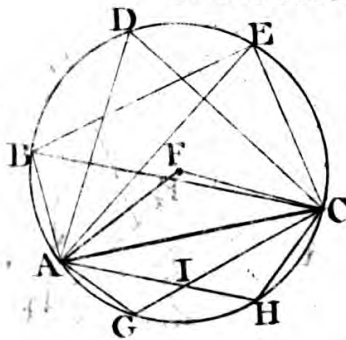
COR. 2. In the same or equal Circles, Angles standing on equal Arks, whether they be at the Center or at the Circumference, are equal to one another.

This is evident from the last; for the Ark AC being equal to CF , the Angle $AEC = CEF$; and, the Angle ABC (eq. half AEC) $= CBF$, half CEF .

And, because the Radii of equal Circles are equal, the Angles on equal Arks, in equal Circles, are also equal. And conversely, equal Angles stand on equal Arks.

THEOREM X. 21. Euclid.

All Angles, which stand on the same Ark, or are in the same Segment of a Circle, and touching the Circumference, are equal to one another.



In the Segment ADC , the Angles ABC , ADC , and AEC , are all equal.

To the Center, F , draw AF and CF , making the Angle AFC .

DEM. Then, the Angle AFC , at the Center, is double AEC , at the Circumference.

For, they stand on the same Ark, AGC , or Chord, AC . But the Angle AFC is double ABC , or any other Angle, ADC , touching the Circumference. - - - Th. 9. Wherefore, the Angles ABC , ADC , &c. being each equal half AFC , are, therefore, equal amongst themselves.

* Theory of Plane Angles, Art. 4. Page 23.

If the Segment be a Semicircle, or less than a Semicircle, it may be demonstrated after this manner.

AGC and AHC are Angles in a lesser Segment, AGHC. The Angle AIC (being external, in respect of the Triangles GAI & ICH) = the Angle G + GAI; and also, to H + HCI; wherefore, the Angle G + GAI = H + HCI. But, the Angle GAI, i. e. GAH = HCI, i. e. HCG; above. Therefore, the Angle AGC = AHC. Q. E. D.

COR. Right Lines, BC and AH, joining the extreme Points of two equal Arks, AB and CH, in the Circumference of a Circle, so as not to cross each, are parallel.

For, having drawn AC) the Arks AB, CH being equal, the Angle BCA is equal to the Angle CAH. - C. 2. 9
Therefore, BC is parallel to AH. - - - Th. 4. 1.

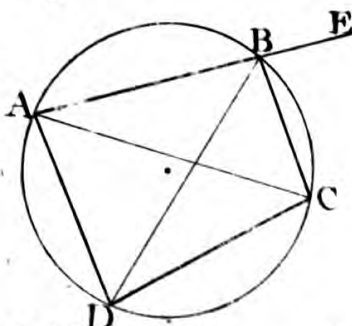
THEOREM XI. 22. Euclid.

In every Quadrilateral, inscribed in a Circle, the opposite Angles are equal to two Right Angles.

ABCD is a Trapezium inscribed in a Circle.

I say, the Angles A and C, also B and D, are, together, equal to two Right ones.

Draw the Diagonals AC and BD,



DEM. Then, in the Triang. ABD, the Angle DAB + ABD added to BDA are equal to two Right ones. - - 10. 1.

But, the Angle BCA = BDA; and ACD = ABD. - 10.

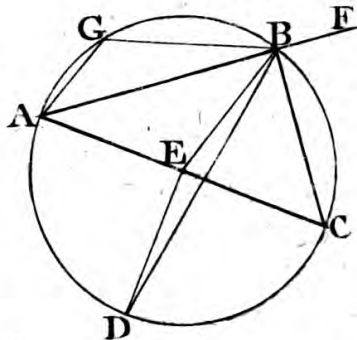
Wh. BCA + ACD, i. e. BCD, + BAD = two Right Angles

COR. 1. The external Angle, CBE, made by producing any Side, AB, of a Quadrilateral inscribed in a Circle, is equal to the opposite Angle, ADC: - by Theo. & 1. 1.

2. Every Parallelogram inscribed in a Circle is a Rectangle.

THEOREM XII. 31. Euclid.

An Angle, at the Circumference, in a Semicircle, is a Right Angle.



In the Circle ABCD, let AC be a Diameter, and E the Center.

To any Point, B, in the Circumference, draw AB and CB.

I say the Angle ABC is a Right Angle.

Draw EB; the Triangles, AEB, BEC, are Isosceles. AE, EB, and EC being equal.

DEM. Then, because $AE = EB$, the Angle $EAB = ABE$ and, because $EB = EC$, the Angle $EBC = BCE$ - 9. 1. wherefore, the Angle $ABC = BAC + ACB$ - Ax. 6. But, if one Angle of a Triangle be equal to the other two, it is a Right Angle. - C. 1. 10. 1. Therefore, ABC is a Right Angle Q. E. D.

Or, if AB be produced, the external Angle CBF is equal to $BAC + BCA$ - Th. 10. 1.

It may be otherwise proved, after this manner.

Draw ED perpendicular to AC, and join BD.

Now, the Angle AED is double the Angle ABD.

And the Angle DEC is double the Angle DBC. - Th. 9.

But, the Angles AED, DEC, are Right ones. - Con.

wherefore ABD, DBC are each half Right - C. 1. 10. 1.

Conf. ABC, i. e. $ABD + DBC$, is a Right Angle.

Or, the Angle ABC at the Circumference, standing on the Ark or Semi-circumference ADC, is equal to an Angle at the Center, AED or DEC, (which are Right Angles) on half that Ark. - Th. 9.

COR.

From this Theorem is deduced that most elegant and expeditious Method for making a Right Angle, or drawing a perpendicular at the extremity of a Right Line; Prob. 10. Method. 1st.

COR. An Angle in a Segment, less than a Semicircle, is obtuse; and an Angle in a greater Segment is acute.

In the Segment ADCB seeing that the Angle ABC is right, ACB, in that Segment, is acute. - - - C3. 10. 1.

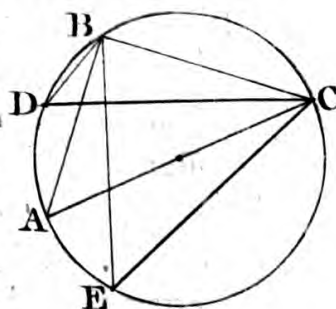
And, the Angle AGB + ACB = two Right Angles - Th. 11.

But ACB is acute; wherefore AGB is obtuse. - Def. 13.

THEO. The Angle in a lesser Segment exceeds, and the Angle in a greater, is deficient of a Right Angle, by the Angle made between the Chord of the Segment and a Diameter, drawn from either extreme of the Chord.

Let AC be a Diameter, and ABC is a Right Angle. Draw two Chords, DC and EC, and join DB and EB.

Then, in the lesser Segment DBC, the Angle DBC is greater than the Right Angle ABC, in a Semicircle. - Ax. 1. And the difference is the Angle DBA,



But, the Angle DBA = DCA. - Th. 10.

Therefore, the Angle DBC, in a lesser Segment, exceeds a Right Angle, by the Angle ACD.

2nd. In the greater Segment EDBC, the

Angle EBC is less than a Right Angle ABC.

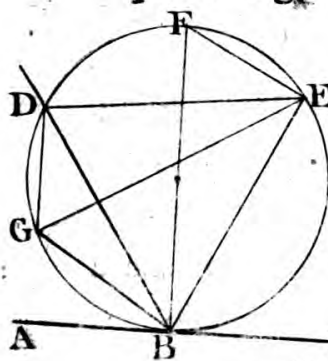
And, the deficiency is the Angle ABE, equal ACE.

Therefore, the Angle, EBC, in a greater Segment, &c.

THEO-

THEOREM XIII. 32. Euclid.

The Angle made between a Tangent and a Right Line, cutting the Circle, drawn from the Point of Contact, is equal to the Angle made in the opposite Segment.



Let ABC be a Right Line, touching the Circle BDE, in B.

Draw at pleasure, from B, the Chord BD.

The Angle ABD is equal to DEB, or any Angle made in the Segment DBE.

And the Angle DBC is equal to DGB, in the other Segment, DBG.

Draw BF perpendicular to AC, and join FE.

DEM. Now BF is a Diameter of the Circle. - - C. 2. 8.
wherefore, FEB is a Right Angle. - - - Th. 12.
But, ABF is a R. Angle (Con.) conf. FEB=ABF, Ax. 9.
But, the Angle FED=FBD (10) Th. DEB=DBA, Ax. 7.

2nd. Let BG be the Chord of the Segment GFEB.
Draw GE, making an Angle, GEB.

Then, the Angle GDB (eq. GEB)=GBA; as above.
And, the Angle G+GDB+DBG=two R. Angles. - 10. 1.
But, the Angle ABD+DBC=two Right Angles. - 1. 1.
and the Angle GDB (eq. GBA)+DBG=DBA. - Ax. 2. 1.
Therefore the remaining Angle, DGB=DBC. - Ax. 7. 1.

Or, in the Quadrilateral DEBG, the opposite Angles,
DGB, DEB, are equal to two Right Angles. - Th. 11.
And the Angle ABD+DBC=two Right Angles. - 1. 1.
But, the Angle DEB=DBA, (as above) Th. DGB=DBC.

THEO-

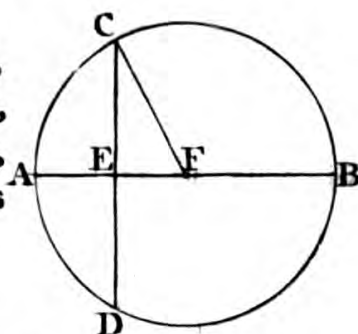
THEOREM XIV. 35. Euclid.

In Circles, if two Chord Lines cut each other ;
the Rectangle, contained under the Segments of
one Line, will be equal to the Rectangle under
the Segments of the other.

1st. When both pass through the Center, it is evident.

For, the Rectangle under the Segments of each, is the
Square of the Radius ; consequently they are equal.

2nd. In the Circle ACBD ; if AB,
passing through the Center, bisects CD,
which does not pass through the Center,
in E ; then, the Rectangle AEB, is
equal to the Square of CE, or ED.

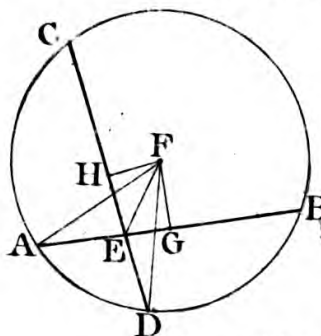


Let F be the Center. Draw CF.

DEM. Then, because AB is bisected, in F, and cut un-
equally, in E ; $AE \times EB + EF^2 = FB^2$ - - 5. 2.
i. e. $= CF^2, = CE^2 + EF^2$. - - - 20. 1.
for, CEF, is a R. Angle (C. 2. 1.) and $CF = FB$. - Ax. 1.
Wherefore, $AE \times EB + EF^2 = CE^2 + EF^2$.
Take away from both, EF^2 , which is common ;
There remains $AE \times EB = CE^2$. i. e. $\square AEB = \square CED$.

3rd. When

3rd. When, neither, of them passes through the **Center**,
and, when neither is bisected, by the other.



From the Center, F, draw FG and FH perpendicular to AB and CD; and join AF, FD, and EF.

Now AB is bisected in G, and cut unequally in E;
wherefore, $AE \times EB + EG^2 = AG^2$ - - - 5.2.
Add, on both sides, FG^2 ;
then, $AE \times EB + EG^2 + FG^2 = AG^2 + FG^2$ - Ax. 6.1:
But $EF^2 = EG^2 + FG^2$; & $AF^2 = AG^2 + FG^2$ - 20.1
Therefore, $AE \times EB + EF^2 = AF^2$.

After the same manner, it may be proved, that the
Rectangle CED + EF^2 is equal to AF^2 .

For, CD is also bisected, in H, and unequally cut, in E.
Wherefore, $CE \times ED + HE^2 = HD^2$. Add HF^2 to both.
Then, $CE \times ED + HE^2 + HF^2$ (equal EF^2) 20.1
is equal to $HD^2 + HF^2$; i. e. equal FD^2 .

But $FD = AF$; wherefore, $FD^2 = AF^2$ - Ax. 1.
consequently, $CE \times ED + EF^2 = AF^2$; (equal FD^2)
But, it was proved, that, $AE \times EB + EF^2 = AF^2$.

Wherefore, taking away EF^2 , from both,
there remains $AE \times EB = CE \times ED$. - - - - Ax. 7.1.
Or, the Rectangle AEB, equal CED. Q. E. D.



THEO-

THEOREM XV.

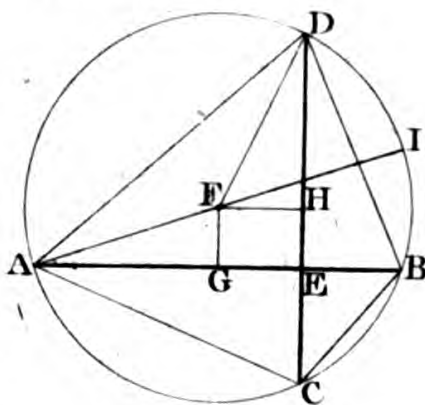
If two Chord Lines, intersect at right angles; the four Squares, of the Segments of those Chords, will be equal to the Square of the Diameter.

And, the Squares of the two opposite Sides of a Quadrilateral, formed by Right Lines, joining the extreme Points of the Chords, are also equal to the Square of the Diameter.

Let the Chords AB and CD cut each other perpendicularly, in E.

Then, the Squares of AE, EB, EC, and ED, are equal to the Square of AI, the Diameter of the Circle.

From the Center, F, draw FG and FH parallel to the Chords CD and AB; and join AF, and FD.



DEM. Now, since AB & CD cut each other perpend. - Hyp.
and, FG, FH, are respectively parallel to CD & AB - Con.
FG and FH are perpendicular to AB & CD. - C.2.4.1.
Then, AB is bisected, in G. (1.3.) and cut unequally, in E.
wherefore, $AE^2 + EB^2 = 2AG^2 + 2GE^2$; - 9.2.
also $CE^2 + ED^2 = 2EH^2 + 2HD^2$ - - - same
Wh. $AE + EB + CE + ED^2 = 2AG + 2GE + 2EH + 2HD^2$
But GFHE is a Parallelogram, by Construction.
wherefore, $FH = GE$, and $FG = EH$ - - - - 15.1.
Conf. the Squares of the four Segments, AE, EB, CE, & ED
are eq. to the Squares of AG, GF, FH & HD, twice taken.
But, $AF^2 = AG^2 + GF^2$; $FD^2 = FH^2 + HD^2$ - 20.1
conf. $2AF^2 + 2FD^2 = 2AG + 2GF + 2FH + 2HD^2$
Wherefore, $AE + EB + CE + ED^2 = 2AF^2 + 2FD^2$;
i.e. $= 4 AF^2$; for, AF is equal to FD.

D d

But

But four times $AF^2 = AI^2$; i. e. of the Diameter. - 4.2
Th. the Squares of the four Segments, AE, EB, CE, & ED,
are equal to the Square of the Diameter.

2nd. Having joined the extremes AD, DB, AC, and CB.
Then, the Squares of AD and CB, together, are equal to
the Squares of AC and DB together.

For, $AD^2 = AE^2 + ED^2$; and, $CB^2 = CE^2 + EB^2$
But, $AC^2 = AE^2 + EC^2$; and $DB^2 = DE^2 + EB^2$
therefore, $AD^2 + CB^2 = AC^2 + DB^2$
consequently, each is equal to the Square of the Diameter.

This extraordinary property of the Circle is otherwise demon-
strated by Mr. Stone; which, notwithstanding it is indisputably
true, does not carry with it that positive conviction; in so much,
that, a young Geometrician would be somewhat at a loss to
perceive it. — It is as follows:

Because, the Angle AED, is a Right one, the Angles EAD, ADE,
are equal to a Right Angle (32.1.) wherefore the Arks, AC, DB,
on which they stand, are, together, equal to half the circumference
of a Circle (26.3.)

Therefore, because the Angle in a Semicircle is a Right Angle,
the Squares of AC and DB are equal to the Square of the
Diameter (47.1.) And, because the Angles at E are Right Angles,
the Squares of AE and EC are equal to the Square of AC, and
the Squares of DE, EB, equal to the Square of DB. Therefore,
the Squares of AE, EC, DE, & EB, are equal to the Squares of AC
& DB; which, were proved equal to the Square of the Diameter.
Therefore those four Squares, are equal to the Square of the
Diameter.

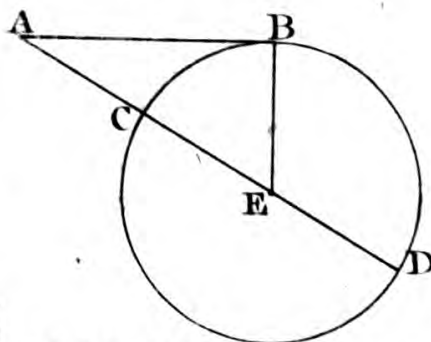
The References are, in this Demonstration, according to Euclid:

SHOL: *It is worthy of observation, that, as, in a Semicircle, the
Sides containing the Right Angle are two Chords perpendicular to
each other, and are equal to the Square of the Diameter; so the
Squares of the Segments of all Chords which cut each other at
right angles, are equal to the Square of the Diameter.*

T H E O-

THEOREM XVI. 37. Euclid.

If, from any Point without a Circle, two Right Lines be drawn, the one a Tangent, the other a Secant (i. e. one touching the Circumference, the other cutting it twice) the Rectangle, under the whole Secant and the external Segment (between the Point, assumed, and the Circle) will be equal to the Square of the Tangent.



A is the assumed Point.

Let AB, touch the Circle, CBD, in the Point B; AD is a Secant, cutting the Circumference in two Points, C and D.

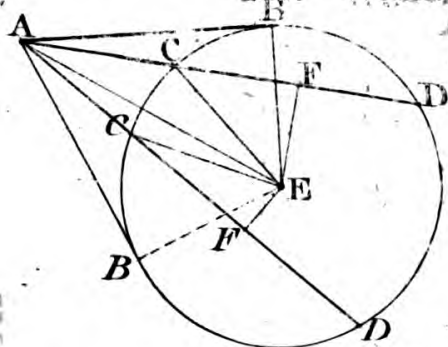
Then, the Rectangle DAC is equal to the Square of AB.

First, when AD passes through E, the Center of the Circle.

Draw EB, from the Center to the Point of contact.

DEM. Now, because CD is bisected, in E, and CA is added to it; $AD \times AC + CE \square = AE \square$ - - 6. 2
 and, $AE \square = AB \square + BE \square$. - - - - 20. 1
 For, ABE is a Right Angle - - - - C. 3. 8
 wherefore, $AD \times AC + CE \square = AB \square + BE \square$.
 But, $BE = CE$; consequently $BE \square = CE \square$. - Ax. B. 2
 wherefore, $AD \times AC + CE \square = AB \square + CE \square$.
 Take $CE \square$ from both, there remains $\square DAC = AB \square$.

2nd. When the Secant falls on either side of the Center.



From the Center, E, draw EF perpendicular to the Secant, AD, or AD, and join the Points CE and EB.

Then, because EF is perpendicular to CD, CD is bisected in F. - - C. 1.1

And, because CD is bisected, in F, and AC is added ;

$AD \times AC + CF^2 = AF^2$ (6. 2.) Add EF^2 to both ;

Then, $AD \times AC + CF^2 + EF^2 = AF^2 + EF^2$ - Ax. 6.1

But, $CF^2 + EF^2 = CE^2$;

and $AF^2 + EF^2 = AE^2$. - - - Th. 20. 1

wherefore, $AD \times AC + CE^2 = AE^2$;

i. e. equal to $AB^2 + BE^2$. - - - 20. 1

But, $BE = CE$; wh. $AD \times AC + BE^2 = AB^2 + BE^2$;

And, if BE^2 be taken from both,

there is left $AD \times AC = AB^2$. Q. E. D.

COR. Hence, it is evident, that the Rectangles under every Right Line, cutting a Circle, from the same Point without the Circle, and the external Segment, are equal.

The Rectangle $DAC = DAC$; for, they are each equal the Square of the Tangent, AB.

2nd. If from the same Point A, without a Circle, two Right Lines AB, AB, are drawn, to the Circle, one on each side, touching the Circumference, those two Tangents are equal.

For, $AB^2 = AB^2$, being each equal to the Rect. DAC

But, equal Squares have equal Sides ;

consequently, AB is equal to AB. - - - Ax. B. 2

3rd. If

3rd. If from any Point, without a Circle, two Right Lines are drawn to the Circle; one of which Lines cuts the Circle, and the other meets the Circumference, in some Point only; and, if the Rectangle under the whole Secant and the external Segment, be equal to the Square of the other Line, that other Line is a Tangent to the Circle, and touches the Circumference in that Point,

THEOREM XVII.

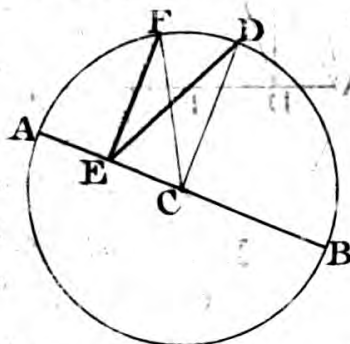
If from any Point, in the Diameter of a Circle, there be drawn two Right Lines to the Circumference; one perpendicular to the Diameter, the other to the middle point of the Ark of the Semicircle; the Squares of those two Lines, together, will be equal to half the Square of the Diameter.

Let AB be a Diameter, C the Center, and D the middle Point of the Semicircle, ADB.

From the Point E, draw EF, perpendicular to AB, and join ED.

I say, the Squares of EF and ED, together, are equal to AC Square twice taken; i.e. to half the Square of the Diameter, AB.

Join CD and CF.



DEM. $CF^2 = EF^2 + EC^2$ - - - - - 20. 1

wherefore, $EF^2 + EC^2 = AC^2$; for $AC = CF$.

But $ED^2 = CD^2 + EC^2$; and $DC = CF = AC$.

Conf. $ED^2 - EC^2 = EF^2 + EC^2$; i.e. $= AC^2$.

wherefore $EF^2 + ED^2 = CF^2 + CD^2$.

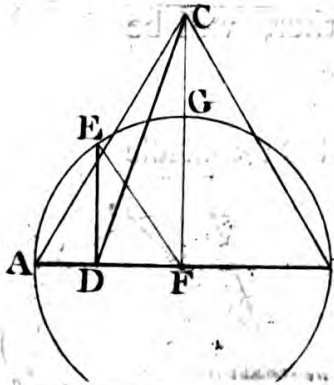
Therefore, $EF^2 + ED^2 = 2AC^2$; i.e. $= AB \times AC$.

THE O-

THEOREM XVIII.

If from the Vertex of an equilateral Triangle, constructed on the Diameter of a Circle, a Right Line be drawn cutting the Diameter; and, from that Point, another Line be drawn, perpendicular to the Diameter, cutting the Circumference; the Squares of those two Lines, together, will be equal to the Square of the Diameter.

From C, the Vertex of the equilateral Triangle ACB, on the Diameter AB, draw CD, at pleasure, cutting the Diameter in D; from which, draw DE perpendicular to AB.



I say, that the Squares of CD, DE, together, are equal to the Square of AB.

If CD be drawn perpendicular to AB, as CF, it is manifest.

For, $CF^2 + FG$ (equal AF) $= AC$ (equal AB) 2 . - - - - - 20.1

Let CF be drawn, perpendicular to AB, and join EF.

DEM. Because CF is perpendicular to AB, and AB is equal to CB, AB is bisected in F; - - - C.3.9.1

wherefore, F is the Center of the Circle. - Def. 20 & 21

Now, $AC = AB$ (Con) and $AC^2 = CF^2 + AF^2$ - 20.1

But, $AF = EF$, and $EF^2 = ED^2 + DF^2$, eq. AF^2 - same; wherefore, $AC^2 = CF^2 + DF^2 + DE^2$.

But, $CD^2 = CF^2 + DF^2$ - - - - - 20.1. consequently, $AC^2 = CD^2 + DE^2$.

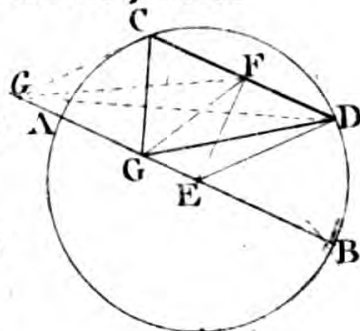
But $AB = AC$ (Con.) Therefore, $AB^2 = CD^2 + DE^2$.

THEO-

T H E O R E M X I X.

If a Chord Line be parallel to a Diameter of a Circle, and, from each extreme of the Chord, Right Lines are drawn to any Point in the Diameter; the Squares of those Lines, together, are equal to the Squares of the Segments of the Diameter, made by that Point.

Let the Chord CD be parallel to the Diameter, AB;
and, to any Point, G, in the Diameter, draw CG and DG.



I say, the Squares of CG, GD, together, are equal to the Squares of AG and GB. Let E be the Center.

Bisect CD, in F; and join FG, FE and ED.

DEM. Then, because CD is bisected, in F,

$$CG^2 + GD^2 = 2FG^2 + 2FD^2 \quad - \quad - \quad - \quad 17.2$$

And, because CD is bisected, EF is perpend. to CD. - 1.

But, CD is parallel to AB; conf. EF is perpend. to AB - 4.1

wherefore, $FG^2 = EF^2 + EG^2$;

$$\text{and, } ED^2 = EF^2 + FD^2. \quad - \quad - \quad - \quad - \quad 20.1$$

conf. $CG^2 + GD^2 = 2GE^2 + 2EF^2 + 2FD^2$ Squares;

i. e. equal to $2GE^2 + 2ED^2$. But AE = ED;

wherefore, $CG^2 + GD^2 = 2GE^2 + 2AE^2$, equal ED^2 , □

$$\text{But, } AG^2 + GB^2 = 2AE^2 + 2GE^2 \quad - \quad - \quad - \quad 9.2$$

therefore, $CG^2 + GD^2 = AG^2 + GB^2$. - Ax. 3.1

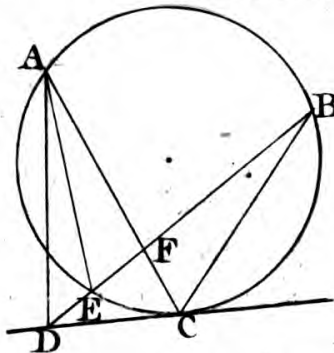
T H E O.

THEOREM. XX.

If two Right Lines are drawn, from any two Points in the Circumference of a Circle, to the same Point in a Tangent to that Circle; those Lines will make the greatest Angle, when they meet in the Point of contact.

Let A and B be the Points assumed, in the Circumference of the Circle ABC.

Draw AC and BC to the Point C, in which a Tangent, DC, touches the Circle; and also, to any other Point, D, draw AD and BD.



If say, the Angle ACB is greater than ADB.

Join AE

DEM. The Angle $AEB = ACB$. - - Th. 10.
standing on the same Ark, AB.

But, the Angle AEB is greater than ADB
for it is equal to $ADB + DAE$ - 10.1.

Therefore, ACB is greater than ADB.

COR. Hence it is evident, that an Angle, AFB, which falls within the Circle, is greater, and an Angle ADB without the Circle, is less, than any Angle, AEB, touching the Circumference, and standing on the same, or an equal, Ark, AB.

From hence may be deduced the following Problem.

Any Right Line being given; and two Points given, or assumed, without the Line; to determine the Point in that Line, to which, if two Right Lines be drawn, from the given Points, they shall contain a greater Angle than any other Right Lines drawn, from those Points, to any other Point in that Line.

A and

E L E M E N T S
O F
G E O M E T R Y.
B O O K IV.

THE fourth Book of Euclid's Elements is not properly elementary, but practical or mechanical; it treats of the inscription and circumscription of right-lined Figures in and about Circles. It is of use in Trigonometry, Astronomy, &c. and also, in Fortification or military Architecture, which seems to depend entirely on it.

As I do not think it proper to deviate from the order of Euclid, in his Books, I have therefore made it the fourth; otherwise, I should have given it a place amongst the other Problems, in practical Geometry, it being entirely problematical. I have given Demonstration of each Proposition, as brief as it will possibly admit of; because, I would not have the Work deficient in any part; which would be sufficient reason, with some Persons, to condemn the whole. Nevertheless I am of Opinion, that great part of it does not require Demonstration; being sufficiently evident from the Construction; especially, as no other part of the Elements is dependant on it.

There are, in some Books extant, sundry other Propositions relative to the inscription and circumscription of right-lined Figures, in and about other right-lined Figures; particularly in Stone's Euclid; and in a curious Tract of practical Geometry by Le Clerc. But, as there is nothing of real utility in them, nor any thing extraordinary to recommend

them to the Curious, I shall not, by useless additions, to this, detain the Reader from matter of greater Importance in the fifth and sixth Books. Indeed this may be passed over entirely, for the present, and proceed immediately to the fifth; there being nothing, in the fourth, necessary to be known previous thereto.

DEFINITIONS.

I have already, in the General Introduction, defined the Terms, to describe, to inscribe, and to circumscribe; a repetition of which would be unnecessary.

1. A Right-lined Figure is said to be inscribed, in a Circle, or to have a Circle circumscribing it, when every Angle of the Figure, touches the Circumference of the circumscribing Circle.
2. A Right-lined Figure is, then, said to circumscribe a Circle, or to have a Circle inscribed, when every Side of the Figure touches the Circumference of the Circle.
3. A Right-lined Figure is said to be inscribed, or to circumscribe a Right-lined Figure; when every Angle of the inscribed one, touches every Side of the circumscribing Figure.
4. So likewise, A Circle is said to be inscribed, in a Right-lined Figure, when it touches every Side of the Figure; or to circumscribe a Right-lined Figure, when the Circumference touches every Angle.

N. B. By inscribing any Figure within a Circle, or any Right-lined Figure within another, is understood, the describing a Figure, like or similar to the given one, the largest possible, to be contained in the other.

5. A Right Line is said to be applied to a Circle, when each extreme is in the Circumference.

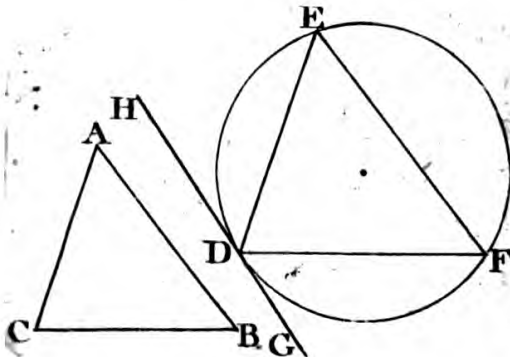
N. B. To inscribe a Right Line in a Circle, is the first Problem of Euclid's Fourth Book, which I think entirely unnecessary, as a Problem, nothing more being required, in the Operation, than to take the given Line, in the Compasses, and placing one point in the Circumference, at pleasure, to cut the Circumference, with the other.

A Right Line, greater than a Diameter, cannot be inscribed in a Circle.

All the Propositions of this Book, follow in the order of Euclid, omitting the First.

PROPOSITION II.

To inscribe a Triangle in a given Circle, equiangular to a given Triangle.



Let ABC be the given Triangle, to be inscribed in the Circle DEF.

Draw at pleasure the right line GH, touching the Circle in any point of its Circumference; as D.

Make the Angle GDF equal to A. i. e. to any Angle of the Triangle; and HDE equal to another, B, and join EF.

DEM. The Angle GDF (equal A) is equal to DEF;
and HDE (equal B) = DFE - - - - - 13. 3
Wherefore, the Angle DEF, being eq. A, and DFE eq. B,
the remaining Angle, EDF, must necessarily be equal to C.
Th. the Triangle DEF is equiangular to ABC. - 10. 1

N. B. By taking the Angles in this order, the Triangle inscribed will be posited as the given one; but any two being taken, the Triangle, inscribed, will be the very same.

Otherwise;

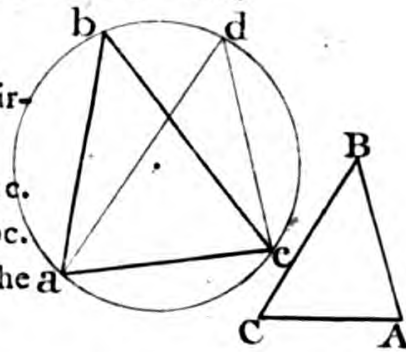
Otherwise :

Draw at pleasure ad , cutting the Circumference in a and d .

Make the Angle, adc , equal to B , & join ac .

Make, cab , equal to the Angle A , & join bc .

The Triangle abc is equiangular to the Triangle ABC .



For, the Angle $abc = adc$ (10. 3.) equal B , by Con.

And, the Angle cab was made equal to A ;

Conf. acb is equal to the remaining Angle C . - 10. 1

Therefore, the Triangle, abc , is equiangular to ABC .

PROPOSITION III.

To circumscribe, that is, to describe or draw, a Triangle, about a given Circle (touching it on every Side) equiangular to a given one.

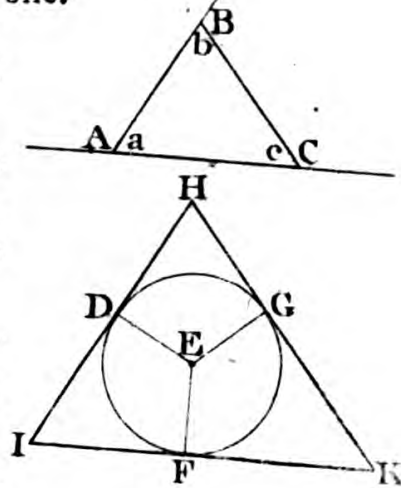
Let ABC be the given Triangle, and DFG the Circle given.

Produce any Side of the Triangle, as AC , both ways; and draw a Radius, DE .

Make the Angle DEF , equal to the external Angle A ; and EFG equal C .

Through the Points D, F , and G ; draw Tangents to the Circle, cutting each other in H, I and K .

Then is HIK the Triangle required.



N.B. Two Angles, DEF , DEG , or FEG , being made equal to any two external Angles of the Triangle, will give the same Triangle, HIK , but differently posited.

DEM:

DEM. Because the Right Lines, HI, IK, and HK, touches the Circle, in the Points D, F, and G; and, from those Points, the Lines DE, FE, and GE are drawn to the Center, the Angles, EDI, IFE, &c. are right ones. - 8.3

And, because the Angles, D, E, F, and I, of every Quadrilateral, are equal to four Right Angles, - 11.3 the Angle $I + DEF =$ two Right ones.

But, the Angle DEF is equal to A, by Construction.

consequently, DIF is equal to BAC. - - 1.1. & Ax. 7.

After the same Manner, the Angle K may be proved equal to ACB; and the Angle H, equal ABC.

Therefore, the Triangle HIK is equiangular to ABC.

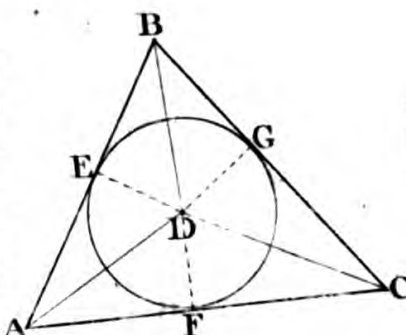
The Reason of this appears obvious, from the two Theorems deduced from the 10th Proposition, Book I. For, since (by the first) all the Angles of every Right-lined Figure are, together, equal to twice as many Right Angles as it has Sides, wanting four; and by the 2nd, all the external Angles are equal to four Right Angles. Also, by the 10th, the three Angles of every Triangle are equal to two Right Angles; and all the Angles, about a Point, are equal to four; (2.1.) consequently, since the three Angles DEF, DEG, and FEG, are respectively equal to the external Angles, A, B, and C, of the Triangle ABC; the Angles H, I, and K are, also, respectively equal to the internal Angles, a, b, and c, of the Triangle.

PROPOSITION IV.

To inscribe a Circle in a given Triangle, (ABC) touching every Side of the Triangle.

Bisect any two Angles of the Triangle, ABC, and CAB, by the Right Lines BD, and AD, cutting each other in D; from which Point, draw a Perpendicular (DE) to any Side (AB) of the Triangle.

With the radius DE, on the Center D, describe the Circle EFG, which will touch every Side of the Triangle, ABC.



Draw DF & DG, perpendicular to the Sides, AC & BC.

DEM. Now, AF is equal to AE - - - C. 2. 16. 3.

and, the Angle EAD is equal to DAF. - - - Con.

Wherefore, in the Triangles AED, AFD, the two Sides AE, AD, are respectively equal to the two Sides AF, AD; and they contain equal Angles.

Therefore DE is equal to DF. - - - 8. 1

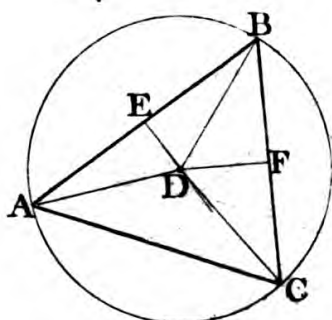
After the same manner, DG may be proved equal to DE, and also to DF.

Wherefore, the three Lines DE, DF, and DG, being perpendicular to the three Sides of the Triangle, and being proved equal among themselves, the Circle, EFG, will pass through the three point E, F, and G; and, consequently, will touch every Side of the Triangle ABC.

P R O-

PROPOSITION V.

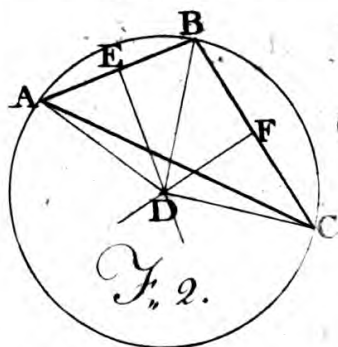
To circumscribe a given Triangle with a Circle;
or, to describe a Circle about a given Triangle.



ABC is the Triangle given.

Bisect any two Sides of the Triangle,
AB and BC, in E and F.

Draw ED and FD, perpendicular to
AB and BC, cutting each other in D,
the Center of the circumscribing Circle.



If the Triangle has a Right Angle, the
Hypotenuse bisected gives the Center.

If an Angle be obtuse, the Center of
a circumscribing Circle falls without the
Triangle; the Operation is the same in
both, (see Fig. 2.)

Join the Points AD, BD and CD.

DEM. Because the Side AB is bisected in E, and DE is
perpend. to AB, the Sides AE, ED, of the Triangle AED,
are equal, respectively, to BE, ED. of the Triangle BED;
and the Angle AED is equal to BED. - - Def. 10.
wherefore, AD is equal to BD - - - - - 9. 1.
for they subtend equal Angles, in the Triangle ABD.

In the same manner may be proved, $CD=AD$, or BD .
Consequently, D is the Center of the circumscribing
Circle, ABC.

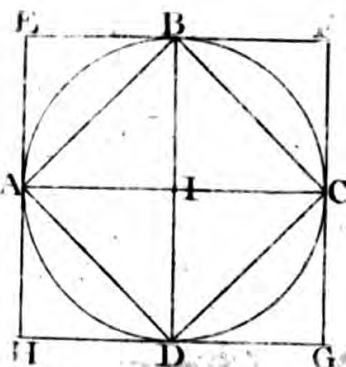
P R O-

PROPOSITION VI.

To inscribe a Square in a given Circle; and to describe a Square about a Circle. ABCD and EFGH.

Draw two Diameters, AC and BD, at Right Angles.
Join the Extremes, A, B, C, D, and a Square is inscribed.

2nd. Draw Tangents through the extreme Points of the Diameters, meeting in E, F, G, and H; or draw Lines, touching the Circle, parallel to each Side of the inscribed Square, and EFGH is a Square circumscribing the Circle.



DEM. The Diameters AC & BD bisect each other, - 16.1
therefore, AI, IC, IB, and ID, are equal;
and they contain equal Angles, $\angle AIB = \angle BIC$ &c. - - Con.
Wh. AB, BC, CD & AD, are all equal amongst themselves.
And, the Angles ABC, BCD, &c being in a Semicircle,
are Right Angles. Therefore ABCD is a Square.

2ndly. Because EF touches the Circle, in the Point B, it is perpendicular to the Diameter BD. - - - 8. 3

And, because HG is perpend. to BD, HG is parallel to EF;

For the same reason, EH is parallel to EG;

Therefore, EFGH is a Parallelogram. - - - Def. 33.

But, AC is perpendicular to BD. (Con.) wherefore, EH is perpendicular to HG, and so is EF to FG and to EH; consequently, EFGH is a Rectangle. - - - Def. 34

But EF, FG, EH, and HG, are each equal to AC, equal BD.

Th. EFGH is equilateral; conf. it is a Square - Def 35

N. B. The Square, EFGH, circumscribing a Circle, is double the Square, ABCD, inscribed in the same Circle.

For the Triangle ABC = half the Rectangle AEFB - - 17. 1

And ADC is equal to half the Rectangle AG;

consequently, the Square EFGH is double ABCD.

F f

P R O.

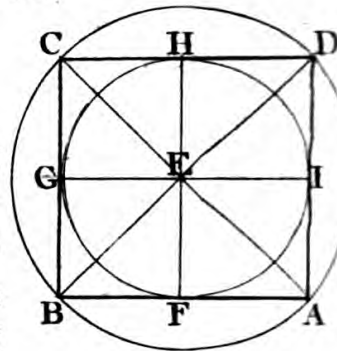
PROPOSITION VII.

To inscribe a Circle in a Square; and to circumscribe
i. e. to describe a Circle about a Square.

Let ABCD be a given Square

Draw the Diameters AC & BD cutting
each other in E.

From the Center E, draw EF perp. to AB.
With the Radius EF, on the Center E,
describe a Circle, which will touch every
Side of the Square ABCD.



Draw EG perpendicular to BC; produce GE to I, and EE to H.

DEM. Now, because ABCD is a Parallelogram, the Diagonals AC & BD bisect each other, in E; - - - 16. 1
And because AB, BC, are respectively equal to AB, AD,
and contain equal Angles; $AC=BD$ - - - 8. 1
consequently their halves are equal. - - - Ax. 1. 1

Then, since AE, EB, and EC are equal, the Triangles
AEB, BEC are Isosceles; wh. AB and BC are bisected by
the Perpendiculars EF & EG, conf. $FB=BG$. - C. 4. 9. 1

But, ABC is a Right Angle, and $AB=BC$. - Def. 35
and since AC is bisected in E, EB is perpendicular to AC
Wh. AEB being a Right Angle ABE, EAB, are half Right;
consequently, BE bisects the Right Angle ABC.

Now since in the Triangles EFB, BGE, the Sides,
FB, BE, are respectively equal to GB, BE; and the Angle
 $FBE=EBG$, the remaining Side $EF=EG$.

After the same manner EH and EI may be proved equal
to EF and EG; conf. the Circle FGH touches every Side
of the Square.

2ndly.

2ndly. Having drawn the Diameters of the Square AC & BD; with the Radius EA or EB, describe the Circle, ABCD.

This needs no further Demonstration.

For, since the Diagonals of a Square are equal, and mutually bisect each other, (16. 1.) the halves EA, EB, EC, and ED are also equal; wherefore, a Circle, ABC, will pass through every Angle of the Square.

N. B. A Circle (FGHI) inscribed in a Square, is equal to half a Circle (ABCD) circumscribing that Square.

For, the Square of $AC = AB^2 + BC^2$; consequently, AC^2 is double BC^2 , equal FH^2 .

But the Areas of Circles are, to each other, as the Squares of their Diameters, - - - - - C. 1. 14. 6. therefore, the Circle ABCD is double FGHD.

PROPOSITION VIII.

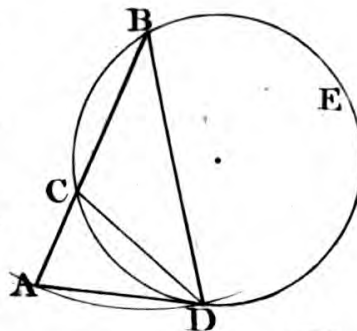
To make an Isosceles Triangle, having its Angles, at the Base, each double the Angle at the Vertex.

Let AB be any given Right Line for one of the equal Sides.

Divide AB in extreme and mean Proportion, in C. (Prob. 35.)

With the Radius AB, on B, describe an Ark, AD.

Make AD equal to CB, and draw BD.



I say; in the Triangle ABD, the Angles A and D, at the Base, are each double the Angle B, at the Vertex.

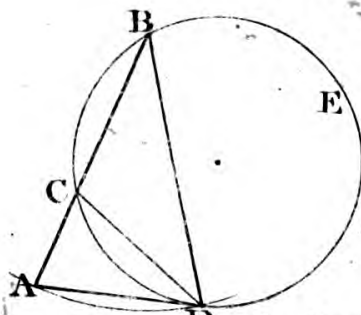
Join CD, and describe a Circle through B, C, and D, (Prob. 40.) or, by the 5th of this.

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DEM. Because the Rectangle $BAC = BC \square$ - - - 11.2.

And $AD = BC$ (Con.) the Rectangle $BAC = AD \square$;

Wherefore, AD , touches the Circle BDE , at D - 16.3



(For A is a Point without, that Circle, from which, two Right Lines are drawn; one (AB) cutting, the other, (AD) touches the Circle;)

Therefore, the Angle $ABD = ADC$. - 13.3

(For, CD cuts the Circle BDE , in the Point of contact, D .)

But, the Angle $ACD = ABD + CDB$, i.e. $= ADC + CDB$;

And, the Angle $BAD = ADB$ (9.1) for $AB = BD$. - Con.

consequently, $BAD = ACD$; wh. $CD = AD$ - C. 3.9.1

But, $AD = CB$ (Con.) wherefore, $CD = CB$; - Ax. 3.1

consequently, the Angle $CDB = CBD$ - - - Th. 9.1

But, the Angle $ADC = CBD$; (proved, above)

wherefore, ADC is equal to CDB . - - - Ax. 3.1

and therefore, ADB , BAD are each double ABD .

The Construction of this particular species of Triangles, in respect of itself, is of little or no use; or, I should have given it a place, amongst the other Problems, in Practical Geometry. But, the inscribing a regular Pentagon in a Circle, according to Euclid, in the next Proposition, depends on it entirely; but, which may, for real use, in practice, be more readily constructed. Nevertheless, for the manner and elegance of its Demonstration, I did not think proper to omit it.

P R O-

PROPOSITION IX.

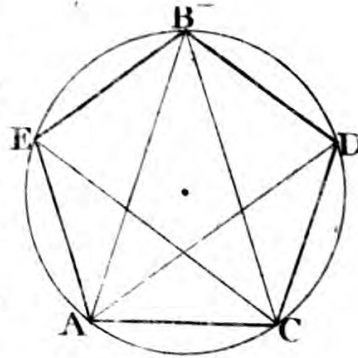
To inscribe a regular Pentagon in a Circle.

ABC is the given Circle.

Inscribe an Isosceles Triangle, ABC, whose Angles BAC, ACB are, each, double the Angle ABC - - Prop. 1 Bisect each of the Angles BAC, ACB by the Right Lines AD and CE, cutting the Circumference in D and E.

Join the Points, A & E &c. by the Right Lines AE, EB, BD, and DC.

AEBDC is a regular Pentagon.



DEM. Now, because each Angle, BAC, ACB, is double the Angle ABC; (Con.) and those Angles are bisected, the Angles, ABC, ACE, ECB, BAD, and DAC, are equal to one another.

But equal Angles stand on equal Arks, - - C. 2. 9. 3 and, equal Arks have equal Chords or Subtenses, - 2. 3. 3 wherefore, the Right Lines AC, AE, EB, &c. are all equal; and consequently, the Pentagon, AEBDC, is equilateral.

I say, it is also equiangular.

For, because the Arks AE, EB, &c. are all equal, the Angles ACE, ECB, &c. are equal.

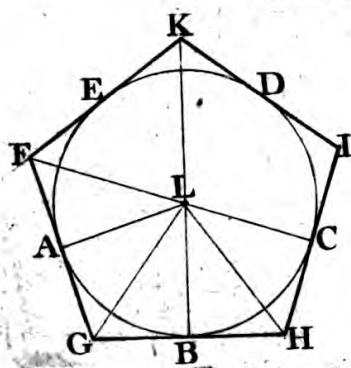
wh. the Angle $EBA + ABC + CBD = EAB + BAD + DAC$. consequently the Angle $EBD = EAC$; and so, of all the rest.

Or, because each Angle of the Pentagon, AEB, EBD, &c. stands on equal Arks, ACDB, FACD, &c; (for each is triple the Ark, AC.)

Therefore, the Pentagon AEBDC is equiangular.

PROPOSITION X.

To describe a regular Pentagon about a Circle.



Let $ABCDE$ be the angular Points of a Pentagon, inscribed in a Circle, through which draw the Tangents $FG, GH, HI, \&c.$ cutting each other, in $F, G, H, I,$ and K , i. e. having, made the Arks $AB, BC, \&c.$ equal, (viz. each equal 72 Degrees) through the Points $A, B, C, \&c.$ draw the Tangents, as before.

The Pentagon $FGHIK$ is equilateral and equiangular.

Let L be the Center of the Circle; draw $AL, BL, \&c.$ also draw $FL, GL, \&c.$

DEM. $AL, BL, \&c.$ are perpendicular to $FG, GH, \&c.$ - 8 3
and they are all equal between themselves - - - Ax. 1. 3

The Angle ALB is equal to BLC , by Construction.

AG is equal to GB , and BH equal to HC - - C.2.16.3

wh. the Triangles $ALG, GLB, BLH, \&c.$ are equilateral and equiangular to each other; for $GB=BH, \&c.$ - 11.1

$AF=AG$, and $CI=CH$, (same) conf. $FG=GH$, - Ax. 1

After the same manner, FK and IK , may be proved equal between themselves, and also to $FG, \&c.$

therefore, $FGHIK$ is equilateral.

And, since the Triangles $ALG, GLB, \&c.$ are equiangular, the Pentagon is also equiangular.

For, the Angles AGL, LGB , also BHL and LHC are equal amongst themselves; consequently, $FGH=GHI$.

After the same manner, the Angles, I and K , may also be proved equal between themselves, and also to $F, G, \& H$.

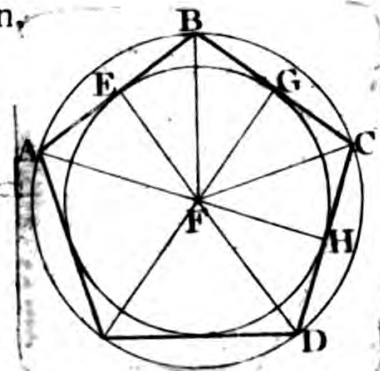
Th. $FGHIK$ is a regular Pentagon, circumscribing a Circle.

Note. If the Circumference of a Circle be divided in five equal Parts, and the Points joined by Right Lines; then if Tangents to the Circle be drawn, parallel to every Chord, a regular Pentagon will be circumscribed.

PROPOSITION XI.

To inscribe a Circle in a regular Pentagon; and, to describe a Circle about a Pentagon.

Bisect any two Sides, AB, and BC, of the Pentagon ABCD, by the Perpendiculars EF and FG, intersecting in F; on which, with the Radius EF or FG, describe a Circle; which, will touch every Side of the Pentagon.



2nd. With the Radius AF or FB (the Point F being found as before, or by bisecting two Angles, ABC and BCD) describe a Circle, which will pass through every Angle of the Pentagon.

DEM. Because $AB = BC$, $EB = BG$. Draw AF, FB, &c. Then, in the Triangles EBF, FBG; $FB^2 = FE^2 + EB^2$. and, $FB^2 = FG^2 + BG^2$; - - - - - 20. 1. consequently, $FE^2 + EB^2 = FG^2 + BG^2$. - Ax. 3. 1. But, $EB = BG$ (Ax. 1. 1.) wherefore, $EB^2 = BG^2$; consequently, $FE^2 = FG^2$; and, therefore $FE = FG$.

In the same manner, FH may be proved equal to FG; F is, therefore, the Center of a Circle inscribed.

2nd. Because AB is bisected, in E, $AE = EB$, & $AE^2 = EB^2$. Add EF^2 , to both; and $AE^2 + EF^2 = EB^2 + EF^2$. But, EF is perpendicular to AB, by Construction; consequently, AEF, FEB are Right Angles. - Def. 10 wh. $AF^2 = FB^2$; and $FB^2 = EB^2 + EF^2$. Consequently, $AF^2 = FB^2$; and therefore, $AF = FB$.

In the same manner, FC, FD, &c. may be proved equal to AF and FB.

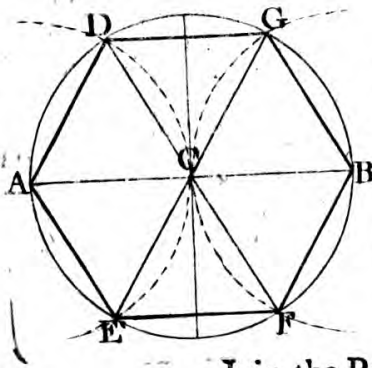
consequently, F is the Center of a circumscribing Circle.

P R O.

PROPOSITION XII.

To inscribe a regular Hexagon in a given Circle.

In the given Circle, AGF , draw a Diameter, AB .



With the Radius of the Circle, AC , on the Centers A and B , describe two Arks, DCE , FCG , cutting the Circumference in the Points D and E , F and G .

Or, having drawn the Ark DCE , only, cutting the Circle in D and E , draw DC & EC . and produce them to F and G .

Join the Points A and D , AE , EF , &c. by Right Lines, which compleats the Hexagon, $ADGBFE$. Q. E. F.

I say, it is both equilateral and equiangular.

DEM. Because C is the Center of the Circle AGF , the Lines DC , AC , and CE are equal; and, because A is the Center of the Circle DCE , AD , AC and AE , are equal; wherefore, ADC & ACE , are equal & equilateral Triangles.

But, the Angles of an equilateral Triangle are, each, equal to one third Part of two Right Angles. - C. 10. 1 (because, the Angles which are opposite to equal Sides are equal; and the three Angles of every Triangle are, together, equal to two Right Angles. - 9 and 10. 1) consequently, $DCA + ACE =$ two thirds of two R. Angles.

Now, because the Right Line EC falls on the R. Line DF , the Angle $ECF + ECD =$ two Right Angles - 1. 1 and, because $DCA + ACE$ (eq. DCE) $=$ two thirds of two Right Angles; ECF is equal one third part of two Right Angles; i. e. equal to ACE , equal ACD .

Again

Again ; because the Right Lines AB, DF, & EG cut one another, in the Point C, the Angles DCG, GCB, & BCF are, respectively, equal to ECF, ACE, and ACD-2.1 Wh. those fix Angles are all equal amongst themselves; conf. they stand on equal Arks; and conf. on equal Chords. Therefore, the Sides, AD, DG, GB, &c. of the Hexagon, are equal to one another.

2nd. Because the Triangles, ADC, CDG, &c. are equilateral, and consequently equiangular, - - - C. 2. 9. 1. the Angle $ADC + CDG = DGC + CGB$, - - Ax. 3. i. e. the Angle $ADG = DGB$; and so of all the rest. Therefore, the Hexagon ADGBFE, inscribed in a Circle, is equiangular; and also equilateral.

COR. The Side of a regular Hexagon, inscribed in a Circle, is equal to the Radius, or Semidiameter of the Circle.

For, the Triangles, ADC, &c. are proved to be equilateral.

Hence, a Duodecagon may be inscribed in a Circle, by bisecting each Ark on the Side of a Hexagon inscribed.

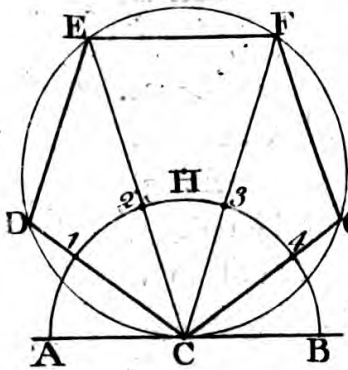
i. e. The Radius of the Circle being applied six times in the Circumference, each Ark bisected gives the Sides of a Decagon.

N. B. Any Polygon may be circumscribed about a Circle, having first inscribed one (of the Species required) by drawing Tangents, through every Angle, or parallel to every Side, of the inscribed Figure.

In Prob. 52, Practical Geometry, is given a general method for inscribing Polygons of every Denomination in Circles; of which Method, there is no Demonstration extant, having consulted some able Mathematicians concerning it.

I shall, here, give another general method, which admits of Demonstration (granting an Ark of a Circle to be divided into any number of equal Parts) which is as follows.

Let DFC be a given Circle, in which it is required to describe a regular Pentagon.



Draw AB, at pleasure, touching the Circle DFC in any Point, C; on which Center, draw a Semicircle AHB, with any Radius at discretion.

Divide the Semi-circumference into five equal Parts, i. e. make the Arks A 1, 1 2, 2 3, &c. each equal 36 Degrees.

Draw C 1, C 2, and produce them till they cut the Circumference, in D, E, F, and G.

Join DE, EF and FG, and it is done.

DEM. Because the Angs DCE, ECF, & FCG are equal - Con. the Arks, and conf. the Chords DE, EF, & FG, are equal, Also, the Angle ACD (eq. DCE by Con.) = DEC. - 13.3 wh. the Triangle CDE is Isotceles; and $CD = DE$. - 9.1

After the same manner CG may be proved equal to FG. consequently DC, DE, EF, FG, and GC are all equal. Therefore, the Pentagon DEFGC is equilateral; and, being inscribed in a Circle, it is necessarily equiangular, each Angle, CDE, DEF, &c. standing on equal Parts of the Circumference, $CGFE = DCGF$.

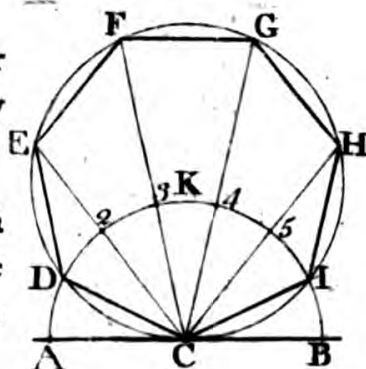
This method of inscribing Poligons is not in Euclid, nor in any Author I am acquainted with. It is neither so operose in the Construction, and is more briefly demonstrated.

Euclids method (for a Pentagon) of inscribing a Triangle similar to another, (which must first be formed) whose Angles at the Base are double of the Vertex, is very ingenious and perfectly Geometrical; but it is liable to great error in the Construction; insomuch, that it is extremely difficult to do it, with accuracy, and tedious in the Operation; whereas, this is both more expeditious and more to be depended on.

After

After the same manner, a Heptagon, or any Polygon whatever, may be readily constructed, as follows.

Draw AB , at pleasure, touching a Circle in C ; on which Center describe a Semicircle, cutting the Tangent in A and B .



Divide the Ark of the Semicircle into as many equal Parts, as the Polygon, required, has Sides, viz. Seven (for a Heptagon) and, through every Division, draw CI , $C2$, &c. cutting the given Circle in D , E , F , &c; which will divide the Circumference into seven equal Parts, in those Points; and, being joined by Right Lines, compleats the Heptagon $CDEFGHI$.

