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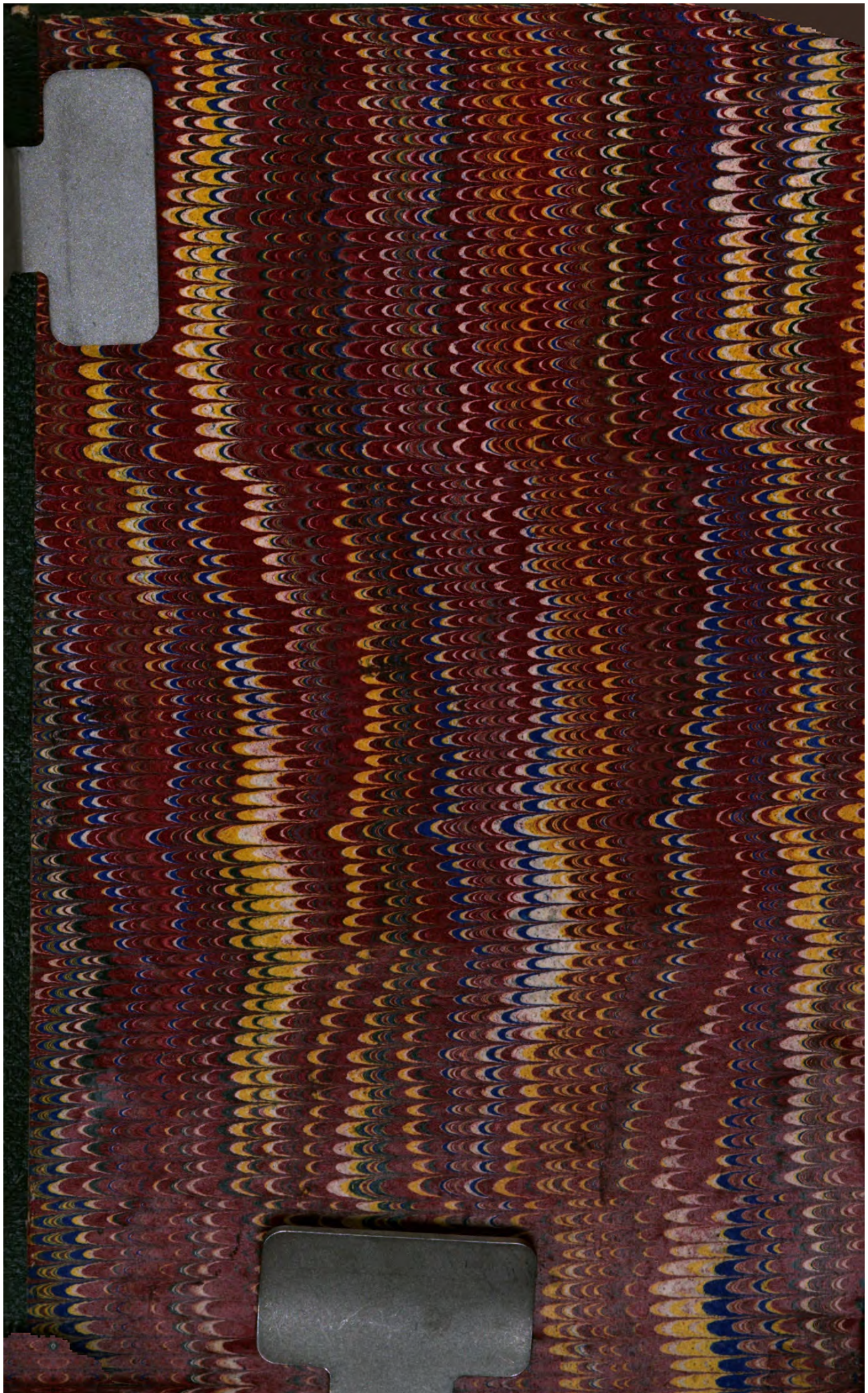
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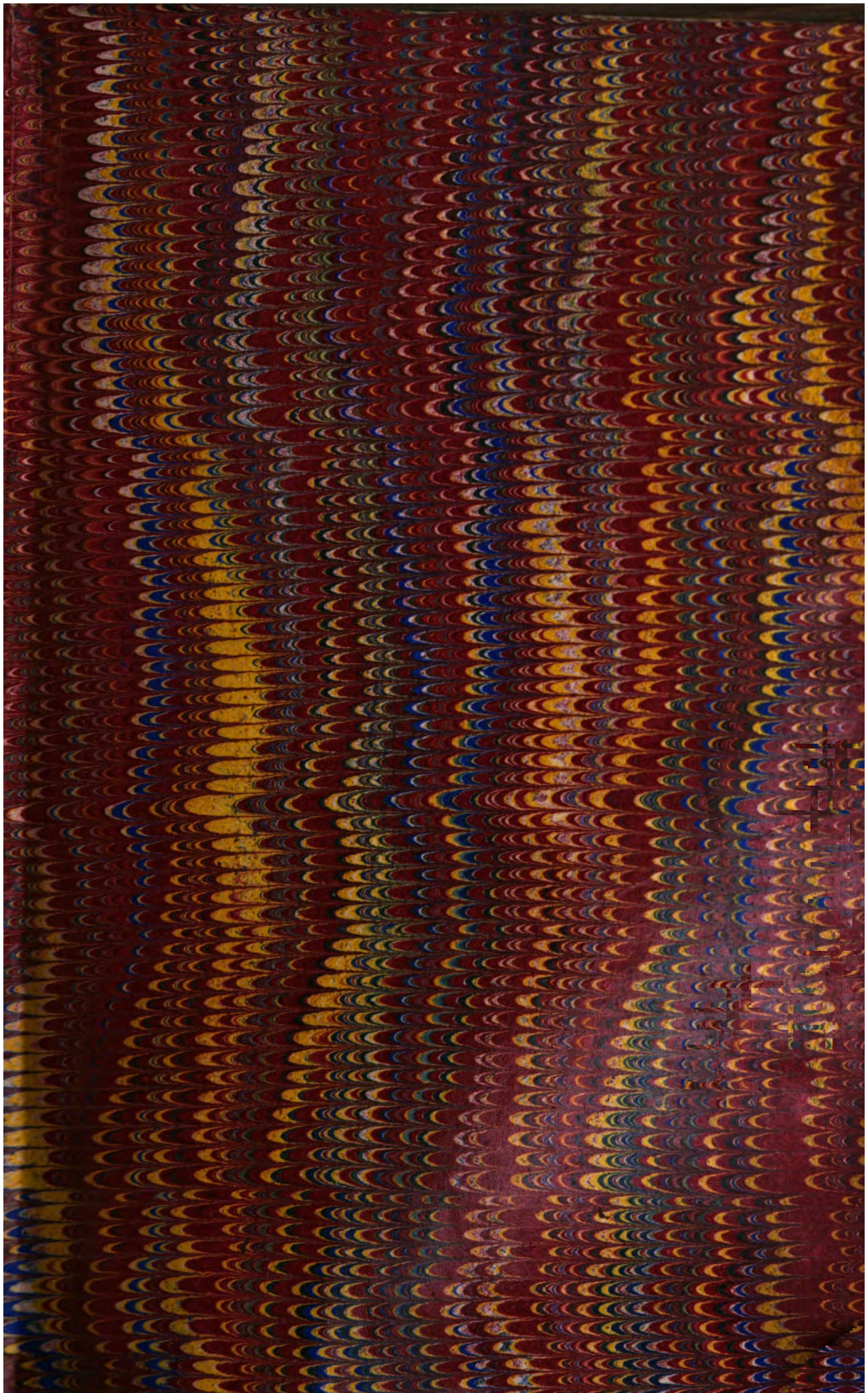
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Baden Powell 15

HARMONICS,
OR
THE PHILOSOPHY
OF MUSICAL SOUNDS.

BY

ROBERT SMITH, D. D, F. R. S.
And Master of Trinity College
In the University of Cambridge.

O Decus Phæbi —————
————— *ô laborum*
Dulce lenimen, mihi cunque salve
Rite vocanti. Hor.

CAMBRIDGE,

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MDCCXLIX.



Scimus musicen, mathesin, atque adeo veram physicam nostris moribus NON abesse à Principis persona: Quæ quidem omnia apud Græcos non laude solùm, sed honore et gloria digna ducebantur.

Epaminondas, Imperator ille insignis, ne dicam summus vir unus omnis Græciæ, philosophiam et musicam egregie didicit. Nam doctus est à Dionysio, qui fuit eximia in musicis gloria. At philosophiæ præceptorem habuit Lysin Pythagoreum, neque prius eum à se dimisit, quam doctrinis tanto antecessit alios, ut facile intelligi posset, pari modo superaturum omnes in cæteris artibus.

Corn. Nep. vit. Epam. sub initio.

TO
HIS ROYAL HIGHNESS
WILLIAM
DUKE OF CUMBERLAND,

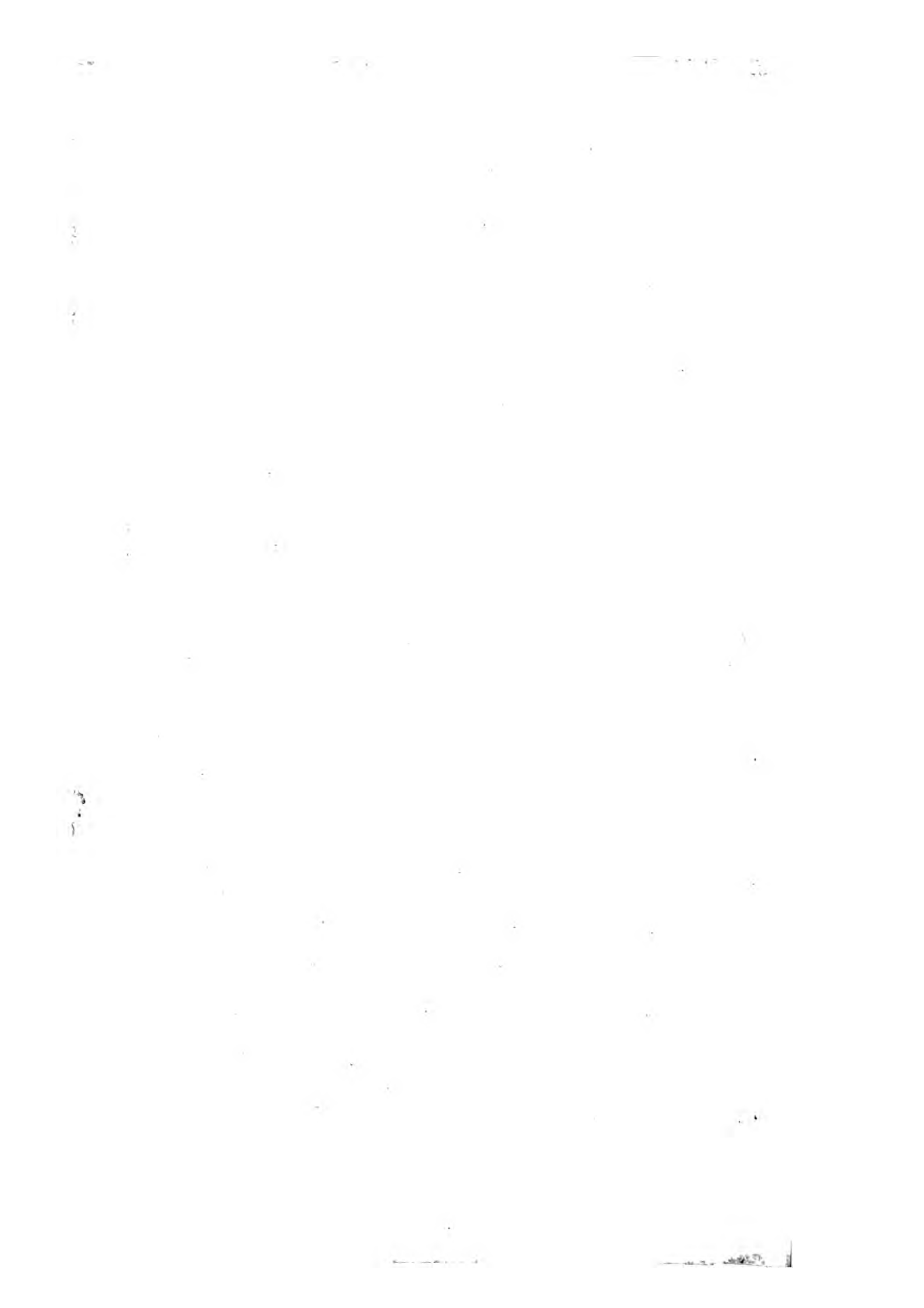
This Philosophical Treatise,
For a lasting Testimony of Gratitude,
Is humbly offered and dedicated,

By His ROYAL HIGHNESS'S

most devoted and

most dutifull servant

ROBERT SMITH.



THE PREFACE.

THE want of an elementary treatise of harmonics, such as might properly have been quoted in support of my demonstrations, has obliged me to begin the following work from the first principles of the science.

The ancient theorists considered no other consonances than such as are perfect, and yet all their musical scales composed of these consonances, have in practice been found disagreeable. The reason is, they necessarily contain some imperfect concords, whose imperfections are too gross for the ear to bear with.

The skill of the moderns has been chiefly employed in the business of tempering the ancient scales, that is, in distributing those grosser imperfections in some of the concords, among all the rest or the greater part of them. By

which means, though the number of imperfect concords be greatly increased, yet if their several imperfections be but as much diminished, the ear will be less offended than before. Because it is the transition from a better harmony to a worse, which chiefly gives the offence; as is evident to any one that attends to a piece of music performed upon an instrument badly tuned. It follows then that the instrument would be better in tune, if all the consonances were made as equally harmonious as possible, though none of them were perfect.

And if this be the true design in tuning an instrument, or tempering a scale of sounds, a theorist ought to begin with the simplest case; and inquire in the first place, whether it be possible for two imperfect consonances to be made equally harmonious; and if so, what must be the proportion of their temperaments or imperfections; and also whether different
con-

consonances require different proportions. These and the like questions being rightly settled, we may then determine in what proportion those grosser imperfections in the ancient scales ought to be distributed, so as to make all the concords equally harmonious in their kind, either exactly or as near as possible.

But as none of the writers that I have seen, have attempted to give us the least notion of the nature and constitution of imperfect consonances, nor of any one property or proportion of their effects upon the ear, except a single conjecture whose contrary is true (*a*), it was not possible for them to determine, from the principles of science, what distribution of those grosser imperfections in the ancient systems, would produce the most harmonious scale of musical sounds.

As this is one of the most difficult and important problems in harmonics,

a 4

in

(*a*) Prop. XIII. coroll. 8.

in order to a scientific solution of it I found it necessary to premise a Theory of Imperfect Consonances (*b*), wherein I have demonstrated as many properties of their Periods, Beats and Harmony as I judged sufficient for solving that problem, and perhaps any other that belongs to harmonics. This theory with its preliminaries and consequences takes up a large part of the present treatise. As to the rest I chuse to refer the reader to the book it self or the Index, rather than trouble him with a further account of it: a short one would be imperfect or obscure, and a perfect one, too long for a preface.

Having been asked more than once, whether an ear for music be necessary to understand harmonics, it may not be amiss to give this answer: That a musical ear is not necessary to understand the philosophy of musical sounds; no more than the eye, to understand that of colours. Our late Professor of Mathematics was an instance of the
(b) Sect. vi.
latter

THE PREFACE. ix

latter case, and the xxth proposition of this treatise affords an instance of the former. For by the solution of that proposition, a person of no ear at all for music may soon learn to tune an organ according to any proposed temperament of the scale; and to any desired degree of exactness, far beyond what the finest ear unassisted by theory can possibly attain to: and the same person, if he pleases, may also learn the reason of the practice.

But though an ear for music is not necessary to understand this treatise, yet those that are acquainted with musical sounds will more readily apprehend many parts of it, and receive more pleasure from them.

In the first scholium to prop. xx, I observed that the winter season had prevented me from tuning an organ by the second table of beats, in order to try what effect the system of Equal Harmony might have upon the ear. But upon telling Mr. *Turner*, one of our organists

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organists at Cambridge, how he might approach near enough to that system, by flattening the major III^{ds}, till the beats of the vth and vith major with the same base, went equally slow, by his great dexterity and skill in tuning he presently put my rule in execution upon a stop of his organ; and affirmed to me, he never heard so fine harmony before, especially in the flat keys; but he added, that the false concords were more intolerable than ever: and no wonder, as their common difference from true concords was then increased from one fifth to one fourth of a tone.

This I mention not so much to confirm my theory, which I did not doubt of, but chiefly as a motive to banish such insufferable dissonance from that noble instrument, and improve its harmony throughout the scale by adding more pipes to the stops, in the manner suggested in this treatise (c).

Nor

(c) Page 200.

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Nor will it be improper to mention a like experiment made by the accurate hand of Mr. *Harrison*, well known to the curious in mechanics by his admirable inventions in watch-work and clock-work, for keeping time exactly both at sea and land: which if duly encouraged and pursued will undoubtedly prove of excellent use in navigation; by correcting the sea-charts, with respect to longitude, as well as the reckonings of a ship, to as great exactness, in all probability, as need be desired.

But in regard to the experiment I was going to mention, he told me he took a thin ruler equal in length to the smallest string of his base viol, and divided it as a monochord, by taking the interval of the major III^d, to that of the VIIIth, as the diameter of a circle, to its circumference. Then by the divisions on the ruler applied to that string, he adjusted the frets upon the neck of the viol, and found the harmony of the consonances so extremely fine, that
after

after a very small and gradual lengthening of the other strings, at the nut, by reason of their greater stiffness, he perfectly acquiesced in that manner of placing the frets.

It follows from Mr. *Harrison's* assumption, that his III^d major is tempered flat by a full fifth of a comma. My III^d determined by theory, upon the principle of making all the concords within the extent of every three octaves as equally harmonious as possible, is tempered flat by one ninth of a comma; or almost one eighth, when no more concords are taken into the calculation than what are contained within one octave. That theory is therefore supported on one hand by Mr. *Harrison's* experiment, and on the other by the common practice of musicians, who make the major III^d either perfect, or generally sharper than perfect: with a design, I suppose, to improve the false concords, though to the manifest detriment of all the rest.

We

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We may gather from the construction of the base viol, that Mr. *Harrison* attended chiefly, if not solely to the harmony of the consonances contained within the octave; in which case the differences between his and my temperaments of the major III^d, VIth and Vth, and their several dependents, are respectively no greater than 4, 3 and 1 fiftieth parts of a comma. And considering that any assigned differences in the temperaments of a system, will have the least effect in altering the harmony of the whole when at the best, I think a nearer agreement of that experiment with the theory could not be reasonably expected.

Upon asking him why he took the interval of the major III^d to that of the VIIIth as the diameter to the circumference of a circle, he answered, that a gentleman lately deceased had told him, it would bring out the best division of a monochord. Whoever was the author of that hypothesis, for so it
must

must be called, he took the hint, no doubt, from observing that as the octave, consisting of five mean tones and two limmas, is a little bigger than six such tones, or three perfect major III^{ds}, so the circumference of a circle is a little bigger than three of its diameters.

When the monochord was divided upon the principle of making the major III^d perfect, or but very little sharper, as in Mr. *Huygens's* system resulting from the octave divided into 31 equal intervals, Mr. *Harrison* told me the major VI^{ths} were very bad, and much worse than the V^{ths}. In which he judged rightly, as I further satisfied my self by trying the experiment upon an organ; and being solicitous to know the reason of that effect, that is, why the V^{ths} and VI^{ths} major when equally tempered should differ so in their harmony, after various attempts I satisfied my curiosity; and this gave me the first insight into the theory of imperfect consonances.

With

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With a view to some other inquiries I will conclude with the following observation. That, as almost all sorts of substances are perpetually subject to very minute vibrating motions, and all our senses and faculties seem chiefly to depend upon such motions excited in the proper organs, either by outward objects or the power of the will, there is reason to expect, that the theory of vibrations here given will not prove useless in promoting the philosophy of other things besides musical sounds.

Such readers as can only dip into this treatise must remember, that by the word Vibration so often repeated, I frequently mean the time of a single vibration; which I notified once for all in Sect. I. Art. 8.

Trinity College,
Cambridge Dec. 31. 1748.

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HARMONICS.

SECTION I.

Philosophical Principles of Harmonics.

1. **S**OUND is caused by the vibrations of elastic bodies, which communicate the like vibrations to the air, and these the like again to our organs of hearing.

Philosophers are agreed in this, because sounding bodies communicate tremors to distant bodies. For instance, the vibrating motion of a musical string puts others in motion, whose tension and quantity of matter dispose their vibrations to keep time with the pulses of air, propagated from the string that was struck. *Galileo* explains this phenomenon by observing, that a heavy pendulum may be put in motion by
A the

the least breath of the mouth, provided the blasts be frequently repeated and keep time exactly with the vibrations of the pendulum; and also by the like art in raising a large bell; and probably he was the first that rightly explained that phænomenon (a).

2. If the vibrations be isochronous the sound is Musical, and continues at the same Pitch; and is said to be acuter, sharper or higher than any other sound whose vibrations are slower; and graver, flatter or lower (b) than any other whose vibrations are quicker.

For while a musical string vibrates, if its tension be increased or its length
be

(a) For he says, in the person of another, il problema poi trito delle due corde tese all'unisono, che al suono dell'una l'altera si muova et attualmente risuona, mi resta ancora irrisolto; come anco non ben chiare le forme delle consonanze et altre particolarità. *Dialogo 1° attenente alla Meccanica*, towards the end.

(b) As the ideas of acute and high, grave and low, have in nature no necessary connection, it has happened accordingly, as Dr. Gregory has observed in the preface to his edition of *Euclid's* works, that the more ancient of the Greek Writers looked upon grave sounds as high, and acute ones as low, and that this
con-

Art. 2. HARMONICS.

be diminished, its vibrations will be accelerated; and experience shews that its sound is altered from what is called a graver to an acuter; and on the contrary. And the like alteration of the pitch of the sound will follow, when the same tension is given by a weight, first to a thicker or a heavier string, and after that to a smaller or a lighter of the same length, as having less matter to be moved by the same force of tension. And these changes in the pitch of the sound are found to be constantly greater or lesser, according as the length, tension, thickness or density of the string is more or less altered (c).

3. There-

connection was afterwards changed to the contrary by the less ancient Greeks, and has since prevailed universally. Probably this latter connection took its rise from the formation of the voice in singing, which *Aristides Quintilianus* thus describes. Γίνεσθαι δὲ ἢ μὲν βαρύτες, κατώθεν ἀναφερομένους τὸ πνεύματι, ἢ δ' ὀξύτερες, ἐπιπολῆς προϊεμένους. pag. 8. Et quidem gravitas fit, si ex inferiore parte (gutturis) spiritus sursum feratur, acumen vero, si per summam partem prorumpat, as *Meibomius* translates it in his notes. pag. 208.

(c) The Greek musicians rightly describe the difference between the manner of singing and talking.

3. Therefore if several strings, however different in length, thickness, density and tension, or other sounding bodies vibrate all together in equal times, their sounds will all have one and the same pitch, however they may differ in loudness or other qualities, and are therefore called Unisons: and on the contrary, the vibrations of unisons are isochronous.

This observation reduces the theory of all sorts of musical sounds to that of the sounds of a single string; I mean with respect to their gravity and acuteness, which is the principal subject of Harmonics (*d*).

4. Con-

They considered two motions in the voice, κινήσεις δύο; the one continued and used in talking, ἡ μὲν συνεχής τε καὶ λογική, the other discrete and used in singing, ἡ δὲ διασημαίική τε καὶ μελωδική. In the continued motion, the voice never rests at any certain pitch, but waves up and down by insensible degrees; and in the discrete motion it does the contrary; frequently resting or staying at certain places, and leaping from one to another by sensible intervals. *Euclid's* Introductio

Art. 4. HARMONICS. 5

4. Consequently the wider and narrower vibrations of a musical string, or of any other body sounding musically, are all isochronous very nearly.

Otherwise, while the vibrations decrease in breadth till they cease, the pitch of the sound could not continue the same; as by the judgment of the ear we perceive it does, if the first vibrations be not too large: in which case the sound is a little acuter at the beginning than afterwards.

5. In like manner, since the pitch of the sound of a string or bell or other vibrating body, does not alter sensibly while the hearer varies his distance from it;

ductio Harmonica. pag. 2. I need not observe, that in the former case, the vibrations of the air are continually accelerated and retarded by turns and by very small degrees, and in the latter by large ones.

(d) *Ptolemy* says, Ἄρμονικὴ μὲν ἐστὶ δύναμις κατὰληπτικὴ τῶν ἐν τοῖς ψόφοις, περὶ τὸ ὄξύ καὶ βαρὺ, διαφορῶν. Harmonics is a power apprehending the differences of sounds, with respect to gravity and acuteness.

it; it follows that the larger and lesser vibrations of the particles of air, at smaller and greater distances from the sounding body, are all isochronous: and consequently that the little spaces described by the vibrating particles are every where proportional to the celerity and force of their motions, as in a pendulum (*e*). And this difference of force, at different distances, causes a difference in the loudness of the sound, but not in its pitch.

6. It follows also, that the harmony of two or more sounds, according as it is perfect or imperfect when heard at any one distance, will also be perfect or imperfect at any other distance: which being a known fact, is mentioned here as a confirmation of the principles of Harmonics.

7. If two musical strings have the same thickness, density and tension, and differ in length only, (which for the future I shall always suppose,) mathematicians

(*e*) See Newton's Princip. Lib. II. Prop. 47.

Art. 8. HARMONICS. 7

thematically have demonstrated, that the times of their single vibrations are proportional to their lengths. (*f*)

8. Hence if a string of a musical instrument be stopt in the middle, and the sound of the half be compared with the sound of the whole, we may acquire the idea of the interval of two sounds, whose single vibrations (always meaning the times) are in the ratio of 1 to 2; and by comparing the sounds of $\frac{2}{3}$, $\frac{3}{4}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{8}{9}$, $\frac{9}{10}$, &c, of the string with the sound of the whole, we may acquire the ideas of the intervals of two sounds, whose single vibrations are in the ratio of 2 to 3, 3 to 4, 3 to 5, 4 to 5, 5 to 6, 8 to 9, 9 to 10, &c.

9. A

(*f*) As a clear and exact demonstration of this curious Theorem depends upon one or two more, of no small use in Harmonics, and requires a little of the finer sort of geometry, which cannot well be applied in few words, I have therefore reserved it to the last Section of this Treatise; which the reader may consult, or, taking it for granted at present, may proceed without interruption; as he likes best.

9. A Musical Interval is a quantity of a certain kind (*g*), terminated by a graver and an acuter sound.

In a ring of bells, for example, the sounds of the first and second bells, counting either from the biggest or the least, terminate a certain interval; those of the first and third a greater interval; those of the first and fourth a greater still; &c. So that the interval increases by degrees, either as the graver of the two sounds descends, or as the acuter ascends; and within the interval of the sounds of the biggest and least bells, the intervals between the sounds of all the rest are contained.

10. Musical intervals are Measures of the Ratios of the single vibrations of the
the

(*g*) See Dr. *Wallis's* preface to *Porphyrie's* comment on *Ptolemy's* Harmonics. Oper. Math. vol. III. *Euclid* says, an Interval is τὸ περιεχόμενον ὑπὸ δύο φθόγων ἀνωμοίων ὀξύτητι καὶ βαρύτητι, what is contained by two sounds different in gravity and acuteness. *Introductio Harmonica*. p. I. *Aristoxenus* defines a musical sound thus, Φωνῆς πλῆσις ἐπὶ μίαν τάσιν ὁ φθόγος, sonus est vocis casus in unam tensionem; and an interval thus, Διάστημα δὲ ἐστὶ τὸ ὑπὸ δύο φθόγων ὠρισμένον, μὴ τῆν

Art. 10. HARMONICS. 9

the terminating sounds, or, *cæteris paribus*, of the lengths of the sounding strings (*b*).

For it is observable in the experiments last mentioned (*i*) and is universally allowed by musicians, that when the lengths of the strings have the same ratio, the interval of their sounds is the same, whatever be their pitch; that if the acuter of the two sounds be raised higher, and consequently the ratio of the lengths of the strings be increased, the interval is increased; and on the contrary, if the acuter sound be depressed lower, that the interval is diminished, and reduced to nothing when the strings have the ratio of equality, whose magnitude is nothing.

FIG.

αὐτὴν τάσιν ἐχούτων; intervallum vero est, quod duobus sonis, non eandem tensionem habentibus, terminatur. And he adds, that it is τόπος δεικτικός φθόγων, ὑψηλῶν μὲν τῆς βαρυτέρας τῶν ὀριζουσῶν τὸ διάστημα τάσεων, βαρυτέρων δὲ τῆς ὑψητέρας; a place capable of sounds, that are acuter than the graver of the two tensions (tones, or sounds) that terminate the interval,* and graver than the acuter of them. pag. 15.

(*b*) Article 7.

(*i*) Art. 8.

FIG. I. Now let the lengths of several strings *A*, *B*, *C*, *D*, &c, be continual proportionals in any ratio. Then since the interval of the sounds of *A* and *B* is equal to that of *B* and *C*, or of *C* and *D*, &c, by adding equal intervals together and equal ratios together, it follows, that the interval of the sounds of *A* and *C*, whose ratio is duplicate of *A* to *B* or of *B* to *C*, is double the interval of the sounds of *A* and *B*, or of *B* and *C*; and that the interval of the sounds of *A* and *D*, whose ratio is triplicate of *A* to *B*, is also triple the interval of the sounds of *A* and *B*, or of *B* and *C* or of *C* and *D*. So that the interval of the sounds of *A* and *C*, is to that of *A* and *D*, as 2 to 3; and the like is evident of any other equimultiples of the proposed ratios and intervals, whatever be their number and magnitude.

II. Therefore musical intervals are proportional to the logarithms of the ratios of the single vibrations of the terminating

Art. I. HARMONICS. 11

minating sounds, or, *cæteris paribus*, of the lengths of the vibrating strings. Because logarithms are numeral measures of ratios; and all sorts of measures of the same magnitudes are proportional to one another. (*k*)

SECTION II.

Of the Names and Notation of consonances and their intervals.

1. FIG. 2. If a musical string CO and its parts DO , EO , FO , GO , AO , BO , cO , be in proportion to one another as the numbers 1 , $\frac{8}{9}$, $\frac{4}{5}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{8}{15}$, $\frac{1}{2}$, their vibrations will exhibit the system of 8 sounds which musicians denote by the letters C , D , E , F , G , A , B , c .

FIG. 3. And supposing those strings to be ranged like ordinates to a right line Cc , and their distances CD , DE , EF ,

(*k*) See M^r. Cotes's *Harmonia Mensurarum*. pag. 1.

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EF, FG, GA, AB, Bc, not to be the differences of their lengths, as in fig. 2, but to be of any magnitudes proportional to the intervals of their sounds, the received names of these intervals are shewn in the following Table; and

$$C..D..E..F..G..A..B..c..$$

$$1.. \frac{8}{9} .. \frac{4}{5} \cdot \frac{3}{4} .. \frac{2}{3} .. \frac{3}{5} .. \frac{8}{15} \cdot \frac{1}{2} ..$$

Mufical Ratios, Interval's Names, Marks, Elements.

$C : c :: 2 : 1$	Cc	Octave	VIII	$3T+2t+2H$
$B : c :: 16 : 15$	Bc	Hemitone	H	
$C : B :: 15 : 8$	CB	VII major	VII	$3T+2t+ H$
$C : D :: 9 : 8$	CD	Tone maj.	T	
$D : c :: 16 : 9$	Dc	7 th minor	7 th	$2T+2t+2H$
$A : c :: 6 : 5$	Ac	3 ^d minor	3 ^d	T+ H
$C : A :: 5 : 3$	CA	VI major	VI	$2T+2t+ H$
$C : E :: 5 : 4$	CE	III major	III	T+ t
$E : c :: 8 : 5$	Ec	6 th minor	6 th	$2T+ t+2H$
$G : c :: 4 : 3$	Gc	4 th minor	4 th	T+ t+ H
$C : G :: 3 : 2$	CG	V major	V	$2T+ t+ H$
$F : B :: 45 : 32$	FB	IV major	IV	$2T+ t$
$B : f :: 64 : 45$	Bf	5 th minor	5 th	T+ t+2H
$D : E :: 10 : 9$	DE	Tone min.	t	
$81 : 80$		Comma	c	T- t

and, except the Tone and Hemitone, are taken from the numbers of the strings or sounds in each interval inclusively; as a Third, Fourth, Fifth, &c, with the epithet of *major* or *minor*, according as the name or number belongs to a greater or smaller total interval; the difference of which results chiefly from the difference of the magnitudes of the tone and hemitone.

2. Hence it is, if the ratio of the single vibrations of any two sounds, or of the lengths of two vibrating strings, be any of those in the first column of the table, that their interval, and the consonance too, retains the name in the third column, whether the intermediate sounds be present or absent.

3. FIG. 3. In the line Cc produced beyond c , if we take the intervals Dd , Ee , Ff , &c, severally equal to the octave Cc , and make the length of the several strings at d, e, f , &c, equal to half the lengths of those at D, E, F , &c, all the intervals within this higher octave cc' , will

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will also consist of major and minor tones and hemitones, ranged in the same order as in the lower octave *Cc*.

Now the names of intervals larger than one or more octaves, are also taken from the number of the strings in them inclusively. Thus the interval *Cd* is called a Ninth, *Ce* a Tenth, *Cf* an Eleventh, *Cg* a Twelfth, &c, with the epithet of *major* or *minor* as before; and are thus denoted, IX or VIII + T, X or VIII + III, XIth or VIII + 4th, XII or VIII + V, &c, the units in the compound marks being constantly one more than those in the simple ones, because the intermediate string at the end of the octave is counted twice.

4. A Comma is the interval of two founds whose single vibrations have the ratio of 81 to 80, and is the difference of the major and minor tones (*l*).

5. When

(*l*) For the ratio of 9 to 8 diminished by the ratio of 10 to 9, is the ratio of 9×9 to 8×10 , or of 81 to 80.

Art. 5. HARMONICS. 15

5. When the single vibrations of two sounds are in any one of the ratios in the table, or in fig. 3 continued, the consonance and its interval too is called Perfect; and Imperfect or Tempered, when that ratio, and interval belonging to it, is a little increased or diminished.

6. Any small increment or decrement of a perfect interval is called its Temperament, and is very conveniently measured by the proportion it bears to a comma.

7. As the addition and subtraction of logarithms answers to the multiplication and division of their absolute numbers, that is, to the composition and resolution of the ratios of those numbers to an unit; so the addition and subtraction of musical intervals answers to the composition and resolution of the ratios of the single vibrations of the terminating sounds, or of the lengths of the vibrating strings: and on the contrary.

In

In the following examples, the composition and resolution of ratios is intimated by the multiplication of their terms, placed, in the form of fractions, upright and inverted, respectively.

$$\left\{ \begin{array}{l} \text{As } \frac{1}{2} = \frac{8}{15} \times \frac{15}{16} = \frac{9}{16} \times \frac{8}{9} = \frac{3}{5} \times \frac{5}{6} = \\ \text{So VIII} = \text{VII} + \text{H} = 7^{\text{th}} + \text{T} = \text{VI} + 3^{\text{d}} = \end{array} \right.$$

$$\frac{5}{8} \times \frac{4}{5} = \frac{2}{3} \times \frac{3}{4} = \frac{8}{9} \times \frac{8}{9} \times \frac{8}{9} \times \frac{9}{10} \times \frac{9}{10} \times \frac{15}{16} \times \frac{15}{16} \\ 6^{\text{th}} + \text{III} = \text{v} + 4^{\text{th}} = 3\text{T} + 2\text{t} + 2\text{H}.$$

$$\left\{ \begin{array}{l} \text{As } \frac{3}{5} = \frac{3}{4} \times \frac{4}{5} \quad \left| \quad \frac{5}{8} = \frac{3}{4} \times \frac{5}{6} \quad \left| \quad \frac{2}{3} = \frac{4}{5} \times \frac{5}{6} \\ \text{So VI} = 4^{\text{th}} + \text{III} \quad \left| \quad 6^{\text{th}} = 4^{\text{th}} + 3^{\text{d}} \quad \left| \quad \text{v} = \text{III} + 3^{\text{d}} \end{array} \right. \right.$$

$$\text{As } \frac{3}{4} = \frac{4}{5} \times \frac{15}{16} \quad \left| \quad \frac{3}{4} = \frac{5}{6} \times \frac{9}{10} \quad \left| \quad \frac{4}{5} = \frac{8}{9} \times \frac{9}{10} \\ \text{So } 4^{\text{th}} = \text{III} + \text{H} \quad \left| \quad 4^{\text{th}} = 3^{\text{d}} + \text{t} \quad \left| \quad \text{III} = \text{T} + \text{t} \right. \right.$$

$$\left\{ \begin{array}{l} \text{As } \frac{5}{6} = \frac{8}{9} \times \frac{15}{16} \quad \left| \quad \frac{3}{5} \times \frac{3}{2} = \frac{9}{10} \quad \left| \quad \frac{2}{3} \times \frac{4}{3} = \frac{8}{9} \\ \text{So } 3^{\text{d}} = \text{T} + \text{H} \quad \left| \quad \text{VI} - \text{v} = \text{t} \quad \left| \quad \text{v} - 4^{\text{th}} = \text{T} \end{array} \right. \right.$$

$$\text{As } \frac{3}{4} \times \frac{6}{5} = \frac{9}{10} \quad \left| \quad \frac{3}{4} \times \frac{5}{4} = \frac{15}{16} \quad \left| \quad \frac{8}{9} \times \frac{10}{9} = \frac{80}{81} \\ \text{So } 4^{\text{th}} - 3^{\text{d}} = \text{t} \quad \left| \quad 4^{\text{th}} - \text{III} = \text{H} \quad \left| \quad \text{T} - \text{t} = \text{c} \right. \right.$$

8. Hence

8. Hence the hemitones and tones major and minor, being the differences of the intervals, III, 4th, V, VI, and of their complements to the octave, may be considered as the Elements that compose the intervals of all perfect concords (*m*).

SECTION III.

Of perfect consonances and the Order of their simplicity.

I. FIG. 4. When a single sound is heard, the series of equal times between the successive pulses of air, that beat on the ear, may be represented by a series of equal parts contained in a right line; as in 02, 03, 04, &c. Consequently when two sounds are heard, two of those lines, as 02 and 03, will
right-

(*m*) The old method of resolving concords into their elements may be seen in Dr. *Wallis's* division of the monochord, or section of the musical canon, as the ancients called it. *Philosoph. Transact.* N^o. 238. or *Abridg.* by *Lowthorp.* vol. 1. p. 698. first edit.

rightly represent the two series of equal times, if the magnitude of the equal parts in one line, be to the magnitude of those in the other, in the ratio of the single vibrations of the sounds: or, the whole lines being supposed equal, if the numbers of aliquot parts in each, as 2 and 3, be severally the same as the least numbers of the vibrations of each sound, made in the same time represented by the line $o2$ or $o3$. (*a*)

2. And the sounds being heard together, if we conceive the two equal and parallel lines that rightly represent them, as $o2$ and $o3$, to coincide throughout, the points that divide the separate lines, will subdivide the combined lines into smaller portions, as in Fig. 5, representing a third series or cycle of times, in which the pulses of both sounds interchangeably succeed one another in beating upon the ear.

3. Such a mixture of pulses, succeeding one another in a given Cycle
of

(*a*) See Art. 12. following.

of Times, terminated at both ends by a perfect coincidence of the pulses, and sufficiently repeated, is the physical cause that excites the idea of a given consonance: Especially when considered as distinct from any other consonance, whose single vibrations having a different ratio from that of the former, will constitute a different cycle, and excite a different idea. But if that ratio be the same, though the absolute times be different, the consonances may be looked upon as the same, in this respect, that their cycles have the same form; the times in both having the same order, and the same proportions; and in this other also, that the interval of the sounds is the same (*b*).

4. This being premised, one consonance may be considered as more or less simple than another, according as the cycle of times belonging to it, is more or less simple than the cycle belonging

(*b*) Art. 10. Sect. 1.

longing to the other. And upon this principle all consonances may be ranged in due order of such simplicity, by the help of the following Rule.

5. *One consonance is Simpler than another in the same Order, as the sum of the least terms, expressing the ratio of the single vibrations, is smaller than the like sum in the other consonance; and when several such sums are the same, these consonances are simpler in the same order, as the lesser terms of their ratios are smaller.*

For the simplicity of a consonance or cycle of times, consists partly in the number of times contained in the cycle, and partly in the different proportions they bear to one another.

FIG. 4. When the numbers of times in different cycles are different, and the times in each cycle are equal to one another, as when we combine the founds 01 and 01, 01 and 02, 01 and 03, 01 and 04, 01 and 05, &c, the cycles of this sort may be ranged in the
order

order of their simplicity above defined, by the order of the numbers of equal times in the cycles, or of the magnitudes of the numbers 1, 2, 3, 4, 5, 6, &c, or of 2, 3, 4, 5, 6, 7, &c, that is, of the sums of the terms of the ratios 1 to 1, 1 to 2, 1 to 3, 1 to 4, 1 to 5, &c.

In the other case, where the numbers of times in different cycles are the same, and the times in each cycle bear different proportions to one another, as when we combine the sounds 01 and 06, 02 and 05, 03 and 04, that cycle is simpler than another, in which the equal times between the pulses of the acuter sound, are less interrupted and subdivided by the pulses of the graver.

Accordingly in the first of these cycles composed of 01 and 06, not one of the 6 equal times between the pulses of the acuter sound 06, is subdivided by any pulse of the graver 01; but in the second cycle composed of 02 and 05, one of the 5 equal times, between the pulses of the acuter sound 05, is

*A table of the Order of the simplicity
of consonances of two sounds.*

Ratios of the vibra- tions.	Order of the simpli- city.	Intervals of the sounds.	Continuation of the table.		
			1 : 15	15	3VIII + VII
1 : 1	1	0	1 : 16	16	4VIII
1 : 2	2	VIII	2 : 15	$16\frac{1}{8}$	2VIII + VII
1 : 3	3	VIII + V	5 : 12	$16\frac{1}{2}$	VIII + 3 ^d
1 : 4	4	2VIII	8 : 9	$16\frac{7}{8}$	T
2 : 3	$4\frac{1}{2}$	V	1 : 18	18	4VIII + T
1 : 5	5	2VIII + III	3 : 16	$18\frac{2}{9}$	2VIII + 4 th
1 : 6	6	2VIII + V	4 : 15	$18\frac{1}{3}$	VIII + VII
2 : 5	$6\frac{1}{3}$	VIII + III	9 : 10	$18\frac{8}{9}$	t
3 : 4	$6\frac{2}{3}$	4 th	1 : 20	20	4VIII + III
1 : 7	7	VI	5 : 16	$20\frac{2}{5}$	VIII + 6 th
3 : 5	$7\frac{2}{3}$	VI	1 : 22	22	2VIII + VI
1 : 8	8	3VIII	3 : 20	$22\frac{2}{11}$	VIII + 7 th
4 : 5	$8\frac{3}{4}$	III	5 : 18	$22\frac{4}{11}$	VII
1 : 9	9	3VIII + T	8 : 15	$22\frac{7}{11}$	
1 : 10	10	3VIII + III	1 : 24	24	4VIII + V
2 : 9	$10\frac{1}{5}$	2VIII + T	9 : 16	$24\frac{2}{3}$	7 th
3 : 8	$10\frac{2}{5}$	VIII + 4 th	1 : 28	28	2VIII + 3 ^d
5 : 6	$10\frac{4}{5}$	3 ^d	5 : 24	$28\frac{2}{7}$	VIII + t
1 : 12	12	3VIII + V	9 : 20	$28\frac{4}{7}$	
3 : 10	$12\frac{1}{3}$	VIII + VI	1 : 30	30	4VIII + VII
4 : 9	$12\frac{2}{3}$	VIII + T	15 : 16	$30\frac{14}{15}$	H
5 : 8	$12\frac{3}{4}$	6 th	32 : 45	$76\frac{31}{38}$	IV
6 : 7	$12\frac{5}{6}$		45 : 64	$108\frac{22}{27}$	5 th

subdivided by one pulse of the graver o_2 ; and in the third cycle composed of o_3 and o_4 , two of the 4 equal times in the acuter sound o_4 , are subdivided by 2 pulses of the graver o_3 . By which it appears, that the first cycle is simpler than the second, and the second simpler than the third; and that the order of simplicity of this sort of cycles, answers to the order of the magnitudes 1, 2, 3 of the lesser terms of the ratios.

6. Now by the first part of the rule above, the integers in the second column of the table, are the several sums of the terms of the opposite ratios in the first, diminished by 1, which alters not the order of their magnitudes, but only makes the series begin with 1, answering to the simplest consonance.

By the second part of the rule, the ratios whose terms have the same sum, as $1 : 6$, $2 : 5$, $3 : 4$, are ranged in the order of their lesser terms 1, 2, 3, or, which alters not the order, of those terms severally diminished by 1, as of

0, 1, 2, or of the fractions $\frac{0}{3}$, $\frac{1}{3}$, $\frac{2}{3}$, whose common denominator 3 is the number of the ratios whose terms have the same sum 7. These fractions either by themselves or the mixt numbers 6, $6\frac{1}{3}$, $6\frac{2}{3}$, made by annexing them to the number 6, may therefore denote the order of the Modes of simplicity of such consonances as have the same Degree of simplicity denoted by 6 or $7 - 1$. And thus the order of the simplicity of all consonances whatever, is denoted by the order of the magnitudes of the integers and mixt numbers in the second column of the table.

7. This series increases from unity in several arithmetical progressions, except that a term or two is here and there omitted, where ratios occur, which being reducible to simpler terms, have been considered before, or else are not Musical Ratios, which are such only whose terms are 1, 2, 3, 5, with their

their powers and products (*d*), or belong not to the intervals in Fig. 3.

For example, writing down all the ratios in due order, whose terms make a given sum, as 1 to 8, 2 to 7, 3 to 6, 4 to 5, I reject the two middlemost for the reasons just mentioned, and place the rest in the first column of the table; which may thus be continued with certainty and order as far as we please.

8. Hence we may distinguish consonances into two sorts, Pure and Interrupted; pure, where none of the equal times between the pulses of the acuter sound, is subdivided by any intermediate pulse of the graver; and interrupted, when any of those equal times are interrupted by one or more pulses of the graver sound.

In the second column of the table, the least simple or lowest mode of each degree of interrupted consonancy, is every where placed above the next inferior degree of pure consonancy, as $4\frac{1}{2}$ above 5.

(*d*) See Sect. IV.

For

For should we depress the mode $4\frac{1}{2}$ to a place next below the degree 5, why not even to a place next below 6? though not below $6\frac{1}{3}$, as being a more complex mode of a less simple degree. But if that were allowable, by parity of reason we ought to depress $4\frac{1}{2}$, $6\frac{1}{3}$, $6\frac{2}{3}$ next below 7, though not below $7\frac{2}{3}$, and likewise $4\frac{1}{2}$, $6\frac{1}{3}$, $6\frac{2}{3}$, $7\frac{2}{3}$ below 8, though not below $8\frac{3}{4}$, and also $4\frac{1}{2}$, $6\frac{1}{3}$, $6\frac{2}{3}$, $7\frac{2}{3}$, $8\frac{3}{4}$ below 9 and 10, and so forth to infinity: which, by depressing all the modes of interrupted consonancy, below all the degrees of pure consonancy, would render them heterogeneal, and incapable of any order or comparison with one another. The table is therefore rightly ordered.

9. Hitherto we have only considered the number and proportions of the times, in the cycle by which a consonance

nance

nance is represented, without regard to the quality of the pulses, as to duration, strength, weakness or other accidents; whereas the pulses of graver sounds are generally stronger and obtuser than those of acuter sounds, and affect the ear differently. Very true; but still this alters not the rational idea of the consonance, as above described, nor does the ear perceive any alteration in the kind of mixture, or in the interval, upon softening or swelling either sound, while the other retains the same strength.

10. It is well known in general, that simpler consonances affect the ear with a smoother and pleasanter sensation, and the less simple with a rougher and less pleasant one. And this analogy seems to hold true according to the order in the table, as far as the ear can judge with certainty. Those that are willing to try the experiment, may readily do it by the help of the third column of the table, shewing the musical
fical

fical intervals answering to the respective consonances. But the analogy will be plainer perceived by intermitting several consonances, and trying it, for example, in this series of all the concords within the octave, VIII, V, 4th, VI, III, 3^d, 6th; but then they should not be tempered as usual, but tuned perfect. And if the experimenter be skilful in melody and composition, he must endeavour, as much as possible, to divest himself of all habitual prepossessions in favour of this or that concord, or succession of concords, acquired from the rules and practice of his art; in order to an impartial judgment of the simple perception of the smoothness and sweetness of each concord, and a fair comparison of such perceptions only.

II. Though nature has appointed no certain limit between concords and discords, yet as musicians distinguish consonances by those names for their own uses, I may do the like for mine;
call-

calling unisons, III^{ds}, v^{ths} and VI^{ths}, and their complements to the VIIIth and compounds with VIII^{ths}, Con- cords, and all other consonances, Dif- cords.

12. If the given times of the single vibrations of any two sounds be V and v , and if $V : v :: m : n$, representing the least integers in that ratio; the length of the cycle of times between the successive coincidences of the pulses of V and v , is $nV = mv$. Because these multiples of V and v are the least of any that can be equal.

For the same reason, if $V : x :: r : s$, in the least integers, the cycle $sV = rx$.

13. Hence the length of the cycle of V and v , is to that of V and x , as n to s ; that is, the cycles of consonances that have a common sound, string or vibration V , are proportional to the Numerators of the fractions $\frac{n}{m} V = v$, $\frac{s}{r} V = x$, expressing the times of the single vibrations, or the lengths of the strings

strings of the other sounds, as in Fig. 3, or to the lesser terms of the ratios in column 1 of the table, of the order of the simplicity of consonances.

14. Consequently, were the degrees of simplicity of consonances to be estimated by the frequency of the coincidences of their pulses, or the shortness of their cycles, as is commonly supposed; the unison, VIII, VIII + V, 2 VIII, 2 VIII + III, &c, whose cycles are but 1 vibration of the base, would be equally simple; and the same may be said of the 5th, VIII + III, 2 VIII + T, &c, whose several cycles are but 2 vibrations of the base; and the same also of all consonances having the same number for the lesser term of their ratios; which shews that the frequency of coincidences is, of it self, too general a character of simplicity and smoothness, and therefore an imperfect one.

SECTION IV.

*Of the ancient Systems of perfect
consonances.*

1. **I**F no other primes but 1, 2, 3 were admitted to the composition of musical ratios, a system of sounds thence resulting could have no perfect thirds; nor any perfect consonance whose vibrations are in any ratio having the number 5, or any multiple of it, for either of its terms, as 5 to 4, 10 to 9, 16 to 15, &c: it being impossible for any powers and products of the given primes 1, 2, 3 to compose any other prime or multiple of it.

2. FIG. 3. The minor tones *DE*, *GA* being thus excluded, and major tones being put in their places, every perfect major III^d will be increased by a comma, as being the difference of the tones (*a*); and every hemitone and
per-

(*a*) Sect. II. Art. 4.

perfect minor 3^d will be as much diminished; because the 4th and 5th, as *CF* and *Fc*, *cG* and *GC*, are perfect, whether 5 be admitted or not, as depending on the primes 1, 2, 3, only.

3. These diminished hemitones being called Limmas, the octave is now divided into 5 major tones and 2 limmas; as represented to the eye in Fig. 6; where the consonances whose vibrations are expressed by such high terms as the powers of 8 and 9, &c, must needs be disagreeable to the ear, according to the foregoing analogy between the agreeable smoothness of a consonance, and the simplicity of the numbers expressing the ratio of its vibrations (*b*): and that in reality they are so, any one will soon find if he pleases to try the following experiment.

4. FIG. 6. Ascending by a perfect 5th and descending by a perfect 4th alter-

(*b*) Sect. III. Art. 10.

Art. 5. HARMONICS. 33

alternately, upon an organ or harpsichord tune the following sounds, from *F* to *C*, *C* to *G*, *G* to *D*, *D* to *A*, *A* to *E*; and the octave *Ff* will then be divided into 5 major tones and 2 limmas; because the differences of those successive v^{ths} and 4^{ths} are major tones.

Then having tuned octaves to every one of those notes, try the consonances that would be perfect if the number 5 were admitted, as thirds major and minor, with their complements to the $viii^{\text{th}}$ and compounds with $viii^{\text{ths}}$; and you will find them extremely disagreeable, or out of tune as we say.

5. But if 5 be admitted among the musical primes, the ratios 10 to 9 and 16 to 15, belonging to the minor tone and the hemitone, are also admitted, and the elements that now compose the octave, are 3 major tones, 2 minor and 2 hemitones, as in Fig. 3.

PROPOSITION I.

A system of sounds whose elements or smallest intervals are tones, major and minor, and hemitones, will necessarily contain some imperfect concords.

6. Whatever be the order of those elements in any one octave, it must be the same in every one; to the end that every sound may have an octave to it, as being the best concord. And in order to have as many perfect v^{ths} as possible, and consequently $viii + v^{\text{ths}}$, which concords are the second best (c), the elements must be ranged in such order, that the contiguous couples shall make as many perfect thirds as possible, both major and minor; these being the intervals which compose the perfect v^{ths} . And that order being rightly determined, we shall have the greatest number of perfect concords of all kinds. Because the complements to the octave, of perfect thirds and v^{ths} , will

(c) See the table of the order of concords in Sect. III. Art. 5.

Art. 6. HARMONICS. 35

will also be perfect, and so will their compounds with any number of VIIIth.

Now it is observable of the seven elements T, T, T, t, t, H, H, which compose an octave, that T and H, T and t are the only couples which make perfect thirds (*d*), all the rest, T and T, t and t, t and H, H and H, making thirds imperfect by a comma, except H and H, which compose an imperfect tone, bigger than the major tone by almost a comma (*e*).

Hence either T and H, or T and t must be the outermost elements in the octave, as in the following table.

For

(*d*) Sect. 11. Art. 5 and 7.

(*e*) Putting $H = \log. \frac{16}{15} = 0.02803$

Then $2H = 2 \times \log. \frac{16}{15} = 0.05606$

And $T = \log. \frac{9}{8} = 0.05115$

Whence $2H - T = 0.00491$

And the Comma = $\log. \frac{81}{80} = 0.00540$

Difference 0.00049

For if the first element in every octave in the system be T and the seventh be H, the seventh in any octave, combined with the first in the next octave, will compose the interval H+T of a perfect minor 3^d, and thus the contiguous octaves will be joined in perfect concord.

Table of the Elements.

	&c	7	1	2	3	4	5	6	7	1&c
Cas. 1.	{	H	T + t		H + T + t				T + H	T
					t + T + H					
Cas. 2.	{	H	T + H		t + T + t				T + H	T
Cas. 2.	{	t	T + t		H + T + H				T + t	T
					t + T + H					
		t	T + H		H + T + t				T + t	T

Likewise if the first element in every octave be T and the seventh be t, here also the seventh in any octave, together with the first in the next octave, will

will compose the interval $t+T$ of a perfect major III^d , and thus the contiguous octaves will again be joined in perfect concord; and in no other case besides those two, as appears by the observation above.

Cas. I. Now if the second element be t , the first joined to it composes the perfect major III^d , $T+t$. And if the sixth element be T , the seventh joined to it will compose the perfect minor third $T+H$.

Two of the seven elements being thus disposed of, at each end of the octave, the contiguous couples of the remaining three, cannot compose perfect thirds in any order different from this, $H+T+t$, or its reverse $t+T+H$; both which being transferred into the interval between those extreme couples, shew, that the elements in the second and third places, compose either the imperfect minor third $t+H$, or the imperfect major third $t+t$.

If H be the second element, as in the third rank of the table, the first couple does now compose the perfect minor third $T+H$, and the last being also $T+H$, as before, the three remaining elements must have this order $t+T+t$, to make perfect thirds of their contiguous couples; and being thus transferred into the interval between those extreme couples, they shew, that the second and third elements do again compose an imperfect minor third $H+t$.

Cas. 2. Here also the sixth element must be T, since no other joined to the seventh can make a perfect third, as $T+t$.

Now if the second element be t , this joined to the first makes the perfect major third $T+t$. And two of the seven elements being thus situated at each end of the octave, the contiguous couples of the remaining three, cannot compose the intervals of perfect thirds in any order different from this,

H+T

H + T + H; which being transferred into the interval between the extreme couples, shews, that the second and third elements do here also compose the interval $t + H$ of an imperfect minor third.

If H be the second element, as in the next lower ranks, then the first couple compose the interval T + H of a perfect minor 3^d, and the last couple being T + t as before, the three remaining elements must have this order, $t + T + H$, or its reverse, H + T + t , for the reason above; and being thus transferred into the middle interval, they shew, that the elements in the second and third places do again compose an imperfect minor third, H + t , or else an imperfect tone H + H; which being joined to the major tone on either side of it, composes an imperfect major third, greater than $t + T$ by almost two commas, as appears by the preliminary observation.

Now any one of those imperfect minor thirds, $t + H$, together with the contiguous perfect major III^d , composes a fifth equally imperfect, and so does the imperfect major third $t + t$ with the perfect minor third next to it. And the complements to the $VIII^{th}$ of these imperfect concords, as well as their compounds with $VIII^{ths}$, are also equally imperfect, which proves the proposition. For having shewn the necessary defects in those six arrangements of the elements, we are freed from the trouble of considering the rest (*f*). Q. E. D.

7. *Coroll.* Of those six arrangements of the elements, the first and fifth in the table are equally good, and better than any of the rest, as producing as many perfect thirds, and a greater number of perfect v^{ths} .

FIG. 7. In order to enumerate them with certainty and ease, if the circumference

(*f*) Mr. *De Moivre's* general corollary to the xvi problem of his *Doctrine of Chances*, gives 210 permutations of these seven things, T, T, T, t, t, H, H.

ference of a circle, be divided into seven arches, *CD, DE, EF, FG, GA, AB, BC*, proportional to *T, t, H, T, t, T, H*, placed in the respective angles at the the center; they and their sums, whether smaller or greater than the circumference, here considered as a continued spiral, will represent all the intervals in a system composed of any number of octaves, and the corresponding intervals in different octaves will be denoted by the same arch and letters: as appears by conceiving the base of the third Figure coiled round into the circumference of a circle, equal to the line *Cc* or *cc'* &c.

In this sense then we have only three major III^{ds}, *CE, FA, GB*, and they all perfect; and four minor thirds, *DF, EG, AC, BD*, the first of which being composed of *t+H*, instead of *T+H*, is too small by a comma; and six fifths, *FAC, CEG, GBD, DFA, ACE, EGB*, all perfect but *DFA*, which being composed of the defective minor third *DF* and
the

the perfect major III^d *FA*, is too small by a comma.

These imperfections being caused by the contiguity of *t* and *H* in the cycle of the elements, cannot be avoided while the hemitones are separated; there being but 3 major tones in the cycle; and if they be joined, as in Fig. 12, the consequences will be worse.

The rest will appear by enumerating the thirds and fifths in the 8th, 9th, 10th, 11th, and 12th Figures, made according to the other five arrangements in the Table of Elements (*g*).

8. Now if any one pleases to try the following experiment, he will find what effect these imperfect fifths and fourths and their compounds with

VIII^{ths},

(*g*) Sir Isaac Newton happily discovered, (Optics Book 1, Part 2, Prop. 3) that the breadths of the seven primary colours in the sun's image, produced by the refraction of his rays through a prism, are proportional to the seven differences of the lengths of the eight musical strings *D, E, F, G, A, B, C, d*, when the intervals of their sounds are *T, H, t, T, t, H, T*:
which

Art. 8. HARMONICS. 43

VIII^{ths}, will have upon the ear; that of the thirds and sixths having been tryed before (*b*).

In Fig. 3, tune upwards from *C* the two perfect v^{ths} *CG*, *Gd*, and the perfect XVIIth, or 2VIII + III, *Ce'*, then downwards the vth *e'a*, and the intermediate fifth *ad* will be too little by a comma, as including the imperfect minor third *df*. And by tuning an eighth below *a* we have the imperfect fourth *Ad*, too large by a comma.

9. The disagreeable effect of this fifth and fourth in every octave, and of their compounds with VIII^{ths}, and also of the third and sixth in every octave and their compounds with VIII^{ths}, and of many more such imperfect concords, when

which order is remarkably regular; but though it agrees best with the prismatic colours, it is not the properest for a system of concords, as producing one major third, two minor thirds and two fifths severally imperfect by a comma. See Fig. 13. N^o. 2.

(*b*) Sect. IV. Art. 4.

when the usual flat and sharp sounds are added to complete the system, has obliged practical musicians, long ago, to distribute that comma, wanting in the fifth da , equally among all the four v^{ths} , CG , Gd , da , ae' , contained in the $xvii^{\text{th}}$ Ce' . Such distribution is therefore called the Participation, or Temperament of the system, and when rightly adjusted is undoubtedly the finest improvement in harmonics.

10. If it be asked why no more primes than 1, 2, 3, 5 are admitted into musical ratios; one reason is, that consonances whose vibrations are in ratios whose terms involve 7, 11, 13, &c, *cæteris paribus* would be less simple and harmonious (*b*) than those whose ratios involve the lesser primes only.

Another reason is this; as perfect fifths and other intervals resulting from
the

(*b*) Sect. III and Table of the order of the simplicity of consonances.

Art. 11. HARMONICS. 45

the number 3, make the Schism of a comma with the perfect thirds and other intervals resulting from the number 5, so such intervals as result from 7, 11, 13, &c would make other schisms with both those kinds of intervals.

11. The Greek musicians, after dividing an octave into two 4^{ths}, with the diazeuctic or major tone in the middle between them, and admitting many primes to the composition of musical ratios, subdivided the 4th into three intervals of various magnitudes, placed in various orders, by which they distinguished their Kinds of Tetrachords (i). Two of them have occurred in this Section. The first, or $\frac{3}{4} = \frac{8}{9} \times \frac{8}{9} \times \frac{243}{256}$, answering to the 4th = T + T + L, in Fig. 6, is *Ptolemy's Genus Diatonum diatonicum*, and results from that division of a Monochord which bears the name of
of

(i) Dr. Wallis has given a table of them in his Appendix to *Ptolemy's Harmonics*, Oper. Math. vol. III. pag. 166.

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of *Euclid's* Section of the Canon; the
second, or $\frac{3}{4} = \frac{8}{9} \times \frac{9}{10} \times \frac{15}{16}$, answering
to the 4th = T + t + H, in Fig. 3, is
Ptolemy's Diatonum intensum.

12. Since the invention of a temperament, all those ancient systems have justly been laid aside, as being unfit for the execution of musical compositions in several parts. But to conclude from thence that the ancients had no music in parts, would be a very weak inference. Because it is much easier for practical musicians to follow the judgment of the ear, which naturally leads to an occasional temperament of any disagreeable concords, than to learn and put in practice the theories of philosophers (*k*): And because we are assured from history, that experience
and

(*k*) It may not be amiss to add the opinion of the famous *Salinas*. Sed unum hoc omnes scire volo, instrumenta quibus antiqui utebantur, consonantias habuisse imperfectas, ut ea, quibus nunc utimur. Neque enim aliter modulatio convenienter exerceri poterat.

and necessity did introduce something of a temperament, before the reason of it was discovered, and the method and measure of it was reduced to a regular theory, as in the following proposition.

SECTION

terat. Quod si de hac consonantiarum imperfectione, neque *Ptolemæus*, neque alius ex antiquis musicis mentionem fecisse reperitur, causam potissimam esse crediderim, quòd ad practicos eam pertinere arbitrentur; quoniam sensu duce solùm, non arte aut ratione semper fieri solita sit: cujus plenissimum et evidentissimum testimonium reperitur apud *Galenum*, libro primo De Sanitate tuenda, capite quinto; ubi magnam esse latitudinem sanitatis ostendere volens, sic inquit: Καὶ τί θαυμαστὸν εἰ τὴν εὐκрасίαν εἰς ἰκανὸν ἐκτείνουσι πλάτῃ ἀπαντες, ὅπου γὰρ καὶ ἐν αὐταῖς λύραις εὐαρμοσίαν, τὴν μὲν ἀκριβεστάτην δὴπερ, μίαν καὶ ἀτμητοῦ ὑπάρχειν εἰκὸς· ἢ μὲν τοι γ' εἰς χρεῖαν ἰούσα, πλάτῃ ἔχει. Πολλάκις γ' οὖν ἠρμόσθαι δοκῶσαν ἄριστα λύραν, ἕτερος μουσικὸς ἀκριβῶς ἐφηρμόσατο· πανταχῶ γὰρ ἡ αἰσθησις ἡμῖν ἐστὶ κριτήριον, ὡς πρὸς τὰς ἐν τῷ βίῳ χρεῖας. hoc est, *Quid mirum, si Eucrasiam in satis amplam latitudinem extendunt universi; quando et in lyris consonantiam ipsam, quæ summa exactissimaque sit, unicam atque inflexibilem esse probabile sit, et quæ in usus hominum venit, certe latitudinem habeat. Sæpe namque, [quam] percommode temperasse lyram videaris, alter superveniens musicus exactius temperavit: siquidem nobis ad omnia vitæ munera sensus ubique judex est.* Ex quibus

SECTION V.

Of the temperaments of perfect intervals and their synchronous variations.

PROPOSITION II.

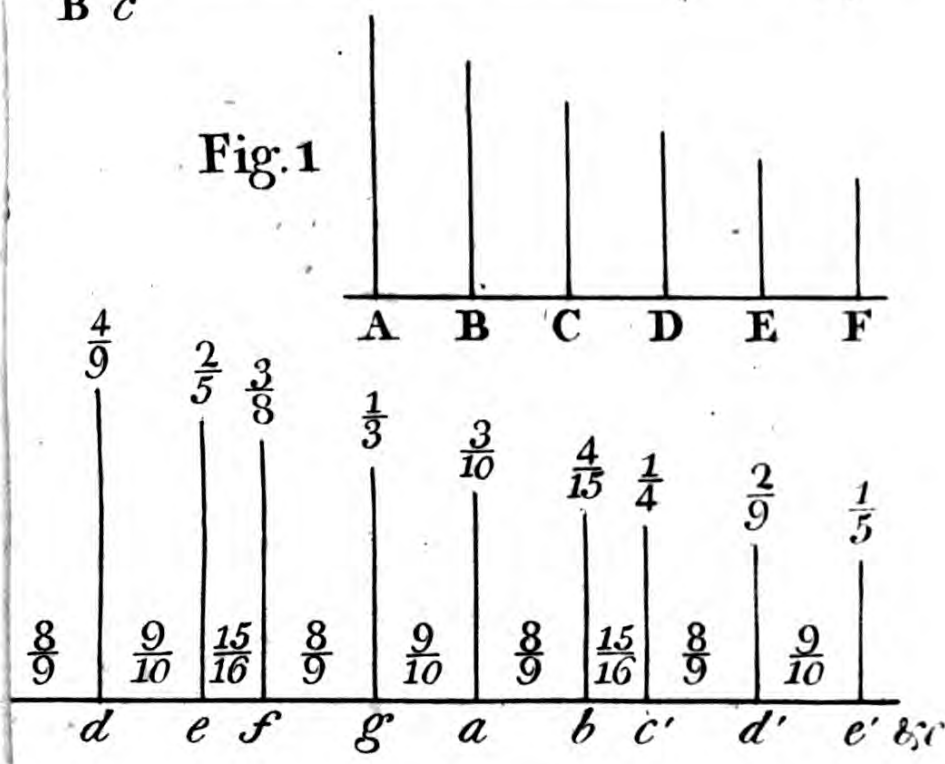
To reduce the diatonic system of perfect consonances to a tempered system of mean tones.

FIG. 13. When the elements are ranged in this order, T, t, H, T, t, T, H, or this, t, T, H, T, t, T, H, which two we shewed to be the best (1), and the arches CD, DE, EF, FG, GA, AB, BC, are proportional to them, let the major 111^d CE, situated between the two hemitones,

Galenī verbis liquido constat, consonantias, quibus in musicis utebantur instrumentis, jam tunc imperfectas esse, quin potius et fuisse semper et semper esse futuras. De Musicā lib. III. cap. 14. True; but did they know, that all the concords cannot be tuned perfect, and why they cannot?

(1) Sect. iv. Art. 7.

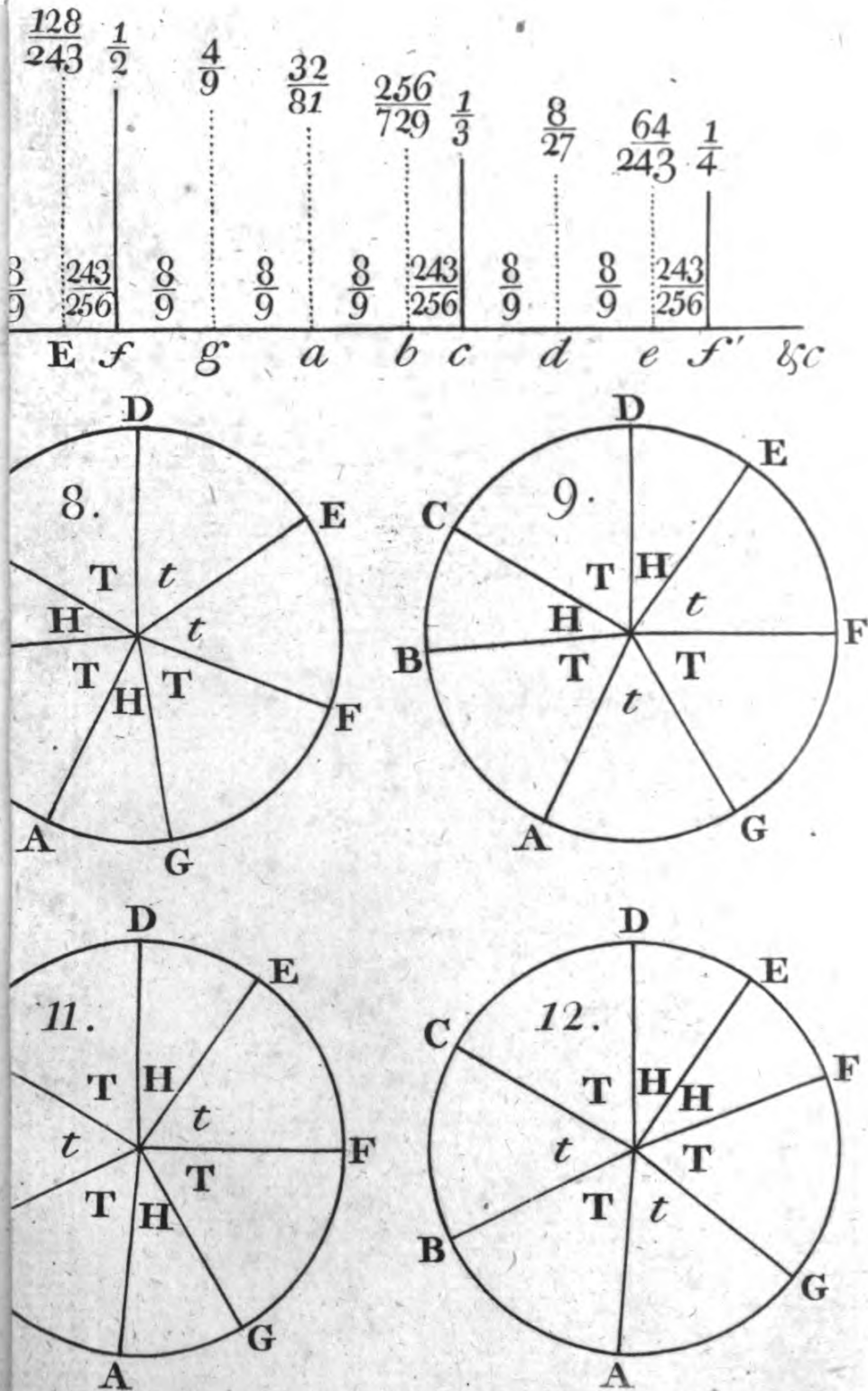
$\frac{8}{15}$ $\frac{1}{2}$ Fig. 2
 B C



1	
1	
2	
3	
4	
5	
6	
7	
8	
2	
3	
3	
4	
3	
5	



Fig. 6.





mitones, be bisected in d ; and let the other two major tones, FG , AB , be diminished at both ends by the intervals Ff , Gg , Aa , Bb , severally equal to half Dd ; and the octave will then be divided into five mean tones and two limmas, each limma being bigger than the hemitone by a quarter of a comma.

For the interval Dd being half the difference between the major and minor tones, CD , DE , is half a comma (m), and therefore the new tone Cd or dE is an arithmetical mean between them. And each of the temperaments Ff , Gg , Aa , Bb , being made equal to half Dd or a quarter of a comma, it appears that every major tone is diminished by half a comma, and that every minor tone is as much increased, which reduces all the tones to an equality. And the limmas bC , Ef apparently exceed the hemitones by a quarter of a comma apiece. Q. E. D.

Coroll.

(m) Sect. II. Art. 4.

D

Coroll. In the system of mean tones every perfect v^{th} is diminished by a quarter of a comma: as will appear by going round the 13^{th} figure, and comparing the tempered v^{ths} , $fC, Cg, g d, da, aE, Eb$, with the perfect ones.

This is usually called the vulgar temperament and might be proved several other ways independent of the first and second propositions (*n*).

P R O-

(*n*) *Salinas* tells us, that when he was at Rome, he found the musicians used a temperament there, though they understood not the reason and true measure of it, till he first discovered it, and *Zarlino* published it soon after; first in his *Dimonstrazioni Harmoniche, Ragionamento quinto, proposta 1^{ma}*, and after that, in his *Institutioni Harmoniche, part. 2. cap. 43*.

After his return into Spain, *Salinas* applied himself to the latin and greek languages, and caused all the ancient musicians to be read to him, for he was blind; and in 1577 he published his learned work upon music of all sorts; where treating of three different temperaments of a system, he prefers the diminution of the v^{th} by a quarter of a comma to the other two, which he says are peculiar to certain instruments. *De Musicâ Lib. III. cap. 22*.

Dechales says, that *Guido Aretinus*, the inventor of the present Scale of musical Notes, was also reputed the inventor of that temperament: Ipse nulla
ha-

PROPOSITION III.

If the five mean tones and the two limmas, that compose a perfect octave, be changed into five other equal tones and two equal limmas, of any indeterminate magnitudes; the synchronous Variations of the limma L, the mean tone M, and of every interval composed of any numbers of them, are all exhibited in the following table, by the numbers and signs of any small indeterminate interval v : And are the same quantities as the variations of the temperaments of the respective perfect intervals.

For

habita ratione toni majoris et minoris, hunc unius quintæ defectum aliis omnibus quintis communicat, et quasi dividit, ita ut nulla deficiat, nisi quarta parte commatis. Hoc systema, quod valde commodum est, dicitur esse *Aretini*. *Cursus Mathem.* Tom. I. pag. 62. *De Progressu Musicæ*, & Tom. IV. pag. 15. But that opinion wants confirmation, especially as *Dechales* makes no mention of the claims of *Zarlino* and *Salinas* to that invention; for it seems they had a dispute about it.

2^d	II^d	3^d	III^d	4^{th}	IV^{th}
L	M	L+M	2M	L+2M	3M
$5v$	$-2v$	$3v$	$-4v$	v	$-6v$
$-5v$	$2v$	$-3v$	$4v$	$-v$	$6v$
L+5M	2L+4M	L+4M	2L+3M	L+3M	2L+2M
VII^{th}	7^{th}	VI^{th}	6^{th}	V^{th}	5^{th}

For since the $VIII^{th} = 2L + 5M$ is supposed invariable, if the variation of L be put equal to $5v$, as in the table, that of $2L$ is $10v$, and that of $5M$, as being the complement of $2L$ to the $VIII^{th}$, is $-10v$; whence the variation of M is $-2v$.

Consequently the variation of the mean 3^d , L+M, is $5v - 2v = 3v$, and that of the III^d , 2M, is $-4v$, and that of the mean 4^{th} , L+2M, is $5v - 4v = v$, and that of the mean IV^{th} , 3M, is $-6v$.

The variations of the intervals in the lower half of the table, are respectively equal to those in the upper half, but have contrary signs; the corresponding in-

intervals being complements to the invariable octave.

For which reason the compounds of every one of those intervals with any number of octaves, have respectively the same variations both in quantity and quality.

And if the sign of the variation of any one interval be changed, the signs of all the rest will also be changed; because their quantities will vanish all together when v or any one multiple of it vanishes.

As to the second part of the proposition, it will appear in Fig. 13, that any variation v of the mean interval $CdEf$ is the same in quantity as the variation of the temperament Ff of the perfect interval $CdEF$: and the like is evident in any other instance. Q. E. D.

Coroll. 1. It is observable in the table, that the variations of all the major mean intervals II^d, III^d, IVth, Vth, VIth, VIIth, have the same sign, and those of the minor intervals the contrary sign.

Coroll. 2. Having extended the circumference $CdEfgabC$ of Fig. 13 into a right line, as in Fig. 14, at the points d, E, g, a, b , that terminate the major mean intervals $II^d, III^d, V^{th}, VI^{th}, VII^{th}$, measured from C , (and the minor too measured from c the other extreme of the octave Cc) place the respective tabular numbers 2, 4, 1, 3, 5, denoting the proportions of their synchronous variations; and in Fig. 15 divide any given line 06 into 6 equal parts, at the points 1, 2, 3, 4, 5; then conceive the 14th Fig. transferred to the 15th five several times, into five parallel positions, so that the several points 1, 2, 3, 4, 5 in each Figure may coincide. And it will be evident, by coroll. 1, that any right line $0vvvv$, drawn from 0 , terminates the synchronous variations, $1v, 2v, 3v, 4v, 5v$, of those mean intervals, measured from their respective origins 1, 2, 3, 4, 5; and that these are also the synchronous variations of the temperaments of the respective perfect intervals, and

and of their complements to the VIIIth and compounds with VIIIths, that is, of all the intervals in the system.

For as to the mean IVth fb , Fig. 14, its contemporary variation in Fig. 15, will be the line $6v$ in the sixth parallel $F6B'F'$, when its temperament $B'6$ or $B'b'$ is taken equal to twice Bb and placed the same way from its origin b' . Because in Fig. 14 the temperament of the perfect IVth FB is $Ff + Bb = 2Bb$.

As want of room in Fig. 15 will not permit the several intervals, even less than one octave, to be represented in their due proportions to $G1$, the quarter of the comma, which is but the 223^d part of an octave, we must conceive them continued far beyond the margin of the paper.

Coroll. 3. When the III^d is perfect, the temperaments of the Vth and VIth are severally $\frac{1}{4}$ of a comma, the former in defect, the latter in excess: and if

either of them be made less, the other will be greater than $\frac{1}{4}$ comma.

FIG. 15, 16. For when the Temperer $Ovvv$ falls upon E , the III^{d} CE is perfect, and the tempered v^{th} CI is less than the perfect v^{th} CG by GI , and the tempered VI^{th} $C3$ is bigger than the perfect VI^{th} CA by $A3 = GI = \frac{1}{4}$ comma.

Hence when Av , any other temperament of the VI^{th} , is less than $A3$ or $\frac{1}{4}$ comma, Gv the corresponding temperament of the v^{th} , is greater than GI or $\frac{1}{4}$ comma: and on the contrary, when Gv is less than GI , the respective Av is bigger than $A3$. And whatever be the magnitudes of these temperaments of the v^{th} and VI^{th} , those of their complements to the VIII^{th} and compounds with VIII^{th} s are the same.

Coroll. 4. When the VI^{th} is perfect, the temperaments of the v^{th} and III^{d} are severally $\frac{1}{3}$ comma, and are both negative.

FIG.

FIG. 16. For when the temperer $Ovvv$ falls upon the line $OHAI$, the temperament of the v_1^{th} vanishes, and those of the v^{th} and III^{d} are GH and EI , and are equal. For the equal lines GI , A_3 and equal triangles GEO , AOE shew, that the line GE is parallel to AO ; whence GH is equal to EI , and the similar triangles IEO , A_3O give $IE = \frac{4}{3} A_3 = \frac{4}{3} \times \frac{1}{4} \text{comma} = \frac{1}{3} \text{comma}$.

Coroll. 5. When the v^{th} is perfect, the temperaments of the v_1^{th} and III^{d} are severally equal to a comma in excess.

For when the temperer $Ovvv$ falls upon the line $OGKL$, the temperament of the v^{th} vanishes, and those of the v_1^{th} and III^{d} are now AK and EL , which are equal, because of the parallelograms $AEGO$, $AELK$; and EL is $= 4 \times GI$ or four quarters of a comma.

Coroll. 6. When the temperer $Ovvv$ falls within the angle AOE , the temp^t. $v^{\text{th}} = \text{temp}^{\text{t}}. v_1^{\text{th}} + \text{temp}^{\text{t}}. \text{III}^{\text{d}}$, that is, the line $Gv = Av + Ev$, or the lines

GI

$GI + Iv = A3 - 3v + Ev$, that is, putting the letter v for the line Iv , $\frac{1}{4}c + v = \frac{1}{4}c - 3v + 4v$, which is evidently true.

Coroll. 7. When the temperer $Ovvv$ falls within the angle EOG , the temp^t. $VI^{\text{th}} = \text{temp}^t.v^{\text{th}} + \text{temp}^t.III^{\text{d}}$, that is, the line $Av = Gv + Ev$, or the lines $A3 + 3v = GI - Iv + Ev$, that is, putting $v =$ the line Iv , $\frac{1}{4}c + 3v = \frac{1}{4}c - v + 4v$, which is true.