The Demonstration is the same as in the preceding Proposition. But, it may be further observed, that, because the equal Angles ACD , DCE , ECF , &c. are at the Center of the Circle AKB , and at the Circumference of the Circle EHC ; and, because the Angle at the Center of a Circle, on any Ark, is equal to an Angle at the Circumference, on twice that Ark (*C. I. g. 3.*) therefore, the Semi-circumference, of one and the whole Circumference, of the other, are divided into the same number of equal Parts, by the Right Lines, CD , CE , &c.

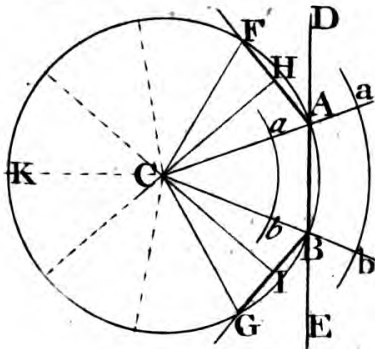
If AC be taken equal to the Radius of the given Circle, it is manifest.

It may be alledged, that the whole Circumference may as easily be divided into any number of equal Parts, as a Semicircumference. 'Tis true it may, but that admits of no other Demonstration, the Sides being all equal by Construction; and is in no wise geometrical.

On account of the singularity of this method, having never seen it in any other work, and because I find it useful in Perspective, in some Cases, I thought proper to insert it.

The Side of any Polygon inscribed in a Circle of any Radius is readily obtained, by making an Angle at the Center, of the number of Degrees, which is the Product, arising from the Division of 360 by the Denomination of the Polygon. e. g. 360 divided by 5 (for a Pentagon) gives 72; for a Hexagon, 60; for an Octagon, 45; for a Nonagon, 40; for a Decagon, 36. A Heptagon does not divide the Number of Degrees exactly; but is nearly, $51\frac{1}{2}$, or more nearly, $51, 43$.

Wherefore, having (by means of a Protractor, or Line of Chords) made an Angle, ACB , at the Center of a Circle, equal 40 Degrees; the Chord AB , of the Ark it stands on, is the Side of a Nonagon inscribed in the Circle FKG ; i. e. AB is one ninth Part of the whole Circumference.



For, let the Angle ACB be constructed or measured on any Ark whatever, as ab , either of a greater or less Radius, the portion AB, of the Circle AKB, will be the same; for AB and ab are each one ninth part of their respective Circumferences.

Hence may be deduced, an universal method for Constructing Poligons of any number of Sides, on a given Right Line, viz. by finding, what Angle the Side of the Polygon, subtends at the Center, as above; which in a Nonagon is 40 Deg. Subtract 40, from 180, two Right Angles; the remainder is 140, half of which is 70.

Construct an Isosceles Triangle, ACB, on the given Line AB, whose Angles, at the Base, A and B, are each of 70 Degrees; the Angle at the Vertex, C, will be 40.

C is the Center of a Circle which will circumscribe a Nonagon, the measure of whose Sides is AB.

With

With the Radius CA or CB describe a Circle, AKB, it will contain AB nine times, applied to the Circumference. For, if the Angle ACB, of 40 Degrees, be repeated 9 times, as in the Figure; it will include all the space of the Circle, equal four Right Angles; and each Angle will be subtended by a Side of the Polygon; which is evident from the Figure.

The three Angles of every Triangle are, together, equal to two Right Angles, i. e. equal to 180 Degrees. (10. 1.) And, since the Angle C, at the Center of the Triangle ACB, is equal to 40 Degrees, consequently, the two remaining Angles, CAB, CBA, at the Base, being equal (9. 1.) they are, each, equal to half the difference between the Angle ACB, and two Right Angles; i. e. of 70 Degrees, each, as by Construction.

Otherwise. The external Angle of every regular Polygon is equal to an Angle at the Center, subtended by a Side.

For, all the external Angles of every Right-lined Figure are equal to four Right Angles (Th. 2. 10. 1.) and so are all the Angles about the Center of a Circle. - - - C. 2. 2. 1

Wherefore, an ordinate or regular Polygon, having all its internal Angles equal, will, likewise, have all its external Angles equal; and, being equal in number to the Angles at the Center, subtended by the Sides, each external Angle is, consequently, equal to an Angle at the Center, subtended by a Side, e. g.

If the Polygon has eight Sides (an Octagon) the external Angle is the 8th part of 360, i. e. 45 Degrees; if a Nonagon, it is 40 Degrees; if a Decagon 36; each being equal to the Angle at the Center, subtended by a Side. Hence.

Let, AB be a Side given for a Nonagon; let it be produced both ways, towards D and E.

At the extremes, A and B, make the Angles DAF, EBG, each equal 40 Deg. and make AF and BG each equal AB.

Bife&

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Bisect AF and BG, in H and I, and draw HC and IC perpendicular to AF and BG, intersecting in C ; which, is the Center of a circumscribing Circle, as before.

It is evident, that the external Angle DAF (equal EBG) is equal to the Angle ACB at the Center.

For, ACB, BCG, and ACF, are congruous Isosceles Triangles; whose Angles at the Base, AB and BG, are 70 Degrees each; two, of which, form an internal Angle of the Nonagon; as $ABC + CBG = ABG$; and, the external Angle, EBG, is the Compliment of two Right Angles; consequently, it is equal to the remaining Angle ACB, of the Triangle ACB, or BCG.

Hence it is also evident; that, if any Right Line, as DE, cuts a Circle, and if from either Point, A or B, of the part AB, within the Circle, another Chord, AF or BG, be drawn, equal AB; the external Angle, DAF or EBG, made by that Chord and the Line DE, is equal to the Angle at the Center of the Circle, subtended by the Chord, AF or BG (equal AB).

From what I have advanced, is deduced the Table, after Prob. 47, Practical Geometry, for constructing Poligons; by dividing the Ark of a Quadrant, or Right Angle, into as many equal parts as the Poligon has Sides.

For, the Ark of a Right Angle being divided into five equal parts, each will be 18 Deg. wherefore, the Angle of a Pentagon being six of those parts, or one added to a Right Angle, i. e. $18 + 90 = 108$; which being subtracted from two Right Angles, or 180 Degrees, the Difference is 72, equal four times 18; which is the Compliment of two Right Angles; consequently, the external Angle of a Pentagon is equal to an Angle at the Center, subtended by a Side.

For a Nonagon, the Ark of a Right Angle is divided into 9 equal parts, each equal 10 Degrees.

Now, the external Angle being 40 Degrees, equal to the Angle at the Center, the Angle of the Nonagon is $50 + 90$ or
five

five ninth parts added to a Right Angle; for, an internal and external Angle are, together, equal to two Right Angles - 1. 1

Therefore a Right Angle is to the Angle of a Nonagon as 9 to 14, difference 5, as in the Table.

The reason of all this is so very obvious, it is almost needless to say more about it; seeing it is manifest, that all the external Angles of any Right-lined Figure, whatever, are, together, equal to all the Angles about a Point, subtended by the Sides. i. e. equal to four Right Angles. And, all the internal Angles of any Right-lined Figure are equal to twice as many Right Angles as the Figure has Sides, wanting four, (Th. 1. 10. 1.) consequently, the external Angles being equal to those four (Th. 2. of the same) are equal to all the Angles at the Center; and, being equal, in number, they are equal in quantity.

Hence, the Angle of any regular Polygon, whatever, may be readily obtained.

For, in a Pentagon, all the internal Angles, together, are equal to six Right Angles; consequently, each Angle is 6 fifths of a Right Angle, i. e. it is equal to one Right Angle and one fifth part of a Right Angle; seeing, there are five Angles in the Figure, and they are, altogether, equal to six Right Angles.

The Angles of a Hexagon are, altogether, equal to eight Right Angles; consequently, each Angle is equal to eight sixths of a Right Angle, or four thirds; i. e. equal to one Right Angle and one third part of a Right Angle.

A Heptagon has all its Angles, together, equal to ten Right Angles; wherefore, each Angle is equal to ten sevenths of a Right Angle, i. e. equal to a Right Angle and three seventh parts of a Right Angle.

All the Angles of an Octagon being equal to twelve Right Angles, it is evident that each Angle is equal to one Right Angle and a half; i. e. to 12 eights, i. e. to six fourths, or three seconds, i. e. one and a half.

To particularize more would be unnecessary; as it is easy, from what I have said, to calculate the quantity of the Angle of any Polygon whatever, by the Proportion it bears to a Right Angle, or to two Right Angles; which must ever be more than the Angle of any Polygon whatever.

For Polygons which have an even number of Sides, it may be observed, that the division of a Right Angle, may be reduced to a lower Denomination; as for a Duodecagon, for instance; all its Angles, together, being equal to 20 twelfths, i. e. 10 sixths, or 5 third parts, i. e. to one Right Angle and two thirds; the Right Angle need be divided into three equal Parts, only, instead of twelve.

Whereas, those which have an odd number of Sides, cannot be reduced lower; but, in forming them, a Right Angle must necessarily be divided into as many parts as the Figure has Sides.

The method of inscribing a Quindecagon in a Circle, according to Euclid, is a matter of mere curiosity; first, to inscribe an equilateral Triangle, and afterward a Pentagon, in the same Circle, in order to get a fifteenth part of the Circumference; which, notwithstanding it admits of perfect Demonstration, is liable to great error in Practice.

PROPOSITION XIII.

To find the Side of a Quindecagon inscribed in a Circle.

Let ABC be a given Circle, in which it is required to inscribe a Quindecagon.

Draw, DG, at pleasure, touching the Circle ABC in A. With any Radius, AD, on A, the Point of contact, describe a Seme-circle, DEFG.

Inscribe the equilateral Triangle ABC; by making the Arks DE, EF and FG, each equal to the Radius AD, and drawing AE, AF to B and C, and joining the Points B and C.

DEM.

E L E M E N T S
O F
G E O M E T R Y.
B O O K V.
O F P R O P O R T I O N.

THE fifth Book of the Elements of Euclid contains the sublime Doctrine of Proportion; which is, psrhaps, the most subtle manner of reasoning, the most brief, solid and convincing, that the Art of Man could devise. It is essentially necessary in demonstrating all the Propositions of the sixth Book, which alone is sufficient to recommend it. But, where we consider its general and extensive use, throughout the Mathematics, it is impossible to be dispensed with. The Doctrine it contains is not only useful in, but, it is the Criterion of almost every mathematical Science; infomuch, that, without the knowledge it teaches, we should not be able to advance one step further.

I must ever be of opinion that the readiest and easiest way of acquiring knowledge, is the best. The Doctrine of Proportion is, in some measure, born with us; it is a portion of Reason, implanted in us by Nature; which does not require a formal Demonstration, so much as barely to be illustrated, by some familiar Examples; and which, indeed, may pass for Axioms, for most of them are really such.

In expressing and comparing Quantities, of any Species, it has been found necessary to make use of certain Symbols or Figures. Which Symbols, Marks or Characters, are entirely arbitrary, and might as well have been applied to signify the Qualities of Things, as Quantity. But, having been accustomed, from our infancy, to express our Ideas of Proportion by Numbers; we no sooner become acquainted with Numbers and the Characters applied to them, but we find it almost impossible, in our Ideas of Magnitude, or Quantity of any kind, to compare or estimate their Ratio, abstracted from Numbers. In respect of Magnitude, as to solid Extension, or Body; we say, when compared, it is twice, thrice, or four times as big as another; so likewise of Extension, as to length, simply, we express it by one, two, or three Yards, Chains, or Miles, &c; and so of any other Quantity whatever.

Hence, the mode of expressing Quantity, by Numbers, is the most natural, simple, and easy; and therefore, it is no wonder, when we find it expressed by other Characters, or by Lines, as it is done by Euclid and all his Commentators, in the fifth Book, that it appears, at first, to those who have not considered it, quite foreign to the purpose; but, in reality, the difference between Lines and Numbers, in expressing Quantity, consists, only, in being more familiarized to one than the other.

Therefore, since Lines, Characters, and Numbers, are only different modes of expressing Quantity, I must needs think that the best, which is the most natural and easy to conceive; besides it is much more convincing; for, unless Lines are divided into the same equal Parts, whereby we may form a judgment by how much one Magnitude, or Quantity of any kind, exceeds another, we have no Idea of their Proportion; otherwise, than

that one does exceed or is more than the other; whereas, Numbers ascertain the Ratio, whether it be equal or unequal.

If Characters (as A, B, C, &c.) are made use of, they only denote Quantity, simply, but in what Ratio, or Proportion, is best expressed by Numbers; algebraic Characters are not so expressive to those who are not well versed in them; they do not convey an Idea adequate to that given by Numbers, neither do they ascertain the Ratio: and, if Lines are divided into equal Parts, to express the Ratio, it is the same as Numbers; because, we only know the Proportion by numbering the Parts.

It is easily known that there is Analogy, or similarity of Proportion, when, in four Quantities, the first (i. e. any one) contains the second (any other) or is contained, the same number of times, as a third contains the fourth, or is contained by it; whether it can be expressed in whole numbers, as once, twice, thrice, &c; or once, twice and a half, a third, a fourth, or any other fractional number whatever.

E. g. 6 has the same Proportion to 24 as 4 has to 16, or 3 to 12, each being contained four times in the other; their Ratio is, therefore, as 1 to 4. 9 has the same Proportion to 24 as 6 to 16; for each is contained in the other, twice and two thirds; the Ratio is as 3 to 8. So likewise, 27 is to 9 as 21 to 7, or 12 to 4; for, each contains the other thrice; therefore, the Ratio is as 3 to 1. 18 is to 12 as 12 is to 8, for they are, each, as 3 to 2. Thus may the Ratios of any commensurable Quantities, whatever, be expressed by Numbers; and, indeed, such as are incommensurable, by approximation, are continually approaching nearer to the truth, till the deficiency is less than any assignable Quantity whatever.

Suppose

Suppose any four Terms or Characters, A, B, C, D, to represent four Quantities, that are analogous in their Ratio, two, and two; i. e. when A (any one) has the same Proportion to B (any other) as C, a third, has to D, the fourth; which is thus written, $A : B :: C : D$ and is read thus; as A is to B, so is C to D. The four Points, dividing the two Pairs, denotes an equal Ratio between them.

The whole business, of this fifth Book, is to shew, what various ways they may be changed, and so ordered amongst themselves, that the Proportion arising, on both sides, may still be equal or analogous; which, will admit of great variety; seeing, the Ratios of several Quantities, that are analogous, may be expressed by the same Numbers; as, 2 to 5 expresses, equally, the Ratio of 6 to 15, of 8 to 20, and of 14 to 35; each being contained in the other, respectively, twice and one half.

Therefore, either by Alternation, Inversion, Composition, Division, converting, or mixing the Terms, the Ratio of each, may, in each change, be expressed by the same Numbers; & consequently, there is always equality of Proportion. Nor indeed, is it possible it should be otherwise, if the Terms, on both sides, be ordered & changed after the same manner.

Wherefore, I am and ever was of opinion, notwithstanding what the learned Dr. Barrow, in his mathematical Lectures, and Dr. Keil, in his Preface, says to the contrary, and the high encomiums which Cunn, in his Preface, bestows on Euclid's Demonstrations of the fifth Book; that, it is involving a thing, in itself not very intricate, if not in darkness and obscurity, at least in perplexity; seeing that, half, or more than half, of the Propositions may be dispensed with, or pass for Axioms; the remainder being sufficient, towards attaining a full Demonstration of the whole.

D E F I-

DEFINITIONS.

Def. I. QUANTITY is whatever may be measured or numbered, estimated or compared, in respect of more or less.

As Magnitude or Body; Extension, or length simply; Weight, or Gravity, Measure, Time, Motion, &c.

N. B. One Surface may be equal, in area, to twice, thrice or four times the area of another Surface; or a Line may have twice or thrice, &c. the length of another Line. Also, if one Body moves, uniformly, through twice, thrice, or four times the Space, through which another Body moves, in equal Time, it will move with twice, thrice, or four times the velocity. All which, come under the Denomination of Quantity, and may be expressed or represented by various Symbols or Characters; or by Right Lines, of twice, thrice or four times the length, one of another; or by Numbers; as 6 contains 3, 2, or $1\frac{1}{2}$, twice, thrice, or four times.

Quantities, being compared together, are of two kinds, *viz.* commensurable, and incommensurable.

Commensurable Quantities are such as have a common measure; that is, such Quantities as may be divided into the same equal Parts, or into Parts of the same magnitude.

Two Quantities are commensurable, when some determinate Quantity may be found, which, being taken, or multiplied, certain numbers of times, is equal to either, without deficiency or excess.

Incommensurable Quantities are such as no other Quantity can measure; i. e. there cannot be found any determinate Quantity, how small soever, which, being multiplied, will be equal to each of the other; but that there will be a deficiency or excess, in one or the other.

N. B.

N. B. Any two Quantities, whose Proportion, to each other, can be expressed by Numbers, are commensurable ; two Quantities, whose Ratio is such as cannot be expressed in Numbers, are said to be incommensurable to each other.

Def. II. An ALIQUOT, or equal Part, is that, when a less Quantity is contained any number of times, precisely, in a greater ; i. e. when a less Quantity, being taken or multiplied any number of times, is equal to a greater.

Thus; an Inch, or a Foot, is an Aliquot part of a Yard or Fathom ; or an Ounce of a Pound, &c ; 3 is an Aliquot part of 9, 12, or 15, &c.

Def. III. MULTIPLE. That Quantity is called a Multiple, in respect of another Quantity, when it contains, exactly, or is equal to the other, being taken any number of times ; then, the less is said to measure the greater.

Thus; a Foot is a Multiple of an Inch, of two, three, four or six Inches, but it is an Aliquot part of a Yard, &c.

A Yard is a Multiple of a Foot or of an Inch, &c. but it is an Aliquot of a Fathom or Furlong, &c.

Def. IV. RATIO, or PROPORTION,* is a mutual habitude or relation of Quantities, of the same kind, in respect to more or less.

Quantities are said to have Ratio to one another, which, being multiplied, can exceed each other.

* This Definition is rejected, by some, as ungeometrical. Professor Simson says, after a long quotation from Dr. Barrow's 18th Lecture, "that he fully believes, the 3rd and 8th Definitions of the 5th Book (the 4th and 6th of this) are not Euclids, but are added by some unskilful Editor." I must freely own, that I am at a loss to conceive why it is they cavil at them ; which, to me, seem both proper and necessary.

Dr. Barrow says, that, "Euclid had, perhaps, no other design in giving this Definition (which he calls metaphysical) than to give a general, though a gross and confused, notion or idea of Ratio to

In order to which, it is necessary that they be of the same species or kind, e. g. A Line (having no breadth) cannot, by multiplying, be compared with, or be equal to the smallest surface. Also a Surface, (having no thickness) cannot, by multiplying, be equal to, or compared with a Solid. So neither can Weight be compared with Time, or Motion, &c. Wherefore, such Quantities as are not of the same kind, are heterogeneous, and can have no Ratio to one another.

N. B. The Ratio of two Quantities is not expressed by the deficiency or excess of the Antecedent to the Consequent; although they may be compared by them, it is not their Ratio.

RATIO, is the Value of the Antecedent Term in respect of the Consequent. (See the next Definition).

Thus, If A is to B as 3 to 5; the Ratio is $\frac{3}{5}$ or 3 fifths of the Consequent. But, if B be taken for the Antecedent, the Ratio is greater, viz. as 5 to 3; i. e. $\frac{5}{3}$ or 5 thirds, which is one, and two thirds of the Consequent; whereas, in the former Ratio, the Antecedent, is two fifths deficient of its Consequent.

to beginners" grant it; was it not necessary, then, to give such a general Idea? which, if it be metaphysical, cannot, I presume, with propriety, be called gross, though it may be confused; as, in my Opinion, is every thing relative to Metaphysics, for, we can only reason, on such abstracted subjects, from Analogy, of imaginary Beings with coporeal; which do not, properly, admit of Analogy.

He says, further, that nothing in Mathematics depends on it, (the 6th) and that, both might well be spared, without any loss to Geometry." I am astonished at such an Assertion; does nothing depend on Proportion, or Ratio, and Analogy of Ratios? can we (as he says we should principally) attend to those which follow, before we know what is meant by Ratio? can we have any notion of any one, from Alternate Ratio to equality of Ratios, before we know what is meant by Ratio, simply? which, in itself is undoubtedly metaphysical; yet, we cannot have any notion of Ratio in abstract; as the Doctor takes great pains (in his 20th Lecture) to convince us, that Ratios are not Quantities; who ever supposed it? at the same time, I find it impossible to form an idea of Ratio, abstracted from Quantity, which is the Essence of Ratios.

A Ratio

Def. V. ANTECEDENT and CONSEQUENT are the Terms of a Ratio.

In every Ratio there must necessarily be two Terms, denoting two Quantities. The Antecedent is that which is first named, and being referred to, or compared with another Quantity, the latter is called the Consequent.

Thus, $A : B$, or $B : A$ signifying, that A has some Ratio to B , or B to A , (which it necessarily must if they be supposed to represent two Quantities of the same kind) A or B is the Antecedent, B or A is the Consequent, according, as they are taken, first or last.

A Ratio has no existence, either physically or metaphysically, without Quantity; the Term, Ratio, implies Quantity; where is the Ratio to exist, if not in Quantity? can we conceive a Ratio between non-existences? the supposition is absurd, and cannot be. Therefore, I look on the Definitions of Ratio, and Analogy of Ratios, to be absolutely necessary in Geometry, that they were given by Euclid, and that his Elements of Proportion would be imperfect without them.

Some Geometers make a distinction between Ratio and Proportion; for, Dr. Barrow, in his 4th, and Mr. Stone, in his 8th Definition, say, "Proportion is a similitude of Ratios," which, if Ratio and Proportion be the same, (as the Doctor affirms, in a Note, annexed) is an absurd Definition; viz. Ratio is a similitude of Ratios. Dr. Keil, in his 8th Definition, calls Proportion Analogy, and says, "Analogy is a similitude of Proportions;" and Professor Simson says, "Analogy, or Proportion, is the similitude of Ratios."

Now, in my opinion, the distinction between them is so very nice, that it requires very acute discernment to discover the difference. What is by them called Proportion is, properly Analogy, i. e. when the Ratios are equal or similar; for, Dr. Keil, in his 4th Definition, says, "Magnitudes are said to have Proportion to one another." Mr. Stone says, "Magnitudes are said to have a Ratio to one another." Dr. Barrow, in his fifth, changes Magnitudes for Numbers, but makes use of the Term Ratio. So likewise, in the sixth Definition; Keil says, "Magnitudes that have the same Proportion;" Simson and Stone say, in the 6th "Magnitudes which have the same Ratio, are called Proportionals;" consequently, they all mean the same thing, by Ratio and Proportion.

Def. VI. ANALOGY,* of RATIOS, is a similitude or equality of Ratios. i. e. When the Antecedent of one Ratio contains its Consequent, equally, as the Antecedent of another Ratio contains its Consequent.

* It was not the Design, of those Geometers who made use of this Term, to define Analogy, in abstract, but only, to explain what is meant by analogy of Ratios (between two and two Quantities) which is a very proper and expressive Term, for equality of Ratios; it was necessary, that some one or other should be made use of. Analogy, applied to Ratios, is, at least to me, very expressive of the thing meant by it; much more so, in my Opinion, than Proportion; which is only another Term for Ratio, or Ratio for Proportion; which, Terms, are synonymous, and equally expressive. To what purpose are all the various changes, by alternation, inversion, &c. if we take away Analogy? all the use that is or can be made of them is, only, to shew, that the result of these changes is Analogy; the chief, if not the only way of reasoning from Proportion, in general; consequently it is not an unnecessary Term.

If we must needs be caviling, there are other Definitions which are much more exceptionable; as the fourth of Euclid, viz. "Magnitudes are said to have a Ratio to one another" &c; in reality, it is not a Definition of any thing, but is only meant to acquaint us, what Quantities can have Ratio to each other; which common sense has amply provided for: who would ever think of comparing Time with Gravity, Measure, &c; or with any corporeal substance, Superficies or Solids?

The fifth is not, properly, a Definition, but intended, only, as a sign of Analogy (see the Remark on it, at the end of this 5th Book). The 9th is less so than either of the former, viz. Proportion (or Analogous Ratio) consists of three Terms, at least. What does this define? nothing; can any Person be at a loss to know, that there must needs be three Terms, who has had Analogy of Ratios defined, where he always finds four? but, one of the four, is sometimes taken twice. I therefore, look on it as superfluous and unnecessary, seeing it is a necessary consequence in Analogy; yet, it is not improper, by way of Nota bene or Remark; but there could be no impediment, to the attaining of the rest, if it was never mentioned. Therefore, it may, very safely, be blotted out of the Vocabulary of Definitions; for, to use Dr. Barrow's own words, Proportion would sustain no loss by it; and I do verily believe, that this was never given by Euclid, as a Definition.

Or, when the Antecedents, of both Ratios, are contained equally in their respective Consequents.

Thus ; 9 has the same Ratio to 3, as 6 to 2 ; for each is as 3 to 1, i. e. each Antecedent contains its Consequent thrice ; therefore, there is the same Ratio or Proportion between them, and they are said to be Analogous.

Also, 6 is to 9 as 2 to 3, for each Antecedent is contained once and a half in its Consequent.

When the second Term has not the same Proportion to the third, as the first has to the second, or the third to the fourth, they are said to be directly or discretely proportional.

Thus, $4:6::8:12$, directly or discretely ; but, $4:6::9:13\frac{1}{2}$ is both directly and continual. (See Def. 7.)

When one Antecedent contains its Consequent, or is contained in it, more than the other Antecedent contains or is contained in its Consequent, though by ever so small a part, the Ratio is unequal ; and they are called, simply, UNEQUAL RATIOS.

Analogy of Ratios is thus expressed ; $A:B::C:D$.

The first and third Terms, A and C, are the Antecedents, and may be taken for any Quantities whatever ; B and D, the second and fourth, are the Consequents.

Wherefore, if A be supposed, in Numbers, to represent 9, and C, 3 ; and, if B be equal 6, it will be thus, $9:6::3:2$; consequently, D will represent 2. But, if B also represents 3, then will D express or signify but 1, viz. $9:3::3:1$.

Again ; if B represent a greater Quantity than A, D will also be greater than C ; thus, $9:12::3:4$; or $9:15::3:5$.

The three first Terms being fixed or known Quantities, and in a certain order, the fourth is a necessary consequence.

N.B. In Analogy of Ratios, there must necessarily be three Terms, seeing that there is similitude or equality of Ratios, (and each Ratio consists of two Terms, the Antecedent and Con-

sequent; Def. V.) But, since the Consequent of the first Ratio may, also, be the Antecedent of the second (as $A:B::B:C$) three Terms are sufficient to constitute Analogy; in which case, it is necessary that they are all of the same species or kind.

But, when there are four Terms ($A:B::C:D$) the first Ratio ($A:B$) may be of a different species from the other ($C:D$) for, there is the same Ratio between a Foot and an Inch, two, three, or four Inches, &c. as between a Pound, and one, two, three, or four Ounces, (Troy Weight) consequently, there may be Analogy of Ratios, between heterogeneous Quantities.

Def. VII. CONTINUAL RATIO is when, in three or more-Quantities, the first is to the second, as the second to the third, as the third to the fourth, &c.; or, more properly, the third is to the fourth, as the second to the third, and that, as the first is to the second.

Thus. In four or more Quantities, A, B, C, D , if C has the same Proportion to D , as B has to C ; and, if B has also the same Proportion to C , as A has to B , they are then in continual Proportion, or geometrical Progression; and are thus writ, or symbolized; $A:B:C:D::$.

N. B. Each Term being twice taken, except the first and last, are, therefore, both Antecedent and Consequent, i. e. the second Term is a Consequent in respect of the first, and the third of the second; but they are, also, Antecedents in respect of the third and fourth Terms.

In continual Ratio, all the Terms are, necessarily, of the same species or kind. i. e. The Quantities must be homogeneous.

If A and C , the two first Terms, or D and C , the two last, are any fixed or determinate Quantities; all the rest are determinable; which, in Numbers, is obvious.

Let A be 4, and B 6; it will then be thus, $4:6:9:13\frac{1}{2}::$ &c. each consequent Term containing its Antecedent once and a half. But if A be to 3, and B 6, it will be $3:6:12:24::$ the Ratio being as 2 to 1, or double.

Or, if A represents 2, and B 6, then $2:6:18:54::$ &c. the Ratio being triple, or as 3 to 1. So that, whatever Ratio there is between the first and second, that Ratio is continued.

Therefore,

Therefore, when the Ratio is increasing (or rather when the Quantity is increasing, for the Ratio is the same, throughout) the Quantity of C, D, &c. is determined either by Addition or Multiplication; as in a decreasing Ratio, (i. e. when the Antecedent is the greater Quantity) by Division or Subtraction.

Def. VIII. PROPORTIONALS are such Quantities as are in the same Ratio; discretely or continual.

Proportionals consist, at least, of three Terms, or Quantities; which are usually symbolized by Right Lines; or by Characters, as $A:B::C:D$, the Ratio being determined by Numbers.

Def. IX. MEAN PROPORTIONAL, and THIRD PROPORTIONAL.

When there are three Terms, only, or, if three Terms are Proportionals, as A, B, C, they are in continual Proportion; thus, if A is to B, as B to C, in equal Ratio; then, the middle Term, B, is a Mean, between the other two; and is, therefore, called a MEAN PROPORTIONAL.

The first and last Terms are the Extremes of the Proportionals; either of which, A or C, when there are but three, is called, a THIRD PROPORTIONAL, in respect of the other two.

Hence it is evident, that, in continual Proportion, of four or more Quantities, each Term, except the first and the last, is a Mean, between its Antecedent and Consequent.

N. B. If two Terms or Quantities are given, and a third be required, in the same Ratio to either of the given Terms as that has to the other, the Quantity found is a third Proportional.

Def. X. EXTREME and MEAN PROPORTION is, when any Quantity is so divided, into two Parts, that, the lesser, the greater, and the whole, are in continual Ratio.

A Right

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A Right Line is said to be divided in extreme and mean Proportion; when, being cut into two unequal Parts, the Ratio of the whole Line, to the greater Segment, is the same as of the greater Segment to the less.

If the Line AB be cut in C. $\overline{A \quad C \quad B}$

Then, if the whole Line, AB, is to AC as AC is to CB; AB is divided in extreme and mean Proportion.

Def. XI. A FOURTH PROPORTIONAL.

If four Terms, A, B, C, and D, are Proportionals; whether they are analogous, only, or in continual Proportion, either of the Extremes, A or D, is a fourth Proportional, in respect of the other three.

N. B. If three Terms or Quantities are given, and a fourth is required, in the same Ratio, to any one of the given Quantities, as the remaining two have, the one to the other; the Quantity, so found, is a fourth Proportional.

Def. XII. DUPLICATE and TRIPLICATE RATIO.

If four, or more Terms, A, B, C, D, are in continual Proportion; the first, A, is said to have, to the third, C, a duplicate Ratio of A to B, the first to the second.

And the Proportion of A to D, the first to the fourth, is triplicate of A to B; and so on.

Def. XIII. ALTERNATE RATIO is the mutation of the two middle Terms, of four proportional Quantities;* i. e. when Antecedent is compared with Antecedent, and Consequent with Consequent; and they are called homologous, corational, or corresponding Quantities.

Thus, $A:B::C:D$, directly. — If $27:18::6:4$.

Then, $A:C::B:D$, alternately — So, $27:6::18:4$.

* Alternate Ratio cannot exist or take place, unless both Ratios are of the same species; seeing, there is no Ratio or comparison between

N. B. When the Product, arising from the multiplication of the two middle Terms, is equal to that of the two Extremes, it is a certain proof of Analogy, consequently, it will be the same, whether they are taken directly or alternately.

Def. XIV. INVERSE RATIO is when the Consequent is taken for the Antecedent, and compared with the Antecedent as a Consequent.

In this Case, the middle Terms become the extremes.

If $A:B::C:D$ directly. — $27:18::6:4$.

Then, $B:A::D:C$ inversely. — So, $18:27::4:6$.

Def. XV. COMPOUNDED RATIO is when the Antecedent and Consequent, being taken or added together, as one Quantity, are compared either with the Antecedent or with the Consequent.

If $A:B::C:D$; then, $A+B:A$, or $B::C+D:C$, or D .

$6:9::2:3$; then, $15:6$, or $9::5:2$, or 3 .

Def. XVI. DIVIDED RATIO is when the difference, between the Antecedent and the Consequent, is compared either with the Antecedent or the Consequent.

If $A:B::C:D$. Then, whether A be greater than B , or B than A ; the difference, being taken, is compared either with A or B ; as also, the difference between C and D , is compared with C or D .

Thus, $A-B$, or $B-A::A$, or $B::C-D$, or $D-C::C$ or D , $15:6::5:2$; then, the difference, $9::6$, or $15::3:2$, or 5 .

between Quantities that are heterogeneous. (see Def. IV.). For, if four Quantities are in analogous Ratio, ($A:B::C:D$), it is not necessary that the first pair (A, B) should be of the same kind as the last (C and D). But, in Alternate Ratio, the Antecedent of one Ratio is compared with the Antecedent of the other, and Consequent with Consequent; wherefore, it is absolutely necessary that both Ratios are of the same kind.

N. B. In the seven following Definitions, it is the same whether they are of the same kind or not; i. e. each pair may be of a different kind and heterogeneous to the other.

Def. XVII.

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Def. XVII. **CONVERSE RATIO** is when the Antecedent is compared with the difference between the Antecedent and the Consequent.

If $A:B::C:D$. Then, whether A exceeds B, or B exceeds A; it will be thus; $A:A-B$, or $B-A$, :: $C:C-D$, or $D-C$.
 $12:27::4:9$; and, 12 or 27 , : 15 (the difference) : : 4 , or 9 : 5 .

Def. XVIII. **MIXED RATIO** is when the Antecedent added to the Consequent, as one Quantity, is compared with the excess, by which the one exceeds the other;

And, vice versa, when the difference, between them, is compared with the sum of both.

If $A:B::C:D$; then, $A+B:A-B$, or $B-A$, :: $C+D:C-D$, or $D-C$. $7:3::21:9$; then, $10:4::30:12$, the Antecedent being the greater Quantity.

Def. XIX. **ORDINATE RATIO of EQUALITY.**

If there be three or more Quantities in one Rank of Order, in any Ratio whatever, and as many in another Rank, in the same Ratio, comparing two and two; i. e. as the first is to the second, in one Rank, so is the first to the second, in the other; and, as the second is to the third, so is the second to the third, &c.; it will then be as the first is to the last, in one Rank, so is the first to the last in the other; or (comparing the Extremes) as the first is to the first, so is the last to the last.

In the two Ranks A, B, C, D, and E, F, G, H;
 If, $A:B::E:F$, and $B:C::F:G$; also, as $C:D::G:H$.
 Then, as $A:C$, or D , :: $E:G$ or H ; also, as $A:E$:: $C:G$, or $D:H$.

Let the first Rank be as $3:7:5:8$; and let E be 9
 Then, F will be 21, G will be 15, and H 24.
 For, as $3:7::9:21$; and as $7:5::21:15$; also, as $5:8::15:24$.
 Then $3:5$, or 8 , as $9:15$, or 24 ; and as $3:9::5:15$, or $8:24$.
 Ratio

Ratio of equality* is of two kinds, ordinate and inordinate.

Ordinate Proportion is what is already defined; or, when there are, at first, but four Quantities, in analogous Ratio, either discretely or continual; and other Quantities are aded, after the same manner, and in equal Ratio.

For example; let the Ratio of $C:D$ be as $A:B$.

Now, if other Quantities be added, on both sides, in such wise, that the Ratio is the same, ordinately, there is still equality.

First, let the Quantities added be Antecedents;

If $A:B::C:D$; then, let E be to A as F to C ;

or, let them both be Consequents, viz. as $B:G::D:H$.

It is manifest, that, in either Case, the Ratio is still the same; being taken progressively.

For, as $E:B$, or $A:G::F:D$, or C to H ;

also, as $A:B$, or $C:D::E:F$, or G to H ; in either.

Def. XX. INORDINATE RATIO of EQUALITY is, when in equal Ratios of two Quantities each; as the Consequent of one Order is to some other Quantity, so is another Quantity to the Antecedent of the other Order.

Then, as the Antecedent, of the first, is to that other Quantity, so is the other Quantity to the Consequent of the other Order.

If $A:B::C:D$; and, if $B:E::F:C$; then, $A:E::F:D$.

$3:4::21:28$; and, $4:14::6:21$; then, $3:14::6:28$.

Again, if $E:G::H:F$; then, $A:G::H:D$.

For, $14:7::12:6$; then, $3:7::12:28$.

* By Ratio of equality is not meant equality of Ratios simply; but, in two Ranks of Quantities, whose Ratios are ordinate or inordinate, the Ratios are equal, at equal Distances from the Extremes.

Def. XXI. COMPOSITION OF RATIOS. *

In any Number of Quantities, and in any Order, whatever; the Ratio of the first to the last is equal to that which is compounded of all the intermediate Ratios:

If A, B, C, and D, represent as many Quantities; whatever is the Ratio of A to B, of B to C, and of C to D; the Ratio of A to D, is equal to the Ratio compounded of them all.

Thus, if A is to B as 3 to 5, B to C as 5 to 8, and C to D as 8 to 6.

Then, $A:D::3:6$; compounded of $\frac{A}{B} \cdot \frac{3}{5}$ of $\frac{B}{C} \cdot \frac{5}{8}$ & of $\frac{C}{D} \cdot \frac{8}{6}$

i.e. Multiply all the Antecedents into one another; $3 \times 5 \times 8 = 120$; and divide the Sum, by that of all the Consequents; $5 \times 8 \times 6 = 240$, then is A to D, as 120 to 240, or as 3 to 6; i.e. as 1 to 2, or $\frac{1}{2}$.

* This last Definition is generally given in the sixth Book, for which I cannot conceive a reason. The Doctrine of Proportion is a distinct subject, and therefore, all which relates to it should be together, in one Book; which, according to Euclid, is the fifth: to which (in the use and application of it, in the sixth and other Books,) we may refer on all Occasions.

Dr. Barrow, indeed, gives nearly the same thing, in both. In Def. 20. B. 5. he says, "any number of Magnitudes being put; the proportion, of the first to the last, is compounded out of the proportion of the first to the second, the second to the third, and the third to the fourth, &c. till the proportion arise."

The proportion, which is between the first and the last, will, and must necessarily, arise, by the composition of all the intermediate Ratios, which is evident from the manner of compounding them; nor is it possible to be otherwise; seeing that, each of the intermediate Quantities is both Antecedent and Consequent. Therefore, the Ratio emerging at last, seeing, it depends entirely on the first and last, must ever be equal to the Ratio of the first to the last; let the intermediate Quantities be ever so many, and in what Proportion soever.

According

According to Euclid, there are six various ways of reasoning, from Proportion, including ordinate and inordinate Ratio in that of Equality; which, in effect, is all one. Inordinate Ratio differs from ordinate, in nothing but the disposal of the last Quantity; for, whether it be considered and compared as an Antecedent or as a Consequent, by placing it first or last; the Ratio, in either case, being the same, it will still be as the first to the last, so is the first to the last; but not, one Quantity, (added) to the other, as in the former case, where the Quantities, added, are both Antecedents, or both Consequents.

But there is another way of reasoning by Proportion, distinct from the rest, which is not in Euclid; viz. mixed Ratio, (Def. XVIII.) that is, comparing the sum of the Antecedent and Consequent with their difference. In respect of Converse Ratio, it differs in nothing from inverse, which includes all Converse Ratios whatever; since, all Ratios, that are analogous, are inversely so, or conversely, which is the same thing. Nor do I see any Reason, why the 15th Definition, of Compounded Ratio, has not a converse, as well as the 16th of Divided Ratio; seeing that, one holds good as well as the other; and is as necessary to be known, it being as frequently made use of as the other.

That the manner, and the order, of the several changes may be seen at one view, I have given them over again; with the converse of each, in the following abstract; keeping the same Ratio throughout the whole. In which five Cases, viz. alternating, inverting, compounding, dividing and mixing, with their converse, together with equality, ordinate and inordinate, is contained all the variety in which the Terms, of analogous Ratios, may be changed and still be analogous; and which, answers almost every purpose, of reasoning from Proportion.

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If $A:B::C:D$, directly, as 3 to 2; thus, $21:14::9:6$.
 Then, $A:C::B:D$, alternately. - - - $21:9::14:6$.
 Also, $B:A::D:C$, inversely. - - - $14:21::6:9$.
 And, $A+B:A$, or $B::C+D:C$, or D . } by Composition.
 $35:21$, or $14::15:9$, or 6 . }
 Also; A , or $B::A+B::C$, or $D::C+D$. } conversely.
 21 , or $14::35:9$, or $6::15$ }
 And, $A-B:A$, or $B::C-D:C$, or D . } by Division.
 $7:21$, or $14::3:9$, or 6 , }
 Also; A , or $B::A-B::C$, or $D::C-D$ } conversely.
 21 , or $14:7::9$, or $6:3$ }
 Again, $A+B:A-B::C+D:C-D$, mixing; $35:7::15:3$.
 or, $A-B:A+B::C-D:C+D$, conversely, $7:35::3:15$.

Ordinate Ratio of equality.

If $A:B::C:D$; and, if $B:E::D:F$; then, $A:E::C:F$.
 $21:14::9:6$; and, $14:42::6:18$; then, $21:42::9:18$.

Inordinate Ratio of equality.

If $A:B::C:D$; and, if $B:E::F:C$; then, $A:E::F:D$.
 $21:14::9:6$; and, $14:42::3:9$; then, $21:42::3:6$.

Thus, having shewn all the variety of changes, and, that the Ratios emerging from each, in numbers, are equal or analogous; there remains, one might reasonable suppose, nothing more to be done. For, after this knowledge once obtained, of what use can it be to perplex the Student, with a tedious Demonstration of what, in Numbers, is clear and manifest; and which, I am persuaded, needs no further Demonstration. Nor do I see, excepting Alternate Ratio, (which is, also, sufficiently evident, in Numbers) that any of the rest require Demonstration; or are made more manifest by it. For, the Ratios of all commensurable Quantities may be expressed by Numbers; and it is sufficiently manifest, that, by Analogy, it must also hold true in respect of Quantities, which are incommensurable.

It

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It is evident, that, since $A:B::C:D$, whether the Ratio of A to B be commensurable or incommensurable, the Ratio of C to D being the same, as A to B, they will still be analogous in every change.

E. g. Suppose the Ratio of A to B and of C to D be as the Side of a Square to the Diagonal; or, as the Diameter of a Circle is to its Circumference, &c.

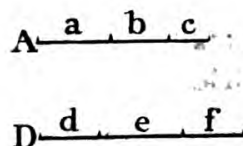
Now, it is manifest, that the Side of one Square is to its Diagonal, as the Side of any other Square is to its Diagonal; consequently, their Ratios are equal, notwithstanding they cannot be expressed in Numbers; for $3:5::9:15$, i. e. $3:5::3:5$; consequently, $3:3::5:5$, &c. Therefore, since the Side of one Square, is to its Diagonal, as the Side of any other Square to its Diagonal; consequently, as Side is to Side, so is Diagonal to Diagonal; also, as the Side added to the Diagonal, is to either Side or Diagonal; so is any other Side added to its Diagonal, to either; likewise as one Diagonal is to the Side, so is the other Diagonal to the Side; or, as the Diagonal, less the Side, to either, so is the Diagonal, less the Side, to either; and thus it must ever be through all the variety of Changes that can be, if they are ordered and changed after the same manner, on both Sides.

In what I have advanced, on the Subject of Proportion, is contained the essence of the whole fifth Book of Euclid; and if it be clearly understood, the Doctrine of Proportion is already acquired; and consequently, all that can be said more of it, by way of Demonstration, is, in a great measure, superfluous. However, not to leave the whole Doctrine of Proportion without a Foundation, lest it should be censured by the scrupulous, I have demonstrated or more fully illustrated, some of the most essential, on which the whole depends; some of the others I have made Axioms, for being self evident they require no Demonstration.

AXIOMS,

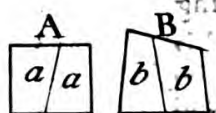
AXIOMS, or self-evident PROPOSITIONS.

Ax. I. One Quantity is to any other Quantity, as all the Parts of one, is to all the Parts of the other.

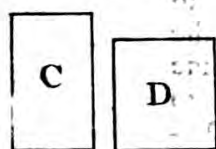


Let A and D be two Quantities, and, let A be divided, any how, into Parts, a, b, c; also let D be divided any how, in d, e, and f.

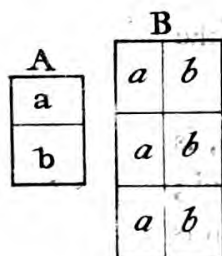
Then, $A : D :: a + b + c : d + e + f$.



II. Equimultiples,* or equal Parts, of equal Quantities are equal.

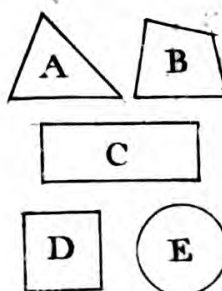


Let A and B be equal Quantities; then, if C and D be equal twice or thrice, &c. of A and B, respectively; $2A = 2B$, or $3A = 3B$, i. e. $C = D$. Also, if A and B be divided into two or more equal Parts, a a, b b, &c; $a = b$ or $\frac{1}{2}A = \frac{1}{2}B$, &c.



III. Any Multiple of the whole is equal to the same Multiple of all its Parts, taken together.

Let B be taken any Multiple of A; then, whatever Multiple B is of A, it is manifest, that it contains the same Multiple of a and b, the Parts of A and B; a, a, a, and b, b, b.



IV. The seventh Proposition of Euclid.

Equal Quantities have the same Ratio to any third Quantity, or to equal Quantities.

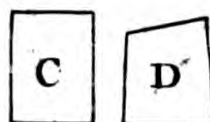
For, if A and B be equal Quantities, they have the same Ratio to a third Quantity, C, or to equal Quantities, D and E; seeing that, either may be taken for the other.

* By Equimultiples is meant, that the Quantities are taken, or multiplied, an equal number of times.

V. The ninth Proposition of Euclid.

Quantities that have an equal Ratio to the same Quantity, or to equal Quantities, are equal.

If A and B have the same Ratio to C, or D; A and B are equal. Also, if A has the same Ratio to C, as B to D; if $C=D$, $A=B$.

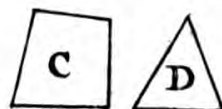


VI. The tenth Proposition of Euclid.

That Quantity which has the greater Ratio to a third Quantity, or to equal Quantities, is the greater Quantity. And, of two Quantities, that, to which a third Quantity has a greater Ratio, is the lesser Quantity.

If C has a greater Ratio to B, than A has to B, C is greater than A.

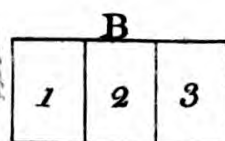
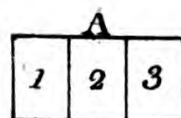
Also, if A has a greater Ratio to D than to C, D is less than C.



VII. The fifteenth Proposition of Euclid.

Quantities have an equal Ratio to their Equimultiples; or to their equal Parts.

Let A and B be two Quantities, in any Ratio whatever; and let them be taken an equal number of times. Or, let them be divided into the same number of equal Parts; The wholes of A and B are equimultiples of those Parts,



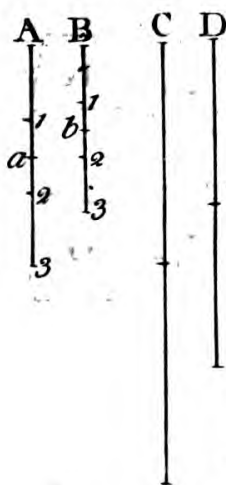
$A : 2A \text{ or } 3A, :: B : 2B \text{ or } 3B$. i.e. $1 : 2 \text{ or } 3 :: 1 : 2 \text{ or } 3$. $A \begin{array}{r} 1\ 2\ 3 \\ \hline \end{array}$

Or, as $\frac{1}{2}A$, or $\frac{1}{3}A$, : $A :: \frac{1}{2}B$, or $\frac{1}{3}B$, : B . $B \begin{array}{r} 1\ 2\ 3 \\ \hline \end{array}$

Also, $A : Aa (\frac{1}{2}A) \text{ or } A1, :: B : Bb (\frac{1}{2}B) \text{ or } B1$.

VIII. Quantities are in the same Ratio, to each other, as their Equimultiples; or, as their equal Parts.

Let

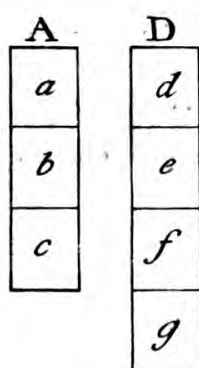


Let A and B be any Quantities, divided into equal Parts, as before, at a and b , or 1, 2, 3; also, let C and D be any equimultiples of A and B.

Then, $A : B :: 2A : 2B$, or as $3A : 3B$; i. e. as $C : D$.

For C and D are any equimultiples of A and B; wherefore, as often as A contains B, or is contained in B; so often 2 A contains 2 B, or is contained in 2 B; i. e. as C contains D, or is contained in D; consequently, $A : B :: C : D$.

Also, as $A : B :: Aa (\frac{1}{3}A) : Bb (\frac{1}{3}B)$
or, as $A1$ to $B1$. ($\frac{1}{3}A$ to $\frac{1}{3}B$.)



IX. If two Quantities be divided into Parts of equal magnitude (consequently equal amongst themselves) then, one Quantity is in the same Ratio to the other, as the number of Parts, in one, to the number of Parts, in the other.

Let A and D be two Quantities, divided into equal Parts, a, b , and c ; d, e, f , and g ;

Then, $A : D :: 3 : 4$; or, as 6 to 8, &c.
i. e. as $a + b + c$ is to $d + e + f + g$.



X. If Quantities are proportional; the first to the second, as a third is to a fourth, &c; then, if the first be any multiple or equal part of the second, the third is an equimultiple or equal part of the fourth, &c.

For Quantities are to each other, as the number of Parts in one, to the number of Parts in the other. - - - - by the 9th.

Therefore, if $A : B :: C : D$; A contains, or is contained in B, as often as C contains, or is contained in D. - - - - Def. 6.
otherwise, the Ratios are not equal, or analogous.

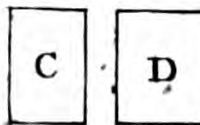
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AXIOM XI. In four proportional Quantities,
i. e. when any one is to another, as a third is A _____
to the fourth, (Def. 6.) whether the first be equal B _____
to, greater, or less than the second, the third is C _____
also equal to, greater, or less than the fourth. D _____

If $A:B::C:D$; then, if A be equal to B, $C=D$;
and, if A be greater, or less than B; C is, also,
greater, or less than D.



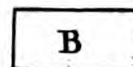
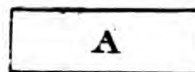
For, A has the same Ratio to B, as C to D ;



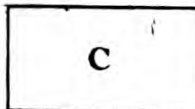
And, a greater Quantity cannot have the same
Ratio to a less, as a less to a greater. - Def. 6.

XII. The fourteenth Proposition of Euclid.

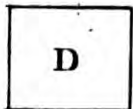
If four Quantities are proportional ; then, if the
Antecedent of one Ratio be greater than the An-
tecedent of the other, the Consequent of the first is
also greater than the Consequent of the other ; if
the first are equal, the other are also equal ; and,
if less, less.



For, $A:B::C:D$; and, if A be equal to,
greater, or less, than B, C is also equal to, greater,
or less than D. - - - - - 12



Wherefore, if $A=C$, $B=D$ (Ax. 5.) and con-
sequently, if A be greater or less than B, C is,
necessarily, greater or less than D.



XIII. The eleventh Proposition, of Euclid.

Ratios that are equal to the same Ratio, or to
equal Ratios, are equal between themselves.

This follows from the third Axiom of Book 1st,
by substituting Ratios for Quantities.

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XIV. The thirteenth Proposition of Euclid.

If four Quantities are proportional, the first to the second, as the third to the fourth; and if the Ratio of the third to the fourth, be greater or less than a fifth Quantity has to a sixth; the Ratio of the first to the second, is also greater or less than the fifth to the sixth.

If $A:B::C:D$; and, if the Ratio of C to D be either greater or less than E to F ; the Ratio of A to B is also greater or less, than E to F .

For, the Ratio of A to B is equal to the Ratio of C to D ; therefore, their Ratios are the same, in respect to any other.

Note. By mistake, the 7th and 8th Axioms are misplaced. But, as either is the other, alternately, it is of no consequence; save only, that the first part of the 8th is, properly, Euclid's fifteenth Proposition.

P O S T U L A T E S.

1. Grant, that equal Ratios may be taken, one for the other.
2. Grant, that any Quantity may be divided into any number of Parts, equal to one another; or as any other Quantity is divided.
3. Grant, that a Quantity may be taken or assumed, in any Ratio to any given Quantity.

The assumption of this last Axiom is not allowed by some Geometers; notwithstanding, to me, it appears full as possible, as that equimultiples, or equal parts of Quantities may be taken; by one or other of which Suppositions, they demonstrate the whole fifth Book.

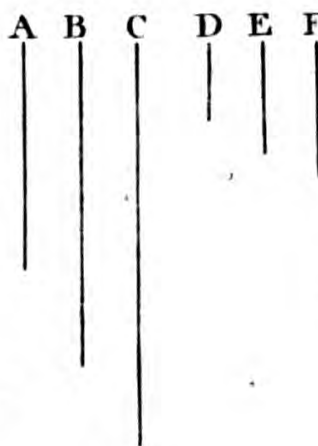
THE.

THEOREM I. 1 Euclid.

If Quantities, which are equimultiples or equal parts of other Quantities, be added into one Sum, or Quantity; the same multiple or part, which one Quantity is of the other, respectively, the whole is of the whole.

Let A, B, and C be equimultiples, or equal parts, of D, E, and F, respectively; i. e. whatever multiple, or part, A is of D, let B be the same multiple, or part, of E, and C of F.

Then, A, B, and C, added together, into one Sum, is the same multiple or part of D, E, and F, added together, as A is of D, B of E, or C of F.



DEM. For, let A be equal to $2D$, or $3D$; then, $B = 2E$, or $3E$, &c.

conf. $A + B + C = D + E + F$, taken twice, or thrice, &c.

But, Quantities are to each other, as the number of Parts, in one, is to the number of equal Parts, in the other; - Ax. 9.

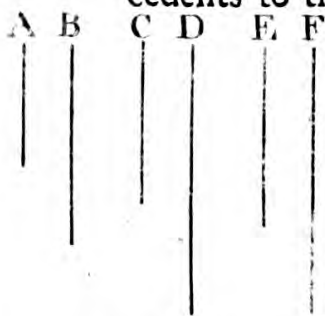
wh. as $A : D$, $B : E$, and $C : F$, $:: A + B + C : D + E + F$.

Consequently, as often as A contains D, or is contained in D, &c. so often the whole of $A + B + C$ contains, or is contained in, $D + E + F$. Therefore, &c. Q. E. D.

For, if either be multiples of the other; the latter is, consequently, equal parts of the former.

THEOREM II. 12 Euclid.

If any number of Quantities, are Proportionals, (i. e. being taken two and two, their Ratios are analogous) then, as any one of the Antecedents is to its Consequent, so is the sum of all the Antecedents to the sum of all the Consequents.



Let A, B, C, D, E, and F be proportional Quantities; of which, let A be to B, as C is to D, and as E to F.

I say, as A is to B, or C to D; so is A, C, and E, added together, to B, D, and F, together.

DEM. Now, $A:B::C:D$, and, as $E:F$, therefore, their Ratios are equal, - - - Def. 6. Wherefore, if $A=B$, $C=D$, and $E=F$ - - Ax. 11. consequently, $A+C+E=B+D+F$. - - Ax. 6. 1. therefore, as $A:B$, or as $C:D$, $:: A+C+E : B+D+F$.

Again, if A be any multiple, or equal part, of B; then, C is an equimultiple, or equal part, of D, & E of F - Ax. 10. wherefore, $A+C+E$ is the same multiple, or equal part, of $B+D+E$, as A is of B, as C of D, and as E of F - Th. 1.

But, Quantities have the same Ratio, to each other, as their Equimultiples, or as their equal Parts. - - Ax. 8. Th. as $A:B$, as $C:D$, or as $E:F$, $:: A+C+E : B+D+F$.

Now, because there are such Quantities, which cannot be equimultiples or equal Parts, one of the other, this Demonstration is not positive, in respect of such Quantities; and, notwithstanding they may be divided into the same number of equal Parts, yet it will be found no easy matter, to give perfect Demonstration, seeing that, equal parts are in the same Ratio to each other as the wholes. Ax. 8th.

Suppose A to B, and C to D, &c. are incommensurable, but in the same Ratio; still, $A+C+E:B+D+F$, &c. $:: A:B$, &c. because, A might either be equal to C, or E, or it might be some Multiple or equal Part of C, and C of E, &c.

Or, suppose A be incommensurable to C, and C to E, &c. provided A and B, C and D, &c. be commensurable, the same thing may be done; as in the Theorem.

But, being incommensurable both ways, there cannot possibly be equality in either; and the Demonstration given (or rather presumed) from a greater or less Ratio, is by no means positive; seeing, it is not ascertained in what Degree they are so.

For, A may have to B, and C to D, &c. the Ratio of the Side of a Square to its Diagonal, or as the Diagonal to the Side, &c. and A to C may be the same; or, as the Diameter of a Circle to the Circumference; and C to E, in extreme and mean Proportion.

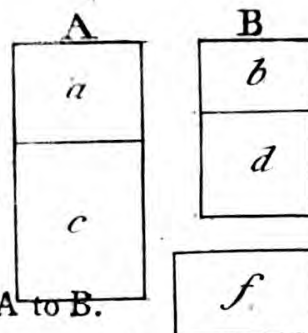
Yet it is easy to conceive, from what has been proved, that (let the Ratio of A to B be what it may) since the Ratio of C to D, and of E to F, &c. is the same, consequently, $A+C:B+D$, or $A+C+E:B+D+F::A:B$, or C to D, &c. and although it is not possible to give direct proof, in such Case, yet we may safely conclude that it is so; and, on that presumption, all which follows will be found positive, let the Ratio be what it may.

THEOREM III. 19 Euclid.

In two Quantities, if a Part taken from one, be in the same Ratio to a Part taken from the other, as one Quantity is to the other; the remainders of those Quantities are also in the same Ratio, as one Quantity is to the other.

Let A and B be two Quantities, in any Ratio whatever, and, suppose any part of A be taken (as a) from A.

Let b be taken, from B, in the same proportion to a , as A to B.



I say, the remaining Part c , is to d , as A to B.

Let any other Quantity (f) be so taken, $a:b::c:f$. Post. 3.

DEM. Now, $a:b::c:f$; and $a:b::A:B$; conf. $c:f::A:B$. - Ax. 13

But, $A:B::a+c:b+d$ (Ax. 1.) and, as $a:b::c:f$. - Post.

wherefore, $a+c:b+f::a:b$; i. e. as $A:B$. - Th. 2

conf. $a+c:b+f::a+c:b+d$; wh. $b+f=b+d$. - Ax. 4

And by taking away b , which is common, $f=d$. - Ax. 7. 1

But, $c:f::a:b$ (Post.) consequently, $c:d::a:b$. - Ax. 4

And, $a:b::A:B$ (Hyp.) Therefore, $c:d::A:B$. - Ax. 13

COR. If one Quantity be to another as a third is to a fourth; and, if a fifth be to the sixth, as the third to the fourth; then, the first and fifth, added together, has the same Ratio to the second and sixth, together, as the third has to the fourth.