Coroll. 8. When the temperer falls any where out of the angle AOG , the temp^t. $III^{\text{d}} = \text{temp}^t.v^{\text{th}} + \text{temp}^t.VI^{\text{th}}$, that is, when it falls beyond the side AO , the temp^t. $EI + Iv = GH + Hv + Av$, or putting the letter v for the line Hv , $\frac{1}{3}c + 4v = \frac{1}{3}c + v + 3v$, which is true: and when the temperer falls beyond the other side OG , the said temp^t. $EL + Lv = Gv + AK + Kv$, that is, putting v for the line Gv , $c + 4v = v + c + 3v$, which is true.

Coroll.

Coroll. 9. The sum of the temperaments of the v^{th} and v_1^{th} is $\frac{1}{2}$ a comma when the III^{d} is perfect; is less than $\frac{1}{2}$ a comma by $\frac{1}{2}$ the temperament of the III^{d} when flattened; and greater than $\frac{1}{2}$ a comma by $\frac{1}{2}$ the temperament of the III^{d} when sharpened.

For in the first case the said sum is $G_1 + A_3$; in the second, it is $G_1 + Iv + A_3 - 3v = G_1 + A_3 - 2v$; and in the third, it is $G_1 - Iv + A_3 + 3v = G_1 + A_3 + 2v$; in which latter cases the temperament of the III^{d} is $4v$.

Coroll. 10. Hence the sum of the temperaments of all the concords is less when the III^{ds} are flattened, than the like sum when the III^{ds} are equally sharpened; but it is the least of all when the III^{ds} are perfect, as in the system of mean tones (a).

Scholium.

From the third and tenth corollaries, I think we might justly pronounce the
 (a) Prop. II. system

system of mean tones to be the best possible, were it evident that equal temperaments cause different concords to be equally disagreeable to the ear (*a*).

But if it shall appear, that the vi^{th} and 3^{d} and their compounds with octaves, are more disagreeable in their kind, than the v^{th} and 4^{th} and their compounds with octaves, all being equally tempered, as in that system; will it not follow, that the temperament of the former Parcel of concords should be smaller than that of the latter, to make them all as equally harmonious as possible, without spoiling the harmony of the iii^{d} and 6^{th} and their compounds with octaves; which
third

(*a*) Mr. *Huygens* has pronounced it the best, in saying that the musicians in the other planets may know perhaps, *cur optimum sit temperamentum in chordarum systemate, cum ex diapente quarta pars commatis ubique deciditur*; *Cosmotheoros* pag. 76; but has given us no reason for his assertion, either in that incomparable book or in his *Harmonic Cycle*; where he only appeals to the approbation and practice of musicians and refers to the demonstrations of *Zarlino*

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third parcel makes up the sum of all the concords in the system.

For if it be the immediate succession of a worse harmony to a better, as in instruments badly tuned, which chiefly offends the ear; it must be allowed, that a system would be the better, *cæteris paribus*, for having all the concords as equally harmonious in their kinds, as the nature and properties of numbers will permit.

In order to resolve those questions upon philosophical principles, and to determine the temperament of a given system, that shall cause all the concords, at a medium of one with another, to be equally, and the most harmonious
in

lino and *Salinas*. But neither of these celebrated authors do any thing more, if I rightly remember, (for I have not the books now by me) than reduce the Diatonic system of perfect consonances to that of mean tones, by distributing the schism of a whole comma into quarters; not at all considering, whether those equal temperaments have the same, or a different effect upon the several concords.

in their several kinds, I found it necessary to make a thorough search into the abstract nature and properties of tempered consonances; and thence to derive their effects upon our organs of hearing: A large field of harmonics hitherto uncultivated.

But before I enter upon it, it will be convenient to finish this section with a determination of the least sum of any three temperaments in different parcels, when any two of them have any given ratio.

PROPOSITION IV.

To find a set of temperaments of the vth, vith and iii^d upon these conditions; that those of the vth and vith shall have the given ratio of r to s, and the sum of all three shall be the least possible.

Part of the 17th and 18th Figures being constructed as before, from *A* towards *K* take $AM : GI :: s : r$, and
through

through the interfection p of the lines AG , $M\Gamma$, draw the temperer $Orst$; I say Gr , As , Et are the temperaments required.

For by the fimilar triangles Grp , Asp , and $G\Gamma p$, AMp , we have $Gr : As :: (Gp : Ap :: G\Gamma : AM ::) r : s$ by construction, as required by the first condition.

Again, in the same line MAC take $AN = AM$, and through the interfection P of the lines AG , $N\Gamma$ produced, draw another temperer $ORST$; and by the fimilar triangles GRP , ASP , and $G\Gamma P$, ANP , we have $GR : AS :: (GP : AP :: G\Gamma : AN \text{ or } AM ::) r : s$ by construction, which likewise answers the first condition; and it is easy to understand, that no other temperers but those two can answer that condition.

Now whatever be the quantity and quality of the given ratio r to s , I say the sum $Gr + As + Et$ is less than $GR + AS + ET$.

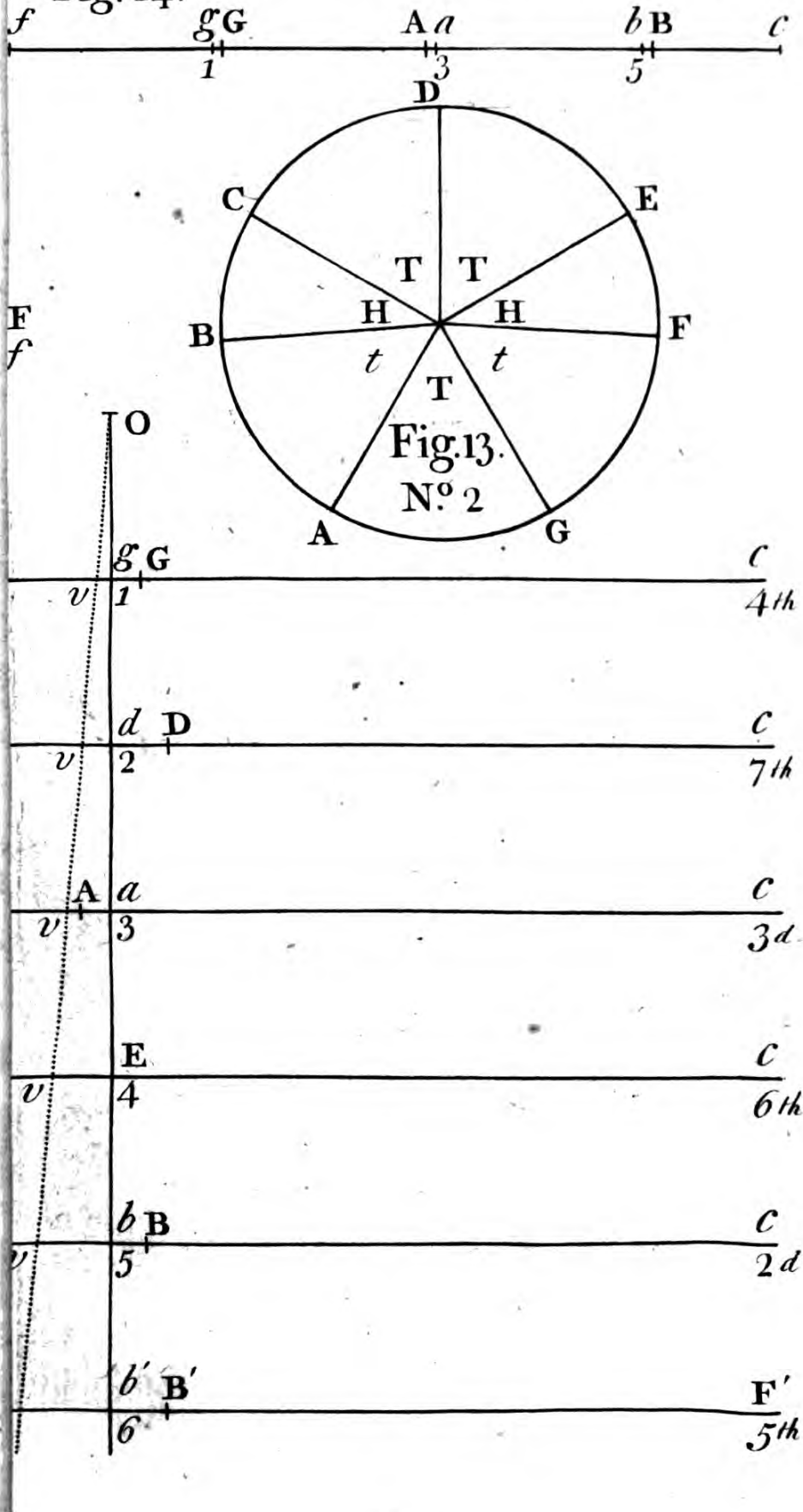
Case

Case 1. Fig. 17. For when r is bigger than s , or the ratio of r to s , or of $G\text{I}$ or A_3 to AM or AN , is a ratio of majority, the temperers Op , OP fall within the angles AOE , AOC respectively; as appears by the construction. Whence, by coroll. 6 and 8. prop. III, $Gr = As + Et$, and $ET = GR + AS$; and therefore $Gr + As + Et : GR + AS + ET :: Gr : ET$, which is a ratio of minority, because Gr is less than GH or EI (*b*) and EI less than ET .

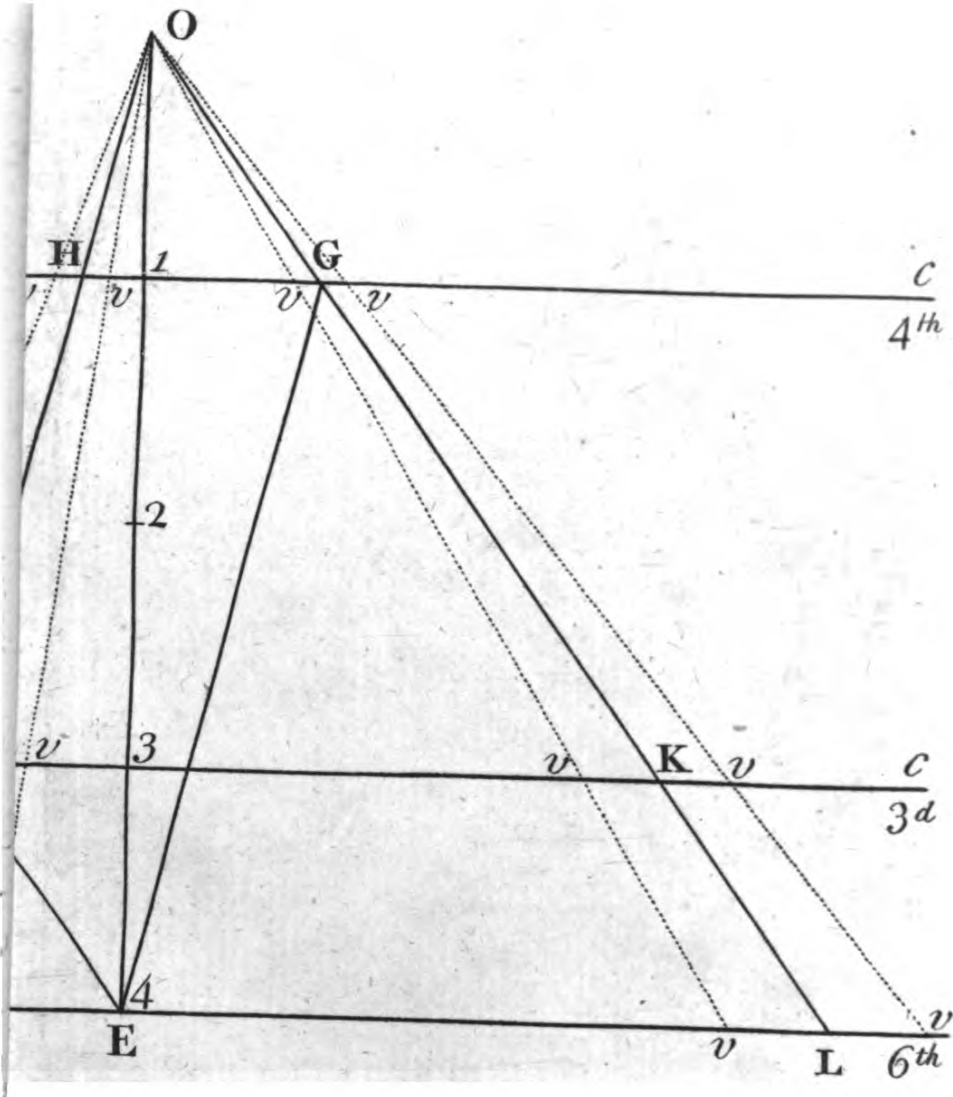
Case 2. Fig. 18. When r is less than s , or the ratio of r to s , or of $G\text{I}$ to AM or AN in a ratio of minority, the temperers Op , OP fall within the angles EOG , AOC respectively; as appears by the construction. Whence, by coroll. 7 and 8. prop. III, $As = Gr + Et$, and $ET = GR + AS$, and therefore $Gr + As + Et : GR + AS + ET :: As : ET$, which is a ratio of minority; because $Gr : As :: r : s :: GR : AS$, whence, as Gr is less

(*b*) See coroll. 4. Prop. III.

Fig. 14.









Prop. IV. HARMONICS. 65

less than GR , so As is less AS , which is less than IT , which is less ET .

Case 3. Fig. 17 and 18. When r to s , or GI to AM or AN , is the ratio of equality, the temperer $Orst$ coincides with the line OE , and $ORST$ is parallel to GA ; whence it is plain, that the sum of the temperaments $GI+As+o$, is less than $GR+AS+ET$, as required.

Q. E. D.

Coroll. Putting c for the comma EL or four GI , when the temp^t. v : temp^t. vi :: r : s , the required temperaments of the v , vi and iii are, $Gr = \frac{r}{3r+s} c$, $As = \frac{s}{3r+s} c$ and $Et = \frac{r-s}{3r+s} c$. And according as r is bigger or less than s , the temperer $Orst$ falls within the angle AOE or EOG .

FIG. 17 and 18. For, $As : Gr :: s : r$, and $As : 3Gr$ or $sK :: s : 3r$, and $As : As + sK$ or $c(a) :: s : s + 3r$. Whence $As = \frac{s}{3r+s} c$, and $Gr = \frac{r}{s} As = \frac{r}{3r+s} c$,
and

(a) See Dem. coroll. 5. prop. III.

E

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 and in the angle AOE , $Et = Gr - As$
 $= \frac{r-s}{3r+s}c$, but in EOG , $Et = As - Gr$,
 by the equations in case 1, 2.

PROPOSITION V.

*To find a set of temperaments of the v^{th} ,
 vi^{th} and iii^{d} upon these conditions;
 that those of the v^{th} and iii^{d} shall
 have the given ratio of t to r , and
 the sum of all three shall be the least
 possible.*

FIG. 19, 20. If t to r be a ratio of mi-
 nority, or of equality, or even of majority
 less than 1 to $\frac{1+\sqrt{33}}{8}$ or 0.843070 &c,
 from E towards I take $EM : GI :: t : r$,
 and through the intersection p of the
 lines MI , GE produced, draw the tem-
 perer $Orst$, and the required tempera-
 ments will be Gr , As , Et .

But if the ratio of t to r be greater
 than 1 to 0.843070 &c, in Fig. 20,
 from E towards L take $EN : GI :: t : r$,
 and through the intersection P of the
 lines

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lines N_1, GE , draw the temperer $ORST$, and the required temperaments will be GR, AS, ET .

And if $t : r :: 1 : 0.843070 \&c$, the required temperaments will be Gr, As, Et , or GR, AS, ET , their sums being equal.

In the first case, Fig. 19, take $EN = EM$, and in the second, Fig. 20, $EM = EN$; and through the intersections P, p of the lines N_1, M_1 with GE , draw two more temperers $ORST, Orst$.

Then by the similar triangles $Grp, Et p$ and G_1p, EMp , we have $Gr : Et :: (Gp : Ep :: G_1 : EM ::) r : t$ by construction, as required by the first condition.

Again, by the similar triangles GRP, ETP and G_1P, ENP , we have $GR : ET :: (GP : EP :: G_1 : EN ::) r : t$ by construction, which also answers the second condition; and it is easy to understand that no other temperers but those can answer that condition.

Case I. Fig. 19. Now when t is to r , and therefore EM or EN to GI in a ratio of minority, the temperers Op , OP fall within the angles AOE , EOG respectively by the construction. Whence, by coroll. 6 and 7 prop. 3, $Gr = As + Et$ and $AS = GR + ET$.

But $Gr : Et :: r : t$, and $Gr : \frac{1}{4} Et$ or $rI :: r : \frac{1}{4} t$, and $Gr : Gr - rI$ or $\frac{1}{4} c :: r : r - \frac{1}{4} t :: 4r : 4r - t$. Whence $Gr = \frac{r}{4r - t} c$, and $Et = \frac{t}{r} Gr = \frac{t}{4r - t} c$, and $As = Gr - Et = \frac{r - t}{4r - t} c$, by the equation in the last paragraph.

Likewise $GR : ET :: r : t$, and $GR : \frac{1}{4} ET$ or $Ri :: r : \frac{1}{4} t$, and $GR : GR + Ri$ or $\frac{1}{4} c :: r : r + \frac{1}{4} t :: 4r : 4r + t$. Whence $GR = \frac{r}{4r + t} c$, and $ET = \frac{t}{r} GR = \frac{t}{4r + t} c$ and $AS = GR + ET = \frac{r + t}{4r + t} c$, by the equation above.

There-

Therefore $Gr + As + Et : GR + AS + ET :: Gr : AS :: \frac{r}{4r-t} : \frac{r+t}{4r+t} :: 4rr + rt : 4rr + rt, + 2rt - tt$, which is a ratio of minority; because t being less than r , tt is less than $2rt$.

Case 2. Fig. 20. When t to r , and therefore EM or EN to GI , is a ratio of majority, the temperers Opt , OPT fall within the angles AOC , EOG respectively; as appears by the construction. Whence, by coroll. 8 and 7 prop. 3, $Et = Gr + As$ and $AS = GR + ET$.

In which case the theorems for the values of Gr , As , Et , GR , AS , ET are the same as before.

Therefore $Gr + As + Et : GR + AS + ET :: Et : AS :: \frac{t}{4r-t} : \frac{r+t}{4r+t} :: 4rt + tt : (4rr + 4rt - rt - tt \text{ or } 4rt + tt, + 4rr - rt - 2tt$, which is a ratio of minority, except either when $4rr - rt - 2tt = 0$, or $\frac{4rr - rt - 2tt}{4tt} = 0$, or

E 3

$\frac{rr}{tt}$

$\frac{rr}{tt} - \frac{r}{4t} - \frac{1}{2} = 0$, which gives $\frac{r}{t} = \frac{1 + \sqrt{33}}{8} = 0.843070 \&c$ (a); or when

$4rr - rt - 2tt$, or $\frac{rr}{tt} - \frac{r}{4t} - \frac{1}{2}$ is negative, and consequently $\frac{r}{t}$ is less than 0.843070 &c (b), or the ratio of t to r is greater than 1 to 0.843070 &c.

In the first case either $Gr + As + Et$ or $GR + AS + ET$, as being equal, are the required temperaments; in the second the latter only, as being less than the former.

Case 3. When $t = r$, we have EM or $EN = GI$; therefore the intersection

p

(a) For supposing $\frac{r}{t} = \frac{x}{1} = x$, we have $\frac{rr}{tt} - \frac{r}{4t} - \frac{1}{2} = xx - \frac{1}{4}x - \frac{1}{2} = 0$. Whence $xx - \frac{1}{4}x = \frac{1}{2}$ and $xx - \frac{1}{4}x + \frac{1}{8} \times \frac{1}{8} = \frac{1}{8} \times \frac{1}{8} + \frac{1}{2} = \frac{1}{8} \times \frac{1}{8} + \frac{32}{8 \times 8} = \frac{33}{8 \times 8}$; whose square roots are $x - \frac{1}{8} = \frac{\pm \sqrt{33}}{8}$; whence x or $\frac{r}{t} = \frac{1 \pm \sqrt{33}}{8} = \frac{1 \pm 5.744562 \&c}{8}$.

(b) For

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p is removed to an infinite distance, and the temperer $Orst$ coincides with $OHAL$. Hence $Gr + As + Et$ becomes $= GH + 0 + EL$, and is to $GR + AS + ET :: 5 : 6$, a ratio of minority, produced by putting $t = r$ in the terms of that ratio in case I or 2. Q. E. D.

Coroll. When the temp^t. v : temp^t. III :: $r : t$, if $\frac{r}{t}$ be bigger than 0.843070 &c, the required temperaments of the v, VI and III are, $Gr = \frac{r}{4r-t}c$, $As = \frac{r-t}{4r-t}c$, $Et = \frac{t}{4r-t}c$. And the temperer $Orst$ falls within the angle AOE or AOC , according as r is bigger or less than t .

But

(b) For since the root 0.843070 &c, when substituted for x , will make the value of $xx - \frac{1}{4}x - \frac{1}{2}$, or of $x - \frac{1}{4} - \frac{1}{2x} = 0$; a smaller number substituted for x , will produce a negative value of the latter, and consequently of the former quantity.

But if $\frac{r}{t}$ be less than 0.84307 &c, they are $GR = \frac{r}{4r+t} c$, $AS = \frac{r+t}{4r+t} c$, $ET = \frac{t}{4r+t}$; and the temperer $ORST$ falls within the angle EOG .

And if $\frac{r}{t} = 0.843070$ &c, their sums are equal and either of them answers the problem.

PROPOSITION VI.

To find a set of temperaments of the Vth, VIth and III^d upon these conditions, that those of the VIth and III^d shall have the given ratio of s to t, and the sum of all three shall be the least possible.

FIG. 21 and 22. From E towards C take $EM : A_3 :: t : s$ and through the intersection p of the lines M_3, AE draw the temperer $Orst$, and the required temperaments will be Gr, As, Et .

For

For by the similar triangles Asp , $Et\phi$ and $A_3\phi$, $EM\phi$, we have $As : Et :: (A\phi : E\phi :: A_3 : EM ::) s : t$ by construction, as required by the first condition.

Again, taking $EN = EM$, through the intersection P of the lines N_3 , AE produced, draw the temperer $ORST$, and by the similar triangles ASP , ETP and A_3P , ENP , we have $AS : ET :: (AP : EP :: A_3 : EN \text{ or } EM) :: s : t$ by construction, which also answers the first condition; and it is plain that those are all the temperers which can answer it.

Now whatever be the ratio of s to t , I say that $Gr + As + Et$ is less than $GR + AS + ET$.

Case 1. Fig. 21. When t is to s , or EM to A_3 in a ratio of minority, the temperers $O\phi$, OP fall within the angles AOE , EOG respectively, as appears by the construction. Whence by coroll. 6 and 7 prop. III, $Gr = As + Et$ and $AS = GR + ET$,

But

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But $Et : As :: t : s$, and $Et : \frac{4}{3}As$
 or $It :: t : \frac{4}{3}s$ and $Et : Et + It$ or
 $\frac{1}{3}c(a) :: t : t + \frac{4}{3}s :: 3t : 3t + 4s$.
 Whence $Et = \frac{t}{4s+3t}c$. And $As =$
 $Et \times \frac{s}{t} = \frac{s}{4s+3t}c$. And $Gr = As + Et$
 $= \frac{s+t}{4s+3t}c$, by the equation in the last
 paragraph.

Again, $ET : AS :: t : s$ and $ET : \frac{4}{3}AS$
 or $IT :: t : \frac{4}{3}s$ and $ET : IT - ET$, or IE
 or $\frac{1}{3}c :: t : \frac{4}{3}s - t :: 3t : 4s - 3t$. Whence
 $ET = \frac{t}{4s-3t}c$ and $AS = ET \times \frac{s}{t} =$
 $\frac{s}{4s-3t}c$.

Therefore $Gr + As + Et : GR + AS$
 $+ ET :: Gr : AS :: \frac{s+t}{4s+3t} : \frac{s}{4s-3t} :: 4ss$
 $+ 4st - 3st - 3tt$, or $4ss + 3st - 2st$
 $- 3tt : 4ss + 3st$, which is evidently
 a ratio of minority.

Case

(a) See Dem. coroll. 4, prop. III.

Case 2. Fig. 22. When t is to s , or EM to A_3 in a ratio of majority, the temperers Op , OP fall within the angles AOE , GOc respectively. Whence, by coroll. 6 and 8 prop. III, $Gr = As + Et$ and $ET = GR + AS$, and $Gr + As + Et : GR + AS + ET :: Gr : ET$, which is plainly a ratio of minority.

Case 3. When $t = s$, or EM or $EN = A_3$, the intersection P vanishes, and the temperer $ORST$ coincides with $OGKL$, as appears by the construction. Whence by the conclusion of the second case, $Gr + As + Et : 0 + AK + EL :: Gr : EL$, a ratio of minority, as before. Q. E. D.

Coroll. When the temp^t. VI : temp^t. III :: $s : t$, the required temperaments of the V, VI and III are, $Gr = \frac{s+t}{4s+3t} c$, $As = \frac{s}{4s+3t} c$, $Et = \frac{t}{4s+3t} c$; and the temperer lies within the angle AOE , whatever be the quantity and quality of the ratio of s to t .

Scho-

Scholium.

These three problems comprehend the solution of a more general one, namely, To find the temperament of any given system of sounds upon these conditions; that the octaves be perfect, that the ratio of the temperaments of any two given concords in different parcels be given, and that the sum of the temperaments of all the concords, be the least possible.

The reason is, that the given ratio of the temperaments of any two concords, determines the position of the temperer of the system, and this the three magnitudes of the temperaments of all the concords, whatever be their number. But if both the given concords be contained in any one of the three parcels above mentioned, the given ratio of their temperaments can be no other than that of equality; and this *datum* is plainly insufficient.

SECTION

SECTION VI.

*Of the Periods, Beats and Harmony
of imperfect consonances.*

DEFINITIONS.

I. Any two sounds whose vibrations have any small given ratio, are called Imperfect Unifons :

II. And the cycle of their pulses is called Simple or Complex, according as the difference of the terms of that ratio is an unit or units :

III. And when a complex cycle is divided into as many equal parts, as that difference contains units, each part is called a Period of the pulses :

IV. And the cycles of perfect consonances are often called Short cycles, to distinguish them from the long cycles of imperfect unifons.

PROPOSITION VII.

In going from either end to the middle of any simple cycle or period of the pulses of imperfect unisons, the alternate lesser intervals between the successive pulses increase uniformly, and are proportional to their distances from that end; and at any distance from it less than half the simple cycle or period, are less than half the lesser of the two vibrations.

Let the vibrations be V and v , and $V : v :: m : n$, the numbers m, n being the least in that ratio; and putting $d = m - n$, we have the complex cycle $nV = mv = nv + dv$ (a), and the period $\frac{n}{d} V = \frac{n}{d} v + v$, which, when $d = 1$, is a simple cycle (b).

FIG. 23, 24, 25, 26. To assist the imagination, let the successive vibrations $V, v, \&c$, be represented by the equal lines

(a) Sect. III. Art. 12.

(b) Def. II.

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lines $AB, BC, CD, \&c$, and the instants of their pulses by the points $A, B, C, \&c$; and the successive vibrations $v, v, \&c$. by the equal lines $ab, bc, cd, \&c$, and the instants of their pulses by the points $a, b, c, \&c$.

Then beginning from the first coincidence of the pulses at A or a , which must happen in less time than that of a cycle, it is observable, that the successive intervals of the pulses are alternately bigger and less; and that the alternate lesser intervals $Bb, Cc, Dd, \&c$, or $V - v, 2V - 2v, 3V - 3v, \&c$, increase uniformly, by the repeated addition of the first and least interval $V - v$, at every equal increment V or v of their distances from A . The alternate lesser intervals are therefore proportional to their distances from the coincident pulses A, a .

Now any assigned distance $3V : nV$
 $:: 3v : nv :: 3V - 3v : nV - nv = dv$,
 by the equation; whence $3V : \frac{n}{d}V ::$
 $3V$

$3V - 3v : v$; consequently if the assigned distance $3V$ or AD be less than half the simple cycle or period $\frac{n}{d}V$, the adjoining interval $3V - 3v$, or Dd is less than half v ; but if bigger, bigger than half v .

And the argument is the same in going backwards from the next coincidence of the pulses at U and w , U and x , &c, their larger and lesser alternate intervals being evidently of the same magnitudes as in going forwards.

FIG. 24. Now if the difference $d = 2$, let the length of the complex cycle be the line AU or $ax = nV = nv + 2v$, and having divided it into two equal parts AX , XU , we have the part or period $AX = \frac{n}{d}V$; which because 2 does not measure n (a), consists of a multiple of V , as AK , and a remainder $KX = \frac{1}{2}V = \frac{1}{2}KL$.

We

(a) By hypothesis.

Fig.17.

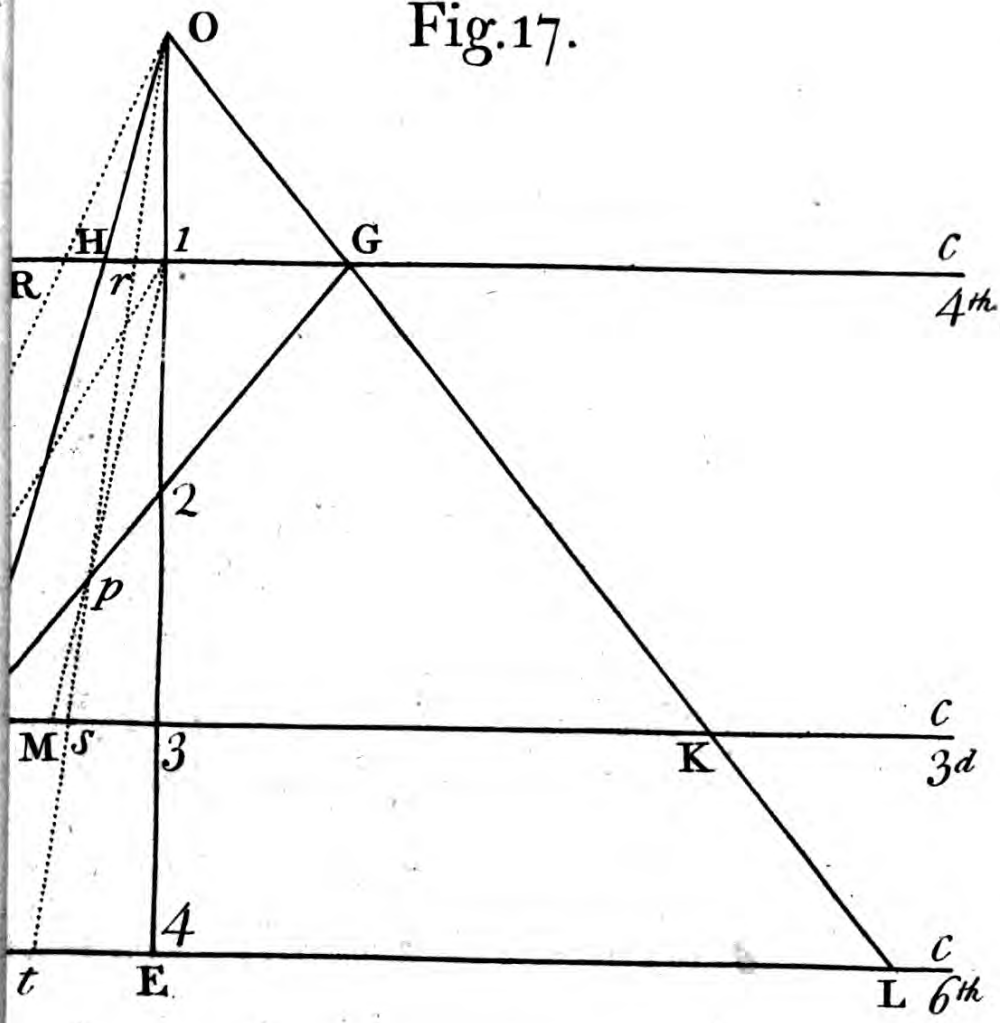
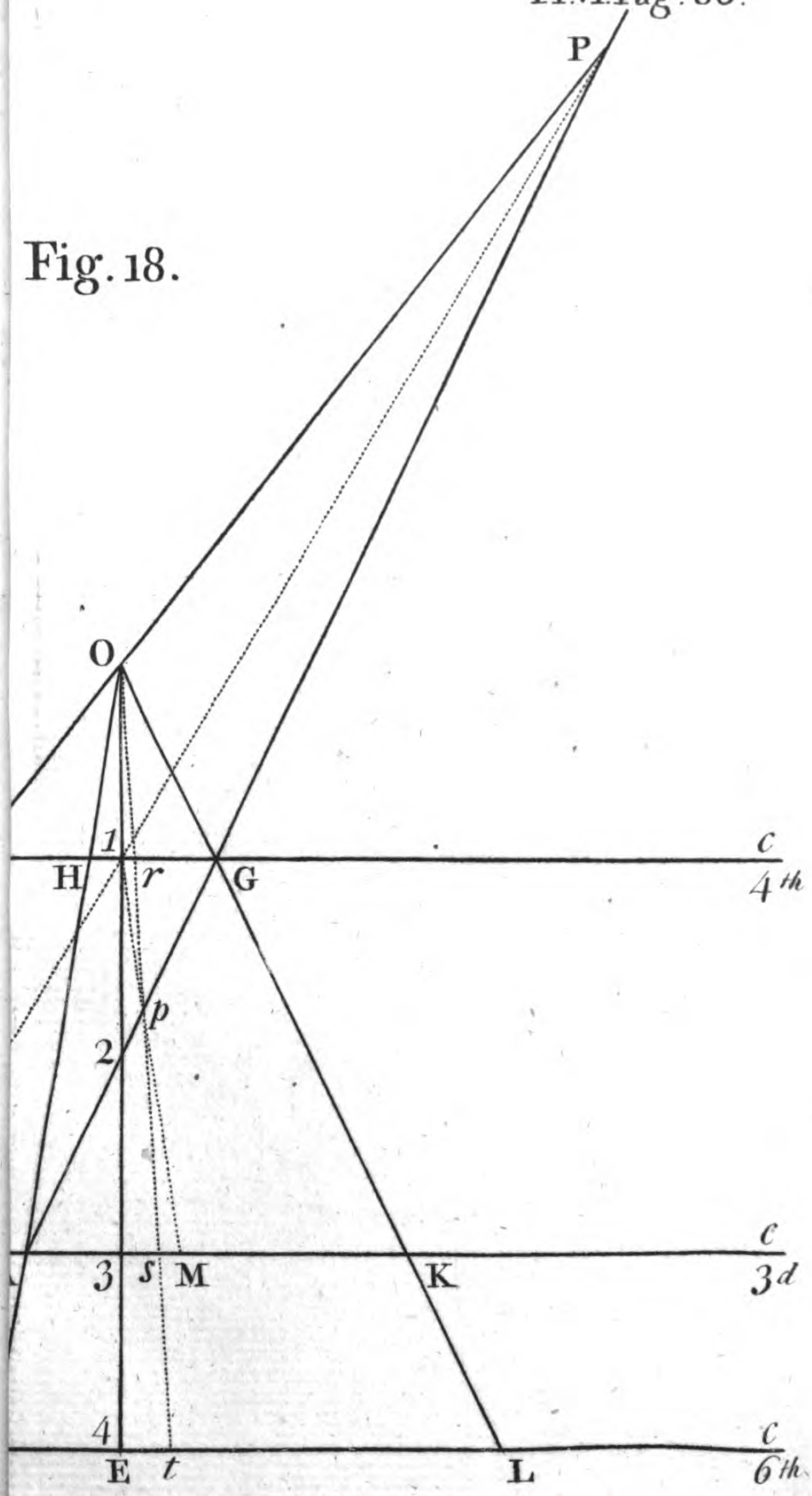




Fig. 18.





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We have also, by the same equation,
 $AX = \frac{n}{2}v + v$, which because 2 does not measure n , consists of a multiple of v (one more than that other of V) as Al , and a like remainder $lX = \frac{1}{2}v = \frac{1}{2}lm$.

Now the distances of the successive pulses of V from the point X are XL , XM , XN , &c, or $\frac{1}{2}V$, $\frac{3}{2}V$, $\frac{5}{2}V$, &c, and those of the successive pulses of v are Xm , Xn , Xo , &c, or $\frac{1}{2}v$, $\frac{3}{2}v$, $\frac{5}{2}v$, &c, and the differences of those respective distances, or the alternate lesser intervals between the successive pulses of V and v , are Lm , Mn , No , &c, or $\frac{1}{2}V - \frac{1}{2}v$, $\frac{3}{2}V - \frac{3}{2}v$, $\frac{5}{2}V - \frac{5}{2}v$, &c; which increase uniformly by the repeated addition of $V - v$ or $\frac{2V - 2v}{2}$ to the first and succeeding intervals.

E di-

Assign any distances XL , XN , or $\frac{1}{2}V$ and $\frac{5}{2}V$; then $\frac{1}{2}V : \frac{5}{2}V :: \frac{1}{2}v : \frac{5}{2}v$
 $:: \frac{1}{2}V - \frac{1}{2}v : \frac{5}{2}V - \frac{5}{2}v$, that is, $XL : XN$
 $:: Xm : Xo :: Lm : No$, or the alternate lesser intervals are proportional to their distances from the periodical point X .

Now any assigned distance $\frac{5}{2}V : \frac{n}{2}V$
 $:: \frac{5}{2}v : \frac{n}{2}v :: \frac{5}{2}V - \frac{5}{2}v : \frac{n}{2}V - \frac{n}{2}v = v$,
 by the given equation, that is, $\frac{5}{2}V : \frac{n}{2}V$
 $:: \frac{5}{2}V - \frac{5}{2}v : v$; consequently if the assigned distance $\frac{5}{2}V$ or XN , be less than half the period $\frac{n}{2}V$ or half XU , the adjoining interval $\frac{5}{2}V - \frac{5}{2}v$ or No , is less than half v ; but if bigger, bigger than half v .

And in going backwards from X , the alternate lesser intervals Kl , Ik , Hi , &c, are respectively equal to Lm , Mn , No , &c, at equal distances on each side of X .

Fig.

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Fig. 25. In like manner if $d = 3$, or $nV = nv + 3v$, having divided this cycle AU or ay into three equal periods AX , XI , IU , that equation gives $AX = \frac{n}{3}V$, which consists of a multiple of V , as AG , and a remainder $GX = \frac{1}{3}V$ (or $\frac{2}{3}V$ hereafter to be considered) $= \frac{1}{3}GH$, whose complement $XH = \frac{2}{3}V$.

The same period AX is also $= \frac{n}{3}v + v$ by the same equation, and therefore consists of a multiple of v (one more than that other of V) as Ab , and a like remainder $bX = \frac{1}{3}v = \frac{1}{3}bi$, whose complement $Xi = \frac{2}{3}v$.

Hence the distances from X of the successive pulses of V are XH , XI , XK , &c, or $\frac{2}{3}V$, $\frac{5}{3}V$, $\frac{8}{3}V$, &c, and those of the successive pulses of v are Xi , Xk , Xl , &c, or $\frac{2}{3}v$, $\frac{5}{3}v$, $\frac{8}{3}v$, &c, and the alter-

nate lesser intervals between the successive pulses of V and v , are $Hi, Ik, Kl, \&c$, or $\frac{2}{3}V - \frac{2}{3}v, \frac{5}{3}V - \frac{5}{3}v, \frac{8}{3}V - \frac{8}{3}v, \&c$; which increase uniformly by the repeated addition of $V - v$ or $\frac{3V - 3v}{3}$ to the first and succeeding intervals.

Assign any distances XH, XK or $\frac{2}{3}V$ and $\frac{8}{3}V$; then $\frac{2}{3}V : \frac{8}{3}V :: \frac{2}{3}v : \frac{8}{3}v, :: \frac{2}{3}V - \frac{2}{3}v : \frac{8}{3}V - \frac{8}{3}v$, that is, $XH : XK :: Xi : Xl :: Hi : Kl$, or the alternate lesser intervals are proportional to their distances from the periodical point X .

Now any assigned distance $\frac{8}{3}V : \frac{n}{3}V :: \frac{8}{3}v : \frac{n}{3}v :: \frac{8}{3}V - \frac{8}{3}v : \frac{n}{3}V - \frac{n}{3}v = v$ by the equation, that is, $\frac{8}{3}V : \frac{n}{3}V :: \frac{8}{3}V - \frac{8}{3}v : v$; so that if the assigned distance $\frac{8}{3}V$ or XK be less than half the period $\frac{n}{3}V$ or half Xl , the adjoining interval

$$\frac{8}{3}V$$

$\frac{8}{3}V - \frac{8}{3}v$ or KL is less than half v ;
but if bigger, bigger than half v .

By doubling the period $AX = AG + \frac{1}{3}V$, we have $AT = 2AG + \frac{2}{3}V = AN + \frac{2}{3}V$, so that NT is $= \frac{2}{3}V$ and its complement $TO = \frac{1}{3}V$. Again by doubling $AX = Ab + \frac{1}{3}v$, we have $AT = 2Ab + \frac{2}{3}v = Ap + \frac{2}{3}v$, so that $pT = \frac{2}{3}v$ and its complement $Tq = \frac{1}{3}v$.