THEOREM IV. 16 Euclid.

If four Quantities, of the same kind, are Proportionals; i. e. taken two and two, in a certain order, they are directly proportional, or analogous in their Ratio; they are also analogous when taken alternately.

If A is to B, as C is to D; then, A is to C, as B to D.

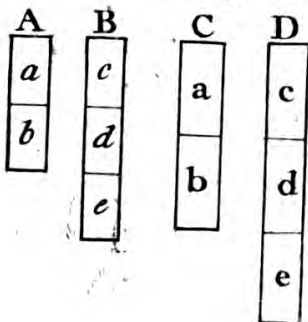
DEM. Now, since $A:B::C:D$; whether A be equal to, greater or less than B; C is also equal to, greater or less than D. - - - - - Ax. 11.

And, if A be equal to, greater, or less than C;

B is also equal to, greater, or less than D. - - - 12.

But, A has the same Ratio to B as C to D; - - Hyp. it follows, then, that A has the same to C, as B to D.

For, whether A and B be equimultiples, or equal parts of C and D; A is in the same Ratio to C, as B to D-Ax. 7. seeing that A is to B as C to D. - - - - - 8.



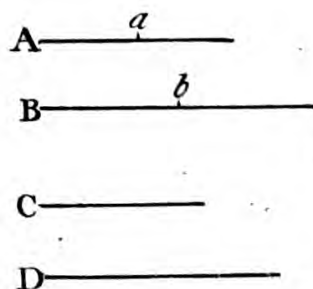
Suppose the Ratio of A to B be as 2 to 3; and, imagine them divided into equal Parts, in a and b; c, d, and e.

Also, suppose C and D divided into the same number of equal Parts, respectively, in a and b; c, d, and e.

Then

Then, since $A:B::2:3$, & $A:B::C:D$; $C:D::2:3$ -Ax. 13
 wherefore, the Parts, a and b , (of C) and c , d , e , (of D)
 are all equal amongst themselves. - - - Ax. 9.
 Consequently, as $a:a$, or $b:b$, :: $c:c$, $d:d$, or $e:e$. - Ax. 4.
 But, a and b are equal parts of A ; and, a and b , of C .
 Also, c , d , & e . are equal parts of B ; and, c , d , and e , of D .
 But, Quantities are, to each other, in the same Ratio as
 their equimultiples, or equal parts. - - - Ax. 8.
 Therefore, as A is to C , so is B to D . Q. E. D.

Again. Suppose the Ratio of A to B ,
 and of C to D , be incommensurable;
 and consequently, cannot be divided into
 Parts, which are all equal amongst
 themselves;



Then, let A and B be divided into the
 same number of equal Parts, suppose they
 are bisected, in a and b . - - - Post. 2

Now, $A:B::C:D$, and $A:B::a:b$, - - - Ax. 8.
 consequently, $C:D::a:b$, - - - Ax. 13.
 But, $A:a::B:b$ (Ax. 7) and, as $a:b::C:D$. - as above.
 Wherefore, by substituting one for the other, - Post. 1.
 consequently, as A is to C , so is B to D . Q. E. D.

THEOREM V.

Quantities, which are analogous when taken directly,
 are also analogous, taken inversely.

If A is to B , as C is to D ; then, B is to A , as D to C .

DEM. For, if $A:B::C:D$; then, $A:C::B:D$, - - by 4.
 i. e. $B:D::A:C$; then, $B:A::D:C$. - - same.

T H E-

THEOREM VI. 18 Euclid.

Quantities which are analogous, taken directly, are also analogous, when compounded.

If A is to B as C is to D.

Then A added to B, is to A or B; as C added to D, is to C or D.

DEM. For, since $A:B::C:D$; then, $A:C::B:D$ - Th. 4.
 wh. as $A:C::A+B:C+D$. (2) conf. $A:A+B::C:C+D$.
 and consequently, as $A+B:A::C+D:C$. - - Th. 5.
 But, $B:D::A:C$; therefore, $A+B:B::C+D:D$ - Ax. 13.

COR. By inversion, A, or B, : $A+B$: C, or D, : $C+D$.

THEOREM VII. 17 Euclid.

Quantities which are analogous, directly, are also analogous when divided.

If, in four Quantities, A, B, C, and D; A is to B, as C to D;

Then, according as A or B is the greater Quantity,

A less B (or B less A) is to A, or B,
 as C less D (or D less C) is to C, or D.

DEM. Now, $A:B::C:D$. (Hyp.) wherefore, $A:C::B:D$ - 4.
 But, as $A:C$, (or B to D) : $A-B$ (or $B-A$) : $C-D$ - 3.
 conf. A (or B) : $A-B$: C (or D) : $C-D$ - - - 4.
 and, conf. $A-B$: A (or B) : $C-D$: C (or D) - - 5.
 Or; - - $B-A$: A, or B, : $D-C$: C, or D.

COR. The Converse of this is first proved, in the third step.

T H E-

THEOREM VIII.

If four Quantities are Proportionals, in a certain order, they are also proportional when mixed; i.e. their Sums and Differences are analogous.

If A is to B as C to D.

Then, $A+B:A-B$ (or $B-A$) $:: C+D:C-D$ (or $D-C$)

DEM. Now, $A:B::C:D$ (Hyp.) then $A:C::B:D$ - Th. 4.

Wherefore, as $A:C::A+B:C+D$ - - - 2.

also, as $A:C::A-B:C-D$. - - - 3.

consequently, $A+B:C+D::A-B:C-D$ - - Ax. 13.

therefore, $A+B:A-B::C+D:C-D$. - Th. 4.

COR. $A-B:A+B::C-D:C+D$, by inversion. - 5.

THEOREM IX. 22 Euclid.

If there be three or more Quantities, and others equal to them in number; which, taken two and two in a certain order (ordinately, or progressively) are in the same Ratio; then will the first be to the last, as the first to the last, in each Order of Quantities.

Also, as the first is to the first, so is the last to the last.

If $A:B::C:D$; and, if $B:E::D:F$; A is to E as C to F.

DEM. Now, $A:B::C:D$; wherefore, $A:C::B:D$ } -Th. 4.

And - $B:E::D:F$; wherefore, $B:D::E:F$ }

Now, since $A:C::B:D$; and, as $B:D::E:F$.

conf. $A:C::E:F$ (Ax. 13.) Therefore, $A:E::C:F$. - 4.

N. B. The last Part is proved first.

M m

COR. I.

Again. If other Quantities, G and H, be added, in the same Ratio to the last Consequents, respectively; as E is to G, so is F to H; it is manifest, that A is to G, as C to H; the first, still, in the same Ratio to the last.

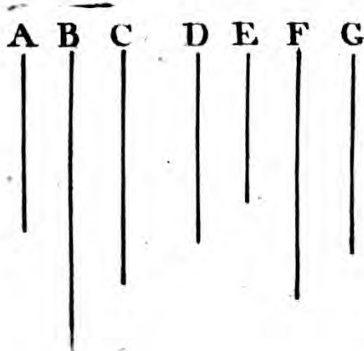
For, as $A:E::C:F$; and, as $E:G::F:H$; by Theorem. wherefore, $A:C::E:F$; and, as $E:F::G:H$; by the 4th. Conf. as $A:C::G:H$ (Ax. 13) th. $A:G::C:H$ - same.

And thus it will ever be, if Quantities be added (in equal Ratio) after the same Order, ad infinitum.

COR. Hence, if the Quantities, added are in equal Ratio to the Antecedents, respectively;
thus, if $A:B::C:D$, and, if E be to A as F to C;
it is manifest, that as E is to B, so is F to D, &c.

THEOREM X. 23 Euclid.

In two Ranks or order of Quantities, if there be three or more in each; which, taken two and two, inordinately, are in the same Ratio; then, as the first is to the last, in one Order, so is the first to the last in the other; as by equality.



Let A, B, C, and D, E, F, be Quantities in such Order, so that, as A is to B, so is E to F; and, as B is to C, so is D to E.

I say, that, as A is to C, so is D to F.

Let any other Quantity, G, be taken so; that, as D is to E, so is F to G; - Post. 3. conf. as D is to F, so is E to G. - - 4.

DEM.

DEM. Then, as before, $A:B::E:F$, wh. $A:E::B:F$. - 4.

Also, as $B:C::D:E$, i. e. as $F:G$, wh. $B:F::C:G$ -same.

Now, $A:E::B:F$; and $B:F::C:G$; wh. $A:E:C:G$ -Ax. 13.

and consequently, as $A:C::E:G$. - - - - Th. 4.

But, $D:F::E:G$, (above) Th. as $A:C::D:F$ - Ax. 13.

If other Quantities be added, inordinately, in equal Ratio, it will still be as the first is to the last, in one Rank, so is the first to the last, in the other. e. g.

Let the Quantities, G and H , be so taken, that, as C is to G , so is H to D ; then, as A is G , so is H to F .

For, as $A:C::D:F$ (above) and, as $C:G::H:D$ - Hyp. consequently, as A is to G , so is H to F ; - - as before.

Inordinate equality may be proved, (after ordinate) by means of the 21st Definition, viz. by composition of Ratios; but, as that Definition is somewhat obscure, the Demonstration arising, therefrom, would be so too.

For, since $A:B::E:F$; and, as $B:C::D:E$; also as $D:G::H:D$; it is manifest, seeing that there is equal Ratios between the two extremes, A and G , H and F ; let the intermediate Ratios be ever so many, and in what order soever, provided there be an equal number in each Rank; and the extreme Ratios are equal, ordinately or inordinately; also, that there be found equal Ratios between the Extremes, in each Rank; two and two, any how situated; the Ratio, compounded of them all, is, as the first to the last, in each Rank, or order of Quantities; by the Definition.

But, the extreme Ratios are equal (ordinately or inordinately) and consequently, the extreme Quantities, of both Ranks, are in equal Ratio.

THEOREM XI. 25 Euclid.

If four Quantities are Proportionals, in a certain order; the greatest and the least are, together, greater than the other two.



Let A, B, C, and D, be four Quantities, of which, let A be the greatest; and let them be, as A to B, so is C to D; consequently, D, is the least Quantity.

For, if the first be greater than the second, the third is greater than the fourth - Ax. 11. and, if the first be greater than the third, the second is greater than the fourth. - 12.

DEM. Because A and C are greater Quantities than B & D; let E and F be taken from A and B, respectively, equal to B and D, respectively.

Now, $A:B::C:D$. But $E=B$, and $F=D$ - Con. wherefore, as $A:E::C:F$. - Ax. 4. conf. $A-E$ (i. e. G) : $A::C-F$ (i. e. H) : C - Th. 7. i. e. $G:A::H:C$; wherefore, $G:H::A:C$ - Th. 4. But, A is greater than C; wh. G is greater than H - Ax. 11. conf. $G+E$ (i. e. A) + D, is greater than $H+F$ (i. e. C) + B. Th. A + D is greater than C + B, the two middle Terms.

In the last ten Theorems, and the Corollaries deducible from them, is contained all the various ways of reasoning, by Proportion, which I conceive to be useful; two of which, the fifth and eighth, are not proved by Euclid. The eighth, indeed, is not mentioned, by him, or any of his Followers, that I am acquainted with; but, I am somewhat surprized, that
none

none have given a Demonstration of Inverse Ratio; which being proved, every Converse Ratio is also proved. Professor Simson is aware of that deficiency, and has very judiciously introduced it, as an additional Proposition (B;) after which, I must think it needless, to add the Converse of divided Ratio (E) unless he had, also, given the Converse of all.

But what, to me, seems very extraordinary is, that Euclid should not demonstrate either; also, that he should prove (if it may be said to be proved) divided Ratio (the 17th) before compounded (the 18th) when the latter is, in some measure, made a Condition of the former; they are, indeed, Converse, each of the other. In the 17th he says, "If Magnitudes, taken jointly or compounded, be proportional, they are also proportional taken separately, or divided." Now, by the tenor of these Premises, *if* one is, the other is, also; and consequently, *if* one is not, the other is not; whereas, the *Condition* is presumed, only; and consequently, there is no proof of either. But where is the necessity of proving it on that Condition, when it may be done as elegantly, solidly, and briefly without it, from the simple Analogy of the two Ratios.

Either I am not clear, in the nature and meaning of compounded and divided Ratio, or those Propositions are not to the purpose.

By Definitions 15th and 16th is to be understood, when Quantities are proportional (simply) they are also proportional compounded or divided; which, Professor Simson has expressly said, in his Definitions of them; and which, I have demonstrated in the 6th and 7th Theorems. Now, his 17th Proposition says, as above; and *vice versa* in the 18th, viz. that, if Quantities are Proportionals, when compounded or divided, they are so, divided or compounded; which, I cannot conceive to be the true meaning, but as I have stated them, in the Theorems, and have demonstrated, on that Hypothesis.

If what I have advanced, on the sublime Doctrine of Proportion, be not sufficient, for any purpose whatever, I will be bold to say, that it is not so in Euclid, or any of his Commentators. The Axioms, which I have given, are gathered from the most easy and simple ideas of Proportion; after the Definitions are digested and clearly understood, there is no Person would hesitate, one moment, to grant every Axiom; they require no proof, being self evident.

That Quantities are in the same Ratio as their Equimultiples, or equal Parts (the 8th) is manifest to the meanest capacity; and this is one of the principal, on which Euclid has founded his whole Theory of Proportion; which amounts to no more than this; that one is to two, three, &c. as, two, is to four, six, &c; or, as three to six, nine, &c. i. e. as one, to two, three, &c. Or, that one Quantity is to any other Quantity, as the half, or third part, &c. of one, is to the half, or third part, &c. of the other.

The 1st Axiom, in the first Book of Elements, of Euclid, (the 3rd of these Elements) expressly says "*Things*, which are equal to the same *Thing*, are equal between themselves." Now, the question is, whether *Ratios* are *Things*? Certainly, if any thing be meant by Ratios, they must come within that Appellation; for, granting the Term to be, in itself, metaphysical; yet, *something* is understood and meant by Ratio, and consequently, *Ratios* are *Things*. Then, the 11th Proposition of Euclid's fifth Book, is as much an Axiom as the first of the first; and, I have accordingly made it so, in the thirteenth. The remainder of the Axioms are as simple, obvious, and self evident; which, with the Postulates, being granted, the whole of the Theorems, I am confident, will be found solidly, yet briefly, and clearly demonstrated.

A critical

A critical Remark on Euclid's fifth Definition, of the fifth Book.

Euclid in his fifth Definition, has given us a Criterion, whereby to determine Analogy of Ratios, rather than a simple Definition of what it is.

He says, "Magnitudes (by which is meant Quantities of any kind) are said to be in the same Proportion, the first to the second, and the third to the fourth, when the equimultiples of the first and third, being compared with the equimultiples of the second and fourth, according to any multiplication; either together exceed the one the other, or together are equal the one to the other, or together are less, the one than the other." This is, literally, according to the old Translation by H. Billinsley in 1570.

Dr. Barrow says, "Magnitudes are in the same Ratio, when the Equimultiples of the first and third, compared with the equimultiples of the second and fourth, according to any multiplication whatsoever, either both together are less than the second and fourth both together, or equal taken together, or exceed one the other together, if those be taken which answer one to the other."

Dr. Keil, from the latin translation of Commandine, has nearly the same words, viz. "the Equimultiples of the first and third (compared as above) are either both together greater, equal, or less than the equimultiples of the second and fourth; if those be taken that answer each other."

Cunn did well to explain the meaning of Dr. Keil's Definition, which is literally this; that, equimultiples of the first and third, taken or added together (i. e. into one Sum or Quantity) are either equal, greater, or less than the equimultiples of the second and fourth. I think, there is great occasion for a comment on Dr. Barrow's; which, to me, is quite unintelligible.

After all that has been said, concerning this Definition, by the learned Doctor and others, either the original Text must be very obscure, or the expounders have rendered it so, for I declare, I cannot think the words bear sense. But, giving the greatest latitude possible to the meaning of the words, they imply, the first and the third taken together, and compared with the second and fourth, together, (i. e. in one Sum) in which Case it amounts to a clear Axiom; for they must, certainly, be either equal, greater, or less, one than the other; after which, the words, "if those be taken which answer the one to the other," mean nothing at all.

Mr.

Mr. Stone says, "they are, together, either deficient, equal or exceed *each* other." They may, indeed, be equal, *each* to the other; but, it is impossible that they can, at the same time, be either greater or less, *each* than the other.

The word, *together*, seems, to me, quite superfluous; or else the tenor of it is, in this place, *at the same time*, or, *by the same multiplication*; which is quite foreign to the common acceptance of it. It is, perhaps, better to omit it entirely, and supply its place with *both* (which Barrow and Keil have added) it is then plain; the first and the third, compared with the second and fourth, will be either *both* greater, *both* equal or *both* less; which is the conclusion all have drawn from it; and then, the words, if those be taken which answer each other, seem necessary.

Mr. Stone in vindicating and illustrating this Definition, shews, that it is clearly deduced from the fourth and fourteenth Propositions. According to the Idea I have of the Definitions of Terms, in any Science, they ought to be defined by their most natural and simple properties; from which the demonstrations of some Theorems are to be obtained; for it seems inconsistent, first to demonstrate several preceding Propositions, by means of a certain Definition, previous to the demonstration of others, from which the Definition, itself, is to be deduced, and demonstrated.

It is true, he says "it is not so simple and plain as the Definition of Numbers, or that which might be given of commensurable Magnitudes." And again: "Euclid could not have given any other, so elegant and general a Definition, that would have taken in incommensurable Magnitudes, as well as Numbers, and commensurable ones.

It is well known, that the Diagonal of a Square is not commensurable to its Side; also, the Circumference of a Circle is incommensurable by its Diameter or Radius; that is, their Ratios cannot be expressed in Numbers; for, the Side of a Square cannot be divided into any number of parts, though ever so small, which will be an aliquot part of the Diagonal, i. e. they cannot both be divided into the same equal parts, or into parts of equal magnitude, without defect; such Quantities are, therefore, incommensurable.

The Ratio, between the Side of a Square, and its Diagonal is very different from that, between the Diameter of a Circle and its Circumference. Yet, the Ratio between the Side of every Square and its Diagonal, or between the Diameter of every Circle and its Circumference, is the same; i. e. their Ratios are equal or analogous.

It is intimated above, and is strongly supported by Dr. Barrow and others, that this Definition comprehends or extends to Incommensurables

commensurables. I am, however, persuaded, that it is no easy matter to determine the analogy of such Ratios by it. The rule it prescribes, or is intended to prescribe, is simply this. Take any Equimultiples of the first and third Terms, in four Quantities, that are presumed to be analogous in their Ratios; i. e. multiply the two Antecedent Terms, equally, the same number of Times, and, either by the same Number or any other, multiply the two Consequents, the third and fourth, or take Equimultiples of them; then, the first and the third, being compared with the second and fourth; if the first exceeds the second, the third will also exceed the fourth, if one be equal, the other is also equal, or they will be both less, than the second and fourth; i. e. the two Antecedent Terms will, at the same time, be either both equal, both greater, or both less than their respective Consequents; and, when this is the case, in all multiplications, *whatever*, we may then conclude that their Ratios are analogous.

Now, if the Ratios are of incommensurable Quantities, which cannot be expressed by Numbers, those Quantities cannot be multiplied by Numbers; and, in Lines, it would require greater accuracy than it is possible for any person to apply, to take Equimultiples of them. It is certain, that being multiplied, mentally, according to this Definition, if they were analogous, the Antecedents would be either greater or less than the Consequents, continually, but never equal; for if, by taking Equimultiples, they could be equal, they may, consequently, also be divided into equal Parts; but, it is well known, they cannot; which, negative Quality, is the distinguishing characteristic of Incommensurables. And, since it is manifest that the Antecedents of Ratios which are not analogous may, by multiplying, be either greater or less than their respective Consequents, but never both equal, at the same time, i. e. by the same multiplication; how, then, is it possible to discover analogy of Ratios by this Criterion? seeing that, in Ratios of commensurable Quantities, which are not analogous, they may be greater or less, but never equal, as they will ever be in incommensurable Quantities that are analogous.

Hence, it appears, to me, that, to produce equality, of the Antecedent to the Consequent, is the only certain and infallible sign of Proportionality, or analogous Ratio, between commensurable Quantities; but, since that Criterion fails, in Incommensurables, there cannot, by its means, be any positive determination concerning Analogy, in such Quantities; unless, by trying *all multiplications*, *whatever*, they are, *at all times*, either *both* greater, or *both* less.

If the Antecedents are either both greater, equal, or less than their respective Consequents (and no Person would look for Proportionality otherwise) it is manifest, that, by multiplying the

Antecedents any number of times, equally, and the Consequents the same, or any other, also equally; they must necessarily be, at the same time, either both greater or both less; but, if they are not analogous, they never can, at the same time, be both equal, i. e. each Antecedent to its respective Consequent; and therefore, no other Sign, but equality, is necessary, or sufficient, to prove Analogy of Ratios.

I am persuaded, that either Euclid, himself, or his Advocates are under some mistake in respect of this fifth Definition; which does not, positively, determine the Analogy of incommensurable Quantities. Therefore, the investigation, by means of it, is vague and imperfect; and consequently, all that is built, entirely on it, is so too; seeing it cannot, as it is presumed, discover Analogy, in incommensurable Quantities; because, equality of the Antecedent to the Consequent (the only certain Sign) cannot, by multiplying, be produced.

It is manifest, that if Equimultiples of incommensurable Quantities are taken, they will still be in the same Ratio; i. e. if the side of a Square and its Diagonal, &c. be equally multiplied; but no multiples of the Side and of the Diagonal can be so taken, that they may be equal to one another; for, it is evident, if they could, they might, also (as observed above) be divided into the same equal Parts, which cannot be. And, for such as are commensurable, it is an unnecessary and unsatisfactory method of determining.

Then, since no sign, but equality of the Antecedent to the Consequent, can be positive, and since that cannot subsist between Incommensurables, although it is certain, that *if* one is, the other must necessarily be so too; but since that, *if*, can never possibly happen, we cannot, on that supposition, determine Analogy of Ratios, in such Quantities.

The Reverend Dr. Barrow, in his 21st Lecture, has made a very learned Defence of this Definition; and, spiritedly, encounters the Detractors of Euclid, viz. Ramus, Tacquet, and Hobbs, in the 22nd. But, having silenced those weak Opponents, he is close put to it by Borellus, in his 23rd and last Lecture; where, after various skirmishes, in which, the Doctor has not always the advantage, he briefly sums up the whole Evidence, in these words, "that, in his judgment, there is nothing extant in the whole work of the Elements, more subtilly invented, more solidly established, or more accurately handled, than the Doctrine of Proportionalities." Nevertheless, if his own Definition of it is to be the Touchstone, I cannot subscribe to his Attestation.

Now, as I have before observed, in the Preamble to this fifth Book, Proportion is, in a great measure, an innate principle; which, being clearly explained, what is meant by it, and explicated by Numbers,

bers, is so very intelligible, that, to any tolerable Capacity, it does not require other Demonstration; and all that is built on that knowledge is, I am confident, as solid and permanent, and as securely established, as by all the Demonstration which can possibly be given; at least, on the foundation of Euclid's fifth Definition.

In respect of commensurable Quantities, to all which Numbers may be applied, whether their Ratios are analogous or not, is readily discovered; since, in Analogy, the Ratios being equal, the same Numbers will express either.

E. g. $56:98::12:21$; which being reduced to their lowest Denomination, the Ratio, of each, is determined. The first pair, 56 and 98, being divided, separately, by 14, gives 4 and 7, the true Ratio of that pair. Now, if the other pair produces the same, they are analogous; 12 and 21, divided singly, by 3, produce, also, 4 and 7; which is, therefore, the true Ratio of both; consequently they are analogous, seeing the Ratios are equal.

If there be taken Equimultiples of Ratios that are analogous, i. e. being multiplied various ways, either all by one Number, (any number whatever) or the first pair by one Number, and the second pair by another; or, if the first and third be multiplied by one Number, and the second and fourth by any other, the Products arising, from every such multiplication, are still analogous, in the Ratio of both Pairs.

Let $A:B::C:D$, in the Ratio of 3 to 5.

Thus, $9:15::3:5$.

So, $27:45::9:15$, being all multiplied by 3.

Also, $18:30::15:25$; the first Pair by 2, and the second by 5.

In both these Cases, the Ratio is still as A to B, or as C to D.

Again, $36:30::12:10$, { the first and third, being multiplied
by 4, are both greater than the second
and fourth, multiplied by 2,

Also, $45:45::15:15$, { the first and third multiplied by 5, the
second and fourth by 3; in which
case they are both equal.

And, being all multiplied by any one Number, as above, it is evident they will both be less. Or, being multiplied by various Numbers, they will both be less; which illustrates the fifth Definition of Euclid.

Now, it is very obvious, in Numbers, from any one, of these various operations, that there is analogy of Ratios between A and B, C and D; and it is full as obvious without. But, provided it is not known, whether the Ratios are analogous or not; no one, nor all of the three first trials are sufficient to determine it, merely from the first and third being, at the same time, both greater or both less than the second and fourth; because, those things may, and very frequently do, happen, where there is not Analogy.

E. g. Let the Ratio of A to B be as 3 to 4; and, $C:D::2\frac{1}{2}:4$.

In whole Numbers, $A:B::3:4$, or $12:16$, and $C:D::11:16$.

Now, let there be taken Equimultiples of A and C, and other equimultiples of B and D. e. g. Let the Antecedents be taken thrice, and the Consequents twice; it will be $3A:2B::9:8$, and $3C:2D::33:32$; by which multiplication, it is evident that the Antecedents are both greater than their Consequents.

But, seeing that the Ratios are not equal, it is impossible to take Equimultiples of the two Antecedents, and also Equimultiples of the Consequents, by which, the Products of the Antecedents shall be both equal, respectively, to their Consequents; and also, because their Ratios are not equal, such Equimultiples may be taken of them, that one Antecedent shall be greater, and the other less, than its respective Consequent.

E. g. Let A and C be taken four times, and B and D thrice, it will then be, $4A:3B::12:12$, and $4C:3D::44:48$;

Again; let A and C be taken 7 times, and B and D 5 times; it will be $7A:5B::21:20$; and, $7C:5D::77:80$.

Either of these operations is sufficient to evince, that the Ratios are not analogous; i. e. A is not to B, as C to D; for, it is impossible that equality of Antecedent to Consequent should happen in both pairs, (i. e. of A to B, and of C to D), unless they are analogous; which is the only indisputable proof of Analogy; for, one Quantity must necessarily be to itself, as any other Quantity is to itself. And since it is not possible for equality to happen, by multiplication any more than by division, between incommensurable Quantities, it is manifest, that Analogy of such Ratios cannot be absolutely determined by that Criterion.

Wherefore, the Demonstration, according to Euclid, in the fifth Book is not positive, in respect either of commensurable or incommensurable Quantities; but is only presumptive, at best, and is very obscure and unsatisfactory; since it is manifest, that, in unequal Ratios, Equimultiples may be so taken (according to Euclid) in which the Antecedents will be either both greater or both less than their Consequents (as it has been shewn above) but, it is not possible they can be taken, so, that both Antecedents shall be equal to their Consequents.

I am therefore of Opinion, that, the Doctrine of Proportion is not properly investigated by Equimultiples; nor can I conceive, that taking multiples of Quantities, in order to prove the equality of their Ratios, is properly geometrical. It seems, to me, more consistent to compare Quantities by themselves, either whole and entire, or by their equal Parts.

But it is manifest, that there are Ratios, which will not admit of division into the same Parts, i. e. into Parts of the same magnitude; yet it is certain, that they may, with equal ease and more propriety, be divided into the same number of Parts, as to take Equimultiples of them; which Method I have therefore adopted, as the most rational and eligible.

E L E M E N T S O F G E O M E T R Y.

B O O K VI.

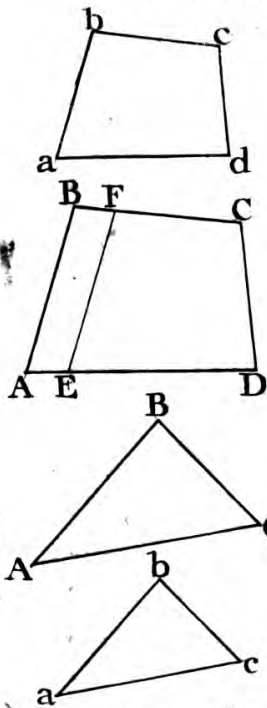
THE sixth Book of Elements is not only the most useful and valuable of the whole, but also the most entertaining and instructive. In it, the Doctrine of Proportion is applied to real use, in finding out the more extraordinary properties of Plane Figures, (particularly Triangles and Parallelograms) by the most subtle and solid reasoning that can be conceived: by means of which, many excellent and extensively useful Problems are found out, and applied to various and most notable purposes.

In it, the Rule of Three, or Proportion, called by way of eminence, the Golden Rule, has its foundation and existence (Theo. 9.) in it, is also demonstrated, with more facility, that, not only the Square of the Hypotenuse is equal to the two Squares of the Sides, of every right angled Triangle, but likewise, that it is so, in all similar Figures whatever, regular or irregular, constructed on the three sides, for the corresponding ones of each Figure. (Theo. 16.) In it, is determined the just proportion which exists between two similar Figures of any kind, and exhibited by two Right Lines (12 and 13) by which means, any Figure, whatever, may be constructed, in any Ratio to a given one, (Prob. 37 and 16 ;) or, the Side of a Square may be determined, whose Area shall be equal to that of any given right lined Figure, whatever. (Prob. 24 and 25.) And, what is more extraordinary, a right lined Figure may be constructed similar to any given Figure, and equal to any other, of different Forms and Denominations, being right lined. (Prob. 18.)

In it, several valuable properties of the Circle are briefly demonstrated; in short, to see and admire all which it contains, I recommend the careful and attentive perusal of it, as the readiest and most certain means, by which its beauties and excellencies can possibly be communicated.

DEFINITIONS.

Def. I. SIMILAR FIGURES are such as have all their Angles equal, each to the other, respectively; and also, the Sides which are opposite to equal Angles, or which lie between equal Angles are proportional.



The Quadrilateral $abcd$, having all its Angles equal, respectively, to the Angles of the Quadrilateral $ABCD$, viz. a equal A , b equal B , &c; and, if, the Side ab have the same Ratio to AB , as bc has to BC , &c. then, $abcd$ is similar, or like, $ABCD$.

If EF be drawn parallel to AD , the Angles at E and F , are equal to A and B , respectively (4. 1.) consequently to a and b ; wherefore, all the Angles of the Quadrilateral $abcd$, are equal to those of $EFCD$; but, seeing the corresponding Sides, ab and EF , bc and FC , &c. are not proportional with cd and CD , $abcd$ is not similar to $EFCD$.

Triangles ABC , abc , having all the Angles of one, equal, respectively, to the Angles of the other, have their Sides necessarily proportional, and are, consequently, similar Figures.

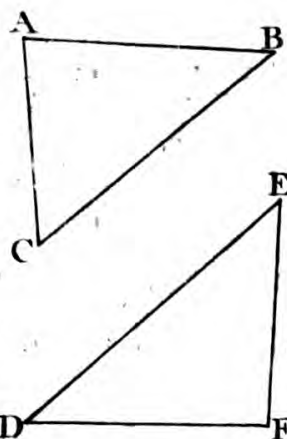
The Sides ab , AB ; bc , BC , &c. being opposite to equal Angles, or which lie between equal Angles, are **HOMOLOGOUS** or corresponding Sides.

N. B. All ordinate Figures, whatever, i. e. such as are equilateral and equiangular, one with another, are similar Figures; as equilateral Triangles, Squares, &c. Also all Circles are similar Figures.

Def. II. In Triangles, and Parallelograms, if there be two Sides, in one, which, with two Sides in another, of the same kind, and about the same Angle, are Proportionals; in such wise, that the Antecedent of one, and the Consequent of the other pair, are in the same Figure, those Figures are said to be RECIPROCAL.

But more properly, the Ratio is reciprocal in such Figures; for, reciprocal Figures is unmeaning.

In the Triangles ABC, DEF; if the Sides AB and BC of the one, and DE, DF, of the other, are Proportionals; and, if AB is to DE, as DF, of the same Triangle, is to BC of the other; then, the Proportionals are reciprocal in those Figures; and they are usually, though improperly, called reciprocal Figures.



Understand the same of Parallelograms.

N. B. Equiangular, or similar, Triangles are not reciprocal; because their Sides are all, respectively, proportional; which is not the Case, in the Figures annexed. For, EF is not necessarily in the same Ratio to AC, as are the other Sides, AB to DE, and DF to BC. Neither can there, in similar Figures, be found the Antecedent of one Pair, and the Consequent of the other, in the same Figure, unless they are congruous. (See Def. 43.)

The Sides of similar Figures are, therefore, directly proportional, and not reciprocally.

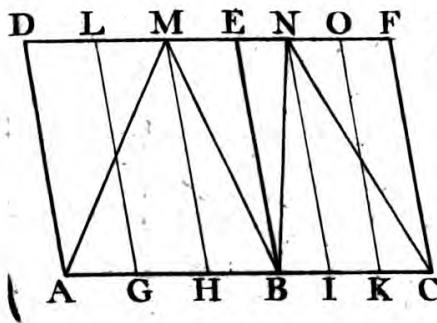
For the Definitions of BASE and ALTITUDE, see Def. 45 and 46 in the general Introduction.

A X I O M. If two, or more, Figures are similar to the same Figure, they are similar between themselves.

T H E.

THEOREM I. 1 Euclid.

Parallelograms, or Triangles, having the same Altitude, are, to each other, in the same Ratio as their Bases.



Let ADEB and BEFC be two Parallelograms, between the same Parallels, AC and DF.

I say, the Ratio of the Parallelogram AE to BF, (i. e. of their Areas) is, as AB to BC, their Bases.

And, the Triangle AMB is to BNC, as the Base, AB is to BC.

Let AB be divided into any number of Parts, equal to each other, in G and H; and suppose BC divided into the same number of equal parts, in I and K (Post. 2. 5.)

Draw, GL, HM, &c. parallel to AD and BE.

DEM. Because AG, GH, and HB are equal to each other; the Parallelograms AL, GM, and MB, are equal. - 18.1.

And, the Pars. BN, NK, and KF, are also equal - same

Now, if $AB = BC$, the parts AG &c. are equal to BI, &c. If AB be greater than BC, the part AG is greater than BI; and, if AB be less than BC, AG is less than BI.

Consequently, the Parallelograms AL, GM &c. are, also, either equal to, greater, or less, than BN, NK, &c. as, AG, &c. is equal to, greater or less than BI, &c.

For, Quantities are in the same Ratio, to each other, as their Equimultiples, or equal Parts. - - Ax. 8.

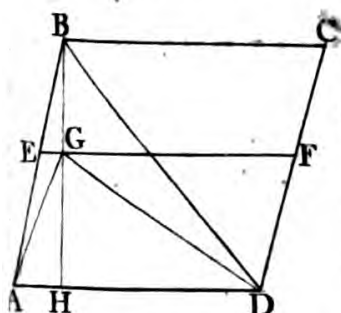
Now,

Now, $AG:AB::BI:BC$ }
 And, Par. $AL:AE::BN:BF$ } in equal Ratio, - Ax. 7. 5.
 Conf. Par. $AL:BN::AG:BI$, i.e. $::AB:BC$, (P. 1. 5.) - 4. 5.
 But, Par. $AL:BN::AE:BF$. - - - same.
 Th. Par. $AE:BF::AB:BC$. - - - Ax. 13. 5.

2nd. Because Triangles are equal to half Parallelograms,
 having the same Base and Altitude; - - - 17. 1.
 the Parallelogram $AE:BF::\Delta.AMB:BNC$ - Ax. 8. 5.
 and, Parallelogram, $AE:BF::AB:BC$ - above;
 conf. the Triangle $AMB:BNC::AB:BC$. - 13. 5.

COR. Parallelograms or Triangles having the same or equal
 Bases, have the same Ratio to each other, as their Altitudes.

For, if their Bases be considered as their Altitudes, and
 their Altitudes as their Bases, it is evident from the Theorem.



Let $ABCD$ and $AEFD$ be two Parallelograms, on the same Base AD .

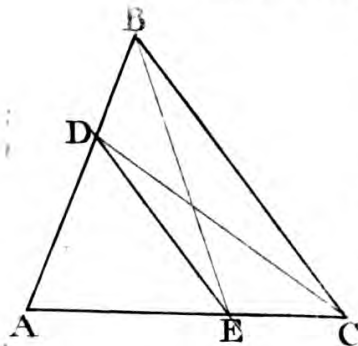
Draw the Perpendicular BH ; - (Prob. 7.)
 BH and GH are their Altitudes. - Def. 46.

Then the Parallelogram $AC:AF::BH:GH$.

For, since all Parallelograms having the same Base and
 Altitude are equal, the Parallelograms, AC , AF , are equal
 to Rectangles, whose Bases are equal to AD , and their
 other Sides equal to BH and GH , respectively. - 18. 1.
 Wherefore, BH and GH being considered as the Bases of
 the Parallelograms AC and AF , and their Altitudes, AD ;
 the Parallelogram $AC:AF::BH:GH$ - - - Theo.
 But, the Triangle ABD is half the Par. $ABCD$, }
 and, the Triangle AGD is half the Par. $AEFD$, } - 17. 1.
 Conf. the Triangle $ABD:AGD::BH:GH$. - Ax. 13. 5.

THEOREM II. 2 Euclid.

If a Right Line be drawn, parallel to any Side of a Triangle; it will cut the other two Sides (produced if necessary) proportionally.



First. In the Triangle ABC, let DE be drawn, parallel to BC; cutting the Sides AB and AC, in D and E. - (Prob. 5.)

Then will AB and AC, be cut proportionally, in D and E.

i. e. AD to DB, as AE is to EC.

Draw BE and DC.

DEM. Because DE is parallel to BC, $\angle DBE = \angle EDC$ - 18.1.

Wh. the Triangle ADE:DBE::ADE:EDC. - Ax.4 5.

But, the Triangle ADE:DBE:: AD : DB; }
and, the Triangle ADE:EDC:: AE : EC. } - Th.1.

Therefore, - - AD : DB :: AE : EC. - Ax.13.5.

Case 2. If ADE be the given Triangle; let BC be drawn (without it) parallel to DE.

Then, if the Sides AD, AE be produced, cutting BC, in B and C; the Sides, AD and AE, of the Triangle, ADE, are proportional to the Segments, DB and EC, of those Sides, produced to B and C.

For, as AD : DB :: AE : EC; - - - as above.
conf. as AD : AE :: DB : EC. - - - 4. 5.

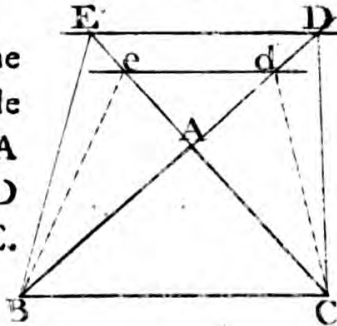
COR. I. If two Sides of a Triangle (AB, AC) are cut proportionally from the same Angle, (A, in D and E; so, that, AD:DB::AE:EC) a Right Line (DE) joining the Points (D and E) will be parallel to the remaining Side of the Triangle. The Converse.

For,

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For, the Sides, AB, AC , being cut, as $AD:DB::AE:EC$;
and the Right Lines, BE, DC , drawn, as before ;
 $\Delta. ADE:DBE::ADE:EDC$; (1.) conf. $DBE=EDC$.
Therefore, DE is parallel to BC ; - - Cor. to 18. 1.

CASE 3. If DE be drawn, beyond the
Vertex A , parallel to BC , the Side
opposite ; and, if the Sides BA and CA
be produced, till they cut DE , in D
and E ; then is AB to AD as AC to AE .



Join BE and DC ; as before.
The Triangle ACD is equal to ABE .

DEM. DE is parallel to BC ; wh. $\Delta DBE=DCE$ - 18. 1.
and, ADE is common to both ; th. $ACD=ABE$ - Ax. 7. 1.
Conf. the Triangle $ABC:ACD::ABC:ABE$ - Ax. 4. 5.
But, - - - $ABC:ACD::AB:AD$;
and, - - - $ABC:ABE::AC:AE$. } Theo. 1st.
Therefore, - - $AC:AE::AB:AD$. - Ax. 13. 5.

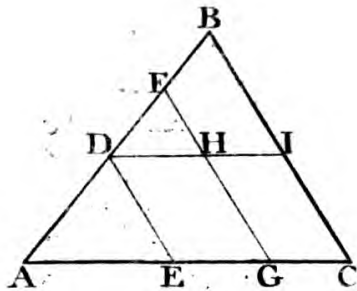
COR. 2. Two Right Lines (BD and CE) cutting each other
(in A) between, or beyond, two parallel Lines (DE and
 EC) are cut proportionally, in that Point, and the Points
in which they cut the Parallels.

For, $AB:AD::AC:AE$, in the 3rd Case, directly ;
and also, in the other two, by composition. - Th. 6. 5.

COR. 3. A Right Line drawn parallel to the Side of a
Triangle (in the first Case) cuts off a Triangle, ADE ,
similar to the whole Triangle ABC ; and, in the other
two Cases, the Triangles ADE and ABC are also
similar. (Def. 1.)

For, the Angle $ADE=ABC$, and $AED=ACB$ - 4. 1.
and, the Angle A is either common, or vertical.

COR. 4. If several Lines be drawn, within a Triangle, parallel to any Side; the corresponding Segments of the other two Sides, cut by the parallel Lines, are proportional; each to the other, respectively.



Let DE and FG be parallel to BC, in the Triangle ABC.

Then, $AD : DF : FB :: AE : EG : GC$.

For, the Segment $AD : DF :: AE : EG$. - Theorem.

Let DI be drawn parallel to AC; then, $DF : FB :: DH : HI$

But, EDHG and GHIC are Parallelograms, - by Constr.

wherefore, $DH = EG$, and $HI = GC$ - - 15. 1.

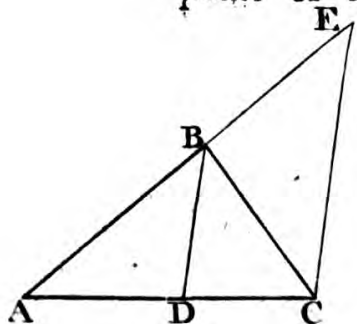
Therefore, $DF : FB :: EG : GC$ (as $DH : HI$) - Ax. 4. 5.

But, $AD : DF :: AE : EG$; th. $AD : FB :: AE : GC$ - 9. 5.

THEOREM. III. 3 Euclid.

If any Angle of a Triangle is bisected by a Right Line, cutting the opposite Side; the Segments of that Side, will be proportional to the two Sides of the Triangle, containing the Angle bisected.

And, conversely, if any Side of a Triangle be cut, in the proportion of the other two Sides; and if the greater Segment be contiguous to the greater Side; then will a Right Line, drawn from the point of section to the opposite Angle, bisect that Angle.



Let the Angle ABC, in the Triangle ABC be bisected by the Right Line BD, cutting the Side AC in D.

I say, AD is to DC, as AB is to BC.

DEM.

DEM. Produce AB indefinite; make $BE = BC$, and draw CE.

Then, because $BE = BC$, the Angle $BEC = BCE$, - 9.1.

(for, the Triangle CBE is Isosceles)

and the Angle ABC (which is external) $= BEC + BCE$ - 10.1.

Now, the Angle $ABD = DBC$, by Construction;

and $BCE = BEC$; consequently, $DBC = BCE$. - Ax. 3.1.

But, the Angles DBC, BCE, are alternate; and since they are equal; consequently, BD is parallel to CE - 4.1.

Wherefore, in the Triangle AEC, the Sides AC, AE, are cut proportionally, by the Right Line BD;

i. e. $AD : DC :: AB : BE$ - - - - - Th. 2.

But, $BE = BC$ (Con.) th. $AD : DC :: AB : BC$. - Ax. 4.5.

2nd. In the Triangle ABC, let the Side AC be so cut, in D, that the Segments, AD, DC, are in the Ratio of AB to BC.

Then, a Right Line, BD, bisects the Angle ABC

(the same Construction remaining as before).

Now, $BE = BC$ (Con.) and, $AD : DC :: AB : BC$ - Hyp.

wh. $AD : DC :: AB : BE$; conf. BD is parallel to CE - 2.

and consequently, the Angle $DBC = BCE$ - 4.1.

But, because (in the Triangle CBE) $BC = BE$, - Con.

the Angle $BCE = BEC$; i. e. AEC . - 9.1.

and, the Angle $ABC = BCE + BEC$. - 10.1.

But, the Angle $DBC = BCE$; and $ABD = AEC$. - 4.1.

Therefore, ABD is equal to DBC, - - - Ax. 3.1.

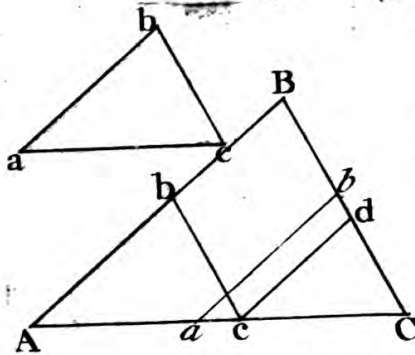
N. B. This Theorem and the foregoing are of extensive utility, and are particularly useful in Perspective.

The third Case of Theo. 2, and the following Corollary, contain the whole Substance of practical rectilinear Perspective.

T H E.

THEOREM IV. 4 Euclid.

In equiangular Triangles, the Sides containing equal Angles are proportional; and, the Sides which subtend equal Angles are homologous, or corresponding.



Let ABC , and abc , be equiangular Triangles, having the Angle a , equal to the Angle A , b equal B , and c equal C .

I say, the Side ab , is in proportion to AB , as ac is to AC , and, as bc to BC .

DEM. Suppose the Angle a applied to, or laid upon the Angle A , and ab upon AB . - - - Post. 5. 1. Then, because the Angle $bac = BAC$, ac will fall on AC , and bc will cut the Triangle ABC , in bc . But, the Angle $ABC = Abc$; and are internal & opposite; wh. bc is parallel to BC (4. 1.) conf. $Ab : bB :: Ac : cC$ - 2. Wh. $Ab : Ab + bB$ (equal AB) :: $Ac : Ac + cC$ (eq. AC) - 6. 5. Therefore, as $ab : AB :: ac : AC$.

Again; suppose the Triangle abc applied to the Angle C , of the Triangle ABC ; (as abC) then will the Side ab (i. e. ab) be parallel to AB . - - - Cor. 1. Th. 2. The rest is as before; i. e. $bC : BC :: aC : AC$. But $aC = Ac$, &c. and, $Ac : AC :: Ab : AB$, - above. consequently, bC (i. e. bc) : $BC :: Ab$ (i. e. ab) : AB .

It is thus demonstrated, according to Euclid.

Let

Let Abc and cdC be equiangular Triangles, so applied, at the Angle c , that the two Sides, Ac and cC , are in a Right Line.

Produce Ab and Cd , meeting in B .

Then, because the Angle $Ac b = ACd$, & $Ccd = CAb$, AB is parallel to cd ; and, CB parallel to cb - 4.1.
 wherefore, $cbBd$ is a Parallelogram; - Def. 33.
 consequently, $bB = cd$, and $Bd = bc$. - 15.1.
 But, $Ab : bB :: Ac : cC$; - Th. 2.
 wherefore, $Ab : cd$ (equal bB) $:: Ac : cC$; }
 also, as $Ac : cC :: bc$ (equal Bd) $: dC$ } Ax. 4 5.
 consequently, as $Ab : cd :: bc : dC$. - 13.5.

COR. 1. Triangles, having all their Sides directly proportional, are equiangular. The Converse; needs no proof.

COR. 2. Triangles, being equiangular, or having all their Sides directly proportional, are similar. - see Def. 13

Consequently, if two Angles, of one Triangle, are equal, respectively, to two Angles of another Triangle, the Triangles are similar; and their Sides proportional.

For, the two remaining Angles are equal - 10.1.

Hence is deduced the 15th Problem, and also the 16th.

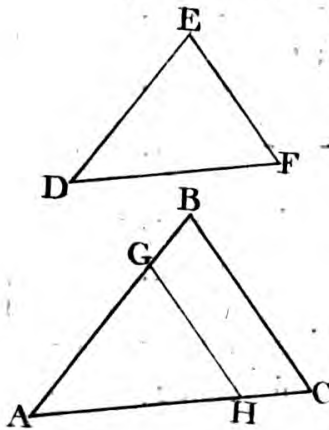
N. B. This Theorem is of universal application. It is the Criterion of Proportion in every branch of the Mathematics; for, wherever there is found equiangular Triangles, seeing that they are necessarily similar, the Proportion of their corresponding Sides, is consequently analogous.

The first Corollary, the Converse of the Theorem, is the 15th Proposition of Euclid.

THE-

THEOREM V. 6 Euclid.

If, in two Triangles, an Angle of one be equal to an Angle of the other, and, if the Sides containing the equal Angles are directly proportional, the Triangles are similar.



In the Triangles ABC, DEF, let the Angle BAC be equal EDF; and, as AB is to DE, so let AC be to DF.

Let the Triangle DEF be applied to the Triangle ABC; the Angle D to the equal Angle A. - - - Post. 5. 1.

Or, from the Angle A (equal D) take AG, equal DE, and AH equal DF, and join GH. - - - (Prob. 3.)

DEM. Now, because $AB : DE :: AC : DF$; - (Hyp.)
consequently, $AB : AG :: AC : AH$. - Ax. 5. 5.
wh. $AB - AG (GB) : AG :: AC - AH (HC) : AH$, - 7. 5.
i. e. $GB : AG :: HC : AH$; or, as $AG : GB :: AH : HC$ - 5. 5.
wherefore, GH is parallel to BC; - - - Cor. 1. 2. 6.
therefore, the Triangle AGH is similar to ABC. - C. 3.
But, the Triangles AGH, DEF are congruous - 8. 1.
Therefore, the Triangle DEF is similar to ABC-Axiom.

The 7th Proposition of Euclid, is of little or no consequence, as it seldom, if ever, occurs; it is rather a critical remark, than a necessary Proposition.

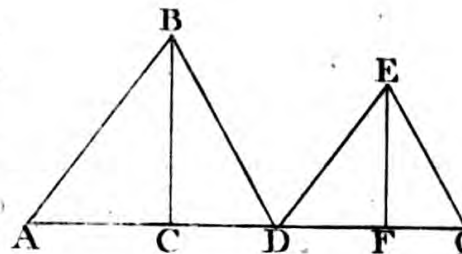
THEOREM VI.

In similar Triangles, the Perpendiculars, from equal Angles, are proportional to the corresponding Sides.

And, corresponding Sides are cut proportionally by the Perpendiculars.

Let, the Triangles ABD, DEG be similar, whose corresponding Sides are AB and DE, AD and DG, BD and EG.

I say, the Perpendiculars BC, EF, from the equal Angles, B and E, are proportional to the Sides.



DEM. For, the Triangles ABC, DEF, also CBD, FEG are similar; because, the Angles, at C and F, are all Right (Def. 11.) therefore equal; - - Ax. 9. 1. the Angle A = EDF, and BDC = G; by Hypothesis; consequently, ABC = DEF, and CBD = FEG. - 10. 1. Wherefore, as AB : DE :: BC : EF. } But, - - AB : DE :: AD : DG. } - Th. 4. Therefore, - BC : EF :: AD : DG. } Also, - - BC : EF :: BD : EG. } - Ax. 13. 5.

2nd. AD and DG are cut proportionally, in C and F, by the Perpendiculars BC and EF.

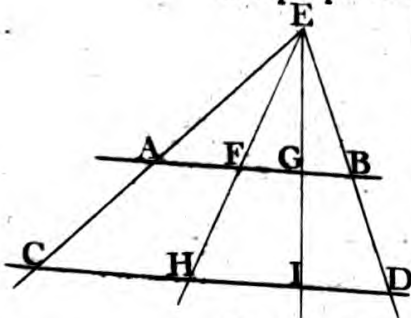
Becausc AC : BC :: DF : EF; } and, - - BC : CD :: EF : FG } - - Th. 4. conf. - as AC : CD :: DF : FG. - - - 9. 5.

COR. If parallel Lines are cut by any number of Lines, proceeding from the same Point, they will be cut proportionally.

P p

And

And, the Lines which proceed from a Point will also be cut proportionally, by the parallel Lines.

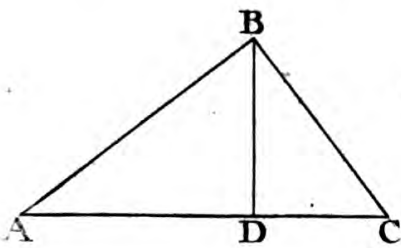


Let AB and CD be parallel Lines.
From any Point, E, at pleasure, if the Right Lines EA, EF, &c. are drawn, cutting the Parallels, in A and C, F and H &c; they will be cut proportionally, in those Points.

For, in the Δ s. AEF, CEH, the Angle EAF = ECH - 4.1.
also, AEF is common to both Triangles;
consequently, the Triangles AEF and CEH are similar.
And, for the same reason, the Triangles FEG and HEI;
also, GFB and IED are, respectively, similar.
Wherefore, $EA : EC :: EF : EH :: EG : EI :: EB : ED$.
consequently, $AF : CH :: FG : HI$, and as $GB : ID$.
Therefore, $CH : HI : ID :: AF : FG : GB$.
Also, $EA : AC :: EF : FH :: EG : GI$, and as $EB : BD$.
Hence is deduced the 36th, a most useful Problem.

THEOREM VII. 8 Euclid.

In a right angled Triangle, if a Perpendicular be drawn, from the Right Angle to the Hypotenuse, it will divide that Triangle into two Triangles, which are similar to each other; and also to the whole Triangle.



In the Right angled Triangle, ABC, and, from the Right angle, B, draw the Perpendicular BD, to AC.

I say, the Triangles ABD, DBC, are similar, to each other, and also to the whole Triangle, ABC.

DEM,

DEM. The Angle $ADB = ABC$ (Con.) and BAD is common;
consequently, the Angle $ABD = BCD$ - - - 10. 1.
Wherefore, the Triangles ABD , ABC are similar - 4.

Again; the Angle $ADB = BDC$ (Def. 10.1)
and $ABD = BCD$ (above) wherefore, $BAD = DBC$ - 10. 1.
Consequently, the Triangle ABD is similar to DBC - 4.
But, the Triangle ABD is similar to ABC - proved
Therefore, the Triangle DBC is similar to ABC - Axiom.

COR. 1. In right angled Triangles, the Perpendicular (BD)
is a mean Proportional between the Segments (AD & DC)
of the Hypothenufe, made by the Perpendicular.

For, the Triangles ABD , DBC , being similar, the Side AD
(of the Triangle ABD) is to BD (of the same) as the Side BD
(of the Triangle DBC) is to DC (of the same)
i. e. $AD : BD :: BD : DC$, - - - Th. 4.

From hence, and from Theorem 12 of the 3rd Book, are deduced the 30 and 31st Problems, in Practical Geometry; which are of most extensive utility.

COR. 2. In right angled Triangles, each Side containing
the Right Angle, is a mean Proportional, between the
adjoining Segment, and the whole Hypothenufe.

AB is a mean Proportional between AD and AC ;
and BC is a Mean, between DC and AC .

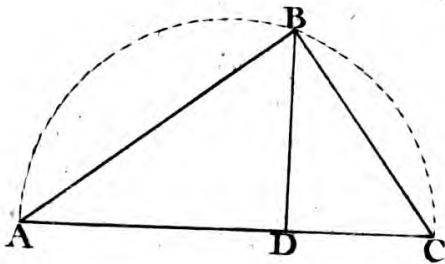
For, $AD : AB$ (of the Tri. ABD) :: $AB : AC$, of the Tri. ABC }
And, $AC : BC$ (of the Tri. ABC) :: $BC : DC$, of the Tri. DBC } - 4.

COR. 3. A Perpendicular, from any Point in the circum-
ference of a Circle, to a Diameter, is a mean Proportional,
between the two Segments of the Diameter, made by the
Perpendicular.

For every Angle, touching the Circumference, in a Se-
micircle, is a Right one (12. 3.) consequently, the Dia-
meter is the Hypothenufe of a right angled Triangle, in-
scribed in the Circle.

From hence, may easily be deduced the following Problems:

PROBLEM I. *The two Sides, or Legs, of a right angled Triangle being given, how to find the Hypotenuse, the Perpendicular, and the two Segments of the Hypotenuse, arithmetically.*



In the right angled Triangle, ABC, is given the two Legs, AB and BC.

1st. To find the Hypotenuse.

$$AB^2 + BC^2 = AC^2 \quad - \quad 20. 1.$$

Consequently, the Squares of AB and BC, being added together, the Square Root, of that Product, gives AC.

2nd. To find the Perpendicular, BD; $AC : AB :: BC : BD$ - 4.

BD is, therefore, a fourth Proportional; AC, AB, and BC; being the three Terms given, and BD is required - Def. 11. 5.

Consequently, $AB \times BC = AC \times BD$ - (See Th. 9.)

i. e. if AB be multiplied by BC, and that Product divided by AC (as in the Rule of Three) the Quotient arising is BD.

For, the Divisor multiplied into the Quotient is equal to the Dividend; i. e. AC multiplied into BD, is equal to AB multiplied into BC; by Theorem 9th.

Therefore, the Perpendicular, BD, is a fourth Proportional.

Hence, a Rectangle, under the two Legs of a right angled Triangle, is equal to a Rectangle under the Hypotenuse and the Perpendicular.

3rd. To find the greater Segment, AD; $AC : AB :: AB : AD$ - 4.

Wherefore, AD, the greater Segment of the Hypotenuse, is a third proportional to AC and AB. - See Def. 9. 5.

Conf. $AB^2 = AC \times AD$; or, the Rect. CAD (see Cor. Th. 9.)

i. e. AB multiplied by itself, and the Product divided by AC, gives AD.

Otherwise; $BC : BD :: AB : AD$.

Conf. AD is a fourth Proportional; and $AB \times BD \div BC = AD$.

4th. To find the lesser Segment, DC; $AC : BC :: BC : DC$; or $AB : BC :: BD : DC$. Also, $AD : BD :: BD : DC$.

When either Segment, AD or DC, is found, the other is a third Proportional, between that Segment and the Perpendicular, BD; conf. $AD \times DC = BD^2$. But, if the Hypotenuse, AC, and either Segment be found, then, $AC - AD$ is equal DC; and $AC - DC = AD$.

PROB. II. *The Perpendicular and either Side being given, how to find the other Side.*

Let AB and BD be given.

Because $AB^2 = AD^2 + BD^2$; - - 20. 1.
consequently, $AB^2 - BD^2 = AD^2$.
i. e. the square of BD being subtracted from the square of AB,
the square Root of the remainder gives AD.
Then, as $AD:AB::BD:BC$; wh. BC is a fourth Proportional:
Conf. $AB \times BD \div AD = BC$. For $AB \times BD = AD \times BC$.

PROB. III. *The Perpendicular and either Segment being given, to find the other Segment, and the two Sides.*

Let AD be the given Segment.

Then, as $AD:BD::BD:DC$. Wh. DC is a third Proportional.
Conf. BD multiplied into itself, and that Product divided by
AD, gives DC. For $AD \times DC = BD^2$.
If DC had been given, it is just the reverse.

2nd. $AD^2 + DB^2 = AB^2$. And $BD^2 + DC^2 = BC^2$ - 20. 1.
Wherefore, if the square of AD be added to the square of BD,
the square Root of that Sum gives the Side AB.

Again; AB being found, BC is a fourth Proportional.
For, $AD:BD::AB:BC$. Wherefore, $AB \times BD \div AD = BC$.
i. e. the sum of AB multiplied by BD, divided by AD gives BC.

PROB. IV. *The two Segments (AD and DC) being given, to find the Perpendicular (BD) &c.*

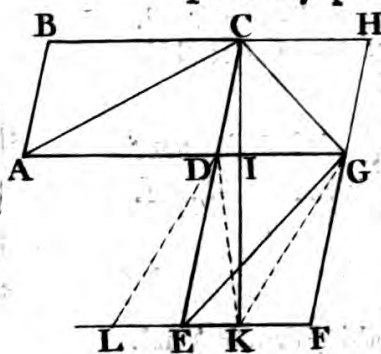
BD is a mean Proportional between the Segments AD & DC.
Wherefore, as $AD:BD::BD:DC$;
consequently, $AD \times DC = BD^2$. - - (see Cor. Th. 9.)
i. e. If AD be multiplied by DC, the square Root of that
Product, is BD.
Hence, the Sides, AB and BC, may be found, as in Problem 3.

These Problems, deduced from this extraordinary and extensive Theorem, are extremely useful in Perspective; to find Vanishing Points and their Distances, &c.

THE O-

THEOREM VIII. 14 & 15 Euclid,
reversed.

Parallelograms, having equal Angles; or, Triangles, having one Angle, in each, equal to one another; and the Sides, containing the equal Angles, reciprocally proportional, are equal.



Let $ABCD$ and $DEFG$ be Parallelograms, having equal Angles; and as the Side AD , of the one, is to DG , of the other, so let DE , of the same, be to DC of the first.

Let the equal Angles, at D , be so placed together, that, the Side AD , of one Parallelogram, and DG of the other, are, in one Right Line; then will DE and DC be also in one Right Line. - 2. 1.

Produce BC and FG , meeting in H , forming a Parallelogram $CDGH$.

DEM. Then, the Par. BD is to DH , as AD is to DG } T. 1.
and, the Parallelm. DF is to DH , as ED is to DC }
But, $AD : DG :: ED : DC$; by Hypothesis;
wherefore, the Parallelograms, BD and DF , have an equal Ratio to the same Parallelogram, DH .

Therefore, the Parallelogram $ABCD = DEFG$ - Ax. 5. 5.

2nd. Having drawn the Diagonals AC and EG ; the Triangle ACD is equal to DEG ; - - - Ax. 4. 1.
for, each is equal to half the Parallelogram BD or DF - 17. 1.

Or (CG being drawn) the Tri. $ACD : DCG :: AD : DG$,
and, - - - the Triangle $DEG : DCG :: ED : DC$.
But, $AD : DG :: ED : DC$ (Hyp.) conf. Tri. $ACD = DEG$,
for, they have an equal Ratio to the Triangle DCG .

Cor. 1.

COR. 1. Equal Parallelograms having equal Angles, or Triangles having one Angle in each also equal, have their Sides about the equal Angles reciprocally proportional.

This, being the converse of the Theorem is manifest, from it; for, by the same construction, it is thus reversed.

Because, the Parallelogram BD is equal to DF, they have an equal Ratio to the Parallelogram DH - - - Ax. 4. 5.

But, Par. BD:DH::AD:DG; and, Par. DF:DH::ED:DC - 1.

Therefore, as AD:DG::ED:DC - - - - - Ax. 13. 5.

Also, the Triangle ACD is equal to half the Par. BD; and DEG is equal to half DF (17. 1.) consequently, ACD=DEG.-Ax. 4. 1. and, their Sides, AD, DC; DE, DG, are reciprocally proportional.

COR. 2. Parallelograms, or Triangles, having their Bases and Altitudes reciprocally proportional, are equal.

For, whether their Angles be equal or not, the Parallelograms are equal to Rectangles having the same Base and Altitude - 18. 1. Consequently, if their Bases (AD and EF) and their Altitudes (CI and IK) are reciprocally proportional, the Parallelograms, BD and DF, or the Triangles, ACD, DEG, are equal - Th. 1.

For, the Perpendiculars (i. e. the Altitudes) of Triangles and Parallelograms, having equal Angles, are in the same Ratio as their Sides; i. e. CI:IK::CD:DE - - - - - Th. 6; or 2.

But, their Bases, EF and AD, are in the same Ratio.

Therefore, CI:IK::EF, or KL (equal DG) : AD; - Ax. 13. 5.

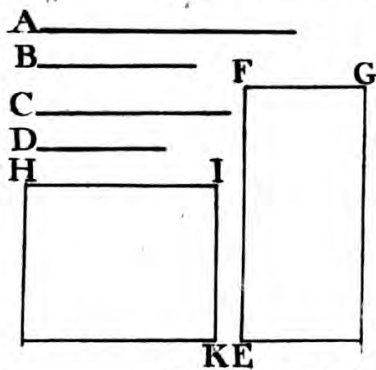
and consequently, whether the Angles, at D, are equal or not, the Parallelogram ABCD=DGKL (equal DEFG.) }

Also, the Triangle DKG=ADC (equal DEG.) } - 18. 1.

T H E O.

THEOREM IX. 16 Euclid.

If four Right Lines are Proportionals, the Rectangle under the two Extremes (i. e. the greatest and the least) is equal to the Rectangle contained under the two Means.

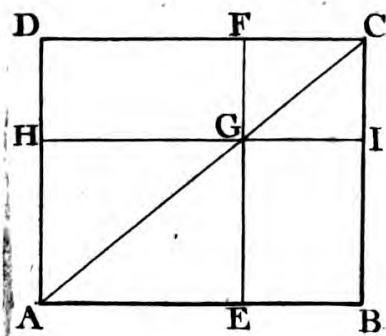


Let the four Right Lines, A, B, C, D, be proportional; and as A is to B, so let C be to D.

Then will the Rectangle under A and D, be equal to that under B and C.

Construct the Rectangles EG and HK, making EF equal A, and FG equal D; also, let HI be equal C, and IK equal B. (Prob. 18.)

DEM. Then, because EG and HK are Rectangles, consequently their Angles are all equal. (Def. 34.)
But, as $EF : IK :: IH : FG$; i. e. as $A : B :: C : D$; wherefore, their Sides are reciprocally proportional.
Therefore, the Rect. EG, is equal to the Rect. HK. - 8.



Otherwise :

Take AB equal to the greatest of the four proportional Lines, and, at either Extreme, B, make a Right Angle, ABC.

Make BC equal to the greatest of the Means; and AE equal to the other.

Compleat the Rectangle ABCD, and draw the Diagonal, AC. - - (Prob. 18.)

Thro' E draw EF, parallel to BC; and through G, where it cuts the Diagonal, draw HI parallel to AB. (Prob. 5.)

DEM.

DEM. Now, because AB, BC, and AE are equal to three of the Proportionals, EG is equal to the fourth;
 for, as AB is to BC so is AE to EG. - - Th. 4.
 And, since the Complements, DG, GB, are equal - 19. 1.
 if HE be added to both, the Rect. ADFE = AHIB - Ax. 6. 1.
 But, AHIB is a Rect. under the two Extremes. }
 And, ADFE is a Rect. under the two Means. } - Con.
 Therefore, the Rectangle under the two Extremes, of four proportional Lines, is equal to a Rectangle under the Means.

Corollary. The 17th Proposition of Euclid.

If three Right Lines are Proportionals, the Rectangle under the two Extremes, is equal to the Square of the Mean.

If the two middle Terms, B and C, had been equal, the Rectangle HK would be a Square. (Def. 35.) See Fig. 1.
 But, as $EF : IK :: IH : FG$; and IH is equal to IK. - Sup.
 Therefore, the Rectangle under the Extremes, of three proportional Lines, is equal to the Square of the Mean.

Or, if (by the 2nd Construction) AE had been equal to BC, the Rectangle ADFE would be a Square.
 Conf. it would be equal to the Rect. AHIB under the two Extremes. For, as $AB : AE$ (eq. BC Sup.) $:: AE : EG$, eq. BI.

From hence, and Prop. 19. 1. is deduced the last method (Prob. 32) for finding a fourth Proportional.

A third Proportional may also be found, by the same; viz. by constructing a Square on either of the given Lines, which is to be the Mean.

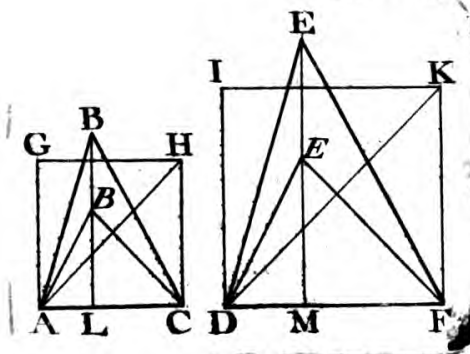
From which Constructions, it is obvious, is deduced that most excellent, Golden Rule, or Rule of Proportion, in Arithmetic.

N. B. If a Right Line be so divided in two Parts, that the Rectangle, under the whole Line and one Segment, is equal to the Square of the other; it is divided in extreme and mean Proportion in that Point.

For, the whole Line, the greater Segment, and the less, are in continual Ratio; consequently, they are three Proportionals.

THEOREM X.

Similar Triangles are proportional to the Squares of their corresponding Sides.



Let ABC , DEF be similar Triangles, having the Angles at A and D equal; the Angle B equal E , and C equal F .

Then, Squares constructed, on corresponding Sides, AC and DF , or any other, have the same Ratio to each other, respectively, as the Triangles, ABC and DEF .

Describe the Squares $AGHC$ and $DIKF$, on the Sides AC and DF , which are homologous; being opposite equal Angles, B and E .

Draw the Diagonals AH , and DK ; also the Perpendiculars BL , and EM , to the Sides AC , and DE .

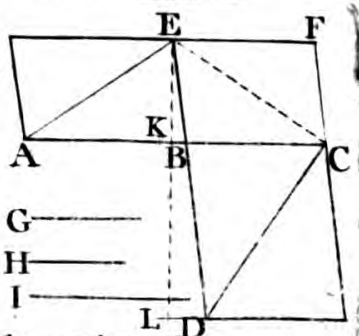
DEM. The $\triangle ABC : AHC :: BL : HC$ (eq. AC , Con) }
 Also, the $\triangle DEF : DKF :: EM : KF$, equal DF . } C. Th. 1.
 But, - as $BL : AC :: EM : DF$; - - - Th. 6.
 wherefore, $\triangle ABC : AHC :: DEF : DKF$ - - Ax. 13. 5.
 conf. - $\triangle ABC : DEF :: AHC : DKF$ - - Th. 4. 5.
 But, $\triangle AHC : \square AGHC :: \triangle DKF : \square DIKF$ - 17. 1. & Ax. 7. 5.
 Wh. $\triangle ABC : \square AGHC :: \triangle DEF : \square DIKF$ - Post. 1.
 Th. $\triangle ABC : DEF :: \square AGHC : DIKF$ - Th. 4. 5.
 That is, the $\triangle ABC : DEF :: AC \square : DF \square$. Q. E. D.

THEOREM XI. 23 Euclid.

Equiangular Parallelograms are, to one another, in a Ratio which is compounded of the Ratio of their Sides.

Let AE and DC be equiangular Parallelograms.

I say, the Ratio of the Parallelogram AE to DC, is equal to the compounded Ratio of their Sides. i. e. of the Ratio of AB to BC, and of EB to BD.



Let the equal Angles, at B, be so placed together, that, the Sides, AB, and BC, also DB and BE, are in a Right Line; and produce the opposite Sides, meeting at F, forming a Parallelogram BEFC.

Take any Right Line, G, at pleasure; and, as BC is to AB, make H to G;

also, as DB is to BE, make I to H. (Prob. 32.)

DEM. Now, the Ratio of G to H is the same as AB to BC; and, the Ratio of H to I is equal to that of EB to BD. - Con. But, the Ratio of G to I, is compounded of the Ratios of G to H, and H to I (Def. 21. 5.) consequently, the Ratio of G to I is that, compounded of the Ratios of the Sides; to which, these are respectively equal.

But, the Par. AE:EC::AB:BC; i.e. as G to H. }
And, the Par. EC:DC::EB:BD; i.e. as H to I. } - Th. 1.

Wh. since the Par. AE:EC::G:H; and Par. EC:DC::H:I.

The Parallelogram, AE: Par. DC::G:I. - Th. 9. 5.

But, G is to I, in the compounded Ratio of the Sides.

Therefore, the Par. AE is to Par. DC in the Ratio, which is compounded of the Ratio of their Sides. Q. E. D.

Qq 2

COR. 1.

COR. 1. Triangles, having one Angle in each, equal to one another, have that Ratio between them, which is compounded of the Ratio of the Sides, containing equal Angles.

For, the Triangle AEB, BCD, having equal Angles, at B, and drawing EC, the Demonstration is in the Theorem.

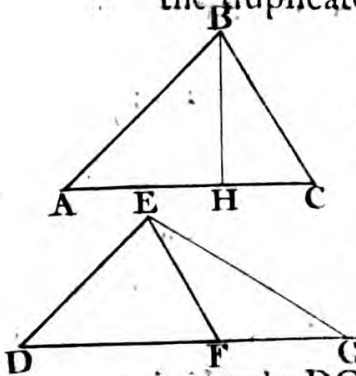
COR. 2. Parallelograms, having equal Angles, or Triangles, which have one Angle in each, equal to one another, have that proportion, to each other, as the Rectangles under the Sides containing the equal Angles.

For, the Sides, containing equal Angles, have the same proportion, to each other, as the Perpendiculars, EK, KL.- Th. 6. Consequently, Rectangles under the Bases and Perpendiculars, are in the same Ratio to each other, as Rectangles under the Bases and adjoining Sides, containing equal Angles, viz. in the Ratio which is compounded of their Sides; by Theorem.



THEOREM XII. 19 Euclid.

The proportion of similar Triangles, to each other, is the duplicate Ratio of their corresponding Sides.



Let the Triangles ABC, DEF be similar and alike situated; i. e. let the Angle A be equal D, B equal E, and C equal F;

The Side AB corresponds with DE, BC with EF, and AC with DF. Produce DF.

Take DG a third Proportional, to DF

and AC, and draw EG;

i. e. make DG to AC as AC is to DF; (Prob. 31.)

Then, the Triangle ABC is to DEF, as DG to DF.

DEM. Because the Triangles ABC, DEF are similar,

consequently, $AB : DE :: AC : DF$. - Th. 4.

But, $DG : AC :: AC : DF$ (Con) conf as $AB : DE :: DG : AC$.

Wherefore, in the Triangles ABC, DEG, the Sides AB and AC, DE and DG, which are about the equal Angles (A and D) are reciprocally proportional.

consequently, the Triangle $DEG = ABC$. - Th. 8.

But, the Triangle $DEF:DEG::DF:DG$ - - Th. 1.

Th. the Triangle $DEF:ABC::DF:DG$ - Ax. 4. 5.

and, by inversion, $ABC:DEF::DG:DF$. - Th. 5. 5.

i. e. in a duplicate Ratio of their corresponding Sides.

Or, if AH be taken a third proportional to AC and DF ;
it will then be, as $AB:DE::DF:AH$, i. e. $AC:DF$ - Con.

Consequently, the Triangle $ABH = DEF$, - Th. 8.

Wherefore, the Tri. $DEF:ABC::ABH:ABC$. - Ax 4. 5.

i. e. as AH is to AC (Th. 1.) viz. in a duplicate Ratio.

COR. If three Right Lines are Proportionals, the Squares
of the first to the second are as the first Line is to the third.

In the Tri. DEF, ABC , and $DEG, DF:AC:DG \div$ - Con.

But, the Triangle $DEF:ABC::DF:DG$. - Theo.

And the Triangle $DEF:ABC::DF^2:AC^2$. - 10.

Therefore, as $DF:DG$ (the first to the third) $::DF^2:AC^2$.

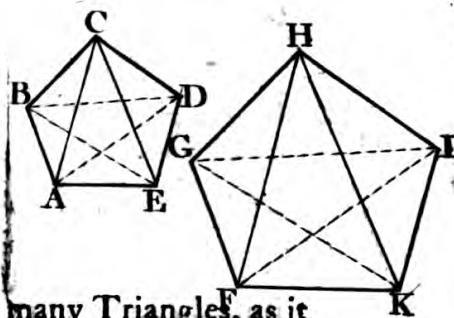
THEOREM XIII. 20 Euclid.

Similar Quadrilaterals, and Polygons, are, by drawing
Diagonals, divided into Triangles, equal in Number;
the corresponding Triangles are similar to each other,
respectively; and proportional to the Polygons.

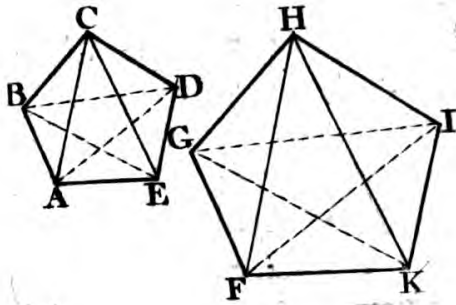
Also, the proportion of the Polygons, to each
other, is in the duplicate Ratio of their cor-
responding Sides.

Let $ABCDE$ and $FGHIK$ be
similar Polygons.

Draw the Diagonals AC, CE , and
 FH, HK , &c. from any correspond-
ing Angles, C and H , to the opposite.



DEM. Every Polygon is divided into as many Triangles, as it
has Sides, wanting two; by drawing all the Diagonals which
can be drawn, from any Angle; as CA, CE , or HF, HK ;
consequently, the number of Triangles, in each, are equal;
seeing the Polygons have an equal number of Sides.



2nd. Because the Polygons are similar, they are consequently equiangular (Def. 1.) and, being similarly posited, the Angle $A = F$, $B = G$, and $C = H$ &c.

Then, because $AB : BC :: FG : GH$, and the Angle $B = G$, the Triangles ABC , FGH are similar. - - Th. 5.

And, because the Angle $BAE = GFK$ and there is taken away, from each, equal Angles BAC , GFH , the remainder, $CAE = HFK$. - - Ax. 7. 1.

But, $AB : AC :: FG : FH$, and $AB : AE :: FG : FK$.

consequently, $AC : AE :: FH : FK$. - - 13.5.

therefore, the Triangles ACE , FHK are similar. Th. 5.

And, because $CD : DE :: HI : IK$, and the Angles D and I are equal ; the Triangles ECD , KHI are similar. - same.

3rd. Because the Triangles ABC and FGH are similar, their Ratio is duplicate of their corresponding Sides ; e. g. of AC to FH . - - Th. 12.

And for the same reason, the Ratio of ACE to FHK is also duplicate of AC to FH ; or, of EC to KH . - same.

Also, the Ratio of ECD to KHI , is duplicate of EC to KH .

But, the Ratio of each Triangle, ABC , &c. to its corresponding Triangle, FGH , &c. is the same, viz. as AC to FH , or EC to KH , &c.

Therefore, as one Antecedent, ABC , is to one Consequent, FGH , so is all the Antecedents ABC , ACE , and ECD , to all the Consequents, FGH , FHK , and KHI , taken together, (2. 5.) i. e. as Polygon to Polygon. Q. E. D.

4th. The Ratio of ACE to FHK is duplicate of AE to FK - 12

But, the Ratio of Polygon to Polygon is equal ACE to FHK .

Therefore, the Ratio of Polygon to Polygon, is duplicate, of AE to FK , of ED to KI ; or, of any other Sides, which are homologous.

4th.

COR. 1. All ordinate or regular Figures whatever, Triangles, Squares, or Poligons of any kind are, respectively, to each other, in the duplicate Ratio of their Sides, or Diagonals.

Because, all regular Figures, of the same Species, are similar. Also, Circles are, to each other, in the duplicate Ratio of their Diameters, &c.

Hence, in all similar Figures whatever, if the ratio of the homologous Sides be known, the proportion of the Figures is known.

For example; If the ratio of the Sides be as 2 to 3, their Areas are as 4 to 9. For, as $2 : 3 :: 3 : 4\frac{1}{2}$; Therefore, as 2 is to $4\frac{1}{2}$, so is one Figure to the other; which, in whole numbers, is as 4 to 9.

COR. 2. The Proportion of all similar Figures, whatever, are as the Squares of their corresponding Sides.

For, as the Triangle ABC or ACE, is to FGH, or FHK, so is Polygon to Polygon.

But, the Triangles, being similar, are, as the Squares of their homologous Sides; viz. as AB to FG, or AE to FK - Th. 10. Therefore, as the Square of AB is to FG, or of AE to FK, &c. so is Polygon to Polygon. - - - Ax. 13. 5.

COR. 3. The Perimeters or Circuits of similar Figures are, to each other, as their corresponding Sides or Diagonals.

For, since each Side AB, BC, &c. has the same Ratio to its corresponding Side FG, GH, &c; also, as Side is to Side, so is Diagonal to Diagonal, AC to FH, &c; consequently, as any one Side, or Diagonal, is to its corresponding Side or Diagonal, so is the sum of all the Sides $AB + BC + CD$, &c. to the sum of all the Sides, $FG + GH + HI$, &c. - 2. 5.

COR. 4. Any similar Figures whatever, described on the Mean and either Extreme of three proportional Lines, have the same Ratio to each other, as the two Extremes; i. e. as the first to the third.

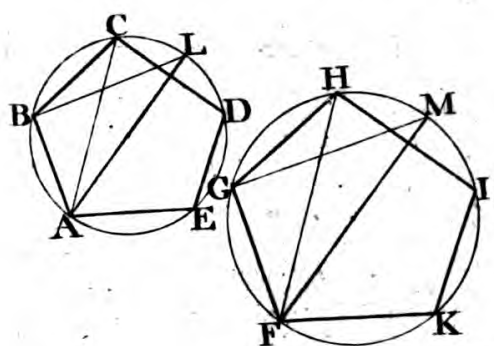
For, they are to each other, in the duplicate Ratio of their corresponding Sides; by the Theorem.

From hence is deduced, an excellent Problem, (37) for finding the Side of any Figure whatever, similar to another, and in any Ratio.

THE O-

THEOREM XIV.

The Perimeters of similar Poligons, inscribed in Circles, are, to each other, in the same Ratio as the Diameters ; and their Areas as the Squares of the Diameters.



Let ABCDE and FGHIK be similar Poligons, inscribed in Circles ; whose Diameters are AL, and FM.

Then, as AL is to FM, so is the Perimeter ABCDE to FGHIK.

Also, as AL square, is to FM square, so is the Area of the Poligon ACE to FHK.

Join AC and BL, also FH and GM.

DEM. Then, the Angle BCA = BLA, and GHF = GMF - 10.3.

But, the Angle BCA = GHF (13) Th. BLA = GMF - Ax. 3.1.

and, the Angle ABL = FGM (for they are R. Angles) - 12.3.

therefore, the Triangles BLA, GMF are similar ; - Th. 4. (for, the three Angles, are respectively equal)

Then, as AB : FG :: AL : FM. - - - 4.

conf. as AB $\square$: FG $\square$:: AL $\square$: FM $\square$. - 10.

But, as AB : FG :: Perimeter ABCDE : Per. FGHIK - 2.5.

Th. the Diameter AL : FM :: ABCDE : FGHIK - Ax. 13.5.

Also, as AB $\square$: FG $\square$:: the Area of ACE : FHK - Th. 10 & 13.

and conf. as AL $\square$: FM $\square$:: the Area of ACE : FHK - Ax. 13.5.

COR. The Circumferences of Circles are proportional to their Diameters, and their Areas to the Squares of their Diameters.

For, the Circumference of a Circle being considered as the Perimeter of a Poligon, of an infinite number of Sides, and, the Diameter may also be considered as a Diagonal ; wherefore, the Demonstration evidently follows from the Theorem.

COR. 2.

COR. 2. Circumferences of Circles are proportional, to the Perimeters of similar inscribed, or circumscribing, rectilinear Figures.

For the Perimeters, of all similar right lined Figures, are in proportion to their corresponding Sides or Diagonals. - C. 3. 13. And, the Circumferences of Circles, are as their Diameters. - Th.

COR. 3. The Areas of Circles are in proportion to that of any similar inscribed or circumscribing Figures.

For, the areas of all similar Figures, are as the duplicate ratio of their corresponding Sides or Diagonals; and, the areas of Circles are as the duplicate Ratio of their Diameters. - Th. 13,

Wherefore, since the Diagonals of similar Figures are in the same Ratio as their Sides, respectively; and also, as the Diameters of circumscribing Circles; consequently, the areas of Circles are, as the areas of similar Poligons, inscribed or circumscribed.

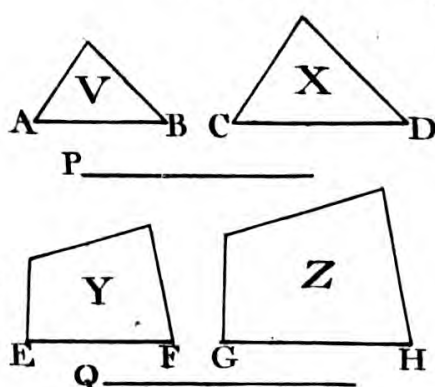
COR. 4. The area of a Circle is to that of any circumscribed right lined Figure, as the Circumference of the Circle to the Perimeter of the circumscribing Figure.

For, the area of a Circle is equal to a Triangle, whose Base is equal to the Circumference, and height, equal to the Radius. And, the Area of a circumscribing Polygon, is equal to a Triangle, whose Base is equal to the Perimeter of the Polygon, and its height equal to the Radius of the inscribed Circle. * Consequently, the Areas of all circumscribing Figures are as their Perimeters; which may be considered as the Bases of Triangles of equal height. - - - - - Th. 1. Therefore, the Area of a Circle is to that of a circumscribed Polygon, as the Circumference of the Circle to the Perimeter of the Polygon.

* See, Theory of Mensuration; Article 7, and 8.

THEOREM XV. 22 Euclid.

If four Right Lines are Proportionals; similar Right lined Figures, described on each Pair, will also be proportional.



Let the Right Lines AB, CD, EF, and GH be proportionals, and let AB be to CD, as EF to GH.

On AB and CD, let there be constructed similar Triangles, V & X; and, on EF and GH any other similar Figures, whatever, Y & Z.

Find P and Q each a third Proportional to AB and CD, and, to EF and GH, respectively. (Prob. 31.)

DEM. Now, the Triangle V is to X, as AB to P; } -Th. 12.
and, the Trapezium Y is to Z, as EF to Q. }
But, as $AB:CD::EF:GH$; and, as $CD:P::GH:Q$ -Con.
wherefore, as $AB:P::EF:Q$ (9.5.) Th. $V:X::Y:Z$.

It is also true, if similar Figures are constructed on the first and third, and others on the second and fourth, of four proportional Lines.

For, since $V:X:Y:Z$, conf. $V:Y::X:Z$. - Th. 4. 5.

COR. Similar Figures being proportional to other similar Figures, their corresponding Sides are Proportionals.

The Converse, needs no proof; seeing it is but reversing the Premises.

T H E-

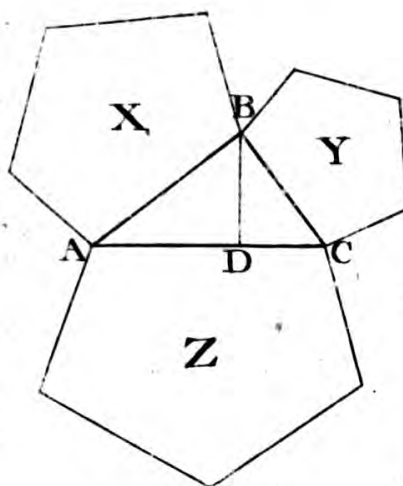
THEOREM XVI. 31 Euclid.

Any similar Figures whatever, described on the three Sides of a Right angled Triangle, and if they are made corresponding Sides of the Figures; then, that which is described on the Side subtending the Right Angle, will be equal to both the others, described on the Sides containing the Right Angle.

Let ABC be a Right angled Triangle; and, let X, Y, and Z be similar, irregular Pentagons, described on the three Sides, AB, BC, and AC, of the Triangle; which are corresponding Sides of the Figures, X, Y, and Z.

I say, that the Figure Z, described on the Hypothenufe, AC, is equal to both X and Y, described on the other two Sides.

Draw the Perpendicular BD.



DEM. In every Right angled Triangle (ABC), since $AD:AB::AB:AC$; and $DC:BC::BC:AC$ - Cor. to 7, Consequently, any Figure, described on AB, is to a similar Figure, described on AC, as AD is to AC. and a similar Figure, described on BC, is to that described on AC, as DC is to AC. - - - 12 and 13. Therefore, since $Z:X::AC:AD$; and $Z:Y::AC:DC$, Z will be to $X + Y::AC:AD + DC$. - Cor. to 3. 5. But, $AC=AD + DC$; therefore, $Z=X+Y$. Q.E.D.

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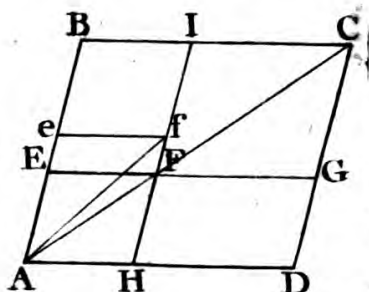
SHOL: *This Proposition, by the Doctrine of Proportion, extends the famous Pythagorean Proposition, viz. the 47th of the first Book of Euclid, and the 20th, of the first, of these Elements, universally.*

Seeing, by Th. 10. of this 6th Book it is demonstrated, that Triangles, and all similar Figures whatever, are in proportion to the Squares of their corresponding Sides, the full and perfect demonstration of this Proposition necessarily follows from thence, and the 20th of the first; this Proposition is therefore, in some measure, unnecessary; but, for the particular beauty and elegance of it, I did not think proper to omit it.

This Proposition is also applicable to the Diagonals of Right lined Figures, as well as to their Sides; and, consequently, to the Diameters of Circles. Wherefore, a Circle described on the Hypotenuse of a Right angled Triangle, for its Diameter, is equal to the two Circles, whose Diameters are respectively equal to its Legs.

THEOREM XVII. 24 Euclid,

In every Parallelogram, those which are about the same diagonal Diameter, are similar to the whole Parallelogram and also to each other. And the two Complements have their Sides reciprocally proportional.



In the Parallelogram ABCD, let AEFH and FICG be Parallelograms, about the Diameter AC.

Then, the Parallelograms EH, IG, are similar to the whole Parallelogram, BD, and to each other.

EG is parallel to AD, and BC; and HI, to AB and DC; by Construction.

DEM.

DEM. Now, because EF is parallel to BC, the Sides AB and AC, of the Tri. ABC, are cut proportionally.

wherefore, $AE : EB :: AF : FC$ - - - Th. 2.

and conf. $AE : AB :: AF : AC$. - - - Conv. 6.5.

And, because HF is parallel to DC, as $AF : AC :: AH : AD$.

wh.as $AE : AB :: AH : AD$; and, as $AE : AH :: AB : AD$ - 4.5

But, $EF = AH$, and $BC = AD$, &c. (15.1.) wherefore, the Sides of the Parallelograms AEFH, and ABCD, have all their Sides proportional.

But, the Angle EAH is common to both; and, the opposite Angles of Parallelograms are equal - - - 15.1.

wherefore, $EFH = BCD$. - - - Ax. 3.

Also, the Angle AEF = ABC, and AHF = ADC. - 4.1.

therefore AEFH is similar to the whole Par. ABCD - Def. 1.

By the same Reasoning, the Parallelogram FICG may be proved similar to ABCD.

For, $CI : CB :: CG : CD$; viz. as CF to CA - as above. they are also equiangular; the Angle at C being common. Consequently, the Par. FICG is similar to AEFH; being both similar to the whole Parallelogram, ABCD - Axiom.

2nd. The Compliments BF, FD are equal. - - - 19.1.

and they are equiangular, (15.1.) for $EFI = HFG$ - 2.1.

Wherefore, as $EF : FG :: HF : FI$, reciprocally - Th 8.

COROLLARY. 26 Proposition of Euclid.

Hence it is evident, that, if on any Angle of a Parallelogram, there be described, or taken away, a lesser Parallelogram, similar, and a like situated, they will have the same common Diameter.

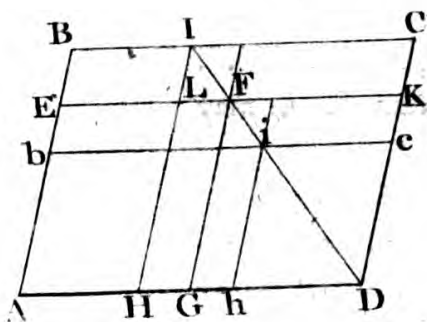
For, the Diameter AF or FC, is common with AC.

If any other Parallelogram AefH, similar to ABCD, be described at the Angle A, not alike situated, they have not the same Diameter AC.

T H E O-

THEOREM XVIII. 27 Euclid.

If a Parallelogram be described on a Right Line, and from it there be taken away a Parallelogram, similar, and alike situated, to one described on half the Line, equiangular to the first; that, which is described on the half Line, is greater than the remaining Parallelogram.



Let ABCD be a Parallelogram described on a given Right Line, AD; and, let AEFH be an equiangular Parallelogram, described on half the Line, AG; also, let HICD be taken away from the Parallelogram ABCD, similar, and alike situated, to the Parallelogram AEFH.

I say, the Parallelogram AEFH is greater than the remaining Parallelogram ABIH.

Draw the Diameter ID, and produce EF to K.

DEM. Then, because the Parallelograms, HC and GK, are about the same Diameter, ID, they are similar. - 17. wherefore, AEFH is similar to HICD. - - Axiom.

Now, HF = FC (19. 1.) and BF = FC - Th. 1. conf. BF = HF (Ax. 3. 1.) and HF is greater than BL. add AL, to both; and AF is also greater than AI. - 8. Therefore, AEFH is greater than the remaining Parallelogram, ABIH, the defect of HICD.

Again,

Again. From the lesser Parallelogram $A b c D$, let there be taken away the Parallelogram $h i c D$, similar to $A E F G$, and situate alike.

Now, because $F D$ is a Diameter, in the Parallelogram $G F K D$, $G i$ is equal to $i K$. - - - 19. 1.
 Add $h i c D$, on both Sides; and $G c = h K$ - Ax. 6. 1.
 But, $G b = G c$. (Th. 1.) wh. $G b = h K$ - - 3. 1.
 Add $G i$, to both; and $A i = G K - F i$,
 consequently, $A F$ (equal $G K$) is greater than $A i$.

Therefore, the Parallelogram $A E F G$, described on half the Line, $A D$, is greater than any other Parallelogram described on the whole Line; being deficient by a Parallelogram, similar and alike situated to that which is described on half the Line. Q. E. D.

Few, if any, who have favoured the World with Treatises on Geometry, have taken notice of this Theorem, or the foregoing, (the 27 and the 24 of Euclid) except those who have trod in his path without stepping the least aside; indeed, it is so very obscurely worded, that it is scarce intelligible; which, Mr. Stone has endeavoured to remedy, with little success. He excuses Euclid, by saying, that he could not have rendered it more clear, in so few words; and therefore, rather than appear tedious, gave it as it is; If the Proposition, itself, be unintelligible, how are we to understand the Premises? Euclid has given some Propositions in more words. I am persuaded that I have made it clearer than Mr. Stone, and in as few words as Euclid makes use of.

How far this Theorem may be of use in the Mathematics I do not pretend to say; but, when it is clearly understood, it will be found to contain something extraordinary in it; inasmuch, that I could by no means dispense with the omission of it.

There are two Problems, following after, in Euclid, which are dependant on it; he also divides a Line in extreme and mean Ratio by it; in the 30th; it is certainly demonstrable, when done, but how it is to be performed, in the operation, I cannot devise. (See, Prob. 11th. B. 2. or Pr. 35. Pr. Geometry.)

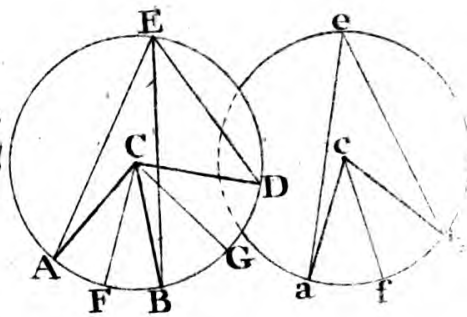
As I do not conceive the Problems to be at all useful to Mechanics, &c. I have not inserted them, amongst the rest, in Practical Geometry.

The 32nd Prop. of Euclid is of little consequence, and has nothing in it worthy of notice.

T H E-

THEOREM XIX. 33 Euclid.

In the same or equal Circles, the Angles, whether at the Center or at the Circumference, are in proportion to the Arks on which they stand.



First, in the Circle AED, the Angle ACB standing on the Ark AB, is to the Angle BCD, on the Ark BD, as AB is to BD.

Let the Arks, AB and BD, be divided into any number of equal Parts, in F and G; and join CE, and CG.

DEM. Now, because the Ark $AF = FB$, and $BG = GD$;
 the Angle $ACF = FCB$, and $BCG = GCD$ - C.2.9.3.
 wherefore, as the Ark $AF : BG :: FB : GD$ { in eq. }
 and, as the Ang. $ACF : BCG :: FCB : GCD$ { Ratio } Ax. 4.5
 For, whether AF be equal to, greater, or less than BG;
 the Angle ACF is also equal to, greater, or less than BCG;
 in the same Ratio. (Th. of Plane Angles.)
 conf. as the Ark $AF : BG ::$ Angle $ACF : BCG$, &c.
 Therefore, as $AF + FB : BG + GD$,
 so is the Angle $ACF + FCB : BCG + GCD$. - 2.5.
 i. e. as the Ark $AB : BD$, $::$ Angle $ACB : BCD$. Q.E.D.

But, the Angle ACB, at the Center, is double AEB,
 at the Circumference; and, BCD is double BED - 9. 3.
 wh. as the Angle $ACB : BCD :: AEB : BED$ - Ax. 8. 5.
 Therefore, as the Ark $AB : BD$, so is the Angle $AEB : BED$.

2nd. Because, in the equal Circles, AEB, a e b, the Radii are equal ; and equal Arks subtend equal Angles-C. 2. 9. 3. conf. as Ark AB : a b :: Angle ACB : a c b ; & AEB : a e b.

If AB and a b be divided into an equal number of Parts, the Demonstration is the same as above.

COR. Sectors of Circles, ACB, BCD, or a c b, are proportional to the Arks AB, BD, or a b.

For, each Sector may be considered as an aggregate of a number of Isosceles Triangles, of equal Bases and equal Legs ; as ACF, FCB, &c. consequently, equal Triangles ; and having equal Altitudes ; each Sector, ACB, BCD, or a c b have, therefore, that Ratio to each other, as the Bases of those Triangles, i. e. as the Ark AB to BD, &c. - - - - - Th. 1.

N. B. None of the following elegant, and some of them elementary, Theorems, are in Euclid ; except the 34th ; which is the 8th Proposition of his thirteenth Book.

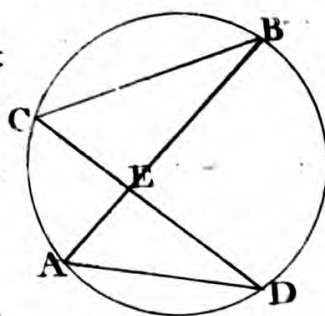
THEOREM XX.

If two Right Lines intersect, within a Circle, and are terminated by the Circumference, the Segments of those Lines are reciprocally proportional.

The two Chords, AB and CD, cut each other in E.

I say, that CE is to AE, as EB is to ED.

Join AD and CB.



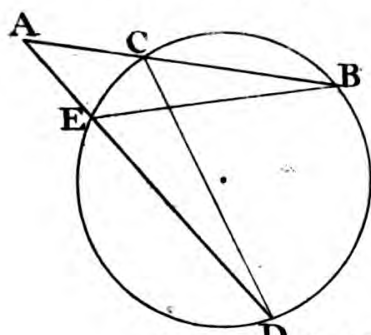
DEM. Now, the Angles AED, CEB are equal. - - - 2. 1.
and ABC = ADC ; also BAD = BCD. - - - 10. 3.
wherefore, the Triangles AED, CEB, are similar. - Th. 4.
Therefore, CE : AE :: EB : ED, by the same. Q E. D.
Conf. CE × ED = AE × EB ; by 9. 6 ; also by 14. 3.

S ;

THE O-

THEOREM XXI.

If two Right Lines are drawn, from the same Point without a Circle, to the opposite and concave part of the Circumference; they will have that proportion to each other, reciprocally, as their external Segments.



From the point A, draw AB and AD, at pleasure, cutting the Circle EBD in the Points E, C, B, and D.

I say, that AB is to AD, as AE is to AC. Join EB and CD.

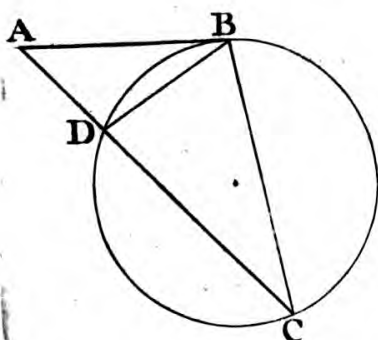
DEM. In the Tri. ABE, ADC, the Angle ABE = ADC - 10.3 and the Angle A is common; wh. AEB = ACD - 10.1. conf. the Triangles ABE, ADC are similar.

Wherefore, $AB : AD :: AE : AC$. - - - Th. 4.

And, $AB \times AC = AD \times AE$, by Th. 9. and, also by 16. 3.

THEOREM XXII.

If from any Point without a Circle, two Right Lines are drawn; one touching the Circle, and the other cutting it; that which touches the Circle is a mean Proportional, between the whole Secant and the external Segment.



From any Point, A, draw AB, touching the Circle at B.

And, from the same Point draw AC, at pleasure, cutting the Circle, in D and C.

I say, that AC is to AB, as AB to AD.

Join BC, and BD.

DEM.

DEM. The Triangles ACB, ABD, are similar. - Th. 4.
 For, the Angle ACB = ABD; - - - 13. 3.
 and BAC is common; wherefore, ADB = ABC - 10. 1.
 Therefore $AC : AB :: AB : AD$. Q. E. D.
 And conf. $AC \times AD = AB^2$. by Cor. Th. 9; also by 16. 3.

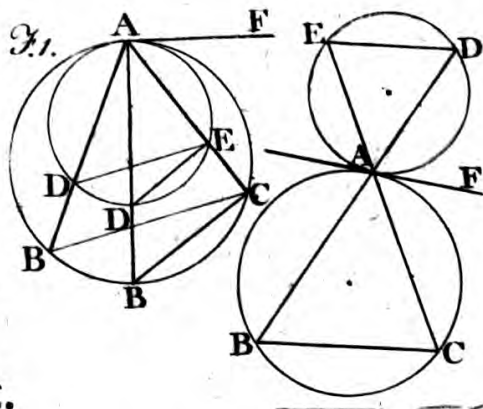
THEOREM XXIII.

If two Circles touch one another, either internally or externally, and if two Right Lines be drawn, from or through the point of Contact, cutting both Circles; those Lines will be cut proportionally, by the Circumferences of the Circles.

In the Circles, ABC, ADE, and from, or through, the Point of Contact, A, draw AB and AC, cutting both Circumferences, in the Points B, C, D, and E.

I say, that AB is to AC, as AD to AE.

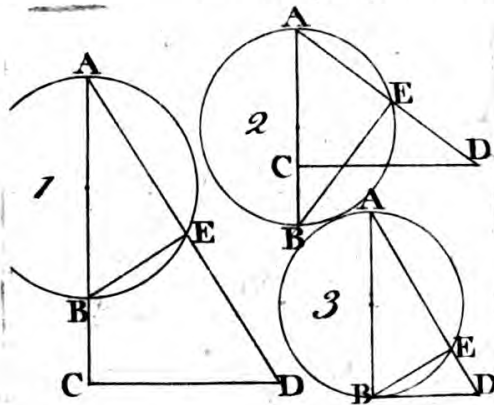
Draw AF, a Tangent to the Circles, at the Point A; and join BC, and DE.



DEM. Then, the Angle ABC is equal to ADE; - Ax. 3. 1.
 For they are each equal to the Angle FAC. - Th. 13. 3.
 And the Angle BAC (Fig. 1.) is common;
 In the second, BAC, DAE, are vertical, th. equal - 2. 1.
 therefore, the Triangles, ABC, ADE, are similar,
 and, consequently, $AB : AD :: AC : AE$. - - Th. 4.

THEOREM XXIV.

If a Right Line be drawn perpendicular to the Diameter of a Circle, whether it be within or without the Circle, or touching the Circumference, and if any Right Line be drawn, from the farthest extreme of the Diameter, cutting the Circumference and the Perpendicular; that Line and the Diameter will be cut proportionally.



Let the Right Line CD be perpendicular to the Diameter AB.

Draw AD, at pleasure, cutting the Circumference, in E, and the Perpendicular, in the Point D.

I say, that AB is to AE, as AD to AC. Join EB.

DEM. Then, because AB is a Diameter, the Angle AEB, (being in a Semicircle) is a Right one; - - - 12. 3. and ACD is a Right one (Con.) th. equal to AEB - Ax 9. and, the Angle DAC is common; conf. ADC = ABE - 10. 1. wherefore, the Triangles AEB, ACD are similar. - 4. 6. Therefore, as AB:AE::AD:AC. Q. E. D.

In Fig. 3. DB and AC are the same; consequently, AB is a mean Proportional between AE and AD.

COR. The Rectangle under any Line, AD, drawn from the extreme A, of the Diameter, to the Perpendicular, DC, and the part of it, AE, within the Circle, is equal to the Rectangle under the Diameter, AB, and the whole, AC, of the Diameter produced to the Perpendicular, (Fig. 1.) For, $AE:AB::AC:AD$; th. they are four proportional Lines consequently, $AE \times AD = AB \times AC$. - Th. 9.

N. B. In Fig. 2nd, the Rectangle under the Means, AB and AC, is under the whole Diameter and a part of it.

In Fig. 3rd, since $AE:AB::AB:AD$; consequently, the Rectangle under AD and AE, is equal to the Square of the Diameter, AB. - Cor. to 9.

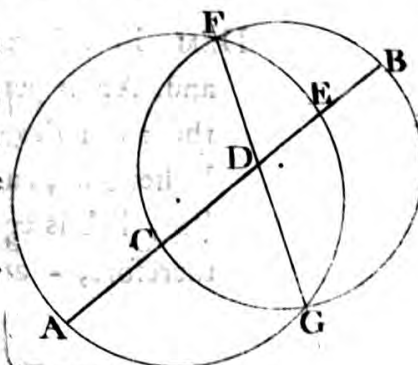
For, $AD \times ED = BD^2$ (16. 3.) } Therefore, }
and, $AB^2 + BD^2 = AD^2$ (20. 1.) } as above. }

THEOREM XXV.

If two Circles cut each other, and a Right Line be drawn cutting both Circles, it will be cut proportionally, by the Circumferences, and a Right Line joining the points of interfection of the Circles.

Join the Points, F and G, in which the two Circles; AFG and FBG, cut each other; and, let any Right Line, AB, cut both Circles, and the Right Line FG, in the Points A, C, D, E, and B.

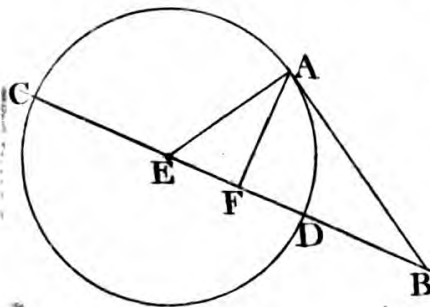
I say, the Line AB is cut proportionally in those Points; viz. as $AC:CD::BE:ED$.



DEM. For, in the Circle AFG, as $AD:DF::DG:DE$ }
And, in the Circle FBG, as $DF:CD::DB:DG$ } - 20.
wherefore, - - as $AD:CD::DB:DE$ - 10. 5.
Therefore, as $AD-CD:CD::DB-DE:DE$ - 7. 5.
That is, as $AC:CD::EB:ED$. Q. E. D.

THEOREM XXVI.

If a Tangent to a Circle cuts any Diameter produced, and a Perpendicular be drawn, from the Point of Contact to that Diameter; the Segments of the Diameter, intercepted between the Center and the Perpendicular, the Circumference, and the Tangent, are proportionals, i. e. the Radius is a mean Proportional betwixt the part, intercepted between the Center and the Perpendicular, and the Point in which the Tangent cuts the Diameter.



Let any Tangent, AB, cut the Diameter, CD produced, in B; and, let AF be perpendicular to CD; and E the Center.

I say, that EF, ED, and EB are in continual Proportion.

DEM. For, because EAB is a Right Angle, - C. 3. 8. 3.
 and, AF is perpendicular to EB; - - - Con.
 the Triangles EAF, EAB are similar. - - - Th. 7.
 Wherefore, as EF : EA :: EA : EB. - - Cor. 2. 7.
 But, EA is equal to ED;
 therefore, - as EF : ED :: ED : EB. - - Ax. 4. 5.

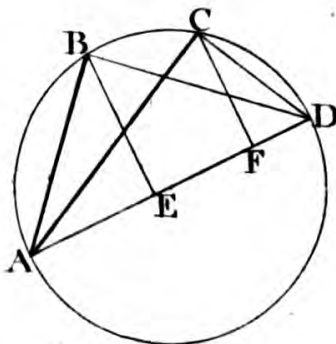
THEO-

THEOREM XXVII.

The Squares of Chord Lines, drawn from one extreme of a Diameter, are in proportion to the Parts of the Diameter, intercepted between the Perpendiculars, from the other extremes of the Chords, to the Diameter.

AB and AC are two Chords, having a common extreme, A.

Draw a Diameter, AD, from the Point A; and, from the other Extremes, B and C, of the Chords, draw the Perpendiculars BE and CF, to the Diameter, cutting it in E and F.



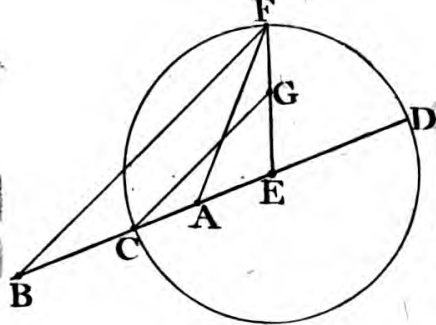
I say, the Square of AB is to AC square, as AE to AF.
Draw BD and CD.

DEM. Now, BE is a mean Proportional between AE & ED;
also, CF is a Mean, between AF and FD - Cor. to 7.6.
for ABD and ACD are Right Angles. - - - 12. 3.
Wh. the Rect. AED = BE²; also AFD = CF² - Cor. 9.6.
But AB² = BE² + AE²; and AC² = CF² + AF²,
conf. AB² = AED + AE²; and AC² = AFD + AF².
But, AED, i. e. AE × ED + AE² = AE × AD; }
and AFD, i. e. AF × FD + AF² = AF × AD; } 3. 2.
wh. AB² = AE × AD; and AC² = AF × AD - Ax. 3. 1.
But, AE × AD : AF × AD :: AE : AF. - - Th. 1.
Therefore, AB² : AC² :: AE : AF. Q. E. D.

THEO-

THEOREM XXVIII.

If any Point, except the Center, be taken in a Diameter of a Circle, and if another Point (in the same Diameter, produced, and on the same side of the Center) be taken; the Distance of which, from the Center, is a third Proportional, to the distance between the Center and the first assumed Point, and the Radius; then will two Right Lines, drawn from those Points to any point in the Circumference, be in the same Ratio, as the internal and external adjoining Segments of the Diameter.



Assume any Point, A, in the Diameter CD; in which Diameter, produced, take the Point B (from the Center, E) a third Proportional to EA and EC (Pr. 31.) and, to any Point, F, in the Circumference, let two Right Lines, AF and BF, be drawn.

I say, that AF is to FB, as AC to CB.

DEM. For, $EA:ED$ (eq. EC) $:: ED:EB$ - - - Con.
 wherefore, $EA:EA+ED :: ED:ED+EB$. - - 6. 5.
 i. e. $EA:AD :: ED:BD$; conf. as $EA:ED :: AD:BD$ - 4. 5.
 Now, $EA:ED :: ED:EB$; and $EF = ED$;
 wh. $EA:EF :: EF:EB$. - - - Ax. 4. 5.
 therefore, the Triangles EFA, EFB are similar - Th. 5. 6.
 for, the Angle FEB is common, & the Sides are proportional.
 Conf. $AF:BF :: EA:EF$; i. e. $:: EA:ED$, or as $AD:BD$;
 that is, as AC to CB. Q. E. D.
 For, $EA:EC$ (eq. ED) $:: EC:EB$ (i. e. as AD to BD) - Con.
 consequently, $EA:EC :: EC-EA:EB-EC$;
 that is, - as $EA:EC :: AC:CB$. - - - 3. 5.
 Otherwise:

Otherwise :

Make EG equal EA, and draw CG.

Then, because EA, EF, are equal EG, EC, respectively, and the Angle at E is common, the Triangles AFE, ECG are congruous ; and $CG = AF$. - - - 8. 1.
But, $EA:EC::EC:EB$ (Hyp.) wh. $EG:EC::EF:EB$
conf. $EG:EC::GF:CB$ (7. 5.) wh. CG is paral. to BF - 2. 6.
and the Triangle ECG is similar to EBF - - - 5.
But, $GF = AC$ (Ax. 7. 1.) for $EF = EC$; and $EG = EA$;
wherefore, as GF, i. e. AC, : CB, or, as EG : EC,
or, $EF : EB :: CG$, i. e. AF, : BF. Q. E. D.

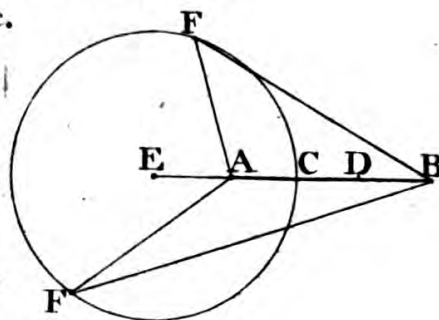
COROLLARY. A PROBLEM.

If any Right Line, AB, be divided at pleasure, in C ; to describe a Circle ; to any Point in the Circumference of which, if Right Lines are drawn, from each extreme of the given Line, they shall have the same Ratio to each other, as the Segments of the given Line.

From the greater Segment, CB, take the less, AC ; i. e. make CD equal AC. Make AE a third Proportional, to BD and CD (equal AC)

With the Radius EC, on the Center E, describe a Circle ;

and to any Point, F, in the Circumference, draw AF & BF.

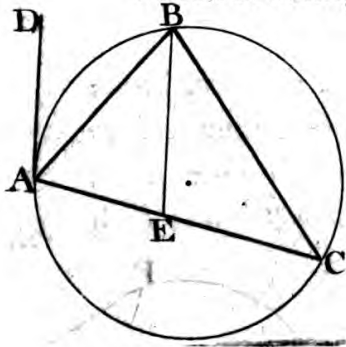


Then, AF is to BF as AC is to CB. - Theorem.

Because, $EA:AC::AC$ (eq. CD) : DB - - - Con.
Conf. $EA:EA+AC$ (i. e. EC) :: $AC:AC+DB$ (i. e. CB)
i. e. as $EA:EC::AC:CB$. - - - Conv. to 6. 5.

THEOREM XXVIII.

If a Triangle be circumscribed by a Circle, and, a Tangent to the Circle be drawn, at any Angle of the Triangle; and, if a Right Line be drawn from the adjacent Angle, parallel to the Tangent, cutting the opposite Side; then, the Side between the Tangent and that Angle, will be a mean Proportional, between the opposite Side, and that Segment which is contiguous to the Tangent.

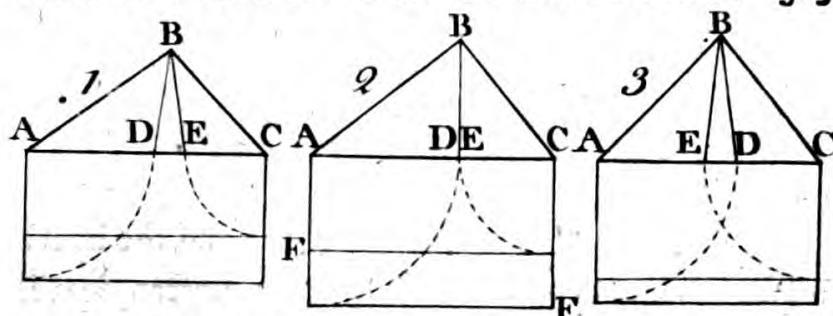


Let AD be a Tangent to the Circle ABCE, at the Point A, (the Angle of a Triangle, ABC, inscribed) and, let BE be drawn parallel to AD, cutting AC, in E.

I say, that AB is a mean Proportional, between AE and AC.

DEM. Now, BE is parallel to AD. - - - Con.
 wherefore, the Angle ABE = BAD - - - 4. 1.
 But, the Ang. ACB = BAD (13 3) conf. ABE = ACB - Ax. 3. 1
 and, in the Δ s, ABE, ABC, the Angle BAC is common.
 wherefore, those Triangles are similar.
 Therefore, as AE : AB :: AB : AC. - Th. 4. 6.
 consequently, AE, AB, and AC, are three Proportionals;
 and therefore, AB is a Mean.

COR. Hence it is manifest, that, what is proved, in Cor. 2. of the 7th, in respect of a right angled Triangle, is general; in every Scalene Triangle; on any Side of which, similar Triangles are constructed. e. g.



In the Triangle ABC, if ABD be made equal to the Angle C, and CBE equal A, the Angles A and C being common; the Triangles ABD, EBC, are similar to the whole Triangle ABC - - - C. 2 4. 6. and consequently, to each other. - - - Axiom.

Wherefore, the square of AB is equal to a Rectangle, under the Side AC, and the Segment AD; and, the square of BC is equal to a Rectangle under AC and EC. - - - - - Cor. to Th. 9.

For because the Triangle ABD is similar to ABC; AD:AB::AB:AC; &, for the same reason, EC:BC::BC:AC Wherefore, AB and BC are, each, a mean Proportional; between the adjoining Segments of AC, and the whole, AC.

If the Triangle be right angled (Fig. 2.) the two Segments AD and EC are equal to the whole AC (for the Points D and E coincide; consequently, the two Squares of AB and BC are equal to the Square of AC.

i. e. AB square is equal to the Rectangle AF, }
and, BC square is equal to the Rectangle FC. - } 20. 1.

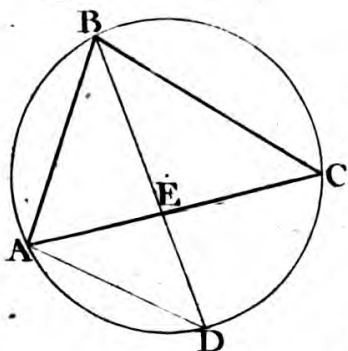
In Fig. 3rd, the Angle at B being acute; consequently, the two Segments AD and EC are greater than the whole AC; but they are, in each Case, subject to one general Demonstration.