Hence the alternate lesser intervals of the pulses of V and v , in going opposite ways to equal distances from X and T , are equal. And in going contrary ways from X towards A , or from T towards U , the alternate lesser intervals are $\frac{1}{3}V - \frac{1}{3}v$, $\frac{4}{3}V - \frac{4}{3}v$, $\frac{7}{3}V - \frac{7}{3}v$, &c, which increase uniformly as before; and $\frac{7}{3}V$ being an assigned di-

stance from X or γ , we have $\frac{7}{3}V : \frac{n}{3}V$
 $:: \frac{7}{3}v : \frac{n}{3}v :: \frac{7}{3}V - \frac{7}{3}v : \frac{n}{3}V - \frac{n}{3}v = v$
 as before. So that if the assigned distance $\frac{7}{3}V$ be less than half the period $\frac{n}{3}V$, the adjoining interval $\frac{7}{3}V - \frac{7}{3}v$ is less than half v ; but if bigger, bigger than half v .

Fig. 26. Lastly when the period AX , $= \frac{n}{3}V$, consists of a multiple of V as AG and a remainder $GX = \frac{2}{3}V$, which remained to be considered, its complement XH is $= \frac{1}{3}V$, and the demonstration would proceed in the same method as before.

Whoever desires a general proof of the proposition for any value of the difference d , need only read the last example over again, with a design to make the proof general; and he will perceive that what has been said of the number 3 as a value of d , *mutatis mutandis*,

tandis, is plainly applicable to any other value. Q. E. D.

Coroll. 1. The periodical points X, γ , &c, always fall within certain values of v that are severally contained within as many values of V , and the number of those points in each complex cycle is $d-1$.

Coroll. 2. The lesser intervals that lye nearest to the periodical points and the points of coincidence, are less than the rest and are $\frac{V-v}{d}$ and all its multiples, whereof the greatest multiplier is d ; as $\frac{V-v}{3}, \frac{2V-2v}{3}, \frac{3V-3v}{3}$ when $d=3$; $\frac{V-v}{4}, \frac{2V-2v}{4}, \frac{3V-3v}{4}, \frac{4V-4v}{4}$, when $d=4$; &c.

Coroll. 3. Some of the alternate lesser intervals of the pulses of imperfect unisons, are the differences of equal numbers of their vibrations, counted from the nearest coincident pulses; and others are the differences of equal numbers

bers of the same parts of their single vibrations, counted from the nearest periodical point.

Coroll. 4. If the vibrations of two couples of imperfect unisons (or of any two consonances) be proportional, the periods and cycles of their pulses, whether simple or complex, will be in the ratio of the homologous vibrations.

Let T and t be the vibrations of one couple, and V and v those of the other; and since $T : t :: V : v :: n + d : n$, the cycles of their pulses are $nT = \overline{n+d} \times t$ and $nV = \overline{n+d} \times v$, and the periods are $\frac{n}{d}T = \frac{n+d}{d} \cdot t$ and $\frac{n}{d}V = \frac{n+d}{d} \cdot v$; and are in the ratio of T to V , or of t to v .

Coroll. 5. If two couples of imperfect unisons have one vibration common to both, the periods and simple cycles of their pulses, are in the inverse ratio of the intervals of their sounds very nearly,

Let

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Let the vibrations of one couple be V and v , and of the other V and x , and let $V : v :: n + d : n$, and $V : x :: N + D : N$. Then the periods of their pulses are $\frac{n}{d} V = \frac{n + d}{d} v$ and $\frac{N}{D} V = \frac{N + D}{D} x$, and are in the ratio of $\frac{n}{d}$ to $\frac{N}{D}$, or of Dn to dN , and are simple cycles when d and D are units.

Now the intervals of the sounds of V and v , V and x , being measures of the magnitudes of the small ratios $n + d$ to n , $N + D$ to N (a), are very nearly in the ratio of $\frac{d}{n}$ to $\frac{D}{N}$ (b), or of dN to Dn , the inverse ratio of the periods.

Coroll.

(a) Art. 10. Sect. 1.

(b) See coroll. 1. prop. 1. of Mr. Cotes's *Harmonia Mensurarum*, or take the following proof of it.

1. *The magnitudes of very small ratios, which have a term common to both, are very nearly proportional to the differences of their terms.*

Fig. 29. Let the lines $ao, bo, co, do, \&c$ be continual proportionals in any very small ratio. Then since $ao : bo :: bo : co :: co : do \&c$, and disjointly $ab : ao :: bc : bo :: cd : co \&c$; the equal quantities

Coroll. 6. The periods and simple cycles of the pulses of any two couples of imperfect unisons, are in the ratio composed of the direct ratio of their single vibra-

ties $\frac{ab}{ao}, \frac{bc}{bo}, \frac{cd}{co}, \&c.$, may expound the equal magnitudes of the ratios ao to bo , bo to co , co to do , $\&c.$

Consequently the magnitude of the ratio ao to co , composed of ao to bo , bo to co , is to the magnitude of the ratio ao to do , composed of ao to bo , bo to co , co to do , as $\frac{ab}{ao} + \frac{bc}{bo}$ to $\frac{ab}{ao} + \frac{bc}{bo} + \frac{cd}{co}$ exactly; or, supposing mo and no to be arithmetical means among the denominators of each sum, as $\frac{ab}{mo} + \frac{bc}{mo}$ to $\frac{ab}{no} + \frac{bc}{no} + \frac{cd}{no}$ very nearly; (because some of these fractional quantities in each sum, are a little bigger, and others about as much smaller, than they were before) that is, as $\frac{ac}{mo}$ to $\frac{ad}{no}$, or as $ac \times no$ to $ad \times mo$, or as ac to ad very nearly; because the ratio no to mo is extremely small in comparison to the ratio ac to ad by hypothesis.

The magnitudes of any other multiple ratios, as well as duplicate and triplicate, are therefore in the ratio of the differences of their terms very nearly, and the nearer, as the ratios are smaller.

Fig. 30. Now let ao to eo and ao to io be any two very small contiguous ratios; and their magnitudes, if

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vibrations, and of the inverse ratio of their intervals very nearly.

Fig. 31, 32. Let the perpendiculars T and t , V and v to the line $T t V v x$ repre-

if commensurable, are multiples of the magnitude of some smaller ratio; in which case the proposition is already proved; because a term of a geometrical progression interposed between the extremes ao, io , will coincide with eo , the third given term; and if their magnitudes be incommensurable, the error in that coincidence will be infinitely diminished by diminishing the ratio of the progression and increasing the number of its terms to infinity. *Q. E. D.*

2. Fig. 30. Hence the magnitudes of any very small ratios ao to eo , po to qo , are very nearly in the ratio

of $\frac{ae}{ao}$ to $\frac{pq}{po}$, or of $\frac{ae}{eo}$ to $\frac{pq}{qo}$, composed of the direct ratio of the differences of their terms, and the inverse ratio of the terms themselves.

For supposing $ao : io :: po : qo$, the magnitudes of these two ratios are equal. Whence the magnitudes of the proposed ratios ao to eo , po to qo , or ao to io , are very nearly as ae to ai (by N°. 1) or as $\frac{ae}{ao}$ to $\frac{ai}{ao}$

$= \frac{pq}{po}$, because $ai : ao :: pq : po$ by construction: Or,

taking $ko : eo :: po : qo$, the magnitudes of the said ratios ao to eo , po to qo , or ko to eo are very nearly as ae to ke (by N°. 1) or $\frac{ae}{eo}$ to $\frac{ke}{eo} = \frac{pq}{qo}$, because $ke : eo :: pq : qo$. *Q. E. D.*

represent the two couples of vibrations, and the intercepted lines Tt , Vv the intervals of their sounds; and taking a vibration $x : V :: t : T$, the intervals Tt , Vx of the sounds of T and t , V and x , are equal (*c*).

Hence, by coroll. 4, the period of the pulses of T and $t : \text{per. } V$ and $x :: T : V$; and by coroll. 5, the $\text{per. } V$ and $x : \text{per. } V$ and $v :: \text{interval } Vv : Vx$ or Tt very nearly. Therefore the $\text{per. } T$ and t , is to the $\text{per. } V$ and v , in the ratio composed of T to V and of the interval Vv to the interval Tt very nearly.

Coroll. 7. Hence the periods of the pulses of imperfect unisons are very nearly equal, when their single vibrations and intervals are proportional, that is, when $T : V :: \text{interval } Tt : Vv$.

By single vibrations in this and the former corollary, we may understand, either those of the graver sounds or of
the

(*c*) Art. 10. Sect. I.

the acuter or promiscuously ; because the ratio T to t and of V to v is supposed very small in comparison with T to V , or t to v .

Coroll. 8. The length of the period of the Least Imperfections in any consonance of imperfect unisons, is the same as that of the period of its pulses.

Fig. 23, 24, 25, 26. For the consonance grows gradually more imperfect in going from the beginning to the middle of any period or simple cycle of its pulses, and gradually less imperfect in proceeding from the middle to the end : the pulses which would be constantly coincident, if the unisons were perfect, being gradually separated by their alternate lesser intervals. And thus the periods of the least degrees of imperfection in the consonance, are the very same as the periods of its pulses.

PROPOSITION VIII.

If either of the vibrations of imperfect unisons and any multiple of the other, or any different multiples of both, be considered as the single vibrations of an imperfect consonance, the length of the period of its least imperfections, will be the same as that of the pulses of the imperfect unisons.

Fig. 23. 27. For instance, if AB and ab be the vibrations of imperfect unisons, $2AB$ or AC and ab will be the vibrations of an imperfect Octave; whose treble is one of the unisons, and whose base is derived from the other by intermitting every other pulse of the series $A, B, C, D, E, \&c.$

Now if this octave were perfect, every pulse of its base would coincide with every other pulse of its treble; but here they are gradually separated by some of the alternate lesser intervals $Cc, Ee, \&c,$ of the imperfect unisons.

The

The intermediate pulses of the treble, which in a perfect octave would bisect the intervals of the pulses of the base, are also gradually separated from the round points which bisect them, by the rest of the alternate lesser intervals of the said unisons. And thus the imperfections of the tempered octave, or the dislocations of the pulses in its successive short cycles (*a*), are every where the same as those of the imperfect unisons, and consequently have the same periods.

The argument is the same if $2ab$ or ac and AB be the vibrations of the tempered octave, as in Fig. 28; and also, if any other multiple of AB or ab , as mAB or mab , be one of the vibrations of the tempered consonance; as appears by supposing $m-1$ pulses of AB or ab , to be so intermitted, as to leave only single equidistant pulses in Fig. 23, 24, 25, 26.

Fig. 34. Now let any different multiples of AB and ab , as $3AB$ and $2ab$,
or

(*a*) Defin. IV.

or AD and ac be the vibrations of the imperfect consonance; and if AB were $=ab$, then would $3AB$ or $AD : 2ab$ or $ac :: 3 : 2$, and all the short cycles of the vibrations AD, ac would be perfect, or their exterior pulses G and g , N and n , &c, would be coincident, as in Fig. 33: because $2 \times 3AB$ or $2AD$ or $AD + DG$ would then $= 3 \times 2ab$ or $3ac$ or $ac + ce + eg$.

But AB in Fig. 34 being bigger than ab , the multiple $6AB$ or AG is also bigger than the equimultiple $6ab$ or ag ; and so the exterior pulses G and g , N and n , &c, which before were coincident, are now separated by some of the alternate lesser intervals Gg, Nn , &c, of the pulses of AB and ab (a): the distances of the pulses G and g , N and n , &c, from A and a , being equimultiples of AB and ab .

For the like reason the interior pulses of the imperfect short cycles are also
sepe-

(a) Prop. VII. coroll. 3.

Fig. 19.

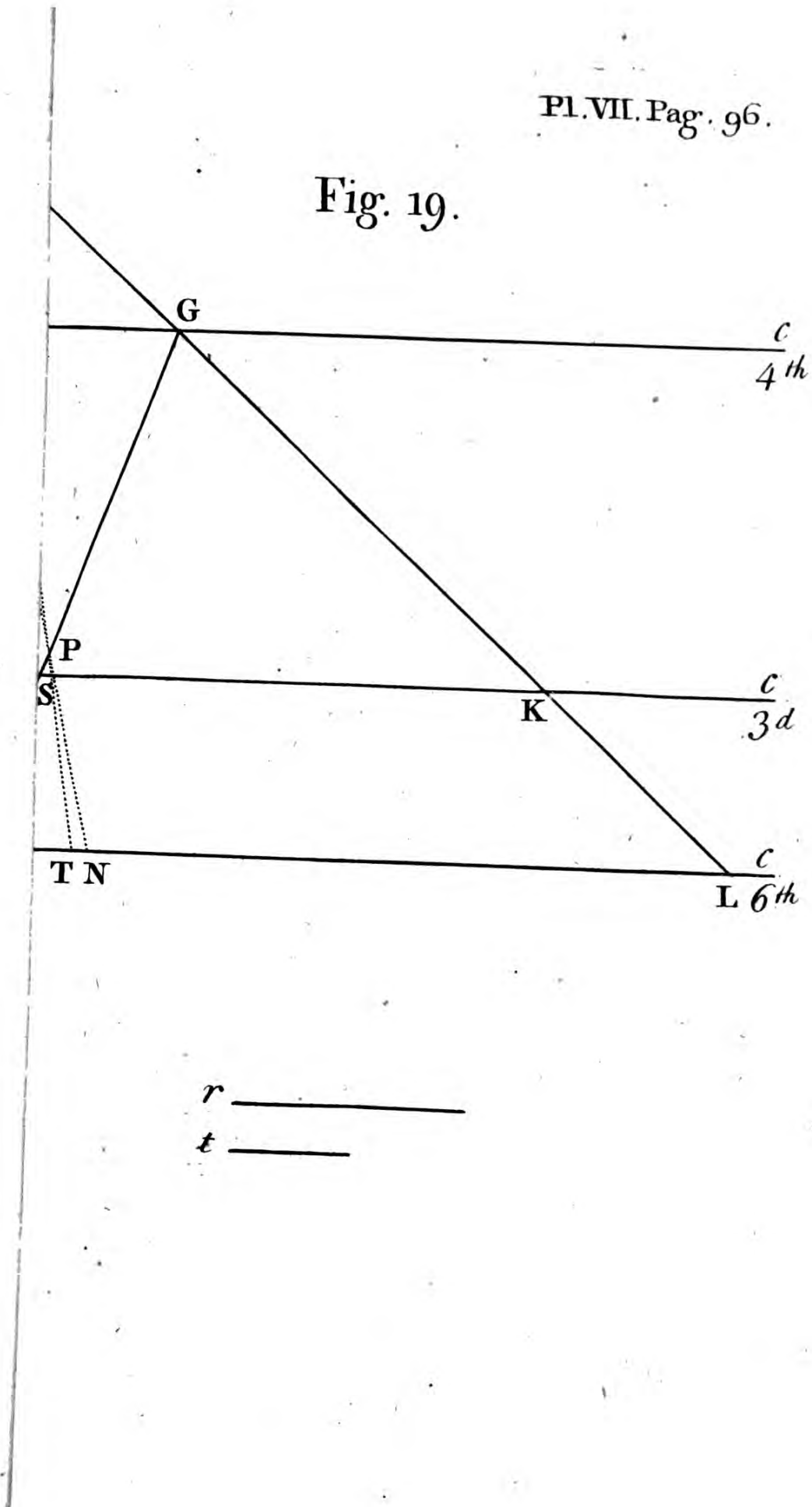
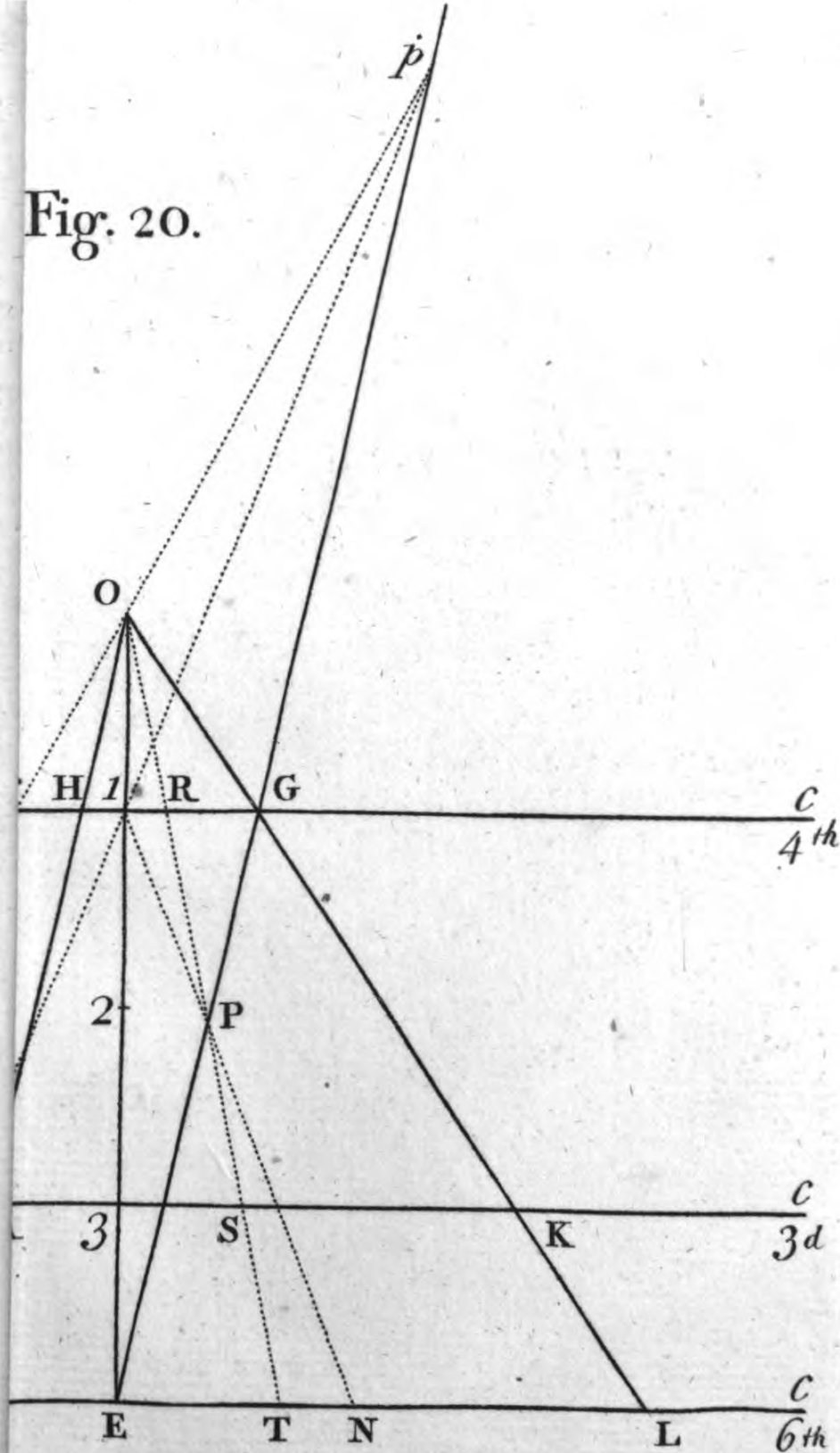




Fig. 20.





Prop. VIII. HARMONICS. 97

separated from the pulses of AB and ab (denoted by round points when different from those of AC and ad) by some of their alternate lesser intervals.

Hence the dislocations of the pulses in all the short cycles, are some of the alternate lesser intervals of the pulses of AB and ab .

For though we began the first cycle from two coincident pulses A, a , yet the argument is the same if we suppose them separated by any one of the alternate lesser intervals; or begin to count the vibrations of the consonance from any two pulses of AB and ab , as Q and r , whose distances from the next periodical point or coincident pulses Z, z , are equimultiples of the vibrations AB, ab , that is, whose interval Qr is an alternate lesser interval of their pulses.

For since the several lengths QX and ry , $X\Delta$ and $y\varepsilon$, ΔK and $\varepsilon\lambda$, &c, of the subsequent short cycles, are

G

equi-

equimultiples of AB and ab , the remaining distances XZ and $y\eta$, ΔZ and $\varepsilon\eta$, KZ and $\lambda\eta$, &c, are also equimultiples of AB and ab ; which shews, that the dislocations of the exterior pulses X and y , Δ and ε , K and λ , &c, and of the interior too, are constantly some of the alternate lesser intervals of the pulses of AB and ab . And thus the period of the least dislocations of the pulses of the tempered consonance, or of the least imperfections in its short cycles, is constantly the same as that of the pulses of AB and ab .

FIG. 35 Shews the same thing, when $2AB$ and $3ab$, or AC and ad , are the different multiples of AB and ab , and their pulses make long simple cycles; and when they make periods, the like is evident by inspection of the pulses about the periodical points X, Y , in Fig. 24, 25, 26, supposing the proper numbers of pulses to be intermitted. And the argument is the same for any other different

ferent multiples of AB and ab , that is for any imperfect consonance whatever. Q. E. D.

Coroll. The same multiples or submultiples of the vibrations of imperfect unisons, will always be the vibrations of other imperfect unisons, whose period is the same multiple or submultiple of the period of the given unisons (a), and whose interval is the same too at a different pitch; because the ratio of the vibrations is the same (b).

PROPOSITION IX.

If either of the vibrations of imperfect unisons, and any aliquot part or parts of the other, be the vibrations of an imperfect consonance, the length of the period of its least imperfections, will be the same part of the period of the imperfect unisons.

Fig.

(a) Prop. vii. coroll. 4. (b) Sect. 1. Art. 10.

FIG. 36. For if the vibrations of imperfect unisons be T and t , and $\frac{n}{m}$ be any irreducible fraction, proper or improper, and the vibrations of a perfect consonance be T and $\frac{n}{m}T$, or t and $\frac{n}{m}t$; those of the imperfect consonance will be T and $\frac{n}{m}t$, or $\frac{n}{m}T$ and t ; which being the same as $m \times \frac{1}{m}T$ and $n \times \frac{1}{m}t$, or as $n \times \frac{1}{m}T$ and $m \times \frac{1}{m}t$, are different multiples of $\frac{1}{m}T$ and $\frac{1}{m}t$; whose period is therefore the same as that of the least imperfections of either of either of the tempered consonances (*a*), and is equal to $\frac{1}{m}$ of the period of T and t (*b*). Q.E.D.

Coroll. 1. Fig. 37. If T and $\frac{n}{m}T$, V and $\frac{s}{r}V$ be the vibrations of any two perfect consonances, the periods of their least im-

(*a*) Prop. VIII.

(*b*) Coroll. Prop. VIII.



Prop. IX. HARMONICS. 101

imperfections when tempered, will be in the ratio composed of the direct ratio of T to V, the inverse ratio of the denominators of the fractions $\frac{n}{m}$, $\frac{s}{r}$, and the inverse ratio of the temperaments.

For if the temperaments be the small intervals of the founds of the vibrations T and t , V and v ; by this proposition, the periods of the tempered consonances, whose vibrations are T and $\frac{n}{m}t$, V and $\frac{s}{r}v$, or t and $\frac{n}{m}T$, v and $\frac{s}{r}V$, are respectively equal to $\frac{1}{m} \times$ Per. of T and t and $\frac{1}{r} \times$ Per. of V and v ; whose ratio is composed of $\frac{1}{m}$ to $\frac{1}{r}$, or r to m and of the other two ratios abovementioned (c).

Coroll. 2. Hence, when the temperaments are equal, and the fractions $\frac{n}{m}$, $\frac{s}{r}$ are the same, or their denominators

tors

(c) Prop. VII. coroll. 6.

tors only, the periods of the consonances are in the ratio of their homologous vibrations, T, V.

Coroll. 3. If $T = V$, that is, if the consonances have a common found, the ratio of their periods is composed of the inverse ratio of the denominators of the fractions $\frac{n}{m}$, $\frac{s}{r}$, and the inverse ratio of the temperaments, by coroll. 1.

Coroll. 4. Consequently when the denominators are the same, the periods are in the inverse ratio of the temperaments.

Coroll. 5. And when the temperaments are the same or equal, the periods have the inverse ratio of the denominators, or the direct ratio of their reciprocals $\frac{1}{m}$, $\frac{1}{r}$.

Coroll. 6. Therefore the periods of simpler consonances are generally longer than those of the less simple when equally tempered; the denominators of the fractions belonging to the simpler

pler consonances being generally smaller (a).

PROPOSITION X.

When either of the sounds of a perfect consonance is tempered flat, the length of the period of its least imperfections is the same as it would have been, if the other sound had been equally tempered flat; and when either of the given sounds is equally tempered sharp, the period will be of another given length; and if the flat and sharp temperaments be equal, the former equal periods will be longer than the latter, in the ratio of the vibrations which terminate the given temperament.

Fig. 38. For let t and $\frac{n}{m}t$ be the vibrations of the given perfect consonance, and let its flat temperament be the

(a) Sect. III. Art. 5 and the Table of the order of the simplicity of consonances, also Fig. 3.

104 HARMONICS. Sect. VI.
 the interval of the sounds of T and t , or
 of $\frac{n}{m}T$ and $\frac{n}{m}t$. Then is the period of T
 and $\frac{n}{m}t = \text{Per. } t$ and $\frac{n}{m}T = \frac{1}{m} \times \text{Per. } T$
 and t (a).

Again if the interval of the sounds
 of t and v , or of $\frac{n}{m}t$ and $\frac{n}{m}v$, be the
 sharp temperament of the same conso-
 nance of t and $\frac{n}{m}t$; then the period of
 t and $\frac{n}{m}v = \text{Per. } v$ and $\frac{n}{m}t = \frac{1}{m} \times \text{Per. } t$
 and v (a).

And the flat and sharp temperaments
 being supposed equal, we have $T : t ::$
 $t : v$, whence $\frac{1}{m} \times \text{Per. } T$ and $t : \frac{1}{m} \times$
 $\text{Per. } t$ and $v :: T : t :: t : v$ (b). Q.E.D.

Coroll. 1. The given vibrations of any
 perfect consonance being t and $\frac{n}{m}t$,
 the period of its least imperfections,
 when tempered flat or sharp by any
 part

(a) Prop. IX. (b) Prop. VII. coroll. 4.

Prop. X. HARMONICS. 105

part or parts of a comma, denoted by $\frac{q}{p}$, will be respectively $\frac{161p+q}{2q} \times \frac{t}{m}$ or $\frac{161p-q}{2q} \times \frac{t}{m}$ very nearly.

For supposing those equal temperaments to be terminated by the vibrations T and t , t and v , we have $T : t :: t : v :: 161p+q : 161p-q$ (c). Whence the

(c) Fig. 38. N°. 2. *If any given part, parts or multiple of the difference ad of the terms ao, do of any small given ratio, be placed in the middle of it, the magnitude of the ratio eo to fo will be the same part, parts or multiple of the magnitude of the ratio ao to do, as ef is of ad very nearly.*

For it would be so very nearly, if ef were placed contiguous to either end of the difference ad , as at ab or cd , (by N°. 1. note (b) pag. 89.) in which case the magnitude of the ratio ao to bo is something less, and that of co to do is something greater, than the magnitude of eo to fo ; which therefore approaches still nearer to the truth, and gives the following algebraical theorem.

If s be the half sum and d the half difference of the terms of any small given ratio, and any fraction, proper or improper, be $\frac{q}{p}$, the magnitude of the ratio $s + \frac{q}{p}d$ to $s - \frac{q}{p}d$, is to that of the given ratio, as $\frac{q}{p}$ to 1 very nearly.

Hence,

106 HARMONICS. Sect. VI.
 the cycle of the pulses of T and t is $\overline{161p+q} \times t = \overline{161p-q} \times T$, and their period is $\frac{161p+q}{2q} t$. Likewise the cycle of the pulses of t and v is $\overline{161p-q} \times t = \overline{161p+q} \times v$, and their period is $\frac{161p-q}{2q} t$. And these periods severally divided by the denominator m , are the periods of the least imperfections of the given consonance, tempered flat or sharp respectively, by prop. IX.

Coroll. 2. Whatever be the situation of a small given temperament, the period of the tempered consonance will be exactly, or else very nearly of the same length. For instance, if the temperament be a comma in dissimilar situations, the ratio of the periods will be

Hence, the ratio of the vibrations of any two sounds, whose interval is any given part or parts of a comma, denoted by $\frac{q}{p}$, is $161p+q$ to $161p-q$ very nearly.

For the terms of the given ratio belonging to the comma are 81 and 80, whence $s = \frac{161}{2}$ and $d = \frac{1}{2}$
 and

be 81 to 80, if $\frac{1}{4}$ comma, 645 to 643 very nearly.

PHÆNOMENON.

In tuning musical instruments, especially organs, it is a known thing that while a consonance is imperfect, it is not smooth and uniform as when perfect, but interrupted with very sensible Undulations or Beats; which while the two sounds continue at the same pitch, succeed one another in equal times, and in longer and longer times while either of the sounds approaches gradually to a perfect consonance with the other; till at last the Undulations vanish and leave a smooth, uniform consonance.

Quick-

$$\text{and } s + \frac{q}{p}d : s - \frac{q}{p}d :: \frac{161}{2} + \frac{q}{p} \times \frac{1}{2} : \frac{161}{2} - \frac{q}{p} \times \frac{1}{2} \\ :: 161p + q : 161p - q.$$

For example, when $\frac{q}{p} = \frac{1}{4}$, that ratio is 645 to 643, and the following logarithms shew how very near that theorem approaches to exactness.

The

Quicker undulations are beats, and are remarkably disagreeable in a concert of strong, treble voices, when some of them are out of tune; or in a ring of bells ill tuned, the hearer being near the steeple; or in a full organ badly tuned: Nor can the best tuning wholly prevent that disagreeable battering of the ears with a constant rattling noise of beats, quite different from all musical sounds, and destructive of them, and chiefly caused by the compound stops called the Cornet and Sesquialter, and by all other loud stops of a high pitch, when mixed with the rest. But if we be content with compositions of unisons and octaves to the Diapason, whatever be the quality of their sounds, the best manner of tuning will render the noise of
their

$$\text{The log. } \frac{81}{80} = 0.00539.503$$

$$\frac{1}{4} \text{ of it } \dots\dots = 0.00134.876$$

$$\text{The log. } \frac{645}{643} = 0.00134.874$$

Fig. 21.

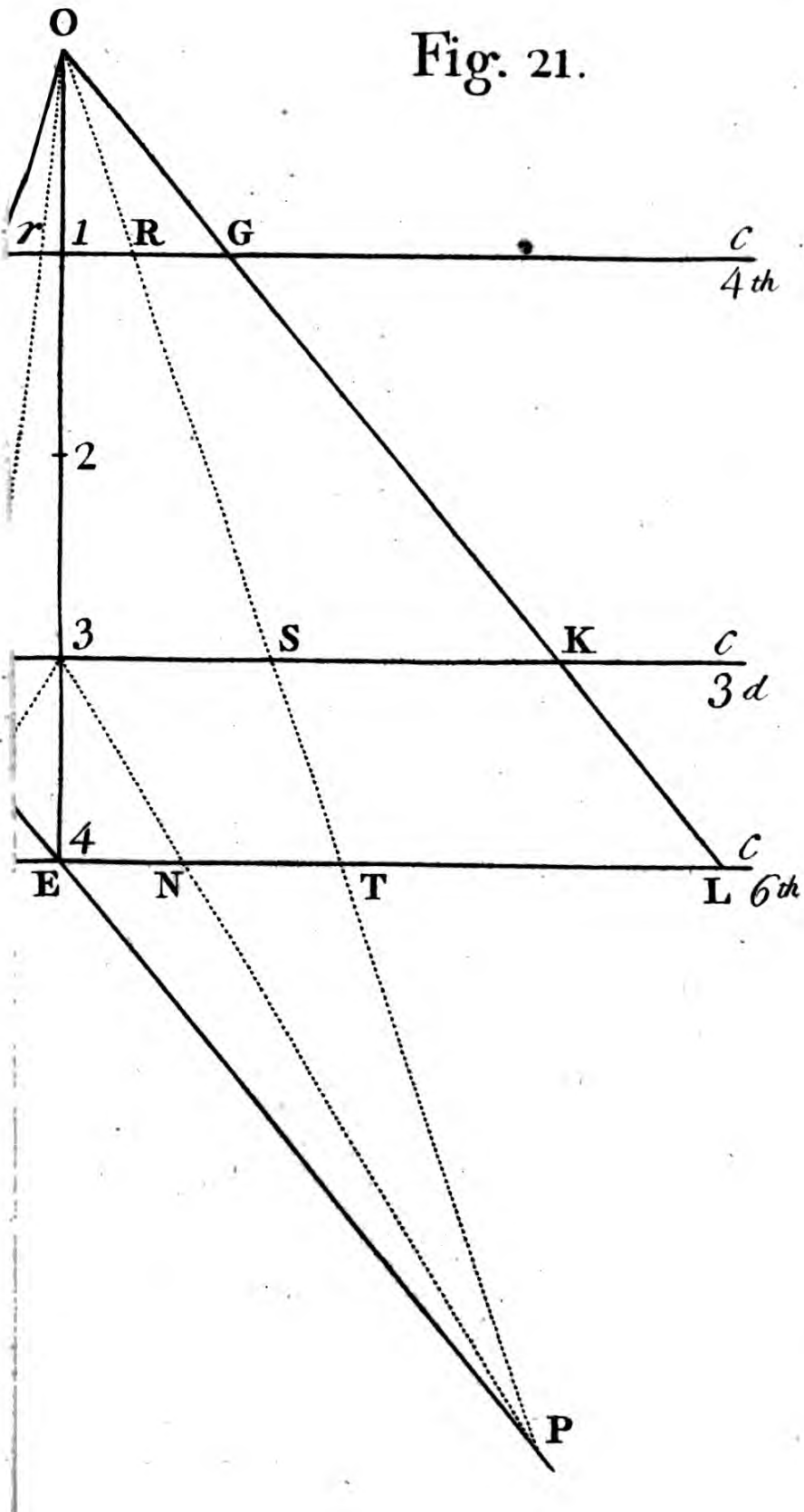
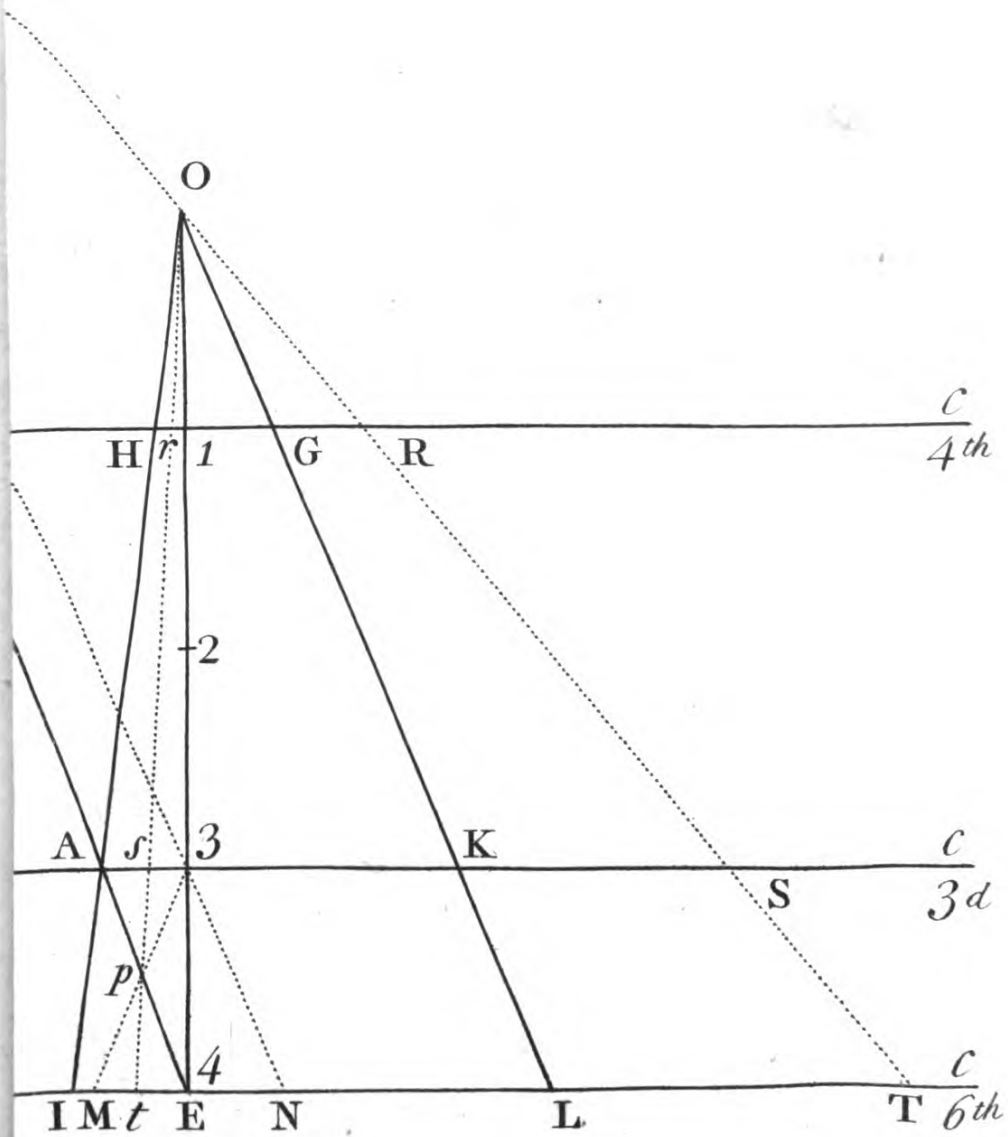




Fig. 22.

s _____
t _____





their beats inoffensive if not imperceptible.

PROPOSITION XI.

The time between any two successive beats of any imperfect consonance, is equal to the periodical time of its least imperfections.

Fig. 23 to 27. 34. 35. Any simple cycle or any period of the pulses of imperfect unisons, contains one more of the shorter than of the longer vibrations, as appears by the equation to it (a): and the short cycle of any imperfect consonance consists of equal numbers of the longer and shorter vibrations of the imperfect unisons (b). Consequently after taking away the greatest equal numbers of short cycles, that can be taken from both ends of the simple cycle or period of the imperfect unisons, some part of another short cycle or two,

as

(a) See the demonstration of Prop. VII.

(b) See pag. 96.

as consisting of unequal numbers of the longer and shorter vibrations of the imperfect unisons, will always remain in the middle. And this part, by interrupting the succession of the short cycles, wherein the harmony of the consonance consists, interrupts its harmony and causes the noise which is called a beat: especially as the interruption is made where the short cycles on each side of it are the most imperfect or confused. Therefore the time between two successive beats, made in the middle of each period, or simple cycle of the pulses of the imperfect unisons, or of the least imperfections of the consonances (*a*), is equal to the time of this period.

And the cause of the beats of imperfect unisons is a like interruption of the succession of their short cycles, in the middle of every period or simple cycle of their pulses, where they are most confused (*b*). Q. E. D.

Scho-

(*a*) Prop. VIII.

(*b*) See the 2^d Schol. following, Note (*c*).

Scholium 1.

The ratio of the numbers of beats made by any two consonances in any equal times, being the inverse ratio of the times between their successive beats, is also the inverse ratio of the periods of their least imperfections; and is therefore determined in all cases by the corollaries to prop. IX, which however are translated hither with that alteration, to prevent the frequent repetition of this observation.

Coroll. 1. If T and $\frac{n}{m} T$, V and $\frac{s}{r} V$ be the vibrations of any two perfect consonances, when they are tempered the numbers of their beats, made in equal times, will be in the ratio composed of the inverse ratio of T to V , the direct ratio of the denominators of the fractions $\frac{n}{m}$, $\frac{s}{r}$, and the direct ratio of the temperaments.

Coroll. 2. Hence when the temperaments are equal and the fractions $\frac{n}{m}$, $\frac{s}{r}$ are the same, or their denominators only, the numbers of beats, made in equal times, are in the inverse ratio of the homologous vibrations T, V: or in the direct ratio of the numbers of these vibrations made in equal times.

Coroll. 3. If $T = V$, that is, if the consonances have a common sound, the numbers of their beats, made in equal times, are in the ratio composed of the direct ratio of the denominators of the fractions $\frac{n}{m}$, $\frac{s}{r}$, and the direct ratio of the temperaments, by coroll. 1.

Coroll. 4. Consequently when the denominators are the same, the numbers of beats, made in equal times, are in the ratio of the temperaments.

Coroll. 5. And when the temperaments are the same or equal, the numbers of the beats, made in equal times, are

are in the direct ratio of the denominators, m, r .

Coroll. 6. Therefore simpler consonances generally beat slower than the less simple do when equally tempered, the denominators belonging to the simpler being generally smaller (*a*).

Coroll. 7. Let the single vibrations of any perfect consonances be t and $\frac{n}{m}t$, and the numbers of them made in one second of time, be N and M respectively; then the number of the beats of that consonance, when tempered flat by any part or parts of a comma denoted by $\frac{q}{p}$, will be $\frac{2q}{161p+q} \times mN$, or $\frac{2q}{161p+q} \times Mn$ in one second of time very nearly; and when tempered sharp by the same quantity, the number will be $\frac{2q}{161p-q} \times mN$, or $\frac{2q}{161p-q} \times Mn$ in one second very nearly.