When the Angles are all equal, the Square of each Side is equal to the Square of the other; for the Triangle will be equilateral.

THEOREM XXX.

If any Angle of a Triangle is bisected, and the bisecting Line cuts the opposite Side; the Rectangle under the two Sides, containing that Angle, is equal to the Square of the bisecting Line, added to the Rectangle under the Segments of the opposite Side.

Also, the Sides of the Triangle, the bisecting Line, and that Line produced till it cuts the Circumference of a circumscribing Circle, are Proportionals.



Bisect the Angle B, of the Triangle ABC, by the Right Line BE, cutting AC, in the Segments AE, EC.

Then, the Rectangle under the Sides AB and BC, is equal to the Square of BE, added to the Rectangle under AE and EC.

2nd. About the Triangle ABC, describe a Circle, ABCD. Produce BE to D, and join AD.

I say, that AB is to BD as BE is to BC.

DEM. For, the Triangles ABD, EBC are similar - Cor. 2. 4. because, the Angles ABD, EBC are equal; - - - Con. and, the Angle ADB is equal to ECB; - - - 10. 3. wherefore, - BAD is equal to BEC. - C. 5. 10. 1. Th. $AB:BD::BE:BC$ (4) conf. $AB \times BC = BD \times BE$ - 9. 6. But, $\square DBE = DEB + \angle B \square$ (3. 2.) & $DEB = AEC$ - 14. 3. Wherefore, the Rectangle $DBE = AEC + EB \square$ - Ax. 6. 1. Consequently, $AB \times BC = EB \square + AE \times EC$. Q. E. D.

The 2nd Part was proved in the fifth Line.

T H E-

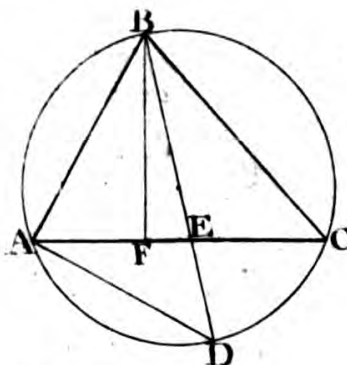
THEOREM XXXI.

The Rectangle, under any two Sides of a Triangle, is equal to a Rectangle contained under the Perpendicular to the other Side, from the opposite Angle, and the Diameter of a circumscribing Circle.

Describe the Circle $ABCD$ about the Triangle ABC . - - (4. 4.)

Draw the Diameter BD ; and BF perpendicular to AC .

Then, the Rectangle under AB and BC , is equal to the Rectangle under BF and BD . Draw AD .



DEM. Now, the Angle $ADB = ACB$; - - 10. 3.

and BAD is equal to BFC ; - - - - Ax. 9. 1.

(for BFC is a R. Angle (Con.) and BAD is Right - 12. 3)

consequently, AED is equal to FBC ;

and the Triangles ABD , FBC are similar;

wherefore, as $FB : BC :: AB : BD$; - - 4. 6.

and therefore, $FB \times BD = AB \times BC$. Q. E. D. - 9. 6.

N. B. If the Angle, ABC , was bisected by the R. Line BD ; the Rectangle under the Sides, AB , BC , is equal to a Rectangle under the Diameter BD and the Segment BE .

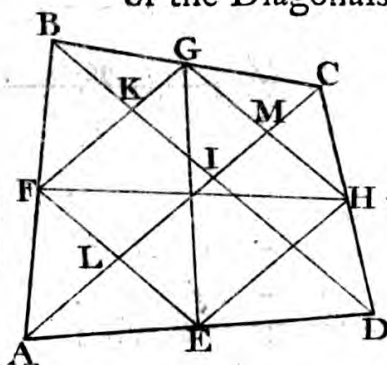
For, AB is to BD , as BE is to BC ; by the 30th.

THE.

THEOREM XXXII.

If all the Sides of a Trapezium are bisected, and the Points of bisection, in contiguous Sides, are joined by Right Lines, the Quadrilateral formed by those Lines is a Parallelogram; and, it is equal to half the Trapezium.

Also, the Sum of the squares of the Diagonals, of the Trapezium, is double the Sum of the squares of the Diagonals of the Parallelogram.



Let the Sides of the Trapezium, $ABCD$, be bisected in the Points E , F , G , and H , and draw EF , FG , GH , and EH .

I say, first, that the Quadrilateral, $EFGH$, is a Parallelogram.

DEM. For, because, in the Triangles ABC , ACD , the Sides AB , BC , and also AD , DC , are cut proportionally, in F and G , E and H , the Right Lines FG and EH are both parallel to the common Base AC ; and, for the same reason, the Lines GH and FE are both parallel to BD ; consequently, they are parallel between themselves; therefore, $EFGH$ is a Parallelogram. (Def. 33. 1.)

Secondly. It is equal to half the Trapezium $ABCD$.

For, in the Triangle ABC , because FG is parallel to AC , and, because AB and BC are bisected in F and G ,

BI is also bisected in K . - - - - 2. 6.

and,

and, for the same reason AI and IC are bisected in L & M;
wherefore, LM is half AC; and consequently, the Par-
allelogram LFGM (on half the Base AC, of the
Triangle ABC, and half its Altitude) is equal to half
the Triangle ABC - - Th. 18. 1. and Pr. 20.
for the same reason the Parallelogram LEHM =
half the Triangle ADC.

conf. the Par. LG + LH = half the Tri. ABC + ADC,
i. e. the Par. FFGH is equal to half the Trap. ABCD.

Thirdly. Having drawn the Diagonals, EG and FH.

I say, that AC square added to BD square, is double
the sum of the Squares, of EG and FH.

For, the Squares of the Diagonals, EG and FH, are
equal to the Squares of all the Sides EF, FG, GH,
and EH. - - - - - Th. 14. 2.

But EF and GH are each equal to half BD; }
and FG, EH, are each equal half AC; } - above.

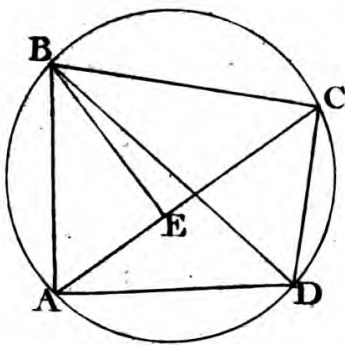
wherefore, the four Squares, of EF, FG, GH, and EH,
are equal to half the Squares of AC and BD - Cor. 4. 2.
and, they are equal to the squares of EG and FH - 2nd.
Therefore, the Squares of AC and BD, are double the
Squares of EG and FH.

This Proposition is, by Stone and some others, given in the
second Book; consequently, there is a necessity for a Lemma,
previous to it; which Lemma, is only a particular Case of the
second Proposition of the sixth Book; which cannot be made
general without that Theorem.

The third Part of which he makes a separate Proposition, may
be demonstrated in the second Book; but I did not think it of
consequence enough to make another Proposition.

THEOREM XXXIII.

In every Quadrilateral inscribed in a Circle, the Rectangle contained under the Diagonals, is equal to two Rectangles under the opposite Sides.



In a Rectangle the thing is manifest - 20.1.

In the Trapezium ABCD, draw the Diagonals AC and BD.

I say, the Rectangle, under AC and BD, is equal to the two Rectangles, under AB and CD, BC and AD.

Make the Angle ABE equal to DBC. (Pr.4)

DEM. Now, $\angle ABE = \angle DBC$ (Con) the Angle $\angle BAC = \angle BDC$ - 10.3
 wherefore, the Triangles ABE, DBC are similar - C.2.4.6.
 Wh. as $AB : BD :: AE : CD$ (4.6) & $AB : AE :: BD : CD$ - 4.5.
 Therefore, $AB \times CD = BD \times AE$ - - - 9.6.

Again. The Triangle ABD is similar to EBC.
 For, the Angle $\angle ABD = \angle EBC$ (because, $\angle ABE = \angle DBC$ (Con.)
 and the Angle $\angle EBD$ is common to both.) - Ax. 7. 1.
 Also, the Ang. $\angle BDA = \angle BCA$ (10.3.) conf. $\angle BAD = \angle BEC$ - 10.1.
 Wherefore, as $BC : BD :: EC : AD$ - - - 4.6.
 consequently, $BC \times AD = BD \times EC$. - - - 9.6.
 But, - - $AB \times CD = BD \times AE$ - proved above.
 and $AE, EC = AC$; wh. $BD \times AE + BD \times EC = BD \times AC$; 2
 Th. $AB \times CD + BC \times AD = BD \times AC$. Q. E. D. - Ax. 3.1.

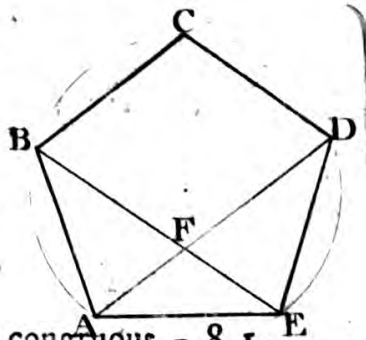
T H E O.

THEOREM XXXIV. 8.13 Euclid.

The Diagonal of a regular Pentagon is to its Side, as a Line divided in extreme and mean Proportion.

In the Pentagon $A B C D E$,
let the two Diagonals, AD and BE ,
be drawn, cutting each other in F .

Then, AD is to AB , or AE ,
as AE is to $AD - AE$.



DEM. The Triangles ABE , ADE , are congruous - 8.1.

For, AB , AE , and ED are equal,

and the Angle $BAE = AED$, - 8.4.

wherefore, the Angle $ABF = AEF = EAF$ - 9.1.

conf. the Triangles ABE , AFE are similar. - C.2.4.6.

Therefore, as $BE:AE::AE:AF$, equal FE . - 4.6.

But, $AE = BF$; for $DE = BC$, and $CD = AB$ - Hyp.

wherefore, BE is parallel to CD , and AD to BC - C.10.3.

consequently, $BCDF$ is a Parallelogram;

wherefore, $BF = CD$ (equal AB , equal AE). - 15.1.

Therefore, as $BE:BF::BF:FE$;

i. e. as $BE:AE$, or AB , :: $AE:FE$.

and consequently, AB , or AE , &c. is equal to the greater Segment of BE , or AD , divided in extreme and mean Proportion.

COR. 1. The Diagonal of a Pentagon is parallel to the opposite Side.

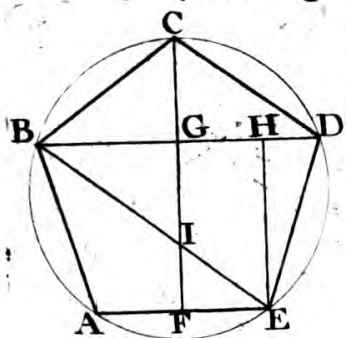
2. Two Diagonals, cutting each other, are mutually divided in extreme and mean Proportion, in the point of intersection.

U u

THE-

THEOREM XXXV.

The Diameter or Perpendicular, of a regular Pentagon, is divided in extreme and mean Proportion, by a Diagonal cutting it at Right Angles.



In the Pentagon $A B C D E$, let the Diameter $C F$ be drawn, and the Diagonal $B D$, cutting $C F$ in G , perpendicularly.

I say, that $C F$ is divided in extreme and mean Proportion, in G .

Draw $E H$ parallel to $C F$; also, the Diagonal $B E$, cutting the Perpendicular in I .

DEM. Now, because $B E$ is parallel to $C D$, - C.1. 34.
 the Angle $C D B$ is equal to $D B E$ - - - 4.1.
 and, $C B D = C D B$ (9. 1.) wh. $C B D = D B E$ - Ax. 3. 1.
 But, the Angle $C G B = B G I$ (Ax. 9) wh. $G I = G C$ - 11.1.
 (for the Tri. $B C G$, $B I G$ are congruous; $B G$ being common)
 Now, since $E H$ is par. to $F G$ (Con) & $A E$ to $B D$; by the last;
 $F G H E$ is a Parallelogram; wherefore, $E H = F G$ - 15.1.
 And, because $E H$ is par. to $F G$; $E H : I G :: B E : B I$ - 4.6.
 (for the Triangles $B G I$, $B H E$ are similar; by C. 3. 2. 6.)

But, $B E$ is to $B I$, as the greater Segment to the less,
 of a Line divided in extreme and mean Proportion. - 34.
 (for, $B I = B C$) $F G = E H$; also, $C G = G I$, proved above;
 wherefore, $F G : G C :: B E : B I$; i. e. as $B I : I E$;
 and therefore, $C F$ is divided in extreme and mean Ratio,
 in the Point G , where it is cut by the Diagonal $B D$.

COR. The Diagonal $B E$, cuts, $F G$, the greater Segment of $C F$, in extreme and mean Proportion; at I .

For $B D$ is par. to $A E$; conf. the Tri. $G B I$, $I E F$ are similar;
 wherefore, as $B I : I E :: G I : I F$ - - - - Th. 4.
 But, $B I : I E :: B E : B I$ (eq. AB) therefore, $F G : G I :: G I : I F$.

Hence, if a Right Line be divided in extreme and mean Proportion, and from either extreme of the greater Segment the measure of the less is set off, it will also be divided in the same Ratio; and the lesser Segment of the first, will be the greater Segment of the other. e. g.

Let AB be divided in extreme & mean Proportion, in C.



Then, if CD be made equal to CB, AC will be divided in the same Ratio; and CD is the greater Segment.

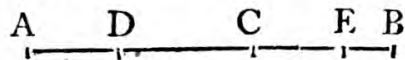
For, since $AB : AC :: AC : CB$, (Hyp.) it will be, as $AC : CB :: AB - AC : AC - CB$, i.e. as $AC : CB :: CD : AD$.

Also. If to the whole Line, divided in extreme and mean Proportion, there be added the greater Segment; then, the whole compounded Line is divided in the same Ratio, and the greater Segment, of the first, will be the lesser.

Let AC be divided in extreme and mean Proportion, in D. Then, if CB be made equal CD, CB will be the lesser Segment of the whole, AB, divided in the same Ratio.

For, since $AC : CD :: CD : AD$; it will be as $AC : CD :: AC + CD : CD + AD$; i.e. as $AC : CD :: AB : AC$.

Hence, if either Segment be given, the other may be found.



Let AC, the greater Segment, be given.

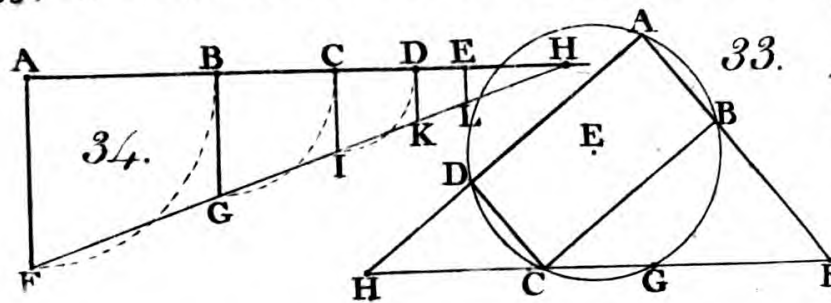
Divide AC in the Ratio required, in D. - (Pr. 35.)

Produce AC; make CB equal CD, and CB is the Segment sought.

If CB, the lesser Segment had been given.

To find the greater; divide CB as before, in E; make CD equal CB, and AD equal CE.

Then, AC is the greater Segment required.



A Demonstration of the 33rd Problem.

FG is equal to CH (Con.) conf. FC is equal to GH - Ax. 6.1.
 wherefore, the Rectangles CFG, GHC, are equal - Ax. B. 2.
 But, $AF \times BF = CF \times FG$; also, $AH \times DH = GH \times CH$ - C. 16.3
 consequently, the Rectangles AFB, AHD, are equal - Ax. 3.1.
 Wherefore, as $AF:AH::DH:BF$, reciprocally - C. 1.8.6.
 But, as $AF:AH::BF:BC$ (Th. 4) wh. $BF:BC::DH:BF$ - Ax. 13.5
 Again. It has been shewn (changing Terms) $DH:BF::BF:BC$
 also, that $BF:BC::AF:AH$; i. e. as $DC:DH$. - Th. 4.
 Therefore, the four Lines, DC, DH, BF, and BC are in
 continual Ratio; and consequently, DH and BF are two
 Means, between DC and BC; equal X & Z, by Construction,

A Demonstration of the 34th Problem.

First, $AH:BH::AB:BC$; i. e. as $AF:BG$ - - Th 4.
 wherefore, $AH:AB::BH:BC$ - - - - 4.5.
 conf. $AH-AB$ (eq. BH): $AB::BH-BC$ (eq CH): BC - 7.5.
 wherefore, $AB:BH::BC:CH$ - - - - 5.5.
 consequently, $AB+BH:BH::BC+CH:CH$. - 6.5.
 i. e. $AH:BH::BH:CH$; and as $CH:DH$, &c.
 But, $AH:BH::AF:BG$; and, $BH:CH::BG:CI$, &c. - 4.6.
 and AH, BH, CH, &c. are in continual Ratio;
 th. $AF:BG:CI:DK$, i. e. $AB:BC:CD:DE,::AH:BH,&c.$ ÷
 But, $AH:AF::BH:BG$, &c. and AH is greater than AF.
 wherefore, DH is greater than DK, and EH than EL;
 consequently, EL may still be taken from EH, *ad infinitum*.
 th. AH is the whole sum of infinite Proportionals, to X and Z.

E L E M E N T S O F G E O M E T R Y.

B O O K VII. The XI. of Euclid.

THE seventh Book, of these Elements, treats first of the Elements or first Principles of plane Solids, (i.e. Solids bounded by, or contained within Planes, only,) viz. of the Positions of Right Lines, and of Planes, to Planes; of the Sections of Planes by Planes, and of Right Lines by Planes.

Secondly, of solid Angles, generated by the intersections of Planes; their Construction and Principles investigated.

Thirdly, of Solids, contained within six parallel Planes, only; their Affections, Proportions, and Properties, are searched out, on which is founded the Theory of Mensuration of Solids.

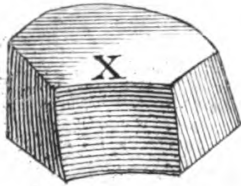
The Doctrine of Solids, contained in this Book and the following, is also of great use throughout the Mathematics; some of which have their very existence in it. The Science of rectilinear Perspective is wholly founded on the Sections of Planes by Planes, and, of Right Lines cut by a Plane, and by parallel Planes; insomuch that, without their assistance, Perspective would be a very imperfect Science; which, to the immortal Fame of Dr. Brook Taylor, is now established on the most permanent Principles.

Every Theorem, of the eleven first, are more or less useful in Perspective; the 5th and 7th are of great use; which, with the 8th and 9th contain the Essence of it, in Theory; as the 10th and 11th, of both Theory and Practice.

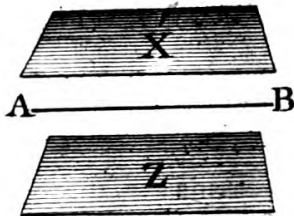
To enumerate every branch of the Mathematics, in which this seventh Book is particularly useful, is needless, was it in my power; the Reader may depend on it, that its Doctrine is, at the same time, entertaining, and of extensive utility.

DEFINITIONS.

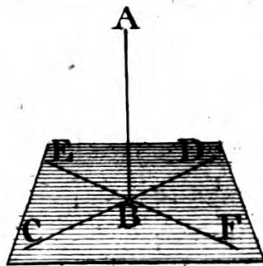
Def. I. A SOLID is a BODY, having length, breadth, and thickness; and is bounded by Planes, or curved Surfaces, or both. As X.



Def. II. A Plane (or Right Line) is parallel to another Plane, which, being produced infinitely, would never meet; but, are every where, equi-distant. As the Plane X, to the Plane Z; or, the Right Line AB, to both X and Z.

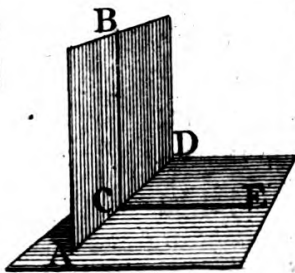


Def. III. A Right Line is perpendicular to a Plane, when it makes equal Angles with every Line drawn in the Plane, to or from that Point in which the Line cuts, or would cut, the Plane.



If AB makes Right Angles with the two Lines, CD and EF, passing through B, in which Point, AB cuts the Plane CDEF; and if the Lines, CD and EF, are both in that Plane; AB is perpendicular to the Plane.

Def. IV. A Plane, cutting another Plane, is perpendicular to the other, when it does not incline to the other on either Side.



If the Plane ABD cuts the Plane AED, in AD; and, if any Right Line, BC, drawn in one, be perpendicular to the other Plane (AED) Those Planes are perpendicular to each other.

N.B. It is usual, to define perpendicular Planes, by Right Lines being drawn perpendicular to their common Section; which, I presume, is taking for granted that their common Section is a Right Line, and which, being granted, there would be no occasion afterwards to demonstrate it, in Proposition 3rd.

Let the young Geometrician take particular notice, that no regard is had to the position of the Plane, or Line, in respect of the Horizon, to which another Line or Plane is said to be perpendicular, but only to their position in respect of each other, for, if they make Right Angles with each other, they are perpendicular, each to the other.

Def. V. If a Plane (or Right Line) be neither parallel nor perpendicular to another Plane, it is said to incline to the other Plane.

The inclination of one Plane to the other is the acute Angle ABC , made by the Section of another Plane, DEC ; cutting both the inclined Planes, BE and BF , perpendicularly; i. e. at Right Angles.

Also, the inclination of a Right Line, AB , to a Plane, BCD , is the Angle ABC made by another Line; drawn in the Plane BCD , from the Point B , in which the inclined Line, AB , cuts the Plane; and passing through another Point, C , in which a Perpendicular, AC , from any Point, A , in the inclined Line, cuts the Plane.

N. B. Two Planes have an equal inclination, to two other Planes when the Angles of their inclination are equal to one another.

Def. VI. A SOLID ANGLE is that which is made by more than two Plane Angles, applied close to each other, at the same Point, so, that two of them are not in the same Plane.

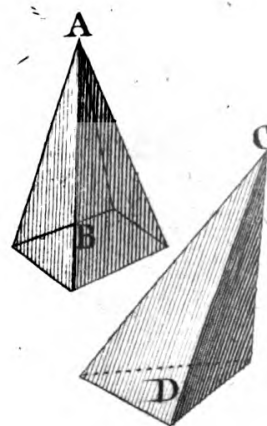
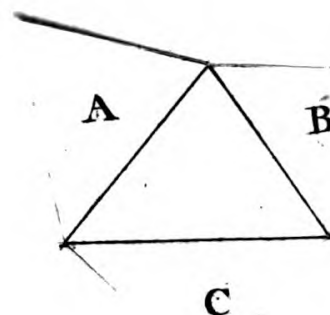
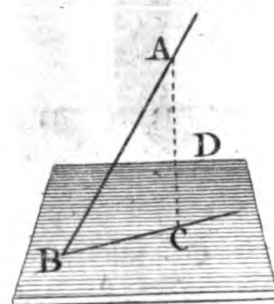
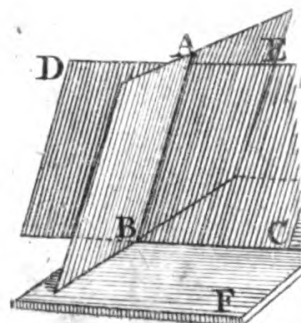
If the three plane Angles A , B , and C , be turned over, they will form a solid Angle, at the Point where their Vertices meet each other.

N. B. All the Plane Angles forming a solid Angle are, together, less than four Right Angles. (See Theorem 12.)

If three Plane Angles, form a solid Angle, any two must be greater than the third; or, three, greater than a fourth, &c. (See Theorem 13.)

Def. VII. A PYRAMID is a SOLID, contained by any number of Planes more than three; of which, one (the Base) may have any number of Sides; all the other Planes are Triangles, meeting in a common Vertex.

As AB , or CD ; A and C are their Vertices; B and D are their Bases.



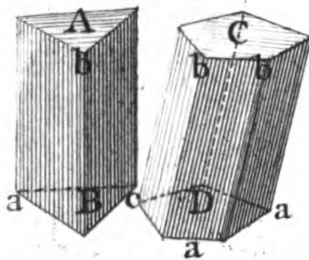
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N. B. Not less than four Planes can form a Solid; and that, must necessarily be a Pyramid, and the Planes are all Triangles.

Therefore a Pyramid may be said to be the first of Plane Solids; nevertheless it is not the simplest.

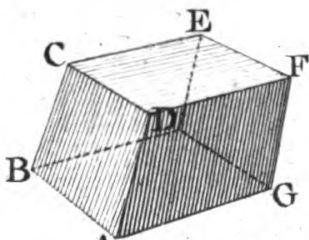
Def. VIII. A PRISM is a SOLID, contained within any number of Planes more than four; of which, two are parallel, equal, and similar Figures (of any number of Sides) all the other Planes are Parallelograms.

As AB; or CD. A and B, C and D are the similar and equal Planes; the others, a b, b c, &c. are all Parallelograms.

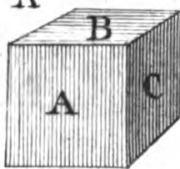


Def. IX. A PARALLELOPIPED is a PRISM, bounded by Six Planes; of which, the opposite Planes are parallel.

If the Plane ABCD be parallel to DEFG, ABDG to DCEF, and ADFG to BCED; the Solid, ACF, is a PARALLELOPIPED.



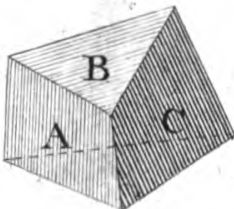
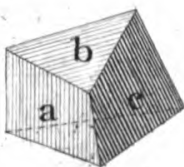
Def. X. A CUBE is a PARALLELOPIPED, whose Planes are all Squares. As A, B, C.



Def. XI. SIMILAR PLAIN SOLIDS are contained within an equal number of plane Figures, similarly situated and similar one to another, respectively, in each Solid.

The Angles are, also, necessarily equal, each to the other, respectively.

If the Plane a, be similar to A, b to B, and c to C; and if all the other Planes, of which each Solid is constructed, be also similar and situate alike; then the Solid a b c is similar to A B C.



All CUBES are similar Solids.

All SOLIDS, contained within an equal number of equilateral and equiangular Planes, are similar.

* A Prism may be conceived to be generated, by the direct motion of any right lined Plane Figure; always parallel to its first position, but not in a continuation of the Plane of the Figure.

A X I O M S.

Ax. I. The first Proposition of Euclid.

Every part of a Right Line is in the same Plane.
i. e. One part of a Right Line cannot be in any Plane, and another part of the Line out of that Plane.

For, if it was possible, a Right Line does not agree with a Plane in every Point; agreeable to Def. 6. of Book 1st.

If the part AB, of a Right Line, ABC, be in the Plane AEC, the whole Line is in the same Plane; for, the two Points, A and B, are in the Plane; and if any Point, D, be not in the Plane, ABD is not a Right Line; seeing that, AB is a Segment of the Right Line AC.

Ax. II. Two Right Lines, being parallel, are in the same Plane, i. e. a Plane, may pass through both Lines.

AB & CD, being parallel, are in the Plane ABCD.

Imagine a Plane, ABEF, revolving on AB as an Axis. Then, since CD is parallel to AB, it is parallel to a Plane passing through AB; wherefore, the Perpendiculars CE and DF are equal (Def. 2.) and, consequently, when the Point C coincides with E, D will coincide with F, and the whole Line CD is in the Plane, ABCD.

Ax. III. The seventh Proposition of Euclid.

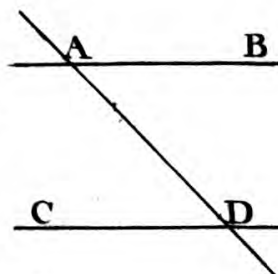
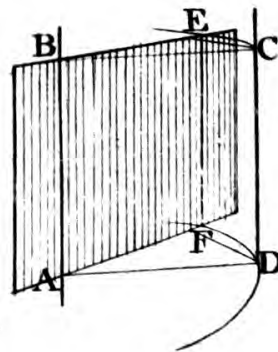
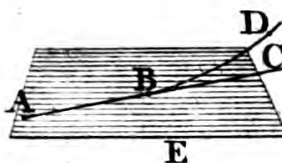
Two Right Lines, being parallel, and cut by another Right Line, are all in the same Plane.

For, if AB be parallel to CD, they are in the Plane ABCD (2nd) consequently, the Points A and D, where AD cuts them both, are in that Plane.

Wherefore, since there are two Points, A and D, of a Right Line AD, in a Plane, the whole Line is in that Plane (Ax. 1.) and, consequently, AD is in the same Plane with the two parallel Lines, AB and CD.

Hence, if two or more Lines, not parallel, are in the same Plane, and are both cut by another Right Line, it is also in the same Plane with them. Consequently, two Right Lines (AB and AD) cutting each other, are in the same Plane.

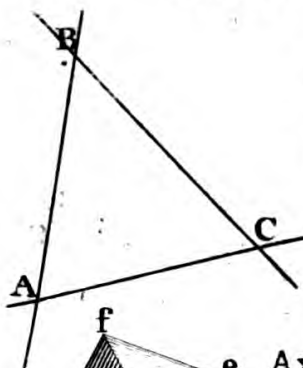
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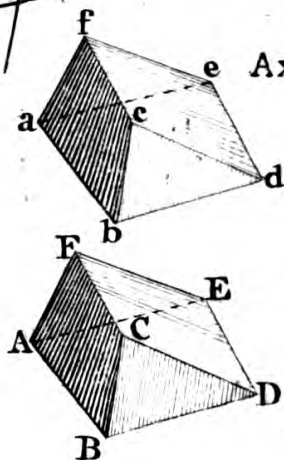
Ax. IV.

Ax. IV. The second Proposition of Euclid.

Three Right Lines cutting each other (not in the same Point) are all in the same Plane; wherefore, the three Sides of every right lined Triangle are in one Plane.



For, the two Right Lines, AB and AC, are in the same Plane (3.) Conf. being both cut by another Right Line, (BC) seeing, the two Points, A and C, are in the same Plane, the whole Line, AC, is also in that Plane - Ax. I.



Ax. V. Solids, contained within an equal number of Planes, which are equal and similar to each other, two and two, (i. e. one in each Solid) respectively; and being similarly situated, in respect of each other, and also of the whole; such Solids are equal in every respect. As ACE and ace.

If the Plane $abde$, be equal and similar to $ABDE$; ac , to AC , and ce to CE ; also afe , to AFE , and bcd , to BCD ; then, the Solid $abcdef$, is equal in every respect, to $ABCDEF$.

The eleventh Book of Elements is, in general, less understood than any of the foregoing; not owing to any deficiency or imperfection in the Demonstrations, but very much so to the imperfection of the Diagrams. It very rarely happens (and which never, yet, has happened) that a Mathematician, who had engaged in a Publication of the Elements of Geometry, was also, a competent Draftsman, in Perspective; for want of which, the Schemes, which are intended to represent Planes cutting Planes, &c. are so badly devised, and worse represented, that it is with the greatest difficulty a just Idea of their meaning can be communicated; nor is it possible, unless the Pupil has very acute Talents, or is assisted in his study of them, by some Apparatus, or an able Tutor.

I flatter myself that I have removed that difficulty, which has hitherto been the greatest impediment, to a clear investigation of the Doctrine of Solids. Where it is necessary, moveable Schemes are adapted for the purpose; and, where they may be dispensed with, the Diagrams are justly, and perspective, represented; so that, the Student must not always expect (as in Plane Geometry) that, Angles which are said to be right, or Lines parallel, &c. are really so, in the Diagram; but, that they are, or would be so, if such a Figure was really constructed, as the Diagram represents. For, unless the Reader can conceive this, it will be very difficult for him to suppose one Line to be greater than another, when it is, absolutely and evidently, less; which, from the Premises given and the foregoing Elements, it is manifest must be greater.

Therefore it is, in some measure, necessary (by way of Postulate) to request, that one Line be equal to another; or at right angles, i. e. perpendicular to another Line, or Plane, &c; that two Right Lines contain an Angle, equal to that which two other Lines contain; which, in the Diagram, is, perhaps, either greater or less. These things must be granted and understood to be so, or no Demonstration, of what is intended, can possibly follow.

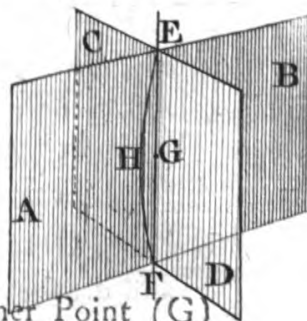
I thought it necessary to apprise the young Student of these preliminaries; otherwise (without a Tutor) he would, frequently, be perplexed and bewildered; and would, very probably, conclude, that there must be some mistake, or error in the Diagram; for it is not easy, at first, to conceive, that one Line should be perpendicular to another, when we see and know, that the Angles they make, are not Right ones; and, being acute, that another Angle, apparently contained within the other, and manifestly less, is a Right Angle; i. e. understood to be so.

THEOREM I. 3 Euclid.

The common Section of two Planes is a Right Line.

Let the two Planes, AB and CD, cut each other, in EF.

I say, that EF is a Right Line.



DEM. The Points E and F, and every other Point (G) in the Line EF, are in both Planes; for, the Line FG is in both Planes; seeing it is their common Section.-Ax. I. Therefore, it is a Right Line; seeing that, a Plane agrees with a Right Line in every Point; - - Def. 6. I. and, a curved Line cannot possibly be in two Planes.

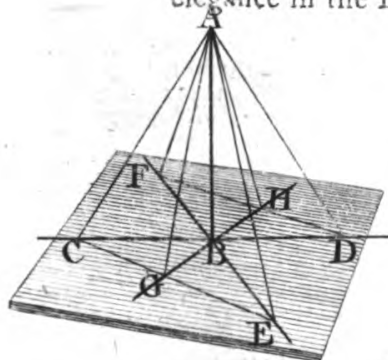
For, suppose the curve Line, EHF, drawn through the Points E and F, in the Plane ABC; if it was their common Section, it must also be in the Plane, CD; and consequently it would not be a Plane, but a convex Surface.

This Theorem might pass for an Axiom, being self-evident.

THEOREM II. 4 Euclid.

If a Right Line stands at right angles, in the Point of intersection of two Right Lines; it will be perpendicular to the Plane, in which those Lines are.

This might also pass for an Axiom; but there is a peculiar elegance in the Demonstration, which cannot be dispensed with.



Imagine the Right Line, AB, standing at right angles to the two Right Lines, CD and EF, in the Point B of their common Section.

Then, AB will be perpendicular to the Plane CEDF, in which the two Lines, CD and EF, are situated,

Make BC equal to BD (at pleasure) and imagine BE, BF, also equal to BC, BD.

AB standing perpendicular to them, imagine the Right Lines AC, AD, AE, and AF, drawn to any Point, A, in AB.

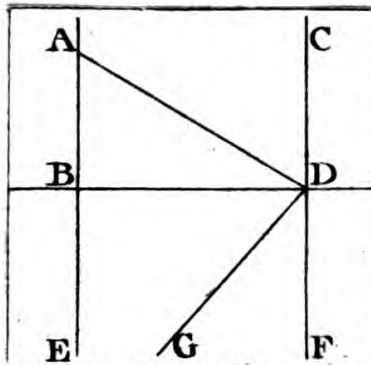
Join CE and DF; and, through B, draw any Right Line, GH, at pleasure, cutting CE and FD, at G and H; and draw AG, AH.

DEM. Then, because CB, BD, BE, and BF are all equal, and AB is common; also, the Angles ABC, ABD, ABE, ABF, are all supposed Right, therefore equal; the Right Lines AC, AD, AE, and AF, are all equal. - 8. 1. wh. the Angle ACE = AEC; and ADF = AFD - 9. 1. And, because the Angle CBE = DBF (2. 1.) and CB, BE are equal respectively, to BD, BF; CE = FD - 8. 1. consequently, the Angle BCE = BDF, - same. Also, the Triangle CAE, being equilateral to FAD*, they are conf. equiangular; & the Angle ACE = ADF - 7. 1.

THEOREM III. 6 Euclid.

If two Right Lines be perpendicular to the same Plane, they are parallel to one another.

This might well pass for an Axiom; for, it is evident, the two Lines may be in the same Plane, which is thus proved.



Let AB and CD be at right angles with the Plane BEFD.

I say, that AB is parallel to CD.

Draw BD, through the points, B and D, in which the Lines, AB & CD, cut the Plane. Also, draw BE and DF, perpend. to BD.

DEM. Because AB and CD are both perpendicular to the Plane BGD, they are at right angles with every Line in that Plane, drawn through the Points B & D, respectively; therefore, to BE and DF (2.) and, consequently, they don't incline to the Plane BGD, on any Side - Def. 3. Wherefore, a Plane may pass through both Lines - Def. 4. But, the Angles ABD and CDB are Right - Def. 3. Therefore, AB is parallel to CD - - - Th. 4.1.

If the Plane BACD be turned up perpendicular, to BGD, the thing is manifest; seeing, the Lines AB and CD, make Right Angles with BE and DF, respectively.

According to Euclid (AD being drawn) the three Lines, AD, BD, and CD, are proved to be at right Angles with DG, supposed perpendicular to BD; but, the process, though strictly true, appears very lame and contradictory in the Figure.

COROLLARY. The 8th Proposition of Euclid.

If two Right Lines be parallel, and if one of them be at right angles with some Plane, the other Line is also at right angles with the same Plane.

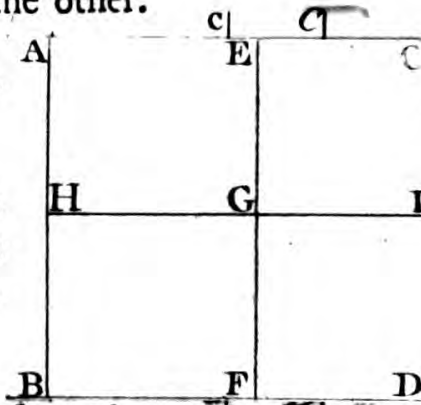
This, being the Converse of the Theorem, is manifest.

THEOREM IV. 9 Euclid.

Right Lines being parallel to the same Right Line, which is not in the same Plane with them, are parallel amongst themselves; i. e. each to the other.

Let AB and CD be each parallel to EF;
AB is parallel to CD.

In EF, take any Point, G, from which draw Right Lines, GH and GI, both at right angles with EF, and cutting the Lines AB and CD respectively.



DEM. By the Premises, EF is not in the same Plane with AB and CD; therefore, raise up EF, out of the Plane, but parallel to it, till CD coincides with CD, or any other Line, c d, parallel to EF.

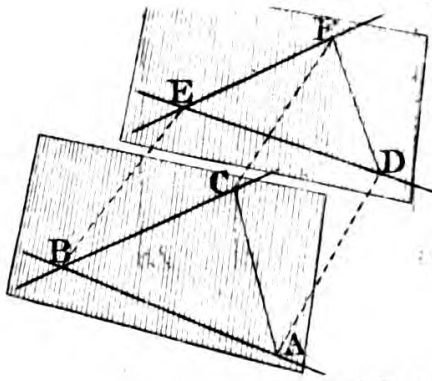
Then, because GH and GI are at right angles with EF, EF is perpend. to a Plane passing through those Lines - Th. 2
But AB and CD are both parallel to EF;
wherefore, they are also at right angles with the Plane HGI, passing through GH and GI - Cor. to Th. 3.
Therefore, they are parallel between themselves; by Th. 3.

When the three Lines are all in the same Plane, as in Theo. 5. Book 1st. it is manifestly an Axiom; and, in this Case, it amounts to little more; seeing that, a Plane may pass through any two Right Lines, which are Parallel to one another. - Ax. 2.

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THEOREM V. 10 Euclid.

If two Right Lines, cutting or meeting each other, be respectively parallel to two other Right Lines, also meeting or cutting one another, though not in the same Plane with the first, they shall contain equal Angles.



Let the two Right Lines, AB and BC, be respectively parallel to DE and EF; viz. AB to DE, and BC to EF; and, suppose they are not in the same Plane.

I say, the Angle ABC is equal to DEF.

Take BA equal to ED, BC equal to EF, and draw the Right Lines AD, BE, & CF.

DEM. Because AB is equal and parallel to DE, - Hyp.
AD is equal and parallel to BE - - - Cor. to 15. 1.
and, for the same reason, BE is equal and parallel to CF;
wherefore, CF is equal and parallel to AD-Ax. 3. 1 & Th. 4.
Join AC and DF, by Right Lines.

Now, because AD is equal and parallel to CF,
AC is equal (and also parallel) to DF; as above.
Then, in the Triangles ABC, DEF, the three Sides, AB,
BC, and AC, of the one, are respectively equal to DE,
EF, and DF of the other;

wherefore, the Angles, of one, are also, respectively, equal
to the Angles of the other. - - - - 7. 1.

Therefore, the Angle ABC is equal to DEF;
being opposite equal Sides, AC and DF. Q. E. D.

From the foregoing Theorems may be deduced the following Problems.

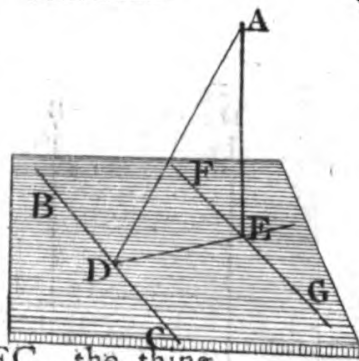
P R O B.

PROBLEM I. 11 Euclid.

From any given Point, to draw a Right Line perpendicular to a Plane, in which that Point is not situated.*

Let A be the given Point, and BEC a Plane.

In the Plane BEC, draw, at pleasure, the Right Line BC, and, from A, draw the Right Line AD, perpendicular to BC. - - - (Prob. 7.)



If AD be perpendicular to the Plane BEC, the thing is already done; but, if it be not, proceed as follows.

From the Point D, draw DE, in the Plane BEC indefinite, also perpendicular to BC; (Prob. 6.) and from A, draw AE perpendicular to DE. (Prob. 7.)

I say, that AE is perpendicular to the Plane BEC.

Through the Point E, draw FG, parallel to BC; (Pr. 5.) consequently, it is in the same Plane with BC. - Ax. 2. because the Point E is in the Plane.

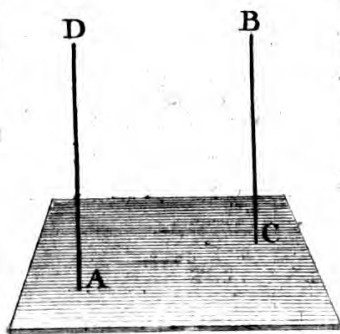
DEM. Now, because AD & DE are both perpendicular to BC, BC is perp. to a Plane passing through AD & DE - Th. 2. But, FG is parallel to BC; by Construction; wherefore, FG is also perpendicular to the Plane ADE. Now, since AE is perpendicular to FG, and, also to DE, it is perp. to the Plane in which those Lines are situated - 2. Therefore, AE is perpendicular to the Plane BEC. Q.E.D.

* Professor Simson says, from a Point given above the Plane.

I am not a little surprized at the expression, from one of his extraordinary Sagacity; because, the same thing holds true however the Plane be situated; or the Point in respect of the Plane, provided it be not in the Plane.

PROBLEM II. 12 Euclid.

From a Point given in a Plane, to draw a Right Line perpendicular to the Plane.



Let A be the given Point, in the Plane AC.

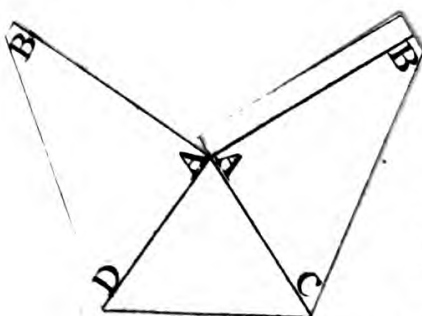
Assume any Point, B, out of the Plane; and, from that Point, draw BC perpendicular to the Plane; by the foregoing.

Then, from the Point A, draw AD, parallel to BC. - - (by Prob. 5. 1.)

AD is perpendicular to the Plane AC.

DEM. Because BC is perpendicular to the Plane - (Prob. 1.)
and AD is parallel to BC; by Construction;
AD is perpendicular to the Plane AC. - - Th. 3.

I believe, that the manual operation, of this Problem, would be easier performed by two Right Angles. Thus



Let A be the given Point, at which a Perpendicular is required, to the Plane ACD.

Apply a Right Angle, BAC, to the Point A, at pleasure, on AC; and, in any Angle (CAD) apply another R. Angle BAD. i. e. raise up the two right angled Triangles

BAC, BAD, on AC and AD, till the Points, B, coincide. Then, AB is perpendicular to the Plane CAD. - Th. 2.

COR. Hence it is manifest, that there cannot be drawn two Lines, from the same Point, perpendicular to a Plane, and on the same Side.

THEO-

THEOREM VI. 14 Euclid.

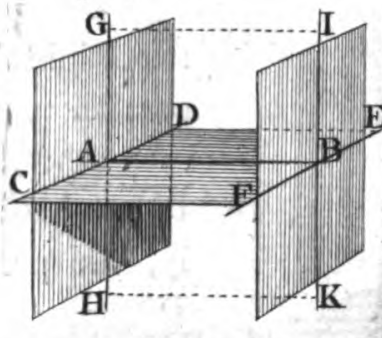
Planes, to which the same Right Line is perpendicular, are parallel to one another.

Let the Right Line AB be perpendicular to both the Planes, CD and FE .

I say, those Planes are parallel.

Imagine a Plane, $CDEF$, to pass through the Line AB , cutting both Planes, in CD , and EF ; which Lines, necessarily pass through the Points A and B , in which the Line AB cuts those Planes, respectively. - Ax. 1. and Th. 1.

At the Points A and B , draw AG and BI , in the Planes $CGDH$ and $FIEK$, perpendicular to CD and EF (Prob. 6.) and produce them to H and K .



DEM. Then, because GH and IK are perpendicular to CD and EF , and these Lines are in the Plane $CDEF$; GH and IK are perpendicular to that Plane; - Th. 2. (Because, AB being perpendicular to both the Planes, $CGDH$ and $FIEK$, it is perpendicular to every Line drawn through the Points A and B ; th. to GH & IK) wherefore, they are parallel to one another; - Th. 3. and, consequently, the two Planes cannot meet; being produced, either way, towards G and I , H and K - Ax. 1.

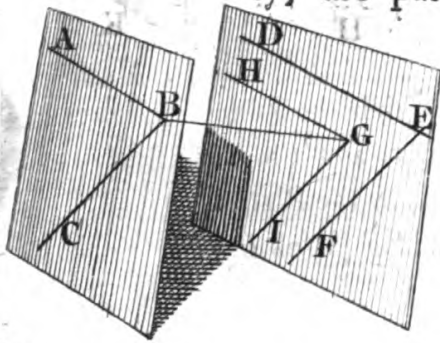
Again. Draw any other Right Line through A & B , in the Planes $CGDH$ and $FIEK$, at right angles, with, or making equal Angles, and inclined the same way towards GH and IK ; as CD and FE .

Then, because the Angle $GAD = IBE$, and AG is parallel to IB ; and those Lines are perpendicular to AB ; CD is parallel to EF . - - - Ax. 10. 1. and consequently, the Planes $CGDH$ and $FIEK$ cannot meet, being produced towards D and E , or F and C .

Th. the Planes $CGDH$ and $FIEK$ are parallel - Def. 2.

THEOREM VII. 15 Euclid.

If two Right Lines, cutting or meeting each other, be parallel to two other Right Lines, also meeting each other, not in the same Plane with the first two; Planes passing through each two Lines, respectively, are parallel.



Let, the Right Lines AB, and BC, meeting at B, be parallel, respectively, to DE and EF, meeting at E, and not in the same Plane with AB, and BC.

The Planes ACB, DFE, drawn through those Lines, are parallel.

From the Point B (in which AB meets BC) draw BG perpendicular to the Plane DFE (Pr. 1. 7.) and, through the Point G, in which BG cuts that Plane, draw, GH and GI, respectively, parallel to ED and EF. (Prob. 5. 1.)

DEM. Then, because BG is perpendicular to the Plane DEF GH, and GI make Right Angles with BG. - Th. 2.
But, AB is parallel to GH, and BC to GI; - - 5,
(for, they are respectively parallel to ED and EF)
And, BGH, BGI are Right Angles; - - Th. 2.
consequently, ABG and CBG are Right Angles - 4. 1.
wherefore, BG is perpendicular to the Plane ABC; - 2.7,
and, consequently, the Plane ACB is parallel to DFE, } 6,
in which the Lines AB, BC, and DE, EF are situated.

T H E.

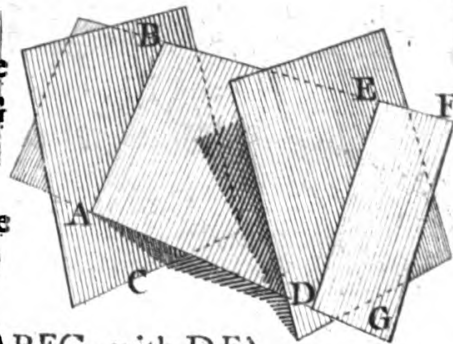
THEOREM VIII. 16 Euclid.

If two, or more, Planes, which are parallel, be cut by another Plane; their common Sections are parallel.

This Theorem might pass for an Axiom, or be deduced from the 6th, as a Corollary; but it is easily demonstrated.

Let the Planes ABC and DEG be parallel; and, imagine a Plane, ABFG cutting them, in AB and DE.

I say, the Sections, AB and DE, are parallel to one another.



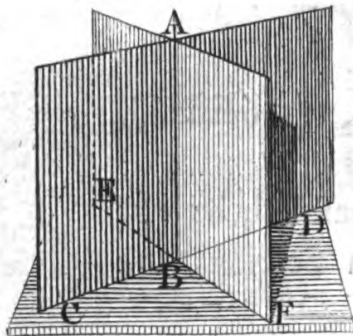
DEM. If AB (being in the same Plane, ABFG, with DE) be not parallel to DE, they will meet, if produced, towards one extreme on the other. - - Def. 7. 1. But, AB is in the Plane ABC; and DE, in DEG - Th. 1. And, the Plane ABC is parallel to DEG. - - Hyp. Therefore, since both Lines are in the Plane ABFG, and are, also, respectively, in the Planes ABC, DEG, AB is parallel to DE; seeing they cannot go out of those Planes, and consequently can never meet. - Ax. 1.

THEOREM IX. 18 Euclid.

If a Right Line be perpendicular to a Plane; every Plane, which passes through that Line, is also perpendicular to the Plane.

Let AB be perpendicular to the Plane CEDF, and let any Plane CAD, or EAF, pass through the Line AB.

I say,



I say, the Planes, CAD and EAF, are perpendicular to CEF.

DEM. Now, because AB is perpendicular to the Plane CEF (by Hypothesis) it does not incline to it on any Side-Def.3.

But, the Line AB is in the Plane CAD; and it is also in the Plane EAF; (by Supposition). consequently, those Planes do not incline, on either Side, to the Plane CEF; seeing that, every Line, BC, BE, BD, and BF, make Right Angles with AB. - - - Th. 2. Th. the Planes, CAD, EAF are perp. to CEF - Def.4.

COROLLARY I. The 19th Proposition of Euclid.

If two Planes, cutting each other, be perpendicular to the same Plane; their common Section is perpendicular to that Plane. The Converse.

For, since the Planes CAD, EAF, cutting each other in the Line AB, are perpendicular to the Plane CEF, AB is perpendicular to CEF.

Because, there cannot be drawn from the same Point, B, on the same side of the Plane CEF, two Right Lines, perpendicular to the same Plane. And AB is in both Planes - - - Th. 1.

Therefore, AB, being in both Planes, CAD, EAF, is perpendicular to the Plane CEF. - - - Cor. to Prob. 2.

COROLLARY II. The 38th Proposition of Euclid.

If from any Point in a Plane, which is perpendicular to another Plane, a Right Line be drawn, perpendicular to that other Plane; it will cut the other in the common Section of the two Planes.

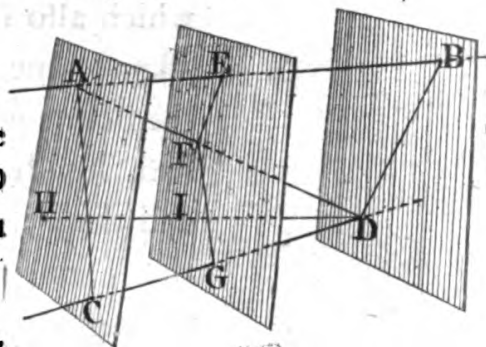
For, since one Plane is perpendicular to the other (Hyp.) every Line drawn from any Point, in one, perpendicular to the other, is wholly in the first Plane; as AB.

Consequently, since it cannot go out of that Plane, (Ax. 1.) it must necessarily cut the other Plane in their common Section.

THEOREM X. 17 Euclid.

Right Lines, being cut by parallel Planes, are cut proportionally.

Let the Right Lines, AB and CD, be cut by the Planes ACH, EG, and BD (being parallel between themselves) in the Points A, E, B, and C, G, D.



I say, as AE is to EB, so is CG to GD.

Join the Points A and C, B and D; and, AD being drawn diagonal-wise, cutting the Plane EG in F, draw EF and FG, joining the Points where they cut the Plane, EG.

DEM. Then, because AB, AD, and BD are three Right Lines, cutting one another, they are in the same Plane ABD; and ADC is also a Plane, by the same - Ax.4.

Now, since the Planes BD and EG are parallel, and are both cut by the Plane ABD;

their common Sections, BD and EF are parallel; }
and, for the same reason, AC is parallel to FG. } Th. 8.

But, in the Triangle ABD, because EF is parallel to BD, the Sides AB, AD are cut, as $AE : EB :: AF : FD$ - 2.6.

And, because FG is parallel to AC, as $AF : FD :: CG : GD$.

Therefore, as AE is to EB, so is CG to GD. - Ax.13.5.

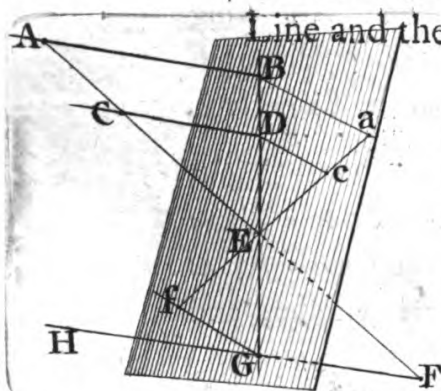
COR. Hence it is manifest, that any Number of Right Lines, proceeding from one Point, are cut proportionally (from their common Point of Section) by parallel Planes.

For, since, $DF : DA :: DG : DC$; 2. 6. & 6. 5. consequently, any other Right Line, DH, is cut in the same Ratio, at H and I.

THE-

THEOREM XI.

If two Right Lines are perpendicular to a Plane, (whether they are on the same Side, or on contrary Sides,) and are both cut by another Right Line, which also cuts the Plane; that Line will cut the Plane, somewhere in a Right Line passing through the Points where the Perpendiculars cut the Plane; and, it will be cut in the same Ratio, by the Plane, and the Perpendiculars, as the Perpendiculars, by that Line; and the Right Line, joining or passing through the Points where the Perpendiculars cut the Plane, will also be cut proportionally, by that Line and the Perpendiculars.



First; let the two Lines AB and CD, (on the same Side) be perpendicular to the Plane a B f; cutting it in the Points B and D.

From any Point A, in AB, draw a Right Line AC, cutting CD, in C, and produce it, till it cuts the Plane at E.

Then, a Right Line, BD, being produced, will pass through the same Point, E; and, they will, both, be cut, in the Points A, C, E, and B, D, E, in the Ratio of the two Perpendiculars; viz. as AB to CD.

DEM. For, because AB and CD are both perpendicular to the same Plane, they are parallel. - - - Th. 3. wherefore, AC and BD are in the same Plane. - Ax. 3. And, because the Points B and D are in the Plane a B f; the Right Line BD, is in that Plane - - - Ax. 1. (for it is the common Section of the two Planes) Conf. every Line (as AC) not parallel to BD, in the Plane AEB, will cut the other Plane, in the Intersection of both. Th. AC produced, will cut the Plane a B f, in BD produced.

2nd. Because CD is parallel to AB, the Sides EA, EB, of the Triangle AEB, are cut proportionally. - - 2. 6.
 wh. as $EC:CA::ED:DB$; conf. $EC:EA::ED:EB$ - 6. 5.
 i. e. $EC:EC+CA::ED:ED+DB$; viz. as $CD:AB$ - 4. 6.

Again. Let FG (on the other Side of the Plane) be perpendicular to the same Plane (a B f) with AB; they are parallel (3.) which is evident, producing either; as FG, to H

Then, if AF be drawn, cutting AB and FG, it will also cut the Plane a B f in the Line BG, joining the two Points, B and G, in which the Perpendiculars cut the Plane.

And, they also, mutually, cut each other, in the proportion of AB to FG.

DEM. Because AB is parallel to FG, a Plane may pass through them both. - - - Ax. 2.
 consequently, AF and BG are both in the same Plane with AB and FG. - - - Ax. 3 and 4.
 Wherefore, in the Triangles, EAB, EFG, the Angles at E being vertical, are equal (2. 1.) the Angles at B and G are right (2. 7.) therefore equal - - - Ax. 10. 1.
 the Angles at A & F, are conf. equal; and also, by Th. 4. 1. Therefore, those Triangles are similar; - - - 4. 6.
 and therefore, as $AB:FG::AE:EF$, and BE to EG.

COR. Hence it is manifest, that if a Right Line, a B, be drawn, at pleasure, through the Point B, in the Plane a B f; and through G, if f G be drawn parallel to a B; a B and f G being made equal to AB and FG, respectively, or in the Ratio of AB to FG; then, a Right Line, drawn through a and f, will cut BG in the same Point E, as before.

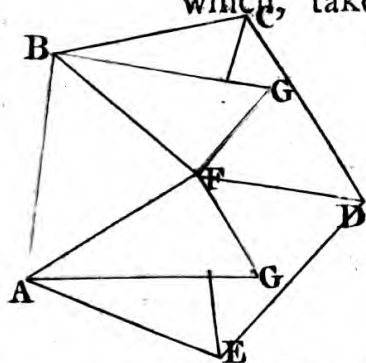
For, because a B is parallel to f G, the Triangles E a B, E f G, are similar; or, because a B is parallel to f G, a f and BG are cut proportionally, in E. - - - Case 3. 2. 6.

And, because the Triangles are similar, $a E:E f::BE:EG$, and, as $a B:f G$. - - - 4. 6.

But, $a B:f G::AB:FG$; conf. $a E:E f$, and $BE:EG::AB:FG$; and therefore, BG is cut in the same Point (E) as before.

THEOREM XII. 21 Euclid.

Every solid Angle is contained under Plane Angles; which, taken together, are less than four Right Angles.



Let $ABCDE$ be a Section of the Planes AFB , BFC , &c. forming a solid Angle, at F ; i. e. let $ABCDE$ be the Base of a Pyramid, whose Vertex is F ; supposed to be elevated out of the Plane of the Base.

DEM. Now, all the Angles AFB , BFC , &c. being in a Plane, are equal to four Right Angles. - - Cor. 2. 2. 1. But, if F be supposed elevated out of that Plane, equal to FG , and perpendicular to AF , BF , &c. and AG , or BG , drawn, AG is longer than AF - - - 12. 1.

For the same reason, all the Right Lines, from each Angle, A , B , &c. of the Base, to the Vertex, at F , elevated perpendicularly, at F (eq. FG) are longer than BF , CF , &c.

But, if two Sides of a Triangle be, respectively, greater than two Sides of another Triangle, and the remaining Sides are equal, they shall contain a less Angle - 14. 1.

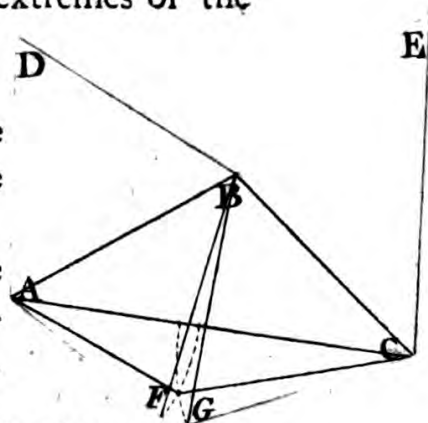
Therefore, the Angles of the Triangles AFB , BFC , &c. being considered as raised out of the Plane $ABCDE$, are less, respectively, than those Angles in the Plane; which, being equal to four Right Angles, consequently, the Angles elevated out of the Plane, are less than four Right Angles.

If the Triangles AGF , BGF , be turned up, on AF & BF , till FG , in one, coincide with FG in the other; and from the Point G , so elevated, (FG being perpend. to the Plane ABD) if Right Lines are drawn to all the other Angles, C , D , and E ; each two will contain a less Angle than those on the Plane, to which they are opposite, (as AGB than AFB) and are, together, less than four Right Angles.

THEOREM XIII. 20 & 22 Euclid.

If any two, of three given Plane Angles, be greater than the third, and the Lines containing the Angles all equal; and if those three Angles are, together, less than four Right Angles, they will form a solid Angle; and three Right Lines, joining the extremes of the equal Sides, will form a Triangle.

Let ABC , ABD , and EBC be three Plane Angles; of which, any two are greater than the remaining Angle; and, the Sides, AB , BC , BD , and BE are all equal; also, the three Angles, together, are less than four Right Angles.



Then, a solid Angle may be formed of those three Angles, ABC , &c. also, of the three Lines, AC , AD , and EC , a Triangle may be formed.

DEM. Turn over the Triangles AFB and BGC (equal ABD and EBC) till the two Sides BF and BG coincide.

Because they are less than four Right Angles (the deficiency being the Angle FBG) it is manifest they will form a solid Angle, at B , their common Vertex; the two lesser Angles, ABF , CBG , being together greater than ABC .

For, it is evident, if they were less, as ABF , CBG , the Sides BF and BG would not meet each other; and conf. with ABC , they cannot form a solid Angle, at B .

Therefore, a solid Angle formed of three Plane Angles, any two are, together, greater than the third. Q. E. D.

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Or, a solid Angle formed of any number of Plane Angles, any one must be less than all the others, added together.

Notwithstanding, if BF and BG be equal to AB & BC respectively, although the Angles (ABF, CBG) are, together, less than ABC , AF & CG , together, are greater than AC ; and consequently, a Triangle may be formed of the three Lines AC , AF and CG ; and therefore, of AC , AD , and EC ; which are greater than AF and AG - 13. 1.

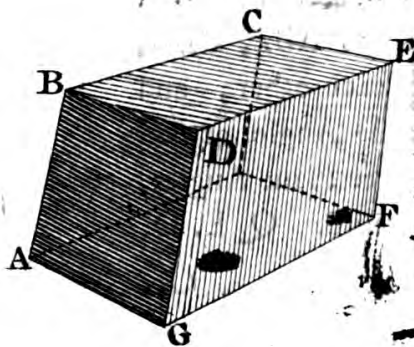
N. B. The last part of this Theorem, which (according to Euclid) seems to imply, that, unless two of the Plane Angles are greater than the third, a Triangle cannot be formed of the three Lines, joining the extremes of equal Sides; whereas, the contrary is manifest. It appears however, to me, of so little consequence, as not worthy of being made a distinct Theorem, of itself.

N. B. 2. To make a Solid Angle of three given Plane Angles, (the 23rd Prop. of Euclid) is to place them so, together, that the Vertices, of all the three, meet in one Point; and, the Sides, which contain the plane Angles, coincide, two and two in one Line, as in the Figure. (see Defin. 6.)

With this Problem, Euclid's Commentators take up more than four Pages; but, for what purpose I can't devise.

THEOREM XIV. 24 Euclid.

If a Solid be contained within six Planes, of which two and two are parallel; the opposite Planes are Parallelograms, equal and similar.



Let $ABEG$ be a Parallelopiped.

The opposite Planes $ABCD$, $DEFG$; $BCEH$, and $ADFG$; also, $ABDH$, and $DCEF$, are, respectively, equal and similar.

DEM.

DEM. The Planes, AC, GE, being parallel, and both cut by the Planes BG, and CF, which are also parallel; their Sections, AB and DG, CD and EF, are parallel amongst themselves. - - - - - Th. 8.

But, they are also cut by the parallel Planes, BE and AF; their Sections, AD, BC, DE, and FG, are conf. parallel.

Therefore, the Figures formed by those Sections are Parallelogms. seeing their opposite Sides are parallel-Def. 33

2nd. Because AB is parallel to GD, and AD to GF, and are in parallel Planes, the Angle BAD is equal to DGF; wh. the opposite Angles, BCD and DEF, are equal-15.1. But $AB = CD$, and also to DG; and $DG = EF$ -same. conf. they are all equal amongst themselves. - Ax. 3. 1. Also, AD, BC, DE, and GF are equal amongst themselves; therefore, the Parallelogram $ABCD = DEFG$, and similar; also, $BE = AF$, and BG to CF, and respectively similar.

THEOREM XV. 25 Euclid.

If a Parallelopiped be cut in two Parts, by a Plane, parallel to any two of its Planes; the two Solids will have that Proportion to each other, as their Bases (on which they are supposed to stand) i. e. as the Parts of any Plane cut by the other; or, as the Segments of the Sides.

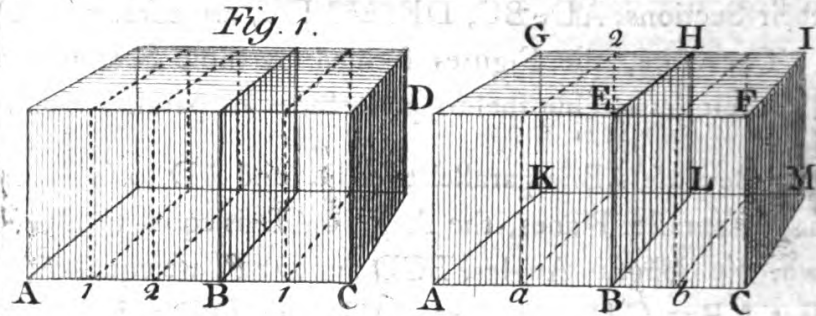
This Theorem is manifestly an Axiom.

For, since the Sections made by parallel Planes are equal, every where, which is manifest, seeing the opposite Planes are equal and similar; consequently, if each Part be again cut, into Parts, by parallel Sections, which are equal amongst themselves (as in Fig. 1.) the Demonstration follows from the 9th Axiom of the 5th Book.

Or,

Or, being incommensurable, they may be divided into an equal Number of Parts, (Fig. 2.) and demonstrated as Theorem 1st.

If the Base (AKMC) or Sides (AC, or DF) are bisected, the Parallelopiped is bisected; and consequently, in whatever Ratio one is divided, (by a Plane BEHL) the other is the same.



DEM. Let the Base AM, or any other Plane, AF, be cut by a Right Line, parallel to its Sides, AK or AD; and imagine a Plane (BH) to pass through BE or BK, parallel to AG; then, $AE : EC$, or $AL : LC :: AB : BC$. - - Th. 1.6. And, seeing the opposite Planes, DGIF and GM, are equal and similar, to AKMC, and AF, respectively; and are all cut by the Plane BEHL, parallel to AG and CI; consequently, the Solid AGHB : BHIC :: AE : BF, or, AL : BM, i. e. as AB : BC.

N. B. The two following Propositions, 26th and 27th of Euclid, are Problems; which, I do not conceive to be at all necessary, or elementary.

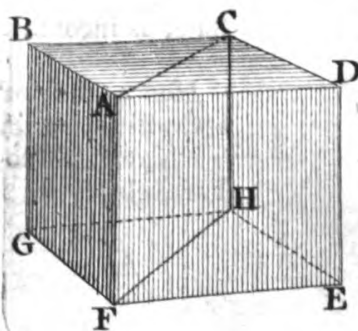
To make Solids, is rather a mechanical Process than geometrical; 'tis sufficient, for the purpose of Demonstration, to suppose that one solid Angle, or one Solid is equal to another; for, an attempt to make them really so, on a Plane, is inconsistent; unless it be to delineate them, perspectiveally so.

THEOREM XVI. 28 Euclid.

If a Parallelopiped be cut by a Plane, passing through the Diagonals of two opposite Planes, it will be cut in two equal Parts.

Let $ABDF$ be a Parallelopiped, and AC, FH , Diagonals of the opposite Planes, $ABCD$ and $EFGH$.

Then, because AF and CH are both parallel to BG , they are parallel between themselves (Th. 4.) wherefore, AF and CH are in the same Plane - Ax. 3.



I say, the Parallelopiped, $ABDF$, is bisected by the Plane $ACHF$.

DEM. For, the Triangle $ABC = ADC$; $FGH = FEH$ - 15. 1.

And the Parallelogram $ABCD$ is equal to $EFGH$ - Th. 14. (wh. the Triangles are all equal amongst themselves.)

But, the Parallelogram $BF = CE$, & $AE = BH$; - Th. 14. also, the Triangles ABC , and FGH ; ADC , and FEH are equal to one another. - - - above.

wherefore, the Prism FBC is equal to CEF - Ax. 5.

(For, they are contained within the same number of Planes which are equal, and similar to each other, respectively)

Therefore, the Plane, $ACHF$, passing through opposite Diagonals, (AC and FH) bisects the Parallelopiped, $ABDF$.

Otherwife.

By the parallel motion of the Parallelogram $ABCD$ along the Right Lines AF , DE , &c. the Parallelopiped is supposed to be generated. (see Note, Def. 8.)

Wherefore, since the Triangles, ABC , ACD , are equal, and the Prism, FBC , CEF , are described in equal Time, that is, by the same motion of both together, consequently, they are equal.

The Definition of a Prism, as it is given by most Authors, is somewhat obscure and unsatisfactory; and, we are left to imagine, that any one of the Planes may be its Base; we may as reasonably conclude, that any Plane of a Pyramid may be its Base, but it is never supposed so, unless they are all Triangles; nor in a Prism, except they are all Parallelograms.

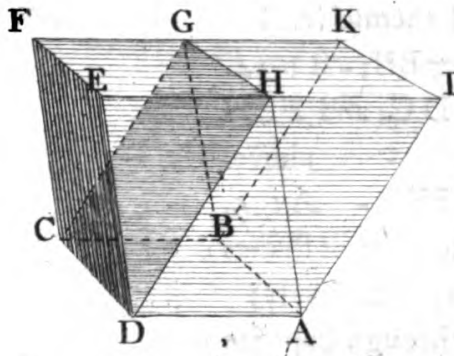
For, to suppose the two equal, similar, and parallel Planes, in a Prism, Triangles or Polygons, and any other Plane to be its Base, is inconsistent.

Hence, the 40th Proposition of Euclid, which, in reality, is but a Corollary to this Theorem, is an absurd Proposition.

Because, I do not conceive, that, unless a Prism be also a Parallelopiped, its Base can be a Parallelogram.

THEOREM XVII. 29, 30, & 31 Euclid.

Parallelopipeds having the same Base,* or equal Bases, and the same Altitude, are equal to one another.



First; let the two Parallelopipeds, AHFC and AIFC, have the same Base, ABCD; and, let their opposite Faces have a common Side, GH.

I say, the Parallelopiped A I C is equal to A K G C.

* By the Base, of a Solid, literally, is understood the Plane or Face, on which it is supposed to rest; but, it is not always considered so, in Geometry,

When any two Faces of Parallelopipeds, are in one Plane, or in parallel Planes; whether they coincide with each other, entirely, or are apart, and distinct from each other (no regard being had to the position of the Plane they are in) the Top, in such Case, is sometimes called the Base.

N. B. Parallelopipeds, having the same Altitude, are supposed to be contained between the same parallel Planes; i. e. two of their opposite Planes, in each Solid, are in the same Plane; or, that there is the same perpendicular distance between them.

DEM. The Parallelopiped AHFC, being cut by a Plane, CGHD, through the opposite Diagonals, DH and CG, is cut into two equal Parts $CAG = DFH$. - Th.16. And, for the same reason, the Plane AHGB bisects the Parallelopiped AIGC; in AIKG equal ADCG. But, $ADCG + DFH = ADCG + AIKG$ - Ax.6.1. Therefore, the Parallelopiped AHFC = AIGC. Q.E.D.

Secondly. Let the Parallelopipeds, AHFC and AMKC, have the same Base, ABCD; their opposite Faces, EFGH and IKLM, are in the same Plane, and have not a common Side.

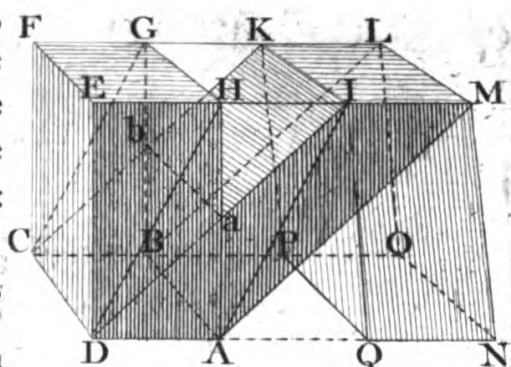
AMKC is equal to AHFC.

Let the space, GHIK, between the upper Faces, be equal to the Base ABCD; conf. equal to EFGH, equal IKLM - Th.14.

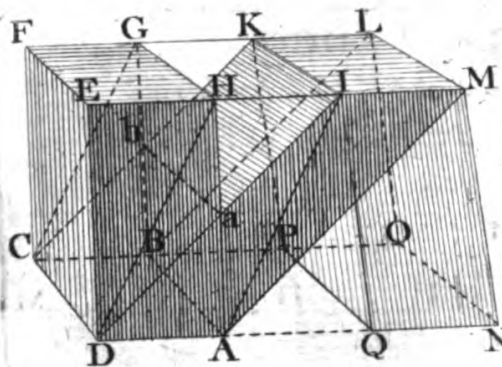
DEM. The Parallelopiped AHFC is equal to AIGC; - proved. and, for the same reason, AIGC is equal to AMKC. Th. the Parallelopiped AMKC is equal to AHFC - Ax.3.1.

Again. Suppose the space, GHIK, be either greater or less than EFGH.

The Triangles, AMH, DIE, are congruous; - 7.1. (for AM is equal to DI, MH = IE, and AH = DE; by 15.1.) And the Sides, AB, DC, EF, GH, IK, & LM, are equal - 14. the Par. AG = DF; ABLM = IKCD; & EFKI = MLGH; conf. the Solid AGLM is equal to DFKI. - Ax. 5. Now, if the Solid, aIKGb (common to both) be taken away, AMKb is equal to DaGC. - Ax.7.1. And, if the common Solid Aa b C D be added, to both; the Parallelopiped, AMKC, is equal to AHFC - Ax.6.1.



The second Part, viz. when they have equal Bases only, and have not one common to both, is readily proved.



Let the Bases, $ABCD$, of the Parallelepiped $AHFC$, be equal to Base $NOPQ$, of the Parallelepiped $NMKP$ (and similar)

Then, the Parallelepiped, $AHFC$, is equal to $NMKP$.

Let them be so situated, that the Planes AE , and NI , are in the same Plane, $DEMN$; and draw AM , DI , BL , and CK .

DEM. Now, because MI is parallel to AD , and LK to BC , and also equal, $AMID$ and $BLKC$ are Parallelograms - C.15.1. And, for the same reason, $AMLB$ and $DIKC$ are Parallelograms; and they are, also, parallel and similar, respectively; wherefore, $AMKC$ is a Parallelepiped. - - Def. 9. (For, AM is parallel to DI , and ML to IK ; wherefore, the Angle $AML = DIK$, &c. - Th. 5.)

But, the Parallelepiped $AHFC$ has the same Base with $AMKC$; and they are contained within parallel Planes; wherefore, they are equal Parallelepipeds. - - proved. Also, the Parallelepiped $NMKP$, has the same Base with $AMKC$ (i.e. the Plane or Face $IKLM$ is common to both.) consequently, they are equal Parallelepipeds. - same. But, the Parallelepiped $AMKC$ is, also, equal to $AHFC$. Therefore, the Par. $NMKP$, is equal to $AHFC$ - Ax.3.1.

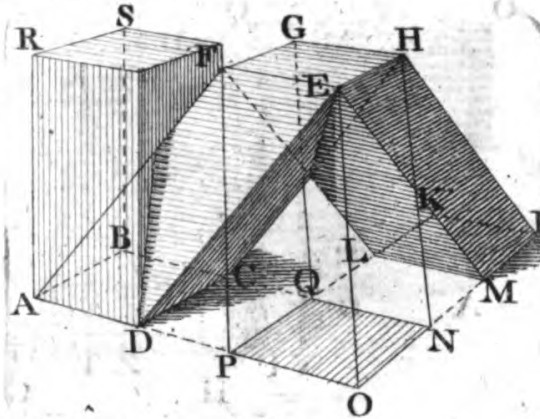
After the Proofs already given, it may, by some, be thought unnecessary to add more; but, there are other Cases, in which there is abundantly more room to dispute the Assertion, than in the preceding; and, as this Theorem is very essential, it cannot be too much enforced.

CASE II.

CASE II. When no more than two Faces of the Parallelopipeds (i. e. two in each) are in the same parallel Planes.

Let AGHD and FHIL be Parallelopipeds; whose Bases, ABCD and IKLM, are equal, and in the same Plane; and the Top, EFGH, common to both; but have no other Plane common.

I say, that the Parallelopiped AGHD, is equal to FHIL.



Produce AD and BC, IM and KL, (being in the same Plane) they will cut each other, in NOPQ, which is a Parallelogram; equal and similar to ABCD, and IKLM.

DEM. Because EFGH is common to both Parallelopipeds, and because the opposite Planes are equal and similar, AC is similar to LI; and, being similarly posited, IK, LM, are parallel to AD and BC; and IM, KL, to AB and DC; therefore, NOPQ, being between the same Parallels, is consequently equal and similar to AC, & LI - 4. & 18. 1. and consequently, to EFGH; and also similarly situated.

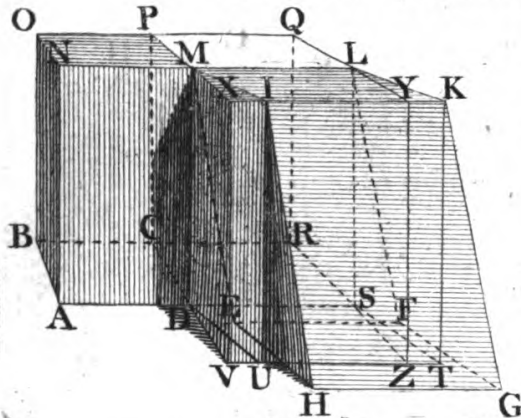
Join EO, FP, GQ and HN.

Then, PFHN is a Parallelopiped; which, being equal to each of the other, AGHD and FHIL (part 1st) they are consequently equal between themselves - Ax. 3. 1.

Also; if the Parallelopipeds, ASFD and FHIL, have their Bases equal; and have no Face common; being between the same parallel Planes, AOI and RGE, they are equal Parallelopipeds.

This is evident, from the foregoing; each being equal to PFHN.

CASE III. When the Parallelopipeds have no Face common, nor similar Figures.



Let the Parallelopipeds AOMD and EMKG, be between the same parallel Planes, ARH and NQI; consequently, they have equal Altitude; and, let the Base, ABCD, be equal to the Base, EFGH, of the other, which are not similar Figures.

I say, the Parallelopiped, EMKG, is equal to AOMD.

DEM. Let the Side NM, of one, and ML of the other, be so placed, at the common Angle M, that NML is a Right Line.

Compleat the Parallelopiped, DMKT; having a common Side, DM, with AOMD.

The Parallelopiped DMKT is equal to EMKG - Case 2nd. (For they have the same Altitude, and IMLK is common)

Produce KI and TU, PM & CD, meeting at X and V. Draw LY and SZ, parallel to MX and DV, cutting IK and TU, at Y and Z. Join VX and YZ.

Then the Parallelopiped VMYZ = UMKT - 1st. conf. the Parallelopiped, VMYZ = HMKG - Ax. 3. 1.

Produce OP and YL, BC and ZS, meeting at Q and R; and, joining QR, there is formed a Parallelopiped DPQS; equiangular to both the Parallelopipeds, AOMD and VMYZ; conf. they are equiangular amongst themselves.

Now, the Parallelogram IMLK = NOPM - Hyp. XMLY (equal IMLK 18. 1.) is equal to NOPM - Ax. 3. 1. consequently, (the Angle PMN being equal XML - 2. 1.) as NM : ML :: XM : MP. - - - 8. 6.

But, the Parallelopiped $AOMD : DPLS :: NM : ML$ }
 And, the Parallelopiped $VMYZ : DPLS :: XM : MP$; } 15.
 i. e. as $NM : ML$. - - - - - above.
 Wherefore, the two Parallelopipeds, $AOMD$ and
 $VMYZ$, have an equal Ratio to $DPLS$;
 and therefore they are equal. - - - - - Ax. 4. 5.
 But, the Parallelopiped $HMKG = VMYZ$ - above.
 Therefore, $HMKG$, is equal to $AOMD$.

COROLLARY I. The 33rd Proposition of Euclid.

Parallelopipeds having equal Altitudes, have that Ratio
 to each other as their Bases.

For, since by this Theorem, all Parallelopipeds, having equal
 Bases and equal Altitudes are equal; and, by the 15th, if a Parallelo-
 piped be cut in two, by a Plane, parallel to its Planes, the two
 Parts have that proportion to each other as their Bases; conse-
 quently, all Parallelopipeds, having equal Altitudes, are to each
 other as their Bases.

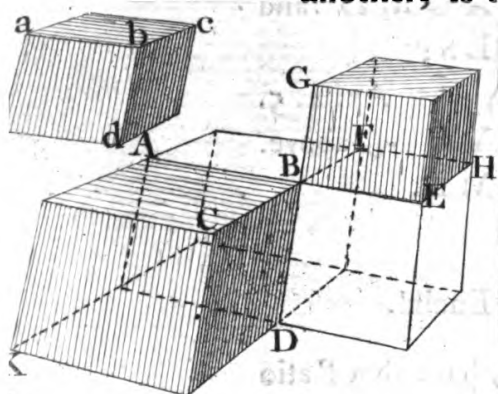
COR. II. Parallelopipeds, having equal Bases, have that
 Ratio to each other as their Altitudes.

For, having equal Bases and equal Altitudes, they are equal;
 consequently, if the Altitude of one, be half, or a third part, &c.
 of another; or, in whatever Ratio the Altitude of one Parallelo-
 piped is to the other, whose Bases are equal; they are in that
 Proportion, to each other, as their Altitudes.

T H E-

THEOREM XVIII. 33 Euclid.

The Proportion of similar Parallelopipeds, to one another, is the triplicate Ratio of their corresponding Sides.



Let ADK and $a c d$ be similar Parallelopipeds; in which, let AB, $a b$, BC, $b c$, and BD, $b d$ be corresponding Sides, in each Solid.

Then the Proportion, which the Solid ADK has to $a c d$, is the triplicate Ratio of AB to $a b$, BC to $b c$, or BD to $b d$.

In the Solid ADK, produce the Sides, AB, CB, & DB.

Make BE equal to $a b$, BF equal $b c$, and BG equal $b d$; draw EH and FH, respectively, parallel to BF and BE; and complete the Parallelopipeds GH, DH, and AFD; equiangular to ADK.

DEM. Because BH is a Parallelogram (by Construction) and the Angle EBF = ABC, (2.1.) (EBG = ABD, and FBG = CBD) also BE = $a b$, and BF to $b c$; BH is equal and also similar to $a c$; for, $a c$ is similar to AC; by Hyp.

And, for the same reason, GE, is equal and similar to $a d$ (which is similar to AD) also, FG is equal and similar to $c d$ (similar to CD) consequently, the Parallelopiped, GH, is equal and similar to $a c d$ (similar to AKD)

Now, AB: $a b$:: BC: $b c$, and as BD is to $b d$ -Def. 11.7. and, the Parallelopiped GH is equal and similar to $a c d$; BE = ab , &c. wh. as AB:BE :: CB:BF, and DB:BG. But, the Parallelopipeds, AFD and DFH, have equal Altitude, with ADK.

wh. they have that Proportion to each other as their Bases; i. e. the Solid ADK: AFD :: Par. AC: AF; i. e. as CB to BF } Cor. 1.
and, the Solid AFD: DFH :: Par. AF: FE; i. e. as AB to BE } of 17.
also, - - DFH: GEH :: Par. DE: EG; i. e. as DB to BG } & 1.6.

wherefore, the Parallelopiped $ADK:AFD:DFH:GEH$ in continual Ratio. (for the Ratio of CB to BF , AB to BE , and of DB to BG , is the same, by Hypothesis.)

Therefore, the Ratio of ADK to $a c d$ (equal GH) is triplicate of ADK to AFD ; i. e. of CB to BF (eq. $b c$) of AB to BE (eq. $a b$) or, of DB to BG (eq. $b d$)-Def. 12.5.

THEOREM XIX. 34 Euclid.

The Bases and Altitudes, of equal Parallelopipeds, are reciprocally proportional.

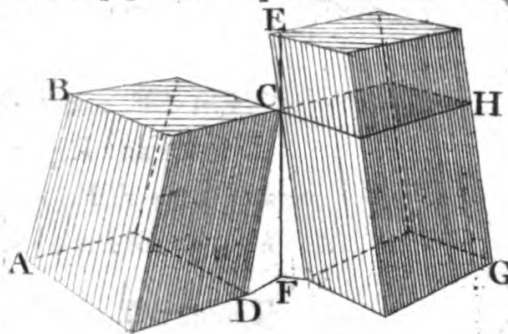
This Theorem will admit of as many Cases as the 17th; but one general Case, being clearly demonstrated, will be sufficient.

Let ABD and FEH be equal Parallelopipeds.

If the Base of one be equal to the Base of the other, their Altitudes are also equal (Th. 17.) because the Parallelopipeds are equal, by Hypothesis. Let them be unequal.

Then as the Base AD is to FG , so is EF to CF , their Altitudes.

Let them be so placed together, that the Angle C , of the one, touches the Side EF , of the other.



Produce the Plane, BC , to H parallel to FG ; and, through C , let EF be drawn, perpendicular to their Bases, AD , FG (which are supposed to be in the same Plane) EF and CF are the Altitudes of the Parallelopipeds.

DEM. Then, $ABD:FCG::AD:FG$, their Bases - Cor. 1.17

And, - $FEG:FCG::EF:CF$, their Altitudes - Cor. 2.17

But, - ABD is equal to FEG ; by Hypothesis.

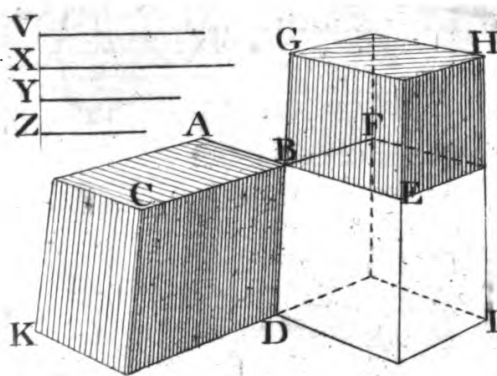
Therefore, - $ABD:FCG$, i. e. $AD:FG::EF:CF$

i. e. as the Base AD , of one, is to FG of the other;

so is EF the Altitude, of the last, to CF of the first.

THEOREM XX.

Parallelopipeds, whose solid Angles are equal, have that Proportion, to one another, which is compounded of the Ratio of their Sides.



Let ADK and BIH be equiangular Parallelopipeds; and let them be so placed together, at the equal Angles B, that, the Side AB of the one, is in the same Right Line with BE, of the other, and CB with BF; consequently, the two Sides BD and BG are in one Right Line.

(It is not material which Plane, EF, GE, or GF, is made the Base.)

On EF, and BD, compleat the Parallelopiped DFI.

Take any Right Line, V, at pleasure; and make X to V as BE to AB; and as BF is to CB, so make Y to X; also, as BG is to BD; make Z to Y. - (Prob. 32.)

Then, as V is to Z, so is the Parallelopiped, ADK to BH.

DEM. The Solids, ADK and DFI, have equal Altitudes; wherefore, they are to each other as AC to EF; -C.1.17. that is, as the Ratio which is compounded of AB to BE, and CB to BF; i. e. as V to Y. - Th.11.6. And, the Solid DFI is to BH, as the Par. DE is to EG - 17. that is, as DB is to BG, i. e. as Y to Z. - 11. 6. But, ADK:DFI::V:Y, and DFI BH::Y:Z - Con. Therefore, the Solid ADK: BH::V:Z - Th. 9. 5. i. e. in the compounded Ratio of AB to BE, CB to BF, and DB to BG.

THE O-

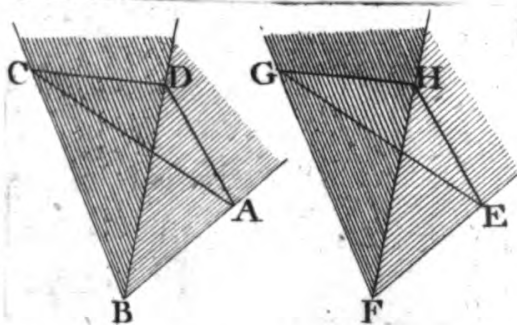
THEOREM XXI.

In solid Angles contained by three Plane Angles, two and two of which (i. e. one in each) being equal, respectively; those Planes, which contain the equal Angles, have equal Inclination to one another.

Let B and F be two solid Angles, each contained by three Plane Angles; viz. ABC equal to EFG, ABD equal to EFH, and CBD equal to GFH.

I say, the inclination of ABD, or CBD, to ABC, is equal to the inclination of EFH, or GFH, to EFG; also, the inclination of ABD to CBD is the same, as EFH to GFH.

In AB take any Point, A, at pleasure, and take EF equal AB; and, in the Planes ABC, ABD; EFG, EFH, draw AC and AD, EG and EH, at right Angles with AB and EF, respectively, and join the Points, C and D, G and H, by Right Lines.



DEM. The Angle $ABC = EFG$ (Hyp.) and $BAC = FEG$ - Con. and $AB = EF$; wh. the Triangles ACB, EGF are congruous; AC is equal to EG , and BC is equal to FG . - Th. 11. 1. Also, the Angle $ABD = EFH$ (Hyp.) & $BAD = FEH$ - Con. wherefore (seeing $AB = EF$) $BD = FH$, and $AD = EH$ - same And, because BC, BD , are respectively equal to FG, FH , and the Angle CBD is equal to GFH ; $CD = GH$ - 8. 1 Lastly, because the Triangle ACD is equilateral to EGH . they are also equiangular; and, the Angle $CAD = GEH$ - 7. 1 But, those Angles are the Inclination of the Planes, ABD to ABC , and EFH to EFG . - Def. 5. Therefore they have equal Inclination. Q. E. D.

After the same manner, it may be proved that any other two have equal Inclination; therefore, &c.

3 B

COR.

370 ELEMENTS OF GEOMETRY.

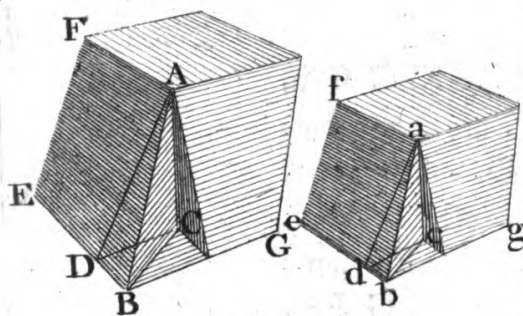
COR. Solid Angles, contained by three Plane Angles, each, of which, two and two (one in each) are equal, respectively, are equal to one another.

Because the Planes have equal Inclination to each other, respectively (by the Theorem) Therefore the Angles are equal. *

THEOREM XXII.

If from equal solid Angles, in equiangular Parallelopipeds (which are not right angled) Perpendiculars be drawn, similarly, i. e. to equiangular Planes; the Perpendiculars (i. e. their Altitudes) will be proportional to the Sides, which are inclined to the Planes on which the Perpendiculars fall.

This Theorem, though very different, apparently, is the same, in Substance, as the 35th of Euclid.



Let BFG , bfg be equiangular Parallelopipeds, and let the Angle A be equal to a ; consequently, B is equal to b .

From the equal Angles A , a , let the Perpendiculars AC , ac , be drawn to equiangular Planes (their Bases) containing the equal Angles $E B G$, $e b g$; cutting them at C and c .

I say, that, as AB is to $a b$, so is AC to $a c$.

Through C and c , draw CD and cd , perpendicular to BE and be , respectively; and join AD and ad .

* Professor Simson says (Prop. B.) "and alike situated" making that a Condition of their equality; whereas, it is not; for, it is impossible, that three Plane Angles, equal respectively to three other Plane Angles (containing a solid Angle) can be so placed, in forming a solid Angle, which will not be equal to the other.

For, they must necessarily have the same Inclination each to the other, respectively, however situated.

DEM. Because AC is perpendicular to the Plane EG - Con.
the Plane, CAD, passing through AC, is perp. to EG-Th.9
and, for the same reason, c a d is perpendicular to e g.

But, CD is perpendicular to BE; and c d to b e; - Con.
wh. BE is perpendicular to the Plane CAD, and b e to cad;
conf. AD is perpendicular to BE, and a d to b e. - Th.2.

Then, in the Triangles BAD, b a d, the Angles ADB,
a d b are right, therefore equal - - - Ax. 10.1.
and the Angle ABD = a b d, (Hyp) conf. BAD = b a d - 10 1
and therefore, as AB : AD :: a b : a d - - - Th.4.6.

But, the Angle ACD = a c d (because, AC is perpen-
dicular to the Plane EG, and a c to e g, ACD, a c d are
Right Angles) and, the Angle ADC = a d c, - 21
for, A D C is the Inclination of the Plane AE to EG;
and, a d c is the Inclination of the Plane a e to e g.-Def.5.
wherefore, the Triangles DAC, d a c are similar;
and, therefore as AC : AD :: a c : a d. - - - Th.4.6.
But, AD:AB::ad:ab (above) Th.AC:AB::ac:ab-9 5.
and, by alternation, AC : a c :: AB :: a b. Q. E. D.

THEOREM XXIII. 36 Euclid.

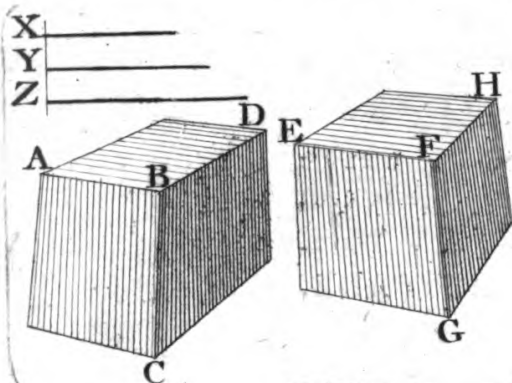
If three Right Lines are Proportionals; a Parallelo-
piped described, or constructed under all three,
for its Sides, is equal to an equiangular Parallelo-
piped, whose Sides are each equal to the Mean.

If the Parallelopipeds be right angled, that which is constructed
under the Mean is a Cube.

Let the three Right Lines X, Y, and Z, be Propor-
tionals; as X is to Y, so let Y be to Z.

3 B 2

And,



And, let ACD be a Parallelepiped, whose Side AB is equal to X, BC equal to Y, and BD equal to Z.*

Also, let EGH be a Parallelepiped equiangular to ACD.

Then, if the Sides EF, FG, and FH, are each equal to Y; * the Parallelepiped ACD is equal to EGH.

DEM. Because, $X:Y::Y:Z$, the Rectangle under X and Z, is equal to the Square of Y, i.e. $X \times Z = Y^2$ - Cor. 9. 6. But, Parallelograms, having equal Angles, are in the same Ratio as Rectangles, whose Sides are equal to the Sides of the Parallelograms, respectively, - - C. 2. 11. 6. wherefore, the Parallelogram AD is equal to EH. But, $BC = FG$; and conf. their Altitudes are equal - 22 Therefore, the Parallelepiped ACD is equal to EGH. For they have equal Bases, and equal Altitudes.

* In those Cases, the Figures being but representations of Solids, on a Plane; if the Lines, BD, and FH, were made equal to Z, and Y, respectively, they could not possibly have that appearance, but they would appear much greater; because they are supposed to recede, from the Planes of the Faces; AC and EG. Therefore, they are supposed to be equal, only; which is the same thing, and holds equally true, in the Demonstration. If one be, the other is a necessary Consequence.

Note. As the Ratio, between two similar Plane Figures, is discovered by a third Proportional, so between Solids, a fourth determines it.

Also, to describe plane Figures, in any given Ratio, a Mean Proportional is found; so in respect of Solids, two Means are requisite (see Prob. 33. or P. 332. B. 6.)

For, since the Ratio between similar Solids is triplicate of their corresponding Sides; consequently, since $AB:DH::BF:AD$; and, if the Ratio of two similar Solids be as AB to AD, AB and DH are in the Ratio of their corresponding Sides.

Hence, similar Solids may be constructed in any Ratio to one another.

THEO-

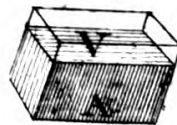
THEOREM XXIV. 37 Euclid.

If four Right Lines are Proportionals; similar Parallelopipeds, and similarly constructed on each Line, are also Proportionals.

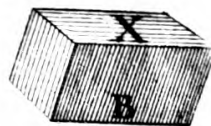
Let A, B, C, and D be Proportionals; as A is to B, so let C be to D.

And let the Parallelopipeds V, X, Y, and Z, be similarly constructed, on A, B, C, and D.

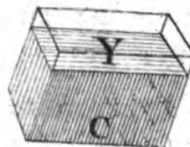
I say, that, as V is to X, so is Y to Z.



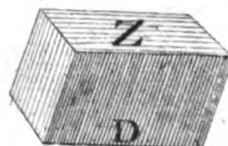
E



F



G



H

DEM. Let E and F be so taken, as $A:B::E:F$ } (Prob. 31)
 And, let G and H be taken, as $C:D::G:H$ }

Then, the Parallelopiped $V:X::A:F$ } - Th. 18.
 And the Parallelopiped $Y:Z::C:H$ }

But, as $A:B::C:D$; wherefore, as $B:E::D:G$; and,
 as $E:F::G:H$; (Ax. 13.5) conf. as $A:F::C:H$ - Th. 9.5.

Therefore, as the Parallelopiped $V:X::Y:Z$. Q. E. D.

It is not necessary that the Parallelopipeds Y and Z should be similar to V and X; for, if V be similar to X, and Y to Z, the Demonstration is evidently the same.

Or, if V be not similar to X, nor Y to Z;

but, let V be similar to Y, and X to Z; it still holds true.

For, since $A:B::C:D$, so $A:C::B:D$ - - Th. 4.5.
 consequently, similar Parallelopipeds, constructed on A and C,
 and others, not similar to them, on B and D, will still be
 Proportionals.

For, since the Parallelopipeds, on A and C, B and D, are,
 as $V:X::Y:Z$, similar; so $V:Y::X:Z$, not similar.

The same holds true, in similar Solids of any kind.

E L E-

E L E M E N T S O F G E O M E T R Y.

B O O K VIII. The XII. of Euclid.

AS the last Book, or Section, treats only of Plane Solids, and determines their Proportions, their Properties and Relation to each other; so, in this, the Doctrine of Solids, bounded by regular curved Surfaces, as the Cylinder, Cone, and Sphere, are considered; the Solution of which is truly admirable.

It must appear, to all who consider it, before they have gone through this Book (or some other on the Subject), a matter not merely of difficulty, but of absolute impossibility, to ascertain the true Measure or Area of the Surface, much less of the solid Contents of a Sphere; to reduce such a Solid to right angled and cubical Measure, with any degree of certainty, seems to be beyond the power of Art, or the reach of finite knowledge.

And yet, nothing more is requisite thereto, than the true measure of its Diameter, which may be acquired; at least of any Sphere formed or constructed by Man. In respect of the Earth itself, its Diameter can only be obtained by its Circumference. It is possible, by various means, to come somewhat near the truth, but it is beyond human abilities to ascertain it, with any degree of absolute certainty.

How much less, then, are we able to come at the true Distances, by which, the magnitudes of distant Planets are reduced to the same Scale of Proportion; viz. by the known
measure

measure of the Earth. For, if the one cannot be had, with certainty, how much more incorrect must be the other, having no other Data to build on?

Although the many ingenious contrivances, by Instruments curiously constructed for that purpose, are worthy of the greatest praise, and redound highly to the Fame of their several Inventors; and notwithstanding all other means, by Transits, &c. may still approach nearer to the truth, or, at least, evince and prove the imperfections of all human Inventions, for the purpose; we must (or may as well) remain satisfied, that we have acquired so much, as to answer various great and notable purposes, in Trigonometry, Navigation, &c; but, that we shall never be able, by any means, to come at the Truth is most certain.

The Affinity between the Pyramid and the Prism, and the Proportion one bears to the other, are, in the first place, determined; by means of which, the Proportion between the Cone and Cylinder, is also determined.

Lastly, of the Sphere; shewing the Proportion it also has to a Cylinder and Cone; as given by Archimedes; for, on that Head, Euclid is totally silent; the 12th Book only determines the Ratio one Sphere has to another, in the last Theorem. To obtain which, he has two Problems, 17th and 18th; the latter, the most prolix of the whole; containing, in four or five full Pages, the most tedious description of the Construction of a Solid, which is not of any other use, whatever; and previous to it, Professor Simson also gives a preparatory Lemma.

The manner, in which the Ratio of one Sphere to another is here investigated, and deduced from the 9th, is, at once, the most solid and convincing; and also shews how the quantity of every Sphere is ascertained, which the other does not; and therefore, what it does is of little use.

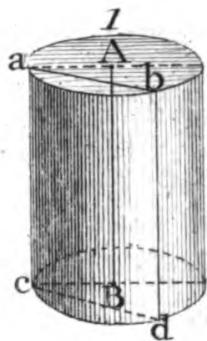
D E F I-

DEFINITIONS.

For the Definition of a Pyramid, see Def. 7th of the 7th; which, notwithstanding it is not used there, is the first of Plane Solids (having the least number of Faces) and as a Prism (Def. 8th) includes all Parallelopipeds whatever, it could not be dispensed with, though but of little use in that Book.

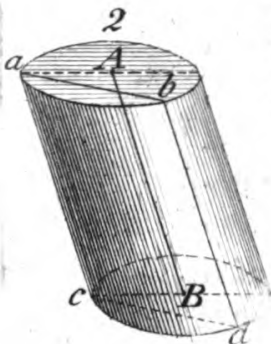
It remains, only, to define such Solids as are bounded by curved Surfaces.

DEF. I. A CYLINDER is a SOLID, having two parallel Plane Surfaces, which are equal Circles; and an uniform convex Surface, bounded, in length, by the circular Planes, and returning into itself in width, without bounds. As AB, or $\overline{AB}$.



DEF. II. The BASES, of a CYLINDER, are the two circular Planes; A and B.

DEF. III. The AXE, of a CYLINDER, is a Right Line passing through the Centers of its Bases. As AB, or $\overline{AB}$.



N. B. A Cylinder may be conceived to be generated by the Direct motion of a Circle (as A, or $\overline{A}$) along its Axe, always parallel to itself.

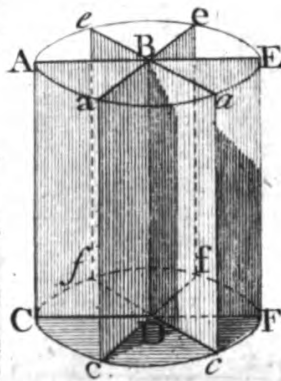
Let the Circle A, or $\overline{A}$, be supposed to be applied, at its Center, to the Right Line AB, or $\overline{AB}$; and being moved, parallel to its first position, at A or $\overline{A}$, to B or $\overline{B}$; the Line AB or $\overline{AB}$, being always in its Center; it will, in that motion, have described the Cylinder AB or $\overline{AB}$.

If any Right Line as $\overline{ab}$, or $\overline{ab}$, be drawn in the Circle, cutting the Circumference, at a, or b, $\overline{a}$ or $\overline{b}$; and, whilst the Circle described the Cylinder (the Right Line $\overline{ab}$, or $\overline{ab}$, being always parallel to its first position) the Point a or b, $\overline{a}$ or $\overline{b}$, will describe a Right Line, a c, or $\overline{bd}$, which is a Side of the Cylinder.

DEF. IV. A RIGHT CYLINDER is one, whose
Axe, AB , is perpendicular to the Planes of its
Bases, A and B .

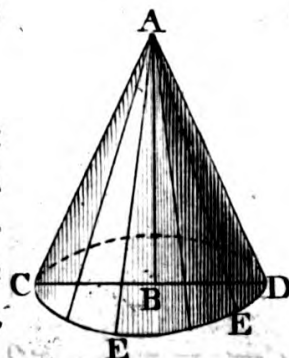
A Right Cylinder may be conceived to be generated by the revolution of a right angled Parallelogram, or Rectangle ($CABD$) on one Side (BD) which remains fixed; and is therefore called its Axe, or Axis; on which the Rectangle revolves, and on which the Solid, revolving, would appear at rest.

The two Sides, AB and CD , being equal, describe the equal Bases, whilst AC describes its Surface.



DEF. V. AN OBLIQUE CYLINDER, is when its
Axe, is inclined to the Bases, (see Fig. 2)

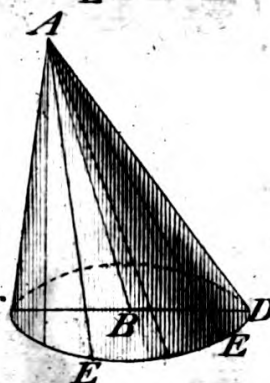
DEF. VI. A CONE is a SOLID, having but
one Plane Surface, which is a Circle; and a
convex Surface, returning into itself; continually
varying, in convexity, from the Circle
which bounds it, till it terminates in a Point.
As ACD , or ACD .



DEF. VII. The BASE, of a CONE, is the circular
Plane CD , or CD .

DEF. VIII. The VERTEX, of a CONE, is the
Point A or A .

DEF. IX. The AXE, of a CONE, is the Right
Line AB , or AB ; from the Vertex A , or A ,
to the Center B , or B , of its Base.

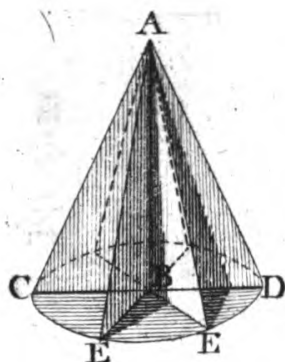


N. B. A Cone may be conceived to be generated thus.

If CD or CD be a Circle, and any Point, A , or A , be taken out of its Plane; then, if a Right Line AB , or AB , be applied from the Point A , or A , to any Point C , or C ; and being fixed, at A , or A , let it be revolved around the circumference of the Circle; in which revolution, it will describe the Cone ACD , or ACD .

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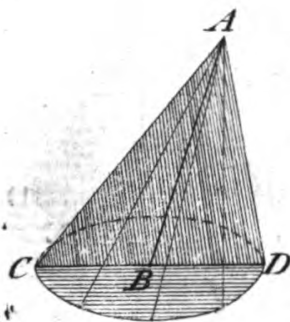
DEF. X. A RIGHT CONE is when the Axe (AB ,) is perpendicular to the Plane of its Base.



A Right Cone may be conceived to be generated by the revolution of a right angled Triangle, ABC , on either Side containing the Right Angle, as AB , which remains fixed; and is therefore called its Axe or Axis, on which it revolves.

The other Side, BC , describes the Base, in its revolution; whilst the Hypothenuse, AC , describes the Surface of the Cone, and prescribes its Bounds.

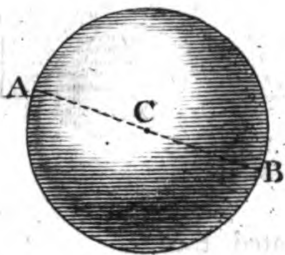
Every Section, by a Plane, through the Axe of a Right Cone, is an Isosceles Triangle, as CAD ; CD is its Base; and a Right Line, AC , or AE , from the Vertex to any Point in the Circumference of its Base, is a Side of the Cone.



DEF. XI. A SCALENE or OBLIQUE CONE is when the Axe, AB , inclines to the Base, CD .

Every Section of an oblique Cone, through its Axe, is a Scalene Triangle; except in one position, when the Axe is at right angles with the Line of Section with the Base; and is, therefore, called a Scalene Cone.

When the Diameter of its Base, CD , is equal to the shortest Side AD ; the Section, CAD , through that Side, perpendicular to the Base, is an Isosceles Triangle, and AC , the longest Side, is its Base; nevertheless CAD is called a Scalene Cone.



DEF. XII. A SPHERE is a SOLID, bounded by one uniform convex Surface, which has no Bounds.

As the Circle is bounded by one Line, its Circumference, so the Sphere is bounded by one Surface.

DEF. XIII. The CENTER of a SPHERE is the middle Point of the Solid; which is equally distant, every way, from its Surface. As C .

DEF. XIV.

DEF. XIV. The DIAMETER, of a SPHERE, is any Right Line passing through its Center, and terminated by its Surface. As AB .

N. B. A Sphere may be conceived to be generated, by the revolution of a Semicircle; as ADB , on its Diameter AB , which remains fixed whilst the Semicircumference revolves; and, in its revolution, describes, or rather prescribes, the Bounds of the Sphere.

The Diameter, AB , being at rest, on which if the Sphere revolved, it would appear at rest, is therefore called an AXE , or Axis of the Sphere.

DEF. XV. A SEGMENT, of a SPHERE, is any portion, cut off by a Plane. As ACB .

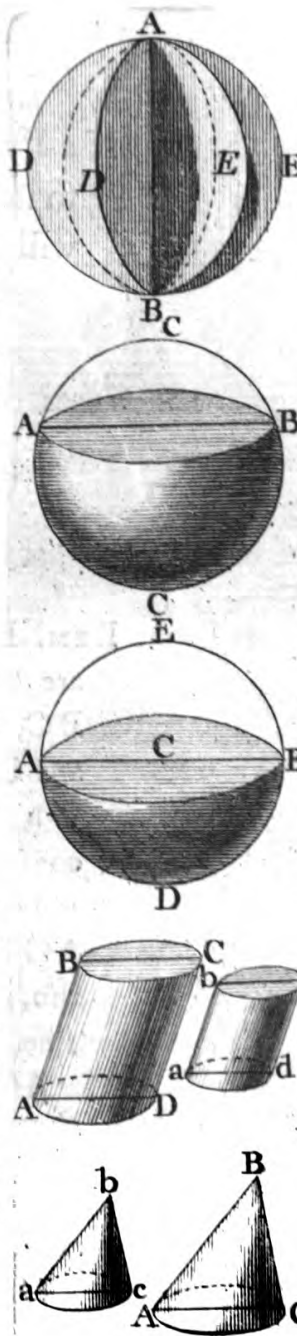
Its Base, AB , is a Circle, made by the Section.

DEF. XVI. A HEMISPHERE is a Segment of a Sphere. When the Plane of the section passes through its Center, each Segment is a Hemisphere. As ADB , and AEB , made by the Plane AB .

DEF. XVII. SIMILAR CYLINDERS, and CONES, are such as have their Axes, and the Diameters of their Bases Proportionals.

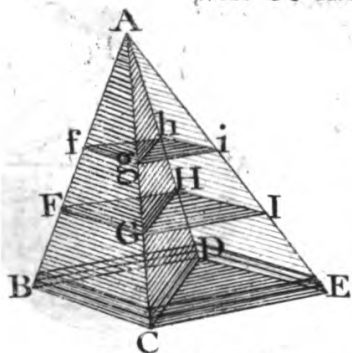
If they are scalene or oblique; Sections through the Axes of Cylinders, perpendicular to their Bases, or equally inclined to their Bases, are similar Parallelograms; as $ABCD$, $abcd$; and, in Cones, they are similar Triangles; as ABC , abc .

All Spheres are similar to one another.



THEOREM I.

If a Pyramid be cut in two parts, by a Plane parallel to its Base; the lesser Pyramid, made by that Section, will be similar to the whole Pyramid.



First; let A be the Vertex, and, BCD the Base of a triangular Pyramid; ABCD.

Let FGH be a Section made by a Plane, parallel to BCD.

I say, the Pyramid, AFGH, is similar to the Pyramid ABCD.

DEM. Because the Planes BCD, FGH are parallel, and they are both cut by the Plane BAC, the Sections, FG and BC, are parallel; for the same reason, the Sections GH and CD, FH and BD are parallel. - - Th. 8.7. Wh. $AF:FB::AG:GC$, and as $AH:HD$ - - 2. 6. conf. $AF:AB::AG:AC$, &c. (6. 5.) i.e. as $FG:BC$, &c. 4. 6. and consequently, the Triangle AFG is similar to ABC, AGH to ACD, and AFH to ABD; also, the Base, FGH, to the Base, BCD; Therefore, the Pyramid AFGH is similar to ABCD.

After the same manner, the Pyramid Afgh may be proved similar to either of the other.

Again, Let CBDE be the Base of a quadrangular Pyramid, and, let GFHI be a Section, parallel to the Base.

It may be demonstrated, in the same manner, that each Triangle AFG, AGI, &c. is similar to ABC, ACE, &c. also, FGH is similar to BCD, and GHI to CDE; consequently, GFHI is similar to CBDE. - 13. 6.

Wherefore, since a Pyramid, whose Base is a Polygon of any number of Sides, may be divided, by diagonal Sections, into triangular Pyramids, the same Demonstration will hold true in every Pyramid whatever.

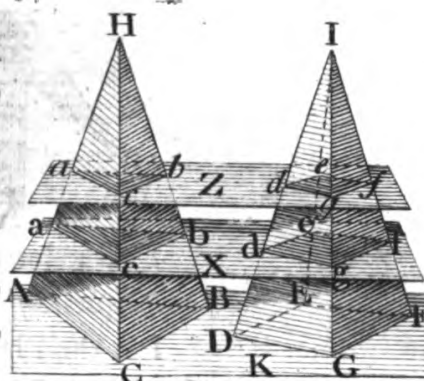
THEOREM II,

Pyramids, having equal Bases and equal Altitudes, are equal. (See the last Figure.)

First; let the Pyramids $ABCD$ and $CADE$ have triangular Bases, BCD and CDE , and a common Vertex, A ; consequently, they have equal Altitude,

I say, they are equal Pyramids.

Let there be made several Sections, $GFHI$, $gfhi$, parallel to the Bases, BCD , CDE , which are in one Plane.



DEM. Then, because they are both cut by a Plane, parallel to their Bases, and the Base $BCD = CDE$; $FGH = GHI$. For, FGH is similar to BCD , and GHI to CDE ; - Th. 1. and each has that Ratio to the other, respectively, which is duplicate of AC to AG ; i. e. of CD to GH - 12. 6.

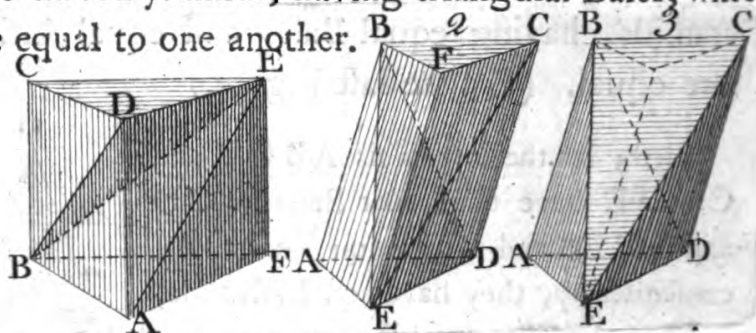
After the same manner, the Section $fg h$, may be proved equal to $gh i$; and conf. every Section, made by Planes par. to $CBDE$, will be equal, each to the other, respectively. Therefore, the Prism $BACD$ is equal to $CADE$.

For, the Sections being equal, every where; conf. if each Pyramid be conceived to be composed of an infinite number of Triangular Prisms, whose height (or thickness) are all equal, and the least that can be conceived, laid on one another, as at BCD , CDE , being each eq. to the contiguous one, the whole is eq. to the whole.

Again. Suppose the Bases of the Pyramids to be different in Figure, as ABC and $DEFG$; their Vertices, H and I , being equally distant from the Plane of their Bases (AKF) consequently, every Section, abc , or abc , and $defg$ or $defg$ (made by the Planes X and Z , parallel to AKF) being in the same Ratio, each to the other, respectively, as the Bases, are equal. Therefore, the Pyramid $AHBC$ is equal to $DEIFG$.

THEOREM III.

Every Prism, having a triangular Base, may be divided, into three Pyramids, having triangular Bases, which are equal to one another.



First; let ACF be a Prism, whose Base ABF, and consequently CED, are right angled Isosceles Triangles.

Let the Planes, ABCD and ADEF, be Squares, and BAF, CDE, Right Angles; conf. BCEF is a Rectangle, under the Side of a Square and its Diagonal (BC and CE).

Draw, AE, BE, and BD (Diagonals in each Face) and imagine Planes to pass thro'. AE & BE, BD & BE - Ax. 4. 7.

The Pyramids, ABEF, AEDB, and BCED, made by the Sections of those Planes, are equal and similar.

DEM. The Triangle ADE = AEF, and ADB = BCD - 15. 1

But, ABCD and ADEF are Squares (by Supposition) and, they have a com. Side, AD; wh. they are equal - Ax. B. 2

Also, BAF is a R. Angle (Sup.) conf. CDE is a Rt one; - 5. 7.

wh. the Tri's. AFB, CED, ADB, BCD, ADE, and AEF are all equal, and similar to one another - 8. 1.

But, AEB and BED (the Planes of the Sections) are each common to two Pyramids; and they are equal and similar to each other, and also to CEB, BEF. - 7. 1.

For, AE and BD are each equal to CE and BF; being all Diagonals; also, AB, DE, CB, and EF are Sides of equal Squares, therefore equal; and BE is common to them all. Wherefore, each Pyramid, ABEF, AEDB, and BCED, being contained by an equal number of Planes, equal and similar to each other, respectively, are equal. - Ax. 5. 7.

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For two Angles in each Pyramid, at A, C, D, and F, are contained by two R. Angles each, and half a R. Angle; as AFE, BFE, & AFB.