For

(*a*) Sect. III. Art. 5 and the Table of the order of consonances: also Fig. 3.

For the time between the successive beats, being equal to the period of the least imperfections of the tempered consonance (*a*), is $\frac{161p+q}{2q} \times \frac{t}{m}$ in the first case, and $\frac{161p-q}{2q} \times \frac{t}{m}$ in the second (*b*); that is (because $t = \frac{1''}{N}$ and $\frac{n}{m} t = \frac{1''}{M}$ by hypothesis) $\frac{161p+q}{2q} \times \frac{1''}{mN}$ or $\frac{161p+q}{2q} \times \frac{1''}{Mn}$ in the first case, and $\frac{161p-q}{2q} \times \frac{1''}{mN}$ or $\frac{161p-q}{2q} \times \frac{1''}{Mn}$ in the second: and the number of beats in one second, being the reciprocal of the time between the successive beats, is the reciprocal of that quantity in each case, as above.

Coroll. 8. All the foregoing corollaries will also follow from this last, by considering

(*a*) Prop. xi, (*b*) Prop. x. Coroll. 1.

(*c*) M. *Sauveur* ayant cherché la cause de ce Phenomene, a imaginé avec une extrême vraisemblance, que le son des deux tuyaux ensemble devoit avoir plus de force, quand leurs vibrations, après avoir été quelque temps séparées, venoient à se réunir et s'accordoient à frapper l'oreille d'un même coup. Histoire de l'Acad.

Prop. XI. HARMONICS. 115

Considering every quantity in those theorems as indeterminate, and by observing that the quantity $\frac{2q}{161p+q}$ or $\frac{2q}{161p-q}$ is as the temperament of the consonance very nearly, by the notes in pag. 91 and 106.

Scholium 2.

Mr. *Sauveur* is the only writer I know of that takes any notice of the beats of consonances. He imagined that they beat at every coincidence of their pulses (*c*), and observing that he could distinguish the beats pretty well when they went no quicker than 6 in one second, and still plainer when they went slower, he concluded that we could not perceive them at all when they went faster than at that rate (*d*);
and

l'Acad. Royale des Sciences, année 1700, pag. 171. 8^{vo}.

(*d*) Donc dans tous les accords où les vibrations se rencontreront plus de 6 fois par seconde, on ne sentira point de battemens, et on les sentira au contraire avec d'autant plus de facilité que les vibrations se rencontreront moins de 6 fois par seconde. *ibid.* pag. 176.

and thence he inferred that octaves and other simple concords, whose vibrations coincide very often, are agreeable and pleasant because their beats are too quick to be distinguished, be the pitch of the sounds ever so low; and on the contrary, that the more complex consonances whose vibrations coincide seldomer, are disagreeable because we can distinguish their slow beats; which displease the ear, says he, by reason of the inequality of the sound (*a*). And in pursuing this thought he found, that those consonances which beat faster than 6 times in a second, are the very same that musicians treat as concords; and that others which beat slower are the discords; and he adds, that

▪ (*a*) Les battimens ne plaisent pas à l'oreille, à cause de l'inégalité du son, et l'on peut croire avec beaucoup d'apparence, que ce qui rend les octaves si agréables, c'est qu'on n'y entend jamais de battemens. *ibid.* pag. 177.

(*b*) En suivant cette idée, on trouve que les accords dont on ne peut entendre les battemens, sont justement

Prop. XI. HARMONICS. 117

that when a consonance is a discord at a low pitch and a concord at a high one, it beats sensibly at the former pitch but not at the latter (*b*).

As Mr. *Sauveur* appeals to numbers, let us see what evidence they produce. The tones and sevenths major and minor being discords, must beat slower than 6 times in one second by his own hypothesis. Then let them beat but 4 or 5 times, and it will follow that the major 1vth and minor 5th cannot beat above once in a second.

For the lengths of the cycles of perfect consonances to a common base, are proportional to the lesser terms of the ratios of their vibrations (*c*), which being but 8 and 9 in the former discords
and

ment ceux que les musiciens traitent de consonances, et que ceux dont les battemens se font sentir, sont les dissonances; et que quand un accord est dissonance dans une certaine octave, & consonance dans une autre, c'est qu'il bat dans l'une, et qu'il ne bat pas dans l'autre. *ibid.* pag. 177.

(*c*) Sect. III. Art. 13.

and 32 and 45 in the latter (*a*), shew, that the latter must beat 4 or 5 times slower than the former, that is, as slow at least as a clock that beats seconds.

But in founding the latter discords upon an Organ, Harpsichord or Violoncello, even at a low pitch, I find their beats are so quick that I cannot count them; or rather they do not beat at all, like imperfect consonances, but only flutter, at a slower or quicker rate according to the pitch of the sounds.

The truth is, this gentleman confounds the distinction between perfect and imperfect consonances, by comparing imperfect consonances (*b*) which beat because the succession of their short cycles is periodically confused and interrupted (*c*), with perfect ones which cannot beat, because the succession of their

(*a*) Table of musical ratios pag. 12.

(*b*) Memoires de l' Acad. 1701, Systeme general, Sect. XII, maniere de trouver le son fixe. pag. 473. 3^o.

(*c*) Dem. of Prop. XI.

their short cycles is never confused nor interrupted.

The fluttering roughness abovementioned is perceivable in all other perfect consonances, in a smaller degree in proportion as their cycles are shorter and simpler, and their pitch is higher; and is of a different kind from the smoother beats and undulations of tempered consonances; because we can alter the rate of the latter by altering the temperament, but not of the former, the consonance being perfect at a given pitch: And because a judicious ear can often hear, at the same time, both the flutterings and the beats of a tempered consonance, sufficiently distinct from each other.

Scholium 3.

In all tempered systems the times of the vibrations of most of the consonances are incommensurable quantities.

In

In the system of mean tones, for instance, the vibrations of the tone are in the ratio of $\sqrt{5}$ to 2, the subduplicate of 5 to 4, as the mean tone is half the III^d. Likewise the vibrations of the vth tempered by a quarter of a comma,

are in the ratio of $\sqrt[4]{5}$ to 1, the subquadruplicate of 5 to 1, as that vth is a quarter of the XVIIth or 2VIII + III.

For as $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{81}{80} = \frac{1}{5}$, so V + V +

V + V - c = XVII; whence $V - \frac{1}{4}c + V$

$-\frac{1}{4}c + V - \frac{1}{4}c + V - \frac{1}{4}c = XVII$. Last-

ly the ratio of the vibrations of two sounds whose interval is a quarter of a

comma, is $\sqrt[4]{81}$ to $\sqrt[4]{80}$, or $\sqrt[4]{3 \times 3 \times 3 \times 3}$

to $\sqrt[4]{2 \times 2 \times 2 \times 2 \times 5}$, or 3 to $2\sqrt[4]{5}$; and

consequently the ratio of the vibrations of any consonance tempered by a quarter of a comma, is also incommensurable, as being composed of the ratio of the vibrations of the perfect con-

nance

nance, and the ratio of the vibrations belonging to its temperament.

The same may be said of any perfect consonance tempered by any aliquot part or parts of a comma; whose vibrations are always incommensurable, because 81 and 80 are not equal powers of any two numbers whatever (a) We may conclude then, that in tempered systems the vibrations of most of the consonances are incommensurable, as resulting from force and motion, which are continued quantities.

Now if the agreeableness of consonances, according to the received principle in Harmonics, be the result of the frequent coincidences of their pulses, and consequently be greater or lesser according as the coincidences are more or less frequent, all the consonances in tempered systems, whose vibrations are incommensurable, ought to be the greatest discords in nature: it being
im-

(a) See coroll. Prop. 2. VIII. Elem.

impossible for their pulses to coincide more than once in an infinite time. For as no two numbers how large soever, can express the ratio of such vibrations, so no multiple of one vibration can ever be equal to any multiple of the other. And yet experience shews that such consonances are much more agreeable than perfect discords whose pulses coincide very often.

We may approach indeed as near as we please, and certainly much nearer than our senses can distinguish, towards the exact magnitude of an incommensurable ratio, by the ratios of whole numbers; but as these will grow larger and larger without bounds, so will the time between the successive coincidences, or the length of the approximating cycle of the pulses: by which I mean the time of either of the incommensurable vibrations multiplied by the heterologous term of the approximating ratio.

Let

Let any man tell us then where we may stop, and which of those cycles it is, whose repetition excites the determinate idea of the consonance.

The like difficulty occurs in approaching gradually even to a commensurable ratio of the vibrations of any perfect consonance. For if either of its vibrations be pretty much altered at once, and then be made to approach by degrees to its former length, the terms of the several approximating ratios will grow larger and larger without bounds and in regular order, except when ratios occur whose terms are reducible; and the cycles of their pulses will accordingly be longer and longer and their coincidences fewer and fewer without limit, those interruptions excepted; and yet the consonance will grow better and better by regular degrees till it arrives at perfection, as is certain by experience. For instance the ratios 30 to 21, 300 to 201, 3000 to
2001,

2001, &c, approach nearer and nearer to 3 to 2, and the v^{th} s whose vibrations are in those ratios grow more and more harmonious, though the cycles of their pulses grow longer and longer to infinity.

It is therefore impossible to account for the phænomena of imperfect consonances upon the principle of coincidences, which indeed is applicable to none but perfect ones. Accordingly Dr. *Wallis* (*a*), Mr. *Euler* (*b*) and others disapprove incommensurable vibrations as impracticable and inharmonious.

But supposing the vibrations V, v of imperfect unisons to be incommensurable

(*a*) It hath been long since demonstrated, that there is no such thing as a just hemitone practicable in music, and the like for the division of a tone into any number of equal parts, three, four or more. For supposing the proportion of a tone or full note to be as 9 to 8, that of the half note must be as $\sqrt{9}$ to $\sqrt{8}$, that is as 3 to $\sqrt{8}$, or as 3 to $2\sqrt{2}$, which are incommensurable quantities; and that of a quarter note as $\sqrt[4]{9}$ to $\sqrt[4]{8}$, which is yet more incommensurate;

Prop. XI. HARMONICS. 125

rable, or $V : v :: \sqrt{p} : \sqrt{q}$, and x to be an indeterminate vibration, and $V : x :: m : n$, and the ratios of the indeterminate numbers m, n to approach gradually to the given ratio of \sqrt{p} to \sqrt{q} ; though the length $nV, = mx$, of the indeterminate cycle of the pulses of V and x , increases without bounds, nevertheless the length $\frac{n}{m-n}V, = \frac{m}{m-n}x$, of the indeterminate period of their pulses tends gradually to a determinate limit $\frac{\sqrt{q}}{\sqrt{p}-\sqrt{q}}V = \frac{\sqrt{p}}{\sqrt{p}-\sqrt{q}}v$. And this is the period of the pulses of the incommensurable vibrations V, v , which excites the determinate idea of the imperfect

menfurate; and the like for any other number of equal parts: which will therefore never fall in with the proportions of number to number. *Upon the imperfection of an Organ.* Phil. Trans. N^o. 242, or Abridgm. vol. 1. pag. 705. edit. 1.

(b) Denique ob nullam sonorum rationem rationalem præter octavas, hoc genus [musicum] harmoniæ maxime contrarium est censendum; etiamfi hebetiores aures discrepantiam vix percipiant. *Tentamen novæ Theoriæ musicæ*, cap. ix. sect. 17. Petropoli. 1739.

perfect unisons, be the complex cycle of their pulses ever so long, infinite or impossible.

I say determinate idea. For though the alternate lesser intervals of the pulses in the several successive periods of V and v , even when commensurate, are not precisely equal (*a*), yet it is highly probable that the ear could not distinguish a repetition of any one period from the succession of them all, and seems agreeable to experience in observing the identity of the tone of imperfect unisons held out upon an organ.

For further illustration I will add an example or two. We shewed above that the vibrations V, v of the mean tone are as $\sqrt{5} : 2 :: 2.23606796 \&c : 2 :: m : n$. Whence the length of the period of the pulses of V and v , is

$$\frac{n}{m-n} V = \frac{2V}{0.23606796 \&c} = 8.47213 \&c \times V;$$

which is a medium between $8V$ and $9V$,

(*a*) Coroll. 2. Prop. vii.

9V, the cycles of the pulses of the major and minor tones, something less than the arithmetical, or even the geometrical mean, but not quite so little as the harmonical mean between them (a).

Again, when V and v are the vibrations of two sounds whose interval is a quarter of a comma, we found $V : v ::$

$3 : 2\sqrt[4]{5}$ or 2.99069756 &c :: $m : n$;
whence the period of the pulses of V

and v is $\frac{n}{m-n}V = \frac{2.99069756 \text{ \&c}}{0.00930244 \text{ \&c}} \times V =$
321.4960 &c $\times V$.

Or thus. In approximating towards the ratio of V to v, or 3 to $2\sqrt[4]{5}$, or 3 to 2.990697, or 3000000 to 2990697 by small numbers (b), the ratios greater than V to v are 322 to 321, 967 to 964, 1612 to 1607, &c. Whence the cycle 321V and the periods $321\frac{1}{3}V$, $321\frac{2}{5}V$, &c, are all too short.

And

(a) See Sect. vii. Def. ii.

(b) See Mr. Cotes's Harmonia Mensurarum, prop. i. Schol. 3.

And the ratios less than V to v being 323 to 322 , 645 to 643 , &c, the cycle $322V$ and periods $321\frac{1}{2}V$, &c, are all too long. Therefore the true period falls between the last mentioned limits, agreeably to the former computation.

From what has been said of imperfect unisons the difficulty vanishes in other imperfect consonances, by observing the reduction of the periods of their imperfections to those of imperfect unisons, as in prop. VIII and IX.

Scholium 4.

Having observed a very strict analogy between the undulations of audible and visible objects, I will here describe it, as an illustration of the foregoing theory of imperfect consonances.

Fig. 39. Let the points a, b, c , &c and α, β, γ , &c represent the places of two parallel rows of equidistant and parallel objects, such as pales, pallisadoes, &c,
and

and let them be viewed from any large distance by an eye at any point z . In a plane passing through the eye and cutting the axes of the parallel objects at right angles in the points $a, b, c, \&c,$ $\alpha, \beta, \gamma, \&c,$ let lines drawn from z through $\alpha, \beta, \gamma, \&c,$ cut the line of the other row in $A, B, C, \&c.$ Then by the similar triangles ABz and $\alpha\beta z, BCz$ and $\beta\gamma z, CDz$ and $\gamma\delta z, \&c,$ we have $AB : \alpha\beta :: (Bz : \beta z ::) BC : \beta\gamma :: (Cz : \gamma z) :: CD : \gamma\delta :: \&c.$ Therefore the antecedents $AB, BC, CD, \&c,$ which are to the equal consequents $\alpha\beta, \beta\gamma, \gamma\delta, \&c$ in the same ratio, are also equal to one another, and are the apparent projections of the consequents upon the line abc of the other row.

Hence supposing m and n to represent the least whole numbers in the given ratio of AB to $ab,$ we have a line $m \times ab = n \times AB,$ equal to the length of the cycle between the apparent coincidences of some of the objects in one row with some in the other; as of α

and a at A , of κ and m at K , &c : and if $m - n$ be not an unit we have a shorter line $\frac{m}{m-n}ab = \frac{n}{m-n}AB = AX$ or XK , equal to the length of the apparent period of their nearest approaches towards coincidences; as on each side of the point X , according to the demonstration of the VIIth proposition.

But if the point z be so situated, that the lines AB and ab or $\alpha\beta$, or Bz and βz , or Cz and γz , &c, which are all in the same ratio, happen to be incommensurable, it will be impossible, mathematically speaking, for more than one couple of objects to appear coincident (*a*), and yet the periods of their apparent approaches will subsist in this case as well as in the other.

Now if the objects be white, or of any colour that reflects more light to the eye than what comes to it from the spaces between them, and their breadth be considerable as usual, the rows will appear

(*a*) See pag. 122. line 3.

Prop. XI. HARMONICS. 131

appear the least luminous about the coincident objects and the periodical points, A, X, K , &c, where the objects of the nearer row hide the whole or some part of those behind them in the remoter row; and the rows will appear gradually more luminous towards the middle of the periods, where the objects will be seen distinct from one another if they be not too broad. And the contrary will happen if the objects in the rows be less luminous than the spaces between them.

Consequently if the spectator stands still and moves his eye from one end of the rows to the other, he will see an alternate succession of light and shade; and while he moves forwards in any transverse direction $\alpha\omega$, and fixes his eye upon a given place of the rows, he will then see an undulation of light and shade, moving forwards quicker or slower according to the celerity of his own motion.

For then the apparent coincidences which were at $A, K, \&c$, and consequently the intermediate periodical points $X, Y, \&c$, will gradually shift from A to $B, \&c$, and from K to $L, \&c$, as is evident from the angular motion of the visual rays about the fixt points or objects $\alpha, \beta, \gamma, \&c, \kappa, \lambda, \mu, \&c$: And this is a known phænomenon.

If the spectator recedes from the rows, the period $\frac{m}{m-n}ab$ will grow longer, and then upon moving transversely, the visible undulations will be broader and slower than before, and at a very great distance from the rows, will become imperceptible; as being changed into an uniform appearance of both rows in the place of one: Quite analogous to the audible undulations of imperfect unisons, as they grow slower and less perceptible while the unisons are approaching to perfection.

The like phænomenon results from two rows of pales that meet in any angle.

PRO-

PROPOSITION XII.

Imperfect consonances of the same sort or name are equally harmonious when their short cycles are equally numerous in the periods of their imperfections.

If all the tempered intervals of any one sort or name be equal to one another in any system, those of any other sort will be every where equal and equally tempered. For instance, if the intervals between any two octaves be equal, the interval between their bases will be equal to the interval between their trebles; which is plain from the ratios of the times C, c, G, g of their single vibrations. For if $C : c :: G : g$, then $C : G :: c : g$. But if the former ratios and intervals be unequal, the latter will be so too: and conversely.

Therefore as it is universally agreed, and with good reason too (*a*), to have all the octaves equal by making them perfect, it appears that all tempered intervals of the same name must have equal temperaments; these being the

(*a*) See Prop. XVI. Schol. 2. Art. 12. diffe-

differences of the tempered and perfect intervals.

Consequently the lengths of the periods of the least imperfections of all consonances of the same name, are in the ratio of the times of the single vibrations of their bases or trebles (*a*), or of the lengths of their short cycles, as being equimultiples of the times of the vibrations. The short cycles are therefore equally numerous in the periods of their imperfections.

And whatever be the quantity of the temperament of any consonances of the same name, as of v^{ths} for instance, I believe no musician ever doubted their being equally harmonious, or agreeable to his ear, in every place of the scale, when well tuned. Q. E. D.

Coroll. Consonances of the same name are equally harmonious when equally tempered.

Scholium.

After an organ had been well tuned by making all the tempered v^{ths} as equally

(*a*) Prop. ix. coroll. 2.

qually

qually harmonious as the ear could determine, I found that the numbers of their beats, made in equal times, were in the inverse ratio of the times of the single vibrations of their bases or trebles, as nearly as could be expected: or that the times between their successive beats, which are equal to the periods of their least imperfections (*a*), were in the direct ratio of those homologous vibrations, or of equimultiples of them, or of the lengths of the short cycles; which therefore were equally numerous in those periods.

PROPOSITION XIII.

Imperfect consonances of different sorts are equally harmonious, in their kind, when their short cycles are equally numerous in the periods of their imperfections.

Fig. 34. The times of the single vibrations of imperfect unisons being represented by *AB* and *ab*, let *AD* and
ac,

(*a*) Prop. xi.

ac , that is $3AB$ and $2ab$ be those of imperfect v^{th} s. And one length of their imperfect short cycle being $2AD = AG$, and the other being $3ac = ag$, their difference Gg is the dislocation of the pulses G, g at the end of the first short cycle $AagG$, measured from the coincident pulses Aa . And the greater of the two dislocations which terminate the several succeeding cycles, is double, triple, &c of Gg (a).

Again, conceiving the pulses c, g, l , &c, to be now intermitted, let AD and ae , that is $3AB$ and $4ab$ be the single vibrations of imperfect 4^{th} s. And the two lengths of their first short cycle $ANna$ being $4AD = AN$ and $3ae = an$, their difference Nn is the dislocation of the pulses N, n at the end of that cycle; and in the several succeeding cycles the greater of the two dislocations is double, triple, &c of Nn .

And the common period AZ or an of those dislocations or imperfections in the short cycles of the v^{th} s and 4^{th} s,

(a) Prop. VII.

is

is the same as the period or simple cycle of the pulses of the vibrations AB , ab of the imperfect unisons (*b*).

Now the two dislocations Gg , Nn , in the first imperfect cycles of the v^{th} and 4^{th} in that period, are in the ratio of AG to AN (*c*), the lengths of the cycles, that is of $2AD$ to $4AD$, or 1 to 2: and the two greater dislocations Xy , Qr , in the last imperfect cycles $Xy\epsilon\Delta$, $Qr\epsilon\Delta$, in the same period AZ , are in the ratio of their distances ZX , ZQ , from this end of it: and this ratio is less than that of ΔX to ΔQ , or 1 to 2. But the two greater dislocations $K\lambda$, $\Pi\sigma$ in the subsequent cycles $K\lambda\epsilon\Delta$, $\Pi\sigma\epsilon\Delta$, of the next period, are in the ratio of ZK to $Z\Pi$, which, on the contrary, is greater than that of ΔK to $\Delta\Pi$, or 1 to 2.

The periods must be conceived to contain a much greater number of short cycles than can be well represented in a scheme. And then, as the corresponding dislocations in the v^{th} and 4^{th} lye farther and farther from Z , the ratio of

(*b*) Prop. VIII. (*c*) Prop. VII. their

their distances and magnitudes will approach nearer and nearer to 1 to 2.

Therefore 1 to 2, or the ratio of the lengths of the short cycles of the v^{th} and 4^{th} , is either the exact or the mean ratio both of the greater and the lesser dislocations in all their corresponding short cycles: because the lesser of the increasing dislocations in any subsequent cycle, is the same as the greater in the antecedent one.

Now while the length AG or ag remains unaltered, imagine the dislocation Gg of the v^{th} to be increased in that ratio of 1 to 2, and then it will be equal to the former magnitude of the dislocation Nn of the 4^{th} , or to Nn in Fig. 35, supposing the pulses C, G, L , &c to be absent. And the first dislocation Bb of the pulses B, b , of the imperfect unisons, being at the same time increased in the same ratio, their period AZ , which is also that of the dislocations in the v^{th} (a), will be diminished in that ratio inverted (b). And thus the present period

(a) Prop. VIII.

(b) Prop. VII. coroll. 5 and the note to it.

period of the imperfect v^{th} s and the former period of the 4^{th} s, are in the ratio of the lengths of their short cycles; which therefore are equally numerous in their respective periods.

And since the greater and lesser dislocations at the ends of the corresponding subsequent short cycles of the v^{th} s and 4^{th} s, are now respectively equal, either exactly or at a medium of one with another, and equally numerous too, the whole periods composed of these short cycles, will be equally harmonious. Because those equal dislocations of the pulses in the corresponding short cycles, are the causes that spoil their harmony: and causes constantly equal will have equal effects.

The conclusion will be the same if the dislocation Nn , in the first cycle of the 4^{th} s in either figure, be contracted to the magnitude of the dislocation Gg belonging to the v^{th} s in the other. For then the new period of the 4^{th} s, being double of the old one (a), will be

(a) Prop. VII. coroll. 5 and the note to it. to

to the old one, or that of the v^{th} , as AN to AG , that is, in the ratio of the lengths of their short cycles, which therefore are equally numerous in these periods: and the dislocations at the ends of the several subsequent short cycles of the 4^{th} , being likewise contracted to the respective magnitudes of those of the v^{th} , the consonances are again made equally harmonious.

And lastly, since either of those consonances is equally harmonious to another of the same name, at any other pitch, when their short cycles are equally numerous in their periods (*a*), it appears that 4^{th} and v^{th} are equally harmonious at any pitches, when their short cycles are equally numerous in their periods. And the like proof is plainly applicable to any other case of these or any other consonances: I mean when their common period, or that of the imperfect unisons, is terminated at first either by coincident pulses or periodical points, as will plainly appear by conceiving a short cycle or two to

(*a*) Prop. XII.

result from a proper intermission of the pulses of imperfect unisons, on each side of such points in fig. 24. 25. Q. E. D.

Coroll. 1. Imperfect consonances are more harmonious in the same order as their short cycles are more numerous in the periods of their imperfections.

For let any two imperfect consonances be supposed equally harmonious; and their short cycles will be equally numerous in their periods, by the proposition. Then if either of the given periods be lengthened, the short cycles will be more numerous in it, and the dislocations of their pulses will be smaller than they were before, and consequently will first increase and then decrease by smaller degrees from one end of the period to the other. And thus the consonance will be more harmonious than it was at first, or than the other given consonance.

And on the contrary, if the period of either of them be shortened, the

number of its short cycles will be diminished, and the dislocations of their pulses will be increased. And thus the consonance will be less harmonious than it was before, or than the other given consonance.

Coroll. 2. Imperfect consonances are more harmonious in the same order, as their temperaments multiplied by both the terms of the ratios of the vibrations of the perfect consonances, are smaller; and are equally harmonious when those products are equal.

Fig. 34. 35. The vibrations of imperfect unisons being AB and ab , and r and s being the terms of any musical ratio, rAB and sab will be the single vibrations of the tempered consonance; and the period of its imperfections will be the same as that of the pulses of AB and $ab(a)$; which period, supposing $AB : ab :: m : n$ in the least integers, and $d = m - n$, is $\frac{n}{d} AB$. Call it p .

And

(a) Prop. VIII.

Prop. XIII. HARMONICS. 143

And since $rAB : sAB :: r : s$, the short cycle of the vibrations rAB, sAB of the perfect consonance, is $rsAB$. Call it c .

Then $\frac{p}{c} = \frac{nAB}{d} \times \frac{1}{rsAB} = \frac{1}{rs} \times \frac{n}{d} = \frac{1}{rs} \times \frac{1}{t}$ by taking $t = \frac{d}{n}$, which being the difference of the terms of the small ratio of AB to ab divided by one of them, is very nearly the measure of that ratio (a), or of the ratio of the vibrations rAB to rab , or of sAB to sab , or of the interval of their sounds (b), which is the temperament of the imperfect consonance.

Therefore in the same order in which the values of $\frac{p}{c}$ or $\frac{1}{rst}$ are greater, or the values of rst are smaller, the corresponding consonances are more harmonious, by corol. 1; and are equally harmonious when the values of rst are equal, by the present proposition.

Coroll.

(a) Note in page 91. N^o. 2. (b) Sect. 1. Art. 10.

Coroll. 3. Consequently imperfect consonances are equally harmonious when their temperaments have the inverse ratio of the products of the terms of the ratios of the vibrations of the perfect consonances.

For when the values of the product $rs \times t$ are equal, the values of t have the inverse ratio of the values of rs .

Coroll. 4. When the products rs of the terms of those musical ratios are equal, the tempered consonances are more harmonious in the same order as their temperaments are smaller; and are equally harmonious if their temperaments be equal.

For if the values of rs or $\frac{1}{rs}$ be equal, the values of $\frac{p}{c} = \frac{1}{rs} \times \frac{1}{t}$ are greater in the same order as those of $\frac{1}{t}$ are greater, or as those of t are smaller; and are equal when the values of t are equal.

Coroll. 5. Therefore consonances of the same name are more harmonious
in

Prop. XIII. HARMONICS. 145

in the same order as their temperaments are smaller; and are equally harmonious when they are equal.

For the vibrations of those consonances are in the same ratio, and the terms of it, being the least possible, are the same, and their product the same also, as in the last corollary.

Coroll. 6. Consonances equally tempered are more harmonious in the same order as the products of the terms of their musical ratios are smaller; and are equally harmonious when those products are equal.

For the values of t being supposed equal, those of $\frac{p}{c} = \frac{1}{rs} \times \frac{1}{t}$ are greater in the same order as the values of $\frac{1}{rs}$ are greater, or as those of rs are smaller; and the former values are equal when the latter are so.

Coroll. 7. Consonances equally tempered are generally more harmonious in the same order as they are simpler,

the pure ones chiefly excepted (*a*), which are more harmonious than some others that are simpler; though seperately considered they follow that order exactly.

This will appear from the last corollary by a series of the products of the terms of the ratios in the first column, compared with the series of numbers in the second column of the table in Sect. III. Art. 5, shewing the order of the simplicity of consonances.

Coroll. 8. Consequently simpler consonances will generally bear greater temperaments than the less simple will; or the less simple ones generally speaking will not bear so great temperaments as the simpler will: contrary to the common opinion. (*b*).

Coroll.

(*a*) Sect. III. Art. 8.

(*b*) Octavarum autem omnium unica est species, eaque perfecta ratione 1 ad 2 contenta. Hoc enim intervallum, propter perfectionem, vix aberrationem à ratione 1 ad 2 pati posset, quin simul auditus ingenti molestia afficeretur. Namque quo perfectius per-

Coroll. 9. The tempered concords in the system of mean tones (*c*) are not equally harmonious in their kinds.

For by coroll. 6, and by inspection of the terms of the musical ratios annexed to the characters of the concords in the first of the tables in the next section, it will appear, that the vth and 4th and their compounds with VIII^{ths}, are more harmonious than the VIth and 3^d and their compounds with equal numbers of VIII^{ths}, as being all equally tempered in that system (*d*).

Coroll. 10. The harmony of those concords is still more unequal in the *Hugenian* system, resulting from a division of the octave into 31 equal intervals (*e*).

Be-

perceptuque facilius est intervallum, eo magis sensibilis fit error minimus; minus autem sentitur exigua aberratio in intervallis minus perfectis. *Tentamen novæ Theoriæ musicæ*, cap. IX. sect. 10. *Petropoli* 1739.

(*c*) Prop. II. (*d*) Prop. III. coroll. 3.

(*e*) See Prop. XVII, scholium.

Because the common temperament of the v_1^{th} and 3^{d} and their compounds with v_{III}^{ths} , which by coroll. 4 and 9, should be smaller than that of the v^{th} and 4^{th} and their compounds with v_{III}^{ths} , to render them equally harmonious, is on the contrary something greater.

Coroll. 11. Imperfect consonances are more harmonious both as they beat slower, and as the cycles of the perfect consonances are shorter.

For the quantities $\frac{p}{c}$ will be greater on both accounts (*a*) and the harmony better (*b*).

Coroll. 12. Consonances to a common base whose vibration is denoted by 1, as in Fig. 3, are more harmonious in the same order as the numbers of their beats, made in equal times and multiplied by the numerators of the fractions $\frac{s}{r}$, denoting the vibrations of the trebles, are smaller.

For

(*a*) Prop. XI.

(*b*) Prop. XII and XIII.

Prop. XIII. HARMONICS. 149

For the vibrations of the base and treble being 1 and $\frac{s}{r}$, and 1 being to $\frac{s}{r}$ as r to s , the short cycle of their vibrations is $s \times 1 = c$: and putting β for the number of beats made in any given time, we have the period p as $\frac{1}{\beta}$: because p is equal to the time between the successive beats (a), which is inversely as β or directly as $\frac{1}{\beta}$. Therefore the indeterminate quantity $\frac{p}{c}$ is as $\frac{1}{\beta c}$ or as $\frac{1}{\beta s}$; that is, the values of the quantity $\frac{p}{c}$ have constantly the same ratio as those of $\frac{1}{\beta s}$ have.

Coroll. 13. And universally, if the vibrations of the bases and trebles of several perfect consonances be indeterminate represented by V and $\frac{s}{r} V$, when they are tempered their harmony

(a) Prop. XI.

ny will be as $\frac{1}{\beta_s V}$; that is, it will be better in the same order as the values of $\frac{1}{\beta_s V}$ are greater, or as those of $\beta_s V$ are smaller; and equally good when they are equal.

For now $sV = c$, because $V : \frac{s}{r}V :: r : s$.

SECTION VII.

Of a system of sounds wherein as many concords as possible, at a medium of one with another, shall be equally and the most harmonious.

DEFINITION I.

The Arithmetical Mean among any number of quantities, is to the sum of them under their given signs, as an unit is to their number; and has the same sign as their sum has: Or if they be expressed by numbers, it is the quotient of their sum divided by their number.

Thus

Defin. I. HARMONICS. 151

Thus the arithmetical mean among the quantities $a, b, c, -d$ is $\frac{a+b+c-d}{4}$.

Coroll. 1. Hence the sum of the excesses of all the greater quantities above their arithmetical mean, is equal to the sum of the defects of all the smaller from the same.

For let the arithmetical mean $\frac{a+b+c-d}{4} = r$, then $a+b+c-d = 4r = r+r+r+r$. Whence if a and b be severally greater than r , we have $a-r+b-r = r-c+r+d$.

Fig. 40. Accordingly if the lines $ao, bo, co, -do$ and ro , be proportional to $a, b, c, -d$ and r , the sum of the parts ra, rb on one side of the point r , is equal to the sum of the parts rc, rd on the other side of it.

Coroll. 2. If any quantity q be added to, or taken from every one of the quantities $a, b, c, -d$, their arithmetical mean will be augmented or diminished by that quantity q .

For

For let $a + b + c - d = 4r$, then r is their arithmetical mean. But $a + q + b + q + c + q - d + q = 4r + 4q = 4 \times r + q$, and therefore $r + q$ is the arithmetical mean among those augmented quantities $a + q, b + q, c + q, -d + q$: and by changing the sign of every q , it appears that $r - q$ is the like mean among the diminished quantities $a - q, b - q, c - q, -d - q$.

Coroll. 3. If every one of the quantities $a, b, c, -d$, be increased or diminished in any given ratio of 1 to n , their arithmetical mean will also be increased or diminished in the same given ratio.

For let $a + b + c - d = 4r$, then r is their arithmetical mean. But $na + nb + nc - nd = 4nr$, and therefore nr is the arithmetical mean among the quantities $na, nb, nc, -nd$.

DEFINITION. II.

The Harmonical Mean among any number of quantities, is the reciprocal of the arithmetical mean among their reciprocals.

Fig. 4. In an hyperbola, for instance, where the ordinates parallel to an asymptote are the reciprocals of their abscisses, measured from the center upon the other asymptote; if an absciss ro be the arithmetical mean among the abscisses ao, bo, co, do , its ordinate rg is the harmonical mean among the ordinates $a\alpha, b\beta, c\gamma, d\delta$ by the definition, rg being the reciprocal of the arithmetical mean ro among their reciprocals, ao, bo, co, do .

Likewise if an ordinate $m\mu$ be the arithmetical mean among the ordinates $a\alpha, b\beta, c\gamma, d\delta$, its absciss mo is the harmonical mean among their abscisses, ao, bo, co, do .

Coroll.

Coroll. 1. The arithmetical mean is greater than the harmonical mean among the same quantities, if they all have the same sign.

For let the line $\beta\epsilon$, produced through the top of either of the ordinates next to $r\epsilon$, cut the rest in f, g, h , and the asymptote ro in e . Then because ro is the arithmetical mean among ao, bo, co, do , the line re is the arithmetical mean among the lines ae, be, ce, de (a); and $r\epsilon$ the arithmetical mean among the proportional lines $af, b\beta, cg, dh$ (b), which, excepting the common ordinate $b\beta$, are severally smaller than the hyperbolic ordinates $a\alpha, b\beta, c\gamma, d\delta$; whose arithmetical mean $m\mu$ is therefore greater than $r\epsilon$ (c), the harmonical mean among the same ordinates.

Coroll. 2. The difference between the arithmetical and the harmonical means among the same quantities, will be very

(a) Defin. I. coroll. 1 or 2. (b) Defin. I. coroll. 3.
 (c) Defin. I.

very small when the differences of the quantities themselves are so.

This will appear by conceiving the ordinates $a\alpha, b\beta, c\gamma, d\delta$ to approach gradually towards one another till they coincide. For then the differences between the hyperbolical ordinates $a\alpha, b\beta, c\gamma, d\delta$, and the lines $af, b\beta, cg, db$, and consequently between their arithmetical means $m\mu, r\varrho$, will gradually decrease to nothing. But $r\varrho$ is also the harmonical mean among those ordinates.

Coroll. 3. By increasing every quantity in any given ratio, the harmonical mean among them will be increased in the same ratio.

For the reciprocals of the increased quantities, and the arithmetical mean among them (*a*), will severally be diminished in that ratio; and the reciprocal of this mean, which is the harmonical mean among the increased quan-

(*a*) Defin. I. coroll. 3.

quantities, will of consequence be increased in the same ratio.

Coroll. 4. Fig. 42, 43. Whatever be the signs of the proposed quantities $a\alpha$, $b\beta$, $c\gamma$, $d\delta$, their harmonical mean $r\varrho$ has always the sign of the sum of their reciprocals $a\sigma$, $b\sigma$, $c\sigma$, $d\sigma$, or of, $r\sigma$, the arithmetical mean among them.

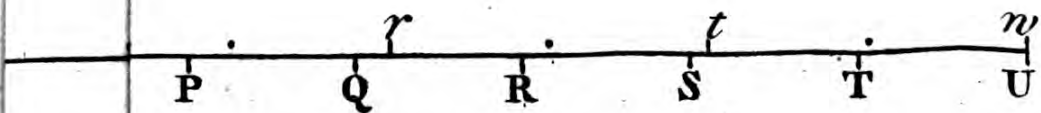
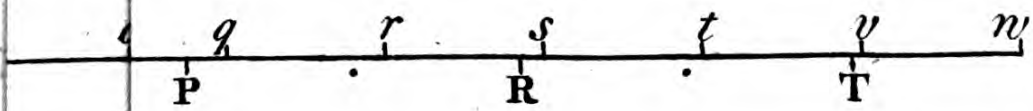
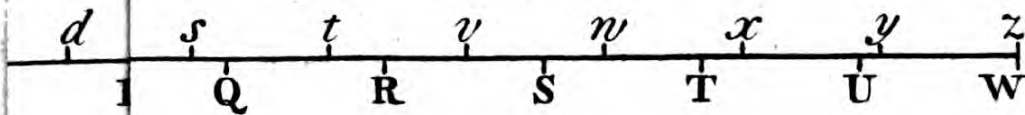
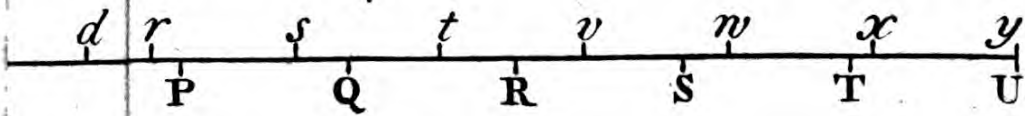
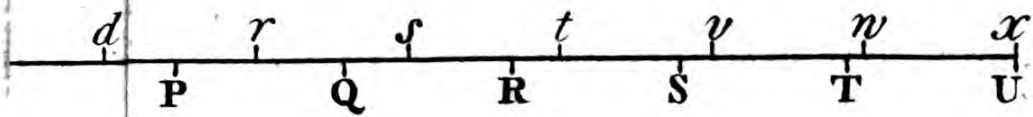
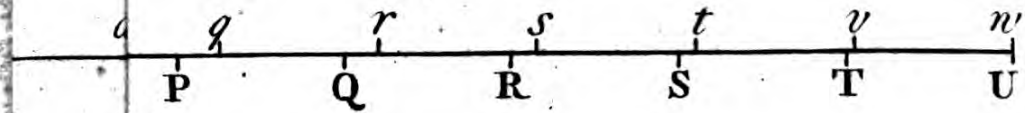
For the reciprocal of each quantity has the sign of the quantity it self, and according as their sum is affirmative or negative, so is their arithmetical mean (b), and so is its reciprocal, or the harmonical mean among the proposed quantities.

PROPOSITION XIV.

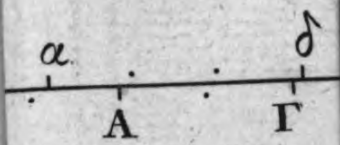
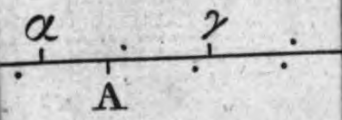
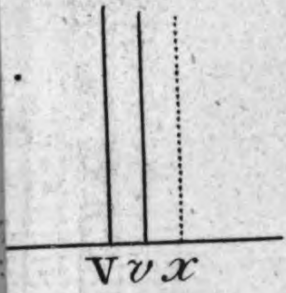
Instead of several imperfect concords resulting from the same perfect one differently tempered, if it be necessary to use but one, let the period of its imperfections be the arithmetical mean among all the periods of those concords, and it will best answer the several purposes of every one.

(a) Defin. 1.

Be-







$$\frac{S}{r}V \quad \frac{S}{r}U$$





Because the excesses of the longer periods above the arithmetical mean are equal, one with another, to the defects of the shorter from the same, and because the arithmetical mean period is longer (*a*) and therefore more harmonious (*b*) than the harmonical mean among the same. Q. E. D.

PROPOSITION XV.

The tempered concords in any one of the parcels derived from the Vth, VIth or III^d (c) whatever be their common temperament, are constantly more harmonious in the same immutable order as the products of the terms of the ratios of the vibrations of the perfect concords are smaller; and those only are equally harmonious which have equal products belonging to them; and no others can be made so, because they cannot have different temperaments while the octaves are perfect. The

(*a*) Def. 2. coroll. I. Sect. VII. (*b*) Prop. XIII. coroll. I. (*c*) In the scholium to prop. III, the concords were distributed into three parcels, which may be seen at one view in Table I pag. 173.

The truth of the proposition appears from prop. XIII. coroll. 6. Q. E. D.

PROPOSITION XVI.

To find the temperament of a system of sounds of a given extent, wherein as many concords as possible, at a medium of one with another, shall be equally and the most harmonious.

Into the second and third columns of the 11^d Table following, transfer every couple of concords in the given system, whose characters can be taken from the different parcels in the 1st Table; omitting all other couples whose characters are both situated in any one of the parcels. And after each couple place the ratio of the temperaments which would make the two concords equally harmonious in their kind (a).

Then will the corollaries to the 14th, 15th and 16th propositions give the temperaments themselves, or the positions of the temperers *Orst*, *Orst*, &c, in Fig. 44, belonging to every one of those ratios.

(a) Prop. XIII. coroll. 3.

Among

Prop. XVI. HARMONICS. 159

Among the temperaments in the three several parcels $Gr, Gr, \&c, As, As, \&c, Et, Et, \&c$, taking three harmonical means GD, AH, EM , and transferring them to Fig. 45, draw three temperers $Def, Oghi, OklM$, and taking Eq the arithmetical mean among the three temperaments EM, Ef, Ei , the temperer $Onpq$ will approach very near to the position required in the proposition.

If greater exactness be desired, among the three temperaments in the three several parcels $GD, Gg, Gk; AH, Ae, Al; EM, Ef, Ei$, taking three harmonical means GD', AH', EM' , and transferring them to Fig. 46, draw three new temperers $OD'ef', Og'H'i', Ok'l'M'$; and taking Eq' the arithmetical mean among the three temperaments EM', Ef', Ei' , the temperer $Gn'p'q'$ will approach still nearer towards the required position.

And by repeating the like construction we may approach as near as we please (a). Q. E. I.

THE DEMONSTRATION.

In combining the concords all the couples whose characters are both in any one of the parcels in Tab. I are omitted; their harmony with respect to one another, or the proportions of their periods being immutable, by reason of their common temperament (*b*).

Fig. 44. Now supposing *Ga* to be the arithmetical mean among all the temperaments *Gr*, *Gr*, &c, of the first parcel of concords, and drawing the temperer *Oabc*, the temperaments *Ab* and *Ec* will be the like Means among *As*, *As*, &c, and *Et*, *Et*, &c (*c*). Whence if the periods of concords were proportional to their temperaments, the best temperer would be *Oabc* (*d*):

Because the period of the v^{th} , for instance, belonging to the temperament
Ga,

(*a*) Want of room in such small plates made it necessary to alter the true proportions of the lines in the figures; otherwise some parts of them would have appeared confused. (b) Prop. xv.

(c) Def. I. coroll. 1. 2. 3. Sect. VII. (d) Prop. xiv.

Prop. XVI. HARMONICS. . 161

$G a$, would then be the arithmetical mean among all the other periods of the v^{th} answering to the several temperaments $G r$, $G r$, &c. And the like may be said of the periods of the 4th and of every other concord in this first parcel, as having the temperaments $G r$, $G r$, &c, common to them all, and likewise of the periods of the several concords in the other two parcels, with respect to their temperaments $A b$, $A s$, &c, and $E c$, $E t$, &c.

But since the periods of given concords to a given base are not directly but inversely proportional to their temperaments (a), the period of the v^{th} , or any other concord, belonging to the arithmetical mean temperament $G a$, is not the arithmetical, but the harmonical mean among the other periods of that concord, answering to the temperaments $G r$, $G r$, &c; and consequently is shorter (b) and therefore less harmonious than

(a) Prop. ix. coroll. 4. (b) Def. 2. coroll. 1. Sect. vii.

than the arithmetical mean period among them (*a*) answering to the harmonical mean temperament *GD*. And what has been said of the periods in that parcel is applicable to those of the other two, with respect to the arithmetical and harmonical mean temperaments *Ab* and *AH*, *Ec* and *EM*.

Hence the arithmetical mean periods belonging to the harmonical mean temperaments *GD*, *AH*, *EM*, would best answer the design of the proposition, if the points *D*, *H*, *M* were all situated in one temperer.

For since the sums of the temperaments terminated at the several temperers *Orst*, *Orst*, &c, are the least that can render the concords in each couple equally harmonious in their kind (*b*), it follows that the sums of all the temperaments *Gr*, *Gr*, &c, in the first parcel, of all the temperaments *As*, *As*, &c, in the second, and of all the temperaments *Et*, *Et*, &c, in the third, taking one
sum

(*a*) Prop. XIII, coroll. I. (*b*) Prop. IV. v. VI.

Prop. XVI. HARMONICS. 163

sum with another, are also the least possible: the sum total being the same in both distributions of the particulars.

The sum of the harmonical temperaments GD , AH , EM being therefore the least possible (*a*), and that of all the corresponding arithmetical mean periods being the greatest (*b*), would render the system of periods, at a medium of one with another, the most harmonious.

But in reality the three harmonical points D , H , M can never fall into any one temperer. Because the harmonical means GD , AH , EM are less than the respective arithmetical means Ga , Ab , Ec (*c*), which are all terminated by one temperer $Oabc$.

Fig. 45. In the solution of the problem we reduced the three temperers $ODef$, $OgHi$, $OklM$ to one, by so drawing the temperer $Onpq$, as to make Eq the arithmetical mean among EM , Ef , Ei , and consequently Ap the like mean

(a) Def. 1 and cor. 1. Def. 2. Sect. VII. among

(b) Prop. IX. cor. 4. (c) Def. 2. cor. 1. Sect. VII.

among AH, Ae, Al and Gn the like among GD, Gg, Gk (a).

Now the differences of the three temperaments in each of those parcels being but small, as will appear by the following calculation (b), the arithmetical means among them will differ but little from the respective harmonical means among the same (c), which would be fitter for the purpose if their extremities D', H', M' could be situated in any one temperer (d). Consequently as the temperer $Onpq$ falls in the middle among the three temperers conceived to pass through the harmonical points D', H', M' , it will nearly answer the several purposes of those three, and approach very near to the situation of the required temperer.

Fig. 46. Hence and by prop. XIV, it appears that a repetition of this last construction, as described in the solution, will give a temperer $On'p'q'$ approaching

(a) Def. 1. coroll. 1. 2. 3. (b) Tab. VI column 2.
 (c) Def. 2, coroll. 2. (d) Prop. XIV.

Prop. XVI. HARMONICS. 165

proaching still nearer to the required situation. Because the latter temperaments EM' , Ef' , Ei' differ less from one another (*a*) and consequently from their arithmetical mean Eq' , than the former, EM , Ef , Ei did from one another and from their arithmetical mean Eq .

And as the same may be said of the temperaments of the other two parcels, it appears that by a further repetition of the same construction, we may find a temperer approaching as near as we please towards the position required in the proposition. Q. E. D.

Coroll. 1. Fig. 45. The comma, or four times the line GI , being the unit, and supposing any three temperaments of different parcels to be given, as $GD=d$, $AH=b$ and $EM=m$, it will be easy to collect, (from the similar triangles under the line OIE , the three temperers $ODef$, $Oghi$, $oklM$, and the three parallels GD , AH , EM ;) that $Gg = \frac{1-b}{3}$ and $Gk = \frac{1+m}{4}$; $Ae = 1-3d$ and

(*a*) Tab. VII at the end of it.

and $Al = \frac{1-3m}{4}$; $Ef = 4d - 1$ and $Ei = \frac{1-4b}{3}$; provided the three temperers be all situated within the angle EOA ; but if OH or OM lyes out of it beyond A or E respectively, the sign of b or m will accordingly be changed in those theorems.

Coroll. 2. Hence we have the three arithmetical mean temperaments,

$$Gn = \frac{1}{3} \times d + \frac{1-b}{3} + \frac{1+m}{4}, \quad Ap = \frac{1}{3} \times \frac{1-3d+b+\frac{1-3m}{4}}{4d-1+\frac{1-4b}{3}+m}, \quad Eq = \frac{1}{3} \times$$

Scholium. I.

Fig. 47 serves to illustrate part of the demonstration of the proposition, by representing to the eye the proportions of the periods of the concords. It is thus constructed. The line AI being parallel to EO , the middlemost hyperbolas $v \frac{1}{5}$, $x \frac{1}{6}$, $y \frac{1}{10}$, $z \frac{1}{12}$, &c, are drawn

to

Prop. XVI. HARMONICS. 167

to the asymptotes AI, A_3 ; and their ordinates to their common absciss A_3 are made proportional to the fractions $\frac{1}{5}, \frac{1}{6}, \frac{1}{10}, \frac{1}{12}, \&c.$

Hence. the periods of the $VI, 3^d, VI + VIII, 3^d + VIII, \&c,$ when tempered by a quarter of a comma, represented by the common absciss A_3 , are proportional to those ordinates (a); and when they have any other temperament represented by the absciss As , their periods are then proportional to the ordinates $sv, sx, sy, sz, \&c.$

And the like construction being made for the other two parcels of concords, the ordinates erected from the intersections r, s, t of any temperer $Orst$, shew the proportions of the periods in the whole system: and these proportions are the same whatever be the unit of the fractional ordinates.

Scho-

(a) Prop. IX. coroll. 5, and Tab. I. pag. 173.

Scholium 2.

In order to calculate the required temperaments of a system of any given extent, it will not be amiss to explain the following tables.

1. According to the solution of the problem, see whether every two characters, each of which lye in the different parcels of concords in Tab. I, be placed over against one another in the second and third columns of Tab. II Part I.

2. Then examine whether the ratios placed after those characters in the 4th column of that table, be rightly deduced from the fractions annexed to the same characters in Tab. I, according to the rule in prop. XIII. coroll. 3.

3. See whether all the temperaments in Tab. IV be rightly deduced from those ratios in Tab. II, by the corollaries to prop. IV, V, VI; and whether the numbers in the first column of each table correspond to the same ratios and concords.

4. Examine whether the reciprocals in Tab. V, of the temperaments in Tab. IV be right, that is, whether the product of the quotient by the divisor, differs from the dividend by less than half the divisor. When a reciprocal is negative, as coming from a negative temperament, I subtract it from 0 and place the remainder in the table instead of the reciprocal it self. Thus at N^o. 10, —6) 26 (= —4.33333 &c, which subtracted from 0 gives $\bar{5}.66667$ to be transferred to Tab. II Part II and there added to the positive reciprocals, for the sake of uniformity in the work; the integer $\bar{5}$ being only negative and the decimals .66667 affirmative. For n being any given integer, the number

$$n - 4 - \frac{1}{3} = n - 5 + \frac{2}{3}.$$

5. See whether the reciprocals in Tab. V be rightly transferred into the respective columns of Tab. II Part II, which is readily done by means of the corresponding numbers in the first column of each table.

6. Cast up the several dozens of reciprocals in Tab. II Part II, and transfer the sums to Tab. III and there cast them up.

7. Tab. VI is thus deduced from Tab. III. By the solution of the problem the fraction $\frac{12}{42.72013}$, = GD in Fig. 44, is the harmonical mean among the temperaments $Gr, Gr, \&c$; because its reciprocal $\frac{42.72013}{12}$ is the arithmetical mean among their reciprocals, as being their sum divided by their number. The same is to be understood of all the other fractions: and as the value of the temperament Eg , computed from coroll. 2. prop. XVI, comes out affirmative; by the coroll. to prop. IV, V, VI it is part of the interval EC of the perfect III^d, and therefore is a negative temperament of that concord, or an affirmative one of its complement to the octave. This is the first approximation towards the required temperament.

8. Tab. VII contains the calculation of Eg' , the second approximation towards the true temperament of the III^d , in a system whose extent is but one octave, and is sufficiently evident from cor. 1, 2, prop. XVI, and Tab. VI. And by a like calculation the values of Eg' , in a system of two and of three octaves, will be found as put down under those of Eg in Tab. VI; care being taken in the operations to continue the quotients in decimals as far as they are just.

9. Therefore the result of the whole is this. As all the parts of musical compositions, setting aside double basses, are generally contained within three octaves, and as their harmony is stronger and better within that compass than it would be in a larger; I choose to make all the concords within every three octaves equally harmonious and no more, be the extent of the system ever so great; and consequently to diminish the III^d by $\frac{1}{9}$ comma, this being
 ing

ing very nearly the value of the last Eg'
 $= 0.11024$ in Tab. VI.

10. Hence in the system of equal harmony the temperaments of the v^{th} , $v1^{\text{th}}$ and III^{d} are $-\frac{5}{18}$, $+\frac{3}{18}$ and $-\frac{2}{18}$ of a comma respectively (*a*) and are proportional to the musical primes 5, 3 and 2. (*b*)

11. In determining these temperaments of the diatonic system, I have regarded no more consonances than the concords. 1. Because the discords are seldom used than the concords. 2. Because the ear is generally less critical in the discords than in the concords. 3. Because a mean temperament among those of the concords and discords too, would
differ

(*a*) Prop. III. coroll. 2, 3.

(*b*) But if any one chooses to have all the concords in 4 octaves made equally harmonious, he will find by continuing the tables, that the III^{d} must be diminished by $\frac{987}{10000}$ of a comma, which being nearly $\frac{1}{10}$ comma, the temperaments of the v^{th} , $v1^{\text{th}}$ and III^{d} will then be $-\frac{11}{40}$, $+\frac{7}{40}$ and $-\frac{4}{40}$ of a comma respectively.

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differ from that of the concords alone, and therefore be less suitable to them.

12. Lastly I have kept the octave untempered. 1. Because it is the simplest and most harmonious of all the concords, both in itself and its multiples. 2. Because some one interval must be kept perfect, in order to determine the variations of the temperaments of the rest (*c*). 3. Because upon several trials of keeping other intervals untempered instead of the octave, many reasons have occurred to me for rejecting every one of them.

13. Does it not follow then, that the system of equal harmony, as above derived from the best system of perfect intervals (*d*), is the best tempered system that the nature of sounds is capable of? (*e*)

(*c*) Prop. III. (*d*) Sect. IV. Art. 7.

(*e*) See Scholium. Prop. III.

TAB. II. PART I.

N ^o	Ratios of the temperaments for equal harmony of the			
I	v	and	VI as	5 : 2
2	v		3 ^d	5 : 1
3	4 th		VI	5 : 4
I	4 th		3 ^d	5 : 2
4	v		III	10 : 3
5	v		6 th	20 : 3
I	4 th		III	5 : 3
4	4 th		6 th	10 : 3
6	VI		III	4 : 3
7	VI		6 th	8 : 3
I	3 ^d		III	2 : 3
6	3 ^d		6 th	4 : 3
		In one	Octave	1 dozen
2	v	and	VI + VIII	5 : 1
8	v		3 ^d + VIII	10 : 1
I	4 th		VI + VIII	5 : 2
2	4 th		3 ^d + VIII	5 : 1
I	v		III + VIII	5 : 3
9	v		6 th + VIII	40 : 3
10	4 th		III + VIII	5 : 6
5	4 th		6 th + VIII	20 : 3
I	VI		III + VIII	2 : 3
11	VI		6 th + VIII	16 : 3
12	3 ^d		III + VIII	1 : 3
7	3 ^d		6 th + VIII	8 : 3
				2 dozen



TABLE I. facing pag. 175.

Contains the characters and terms of the ratios of the times of the vibrations of all the concords.

1 st Parcel.	2 ^d Parcel.	3 ^d Parcel.
V. $\frac{2}{3}$	VI. $\frac{3}{5}$	III. $\frac{4}{5}$
4 th . $\frac{3}{4}$	3 ^d . $\frac{5}{6}$	6 th . $\frac{5}{8}$
V + VIII. $\frac{1}{3}$	VI + VIII. $\frac{3}{10}$	III + VIII. $\frac{2}{5}$
4 th + VIII. $\frac{3}{8}$	3 ^d + VIII. $\frac{5}{12}$	6 th + VIII. $\frac{5}{16}$
V + 2VIII. $\frac{1}{6}$	VI + 2VIII. $\frac{3}{20}$	III + 2VIII. $\frac{1}{5}$
4 th + 2VIII. $\frac{3}{16}$	3 ^d + 2VIII. $\frac{5}{24}$	6 th + 2VIII. $\frac{5}{32}$
V + 3VIII. $\frac{1}{12}$	VI + 3VIII. $\frac{3}{40}$	III + 3VIII. $\frac{1}{10}$
4 th + 3VIII. $\frac{3}{32}$	3 ^d + 3VIII. $\frac{5}{48}$	6 th + 3VIII. $\frac{5}{64}$
&c.	&c.	&c.
&c.	&c.	&c.

TAB. II. PART II.

N ^o	Reciprocals of the temperaments of the		
	v, 4 th & Comp.	vi, 3 ^d & Comp.	iii, 6 th & Comp.
1	3.40000	8.50000	5.66667
2	3.20000	16.00000	4.00000
3	3.80000	4.75000	19.00000
1	3.40000	8.50000	5.66667
4	3.70000	5.28571	12.33333
5	3.85000	4.52941	25.66667
1	3.40000	8.50000	5.66667
4	3.70000	5.28571	12.33333
6	3.57143	6.25000	8.33333
7	3.72727	5.12500	13.66667
1	3.40000	8.50000	5.66667
6	3.57143	6.25000	8.33333
Sums	42.72013	87.47583	126.33334
2	3.20000	16.00000	4.00000
8	3.10000	31.00000	3.44444
1	3.40000	8.50000	5.66667
2	3.20000	16.00000	4.00000
1	3.40000	8.50000	5.66667
9	3.92500	4.24324	52.33333
10	5.20000	2.36364	5.66667
5	3.85000	4.52941	25.66667
1	3.40000	8.50000	5.66667
11	3.84211	4.56250	24.33333
12	3.25000	13.00000	4.33333
7	3.72727	5.12500	13.66667
Sums	43.49438	122.32379	144.44445

TAB. II. PART I.

No	Ratios of the temperaments for equal harmony of the		
2	v + VIII	VI	5 : 1
8	v + VIII	3 ^d	10 : 1
13	4 th + VIII	VI	5 : 8
3	4 th + VIII	3 ^d	5 : 4
5	v + VIII	III	20 : 3
9	v + VIII	6 th	40 : 3
10	4 th + VIII	III	5 : 6
1	4 th + VIII	6 th	5 : 3
1	VI + VIII	III	2 : 3
6	VI + VIII	6 th	4 : 3
12	3 ^d + VIII	III	1 : 3
1	3 ^d + VIII	6 th	2 : 3
			3 dozen
8	v + VIII	VI + VIII	10 : 1
14	v + VIII	3 ^d + VIII	20 : 1
3	4 th + VIII	VI + VIII	5 : 4
1	4 th + VIII	3 ^d + VIII	5 : 2
4	v + VIII	III + VIII	10 : 3
15	v + VIII	6 th + VIII	80 : 3
16	4 th + VIII	III + VIII	5 : 12
4	4 th + VIII	6 th + VIII	10 : 3
12	VI + VIII	III + VIII	1 : 3
7	VI + VIII	6 th + VIII	8 : 3
17	3 ^d + VIII	III + VIII	1 : 6
6	3 ^d + VIII	6 th + VIII	4 : 3
In two		Octaves	4 dozen

TAB. II. PART II.

N ^o	Reciprocals of the temperaments of the		
	v, 4 th & Comp.	vi, 3 ^d & Comp.	iii, 6 th & Comp.
2	3. 20000	16. 00000	4. 00000
8	3. 10000	31. 00000	3. 44444
13	4. 60000	2. 87500	8. 33333
3	3. 80000	4. 75000	19. 00000
5	3. 85000	4. 52941	25. 66667
9	3. 92500	4. 24324	52. 33333
10	5. 20000	2. 36364	5. 66667
1	3. 40000	8. 50000	5. 66667
1	3. 40000	8. 50000	5. 66667
6	3. 57143	6. 25000	8. 33333
12	3. 25000	13. 00000	4. 33333
1	3. 40000	8. 50000	5. 66667
Sums	44. 69643	110. 51129	122. 11111
8	3. 10000	31. 00000	3. 44444
14	3. 05000	61. 00000	3. 21053
3	3. 80000	4. 75000	19. 00000
1	3. 40000	8. 50000	5. 66667
4	3. 70000	5. 28571	12. 33333
15	3. 96250	4. 11688	105. 66667
16	6. 40000	1. 88235	3. 33333
4	3. 70000	5. 28571	12. 33333
12	3. 25000	13. 00000	4. 33333
7	3. 72727	5. 12500	13. 66667
17	3. 14286	22. 00000	3. 66667
6	3. 57143	6. 25000	8. 33333
Sums	44. 80406	168. 19565	188. 98830

TAB. II. PART I.

N ^o	Ratios of the temperaments for equal harmony of the		
8	v	VI + 2VIII	10 : 1
14	v	3 ^d + 2VIII	20 : 1
2	4 th	VI + 2VIII	5 : 1
8	4 th	3 ^d + 2VIII	10 : 1
10	v	III + 2VIII	5 : 6
15	v	6 th + 2VIII	80 : 3
16	4 th	III + 2VIII	5 : 12
9	4 th	6 th + 2VIII	40 : 3
12	VI	III + 2VIII	1 : 3
18	VI	6 th + 2VIII	32 : 3
17	3 ^d	III + 2VIII	1 : 6
11	3 ^d	6 th + 2VIII	16 : 3
			5 dozen
14	v + VIII	VI + 2VIII	20 : 1
19	v + VIII	3 ^d + 2VIII	40 : 1
1	4 th + VIII	VI + 2VIII	5 : 2
2	4 th + VIII	3 ^d + 2VIII	5 : 1
1	v + VIII	III + 2VIII	5 : 3
20	v + VIII	6 th + 2VIII	160 : 3
21	4 th + VIII	III + 2VIII	5 : 24
5	4 th + VIII	6 th + 2VIII	20 : 3
17	VI + VIII	III + 2VIII	1 : 6
11	VI + VIII	6 th + 2VIII	16 : 3
22	3 ^d + VIII	III + 2VIII	1 : 12
7	3 ^d + VIII	6 th + 2VIII	8 : 3
			6 dozen

TAB. II. PART II.

N ^o	Reciprocals of the temperaments of the		
	v, 4 th & Comp.	vi, 3 ^d & Comp.	iii, 6 th & Comp.
8	3. 10000	31. 00000	3. 44444
14	3. 05000	61. 00000	3. 21053
2	3. 20000	16. 00000	4. 00000
8	3. 10000	31. 00000	3. 44444
10	5. 20000	2. 36364	5. 66667
15	3. 96250	4. 11688	105. 66667
16	6. 40000	1. 88235	3. 33333
9	3. 92500	4. 24324	52. 33333
12	3. 25000	13. 00000	4. 33333
18	3. 91429	4. 28125	45. 66667
17	3. 14286	22. 00000	3. 66667
11	3. 84211	4. 56250	24. 33333
Sums	46. 08676	195. 44986	243. 09941
14	3. 05000	61. 00000	3. 21053
19	3. 02500	121. 00000	3. 10256
1	3. 40000	8. 50000	5. 66667
2	3. 20000	16. 00000	4. 00000
1	3. 40000	8. 50000	5. 66667
20	3. 98125	4. 05732	212. 33333
21	8. 80000	1. 51724	2. 16667
5	3. 85000	4. 52941	25. 66667
17	3. 14286	22. 00000	3. 66667
11	3. 84211	4. 56250	24. 33333
22	3. 07692	40. 00000	3. 33333
7	3. 72727	5. 12500	13. 66667
Sums	46. 49541	296. 79147	302. 81310

TAB. II. PART I.

N ^o	Ratios of the temperaments for equal harmony of the		
1	V + 2VIII	VI	5 : 2
2	V + 2VIII	3 ^d	5 : 1
23	4 th + 2VIII	VI	5 : 16
13	4 th + 2VIII	3 ^d	5 : 8
4	V + 2VIII	III	10 : 3
5	V + 2VIII	6 th	20 : 3
16	4 th + 2VIII	III	5 : 12
10	4 th + 2VIII	6 th	5 : 6
12	VI + 2VIII	III	1 : 3
1	VI + 2VIII	6 th	2 : 3
17	3 ^d + 2VIII	III	1 : 6
12	3 ^d + 2VIII	6 th	1 : 3
			7 dozen
2	V + 2VIII	VI + VIII	5 : 1
8	V + 2VIII	3 ^d + VIII	10 : 1
13	4 th + 2VIII	VI + VIII	5 : 8
3	4 th + 2VIII	3 ^d + VIII	5 : 4
1	V + 2VIII	III + VIII	5 : 3
9	V + 2VIII	6 th + VIII	40 : 3
21	4 th + 2VIII	III + VIII	5 : 24
1	4 th + 2VIII	6 th + VIII	5 : 3
17	VI + 2VIII	III + VIII	1 : 6
6	VI + 2VIII	6 th + VIII	4 : 3
22	3 ^d + 2VIII	III + VIII	1 : 12
1	3 ^d + 2VIII	6 th + VIII	2 : 3
			8 dozen

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TAB. II. PART II.

N ^o	Reciprocals of the temperaments of the		
	v, 4 th & Comp.	vi, 3 ^d & Comp.	iii, 6 th & Comp.
1	3.40000	8.50000	5.66667
2	3.20000	16.00000	4.00000
23	6.20000	1.93750	3.18182
13	4.60000	2.87500	8.33333
4	3.70000	5.28571	12.33333
5	3.85000	4.52941	25.66667
16	6.40000	1.88235	3.33333
10	5.20000	2.36364	5.66667
12	3.25000	13.00000	4.33333
1	3.40000	8.50000	5.66667
17	3.14286	22.00000	3.66667
12	3.25000	13.00000	4.33333
Sums	49.59286	99.87361	48.18182
2	3.20000	16.00000	4.00000
8	3.10000	31.00000	3.44444
13	4.60000	2.87500	8.33333
3	3.80000	4.75000	19.00000
1	3.40000	8.50000	5.66667
9	3.92500	4.24324	52.33333
21	8.80000	1.51724	2.16667
1	3.40000	8.50000	5.66667
17	3.14286	22.00000	3.66667
6	3.57143	6.25000	8.33333
22	3.07692	40.00000	3.33333
1	3.40000	8.50000	5.66667
Sums	47.41621	154.13548	101.61111

TAB. II. PART I.

No	Ratios of the temperaments for equal harmony of the		
8	V +2VIII	VI +2VIII	10 : 1
14	V +2VIII	3 ^d +2VIII	20 : 1
3	4 th +2VIII	VI +2VIII	5 : 4
1	4 th +2VIII	3 ^d +2VIII	5 : 2
10	V +2VIII	III +2VIII	5 : 6
15	V +2VIII	6 th +2VIII	80 : 3
24	4 th +2VIII	III +2VIII	5 : 4 ⁸
4	4 th +2VIII	6 th +2VIII	10 : 3
22	VI +2VIII	III +2VIII	1 : 12
7	VI +2VIII	6 th +2VIII	8 : 3
25	3 ^d +2VIII	III +2VIII	1 : 24
6	3 ^d +2VIII	6 th +2VIII	4 : 3
	In three	Octaves	9 dozen

TAB. III.

The numbers and sums of the - - - -		
Dozen	No	v, 4 th & Comp.
In 1 VIII ^{ve} . 1 st	12	42.72013
2 ^d	12	43.49438
3 ^d	12	44.69643
4 th	12	44.80406
In 2 VIII ^{ves} .	48	175.71500
5 th	12	46.08676
6 th	12	46.49541
7 th	12	49.59286
8 th	12	47.41621
9 th	12	53.22812
In 3 VIII ^{ves} .	108	418.53436



C

TABLE IV. Contains all the different temperaments for making every two concords, in the different parcels in 3 Octaves, equally harmonious.

No	V	VI	III	No	V	ments
	4 th &c	3 ^d &c	6 th &c		4 th &c	omp.
1	$\frac{5}{17}^c$	$\frac{2}{17}^c$	$\frac{3}{17}^c$	14	20	66667 00000
2	$\frac{5}{16}$	$\frac{1}{16}$	$\frac{4}{16}^c$	15	80	00000 33333 66667
3	$\frac{5}{19}$	$\frac{4}{19}$	$\frac{1}{19}$	16	5	33333 66667
4	$\frac{10}{37}$	$\frac{7}{37}$	$\frac{3}{37}$	17	7	
5	$\frac{20}{77}$	$\frac{17}{77}$	$\frac{3}{77}$		In tv	44444 33333
6	$\frac{7}{25}$	$\frac{4}{25}$	$\frac{3}{25}$	18	35	66667 33333
7	$\frac{11}{41}$	$\frac{8}{41}$	$\frac{3}{41}$	19	40	33333 33333
	In one Octave			20	160	21053 66667
8	$\frac{10}{31}$	$\frac{1}{31}$	$\frac{9}{31}$	21	5	33333 66667
9	$\frac{40}{157}$	$\frac{37}{157}$	$\frac{3}{157}$	22	13	66667 10256
10	$\frac{5}{26}$	$\frac{11}{26}$	$\frac{-6}{26}$	23	5	33333 16667
11	$\frac{19}{73}$	$\frac{16}{73}$	$\frac{3}{73}$	24	5	33333 18182
12	$\frac{4}{13}$	$\frac{1}{13}$	$\frac{3}{13}$	25	25	58333 16667
13	$\frac{5}{23}$	$\frac{8}{23}$	$\frac{-3}{23}$		In th	

D

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TAB. II. PART II.

N ^o	Reciprocals of the temperaments of the		
	v, 4 th & Comp.	vi, 3 ^d & Comp.	iii, 6 th & Comp.
8	3. 10000	31. 00000	3. 44444
14	3. 05000	61. 00000	3. 21053
3	3. 80000	4. 75000	19. 00000
1	3. 40000	8. 50000	5. 66667
10	5. 20000	2. 36364	5. 66667
15	3. 96250	4. 11688	105. 66667
24	13. 60000	1. 28302	2. 58333
4	3. 70000	5. 28571	12. 33333
22	3. 07692	40. 00000	3. 33333
7	3. 72727	5. 12500	13. 66667
25	3. 04000	76. 00000	3. 16667
6	3. 57143	6. 25000	8. 33333
Sums	53. 22812	245. 67425	172. 07164

TAB. III.

----- reciprocals of the temperaments	
vi, 3 ^d & Comp.	iii, 6 th & Comp.
87. 47583	126. 33334
122. 32379	144. 44445
110. 51129	122. 11111
168. 19565	188. 98830
488. 50656	581. 87720
195. 44986	243. 09941
296. 79147	302. 81310
99. 87361	48. 18182
154. 13548	101. 61111
245. 67425	172. 07164
1480. 43123	1449. 65428

TAB. VI.

*The values of E_q and E_q' in Fig. 45 and 46, ---
wards the temperament of the 111^d , for ---
equally and the most harmonious.*

In 1 Octave.

$$\frac{12}{42.72013} = 0.2808980 = GD = d$$

$$\frac{12}{87.47583} = 0.1371808 = AH = b$$

$$\frac{12}{126.33333} = 0.0949868 = EM = m$$

In 2 Octaves.

$$\frac{48}{175.71500} = 0.2731696 = GD = d$$

$$\frac{48}{488.50656} = 0.0982587 = AH = b$$

$$\frac{48}{581.87720} = 0.0824916 = EM = m$$

In 3 Octaves.

$$\frac{108}{418.53436} = 0.2580433 = GD = d$$

$$\frac{108}{1480.43123} = 0.0729517 = AH = b$$

$$\frac{108}{1449.65428} = 0.0745005 = EM = m$$

TAB. VI.

--- being the first and second approximations to-
 --- making all the concords in 1, 2 or 3 octaves

Hence

$$Ef = 4d - 1 = 0.1235920$$

$$Ei = \frac{1-4b}{3} = 0.1504256$$

$$EM = m = 0.0949868$$

$$3) 0.3690044$$

$$Eq = 0.1230015$$

$$Eq' = 0.122233. \text{ in 1 Octave,}$$

See Tab. VII.

$$Ef = 4d - 1 = 0.0926784$$

$$Ei = \frac{1-4b}{3} = 0.2023217$$

$$EM = m = 0.0824916$$

$$3) 0.3774917$$

$$Eq = 0.1258306$$

$$Eq' = 0.124719. \text{ in 2 Octaves.}$$

$$Ef = 4d - 1 = 0.0321732$$

$$Ei = \frac{1-4b}{3} = 0.2360634$$

$$EM = m = 0.0745005$$

$$3) 0.3427371$$

$$Eq = 0.1142457$$

$$Eq' = 0.11024.. \text{ in 3 Octaves.}$$

TAB. VII.

The computation of $E q'$ in Fig. 46, being the ---
 ment of the III^d , for making all the concords ---

$$\frac{1}{GD} = \frac{1}{d} = \frac{42.72013}{12} = 3.560011$$

$$\frac{1}{Gg} = \frac{3}{1-b} = \frac{3}{0.8628192} = 3.476974$$

$$\frac{1}{Gk} = \frac{4}{1+m} = \frac{4}{1.0949868} = \underline{3.652913}$$

$$3) 10.689898$$

$$\text{Arith. mean } 3.563299$$

$$GD' = d' = \frac{1}{3.563299} = 0.280631$$

$$\frac{1}{AH} = \frac{1}{b} = \frac{87.47583}{12} = 7.289653$$

$$\frac{1}{Ae} = \frac{1}{1-3d} = \frac{1}{0.1573060} = 6.357031$$

$$\frac{1}{Al} = \frac{4}{1-3m} = \frac{4}{0.7150396} = \underline{5.494096}$$

$$3) 19.240780$$

$$\text{Arith. mean } 6.413593$$

$$AH' = b' = \frac{1}{6.413593} = 0.155919$$

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TAB. VII.

---second approximation towards the tempera-
 ---in 1 octave equally and the most harmonious.

$$\frac{1}{EM} = \frac{1}{m} = \frac{126.33333}{12} = 10.527777$$

$$\frac{1}{Ef} = \frac{1}{4d-1} = \frac{1}{0.1235920} = 8.091138$$

$$\frac{1}{Ei} = \frac{3}{1-4b} = \frac{3}{0.4512768} = \frac{6.647805}{3) 25.266720}$$

Arith. mean 8.422240

$$EM' = m' = \frac{1}{8.422240} = 0.118733$$

Hence $Ef' = 4d' - 1 = 0.122524$

$$Ei' = \frac{1-4b'}{3} = 0.125441$$

$$Em' = m' = 0.118733$$

$$3) 0.366698$$

$$Eq' = 0.122233$$

PROPOSITION XVII.

A system of rational intervals deduced from dividing the octave into 50 equal parts, and taking the limma $L = 5$ of them, the tone $T = 8$ and consequently the lesser 3^d $L + T = 13$, the greater III^d $2T = 16$, the 4th $L + 2T = 21$, the vth $L + 3T = 29$, &c, according to the table of elements (a), will differ insensibly from the system of equal harmony: I mean with regard to the harmony of the respective consonances in both.

For since $L = 5$, $T = 8$ and the III^d $2T = 16$ and the VIII = $5T + 2L = 50$, we have the III^d $2T : VIII :: 8 : 25$; whence the III^d $2T = \frac{8}{25} VIII = \frac{8}{25} \log. 2 = \frac{8}{25} \times 0.30102.99957 = 0.09632.95962$, which subtracted

(a) Prop. III. pag. 52.

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ed from the perfect $\text{III}^d = \log. \frac{5}{4} = 0.09691.00130$, leaves the temperament $0.00058.04168$, which is to the comma $c = \log. \frac{81}{80} = 0.00539.50319$ as 4 to 37 very nearly (*b*). Hence the temperament of the III^d is $-\frac{4}{37}c$, and those of the v^{th} and VI^{th} as in Tab. I, by prop. III. coroll. 1. 2. 3.

TABLE I.

$\text{T} : \text{L} :: 8 : 5$ $\text{VIII} = 50$	The system of equal harmony.	Ratios of the temperaments and of the beats made in the same time (<i>c</i>).
$\text{V} - \frac{1}{4}c - \frac{1}{37}c$	$\text{V} - \frac{1}{4}c - \frac{1}{36}c$	$\frac{1}{4} + \frac{1}{37} : \frac{1}{4} + \frac{1}{36} :: 369 : 730$
$\text{VI} + \frac{1}{4}c - \frac{3}{37}c$	$\text{VI} + \frac{1}{4}c - \frac{3}{36}c$	$\frac{1}{4} - \frac{3}{37} : \frac{1}{4} - \frac{3}{36} :: 75 : 74$
$\text{III} - \frac{4}{37}c$	$\text{III} - \frac{4}{36}c$	$\frac{4}{37} : \frac{4}{36} :: 36 : 37$

Now though the concords of the same name in this system and that of equal

(*b*) See an example of the like reduction in the next Scholium. (*c*) Prop. XI. coroll. 4.

qual harmony are not exactly equally harmonious (*d*), and though the difference of an unit in the largest numbers of beats made in a given time may be distinguished by counting them; yet when the numbers are no larger than those in the table, the difference in the harmony of the concords will be deemed insensible by proper judges; which are those only that have carefully attended to the beats of concords in tuning instruments. But any one else may be satisfied experimentally, by causing two concords to the same base to beat as in the table. Q. E. D.

Scholium.

In like manner if $T=5$ and $L=3$, then the octave $5T + 2L$ is $= 31$ and the temperament of this system, which *Hugenius* has adopted (*e*), will be found as in the third column of the next table.

TAB.

(*d*) Prop. XIII. coroll. 5.

(*e*) *Cyclus harmonicus*, at the end of his Works, or *Histoire des Ouvrages des Sçavans*, Octob. 1691, pag. 78.

TABLE II.

T : L :: 2 : 1 VIII = 12	T : L :: 3 : 2 VIII = 19	T : L :: 5 : 3 VIII = 31
$v - \frac{1}{4}c + \frac{3}{19}c$	$v - \frac{1}{4}c - \frac{3}{35}c$	$v - \frac{1}{4}c + \frac{1}{110}c$
$vi + \frac{1}{4}c + \frac{9}{19}c$	$vi + \frac{1}{4}c - \frac{9}{35}c$	$vi + \frac{1}{4}c + \frac{3}{110}c$
III + $\frac{12}{19}c$	III - $\frac{12}{35}c$	III + $\frac{4}{110}c$

On the contrary, if from the given temperament of a system it be required to find the ratio of T to L, we may proceed as follows. Let it be proposed to approximate to the system of equal harmony, where $2T = III - \frac{1}{9}c$ (*f*); then since $5T + 2L = VIII$, we have $2L = (VIII - 5T) = VIII - \frac{5}{2} \times III - \frac{1}{9}c$, whence $T : L :: III - \frac{1}{9}c : VIII - \frac{5}{2} \times III - \frac{1}{9}c$.

To

(*f*) Prop. xvi. Scholium 2. Art. 9.

To find this ratio, we have the III
 $= \log. \frac{5}{4} = 0.09691.00130$ and the
 comma $c = \log. \frac{81}{80} = 0.00539.50319$
 and $\frac{1}{9} c = 0.00059.94480$. Whence
 $2T = \text{III} - \frac{1}{9} c = 0.09631.05650$ and
 $\frac{5}{2} \times \text{III} - \frac{1}{9} c = 0.24077.64125$ and
 the VIII $= \log. 2 = 0.30102.99957$
 and $2L = \text{VIII} - \frac{5}{2} \times \text{III} - \frac{1}{9} c =$
 $0.06025.35832$, and lastly $T : L ::$
 $963105650 : 602535832$.

Now the quotients of the greater term of this ratio divided by the lesser and of the lesser divided by the remainder and of the former remainder by the latter &c, are 1, 1, 1, 2, 24, &c. Whence the ratios greater than the true one are 2 to 1, 5 to 3, 8 to 5, &c, and the lesser are 3 to 2, 11 to 7, &c (g).

Hence taking T to L successively in those ratios, by the method used in the
 de-

(g) See Mr. Cotes's *Harmonia Mensurarum*, Schol. 3. prop. 1.

demonstration of the proposition the temperaments of the approximating rational systems will be found as in the tables. By which we see how much and which way they differ from that of mean tones, as well as from that of equal harmony in Table I.

SECTION VIII.

Of complete and defective scales of musical sounds.

1. Fig. 48. Conceiving the circumference of a circle to represent an octave and to be divided into 31 equal parts, of which the tones $AB, CD, \&c,$ are 5 apiece and the limmas BC, EF are 3, take the interval AA^b towards the graver sounds, and AA^* towards the acuter, each equal to the lesser limma $AB - BC = 2$ of those parts; and do the like at all the other primary sounds, B, C, D, E, F, G . Then will the addition of these 7 couples of secondary sounds

sounds A^b and A^* , B^b and B^* , C^b and C^* , &c, produce 2 secondary systems, $A^b, B^b, C^b, D^b, E^b, F^b, G^b$, and $A^*, B^*, C^*, D^*, E^*, F^*, G^*$, each like the first; every sound of each being equally depressed below, or raised above the corresponding sound in the primary system A, B, C, D, E, F, G .