The remaining two, in each, meeting at B and E, are each contained by two half right Angles, (AEF, BEF, &c.) and an Angle AEB, equal DBE, (each common to two Pyramids) each equal CEB, or EBF; being contained by the Diagonal of a Square, (CE) and the Hypothenufe (BE) of a right angled Triangle, under the Side and Diagonal, BC, and CE.

2nd. Let ABCE be a triangular Prism; i. e. whose Bases, ADE and BCF, are Triangles; suppose them Scalene, and the Sides AB, &c. of the other Faces, not perpendicular to them; the Prism is conf. oblique in every respect. The Prism ABCE, may be divided into three eq. Pyramids.

Draw the Diagonals, BE, BD, & EC; and suppose a Plane to pass thro'. BE & BD, and, another, thro'. BE & EC - Ax. 4.7

The Prism will be divided into three Pyramids; ABDE, BCEF, and BECD.

I say, those Pyramids are equal to one another.

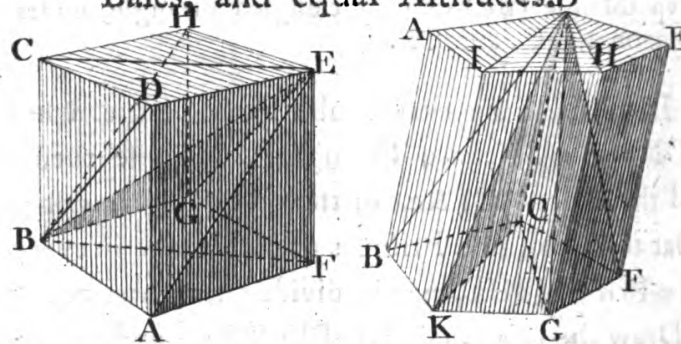
DEM. The Pyramids, ABDE and BCEF, have equal and similar Bases, ADE & BCF, and equal Altitudes; Def. 8.7. Consequently, every Section, made by parallel Planes, at equal distances from their Bases, are equal, and also similar; Therefore, the Pyramids are equal. - - Th. 2.

Again. Suppose the Pyr. BCEF (eq. ABDE) taken away; the Face BCE is common to the Pyramid BECD (see Fig. 3) Then, in the Pyramids ABDE, BECD; because ABCD is a Parallelogram (Def. 7.) it is divided by the Diagonal, BD, into congruous Triangles, ABD and BCD. - 15.1. Let them be considered as the Bases of the two Pyramids; E being their com. Vertex, they have, conf. equal Altitude; wh. every Section, parallel to ABCD, will be equal, and also similar; and consequently, the Pyramids are equal - Th. 2.

If ABDE be supposed removed (Fig. 2.) then the Pyramids BCEF, & BECD are equal; for their Bases CEF, & CED are equal (15.1.) and B is their common Vertex; Therefore, they are all equal to one another. Q. E. D.

THEOREM IV.

Every Prism is triple of a Pyramid, having equal Bases, and equal Altitudes.



First. Let the Prism ACHF be a Cube.

Then, a Plane drawn through the opposite Diagonals CE and BF, will cut the Cube into two equal triangular Prisms; whose Faces FD and AC, &c. are Squares, and the Angles BAF, CDE, CHE, and BGF, are R. Angles.

Draw, AE, EB, and DB; dividing the hither Prism, ABCE, into three equal and similar Pyramids ABEF, AEDB and BCED, (as in Case 1st of the 3rd.)

Also, draw EG. BEFG is a Pyramid, equal and similar to ABEF; and the Prism ACHF is triple of the Pyramid ABEFG.

DEM. For, the Base BFG is equal and similar to ABF - 15.1. and the Face BEG is congruous with AEB; - Ax. 7.1. (for EG bisects the Square FH, and AE bisects FD) But, BEF is common; and BEG, AEB are congruous - 7.1. (for, BG is equal to AB, EG = AE, and BE is common) wh. the Solid, BEFG, is contained within four triangular Planes, equal and similar to those of AEBF; and consequently, they are equal Pyramids. - Ax. 5. 7.

But, the triangular Pyramid, ABEF, is equal one third of the Prism, ABCE; and BEFG is equal one third of BHF, equal CAE. - - - Cor. 1. 3,

Therefore, the quadrangular Pyramid, $ABEFG$, is equal one third of the Cube ACE ; or, the Cube triple of the Pyramid, on the same Base, $ABGF$, and having the same Altitude, EF .

2nd. After the same manner, as in Case II. of the last, every triangular Prism may be proved triple of a Pyramid, having the same, or an equal Base and Altitude.

Wherefore, since every Prism, having a quadrangular Base, may be divided into two triangular Prisms; and each triangular Prism, into three equal Pyramids having triangular Bases; conf. a Pyramid on the whole quadrangular Base of the Prism, and having equal Altitude, is equal one third of the Prism, or the Prism triple of the Pyramid.

3rd. Let the Base, $AIHED$, of the Prism, $AKFD$, be a Pentagon; and let the Diagonals, CG, CK, DH, DI , be drawn.

Then, because CD is parallel to AB and EF ; and, GH is parallel to EF , and IK to AB - - - Def. 33.

DC is parallel to IK and GH . - - - Th. 4. 7.

wh. Planes may pass through $DC \& GH, DC \& IK$ - Ax. 2.

and conf. the Prism, $AKFD$, will be divided into three triangular Prisms, by the Planes, $DCKI$ and $DCGH$; viz. AKD, KDG , and DGE .

Draw the Diagonals BD, DF, DG , and DK ; $BDFGK$ is a Pyramid having the same Base, $BCFGK$, and Altitude as the Prism.

But, $BDCK$ is a Pyramid, having the same Base, BCK , and Altitude, as the Prism AKD ;

AKD is therefore triple of the Pyramid, $BDCK$.

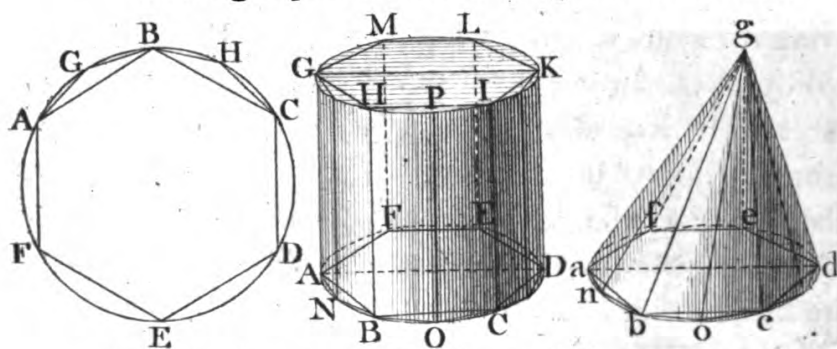
For, the same reason, $KIDHG$ is triple of the Pyr. $KDGC$; and the Prism DGE is triple of the Pyramid $CDFG$.

But, the Pyramid $BDFGK$ is equal to the three Pyramids $BDCK, KDGC$, and $CDFG$; and the Prism $AKFD$ is equal to the three Prisms $AKD, KIDHG$, and DGE .

Th. the Prism $AKFD$ is triple the Pyr. $BDFGK$. Q.E.D.

THEOREM V.

Every Cone is equal to the third part of a Cylinder, having equal Bases and equal Altitudes.



Let the Circle ACE be considered as the Base of a Cylinder or Cone; and, let a regular Polygon, ABCDEF, of any number of Sides be inscribed; it is manifest, that the Polygon is less than the Circle.

Let the Arks, AGB, &c. be bisected, as at G; and draw AG and BG, &c. This Polygon will be greater than the first, but it is also less than the Circle.

Wherefore, it is evident, that, the more the Sides of the Polygon are multiplied, the nearer it approaches to the Circle, till the difference is less than any assignable Quantity whatever. The difference is the Segments AG, GB, &c.

And, by the same reasoning, every Polygon, circumscribed, will be greater than the Circle; consequently, they will at last end in the Circle; i. e. the Perimeter of the Polygon, in the Circumference of the Circle, and their Areas will be equal, or the difference will be less than any other Quantity.

Let AGKD be a Cylinder whose Base is AD; and agd a Cone, whose Base, ad, is equal to the Base AD, of the Cylinder; and let them also have equal Altitudes.

I say, the Cylinder AGKD, is triple of the Cone, agd.

DEM.

DEM. Let there be regular Poligons, of any number of Sides, inscribed; as $ABCDEF$, and $abcdef$, whose Sides $AB, BC, \&c.$ $ab, bc, \&c.$ are equal; consequently, they are equal Poligons.

Let a Prism be constructed on the Poligon, $ABCDEF$, whose Sides $AG, BH, CI, \&c.$ are in the Surface of the Cylinder; and from every Angle of the Poligon, $a, b, c, \&c.$ let there be drawn Right Lines $ag, bg, \&c.$

Then, $abgdc$ is a Pyramid, whose Base is equal to the Base of the Prism (by Construction); and, having equal Altitudes, the Prism, $AMKC$, inscribed in the Cylinder, is triple of the Pyramid, inscribed in the Cone. - Th. 4.

Again. Let the Arks $AB, BC, \&c.$ be bisected, at N and O ; and let a Prism be inscribed, having twice the number of Sides.

Also, let the Pyramid, inscribed in the Cone, have the same number of Sides, as the Prism; their Bases are equal; and, consequently, the Prism is triple of the Pyramid - 4.

And so it must always be; till, by multiplying the Sides of the Prism, its Surface will be the same as the Surface of the Cylinder, and the Surface of the Pyramid as the Cone; the Bases of the Prism and Pyramid will be the same as the Bases of the Cylinder and Cone.

But, they are equal, and they have equal Altitudes.

Therefore, the Cylinder is triple of the Cone; or, the Cone equal to one third of the Cylinder. Q. E. D.

THEOREM VI.

Prisms, or Pyramids, having equal Altitudes, have that Ratio to each other, as their Bases.

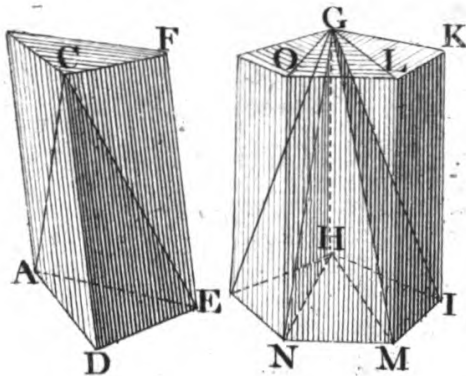
First. Let the Prisms have quadrangular Bases.

If they are Parallelopipeds, it is already proved - Cor. 1. 17. 7.

For, every Parallelopiped (is a Prism; Def. 7. 7.) may be divided, by a Plane passing through its opposite Diagonals, into two equal and similar triangular Prisms; consequently their Bases and Altitudes are equal. - - - - - Th. 16. 7.

And it is manifest, that if the Base of one be double, or triple, of the other, seeing they have equal Altitudes, they are as their Bases; i. e. the Prism, whose Base is double or triple of the others Base, is double or triple of the other Prism; and consequently, whatever ratio one Base has to the other, the Prisms have necessarily the same. (For, whether the Bases be similar Figures or not, being equal, it is the same; as it is proved, in Case the 3rd of the 17th, being Parallelopipeds.)

Also, if their Bases be equal, they are as their Altitudes - Cor. 2. 17. and, if the Prisms are equal, their Bases and Altitudes reciprocate, (Th. 19.) Consequently, every triangular Prism is to another, of equal Altitude, in the same ratio as their Bases; seeing, they are each half a Parallelopiped or Prism, whose Base is double the Base of the triangular Prism, and having equal Altitudes.



Let the Prisms AFD and EGKM have equal Altitudes; the one on a triangular Base, the other pentagonal.

Let the Diagonals GL, GO, HM, and HN be drawn, between opposite Angles in the equal Pentagons; and imagine Planes OGHN, and GLMH, drawn

through those Diagonals, dividing the Prism, EGKM, into three triangular Prisms, EGN, NGM, and MGI.

DEM. Then; because the triangular Prisms, AFD and EGN, have equal Altitudes, they have that Ratio to each other, as their Bases ADE to EHN. - proved above.

i. e. the Prism $AFD:EGN::ADE:EHN$ }
 And, the Prism $AFD:NGM::ADE:NHM$ } their Bases.
 also, the Prism $AFD:MGI::ADE:MHI$ }
 Wh. as the Base $ADE:EHN + NHM + MHI$ }
 so is the Prism $AFD:EGN + NGM + MGI$ } - 2. 5.
 i.e. as the Base $ADE:EHIMN::$ the Prism $AFD:EGKM$.

2nd. Because the Prism AFD is triple the Pyramid $ACED$;
 and the Prism $EGKM$ is triple of the Pyramid $NEGIM$ -4.
 Also, as Base $ADE:EHIMN,::$ the Prism $AFD:EGKM$
 conf. as Base $ADE:EHIMN,::$ the Pyr. $ACED:NEGIM$

COR. 1. Cylinders, having equal Altitudes, have the same Ratio to each other as their Bases.

For, similar Poligons, inscribed in Circles, are, to each other, as the Squares of their Diameters. - - - Th. 14. 6.

And it has been proved, that all Prisms, having equal Altitudes, are in the Ratio of their Bases, to each other; consequently, Cylinders are to each other in the same Ratio.

Because, every Cylinder, whether it be right or oblique, is equal to a Prism, whose Base is the same, or equal to the Base of the Cylinder; i. e. having its Sides multiplied continually till it ends in the Surface of the Cylinder; and its Base, in the Base of the Cylinder.

COR. 2. Cones, having equal Altitudes, are to each other, in the Ratio of their Bases.

Because, a Cone is equal to one third of a Cylinder, having equal Bases and Altitudes; and Cylinders are in that Ratio to each other; therefore Cones have the same Ratio as Cylinders.

COR. 3. Prisms and Pyramids, having equal Bases, are to each other as their Altitudes.

Because Parallelopipeds are Prisms; and it has been proved, that Parallelopipeds are to each other in that Ratio; and consequently, all Prisms, having equal Bases, are as their Altitudes.

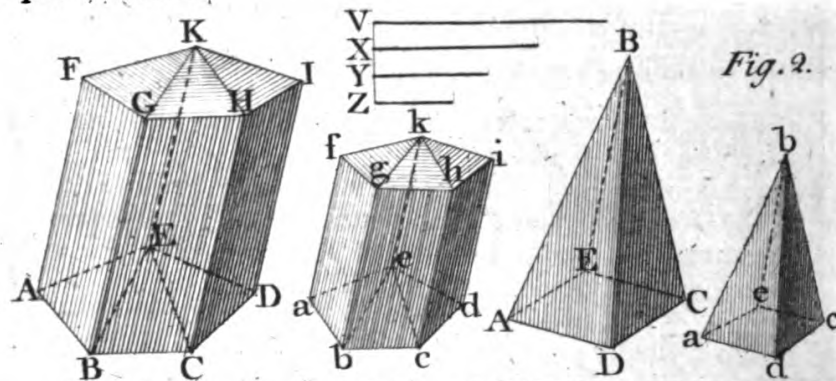
COR. 4. Cylinders and Cones, having equal Bases, are to each other as their Altitudes.

Because, there is the same Ratio between Cylinders and Cones, as between Prisms and Pyramids, having equal Bases. Therefore, &c.

THEOREM VII.

Similar Prisms and Pyramids are, to one another, in the triplicate Ratio of their corresponding Sides. *

In respect of Prisms (being also Parallelopipeds) this property is already proved (in Theo. 18th of the 7th) and consequently, in Prisms having triangular Bases; seeing, every such Prism is half a Parallelopiped, whose Base is double of the Prism and having equal Altitudes.



Let the Pentagonal Prisms AKD , akd be similar.

Take any Right Line, V ; and as $a b$ is to $A B$, &c. make X to V ; and as X is to V , so make Y to X ; also, as Y is to X , or X to V , make Z to Y .

I say, that, as V is to Z , so is the Prism AKD to akd .

Let them be divided by diagonal Planes, BK , KC , bk , kc , passing through the opposite Diagonals; dividing the Prisms into three triangular Prisms, each.

* I cannot conceive the reason, why Euclid confines this proportion, and the following, to Pyramids only; and, to such, only, as have triangular Bases; seeing that, it holds equally true in Prisms; and also in Pyramids, having polygonal Bases, of any kind.

DEM.

DEM. The Poligons ABCDE, abcde, are similar - Hyp. wh. the Triangles ABE, a b e, &c. are similar. - 13. 6. and conf. the Prisms AKB, akb, &c. are similar-Ax. 11. 7 (For, as $AB:ab$, or as $BC:bc$, $::BG:bg$, and $GK:gk$; consequently, the diagonal Planes, BK and bk, KC and kc, are similar)

Wherefore, since each triangular Prism is equal to half a Parallelopiped, whose Base is double of the triangular Prism, and Altitudes equal (above) they have that ratio to each other, which is triplicate of their corresponding Sides, i. e. as V to Z. (Ax. 8. 5.) - - - Th. 18. 7. But the Prism $AKB:akb::BKC:bkc$, and as $CKD:ckd$. wh. as $AKB:akb::AKB+BKC+CKD:akb+bkc+ckd$.* i. e. as the Prism AKB : akb :: the Prism AKD : akd. But, $AKB:akb::V:Z$, Th. the Prism AKD : akd :: V : Z.†

2nd. Let the Pyramids ABCD, abcd be similar. (Fig. 2.)

The Pyramids ABCD, abcd, being similar, are each equal to the third part of a Prism on the same Base and Altitude.

But, Quantities are in the same Ratio, to each other, as their Equimultiples or equal Parts. - - Ax. 8. 5. and the triples of the Pyramids ABCD, abcd, are as V to Z. therefore, the Pyramid ABCD : abcd :: V : Z. Q. E. D.

COR. Similar Cylinders are, to one another, in the triplicate Ratio of their Diameters or Sides.

For, they are in the same Ratio, to each other, as similar Prisms inscribed; or, as Prisms, whose Bases are equal to the Bases of the Cylinders, and having equal Altitudes.

COR. 2. Similar Cones are, to each other, in the triplicate Ratio of the Diameters of their Bases, or of their Axes.

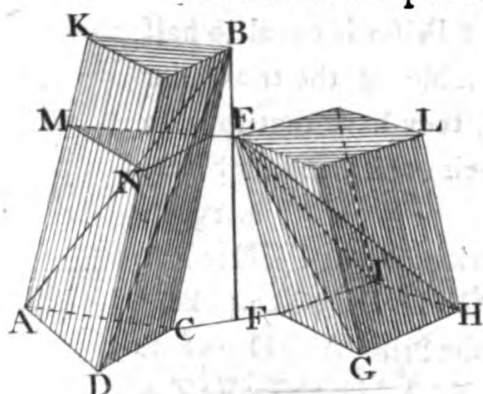
Because a Cone is equal to the third part of a Cylinder, on equal Bases and Altitudes.

* By Theo. 2nd of the 5th.

† By Axiom 13 of the 15th.

THEOREM VIII.

Equal Prisms, or Pyramids, have their Bases and Altitudes reciprocally proportional.



If the Prisms are Parallelopipeds it is already proved. - - 19.7.

Let the triangular Prism AKC be equal to the quadrangular Prism FLG; whose Bases, ADC, FGHI, are in the same Plane; to which, Draw the Perpendicular BF. BF and EF are their Altitudes.

I say, the Base, ADC, is to the Base, FGHI, as EF is to BF.

Let the Plane of the Top, LE, be produced, cutting the triangular Prism, at EMN.

DEM. The Prism AMC:AKC::EF:BF, their Altitudes.
And - - - AMC:FLG::ADC:FGHI, their Bases.
But, the Prism FLG is equal to AKC; by Hypothesis, consequently, AMC:FLG::EF:BF, i.e. as ADC:FGHI
Therefore, as ADC:FGHI::EF:BF. Q. E. D.

2nd. Draw the Diagonals AB, DB; EG, EH, and EL.

The Pyramid ABCD is equal to one third of AKC -4.
And, the Pyramid FGEHI = one third part of FLG.
But, Quantities are in the same Ratio as their Equimultiples; wherefore, the the Pyramid ABCD is equal to FGEHI.

But, they have the same Bases and Altitudes, as the Prisms, AKC and FLG.

Th. their Bases and Altitudes are reciprocally proportional.

COR.

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COR. Equal Cylinders, or Cones, have their Bases and Altitudes reciprocally Proportional.

Because, Cylinders are equal to Prisms whose Bases and Altitudes are equal; and Cones to Pyramids.

In these eight Theorems, and their Corollaries, with the Corollary to the next, is contained the whole of Euclid's 12th Book; those which follow, are select Theorems from Archimedes; which shews how the surface of the Sphere may be ascertained: also, of any Segment, or Portion of its Surface, intercepted between parallel Circles, with great accuracy.

The next Theorem, which is not from Archimedes, shews, in a brief and elegant manner, the Ratio between the Sphere and a circumscribing Cylinder; and consequently, if the quantity of the Cylinder be known, or attainable, the quantity of the Sphere may also be obtained. According to Archimedes, it is determined by the Cone, in the 17th of this Book; which, though perfectly true, is not so brief; and though perhaps more geometrical, yet not more convictive and satisfactory.

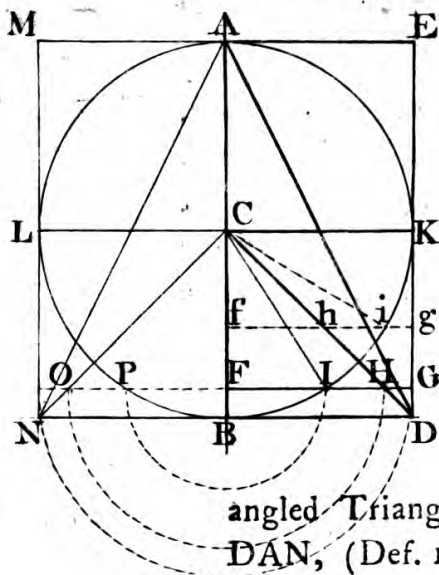
In the following is contained all that is really useful or valuable, in respect of the Cylinder, Cone, and Sphere, nor is it possible, I think, to obtain the Area of the Surface of a Sphere in a more concise manner.

In short, without the following Theory, I cannot but think the Doctrine of Solids imperfect; seeing that, the Quantity of a Sphere, the most perfect Solid, cannot, by Euclid, be obtained, in respect either of its Superficies or solid Contents; for want of which, Artificers are frequently at a loss, in measuring a Dome or other spherical Object.

THEOREM IX.

Every Sphere is equal to two thirds of a circumscribing Cylinder; i. e. of a Cylinder whose Diameter and Altitude are equal, each to the Diameter of the Sphere.

Let AB be the Diameter of a Sphere; bisect AB, at C.



On C, describe the Semicircle AKB; and, on AB, construct the Rectangle AEDB, circumscribing it, and draw AD.

Then (AB remaining fixed) imagine them all revolved about AB, as an Axis; the Semicircle, AKB will describe a Sphere, AKBL, (Def. 14.)

The Rectangle, AEDB, will describe a Right Cylinder, DEMN, containing the Sphere (Def. 4.) and, the right-angled Triangle, ADB, will generate a Right Cone, DAN, (Def. 10) whose Base, DN, is the same as the Cylinder, and equal to the Diameter of the Sphere.

The Cone, DAN, is equal to one third part of the Cylinder. - - - - - Th. 5.

I say, that the Sphere, AKBL, is equal to the remaining two thirds.

Draw CD; and FG (at pleasure) parallel to CK; cutting CD at H, and the Circumference of the Sphere at I; and draw CI.

DEM. In the Rt. angled Tri. CFI, $CI^2 = CF^2 + FI^2$ - 20.1
and, the Areas of Circles are as the Squares of their
Diameters. - - - - - C. 1. 14. 6.
wherefore, a Circle, described on CI, is equal to two
Circles, on CF and FI.

Now, in the revolution of the Square CKDB, there will be described the Cylinder KDNL; the Ark BIK will describe a Hemisphere; and, CD will describe the Surface of a Right Cone, DCN; also, the Points G, H, and I, will describe Circles, with the several Radii FG, FH, and FI; FG is equal to CI (equal CK) and FH is equal to CF. - - - 9. 1.
(for, the Angle CFH is Right, and the Angles FCH, FHC, are half Right; seeing that, the Diagonal, CD, bisects the Angles BCK, KDB)

Wherefore, the Circle described by FH (equal CF) is equal to the difference between the Circles described by FG (equal CI) and FI.

Consequently, the Circle, described by FH, is equal to the Annulus or Ring, described by GI;

and, consequently, every Section (FG or fg) of the Cone DCN, by a Plane parallel to its Base, is equal to a Section of the concave Solid, remaining when the Sphere is supposed to be taken out of the Cylinder.

Wherefore, the Cone, DCN, is equal to the Solid described by the mixed Triangle KIBD.

But, the Cone, DCN = one third of the Cylinder KDNL, conf. the concave Solid (taking away the Hemisphere, KBL, out of the Cylinder, KDNL) is equal to one third of that Cylinder; and consequently, the Hemisphere, KBL, is equal to two thirds of the Cylinder.

But, the Cone, DCN, is half the Cone DAN; and, the Cylinder, DKLN, is half the Cylinder DEMN; C. 4. 6. also, the Sphere, AKBL, is double the Hemisphere, KBL. Therefore, the Sphere AKBL, whose Diameter, AB or KL, is equal to the Diameter and Altitude of the Cylinder, DEMN, is equal to two thirds of the Cylinder. Q. E. D.

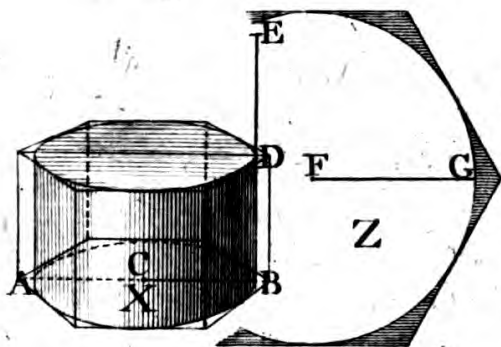
COROLLARY. The 18th Proposition, of Euclid.

Spheres have that Proportion to one another, which is triplicate of their Diameters;

For, all Cylinders, being similar, are in that Ratio to one another; and consequently, Spheres, inscribed in Right Cylinders whose Diameters and Altitudes are equal, respectively, to the Diameter of the Spheres, being two thirds of such Cylinders, are in the same Ratio to each other as the Cylinders; i. e. in the triplicate Ratio of their Diameters.

THEOREM X.

A Circle, whose Radius is a mean Proportional between the Diameter of the Base, and the Side of a Right Cylinder, is equal to the cylindrical Surface.



Let AB be the Diameter of the Base of the Cylinder AD, and let BD be its Side.

Take FG a mean Proportional, between AB and BD, (Prob. 30) and on FG, describe a Circle; let C be the Center of the Base of the Cylinder; produce BD, and make DE equal to BD.

Then, CB, FG, & BE are Proportionals; for, $CB : AB :: BD : BE$, - - - Ax. 8. 5. wherefore, $CB \times BE = AB \times BD$ - - - 9. 6. i. e. to FG^2 ; for $AB \times BD = FG^2$. - Cor. to 9. 6.

Let similar Polygons (X and Z) be circumscribed about the Base of the Cylinder, and the Circle whose Radius is FG; and let a Prism, be supposed to circumscribe the Cylinder.

DEM. A Triangle, whose Base is equal to the Perimeter of the Polygon, and its Altitude equal CB, is equal to the Polygon;* and a Rectangle under the Perimeter and BD, a Side of the Prism, is equal to the Superfices of the Prism-1.2. Also, a Triangle under the Perimeter and BE is equal to a Rectangle under the Perimeter and BD, half BE.

* This follows from the 18th of the first. For, all the Sides of the Polygon are equal; and all Triangles having equal Bases and Altitudes are equal. (See Art. 7. P. 7. in the Appendix).

But, a Triangle, under the Perimeter and CB , is to a Triangle under the same Perimeter & BE , as CB is to BE -1.6
 conf. the Polygon is to the Surface of the Prism, as $CB:BE$.
 But, CB to BE is duplicate of CB to FG - Def.12.5.
 wherefore, the Polygon X is to the Superficies of the Prism,
 in the Duplicate ratio of CB to FG .

But, the Polygon X is to Z in the Duplicate ratio of their
 Sides, or Perpendiculars, CB to FG , i. e. as CB to BE .
 wh. the Polygon X has the same Ratio to Z , as X has to the
 Superficies of the Prism, AD ; conf. they are equal-Ax.5.5

After the same manner, it may be proved that the Super-
 ficies of any other Prism, whose Faces being multiplied
 infinitely, till they end in the Surface of the Cylinder, AD ,
 is equal to the Polygon circumscribing the Circle, Z , whose
 Sides being multiplied, end at last in the Circle.

Therefore, the Circle, Z , whose Radius is a Mean, be-
 tween AB & BD , is equal to the Surface of the Cylinder, AD .

COR. 1. The Superficies of a Right Cylinder is to its Base,
 as the Side of the Cylinder to the fourth part of the
 Diameter of the Base.

For the Superficies of the Cylinder is equal to the Circle,
 whose Radius is FG ; and that Circle, is to the Base of the
 Cylinder in the duplicate ratio of FG to CB ; i. e. of BE
 to CB ; i. e. of BD to half CB .

COR. 2. The Superficies of a Cylinder whose Side is equal to
 the Diameter of its Base, is equal to four times the Area of
 the Base; conf. if the Side be a fourth part of the Dia-
 meter the cylindrical Superficies, and the Base are equal.

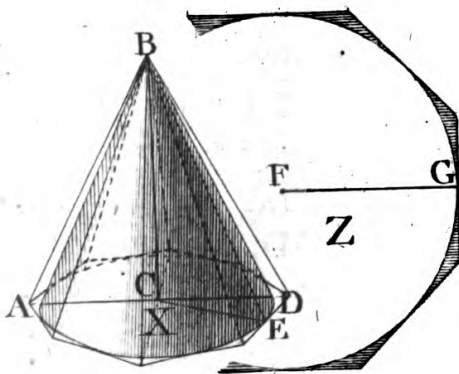
COR. 3. The Superficies of a Right Cylinder is equal to
 a Rectangle, under its Circumference and Side.

COR. 4. Cylindrical Surfaces, of equal height, are to each
 other as the Diameters of their Bases; and, having equal
 Bases, they are as their Altitudes.

COR. 5. The Surfaces of similar Cylinders have that Ratio
 between them; which is duplicate of their Diameters.

THEOREM XI.

A Circle, whose Radius is a mean Proportional between the Side of a Right Cone and the Radius of the Base, is equal to the Superficies of the Cone.



Let ABD be a Right Cone, the Diameter of whose Base is AD, and C its Center.

Take FG, a mean Proportional between the Side, BD, and the Radius CD (equal BE and CE) and on FG describe a Circle.

Let regular Poligons (X and Z) be circumscribed about the Base of the Cone and the Circle, whose Radius is FG, of the same kind; and suppose a Pyramid, on the Polygon X circumscribing the Circle.

DEM. Because $CE : FG :: FG : BE$; by Hypothesis.

CE to BE is duplicate of CE to FG. - Def. 12. 5.

But, as CE is to BE, so is the Triangle, whose Altitude is CE, and its Base, the Perimeter of the Polygon X, to the Triangle under BE and the same Perimeter. - Th. 1. 6.

Also, as CE is to BE so is the Triangle under CE and the Perimeter of the Polygon X, to a Triangle, under FG and the Perimeter of Z. - Th. 6. and 12. 6.

But, a Triangle, under CE and the Perimeter of X, is equal to the Polygon X; and, the Triangle under FG and the Perimeter of Z is equal to that Polygon; also, the Triangle under BE and the Perimeter of X, is equal to the Superficies of the Pyramid.*

* This also follows from the 18th of the first Book of Elements, and Ax. 1. 1. (See the Reference to the last).

Wherefore,

Wherefore, the Poligon X has the same Ratio to the Poligon Z, and to the Superficies of the Pyramid; consequently, the Poligon Z is equal to the Superficies of the Pyramid. - - - - - Ax. 5. 5.

After the same manner it may be proved, that the Superficies of any other Pyramid (whose Base is a regular Poligon, similar, to one circumscribing the Circle, whose Radius is FG, till by multiplying the Sides, equally, the Perimeters, of both, fall into the Circumferences, of the Circles, and the Superficies of the Pyramid into the Cone) is equal to the Poligon Z; consequently, the Superficies of the Cone is equal to the Circle whose Radius is FG, a mean Proportional between the Side BD & the Radius CD.

COR. 1. The Superficies of a Right Cone is to its Base, as the Side of the Cone to the Radius of the Base.

For, they are, respectively, equal to Triangles, whose Bases are equal to the Circumference of the Base, and Altitudes equal to BD and CD respectively; consequently, they are to each other, as BE to CE. - - - - - I. 6.

COR. 2. The Superficies of a Right Cone is equal to a Triangle, whose Base is equal to the Circumference of the Base of the Cone, and its Altitude equal to a Side of the Cone.

Hence it is manifest, that the Surfaces of Right Cones have all the same Properties as Triangles.

If their Sides be equal, they are, to each other, as the Diameters of their Bases; and having equal Bases, they are as their Sides; or as their Axes.

If they are similar, the Ratio between their Surfaces is duplicate of their Sides, or Diameters of their Bases.

If they are equal, their Sides and Diameters are reciprocally Proportional.

COR. 3.

COR. 3. The Superficies of a Right Cylinder, is to a Right Cone, having equal Bases and Altitudes, as the Side of the Cylinder, to half the Side of the Cone.

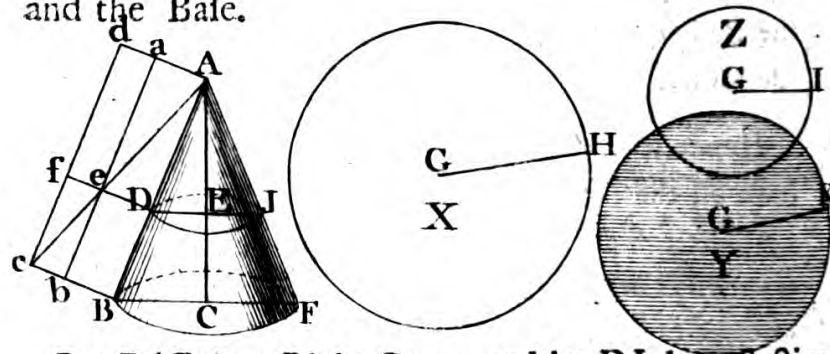
For, the Superficies of the Cone is to its Base, as the Side to the Radius; or, as half the Side to half the Radius. And the Superficies of the Cylinder, to its Base, is as the Side to half the Radius; therefore, the Surface of the Cone is to the Surface of the Cylinder, as the Side of the Cylinder to half the Side of the Cone.

COR. 4. An Equilateral Cone, i. e. whose Side is equal to the Diameter of its Base, has its Superficies double of its Base; and if the Vertex be right angled, the Ratio is as the Diagonal of a Square to its Side.



THEOREM XII.

If a Right Cone be cut by a Plane, parallel to its Base; a Circle whose Radius is a mean Proportional, between part of the Side (betwixt the Base and the Section) and the Radius of that Section added to the Radius of the Base, is equal to the conical Superficies, which is between the Section and the Base.



Let BAF be a Right Cone; and let DJ be a Section, parallel to the Base, BF.

Take GH a mean Proportional, between AB, and BC; and GI a Mean between AD and DE; on which, describe Circles, X and Z.

The Circle X, on GH, is equal to the whole Superficies;
and, that on GI = to the Surface of the Cone, DAJ - 11.
and, $AB \times BC = AD \times DE, + DB \times BC, + DB \times DE.$ *

Take GK a Mean, between DB and DE + BC; on
which, describe a Circle; let it be called Y.

I say, that Circle (Y) is equal to the conical Super-
fices BDJF; between the parallel Circles.

DEM. Because GH is a Mean, between AB, & BC - by Con.
the Square of GH = $AB \times BC$; i.e. to the Rect. AB cd; }
For the same reason, GI = $AD \times DE$; i.e. to AD ea } 9.6
And, because GK is a Mean, betwixt BD and DE + BC;
GK = $BD \times BC, + BD \times DE$; (equal Bf + Be)
i.e. to a Rectangle under BD and DE + BC.

Consequently, $GH^2 = GI^2 + GK^2$.

(Because, the Rectangle equal to GH Square, is equal to
two Rectangles, which are respectively equal to the two
Squares, of GI and GK)

But, Circles are to each other, as the Squares of their
Diameters, and consequently as their Radii - C.1. 14. 6.
And the Circle X is equal to the whole conical Surface;
also, the Circle Z is equal to the Surface DAJ.

Therefore, the Circle Y (being equal to the Difference
between the two, X and Z) is equal to the Superficies
BDFJ; between the Section, DJ, and the Base.

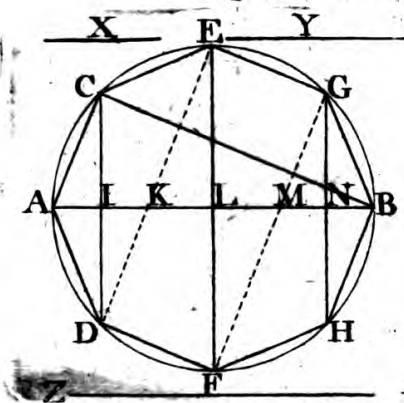
* This is deducible from the second Book of Elements; having
formed a Construction.

Let ABcd be a Rectangle, under AB and BC; Ac being
drawn, and Df parallel to Bc, cutting it at e; and ab (through e)
parallel to AB.

Then, the whole, Bd (equal $AB \times BC$) = Da (equal $AD \times DE$)
+ Bf (equal $BD \times BC$) + ed (equal $BD \times DE$, i.e. Be, 19. 1.)

THEOREM XIII.

If a regular Polygon, having an equal number of Sides, be inscribed in a Circle, and if a Diameter (which is also a Diagonal of the Polygon) be drawn, and another Diagonal, from either extreme of the Diameter to the adjacent Angle; the Rectangle, under the Diameter and that Chord, is equal to a Rectangle under a Side of the Polygon and all the parallel Chords, joining two and two contiguous Angles.



In the Circle AGF, describe a regular Octagon, ACEGBHFD; draw the Diameter AB, and the Diagonal BC; also, draw CD, EF, and GH.

I say, the Rectangle, under AB and BC, is equal to a Rectangle under AC (or AD) and CD, added to EF, added to GH.

Join DE and FG.

DEM. Because $CE = DF$, and $EG = FH$, the Chords CD, EF, and GH, are all parallel - Cor. to 10. 3. And, for the same reason, AC, DE, FG, & HB are parallel; conf. the Triangles ACI, IDK, KEL, &c. are similar. Wh. $AI:IC::IK>ID,::KL:LE,::LM:LF, \&c.$ - Th. 4. 6. i. e. as $AI:IC::AB:CD + EF + GH$, - 2. 5. for, the Diameter, $AB = AI + IK + KL, \&c.$ and $CD + EF + GH = CI + ID + EL, \&c.$ - Ax. 1. 1. But, as $AI:IC::AC:CB$ (7. 6) for ACB is a R. Angle - 12. 3. Consequently, $AC:CB::AB:CD + EF + GH$ - Ax. 13. 5. Therefore, $AC \times CD + EF + GH = CB \times AB$ - Th. 9. 6.

COR. Hence it is manifest, that if in any Segment of a Circle, as GAH, a Polygon be inscribed, on GH, whose Sides are equal, and equal in Number; a Rectangle, under the Chord CB and AN (part of a Diameter, perpendicular to GH) is equal to a Rectangle under a Side AC, and the Chords, $CD + EF + GN$, half the Base.

THEOREM XIV.

If a regular Polygon be inscribed in a Circle, (as in the foregoing Theorem) and Chord Lines be drawn, (as there described); a Circle, whose Radius is a mean Proportional between the Diameter (AB) and the Chord CB, is equal to all the conical Superficies described by the revolution of the Polygon, on the Diameter or Axis, AB.

The same Construction remaining, as in the former Figure.

Find X a mean Proportional between AC and CI; also Y, a Mean, between CE and CI added to EL.

DEM. Now, AC, CE, EG, & GB are equal, by Construction.

And, $AB \times BC = AC \times CD + EF + GH$ (Th. 13.) i. e. to $AC \times CI, + CE \times CI + EL, + EG \times EL + GN, + GB \times GN$

Find Z, a mean Proportional between AB and BC; the Square of Z is equal to the Squares of X and Y, taken twice; i. e. to $AC \times CD, + CE \times EF, + EG \times GH$ - 13.

On X and Y describe Circles; the Circle X is equal to the Surface of the Cone DAC, - - - Th. 11.

and, the Circle Y is equal to the conical Surface ECDF - 12.

But, Circles are in the same Ratio as the Squares of their Diameters, (C. 1. 14. 6.) wherefore, a Circle, described on Z, is equal to X and Y, taken twice.

But, the Cone $DAC = GBH$; and $ECDF = EGHF$; for, CD is equal to GH, and CE to EG, &c.

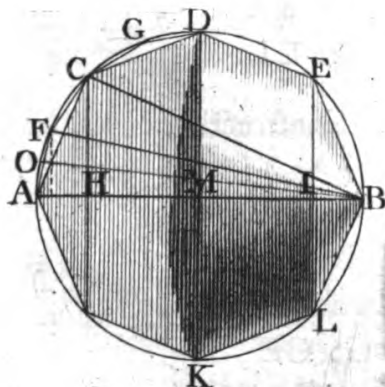
Therefore, a Circle (Z) whose Radius is a mean Proportional between AB and BC, is equal to all the conical Superficies, described by the revolution of ACEGB, on AB.

COR. After the same manner it may be proved, that a Circle, whose Radius is a mean Proportional between CB & AN, is equal to all the conical Superficies described by ACEG on AN, in the Segment GAH.

THEOREM XV.

The Surface of a Sphere is quadruple of the greatest Circle, made by a Section through its Center ; i. e. to a Circle whose Radius is equal to the Diameter of the Sphere.

Let $ADBK$ be a Circle, the Section of a Sphere through its Center.



Let there be inscribed a regular Polygon, having an even number of Sides ; draw the Diameter, AB , and the Diagonal BC , to the adjacent Angle.

Imagine the Circle $ADBK$, with the Polygon, revolved on AB ; the Circle will generate a Sphere (NB. Def. 14.) and the Sides of the Polygon AC , CD , &c. will generate Right conical Surfaces, inscribed in the Sphere.

AC and EB will describe equal Cones (for they are equal, and equally inclined to the Axe AB) ; CD and DE will also describe equal

Surfaces ; for they are equal, and CH is equal to EI .

The whole conical Superficies, is less than the Superficies of the Sphere ; for the Arks, AFC , CGD , &c. are greater than the Chords AC , CD , &c. ; also, a mean Proportional, between AB and BC , is less than the Diameter, AB .

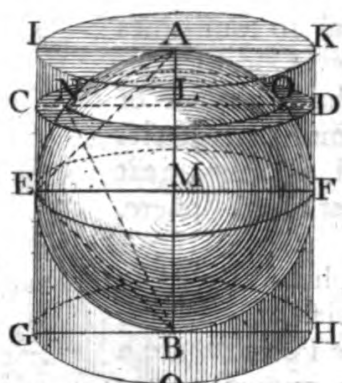
Let the Arks, AC , CD , &c. be bisected, at F and G ; and draw AF , FC , &c.

It is manifest, that the conical Surfaces, described by the revolution of $AFCGD$, will be greater than those described by AC , CD ; but they are less than the Surface of a Hemisphere, described by the Ark ACD ; and, a mean Proportional between AB and BF , is greater than a Mean, between AB and BC , but it is also less than the Diameter AB .

By the same reasoning, i. e. by multiplying the Sides of the inscribed Polygon, the Superficies of Cones, inscribed in the Sphere, will, at last, end in the Surface of the Sphere ; when the Sides, of the Polygon end in the Circumference of the Circle. For, if AF be bisected, at O , the Chords AO , OF deviate but very little from the Arks they subtend ; and a mean Proportional between AB and BO , will differ very little from the Diameter AB ; and consequently, it will at last be the same as the Diameter.

T H E O R E M XVI.

If a Sphere be inscribed in a Cylinder, and being both cut by Planes, parallel to the Bases of the Cylinder; each Segment of the spherical Surface will be equal to the portion of the cylindrical, intercepted between the same parallel Planes.



Let the Sphere, AEBF, be circumscribed by the Cylinder GIKH, and let them be both cut by the Planes CD, EF, parallel to the Bases, GH, IK, of the Cylinder.

I say, the spherical Surface NAO, or ENAOF, is equal to the cylindrical Surface ICDK, or IEFK; also, the spherical Surface NEFO, is equal to the cyl. Surface, CEFD.

Let AB, be the Axe of the Cylinder, and Sphere; and let the Diameters CD, EF, of the Cylinder, be drawn, cutting the Axe perpendicularly, at L and M, and the Surface of the Sphere at N and O, E and F. Draw AN, NB, and AE, EB.

DEM. Then because ANB is a Right Angle (12. 3.) AN is a mean Proportional between AL and AB; i. e. between CI and CD. - - - - - C. 2. 7. 6.

Also, because AEB is a Right Angle, AE is a Mean, between AM and AB, i. e. between EI and EF. - same.

But, the Circle, whose Radius (AN) is a mean Proportional, between CI and CD, is equal to the cylindrical Superficies, ICDK. (10.) And, it is equal to the Superficies of the spherical Segment NAO. Therefore, the Superficies of the two Segments, of the Cylinder and Sphere, are equal.

After the same manner it may be proved, that the Superficies IEFK, of the Cylinder, is equal to the Surface of the Segment, ENAOF.

Conf. the Surface of the Cylinder, CEFD, between the parallel Circles, CD and EF, is equal to the Segment of the spherical Surface, NEFO, between the same parallel Planes.

And, it is also manifest, that the remaining Superficies, EGQHF, of the Cylinder, is equal to EBF of the Sphere.

From this and the foregoing most excellent Theorems, is deduced a certain Rule for ascertaining the true Area of the Surface of any Segment of a Sphere or other Portion, contained between parallel Circles.

COR. The Segments of a spherical Surface, intercepted between parallel Circles, have the same Ratio, between themselves, as the Segments of the Axe, or Diameter (AB) cut by these Planes.

For, each Segment, or portion of the spherical Surface, NAO, NEFO, and EBF, is respectively equal to the corresponding portion of the Cylinder, ID, CF, and EH; and those Portions are, to each other, as IC to CE, to EG, i. e. as AL:LM:MB; consequently, the Surfaces, NAO, NEFO, and EBF, are to each other, as AL, LM, and MB.

THEOREM XVII.

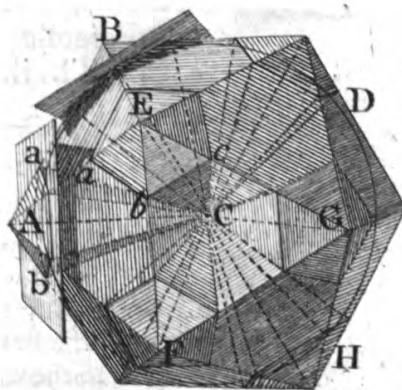
A Sphere is equal, in its solid Contents, to a Cone, whose Base is equal to the Surface of the Sphere, and its Altitude to the Radius.

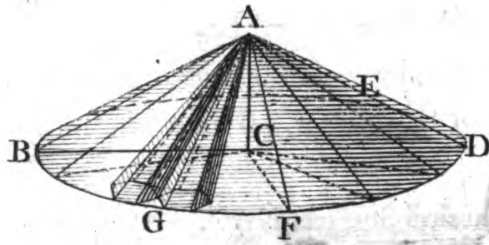
Imagine the Sphere ADF circumscribed by any regular Body whatever; and, from every Angle of the circumscribing solid suppose Right Lines, AC, EC, &c. drawn to the Center, C; there will be formed as many Pyramids, as the Solid has Faces, whose common Vertex is at the Center of the Sphere; which, together, is greater than the Sphere.

Imagine further, that every Angle, A, B, E, &c. of this Solid, be cut off by a Plane, touching the Sphere, and perpendicular to the Right Line from the Angle cut off. It is evident, that there will be formed another regular solid, containing the Sphere, touching every Face; which Solid is less than the first, but it is greater than the Sphere; and, by drawing Lines from every Angle, as before, there will be formed as many Pyramids as this Solid has Faces.

By proceeding thus, *ad infinitum*, it is manifest that the exterior Surfaces of this Solid, which constitute all the Bases of the innumerable Pyramids, must, at last, end in the Surface of the Sphere; and consequently, the Pyramids, or whole Solid, is the same as the Sphere; or, the difference less than any given Quantity.

Then, let BEF be a Circle, and C its Center. If the Radius BC be equal to the Diameter of the Sphere, the Circle BEF is equal to the Surface of the Sphere - Th. 15.





Make AC (perpendicular) equal to its Radius, and imagine the Right angled Triangle ABC revolved around; the Hypothenufe, AB, will describe the Surface of a Cone, BADF.

I say, the Cone is equal to the Sphere.

DEM. For, because the Base of the Cone is equal to the Surface of the Sphere, it may be divided into as many plane Figures, as the circumscribing Solid has Faces; each equal, respectively, to the other.

Consequently, if Right Lines be drawn from every Angle of those Figures, to the Vertex (as at G) there will be formed as many Pyramids, as the Solid has Faces.

But, they are, respectively, equal to them, by Hypothesis; and they have, also, equal Altitudes; i. e. the Radius of the Sphere. Conf. each Pyramid is equal to its corresponding one in the Solid; and, conf. all the Pyramids contained in the one are equal to the other. But, Cones and Pyramids, whose Bases are equal to one another, are equal. Th. the Cone, BADF, is equal to the Sphere ADF.



COR. 1. Sectors of Spheres, * are equal to Cones, whose Bases are equal to the spherical Superficies of the Sectors, and the Altitude of the Cone to the Radius of the Sphere.

For, since the whole Sphere is equal to a Cone, whose Base is equal to the whole Surface, and Altitude to the Radius; conf. a Cone having the same Altitude will be to the other, as the Base of one to the Base of the other; i. e. as the portion of the spherical Surface to the whole; and, conf. the Sector is equal to such a Cone.

COR. 2. A Hemisphere is double of a Cone, having the same Base and Altitude.

For, because the Surface of a Sphere is equal to four large Circles; the Surface of the Hemisphere is double its Base.

But, a Cone whose Base is equal to the spherical Superficies of the Hemisphere, and equal Altitude, is equal to the Hemisphere; consequently, if the Base of the Cone be equal to the Base of the Hemisphere, it will be half the Hemisphere.

COR. 3. A Segment of a Sphere is equal to the Difference between a Sector, whose spherical Superficies is the Surface of the Segment, and a Right Cone, whose Base is equal to the Base of the Segment, & its Side, to the Radius of the Sphere.

* By Sector of a Sphere is to be understood, a Portion containing a Segment, and a Right Cone on the Base of the Segment, whose Side is the Radius of the Sphere.

A N
A P P E N D I X;
O N T H E T H E O R Y
O F M E N S U R A T I O N.

M E N S U R A T I O N O F S U P E R F I C I E S.

THE whole Theory of Mensuration of Superficies, consists in finding a Rectangle equal to the given Figure.

In Theo. 17, Book 1st. it is demonstrated, that every Triangle is equal to half a Parallelogram, on the same Base and Altitude; and, in Theo. 18th. is shewn and demonstrated, that, all Parallelograms or Triangles, having the same or equal Bases and the same Altitude, are equal. Consequently, since Rectangles are Parallelograms (Def. 33 and 44) and it is, there, fully demonstrated that they are all equal, having equal Bases (i. e. having any two Sides equal) and the perpendicular distance between those Sides and the opposite also equal, it is evident, that the measure of one is also the measure of the other. And, because Triangles on equal Bases with Parallelograms, and being of equal height are equal half such Parallelograms, the Area of a Triangle is readily obtained.

Hence is deduced the general Rule (well known to all Surveyors or Artificers, concerned in measuring superficies) for finding the Area of any triangular plane Figure; which is, to multiply the Base, i. e. any one Side, by half its height, from that Side, or half its Base by the whole height; the reason for which is, in those Theorems, clearly accounted for; and, in that consists the whole of superficial Mensuration.

In Prob. 20th is shewn how to construct a Rectangle, equal to a Triangle, on those Principles; and in the 22nd it is further extended to a Trapezium; or to any Quadrilateral, whatever, regular or irregular.

The 21st shews how to construct a Parallelogram, under any Angle, equal to a Rectangle, on the same Base; and consequently, by changing the Premises, a Rectangle may be formed equal to any Parallelogram whatever.

If the construction of those Figures be well understood; practical mensuration is easily attained; which consists (as I have observed above) in finding a Rectangle, i. e. in knowing how to take the dimensions of the two Sides of a Rectangle, equal to any given Figure; for, the multiplication of any two Numbers being applied to measure, always denotes, or produces the Area of, a Rectangle under such dimensions; which I shall, in the first place, endeavour to make clear and intelligible.

Most People, at the first thought on these matters, imagine, that if two Figures (of any species whatever) have equal Circuit, i. e. if the measure of all the sides, of each Figure, in one sum, be the same, they have equal Areas; than which, nothing is more false, as will be made appear.

A Circle contains the greatest Area of any Plane Figure, having an equal Circumference or Perimeter. If a Circle be depressed, though ever so little, it becomes elliptical; in which Case, it loses of its dimensions; and the more it is depressed, i. e. the more excentric an Ellipsis, having an equal Periphery, or Circumference with a Circle, the less is its Area to that of the Circle. Consequently, if a cylindrical Vessel be bulged, or if its Sides are depressed till it becomes elliptical, it loses of its measure; and the more the Sides are depressed, the more it loses; because it is evident, that, if they are pressed quite flat, it loses the whole internal Area.

So likewise, of right lined Figures; the more the Sides are multiplied, and the nearer it approaches, in figure, to a Circle, the greater area it contains. Consequently, regular Poligons, of a greater number of Sides, contain a greater Area, than those of fewer Sides and having equal Perimeters.

Hence

O F M E N S U R A T I O N.

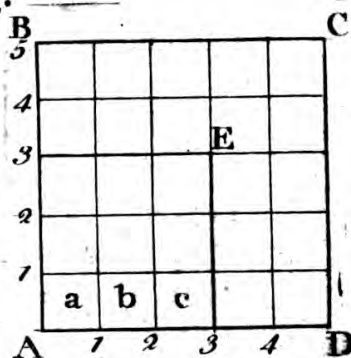
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Hence it is plain, that a Rectangle contains a greater Area than any other Quadrilateral whatever; if the measure of their Sides, in one Sum, be equal. But, a Square, which is the most perfect Rectangle, contains a less Area than a regular Pentagon; and a Pentagon contains less than a Hexagon; a Hexagon less than a Heptagon, and so on to a Circle, whose Perimeters are all equal; so that the Square and Circle, whose Perimeter and Circumference are equal, differ greatly in Area, as shall be illustrated hereafter to demonstration.

Now since, in Mensuration, a Rectangle is the standard or criterion, by which the Area of all Plane Figures, as well as all other superficial Contents, are ascertained; I shall, in the first place, shew, and account for, the methods of taking the dimensions of various Figures; which measures, being multiplied into each other, will give the true Area; each Figure being equal to a Rectangle under those Dimensions.

1. The area of a Square is obtained, by multiplying the measure of one Side into itself. e. g.

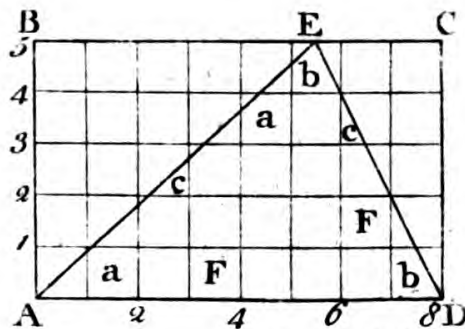
Let the Side of the Square ABCD be five feet; the measure of a Foot being represented by each small division on its Side, as 1, 2, 3, 4, 5. Now, if from these divisions, lines are drawn, both ways, parallel to the Sides; it will be divided into as many less Squares, as the Side multiplied by itself produces; viz. 5 times 5, equal 25; which, is the number of small



Squares contained in the large Square, ABCD; each small square a, b, c, &c. being supposed one Foot, on each Side, is, therefore, called a square Foot. So, the Square AE, containing three feet on each Side, is a Square Yard; its Area is nine square Feet.

But, when a Rectangle measures more, or is longer, on one Side than the other; then, either Side multiplied by the other, not opposite, gives its Area or superficial Measure. Whereas, in a Square, the Sides are all equal; and consequently, the measure of one is also the measure of any other; therefore, the Side is said to be multiplied into itself, to produce its Area.

T H E T H E O R Y



2. If the Rectangle ABCD measures 8 Feet on one Side (AD) and 5 Feet on the other (AB) the opposite Sides are the same (15.1.) and if Lines are drawn through the divisions (as in the Square) parallel to the Sides, it will be divided into 5 times 8, equal 40; the number of square Feet, contained in the Rectangle.

Hence, it is evident, that, in Mensuration, any two Numbers denoting Measure, in Inches, Feet, Yards, &c. being multiplied into each other, gives the area of a Rectangle under these Dimensions, in square Inches, Feet, &c. and consequently, all multiplication of Measure, denotes the Figure, under such dimensions, to be right angled. Wherefore, the multiplication of Lines, in Geometry, implies a Rectangle constructed on two Lines.

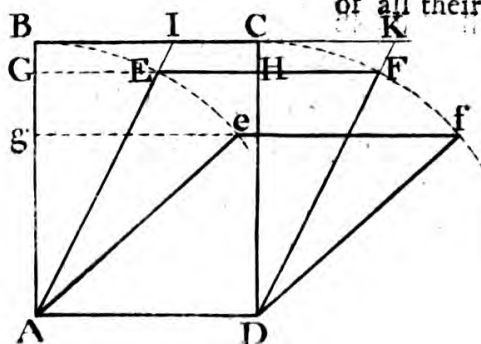
3. To any Point, E (in BC) let AE, DE be drawn, forming a Triangle, AED.

It is obvious, that the Triangle AED contains several whole and entire Squares, F, F, &c. and some Pentagons, as a, a; some Trapezia, as b b, and some Triangles, as c c, &c.

Now, it would be no easy matter to ascertain how many entire Squares all those irregular Figures are equal to; for there are but 12, entire, 4 Pentagons, 3 Trapezia, and 7 Triangles.

But, since (by Th. 17) we have full conviction, that the Triangle AED is equal to half the Rectangle ABCD; and the Rectangle ABCD contains 40 small Squares; consequently, all those irregular Figures are equal to 20 Squares; which, added to the 12 entire ones, make 32, the true Area of the Triangle AED.

4. Next, I will shew, that two Parallelograms, may have the Sum of all their Sides equal, and differ greatly in Area.



The Rect. ABCD, and Par. AEFD, have their Sides, AD, BC, and EF, equal; for, AD is common to both; consequently, BC and EF, being both equal to AD (15.1.) are equal to each other - Ax. 3. $AB = AE$, and $DC = DF$ (N.B.1, Def.20.) and conf. $AB + BC + CD + AD$ is equal to $AE + EF + FD + AD$.

O F M E N S U R A T I O N . 5

But, the Area of the Par. AEFD is not equal to the Area of the Rect. ABCD. For, if AE be multiplied into AD, it will produce an Area equal to the Rect. ABCD, because, $AB=AE$.