2. These three diatonic systems compose a scale of 21 sounds in the octave, which is sufficiently complete for musical compositions. But if it happen now and then that others are wanted, they are always situated at the points that bisect the lesser limmas AA^* , BB^b , CC^* , DD^b , &c, and by analogy may be thus denoted A^{**} , B^{bb} , &c, signifying twice the distance of a single flat or sharp sound from the primary sounds A, B , &c.

3. The 7 Diesis or smallest intervals A^*B^b , C^bB^* , C^*D^b , D^*E^b , &c, situated in the middle of the primary tones and limmas, are severally equal to $\frac{1}{5}$ of the
tone

tone or $\frac{1}{31}$ of the octave. This is the *Hugenian* system.

4. Fig. 49. In like manner conceiving the circumference of a circle to be divided in 50 equal parts, as in the approximation to the system of equal harmony (*a*), the tone *AB* is 8 of them and the greater limma *BC* is 5, and the lesser $AA^* = AB - BC$ is = 3; and each of the 5 greater dieses $A^*B^b, C^*D^b, \&c$, within the primary tones is 2, and each of the lesser C^bB^*, F^bE^* , within the two primary limmas is 1 of those 50th parts. And the system composed of these 21 sounds in the octave is likewise complete enough for musical compositions.

5. Hence the *Hugenian* diesis is to the lesser of these dieses as $\frac{1}{31}$ to $\frac{1}{50}$, or as 50 to 31, and is to the greater as 50 to 62, or as 5 to $6\frac{1}{5}$.

6. The several columns in the first table of concords (*b*) contain the notes that

(*a*) Prop. xvii. (*b*) Plate 19.

that make the six concords to the seven primary base notes *A, B, C, D, E, F, G*, placed in the lowest transverse row, and also to the 5 secondary base notes *B^b, C[♯], E^b, F[♯], G[♯]* in the same row continued; an organ or harpsichord having generally no more than those 5.

Consequently when a composer uses any of these other secondary notes *A^b, A[♯], C^b, B[♯], D^b, D[♯], F^b, E[♯], G^b*, which in the table and Fig. 48 or 49 are inclosed in circles, the performer is obliged to take the note next to it, as *G[♯]* instead of *A^b*, *B^b* instead of *A[♯]*, *B* instead of *C^b*, &c, whose interval is a diesis.

7. Hence each concord in the table between any base note and the note used instead of the true one in the circle, is either too flat or too sharp by a diesis, according as the mark ^b or [♯] is annexed to the true but absent note inclosed in the circle.

8. These false concords render some of the Keys, or Modes of harmony, more imperfect than others with respect to
the

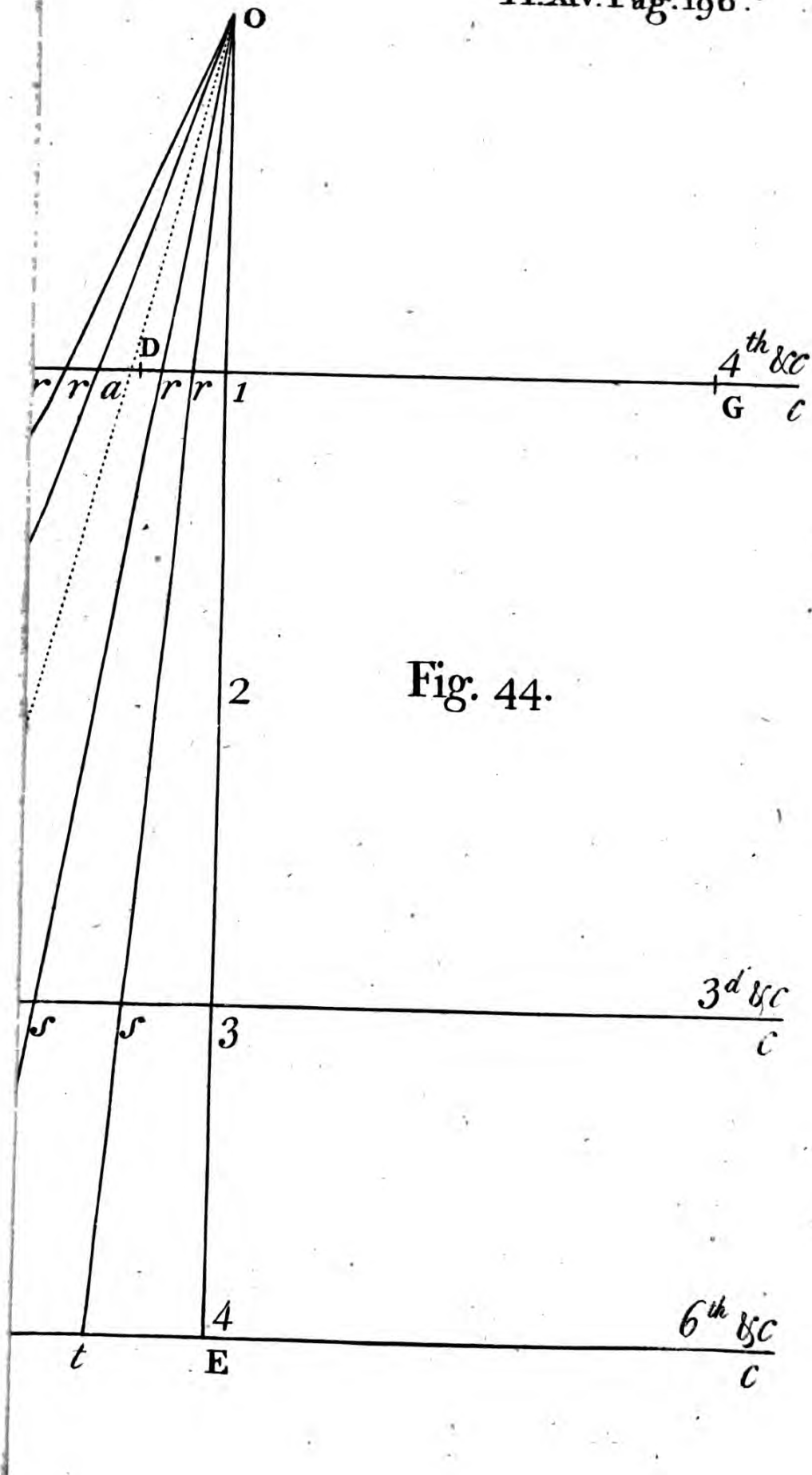
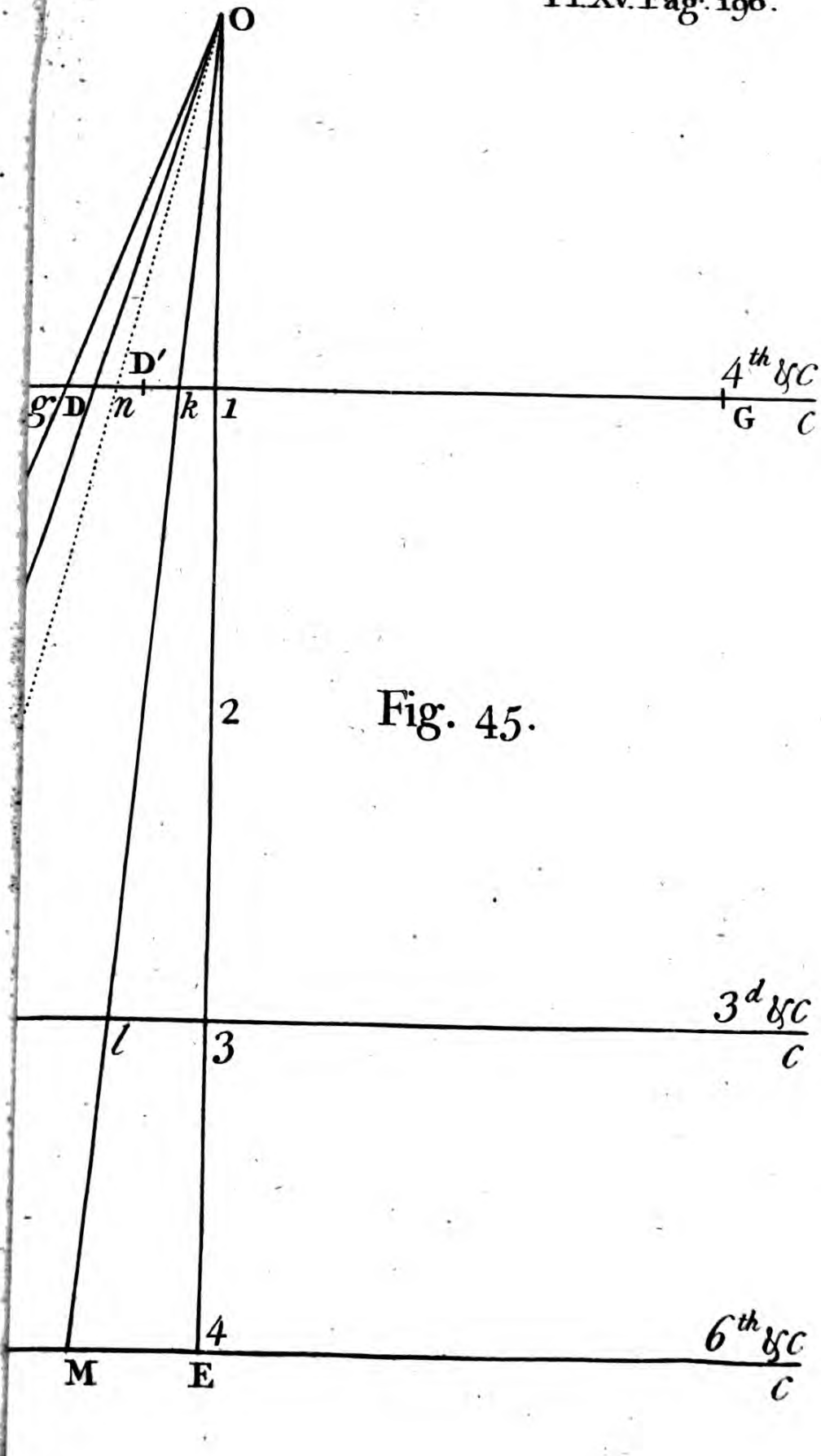


Fig. 44.





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the concords, in the same order as the numbers 1, 2, 3, placed under the base notes, increase in magnitude. For one false concord makes a mode less imperfect than two, and two less than three, and I do not here consider the false discords.

9. Hence it is observable, that the modes both flat and sharp, whose bases *A, D, E, G*, are the notes of the four open strings of the violin, are all free from false concords, though not all from false discords.

As to the rest, perhaps I need not observe, that the mode whose base is *B*, for instance, with a flat 3^d above it, is as perfect with respect to the concords as the like mode whose base is *A*; or that the degrees of imperfection 1, 2, 3, relate only to the imperfect concords in the columns above them.

10. The design of placing but one secondary sound instead of two within each primary tone, is to facilitate the

practice of music and the structure of instruments. And the reason for choosing these five B^b , C^* , E^b , F^* , G^* , rather than any other, was, I suppose, to make as many perfect modes as possible with that number of sounds.

Now the numbers in Tab. 3, placed below the five couples of secondary notes, shew how often each note is found among the concords to the seven primary bases A , B , C , D , E , F , G , whose modes should first be regarded; and that note of each couple, placed within the primary tones in Fig. 48 or 49, which occurs seldomer than the other, is justly excluded by the circle, as being generally less used.

But A^b and G^* occur equally often, and musicians have excluded A^b and admitted G^* , as in the first table of concords.

On the contrary, if G^* were excluded and A^b admitted, as in the second table of concords, two of the modes,
whose

whose common base is E^b , would be freed from the false 4th, and the whole number of false concords, reckoned up in the last column, would be but 15 instead of 16 in Tab. 1.

11. The numbers in the 4th table placed under the notes excluded in the first, shew how often each of these notes is found in this whole table of concords to the 12 bases; by which it appears that A^b and D^* are oftenest wanted; and that an addition of these two to an organ or harpsichord, would reduce the 16 false concords in the first table to 10, by making 6 of them as good as any in the scale.

For the like reason the 5th table shews, that an addition of a couple of notes would also reduce the 15 false concords in the second table to 10.

12. We may observe of those two systems of rational intervals, that the *Hugenian* is simpler than the other, and has this advantage too in a defec-

tive scale, that the diesis being something smaller (*c*), the false concords are something better than in the other, though still extremely bad: but in those respects the system of 12 equal parts in the VIII far excels them both.

13. So that considering the extreme disagreeableness of the imperfect modes in any defective scale, in comparison to the perfect ones when tuned in the best manner, and what a charming instrument an organ would be if those defects were supplied, I propose to do it as follows, without adding a single key to the usual number, or splitting any one of them.

Let every octave in an organ be furnished with seven couples of pipes, answering to the seven couples of secondary notes $A^{\ast}B^b$, C^bB^{\ast} , $C^{\ast}D^b$, $D^{\ast}E^b$, &c, which terminate the 7 diesis, and by means of seven stops, let either of the pipes in each couple be made to
found

(*c*) Art. 5. Sect. VIII.

found by the same key while the other is silent; and since both the pipes in any couple are very seldom used in the same piece of music, the organist before he begins to play, may presently adjust the seven stops for sounding such secondary notes as he foresees he has most occasion for.

If such an addition of pipes to the present number be thought too expensive, or too cumbersome in the instrument, some of the least useful pipes may be omitted in each octave, and the expence and incumbrance of the rest may well be balanced by rejecting some entire stops of little or no use, especially the Cornet and Sesquialter, which only add a noise to an organ and destroy its harmony.

The like contrivance is easily applicable to a harpsichord by adding more strings, and possibly without such addition.

14. Another great convenience in such complete scales would be the transposition of music into different keys, but little higher or lower than the given one, and yet equally perfect; the frequent want of which in the present instruments is a great disadvantage to the voices of the fingers.

SECTION IX.

Methods of tuning an organ and other instruments.

PROPOSITION XVIII.

To find the pitch of an organ.

By the following experiment made upon our organ at Trinity College, I found that the particles of air in the cylindrical pipe called *d*, or *d-la-sol-re* in the middle of the open diapason (*a*) made 262 vibrations, or returns to the places

(*a*) See the notation, Tab. IV. Plate XX.

places they went from, in one second of time. And this number of vibrations is what I call the pitch of the organ.

Having suspended a brass weight of seven pounds averdupois at one end of a copper wire, commonly used for some of the lowest notes of a harpsichord, I lapped the other end round a peg, taken from a violin, that turned stiffly in a hole made in the wainscot near the organ. Then by turning the peg to and fro I lengthened or shortened the vibrating part of the wire, till it sounded a double octave below the sound of the pipe *d* abovementioned. Then having measured the length of the vibrating part of the wire, while stretched by the weight, from the loop below to the under side of the peg, and added to it the semidiameter of the peg, I cut the wire, first at the point of contact with the side of the peg, and then at the loop below; for I found no need of a bridge either above or below. And

having repeated the experiment with another piece of wire taken from the same bunch, I found no sensible difference either in the length or weight of the vibrating part; the length being 35.55 inches and the weight 31 grains troy; and the seven pounds averdupois that stretched it, was equal to 49000 grains troy, allowing 7000 for each pound.

Hence by a Theorem hereafter demonstrated (*b*), the number of semivibrations, forwards and backwards together, made by the wire in one second was 131, and the number of such vibrations made by the air in the pipe *d*, two octaves higher, was 4×131 (*c*), and the number of complete vibrations was 2×131 or 262. Q. E. I.

Scholium.

I made that experiment in the month of September, at a time when the thermometer

(*b*) Prop. xxiv. coroll. 1, 2. (*c*) Sect. I. Art. 3 and 7.

mometer stood at temperate or thereabouts.

But upon a cold day in November I found by a like experiment, that the same pipe gave but 254 complete vibrations in one second; so that the pitch of its sound was lower than in September by something more than $\frac{1}{4}$ of a mean tone.

And upon a pretty hot day in August I collected from another experiment, that the same pipe gave 268 complete vibrations in a second of time; which shews that its pitch was higher than in November by almost half a mean tone.

By some observations made upon the contraction and expansion of air, from its greatest degree of cold in our climate to its greatest degree of heat (*d*), compared with Sir *Isaac Newton's* theory of the velocity of sounds, I find also

(*d*) See Mr. *Cotes's* xvth Lecture upon Hydrostat. and Pneumat. towards the end.

so that the air in an organ-pipe may vary the number of its vibrations made in a given time, in the ratio of 15 to 16; which answers to the major hemitone or about $\frac{7}{12}$ of the mean tone and agrees very well with the foregoing experiments.

Coroll. In order to know when the pitch of an organ varies, and when it returns to the same again, it is convenient to keep a thermometer constantly in the organ-case.

PROPOSITION XIX.

The pitch of an organ and the temperament of the v^{th} being given, to find the numbers of beats that every v^{th} will make in a given time.

The musical notes in Tab. 1, 2, Plate 20, shew all the v^{th} s of different names in a complete scale of sounds.

Let v to 1 be the given ratio of the times of the single vibrations of the base
and

and treble of the given tempered v^{th} , and N be the given pitch of the organ, that is, the number of complete vibrations made in 1" by the sound d in the middle of the Tables, and β the number of beats made in a given time by the v^{th} above d , to be found by coroll. 7: prop. XI. —

Then if a series of v^{ths} equally tempered, ascended continually from d , as in Tab. 3, the numbers of their beats would continually increase in the ratio of 1 to $v(a)$ and therefore would be $\beta, \beta v, \beta v^2, \beta v^3, \beta v^4, \beta v^5, \beta v^6, \beta v^7, \beta v^8, \beta v^9$.

Now as often as these v^{ths} are depressed by an octave, as in Tab. 1 and 2, so often must these beats be divided by 2 (a); which changes that series of beats into this, $\beta, \frac{\beta v}{2}, \frac{\beta v^2}{2}, \frac{\beta v^3}{4}, \frac{\beta v^4}{8}, \frac{\beta v^5}{8}, \frac{\beta v^6}{16}, \frac{\beta v^7}{16}, \frac{\beta v^8}{32}, \frac{\beta v^9}{32}$.

In

(a) Prop. XI. coroll. 2.

In like manner, if the former series of beats were continued downwards from d , as in Tab. 3, the numbers of their beats would continually decrease in the ratio of v to 1 or of 1 to $\frac{1}{v}$, and

therefore would be, $\frac{\beta}{v}$, $\frac{\beta}{v^2}$, $\frac{\beta}{v^3}$, $\frac{\beta}{v^4}$, $\frac{\beta}{v^5}$, $\frac{\beta}{v^6}$,
 $\frac{\beta}{v^7}$, $\frac{\beta}{v^8}$, $\frac{\beta}{v^9}$, $\frac{\beta}{v^{10}}$.

Now as often as these v^{ths} are raised by an octave, as in Tab. 1 and 2, so often must these beats be multiplied by 2; which produces this series of beats,

$\frac{\beta}{v}$, $\frac{2\beta}{v^2}$, $\frac{2\beta}{v^3}$, $\frac{4\beta}{v^4}$, $\frac{8\beta}{v^5}$, $\frac{8\beta}{v^6}$, $\frac{16\beta}{v^7}$, $\frac{16\beta}{v^8}$, $\frac{32\beta}{v^9}$, $\frac{32\beta}{v^{10}}$,

Q. E. I.

Scholium

For example, be it proposed to calculate the number of beats, in Tab. 1, Plate xx, which every v^{th} in the system of mean tones, will make in 15 seconds of time.

Here

Prop. XIX. HARMONICS. 209

Here the temperament of the v^{th} is $\frac{1}{4}$ of a comma, and supposing the interval of the perfect $v^{\text{th}} = \log. \frac{3}{2} = 0.17609.13$, we have the comma $c = \log. \frac{81}{80} = 0.00539.50$, and $\frac{1}{4} c = 0.00134.88$, and the $v^{\text{th}} - \frac{1}{4} c = 0.17474.25$; which is the logarithm of the number 1.4953 or $\frac{1.4953}{1}$, that is, of the ratio of 1.4953 to 1, which in the solution of the problem we represented by v to 1.

Now when the thermometer stood at temperate, the pitch of our organ at Trinity College, or the number of complete vibrations made in 1 second by the air in the pipe denoted by d , in the middle of our table, was 262 (c).

Hence to find the number of beats of the v^{th} above d , when tempered flat by $\frac{1}{4}$ comma, in coroll. 7. prop. XI, we
have

(c) Prop. xviii.

have $\frac{n}{m} = \frac{2}{3}$, $N = 262$ and $\frac{q}{p} = \frac{1}{4}$, and the required number in $15'' = 15 \times \frac{2q}{161p+q} \times mN = \frac{15 \cdot 2 \cdot 1 \cdot 3 \cdot 262}{161 \times 4 + 1} = \frac{1572}{43} = \beta$, whose logarithm is 1.56298.41; to which adding continually the log. of v , we

Abacus I.

0. 17474. 25	v	N ^o of beats.
1. 56298. 41	$\beta = 36, 558$	$\beta = 37$
1. 73772. 66	$\beta v = 54, 667$	$\frac{1}{2} \beta v = 27$
1. 91246. 91	$\beta v^2 = 81, 747$	$\frac{1}{2} \beta v^2 = 41$
2. 08721. 16	$\beta v^3 = 122, 24$	$\frac{1}{4} \beta v^3 = 31$
2. 26195. 41	$\beta v^4 = 182, 79$	$\frac{1}{8} \beta v^4 = 23$
2. 43669. 66	$\beta v^5 = 273, 34$	$\frac{1}{8} \beta v^5 = 34$
2. 61143. 91	$\beta v^6 = 408, 73$	$\frac{1}{16} \beta v^6 = 26$
2. 78618. 16	$\beta v^7 = 611, 20$	$\frac{1}{16} \beta v^7 = 38$
2. 96092. 41	$\beta v^8 = 913, 95$	$\frac{1}{32} \beta v^8 = 29$
3. 13566. 66	$\beta v^9 = 1366, 7$	$\frac{1}{32} \beta v^9 = 43$

Prop. XIX. HARMONICS. 211

we get the logarithms of $\beta v, \beta v^2, \beta v^3$ &c, as in the first Abacus, and thence the corresponding numbers, which divided

Abacus 2.

$\bar{1}. 82525. 75$	$\frac{1}{v}$	N ^o of beats
$1. 56298. 41$	$\beta = 36, 558$	$\beta = 37$
$1. 38824. 16$	$\frac{\beta}{v} = 24, 448$	$\frac{\beta}{v} = 24$
$1. 21349. 91$	$\frac{\beta}{v^2} = 16, 349$	$\frac{2\beta}{v^2} = 33$
$1. 03875. 66$	$\frac{\beta}{v^3} = 10, 933$	$\frac{2\beta}{v^3} = 22$
$0. 86401. 41$	$\frac{\beta}{v^4} = 7, 3116$	$\frac{4\beta}{v^4} = 29$
$0. 68927. 16$	$\frac{\beta}{v^5} = 4, 8896$	$\frac{8\beta}{v^5} = 39$
$0. 51452. 91$	$\frac{\beta}{v^6} = 3, 2699$	$\frac{8\beta}{v^6} = 26$
$0. 33978. 66$	$\frac{\beta}{v^7} = 2, 1867$	$\frac{16\beta}{v^7} = 35$
$0. 16504. 41$	$\frac{\beta}{v^8} = 1, 4623$	$\frac{16\beta}{v^8} = 23$
$\bar{1}. 99030. 16$	$\frac{\beta}{v^9} = 0, 9779$	$\frac{32\beta}{v^9} = 31$
$\bar{1}. 81555. 91$	$\frac{\beta}{v^{10}} = 0, 6540$	$\frac{32\beta}{v^{10}} = 21$

vided by the proper powers of 2, as directed in the solution of the problem, give the latter part of the set of beats opposite to the pitch 262 in Tab. 1.

The log. of v subtracted from 0 gives the log. of $\frac{1}{v}$, which log. continually added to the log. of β , gives the logarithms of $\frac{\beta}{v}$, $\frac{\beta}{v^2}$, $\frac{\beta}{v^3}$, as in the 2^d Abacus. And these logarithms give the numbers themselves, which multiplied by the proper powers of 2, as above directed, give the former part of the same set of beats opposite to the pitch 262 in Tab. 1.

The superior sets of beats are designed for tuning the same or different organs, when their pitch is higher than this by 1, 2, 3 or 4 quarter-tones, as noted at the beginning of the table, and may be found by the continual addition of the logarithm of a quarter-tone to the logarithms in the Abacus; and the first inferior set of beats may be found

H

VIII

VIII

VIII

VIII

4th 8c

C

r

G

Fig. 47.

v

x

VIII

y

VIII

z

3^d 8c

C

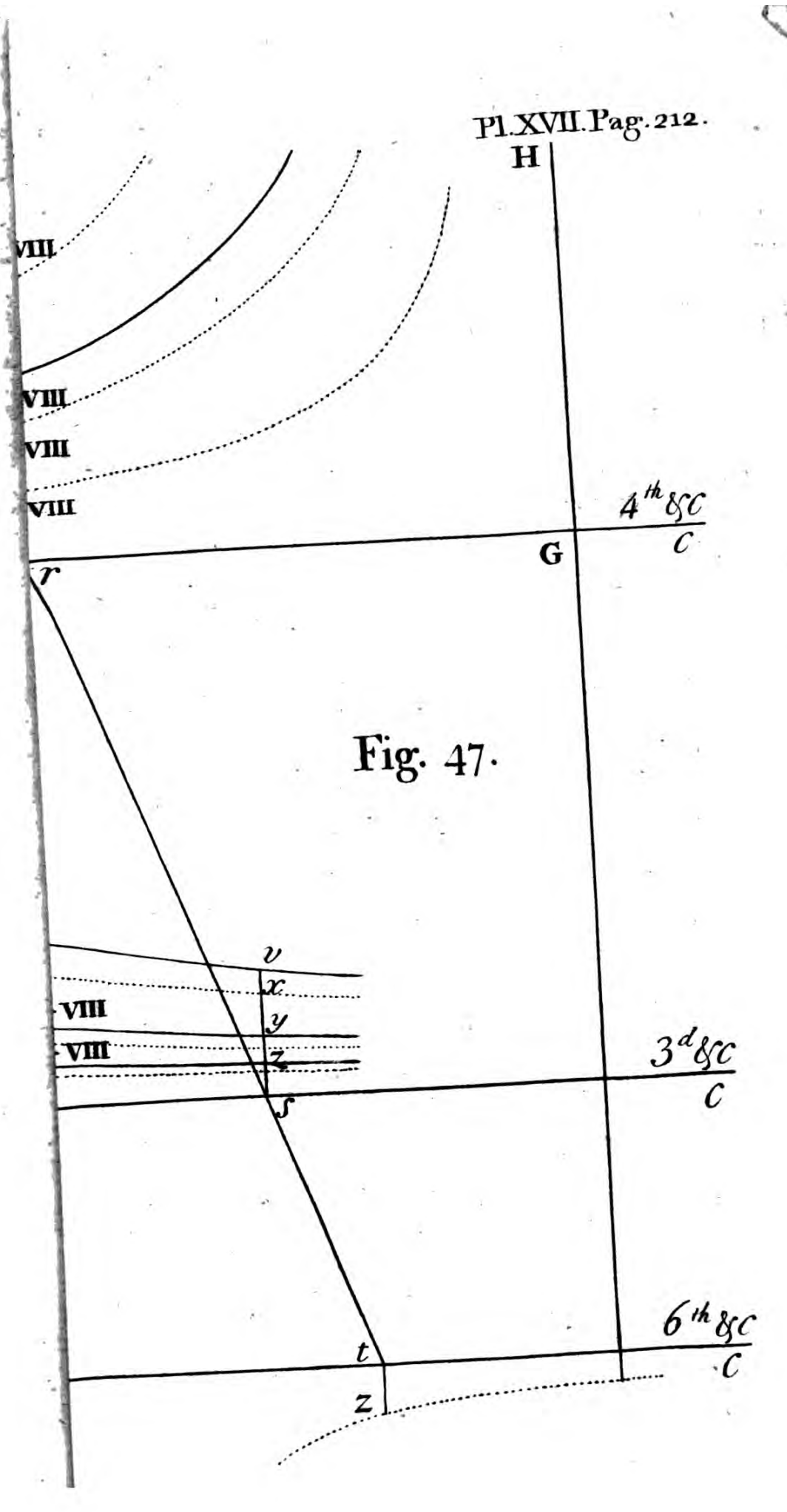
s

6th 8c

C

t

z





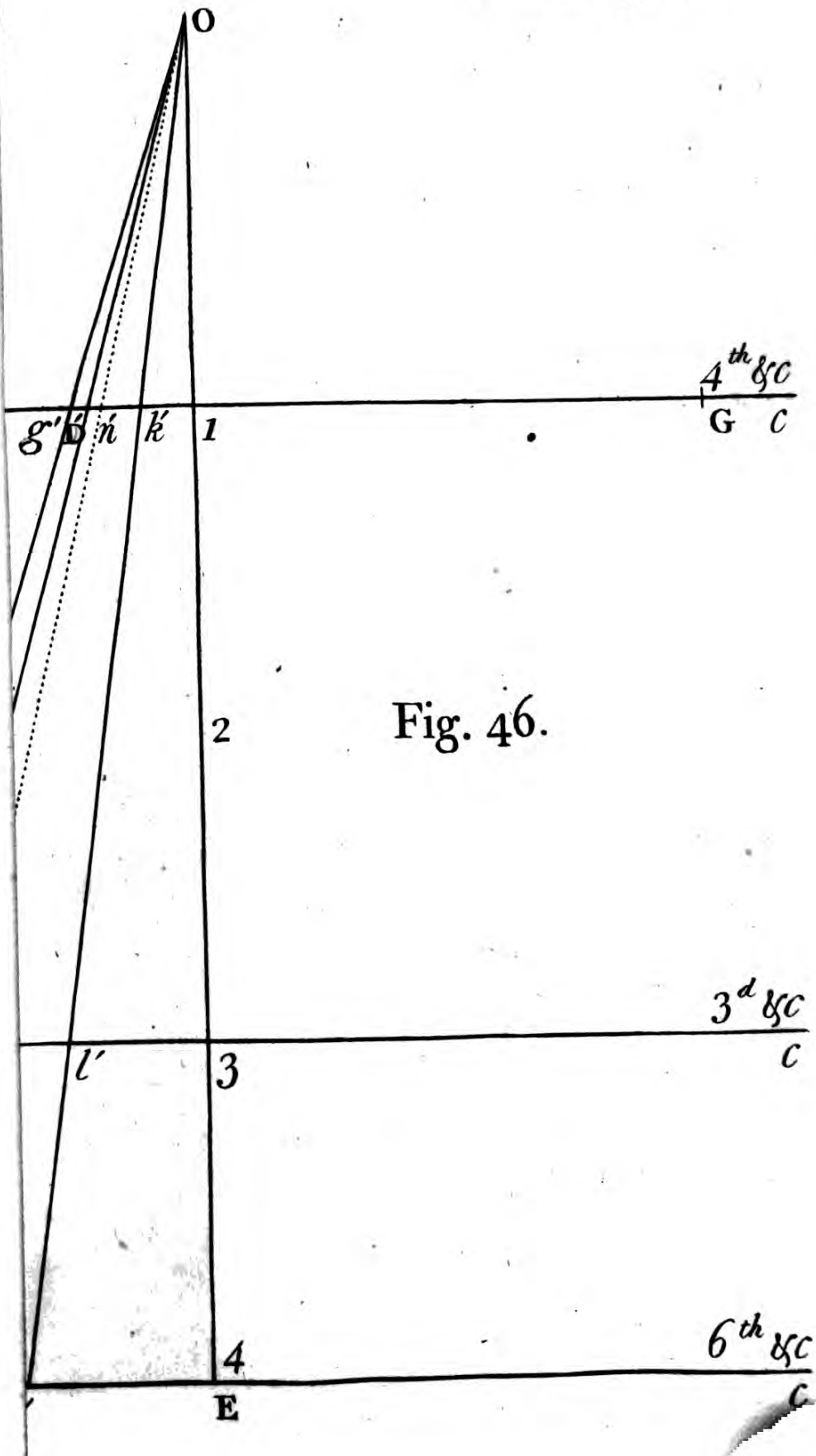


Fig. 46.



Prop. XIX HARMONICS. 213

found by the subtraction of the log. of a quarter-tone from the said set of logarithms, or by the addition of its arithmetical complement: remembering to divide and multiply the corresponding numbers by the same powers of 2 as before in the Abacus.

And as the $\frac{1}{4}$ tone is $\frac{1}{8}$ of the III^d, its logarithm is $\frac{1}{8} \log. \frac{5}{4} = 0.01211.38$, which continually added to the logarithm of 262, gives the successive logarithms of the higher pitches 269, 277, &c, in the first column of the table, over against the corresponding superior sets of beats; and subtracted from the same logarithm of 262, gives the log. of the lowest pitch 255, over-against the lowest set of beats.

From the given temperament of the system of equal harmony (*d*) the beats of all the v^{ths} may be calculated by the same

(*d*) Prop. XVI, Schol. 2, Art. 10.

same method; and will be found as in Tab. II. Plate xx.

Coroll. 1. Supposing the middlemost notes d , in the first and second tables, to be unisons, the numbers of the beats in a given time, of any two corresponding v^{ths} are very nearly in the given ratio of their temperaments $\frac{9}{36} c$ and $\frac{10}{36} c(d)$, or as 9 to 10. For they would be exactly in that ratio if their several base notes were exactly unisons (e); and the difference of their pitches, at the distance of the tenth v^{th} from the middle note d , is but ten times the difference of the temperaments, or $\frac{5}{18} c$; which produces the difference of but 1 beat in 290 in the extreme v^{ths} in the tables, and less in the rest in proportion to their distances from the note d (f).

Coroll. 2. In the system of equal harmony the ratio of the numbers of beats
of

(d) Tab. I. prop. xvii. (e) Coroll. 4. Prop. xi.
(f) Coroll. 2. Prop. xi. and the note in pag. 106.

Prop. XX. HARMONICS. 215

of the III^d and vth above the same base note, is 2 to 3 in a given time; as being compounded of the ratio of their temperaments $\frac{2}{18}c$ and $\frac{5}{18}c$, and the ratio of the denominators of the fractions $\frac{4}{5}$ and $\frac{2}{3}$ denoting the ratio of the single vibrations of the trebles, that of the common base being 1 (*g*).

Coroll. 3. For the same reason the beats of the vth and vith above the same base note are isochronous in the system of equal harmony, whereas in that of mean tones they are in the ratio of 3 to 5 in a given time.

PROPOSITION XX.

To tune an organ by a given table of beats.

Case 1. If the Pitch of the organ be given (*a*), look for it, or the nearest to it, in the first column of the given table

(*g*) Coroll. 3. Prop. xi. (*a*) Prop. xvii.

ble, Plate XX, and over against it you have the proper set of beats for tuning the organ, at a time when the thermometer has the same height as it had when the pitch was found by experiment.

At that time flatten the treble of the v^{th} above $d(b)$ more or less, till the number of its beats, made in 15 seconds, agrees with the tabular number placed over that v^{th} in the proper set.

From the treble of that $v^{\text{th}} d a'$ tune downwards the octave $a a$, so as to be quite free from beats, and repeat the like operation upon the next ascending $v^{\text{th}} a e$, and the like again upon the next till you have tuned all the sharp notes in your organ.

Then going backwards from d , sharpen the base of the v^{th} below it, more or less, till the number of its beats, made in 15 seconds, agrees with the tabular number, placed over this v^{th} in the proper set.

From

(*b*) See the notation Tab. iv. Plate xx.

From the base of the v^{th} dG , tune upwards the octave Gg and repeat the like operation upon the next descending v^{th} gc , and the like again upon the next, till you have tuned all the flat notes in your organ.

This being done, let all the other sounds be made octaves to these, and the stop will be exactly tuned according to the temperament in the given table; that is, all the v^{ths} will be equally tempered, and consequently equally harmonious (c), and so will all the vi^{ths} , and every other set of concords of the same name, which answers the design of tuning by a table of beats.

The time may be measured either by a watch that shews seconds, or a pendulum-clock, or a simple pendulum that vibrates seconds. And the person that counts them must give a stamp with his foot at the beginning and end of the 15 seconds; and another person that

(c) Prop. XIII. coroll. 4.

that counts the beats, at the first stamp should say, None, one, two, three, &c ; otherwise if he begins with saying one, he must count one beat more between the stamps than the tabular number, as properly signifying the number of intervals between the beats.

Case 2. If the pitch of the organ be not given, you may try to tune by that set of beats in Tab. I, which by any kind of estimation you think is the nearest to the proper set; and having tuned upwards any four of the successive v^{ths} , you must attend to the III^{d} or x^{th} between the base of the first and the treble of the fourth.

If that III^{d} or x^{th} be perfect, you have hit upon the proper set in this table, and consequently have found the pitch of the organ, placed opposite to it, which directs you to the proper set in the other table in case you choose to tune by it.

But if that concord be sharper than a perfect III^{d} or x^{th} , the four v^{ths} are not

not flat enough, and must be flattened a little more, till their beats agree with the numbers of the next superior set, which if the III^d or Xth comes out perfect, is the set required; if not, you must repeat the like trial upon the set next superior to that, till you find one that produces a perfect III^d or Xth.

On the contrary, if at the first trial the III^d or Xth comes out flatter than a perfect one, the four v^{ths} are too flat, and their trebles must be sharpened, till their beats agree with the next inferior set and bring out a perfect III^d or Xth, either at this or some other trial upon the successive inferior sets.

Or the required pitch of the organ may be found somewhat quicker by Tab. II. Tune upwards any three of the successive v^{ths}, except where two octaves intervene, by that set of beats which you think most likely to belong to the unknown pitch; then if the beats of the first v and of the VI above the assumed base be isochronous, you have

hit upon the proper set and the required pitch over against it (*d*).

But if that v^{th} beats slower than the $v\text{I}^{\text{th}}$, make each of the three v^{th} s beat quicker according to the next superior set of numbers; and on the contrary, if the v^{th} beats faster than the $v\text{I}^{\text{th}}$, make each of the three v^{th} s beat slower according to the next inferior set, till you find one that makes the v^{th} and $v\text{I}^{\text{th}}$ beat equally fast.

And as the production of perfect III^{ds} , in tuning by the first table, is a sure test of the exactness of the performance, so that of isochronous beats of the v^{th} and $v\text{I}^{\text{th}}$ above the same base, is as sure a test of exact tuning by the second table; and each of them must be constantly attended to in the progress of the respective methods of tuning.

If you choose to tune the organ according to the *Hugenian* system, the set of beats in Tab. I, next below that which answers to the pitch, found by
any

(*d*) Prop. XIX. Schol. coroll. 3.

any of the foregoing methods, will serve your purpose.

For the *Hugenian* v^{th} , having its temperament $-\frac{1}{4}c + \frac{1}{110}c$ smaller than $-\frac{1}{4}c$, in the ratio of 53 to 55 (*e*), beats slower than the tabular v^{th} in that proportion, which is but very little slower than it would do, if its pitch were depressed by $\frac{1}{4}$ of a mean tone. Q. E. F.

Scholium I.

Since our organ at Trinity College was new voiced, and by altering the disposition of the keys was depressed a tone lower, and thereby reduced to the Roman pitch, as I judge by its agreement with that of the pitch-pipes made above 30 years ago; by the help of such a pipe one may know by how many quarter-tones the pitch of any other organ is higher than that of ours, and thus determine the proper set of
beats

(*e*) Prop. xvii. Schol. Tab. 2.

beats for tuning it (*e*); allowance being made for the season of the year (*f*).

At a time when the thermometer stood at Temperate, as it did also when the pitch of our organ was found to be 262, I assisted at the tuning of the vth of the open Diapason by the set of beats opposite to that pitch in Tab. I; and upon examining the III^d and xth I found them all perfect: a manifest proof of the theory of beats and of the certainty of success in tuning by it.

At that time the whole organ was tuned to the open diapason, and is now universally allowed to be much more harmonious than before, when the major thirds were sharper than perfect ones; and its harmony, I doubt not, is still improveable by flattening those thirds still more according to the system of equal harmony. But at that time I had not finished the calculation
of

(*e*) See Column 2, Tab. I, 2, Plate xx.

(*f*) Scholium Prop. xviii.

of it, and to repeat the tuning of the organ over again, would be troublesome and improper at the present season, when cold and damp weather is coming on very fast.

For the properest times for tuning an organ seem to be from the latter end of August to the middle of October, when the air being dry, temperate and quiet, will keep nearer to the same degree of elasticity for a given time. Because a very small alteration in the warmth of moist air will suddenly and sensibly alter its elastic force and thereby the pitch of some of the pipes before the whole stop can be accurately tuned.