But, the Par. AEFD is equal to the Rect. AGHD only. For, it is demonstrable, that, the Triangle AGE=DHF. - 7. 1. Consequently, if DHF be taken away, and its equal, AGE, be added, the Rectangle AGHD is equal to AEFD.

Again, if AE be produced to I, and DF to K (in BC produced) Then, the Parallelogram AIKD is equal to the Rect. ABCD. For, the Triangle ABI=DCK; wherefore, taking away DCK, and adding an equal, ABI, the thing is manifest.

Now, if the Rectangle ABCD be supposed to be depressed, to AEFD, it is evident that it has lost of its dimensions, considerably; and if it be depressed lower, to ef, it still loses more; notwithstanding the Perimeter, AefD, remains the same, equal ABCD. Consequently, ABCD, contains a greater Area than any other Parallelogram, having an equal Base, and equal Perimeter.

Therefore, if ABCD be depressed, i. e. if the Angles are not Right ones, it will contain a less Area; for, if AB and CD deviate ever so little from a Perpendicular, BC must necessarily fall lower. Consequently, its Altitude being less, and the Base remaining the same, its Area is less. - - - Cor. 1. 6.

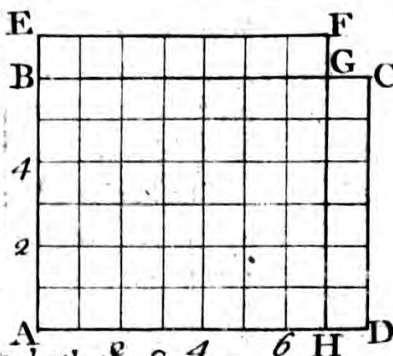
5. It may seem strange, to some, that a Square should contain a greater Area than any other Rectangle, having equal Perimeters.

Suppose the Rectangle ABCD to measure, E on one Side (AD) 8 feet, and on the other Side (AB) 6 feet; its Area is $8 \times 6 = 48$; and its Perimeter is $8 + 8 + 6 + 6 = 28$.

Let a Square, AEFH, be described on AH, equal 7 feet. Then, the measure of its Sides, in one sum, is 7×4 , or 4 times $7 = 28$, the same as the Rectangle.

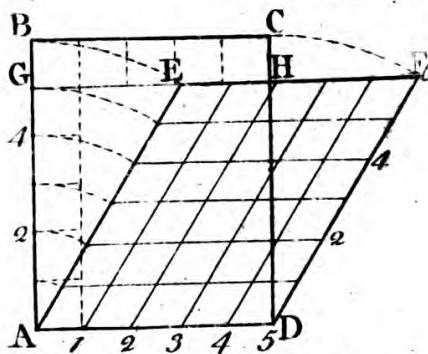
Now, the Rectangle ABGH is common to both the Square, AEFH, and the Rectangle ABCD.

But the Rectangle, BEFG, which is the remaining part of the Square, contains 7 small squares; whereas, the Rectangle DCGH, the remaining part of the Rectangle, ABCD, contains but six.



The area of the Square is 7×7 ; or 7 times 7 = 49, whereas the area of the Rectangle is but $8 \times 6 = 48$; one 49th part less, in its Area, than the Square; which is accounted for by the Figure; and from the 11th Theorem, Book 5th.

Hence, it is plain, that a Square is the most perfect of Rectangles, or other Parallelograms; as a Circle, of all Plane Figures. And, since no other Figure, but Rectangles, can answer the purposes of Mensuration, in producing the true Area; it is certainly most consistent, that all superficial Measure should be reduced to a standard measure by the most perfect of Rectangles, a SQUARE.



6. In the Parallelogram AEFD, let AE be divided into the same equal parts as AB or AD; and if, through those divisions, Lines are drawn parallel to the Sides, AD and AE, it will be divided into as many lesser Parallelograms as the Rectangle, ABCD, contains Squares.

But, those lesser Parallelograms are Rhombuses (Def. 36) each being in the same Ratio to a Square, of an equal Perimeter, as the whole Parallelogram AEFD to the Rectangle ABCD; consequently, they do not ascertain its Area.

The Area, therefore, of a Parallelogram (except it be a Rectangle) is not obtained by the measure of its two Sides, AD, AE, or DF, but by the measure of one Side, as AD, only, and its perpendicular Altitude, DH; which, being multiplied together, produce the Area of a Rect. AGHD; to which, the Par. AEFD, is equal; as it is evident from the Figure.

The Par. AEFD = AGHD, and is deficient of the Rect. ABCD, by the Rectangle GBCH, equal one sixth part of ABCD (1. 6.) GB being one sixth part of AB.

N. B. Either Side of a Parallelogram may be taken, in order to obtain its Area, but not both; unless it be a Rectangle.

If DF be taken, then, the perpendicular distance between that Side and its opposite, AE, will also give the true Area.

7. The

7. The Area of regular Poligons is readily obtained.

The general Rule for measuring Poligons is, to multiply half the Perimeter, i. e. half the Sum of all the Sides, by a Perpendicular from the Center to any Side; which implies, that a Rectangle of those Dimensions is equal to the Polygon; the truth of which I shall endeavour to make appear; as follows.

Let ABDEFGH be a regular Heptagon, whose Center is C.

Draw AC, BC, &c. dividing it into seven equal, Isosceles Triangles.

Draw a Right Line, af, indefinite; in which, take ab, bd, &c. each equal to a Side of the Polygon; and, on the Bases ab, bd, &c. construct the Triangles, acb, bld, dke, and ehf, equal and similar to the Triangle ACB in the Polygon; by Prob. 12. d

And, if a Right Line, ch, be drawn through their Vertices, it will be parallel to af (C. 18. 1.) and, there will be constructed three more Triangles cbl, &c. equal and similar to the other; and consequently, the Trapezium, achf, is equal to the Heptagon AGEB: for, it contains seven Triangles, equal to those in the Polygon.

Draw ci perpendicular to ab; the Triangle acb being Isosceles, the Base, ab, and consequently, the Triangle, acb, is bisected, by the Perpendicular ci. - - - C. 3. 9. 1.

Produce ch; and, make hg equal ai and draw fg, which will be parallel to ci, and also equal. - - - 15. 1.

wherefore, the Tri. fgh, is equal to the Tri. aci. - 7. 1.

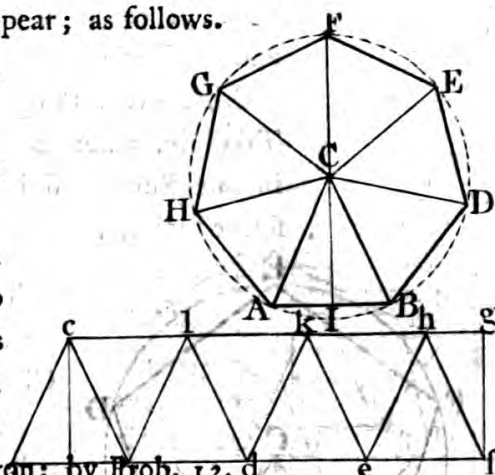
consequently, the Rectangle icgf is equal to the Trap. achf; (which was made equal to the Polygon.) - - - Ax. 6 and 7.

But, the Side if, equal cg, of the Rect. icgf, is equal to IBDEF, half the Perimeter of the Polygon; and ci, equal gf, the other Side of the Rectangle, is equal to CI, a Perpendicular of the Polygon; by Construction.

Consequently, half the Sum of the Sides, AB, BD, &c. (equal if) multiplied by the Perpendicular CI (equal ci) gives the Area of the Heptagon ABDEFGH.

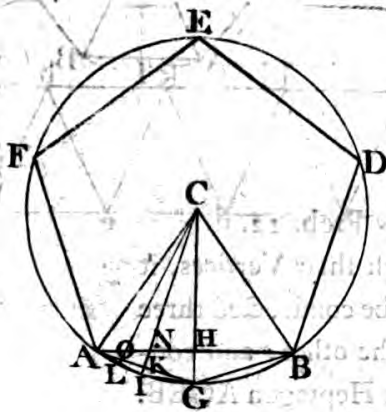
Which, Rule, holds good for all regular Poligons whatever.

8. From



8. From what has been advanced, concerning Polygons, we do with certainty infer, that a Triangle whose Base is equal to the Circumference of a Circle, and its Altitude equal to the Radius, is equal to that Circle; and consequently, half the Circumference multiplied by the Radius gives the Area of a Circle; for it is equal to a Rectangle, under the femi-Circumference and the Radius.

Also, every Polygon, circumscribing a Circle, is equal to a Triangle, whose Base is equal to the measure of all the Sides, in one Sum, and its Altitude equal to the Radius of the inscribed Circle.



The Pentagon ABDEF, it is evident, is less, in its Area, than the Circumscribing Circle; by the five Segments AGB, BD, &c.

Bisect the Ark AGB and draw CG, which will be perpendicular to AB; and draw the Chords AG, GB, which are the Sides of a Decagon, inscribed in the same Circle; and, if the Side AG be bisected, and CK drawn, it will be perpendicular to AG. - Cor. 3. 9. 1.

The two Sides AG, GB, of the Decagon, are greater than the Side AB, of the Pentagon (13. 1.) and, the Perpendicular CK, of the Decagon, is greater than the Perpendicular CH, of the Pentagon; for, CN is longer than CH - - - C. 2. 12. 1.

And CK is longer than CN; consequently, the Area of the Decagon, whose Perimeter is ten times AG or GB, and its Perpendicular is CK, must be greater than the Pentagon, whose Perimeter is five times AB, and its Perpendicular is CH.

The difference is five Isosceles Triangles, equal ABG; whose Base is AB, the Side of the Pentagon, and the equal Legs or Sides, AG, and GB, are the Sides of the Decagon.

Again, the Area of the Decagon is less than the Circle, by the ten Segments AIG, GB, &c.

CK, produced, bisects the Ark AIG at I; draw the Chords AI and IG; which are the Sides of a Polygon of twenty Sides, whose Perpendicular is CO.

This

This Polygon is greater than the Decagon, by ten Isosceles Triangles, equal AGI, in the Segment AG or GB; and it is still less than the Circle by twenty Segments AI or IG.

9. From all which, it is clear, that every Polygon, inscribed in a Circle, is less than the Circle, by as many Segments as the Polygon has Sides; and consequently, every Polygon, circumscribed, is greater than the Circle.

But, the greater the number of Sides of the Polygon, the nearer it is to the Area of the Circle, i. e. to the Circle itself; and consequently, it must at last end in the Circle; that is, the Perimeter of the Polygon, will be equal to the Circumference of the Circle, and the Perpendicular (from the Center) to the Radius.

For, if AI be again bisected in L; the Sides AL, LI, of a Triangle, in the Segment AI, are greater than the Chord AI, which is the Base of that Triangle; yet, they are less than the Arcs of the Segments AL, LI; but they must, at last, by multiplying them, end in the Circumference.

So likewise, the Perpendicular CH, of the Pentagon, is less than CK, the Perpendicular of the Decagon, which is also less than the Perpendicular CO, of the Vigintagon. And, if AL is again bisected, and a Perpendicular drawn to the Chord AL, it will be greater than the Perpendicular CO; and will at last be equal to the Radius of the Circle.

Wherefore, the Circle being conceived to be formed of an infinite number of Isosceles Triangles, whose common Vertex is the Center, and whose Bases are in the Circumference, and which are, altogether, equal to it; consequently, a Rectangle under the Semi-circumference and the radius of the Circle; or, under half the Perimeter of the innumerable sided Polygon (equal to the Circumference) and its Perpendicular (equal the Radius) is equal to the Circle.

And consequently, a Triangle whose Base is equal to the Circumference, and its Altitude equal to the Radius, is also equal to the Circle; for every Triangle is equal to half a Rectangle of the same Base and Altitude (17. 1.) Therefore the Triangle, is

b equal

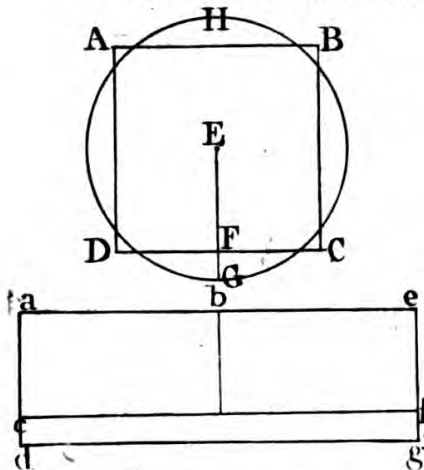
equal to a Rectangle under half its Base, and the Altitude. Hence, the Rule for measuring a Circle is, to multiply half the Circumference by the Radius.

From hence it is clear, that, of all Figures which have an equal Perimeter or Circumference; i. e. whose Bounds are equal to the same Right Line; a Circle contains the greatest Area.

10. I have already shewn, that a Square contains a greater Area than any other Rectangle, having an equal Perimeter, or the measure of all their Sides equal. I will next shew, how much it is less than a Circle, having equal circuit.

Suppose the Sides of the Square, ABCD, be each equal to a fourth part of the Circumference of the Circle, GH; then the four Sides are equal to the whole Circumference.

The Area of the Circle is greater than the Area of the Square.



From what has been said, concerning Polygons and Circles, it is evident; that the Circle, GH, is equal to a Rectangle, two opposite Sides of which, are equal to the Circumference; the other are each equal to the Radius EG.

But, the Square is equal to a Rectangle, two opposite Sides being equal; each to two sides of the Square, i. e. equal half the circumference of the Circle; and the other two Sides, each equal to the Perpendicular, EF, only; which, being considerably less than the Radius EG; consequently, a Rectangle, a e g d, whose Sides a e, d g, are each equal half the Circumference, GH (equal a e, c f, of the Rectangle a e f c) and the Side e g (equal a d) is equal the Radius, EG (but, the Sides a c, e f, each equal the Perpendicular EF, only) will have a greater Area than the Rectangle a e f c; and the difference is the Rectangle c f g d.

The Circumference of a Circle is to its Diameter, nearly, as 22 is to 7; which, is near enough, for measuring any Circle that can be formed. But more exact, the Diameter being 113, the Circumference will be 355. In both, the Circumference is too much; the last, by little more than one five millionth part of the Diameter.

Suppose then, by the last, the Diameter, GH, to be 17, the Circumference will be 53,4, very near. The Side of the Square, ABCD, being a fourth part of the Circumference, is 13,35, the Perpendicular, EF, half the Side, is 6,675; and the Radius of the Circle EG, is 8,5; the Area of each is as follows.

$\begin{array}{r} AB + BC = 26,7 \\ \times \text{ by } EF = 6,675 \\ \hline 1335 \\ 1869 \\ 1602 \\ 1602 \\ \hline \end{array}$	$\left. \begin{array}{l} \text{half the Circumference } 26,7 \\ \times \text{ by the Radius, EG, } = 8,5 \end{array} \right\}$ $\begin{array}{r} 1335 \\ 2136 \\ \hline \end{array}$
Area 178,2225 of the Square	Area of the Circle $\begin{array}{r} 226,95 \\ 178,2225 \\ \hline \end{array}$

difference 48,7275

A regular Pentagon, having the same Perimeter as the Square, is greater than the Square, but it is less than the Circle; also a Hexagon, is still greater than a Pentagon, of equal Perimeter. A Heptagon, is still greater than a Hexagon; and an Octagon, greater than a Heptagon; so that, the more the Sides are multiplied, the nearer it approaches to the Circle; for, the Perimeter of the Polygon, by multiplying, will end, or fall, at last, into the Circumference of the Circle.

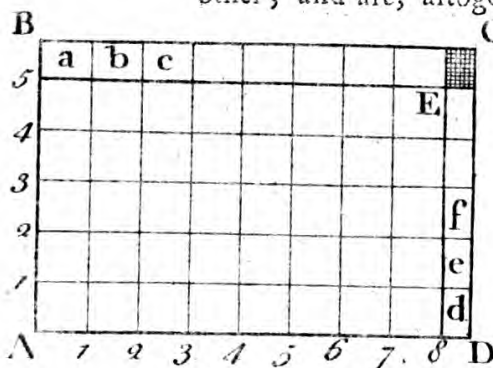
11. From what has been advanced, it is manifest that the whole theory of Mensuration, of Superficies, consists in finding the sides of a Rectangle, equal to the given Figure; since the Multiplication of any two measures (into each other, or one by the other) denotes a Rectangle of those dimensions.

Having shewn how the measures of Plane Figures, from a Square to a Circle, are taken, so as to produce their Areas; I shall, in the next place, account for the multiplication itself, by which, the true Area is ascertained.

Suppose, then, a Rectangle ABCD, whose Sides, AB and CD, measure, by some Scale of equal Parts, 5 and 9 twelfths, each; and the other Sides, AD and BC, each 8 and 6 twelfths.

Through the Divisions 1, 2, 3, 4, &c. on the Sides AB and AD, which may represent the measure of a Foot or 12 Inches, each (for, you may suppose any measure whatever, to be represented by them) if Lines be drawn, parallel to the Sides of the Rectangle, as in the Figure, it will be divided into as many whole

and entire Squares, as the whole numbers produce ; viz. the Rectangle AE, whose Sides, A 8, A 5, being multiplied, 5 times 8, is equal to 40. The fractional parts a, b, c ; d, e, f, &c. produce 8 more Integers or Squares, and 7 eighths of another ; for, the 8 small Rectangles a, b, c, &c. to E, are each 3 fourths of a Square ; being the full measure on one Side, and but 9 twelfths on the other ; and are, altogether, equal to 6 entire Squares.



The 5 smaller Rectangles d, e, f, &c. to E, being but 6 twelfths (half the Integer) on one side, are each equal to half a Square ; and make, together, two and a half ; and the small Rectangle EC, being 3 fourths of one of these, or half one of the other, is equal 3 eighths of a Square.

$$AE = 40$$

$$BE = 6$$

$$DE = 2 - 6$$

$$CE = 0 - 4\frac{1}{2}$$

$$ABCD \ 48 - 10\frac{1}{2}$$

Area of the Rectangle ABCD, in square measure, by that Scale.

The large Rectangle, AE, is equal to 40 Squares ; to which, if the Rectangles BE, DE, and EC be added ; BE=6, DE=2 $\frac{1}{2}$, and EC= $\frac{3}{8}$ of a Square ; together, they are equal to 8, and 7 eighths ; which, added to 40, makes 48 and 7 eighths ; the sum total of the

12. Next, I will shew how the Product arises, by multiplying Feet and Inches, together ; commonly called Cross Multiplication.

The measures of the two Sides of the Rectangle, AB and AD, 8 - 6, and 5 - 9, being placed one under the other ('tis not material which is first, or uppermost) as in the margin, we begin with the Feet, in the multiplier, and say, 5 times 6 is 30.

$$8 - 6$$

$$5 - 9$$

$$42 - 6$$

$$6 - 4 - 6$$

$$48 - 10 - 6$$

$$2 \text{ feet and } 6 \text{ Inches.}$$

Now, it is evident, from the Figure, that this is Feet, one way, and Inches, the other ; and is the Rectangle DE, 5 feet long, E 8, by 6 Inches, 8 D.

Feet multiplied by Inches, or Inches multiplied by Feet, produce Inches ; i. e. the twelfth part of a square Foot ; 30 Inches is, therefore, equal to 2 feet and 6 Inches.

'Tis the same, if you begin with the 9 inches, first.

Set

Set down the 6 Inches, under Inches, and carry the two Feet forward to the whole Numbers, and say, 5 times 8 is 40; which being Feet both ways, they consequently produce square Feet, and is the large Rectangle, AE. The two Feet, arising from the Rectangle BE, being added, make 42 Feet; which, set down under the Feet.

Next, take the Inches, in the multiplier, and say, 9 times 6 is 54.

Now, these are Inches both ways (the small Rectangle EC) and produce only Parts, or square Inches, i. e. the 144th part of a square Foot. For if the Rectangle EC be divided, on each Side, into Inches (viz. 9 and 6) and lines be drawn through them, as in the large Rectangle AE, it will contain 54 square Inches, which are called Parts; twelve of which are called an Inch; for, it must be observed, that what is here meant by an Inch, is not a square Inch, as by a Foot is understood a square Foot; but it is the 12th part of a Square Foot, equal 12 square Inches, or Parts.

Wherefore, the small Rectangle EC, which contains 54 square Inches, it is evident, is equal to 4 Inches and 6 Parts only; since twelve of these Parts are but one Inch, and 4 times 12 is 48, there remains 6 Parts, or half an Inch; which, set down, one place to the right hand of Inches, and carry forward the Inches to the next place.

Lastly, say 9 times 8 is 72, which is the Rectangle BE, 8 Feet one way (5 E) by 9 Inches the other (5 B) and produces Inches; to which, add the 4 Inches from the last place, it makes 76 Inches, equal 6 Feet 4 Inches; set them down, as above, Feet under Feet, and Inches under Inches; which, added into one Sum, makes 48 Feet, 10 Inches, and 6 Parts (equal half an Inch) the Area of the whole Rectangle ABCD, in square Feet, Inches, and Parts.

Thus, multiplying Feet and Inches together, the Area of a Rectangle is obtained, whose Sides are equal to such Dimensions. 'Tis needless to proceed further; as it is evident, if Parts were taken, they would be accounted for, after the same manner.

13. As there are many Persons, who measure accurately by Feet and Inches, yet are unacquainted with Decimals; I shall, briefly, explain the manner of multiplying Decimals; which is much easier, and better adapted to any Quantity whatever.

Multiplication, by Feet and Inches, denotes the fraction of a Foot to be in twelfth parts, which are called Inches; and these are again divided into twelve, called parts, and so on to Seconds and Thirds; each of which is a twelfth part of the foregoing. Whereas, the Decimal Fraction implies the Integer, whatever it be, to be divided into tenths, hundredths, and thousandth Parts, and so on, by tens, *ad infinitum*, as the other by twelves, and are therefore called Duodecimals.

Multiplication, in Decimals, is performed in the same manner as whole Numbers; seeing, that the places increase regularly, by tens, from the Decimal to the Integer, as in whole Numbers.

Observe the following Example; by which, the Area of the Rectangle ABCD is produced, the same as by Feet and Inches.

5,75 8,5 2875 4600 48,875 	The measures of the Sides of the Rectangle, in Decimals, are as in the margin. AB, 5 feet 9 Inches, is, in Decimals, 5 Feet and 75 hundredth parts of a Foot, equal 9 Inches; and AD, 8 Feet and 6 Inches, is 8 feet, and 5 tenths of a Foot, equal 6 twelfths. I place the least Side, AB, first, because it has more places of Figures, and would occasion another Line, if placed otherwise, the Product would be the same.
--	---

In the operation, the Figures are multiplied together, as in common Arithmetic, without any regard to the places of Decimals or Integers; and, when added together, as many places of Decimals as there are in both the Multiplicand and Multiplier, viz. three, are taken from the right hand of the Product; the remaining two are Integers, or Feet; which produces 48 Feet, and 875 thousandth parts of a Foot, equal 7 eighths of a Foot, equal 10 Inches and a half. For, 500 is half a Thousand, therefore equal to 6 Inches; and the remaining 425 is 3 eighth parts of a Thousand, therefore equal to 4 Inches and a half, which is 3 eighth parts of a Foot.

The value of a Decimal Fraction is easily obtained, as follows.

Let the Integer be whatever it may; multiply the Decimal by the number of parts of the next inferior denomination of the Integer; which, in this Case, is Inches, or the twelfth part of a square Foot, to which I would reduce the Decimal 875.

Multiply.

Multiply it by 12, and take away from the Product, as many places as you had at first, there is left 10 Inches; and the Decimal, 500, are now thousandth parts of an Inch; consequently, equal to half an Inch, or 6 Parts, as a second operation evinces; 12 being, again, the next inferior Denomination, as twelve Parts are equal to one Inch.

875
12
—
10,500
12
—
6,000

If the original standard measure of a Foot was, instead of Inches, divided into tenth parts of a Foot; or, which is the same thing, if an Inch had been made a tenth part of a Foot, instead of a twelfth part, and the Inches subdivided into tenths instead of eighth parts, Mensuration would have been greatly facilitated, by decimal Parts instead of Duodecimals, or multiplication of Feet and Inches. The measures being taken, at once, in Decimals (which, by a common Rule, is difficult to do, accurately, and loss of time to reduce some measures to a Decimal) would be more useful in all Cases whatever.

I am not a little surprized, that there has been no attempt to make it universal, as every Person, who has experienced it, must give the preference to Decimals, which is infinitely preferable to any other manner of dividing.

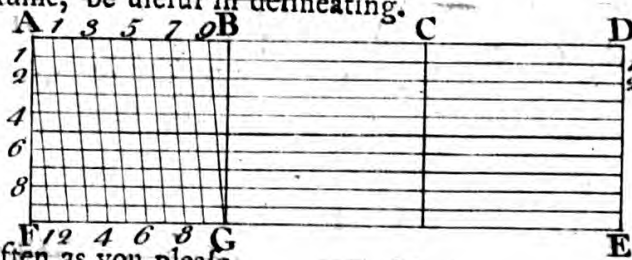
14. I shall, here, shew how to make a Scale, and divide it into Decimal Parts, which may, I presume, be useful in delineating.

Let AB be a determined Measure representing one Foot, or any other measure required.

Produce AB, and repeat the measure of AB, to C and D, as often as you please.

Draw AF and DE perpendicular to AD; on either of which, make, at pleasure (i. e. with any opening of the Compasses) ten equal divisions, as on AF; and, through them, draw lines parallel to AD, as 1, 1 and 2, 2, &c. in all, eleven lines.

Divide AB and FG into ten equal parts (Prob. 36) and draw A 1, 1 2, &c. diagonal-wise, as in the Figure, and proceed in the same manner, to 9 G, parallel to each other.



By

By which means, the length or measure given, AB , is divided into a hundred equal parts; each interval, from AF to $A1$, the first Diagonal, at 1, 2, 3, &c. expressing all the Units from 1 to 10.

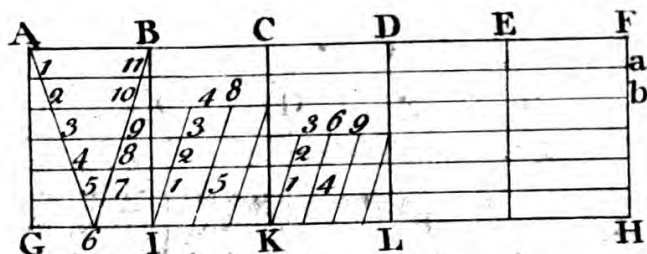
Between each of the Diagonals, as $A1$, and 12 , &c. the intervals are ten of those hundredth parts, being every where equal to the Divisions on AB .

In the same manner, any Measure whatever, either greater or less than AB , may be accurately divided into Decimal Parts.

After such plain directions, in the Construction, the application of it, in Practice, would be impertinent and unnecessary.

15. In the same manner, a Scale may be divided into Duodecimals; by taking 12 Divisions on AF , and dividing the given Measure, AB , into twelve, and proceeding as directed above.

Duodecimals is seldom practised, further than the division of a Foot into Inches, which are the standard Measure, on all common Rules; therefore, to make a Scale of Feet and Inches, to any determinate measure, observe the following directions.



Let AB be the determinate measure of a Foot, according to the proportion you intend to make a Drawing, or Design.

Produce AB , and repeat the measure of AB as often as you think necessary, as C, D, E, F .

Draw AG and FH , at right angles to AF , and make six equal divisions on AG , at discretion; through which, draw $1a$ and $2b$, &c. parallel to AF , and also BI and CK , &c. parallel to AG .

Bisect GI , or AB , and draw the Diagonals, $A6$ and $B6$; and the given Measure, AB , is divided into twelve equal parts, as they are numbered from the Perpendicular AG , on the Diagonals.

N. B. It may be done with only four, or three Spaces, as between IK and KL ; but I think it more applicable, in Practice, with six; which is left to every Person's own discretion; the Divisions are the same in them all.

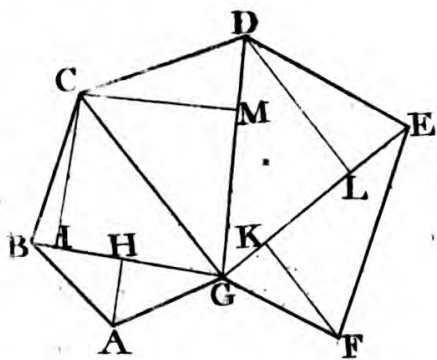
Having

Having illustrated and explained the whole rationale of mensuration of plane Superficies, and shewn, how the Dimensions of all regular Figures are taken, in order to find Rectangles equal to them; I shall just give one specimen of irregular Figures; for, I do not intend this as a Treatise on Mensuration, but only a brief Theory, to shew how it is founded in Geometry; which, being well understood and digested, it is easy to be applied in all Cases whatever, which may occur, in Practice.

Mensuration of Superficies, applied to various kinds of Artificers Work, is still the same in Theory, but much easier, in Practice, than Surveying Land; on account of the irregularity of its Figure, and unevenness of its Surface. The former depends, chiefly, on the mode, or custom of taking their dimensions, in order to reduce the Work to square measure; which is not so easy to describe, but requires very little application under a Proficient. For, a Person ever so well versed in Theory, being a stranger to practice, would differ in his estimate, for no other reason but taking his dimensions different; in which they are not so very exact, but give and take all reasonable allowances; insomuch, that two Measurers, of long practice, taking their dimensions, separately, would differ considerably in their Sums total.

16. ABCDEFG is an irregular Heptagon; having 7 Sides.

Draw the Diagonals BG, CG, GD and GE, or otherwise, at discretion, dividing the given Figure into Triangles, as ABG, BCG, &c. in each Triangle, draw Perpendiculars, as AH, CI, &c. The measure of a Perpendicular and half the Base on which it falls, in Feet and Inches, or Decimal Parts, being multiplied into one another, gives the Area of each Triangle separately; which, added into one Sum, is the Contents of the whole Figure (Ax. 2. 1.)



Or it may be done somewhat readier, by taking any two contiguous Triangles, as ABG, BCG, which together, make a Trapezium, ABCG; unless it should happen to be a Parallelogram.

The Diagonal, BG, is then a common Base to the two Triangles and the Area, of both, is obtained at once, by multiplying the Diagonal, BC, by half the Sum of the two Perpendiculars (Prob. 22.)

The Area of any other two contiguous Triangles, as GDE, GEF, forming a Trapezium, is had after the same manner; and if another Triangle remains, as GCD, its Area, being obtained, must be added to the Areas of the two Trapeziums, into one Sum; which is the true Area of the Heptagon ACDF.

In this Process it may be observed, that not one Side of the Figure is measured, but only the Diagonals and Perpendiculars, all which fall within the Figure. The reason is obvious; because the Sides are not at right angles with each other; or if some of them were, it would be of no use, unless they formed a Rectangle, or other Parallelogram, with the adjacent Diagonal, which is seldom the Case. Whereas, the Perpendiculars making Right Angles with the Diagonals, it is easy to account for the true Area being obtained by them; from 17 and 18th of the 1st of Elements.

N. B. If the Figure to be measured be a large Field, the measures are taken in Chains and Links, &c.

As it must be obvious to every one, who has gone thus far, that to reduce any irregular Figure to a Triangle, having an equal Area to the Figure, is the principal business of Mensuration of Superficies; I would advise the learner to be clear in the Construction, and also in the Demonstrations, of the 20, 21, 22, and 24th Problems. In respect of the 26th and 27th, I shall add further; that not only a Triangle, or Trapezium, but almost any right lined Figure may be divided into two equal parts by a Right Line, from any determined Point, in any Side; as follows.

17. ABCDEF is a Hexagon, having an internal Angle, at D.

Draw the Diagonal AD, dividing it into two Trapeziums.

Bisect AD, and divide each Trapezium, AC and AE, into two equal parts, by the Right Lines GH and GI, (Prob. 27.)

The whole Figure is divided into equal parts; $HAI = CGE$,

Draw HI, and GB parallel to it; then, a Right Line, BI, divides the whole Figure equally.*

Now,

Now, BI bisefts the hexagonal Figure ACF ; but, the Points B and I are accidentally obtained.

Fig. 2. Let K be the given Point, from which a Right Line is required to be drawn, bisefting the given Figure.

Draw BF , and IH parallel to it, and draw FH ; which also bisefts the Figure.

Draw KH , and FL parallel to it, and join KL . Ifay, KL divides the whole Figure equally in two.

The proof is in the 18th of the 1st of Elements, the same as in the 27th Problem; for, the Triangle $BHF = BIF$, and $HLK = HFK$.

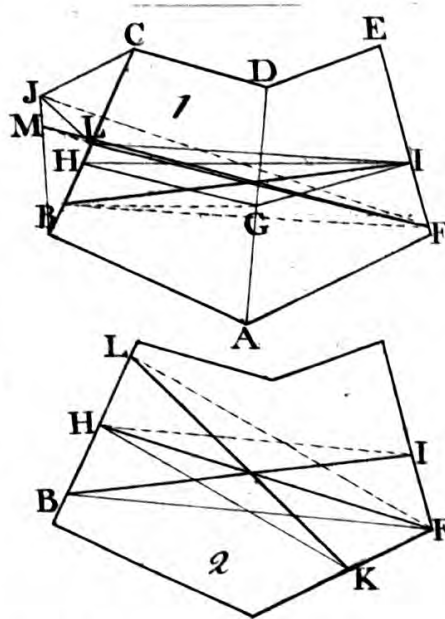
After the same manner it may be done from any Point whatever, in any Side.

If the Figure had more Angles, as BJC (Fig. 1.); a Right Line may be drawn, after the same manner as BI , dividing the whole Figure equally; having perhaps two or three more operations. For, by bisefting BC , and drawing JL , the Point, where BI cuts BC , may, by one more operation, be brought to L (as KL) and then by joining KJ and drawing LM parallel, KM , will biseft the whole Figure.

18. Now, seeing that all irregular Figures are reduced, to Triangles, in order to obtain their Areas; I will shew how the Area of any right-lined Triangle may be obtained without having a Perpendicular, by the measures of its Sides only.

In surveying Land, it may so happen, that, on account of Bogs, Water, Trees, &c. it would be very difficult, if not impossible, to obtain a Perpendicular, within the Figure, when its Sides are accessible. In such Case, the Perpendicular may be found arithmetically, deduced from the 13th of the 2nd Book; or, the Area may be had without the Perpendicular, by the following Rule.

* Let it be observed, that if GB had fallen on this Side of the Angle (which is very probable) it would then be necessary to have recourse to the third operation (Fig. 3.) in the 27th Problem; by which, the Line, BI , bisefting the Figure, would be obtained on this Side of the Angle B .



DEM. Because $AE=AI$; $BE=BK$; and BF equal DK - Con.

AF is equal to half the sum of the Sides, AB , BD , and AD .

Also, AE , EB , and BF , are respectively, the Differences of those Sides, subtracted from the half of their Sum;

(for, $AE=AF-BD$; $EB=AF-AD$; and, $BF=AF-AB$.)

Draw FG perpendicular to AF , produce AC to G , and draw GB .

GB will bisect the external Angle, DBF , of the Triangle.

Draw GH perp. to BD ; the Trap. $FGHB$ is similar to $EBKC$.

For, the Angles at F and H , E and K are Right; by Construction;

wh. the Angle $FBH+FGH=$ to two Right Angles - Th. 1. 10. 1.

and, $FBH+HBE$ are equal to two Right Angles - - 1. 1.

conf. $ECK=FBH$ (for, $HBE+ECK=2$ Right Angles) Ax. 7. 1.

But, BC bisects the Angle ECK , by Construction; & $CE=CK$

wherefore, BG bisects the Angle FBH ; and BH is equal to BF ;

and, consequently, the Triangles CBE , BGF are similar - 13. 6.

Also, because FG is parallel to CE , the Triangles ACE , AGF

are similar; (2. 6.) and therefore, $CE:AE::FG:AF$ - 4. 6.

Also, because BCE , BGF are similar, $CE:EB::BF:FG$.

and consequently, $CE \times CE \times AF \times FG = AE \times EB \times BF \times FG$.

Now, because FG is on both Sides, let it be rejected, and

substitute AF , on both Sides; and consequently, it will be

$CE \times CE \times AF \times AF$, i. e. $CE \square \times AF \square$ is equal to

$AE \times EB \times BF \times AF$; i. e. to AF , the half Sum of the

Sides, multiplied into the three Differences AE , EB , and BF .

But, AF multiplied by CE is equal to the Area of the Triangle;

conf. it is the square Root of $CE \square \times AF \square$; which is equal to

AF , multiplied into the three Differences. Q. E. D.

Mensuration of curved Surfaces, convex or concave.

18. The Areas of the Superficies of regular curved Surfaces, are easily deduced from the 10th, 11th, 12th, the 15th and 16th Theorems, Book the 8th.

By the 10th, the curved Superficies of a Cylinder is equal to a Circle, whose Radius is a mean Proportional between its Side and Diameter.

Or, it is obvious, from the 5th, that it is equal to a Rectangle, whose Sides are its Circumference and Height.

19. By

19. By the 11th it appears, that the conical Superficies of a Right Cone, is equal to a Circle, whose Radius is a mean Proportional between its Side and the Radius of its Base.

Or, it is equal to a Triangle, whose Base is equal to the Circumference of the Base of the Cone, and its Altitude, to the Side of the Cone.

20. By the 12th it is manifest, that the conical Superficies, of the Frustrum of any Right Cone, is equal to a Circle, whose Radius is a mean Proportional between the Side of the Frustrum, and the Radius of the Base, added to the Radius of the opposite Circle.

And every Circle is equal to a Triangle, whose Base is equal to its Circumference, and its Altitude to the Radius. Art. 8th.

Or, the conical Superficies is equal to a Rectangle, whose Sides are equal to the Side of the Frustrum, and half the Sum of the Circumference of the Base, added to half the Circumference of the opposite Circle.

21. The Area of the Surface of a Sphere is equal to a Circle, whose Radius is equal to the Diameter of the Sphere. - 15.8.

And the Area of any spherical Segment is equal to a Rectangle, one Side of which is equal to the Circumference of the Sphere, the other to the height of the Segment.

For, it is equal to a cylindrical Surface of those Dimensions - 16.8.

The same holds true of any portion of the Surface of a Sphere, intercepted between parallel Planes.

The spherical Surface, intercepted between parallel Planes, is therefore equal to a Rectangle, whose Sides are the Circumference of a large Circle of the Sphere, and the perpendicular distance between the parallel Circles.

These Rules, respecting curved Surfaces, being clearly understood, will be found extremely useful to Artificers, in measuring circular Halls, or Rotundas of any kind; and Domes, entire; or where a part is deficient by means of a Lanthorn, or otherwise, at the Top; the Surface between any two parallel Circles being determined by the last.

MENSURATION OF SOLIDS.

AS, in Mensuration of Superficies, the whole Business is to find a Rectangle equal to the Figure proposed; and to determine how many Squares of a certain Dimension, the Figure is equal to; so, Mensuration of Solids consists in determining how many cubical Feet, &c. are contained in the proposed Solid.

In order to which, it is necessary to know the affinity and proportion between one Solid and another, as contained in the 7th and 8th Books of these Elements; and particularly Parallelopipeds, to which all other Solids must necessarily be reduced, in Mensuration; a right angled Parallelopiped being of the same importance, in respect of Solids, as the Rectangle amongst Plane Figures, Also, as a Square is the Criterion of superficial measure, so a Cube, being the most perfect Parallelopiped, is the standard, by which the Quantities of Solids are compared and estimated.

For, the Definition of a Cube, see Def. 10. 7.

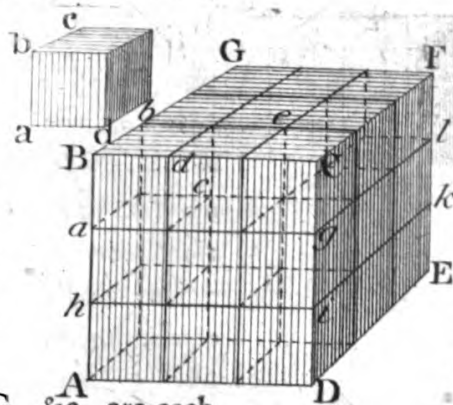
1. Let AGE be a Cube, whose Sides, AB, &c. are each equal to 3 feet; also, let *a c d* be a Cube, whose Side is one foot, or inch, &c.

Now, suppose the Cube *a c d* to be the Unit of measure; it is required to know, how oft the Unit, *a c d*, is contained in AGE.

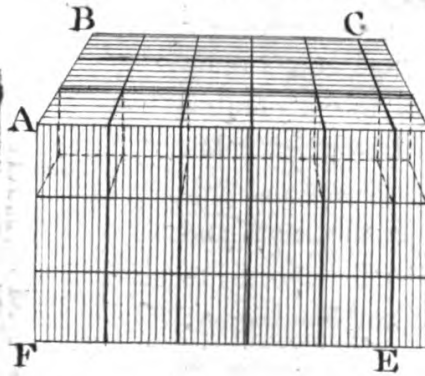
It is evident, seeing, the Measure of AB, BG, &c. are each equal to three times *ab*, *bc*, &c. that the Face ABCD, contains the Square *bd*, 9 times; every other Face, BCFG, or CDEF, the same; therefore, each small Cube *abc*, *def*, *efg*, &c. made by Sections of Planes through *ag*, *gl*, &c. as in the Figure, are equal to *a c d*. Wherefore, since each Surface contains 9 Squares, as BF; and if *Ba*, *Cg*, &c. be one foot in thickness, consequently, the Solid *a G l*, contains the small Cube, *a c d*, 9 times.

But, AB is equal to three times *ab*; wherefore, the whole Cube AGE contains the small one 27 times.

For, if it be supposed to be cut, by parallel Planes, through *agl* and *bik*, parallel to the Top and Base, the Parts *a G l*, *a i l*, and *b D k* are equal, seeing that their Surfaces are equal;
and



and being also of equal thickness, Cg , gi , iD ; consequently, BgF , ail , and bDk are equal (15. 7.) But, $BgFG$ is equal to 9 times acb ; therefore, AGE is equal to three times 9, = 27.



2. It is manifest, that if the Solid ACE (being a Parallelopiped) was longer, equal 5 times ab , having equal thickness, it would contain the small Cube, acb , 45 times, i.e. five times 9; and thus it will increase, as often as the measure ab is added in length.

Suppose half ab be added; it will contain half 9 Cubes, i. e. $4\frac{1}{2}$; seeing that the Surface of the End, or Section through

CE , contains 9 Squares, and the Solid CE has but half a Cube, i. e. half ab in thickness. Therefore, the whole Solid ACE contains the Cube acb 49 times and a half.

Hence, the measure of a right-angled Parallelopiped is to multiply the Side AD by AB , which gives the Area of one Surface (the Top, or Base) and, that Product being multiplied by its height or thickness, AF , i. e. by the number of times it contains ab (as, in this Case, three times) the Product of that multiplication is the Solid Contents of the Parallelopiped AFH .

But, all Parallelopipeds, having equal Bases and Altitudes, are equal (17. 7.) Therefore, whether it be right or acute angled, having obtained the Area of any one Face, in square measure (Art. 12.) that Product being multiplied by the perpendicular Distance, between that Face and its opposite, gives the solid Contents of the Parallelopiped. e. g.

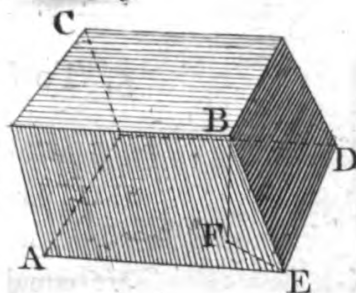
3. Let $ABCD$ be an acute angled Parallelopiped whose Base AD

$$\begin{array}{r}
 7-9 \\
 5-4 \\
 \hline
 38-9 \\
 2-7 \\
 \hline
 41-4 \\
 3-8 \\
 \hline
 124-0 \\
 27-6-8 \\
 \hline
 151-6-8
 \end{array}$$

is 7 Feet, 9 Inches, on one Side, AE ; the other, ED , 5 Feet, 4 Inches, its Area in square measure, is 41 Feet, 4 Inches.

The height of the Parallelopiped, BF , being 3 Feet, 8 Inches, it is obvious, that the whole Solid contains 3 times 41-4, and $\frac{2}{3}$; i. e. as often as the measure of a Foot is contained in the height, BF , so often the Solid contains cubical Feet, as its Base contains square Feet.

But, the Perpendicular BF, is 3 Feet and 8 Inches; consequently, the Solid will contain 41-4, 3 times and two thirds, (8 Inches, being two thirds of a Foot.) Wherefore, 41 Feet, 4 Inches multiplied 3 times, and two thirds, equal 151 Feet, 9 Inches, and 8 Parts, is the true Area, or solid Contents of the Parallelop. ABCD.



Note. What is here meant by an Inch, in solid measure, is the twelfth part of a Cube Foot; viz. a Foot square, and one Inch in thickness; or, that Quantity in any other Figure. And the Part, in Solid measure, is 12 cubical Inches; or 12 Inches in length, and one Inch in breadth and thickness: a cubical Inch is, consequently, a Second of a cubical Foot.

It is obvious, that the Side BG, being inclined to the Base AD, is longer, than the Perpendicular BF, and consequently cannot give its true Area, for (joining G F) BFG is a right angled Triangle, of which BG is the Hypothenufe, therefore it is longer than B F (12. 1.)

Hence it is manifest, that all Parallelopipeds, or Prisms whatever, (for Parallelopipeds are Prisms) having equal height, have that Proportion to each other, which is between their Bases. And, having equal Bases, they are consequently, as their Altitudes.

The Rule, therefore, for measuring any Prism whatever, is to multiply the superficial Area of its Base, by its perpendicular height.

And consequently, the same Rule is applicable for Cylinders-5.8.

4. Every Pyramid is equal to the third part of a Prism having equal Bases and equal Altitudes. - - - Th. 4. 8.

Also, every Cone is equal to the third part of a Cylinder - 5.8.

And consequently, Pyramids and Cones, having equal Bases and Altitudes, are equal.

Hence the Contents of Pyramids and Cones are obtained; viz. by multiplying the Areas of their Bases by one third of their height,

Or, if multiplied by the whole height, it gives the Contents of a Prism or Cylinder, of equal height; consequently, a Pyramid, or Cone, is one third part of such a Prism, or Cylinder.

5. To find the solid Contents of a Sphere.

Having obtained its Diameter; find the Area of a Circle of that Diameter (Art. 8.) which being multiplied, by the Diameter, gives the Contents of a Cylinder, whose height and Diameter are equal.

The Sphere is equal to two thirds of such a Cylinder-Th. 9. 8.

Or, having obtained the Area of a large Circle, of the Diameter of the Sphere; multiply the Product by two thirds of the Diameter.

Otherwife. Every Sphere is equal to a Cone, whose Base is equal to the Surface of the Sphere, and its Altitude to the Radius - 17.8.

Therefore, having obtained the Area of its Surface, multiply that Area by one sixth part of the Diameter.

See an Example, in each, in the Margin.

Let the Diameter of a Sphere be 15; the Circumference of a Circle, of that Diameter, is 47,14, nearly; by the Ratio of 7 to 22.

Then, by Art. 8. the Semicircumference (23, 57) being multiplied by the Radius (7,5, half 15) gives the Area of a great Circle of that Sphere, equal 176,775; which multiplied by 10 (two thirds of the Diameter) gives 1767,75, or three fourths of the Integer.

117 85
164 99
176,775
1767,75

—————
3 fourths.

—————
Multiplying by 10, is only giving one place more of Integers; seeing that, a Cypher, being added, makes no difference in the Decimal; 750 thousandth parts, being equal to 75 hundredth parts; i. e. equal to 3 fourths.

Secondly. 15 being the Diameter of the Sphere, find the Area of its Surface (Art. 21.) which is equal to a Circle whose Radius is equal to the Diameter of the Sphere.

The Circumference of every Circle has the same Ratio to its Diameter (Cor. 1. 14. 6. El.) consequently, the Circumference of a Circle, whose Area is equal to the Surface of the Sphere, is double the Circumference of the Sphere; half of which, is equal to the whole of the other, equal 47,14.

Multiply

Multiply 47,14 by 15, the Diameter of the Sphere, 47,14
 the Product is the Area of the Surface of the Sphere. 15,

Now, since the Contents of the Sphere is equal —————
 to a Cone, the Area of whose Base is 707,1; 707,1
 and its Altitude equal to its Radius, viz. 7,5; 2,5
 and a Cone is equal to one third of a Cylinder of —————
 the same Base and Altitude, therefore, 707,1, mul- 35355
 tiplied by 2,5, one third of 7,5, will produce the 14142
 same Contents, as before, viz. 1767,75; as in —————
 the Margin. 1767,75

6. The solid Contents of a Segment of a Sphere is obtained from the 17th of the 8th, Cor. 3rd.

The solid Contents of the Sphere being equal to a Cone, whose Base is equal to the Surface of the Sphere, and its Altitude to the Radius, as above; and consequently, the Contents of any Sector of a Sphere, being equal to a Cone whose Base is equal to the spherical Surface, and its Altitude to the Radius; (Cor. 1.) it is manifest, that a Segment of the Sphere having the same portion of the Spheres Surface, as a Sector, is equal to the Difference between the Sector and a Right Cone on the Base of the Segment, and, its Side equal to the Radius of the Sphere.

Thus, having shewn how the solid Contents of regular geometrical Solids are obtained, the application of it, in complex ones, will not be difficult; being well versed in the Theory, and having a solid foundation, for the whole, in Geometry.

7. In respect of Guaging; are not almost all large Vessels, either Cylinders or Frustrums of Cones? and consequently they are measured on that Principle; allowing so many Gallons, Quarts, &c. to a cubical Foot, as it is easily known to contain.

Barrels, of all kinds, are measured as two Frustrums, whose common Base is the Diameter at the Bung; but being almost wholly curved, from the Head to the middle, it will be truer to consider it as four, or six Frustrums, two and two, of which, are equal; or the middle part as a Cylinder.

8. The true solid Contents of irregular Solids cannot be obtained, otherwise than by covering them with Water, or other Fluid, in some regular Vessel; and by taking it out, the Difference, in the fulness of the Vessel, is its Contents.

In measuring the Trunks of Trees, &c. which are, generally conical, it is usual to take the Girt in the middle, in order to reduce them to right angled Parallelopipeds; which, if the intention be to produce the real solid Contents, is very erroneous. It is well known, that a Circle contains the greatest Area of any other plane Figure, having an equal Perimeter; and in Art. 10. is shewn the great difference between a Circle and a Square, of equal circuit. Consequently, a Cylinder would be to a right angled Parallelopiped, of equal length, and two of its opposite Faces, (i. e. its Ends) Squares, whose Sides are, each, a fourth part of the Circumference of the Base, of the Cylinder, as the Circle is to the Square; the difference of which is very considerable. (See Art. 10. of Superficies.)

And also (being the Frustrum of a Cone) if the Girt be taken in the middle, in order to reduce it to a Cylinder, it is also erroneous; seeing, it is obvious, that the Frustrum of the Cone is more than such a Cylinder; the excess of the greater End being considerably more than the deficiency of the other. But, as I do not intend to treat at large on those matters, and having gone beyond my first design, I shall only observe, that, in this Case, the Frustrum of a Cone is equal to the Difference, between the whole Cone and the lesser Cone, supposed to be cut off, from the Vertex.

FINIS, of MENSURATION.

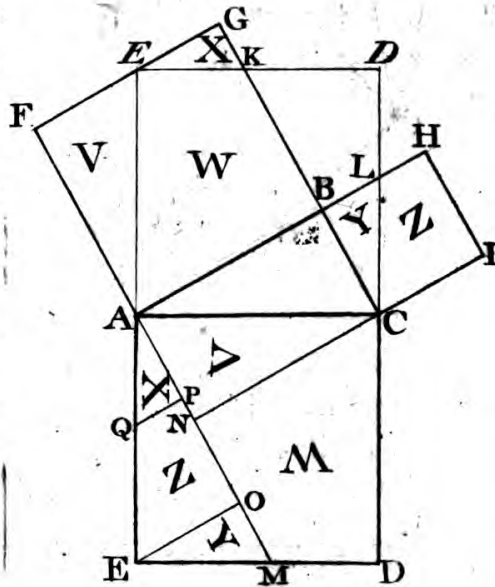
A mechanical and ocular Demonstration of the 20th Theorem, Book 1st.

In Theo. 20. of the first Book of Elements, it is demonstrated, that the Square of the Hypotenuse, of every right-angled Triangle, is equal to both the Squares of the two Sides, containing the Right Angle. It may be entertaining to some, and conviction to others, to give ocular Demonstration of it; by cutting the two Squares of the Sides, in such wise, that they shall exactly cover the Square constructed on the Hypotenuse.

On the right-angled Triangle ABC , describe the three Squares, $ACDE$, $AFGB$, and $BHIC$.

Let the Square $AEDC$, of the Hypotenuse, be supposed to be turned over, on AC , in the inverted position $AEDC$, cutting the two other Squares; BF , in AE , EK , and, the Square BI , in CL .

The two Trapeziums, W and Z , and the three Triangles V , X , and Y , will exactly coincide with the same Figures, as they are described on the Square $ACDE$.



Or, having drawn the Squares as directed, produce EA to E' , and DC to L ; and draw $E'K$, perpendicular to AE . Also, produce FA to M , and IC to N ; draw EO perpendicular to AM , make OP equal BC , (equal EC) and draw PQ parallel to EO .

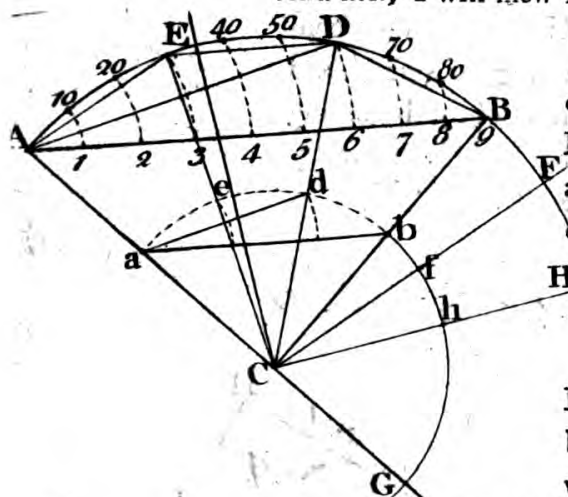
The Lines being drawn on the three Squares, as they are described, in the Figure annexed, on strong Paper or Pasteboard, and the two Squares, FB and BI , being cut, carefully, with a fine thin edged knife, or otherwise, into five pieces, as in the Figure, they will, each, coincide with the corresponding Figures in the Square $ACDE$,

If the position of the Letters, V , W , X , &c. be regarded, there needs no other directions in respect of placing them.

The

The Line of Chords, and its Use, explained.

As the Line of Chords, and its Use, is better known than understood, I presume, a Description of it may not be unnecessary. And first, I will shew how to construct a Line of Chords.



With any Radius, at Discretion, as AC, on C describe the Ark ADB, of 90 Degrees; i. e. make ACB a Right Angle; and on the Center, C, with any Radius, AC, draw ADB; which will be a fourth part of the circumference of a Circle.

Divide the Ark ADB into 9 equal Parts.

First, make AD equal AC and divide DB into three equal parts, and AD will be equal six of those divisions, each of which will be 10 Degrees of that Circle.

Draw the Chord Line AB; and, from the extreme A, as a Center, transfer all the Measures on the Ark, to the Chord AB, as in the Figure. Each Division being subdivided, into ten, and transfered in the same manner, AB will then be a Scale for any number of Degrees, from one to ninety; which gives a Right Angle at C, the Center of the Circle, of which, AC is the Radius.

N. B. This Line of Chords is upon every common Scale, in a Case of Instruments, for drawing; the Use, of which, is the same as the Protractor, described in the Theory of Plane Angles; viz. to measure any Plane Angle; or, to lie down any Angle required, on a Plane, the measure of the Angle, that is, the number of Degrees it contains, being known.

In applying the Line of Chords to use, take 60 Deg. from A, on the Line AB, with your Compasses; and, with that Radius, on any Point, C, of the Line AC (at which Point an Angle is required, with that Line, of some known measure) describe the Ark AEB, cutting AC in A; from which Point, set off the number of Degrees, on the Ark, which is the Measure of the Angle required.

e. g. Suppose an Angle of 35 Degrees is wanted, take 35 Deg. from the Point A on the Line of Chords, and set it off from A to E, draw EC; then, ACE is an Angle of 35 Deg.

If an Angle of 60 Degrees was required; make AD equal to the Radius, and draw DC; then is ACD an Angle of 60 Degrees. For ADC is an Equilateral Triangle, whose Angles are all equal, and consequently, the Chord, AD, is equal to the Radius AC-11.4.

If a Right Angle be required; from A make AB, equal to the whole Line of Chords of 90 Degrees, and draw BC, which will be perpendicular to AC.

Thus, may any Angle, less than 90 Degrees be readily obtained. But, if an obtuse Angle is wanted, as 105 Degrees, with the Line AC, at the Point C; having first set off 90 Degrees, from A to B, or 60, to D; take the remainder BF, 15 Degrees or DF 45, and draw FC, making the Angle ACF, 105 Degrees. Thus may any Angle whatever be described.

Or, an obtuse Angle may be thus taken, at once, if there be room on your Paper.

If an Angle ACF, of 105, or ACH, 125, be required; subtract the number of Degrees from 180, the Sum of two Right Angles, the Difference is 75, or 55.

Produce AC towards G; with the Radius AC, equal AD, (i. e. to 60 Degrees, on the Line of Chords,) describe the Ark GHF, and set off GF, 75, or GH 55 Degrees, and draw CF, or CH, making the Angle required.

After the same manner an Angle may be measured; by describing an Ark, on its Vertex, with the Radius of 60 Degrees, cutting both Sides; as AE, cutting AC and CE, of the Angle ACE; then, the measure of the Chord Line, AE (applied to AB) shews the number of Degrees that Angle contains.

The Reason of all which is so very obvious, it needs no further explanation; but, be sure that you always take your measure, or number of Degrees on the Line of Chords, beginning at the Point A; because it is evident, that they are continually diminishing, from A to B, although they are equal on the Ark; for the Divisions on AB are the Chords of each Ark, set off from A towards B; each of which, is still less in proportion to the Ark.

The Chord of 10 Degrees, it is evident, deviates very little from a Right Line; the Chord of 30 Degrees deviates considerably, and AD, the Chord of 60, still more; for the Chord AD, of 60, and the two Chords of 30 make a Triangle, AED; of which, it is manifest, that, the two Sides AE, ED, each of 30, Degrees is greater than AD, the Chord of 90. (13. 1. El.)

If it should be asked, why 60 Degrees, particularly, is taken for Radius, when we begin the Operation of taking or describing an Angle, by the Line of Chords? the Reason is plain; because the Chord of 60 Degrees is always equal to the Radius; and, the Side, AD, of the Equilateral Triangle ADC, is the Side of a regular Hexagon inscribed in a Circle, of that Radius; and consequently, it is equal to a sixth Part of the Circumference, or 60 Degrees. Hence it is clear, that no other Radius can answer to that Line of Chords.



F I N I S.

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At the Author's House, in Poland-Street, Oxford-Road, the third from Broad-Street on the Left, Youth are taught Geometry and Mensuration; Trigonometry, plane and spherical; Projection of the Sphere, &c. Architecture and Perspective; also the Use of the Globes, and Explanation of the Orrery, &c.

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