For that reason constant care must be taken not to heat the pipes by touching them oftener than is needful; not to stay too long at a time in the organ-case; not to tune early in the morning, but rather towards the evening, when the air is drier and its declining warmth is kept at a stay by the warmth
of

of the persons about the organ; in a word not to proceed with the tuning whenever the temperature of the air is sensibly altered, which is discoverable either by a thermometer in the organ-case, or by the present number of beats made by a given pipe, in a given time, compared with a former: and this pipe should be of a smaller sort, as being subject to alter its pitch sooner than the larger ones do; an alteration of the elasticity and density of air in a slender column, being sooner effected by heat, cold or moisture than in a grosser.

But these and the like cautions may sooner be learned by a little practice than by any description, and if not altogether necessary, will however contribute to the accuracy of tuning by so nice a method; which is plainly capable of any desired degree of exactness, provided the blast of the bellows be uniform: because the numbers of the beats of each concord may be taken larger, in proportion to any larger time
de-

designed for counting them, if the tabular numbers be thought too small.

I will add but one observation more. After tuning an organ according to any new system whatever, we must be cautious of judging too hastily of it. Some musicians here, who had constantly been used to major thirds and consequently major sixths tuned very sharp, could not well relish the smoother harmony of perfect thirds and better sixths in the organ newly tuned. At first they thought it dull and lifeless, till after a little use they became better satisfied with it, and after a longer use they could not bear the harshness of other organs tuned in the former manner.

It is therefore necessary to have equal experience in different objects of sense, in order to judge impartially, which of the two is more grateful than the other: as is evident in many instances, besides music, which any thinking person cannot miss of.

Scho-

Scholium 2.

The following table shews the numbers of beats, made in 15 seconds, by the several concords to the base note D, or D-fol-re

TAB. I.

The beats in 15" of all the -----

VI + 2VIII. $\frac{3}{20}$	80	120	133 $\frac{1}{11}$
V + 2VIII. $\frac{1}{6}$	40	36	34 $\frac{8}{11}$
III + 2VIII. $\frac{1}{5}$	13	0	4 $\frac{1}{4}$
VI + VIII. $\frac{3}{10}$	40	60	66 $\frac{6}{11}$
V + VIII. $\frac{1}{3}$	20	18	17 $\frac{4}{11}$
III + VIII. $\frac{2}{5}$	13	0	4 $\frac{1}{4}$
VI. $\frac{3}{5}$	20	30	33 $\frac{3}{11}$
V. $\frac{2}{3}$	20	18	17 $\frac{4}{11}$
III. $\frac{4}{5}$	13	0	4 $\frac{1}{4}$
System of	equal harm.	mean tones	M. Huy- gens.
Column	I	2	3

Prop. XX. HARMONICS. 227

D-sol-re, at the Roman Pitch, and likewise the proportions of the beats of the same concords to any other base note; with this design, that persons wanting leisure or proper qualifications for examining

TAB. I.

---- concords to the base D-sol-re.

$3^d + 2^{VIII}. \frac{5}{24}$	96	144	160
$4^{th} + 2^{VIII}. \frac{3}{16}$	107	96	93
$6^{th} + 2^{VIII}. \frac{5}{32}$	83	0	27
$3^d + VIII. \frac{5}{12}$	48	72	80
$4^{th} + VIII. \frac{3}{8}$	53	48	46
$6^{th} + VIII. \frac{5}{16}$	42	0	14
$3^d. \frac{5}{6}$	24	36	40
$4^{th}. \frac{3}{4}$	$26\frac{2}{3}$	24	23
$6^{th}. \frac{5}{8}$	$20\frac{4}{5}$	0	7
System of	equal harm.	mean tones	M. Huy- gens.
Column	I	2	3

aming the principles and conclusions requisite to determine the system of equal harmony, may yet form some judgment of its advantages and disadvantages, when compared with the system of mean tones and that of Mr. *Huygens*; upon this allowed principle, that, *cæteris paribus*, concords to the same base are more or less disagreeable in their kind, for beating faster or slower respectively.

To assist the reader's judgment I have added the following observations resulting from inspection of the table.

1. The beats in column 1 differ less from one another than those in column 2 and 3 do, agreeably to the Name of the system.

2. The beats of the $v, v + VIII, v + 2VIII$ in col. 1, are a little quicker than those in col. 2, in the ratio of 10 to 9, and quicker than those in col. 3, in a ratio something greater. But the beats of the $VI, VI + VIII, VI + 2VIII$ in col.

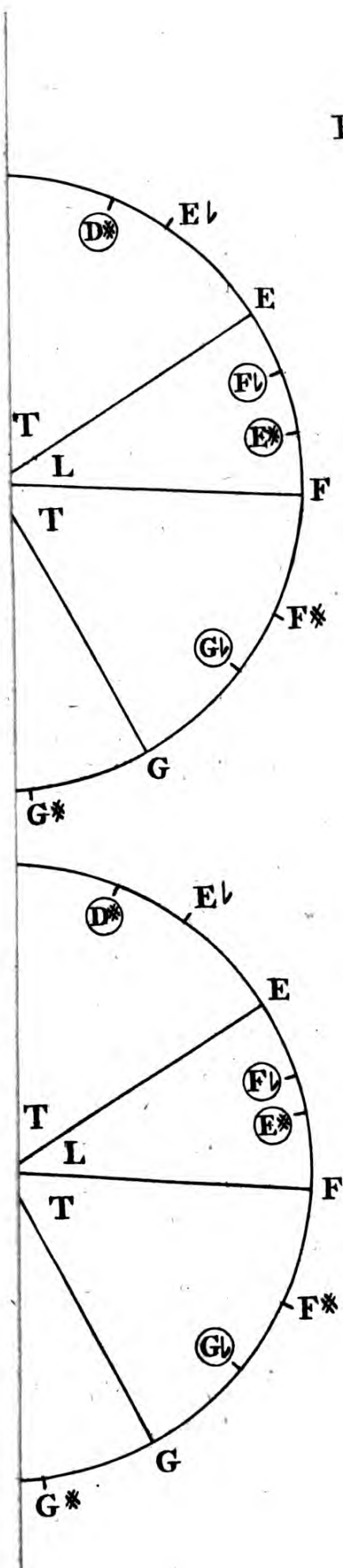




Table of Concords.

D	E	G	(A*)	C	(D*)	(E*)	3
(D♭)	(E♭)	(G♭)	A	(C♭)	D	E	4
C	D	F	G*	B♭	C*	(D*)	1
B♭	C	(E♭)	F*	(A♭)	B	C*	1
A	B	D	(E*)	G	(A*)	(B*)	4
(A♭)	(B♭)	(D♭)	E	(G♭)	A	B	3
F	G	B♭	C*	(E♭)	F*	G*	
2		2	2	3	2	3	16

Tab. 4.

(G*)	(A♭)	(A♭)	(A*)	(C♭)	(B*)	(D♭)	(D*)	(E*)	(G♭)
2	2	3	2	1	1	2	3	2	2

Table of Concords.

D	E	G	(A*)	C	(D*)	F	3
(D♭)	(E♭)	(G♭)	A	(C♭)	D	(F♭)	4
C	D	F	(G*)	B♭	C*	(E♭)	1
B♭	C	(E♭)	F*	(A♭)	B	(D♭)	1
A	B	D	(E*)	G	(A*)	C	4
(A♭)	(B♭)	(D♭)	E	(G♭)	A	(C♭)	3
F	G	B♭	C*	(E♭)	F*	(A♭)	
1		2	3	2	2	3	16

(D*)	(F♭)	(E*)	(G♭)	(G*)
2	1	1	2	3



col. 1 are much slower than those in col. 2 and 3, in the ratio of 2 to 3 and more.

3. These quick beats of the vi^{th} in col. 2 and 3 have the disadvantage to be doubled in every superior octave, whereas the quick beats of the v^{th} in col. 1 remain the same in the second octave and are only double in the third &c.

4. The beats of the iii^{d} and compounds are quicker indeed in col. 1 than those in col. 2 and 3; but being slower than the beats of the other concords in all the systems, they can scarce be so offensive as these will be.

5. Likewise the beats of the 6^{th} and compounds in col. 1, being slower than those of the 4^{th} and 3^{d} and compounds with the same number of $viii^{\text{ths}}$ in all the systems, can hardly offend the ear so much as the quicker beats of these other concords will.

6. The sums of the beats of all the concords to the note D-sol-re in both

Q

the

the col. 1, 2, 3, are respectively 759, 702, 805, whose proportions with respect to the same or any other base note are 38, 35, 40 very nearly. The small excess of the first sum above the second arises chiefly from the beats of the III^d and 6th with their compounds, which

TAB. II.

The order of the harmony ----

VI + 2VIII	240	360	399 $\frac{3}{11}$
V + 2VIII	40	36	34 $\frac{8}{11}$
III + 2VIII	13	0	4 $\frac{1}{4}$
VI + VIII	120	180	199 $\frac{7}{11}$
V + VIII	20	18	17 $\frac{4}{11}$
III + VIII	26	0	8 $\frac{1}{2}$
VI	60	90	99 $\frac{9}{11}$
V	40	36	34 $\frac{8}{11}$
III	52	0	17
System of	equal harm.	mean tones	M. Huy- gens.
Column	I	2	3

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which in all probability are inoffensive, as we said above.

But a completer rule for comparing the harmony of imperfect concords to a given base, appears in the second table; *That concord to which a smaller number corresponds, being more harmonious*

TAB. II.

----- of all the concords.

3 ^d + 2VIII	480	720	800
4 th + 2VIII	321	288	279
6 th + 2VIII	415	0	135
<hr/>			
3 ^d + VIII	240	360	400
4 th + VIII	159	144	138
6 th + VIII	210	0	70
<hr/>			
3 ^d	120	180	200
4 th	80	72	69
6 th	104	0	35
<hr/>			
System of	equal harm.	mean tones	M. Huygens.
Column	I	2	3

nious, in its kind, than any other to which a larger number corresponds; which affords two or three more observations.

7. In col. 1 and 3, the III^{ds} become more harmonious by the addition of VIII^{ths}.

8. In all the systems, the $v + VIII$ is more harmonious than the v and $v + 2VIII$.

9. In col. 1, the v^{th} is more harmonious than the III^d and VIth, the $v + VIII$ than the III + VIII and VI + VIII, and likewise the 4th and compounds than the 6th and 3^d and compounds with equal numbers of VIII^{ths}.

10. It may be objected to the system of equal harmony that the beats of the v^{ths} are not only a little quicker, but something stronger and distincter than those of the other concords; which deserves to be considered. On the other hand it should be considered too, whether those very quick and less distinct beats of the VIth and compounds, have
not

not a worse effect in destroying the clearness of their harmony.

These are the principal advantages and disadvantages that occur in comparing these systems. For as to the false concords being something worse in the system of equal harmony, than in the other two (*d*), this is no objection to the system, but only to the application of it to defective instruments; and I have shewn above how to supply their defects, without the least inconvenience to the performer (*e*).

I shall only observe, that the first table was calculated from the temperaments of the systems in prop. xvii and scholium, by the corollaries to prop. xi; and that the numbers in that table multiplied by the numerators of the known fractions, annexed to the characters of the corresponding concords, produce the corresponding numbers in the second table, according to coroll. 12, prop. XIII.

(*d*) Sect. viii. Art. 12. (*e*) Sect. viii. Art. 13.

Scholium 3.

In transferring the foregoing method of tuning from an organ to a harpsichord, as the scale of it is defective, the readiest way is to begin at *Elami* flat in the base (*e*), and thence to ascend by two successive v^{th} s and descend by a perfect $viii^{\text{th}}$ alternately, till you arrive at *g-sol-re-ut* sharp in the treble; and as you go on, to compare the treble of every third and fourth v^{th} with the base of the first, as above directed in tuning an organ. But since the beats of the concords die away too soon for counting a sufficient number of them in a given time, we must be content with estimating and correcting their degrees of celerity by the judgment of the ear; in doing which those that are conversant in tuning organs will find less difficulty than others unacquainted with that instrument.

And

(*e*) See Tab. v. Plate xx.

And herein this observation is of use, that the higher concords of the same name must be made to beat quicker than the lower, in the same ratio as the vibrations of their sounds are quicker; that is, if a tone higher in the ratio of 10 to 9 or 9 to 8 nearly, if a 111^d higher in the ratio of 5 to 4, if a 4th higher, of 4 to 3, if a vth higher, of 3 to 2, &c, which the ear must determine by estimation.

But if a machine were contrived, as it easily might, to beat like a clock or watch, any given number of times in 15 seconds, between 20 and 53 or thereabouts; by setting it to beat according to any given number in the table for tuning an organ, and by comparing its beats with those of the corresponding vth, the ear would determine immediately and exactly enough, whether they were isochronous or not: and thus a harpsichord might be tuned almost to the same exactness as an organ; and the tuning of an organ might

be performed much quicker, by the help of such a machine, than by counting the beats as above.

As the known method of tuning an instrument by the help of a monochord is easier than any other to less skilful ears, and pretty exact too, if the *apparatus* to the monochord be well contrived, it may not be amiss to shew the manner of dividing it, according to any proposed temperament of the scale.

PROPOSITION XXI.

To find the parts of a given monochord, whose vibrations shall give all the sounds in an octave of any proposed tempered system.

Let the system of equal harmony be proposed, and let the several parts of the monochord be measured from either end of it, and be to the whole, in the ratios of the several numbers in the 3^d column of the table, to 100000; I say the vibrations of the parts so found, and of the whole, will give all the sounds

Prop. XXI. HARMONICS. 237

found in an octave of the proposed system, as denoted in the first column of the table. Q. E. I.

For in the scholium to prop. xvii we had $2T = 0.09631.05650$ and $2L = 0.06025.35832$; whence we have

$$\begin{array}{r} T = 0.04815.52825 \\ L = 0.03012.67916 \\ \hline T - L = l = 0.01802.84909 \\ L - l = d = 0.01209.83007 \end{array}$$

From these logarithms of the tone, limma major and minor and the diesis, and from the logarithm $4.69897.00043$ of the number 50000, the uppermost in the table, all the logarithms below it will be found by the following additions: where the musical notes in column 1, are supposed also to represent the logarithms over against them:

$$\begin{array}{lll} c + d = B^*, & c + l = c^b, & c + L = B \\ B + l = B^b, & B + L = A^*, & B + T = A \\ A + l = A^b, & A + L = G^*, & A + T = G \\ G + l = G^b, & G + L = F^*, & G + T = F \\ F + d = E^*, & F + l = F^b, & F + L = E \\ \&c & \&c & \&c \end{array}$$

till

till you come down to C, which comes out 5.00000.00000 and shews that the logarithms of the principal notes B, A, G, F, E, D are right; and those of the secondary notes will be right too, if the operations in the addition be right.

The corresponding numbers in column 3, which may be found by the tables of logarithms, shew the required parts of the monochord; as a very little reflection will satisfy any one that understands the common properties of logarithms, and attends to the intervals of an octave in Fig. 49 described in Sect. VIII, but not divided as there into 50 equal parts, which is only an approximation to the system proposed. Q. E. D.

Scholium.

The numbers in the 4th and 5th columns of the table shew the parts of a monochord, whose vibrations will give
the

The division of a Monochord.

I	2	3	4	5
c	4. 69897. 00043	50000	50000	50000
B*	4. 71106. 83050	51412	51200	51131
c ^b	4. 71699. 84952	52119	52245	52278
B	4. 72909. 67959	53592	53499	53469
B ^b	4. 74712. 52868	55863	55902	55914
A*	4. 75922. 35875	57441	57243	57179
A	4. 77725. 20784	59876	59814	59794
A ^b	4. 79527. 05693	62412	62500	62528
G*	4. 80737. 88700	64177	64000	63942
G	4. 82540. 73609	66897	66874	66866
G ^b	4. 84343. 58518	69733	69877	69924
F*	4. 85553. 41525	71702	71554	71506
F	4. 87356. 26434	74742	74767	74776
E*	4. 88566. 09441	76853	76562	76467
F ^b	4. 89159. 11343	77910	78125	78196
E	4. 90368. 94350	80110	80000	79964
E ^b	4. 92171. 79259	83506	83593	83621
D*	4. 93381. 62266	85865	85599	85512
D	4. 95184. 47175	89504	89443	89422
D ^b	4. 96987. 32084	93298	93459	93512
C*	4. 98197. 15091	95934	95702	95627
C	5. 00000. 00000	100000	100000	100000
	System of	equal harmony	mean tones	Mr. <i>Huygens</i> .

the sounds of the opposite notes in the system of mean tones and that of Mr. *Huygens*, who has shewn how to find the last column of numbers in his *Harmonic Cycle*. And as all the measures in the 3 systems may be taken and marked upon the sounding-board of the same monochord, the different effects of those systems upon the ear, may be easily tried and compared together, provided the tone of the monochord be good and the divisions accurate, and the moveable bridge does not strain it in one place more than in another.

SECTION X.

Of occasional temperaments used in concerts well performed upon perfect instruments.

By a perfect instrument I mean a voice, violin or violoncello, &c, with which a good performer can perfectly express any sound which his ear requires.

PRO-

PROPOSITION XXII.

The several parts of a concert well performed upon perfect instruments, do not move exactly by the given intervals of any one system whatever, but only pretty nearly, and so as to make perfect harmony as near as possible.

For instance, if the base be supposed to move by the best scale of perfect intervals (*a*), the other part or parts cannot constantly move by it too, without making some of the concords imperfect by a comma (*b*), which would grievously offend the performers (*c*). Consequently if they are pleased, those intervals are occasionally tempered by the upper part or parts, which therefore do not move by the same scale which the base is supposed to move by.

Likewise if the base be supposed to move by the scale of mean tones and
lim-

(*a*) See Sect. II. Art. 1. (*b*) Prop. 1. and coroll.
(*c*) Sect. IV. Art. 9.

limmas (*d*), the other part or parts cannot constantly do so too, without making about two thirds of all the concords imperfect by a quarter of a comma (*e*). But whenever concords are held out by good performers, they seem to me to be always perfect. And if so, the upper part or parts cannot move by the scale of mean tones, which the base is supposed to move by. And the argument is the same if the base be supposed to move by any other scale of tempered intervals: and that it cannot constantly move by perfect ones, I shall shew in the next scholium.

What has been said of perfect and imperfect concords, is applicable to discords too, a good ear being critical in both. Now the reason why the best performers acquire a habit of making perfect harmony, as near as possible, is plainly this. When the harmony is
made

(*d*) Prop. II:

(*e*) Prop. III. coroll. 3.

made perfect they are pleased and satisfied, though the several parts do not move by perfect intervals. For the passing from one sound to the next, whether by a perfect or an imperfect interval, being nearly instantaneous, is almost, if not altogether imperceptible, and therefore is indifferent to the performer. But the succeeding consonance is long enough held out to give him pleasure or pain, according as he makes it perfect or imperfect. Q. E. D.

Coroll. 1. Cæteris paribus the harmony will be the same whatever be the scale which the base moves by; but the sum of the occasional temperaments will be the least possible, if it moves by the scale of mean tones and limmas (*g*), and but very little bigger, if it moves by the scale of equal harmony (*b*).

Coroll. 2. The proposition holds true though some of the instruments be imperfect; as when the thorough-base is play-

(*g*) Prop. III. cor. 10. (*b*) Prop. xx. Schol. Art. 6 and Prop. xi. coroll. 4.

played upon an organ or harpsichord: because the performers of the upper parts are more attentive to make perfect harmony with the base notes, than with the chords to them. Consequently those parts do not move by the tempered scale of the thorough-bass.

Coroll. 3. Cæteris paribus the same piece of music well performed upon perfect instruments, is more agreeable than it would be if it were as well performed upon imperfect ones, as an organ, &c.

For nothing gives greater offence to the hearer, though ignorant of the cause of it, than those rapid, piercing beats of high and loud sounds, which make imperfect consonances with one another. And yet a few slow beats, like the slow undulations of a close shake now and then introduced, are far from being disagreeable.

Coroll. 4. Therefore the harmony of a concert will be smoother and distincter, and generally more pleasing, for
taking

taking the chords of the thorough-bass as near as can be to the bass notes, and no more of them than are necessary, and these few upon the softer and simpler stops of an organ.

Because the beats will then be fewer, slower and softer, and so the voices and other instruments will appear to greater advantage.

Coroll. 5. It appears also from the reasons above, that no voice-part ought ever to be played on the organ, unless to assist an imperfect singer, and keep him from making worse concords with the bass and other parts than the organ it self does.

Scho-

(a) Aio itaque, si quis canat deinceps sonos, quos Musici notant literis C, F, D, G, C, per intervalla consona, omninò perfecta, alternis voce ascendens descendensque; jam posteriorem hunc sonum C, toto commate, quod vocant, inferiorem fore C priore, unde cani cœpit. Quia nempe ex rationibus intervallo-
rum istorum perfectis, quæ sunt 4 ad 3, 5 ad 6, 4 ad 3, 2 ad 3, componitur ratio 160 ad 162, hoc est 80 ad 81, quæ est commatis. Ut proinde, si novies idem cantus repetatur, jam propemodum tono ma-
jore

Numbers of

43	32	47	35	53
41	31	46	34	52
40	30	45	34	50
39	29	44	33	48
38	28	42	32	47
37	28	41	31	46

Musical notation for the first table, consisting of three staves. The first staff has a flat sign (b) followed by a note (o) and a sharp sign (#) followed by a note (o). The second staff has a sharp sign (#) followed by a note (o) and a sharp sign (#) followed by a note (o). The third staff has a flat sign (b) followed by a note (o) and a sharp sign (#) followed by a note (o).

Numbers of

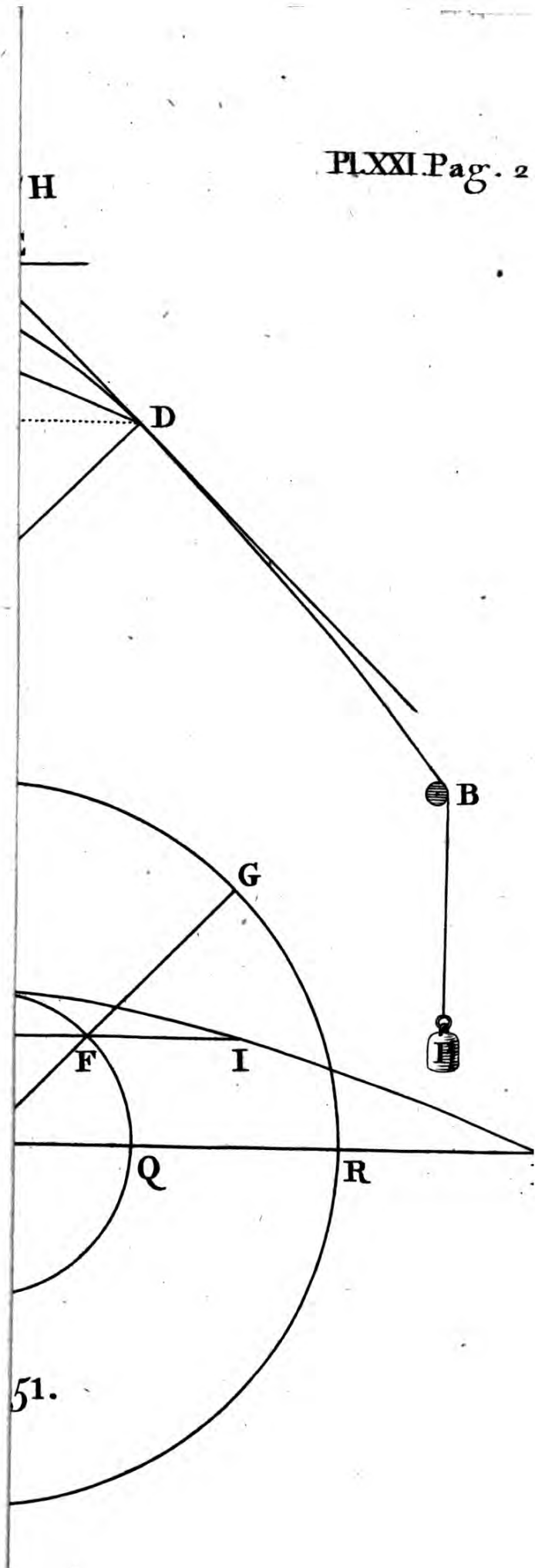
39	29	43	32	48
38	28	42	31	47
37	27	40	30	45
36	26	39	29	44
35	26	38	29	43
34	25	37	28	42

Musical notation for the second table, consisting of three staves. The first staff has a flat sign (b) followed by a note (o) and a sharp sign (#) followed by a note (o). The second staff has a sharp sign (#) followed by a note (o) and a sharp sign (#) followed by a note (o). The third staff has a flat sign (b) followed by a note (o) and a sharp sign (#) followed by a note (o).

Tab. v.

Musical notation for Tab. v, showing a grand staff with treble and bass clefs. The notation includes notes and accidentals (sharps and flats) on both staves.







Scholium.

Mr. *Huygens* observed long ago, that no single voice, or perfect instrument, can always proceed by perfect intervals, without erring from the pitch at first assumed (*a*). But as this would offend the performer, he naturally avoids it by his memory of the pitch, and by tempering the intervals of the intermediate sounds, so as to return to it again (*b*).

This is also confirmed by what we are told of a monk (*c*), who found, by sub-

jore, cujus ratio 8 ad 9, descendisse vocem, tonoque excidisse oportet. *Cosmotheoros* lib. i. pag. 77.

(*b*) Hoc verò nequaquam patitur aurium sensus, sed toni ab initio sumpti meminit, eodemque revertitur. Itaque cogimur occulto quodam temperamento uti, intervallaque illa canere imperfecta; ex quo multo minor oritur offensio. Atque hujusmodi moderamine ferè ubique cantus indiget; uti colligendis rationibus, quemadmodum hic fecimus, facile cognoscitur. *ibid.*

(*c*) Methode generale pour former les Systême temperés de musique. *Mem. de l' Acad. des Scienc.* Ann. 1707. pag. 263. 8^{vo}.

subtracting all the ascents of the voice in a certain chant from all its descents, that the latter exceeded the former by two commas: so that if the ascents and descents were constantly made by perfect intervals, and the chant were repeated but four or five times, the final sound, which in that chant should be the same as the initial, would fall about a whole tone below it. But finding that the voices in his quire did not vary from the pitch assumed, he concluded that the musical ratios, whereby he measured those successive ascents and descents, were erroneous. But had he but known Mr. *Huygens's* remark, it would have solved his difficulty.

This

(d) Stetti lungo tempo perplesso intorno à queste forme delle consonanze, non mi parendo che la ragione, che communemente se n'adduce da gli autori, che fin quì hanno scritto dottamente della musica, fosse concludente à bastanza. Dicono essi &c. *Discorsi attenenti alla Meccanica, Dialogo 1°*, towards the end.

(e)

This was not the first time that the truth of those musical ratios had been called in question. For *Galileo* observes that the reason commonly alledged for it, appeared to him insufficient (*d*). At last indeed he hit upon a couple of experiments which gave him satisfaction (*e*), but a scientific proof was still wanting till Dr. *Taylor* published his theory of the vibrating motion of a musical chord (*f*), which has since been cultivated by several able mathematicians (*g*), and being the principal foundation of Harmonics, deserves to be further considered in the next section.

SECT.

(*e*) Ibidem.

(*f*) *Methodus incrementorum*, prop. 22, 23, and *Philos. Transf.* n° 337, IV. or *Abridg.* by Jones Vol. 4. p. 391.

(*g*) *Commentarii Acad. Petropol.* Tom. III.
 Comment on the *Principia* Vol. 2. pag. 347.
 Mr. *Maclaurin's Fluxions*, Art. 929.

SECTION XI.

Of the vibrating motion of a musical chord.

PROPOSITION XXIII.

When a musical chord vibrates freely, the force which urges any small arch of it towards the center of its curvature, is to the tension of the chord, in the ultimate ratio of the length of that arch when infinitely diminished, to the radius of its curvature.

I suppose the chord to be uniform and very slender, or rather to be a mathematical line, flexible by the least force and elastic; and its tension or quantity of elastic force to be measured by a weight, which if hung to one end of it, would distend it to the same length which it has when it vibrates freely, by the force of its elasticity.

Fig.

Fig. 50. Let ACB represent such a chord, fixed at the points A and B , CD any arch of it, CE and DE tangents at C and D ; in either of which as EC produced if need be, take EF equal to ED , and draw FG perpendicular to FE , and DG to DE , and joining DF produce GE towards H .

Then imagine the chord to keep its curvature while a force applied at E or H draws the tangents EC , ED and these the points of contact C , D , so as to keep them in *æquilibrio*. And since the elastic forces at C and D are each equal to the force of tension, the direction of the third force at E will bisect the angle CED under the other two directions, and consequently will coincide with the line GEH , agreeably to the construction of the equal triangles EDG , EFG .

Hence the three forces at H , C and D , which would keep the point E at rest, are proportional to the sides DF , FG , GD of the isoscelar triangle DFG , to which their several directions EH , EC , ED are

perpendicular; because that triangle is similar to any other, as EDI , whose sides are either parallel to, or in part coincident with those directions, and therefore proportional to the forces acting in them, by the known theorem in Mechanics (*a*).

Now suppose CL and DM to be the *radii* of the curvatures of the chord at the points C, D , and the curve LM to be the *locus* of all the centers of the curvatures at every point of the arch CD . Then conceiving the point D to move up to C , and consequently M and G up to L , the limit of the variable ratio DF to FG , of the said forces, will be that of the evanescent arch CD to CL the *radius* of its curvature. And a force constantly equal and opposite to the former of the two, is that which urges the vanishing arch CD in the direction EG , which ultimately coincides with CL ; and the latter was the force of tension. Q. E. D.

Coroll.

(*a*) See Theorem 33 of Keill's Physics.

Coroll. When a musical chord vibrates freely, the forces which accelerate its smallest equal arches, are constantly proportional to their curvatures very nearly, provided the latitude of the vibrations be very small in proportion to the length of the chord.

For the force of its tension being then very nearly invariable, the forces which accelerate its smallest equal arches are very nearly in the inverse ratio of the *radii* of their curvatures (*b*), which is the same as the direct ratio of the curvatures themselves.

DEFINITION
OF THE HARMONICAL CURVE.

Fig. 51. *Let C be the common center of any two circles DF, EG, and CDE, CFG any two semidiameters, and of either of the included arches as DF, let FH be the sine, in which produced both ways, let the lines HI and HK be severally equal to the other arch EG; then*

(*b*) By the present Proposition.

then while the semidiameter CFG moves round the center C and carries with it the line $IFHK$, parallel to it self and constantly equal to twice the arch EG , the extremities I, K will describe a curve whose vertex is D and axis DC , and whose base ACB is equal to the semicircumference of the circle EG .

Coroll. 1. Fig. 52. Drawing FL perpendicular to the base ACL , a line KP , perpendicular to the curve at K , will be parallel to EL .

For drawing KN perpendicular to the base, let the radius CFG go forwards a little into the place Cfg , and carry the line $KHFI$ into the place $khfi$, cutting KN in O and FL in r . Then since $HK = EG$ by the definition, and also $hk = Eg$, their difference Ok is $= Gg$. Now by the similar triangles CLF and $Fr f$, CFf and CGg , OKk and NPk , we have $CL : CF :: Fr : Ff$, and $CF : CG :: Ff : Gg$, and *ex æquo* $CL : CG$ or CE

::

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$:: (Fr : Gg :: OK : Ok ::) NP : NK$.
 Consequently the right angled triangles CLE , NPK are equiangular, and the perpendicular KPM is parallel to the line EL .

Coroll. 2. At any point K the radius of curvature $KM : LE :: LE quad. : KN \times CE$.

For drawing fl parallel to FL ; another line kM , perpendicular to the curve at k , will be parallel to El , by coroll. 1; consequently if the arch Kk be infinitely diminished, either of the coinciding perpendiculars KM , kM will be the radius of the curvature at K .

In the line El take $Es = EL$ and joining Ls , the triangles LEs and KMk , CEL or CEl and sLl , are ultimately equiangular.

Now KN or $FL : LC :: fr : rF$ or KO ,
 and $LC : LE :: PN : PK :: KO : Kk$,
 and *ex æquo*, $KN : LE :: fr : Kk$;

But $CE : LE :: sL : Ll$ or fr , therefore *componendo*, $KN \times CE : LE quad. :: (sL : Kk ::) LE : KM$.

Coroll.

Coroll. 3. Hence if the ratio of the circles CEG , CDF be vastly great, the curvature at any point K will be extremely small, and its radius $KM : CE :: CE : KN$ very nearly; because the lines LE and CE will be very nearly equal.

Coroll. 4. Upon the same supposition, the very small curvatures at any points D , K are very nearly in the ratio of their distances DC , KN from the base AB .

For when CE and consequently AB is given, the curvature at K , being reciprocally as its radius KM , is directly as KN by coroll. 3.

Coroll. 5. Fig. 53. While the greater circle remains let the lesser be diminished, and the curve $AKDB$ will be changed into another $A\kappa\delta B$ of the same species, and every ordinate to the common base will be diminished in the same ratio, that is, $NK : N\kappa :: CD : C\delta$.

Fig. 52. For while any arch EG equal to HK or CN is given in magnitude, let the other radius CD or CF be diminished,

ed, and because the triangle CFH retains its *species*, the line CH or NK is diminished in the same ratio with CF or CD .

Coroll. 6. Fig. 53. When the axes CD , $C\delta$ of two curves are very small in comparison to their common base AB , the curvatures at the tops of any two coincident ordinates NK , $N\kappa$, are in the ratio of the ordinates.

For if $\kappa\mu$ be the radius of curvature at κ , by coroll. 3 we have $KM \times KN = CE \text{ quad.} = \kappa\mu \times \kappa N$; whence $\kappa N : KN :: KM : \kappa\mu$, that is, as the curvature at κ to the curvature at K .

Coroll. 7. Hence, supposing the curve $AKDB$ to have the elasticity and tension of a musical chord, it will vibrate to and fro in curves very nearly of the same species with the given curve $AKDB$, provided none of the vibrations be too large.

For let the first effort of the tension reduce that curve into some other, as $A\kappa\delta B$, in the first moment of time;
and

and since the ordinates DC, KN are in proportion as the curvatures at D and K by coroll. 4, and these curvatures as the accelerating forces at D and K (c), acting in the directions DC, KM or KN very nearly, and these forces as the velocities generated by them in that time, and the velocities as the nascent spaces $D\delta, K\kappa$; *alternando*, we have $DC : D\delta :: KN : K\kappa$ and *dividendo*, $DC : \delta C :: KN : \kappa N$. Consequently by coroll. 5 the curve $A\kappa\delta B$ is very nearly of the same species with $AKDB$. And in the next moment it will be changed into another of the same species, and so on, till every point of the chord be reduced to the base AB at the same instant. And by the motion here acquired it will be carried towards the opposite side of the base, till by the opposition of the tension, it shall lose all its motion by the same degrees, and in the same curves, by which it was acquired; and thus

(c) Coroll. PROP. XXIII.

thus the chord will continually vibrate in curves of the same species as the first, neglecting the small difference in the directions KN , KM , and the resistance of the air.

Coroll. 8. The small vibrations of a given musical chord are isochronous.

For if the chord at the limit of its vibration assumes the form of the harmonical curve, it will vibrate to and fro in curves of that species by coroll. 7, and its several particles, being accelerated by forces constantly proportional to their distances from the base AB (d), will describe those unequal distances in equal times, like a pendulum moving in a cycloid.

If the chord at the limit of its vibration assumes any other form, it will cut an harmonical curve, equal in length to it, in one or more points, as A , K , L , B in Fig. 54; and the intercepted parts of the chord will be more or less incurvated towards AB than the corresponding parts of the curve, according as they fall

(d) Cor. prop. xxiii and cor, 6. Defin. curve.

fall without or within them; and will accordingly be accelerated by greater or smaller forces than those of the corresponding parts of the curve (*e*). Therefore, supposing the chord and curve to differ in nothing but their curvatures, the difference of the curvatures of the corresponding parts will be continually diminished by the difference of their forces, till the parts coincide either before, or when they arrive at the base *AB*. And thus the times of the several vibrations of the chord will be the same as those of the curve, and therefore equal to one another.

Coroll. 9. The Figure contained under the harmonical curve and its base, is of the same species as the Figure of Sines.

Fig. 52. For supposing the circle *DFQ* to grow bigger till it becomes equal to *EGR*, the figure *AKDB* will become a figure of sines. Because any ordinate *KN* to the absciss *AN* or arch *GR*, being constantly equal to *FL*, will
then

(*e*) Cor. prop. xxiii.

then be equal to the sine of the arch GR ; and thus every ordinate as KN is increased in the given ratio of CF to CG , or CD to CE . And on the contrary the several ordinates in the said figure of sines diminished in that constant ratio of CE to CD , are the ordinates in the figure $AKDB$ of a musical chord in motion.

PROPOSITION XXIV.

The vibrations of a musical chord stretched by a weight, are synchronous to those of a pendulum, whose length is to the length of the chord, in a compound ratio of the weight of the chord to the weight that stretches it, and of the duplicate ratio of the diameter of a circle to its circumference.

Fig. 54. If P be the weight that stretches the chord ADB , and DM be the radius of its curvature at the vertex D , the force that urges any small particle Dd towards C is $= \frac{Dd}{DM} \times P$ by prop. xxiii. And

And since Dd vibrates like a pendulum (d), if it were suspended by a string $OP=DC$ in a cycloid $QPR=2DC$ or DCF , and were urged at the highest points Q, R by a force acting downwards like that of gravity, but equal to the said force $\frac{Dd}{DM} \times P$, which urges Dd at the limits D, F of its vibrations; the times of those oscillations and of these vibrations would be equal to one another. Because the forces being also equal at all other equal distances of the particle from P and from C , would impel it through equal parts of the equal lines QP, DC in equal times.

Again putting p for the weight of the chord ADB or ACB , the weight of its particle Dd is $= \frac{Dd}{AB} \times p$.

Hence if another string L be to the string OP or DC , as this latter weight $\frac{Dd}{AB} \times p$ is to the former $\frac{Dd}{DM} \times P$, equivalent

(d) Coroll. 9. Def. of the curve.



valent to the force at D ; and the particle Dd be again suspended by the string L in another cycloid of the length $2L$; since at the highest points of this cycloid the particle is urged downwards by the whole force $\frac{Dd}{AB} \times p$ of its own gravity, its oscillations will be synchronous to those of the former pendulum (e). Because we took their lengths in the ratio of the forces that act upon them at the highest points of the cycloids, that is, $L : DC :: p \times DM : P \times AB$; which two ratios compounded with DC to AB , give $L : AB :: p \times DM \times DC$ or $p \times CEq$ (f) : $P \times ABq$, which was to be proved. For CEq is to ABq in the duplicate ratio of the diameter to the circumference, by the definition of the curve; and we shewed above, that every particle of the curve vibrates in the same time with the middlemost. Q. E. D.

Coroll.

(e) Coroll. Theor. 4 De Motu Pend. in Mr. Cotes's Harmon. Mensurarum. (f) Coroll. 3. Defin.

Coroll. 1. The time of a femivibration, forwards or backwards, of the chord AB measured by inches and decimals, is $\frac{113}{355} \sqrt{\frac{p}{P}} \times \frac{AB}{39.126}$, and its reciprocal is the number of such vibrations made in one second.

For the length of a pendulum that vibrates forwards or backwards in one second, is 39.126 inches in the latitude of London, and the diameter to the circumference of a circle as 113 to 355 very nearly, and the times of the vibrations of pendulums are in the subduplicate ratio of their lengths. Whence putting t for that of the pendulum L , we have $t'' : 1'' :: \sqrt{L} = \frac{113}{355} \sqrt{\frac{p}{P}} \times AB : \sqrt{39.126}$, and $t'' = \frac{113}{355} \sqrt{\frac{p}{P}} \times \frac{AB}{39.126}$, and $\frac{1}{t} = \frac{355}{113} \sqrt{\frac{P}{p}} \times \frac{39.126}{AB}$, the number of femivibrations made in one second.

Coroll. 2. Supposing the last number to be n , we have the logarithm of $n^2 = \log. \frac{P}{p \times AB} + 2, 58676.52698$, which gives n very expeditiously. For

For the logarithm of $\sqrt{\frac{355}{113}} \times 39.126$
 $= 2,58676.52698.$

Coroll. 3. If the lengths and tensions of two chords be equal, the times of their vibrations are in the subduplicate ratio of their weights, by coroll. 1.

Coroll. 4. If their lengths and weights be equal, the times of their vibrations are reciprocally in the subduplicate ratio of their tensions, by coroll. 1.

Coroll. 5. If their tensions be in the ratio of their weights, the times of their vibrations are in the subduplicate ratio of their lengths, by coroll. 1.

Coroll. 6. The weights of cylindrical chords are in a compound ratio of their specific gravities, lengths and squares of their diameters, that is, p is as $s \times AB \times d^2$; whence t is as $AB \times d \sqrt{\frac{s}{p}}$, by coroll. 1.

Coroll. 7. Hence, if the tensions and diameters of homogeneal chords be equal

qual, the times of their vibrations are in the ratio of their lengths.

Coroll. 8. If the tensions and lengths of homogeneal chords be equal, the times of their vibrations are in the ratio of their diameters.

Coroll. 9. If the tensions of similar chords be as their specific gravities, the times of their vibrations are in the duplicate ratio of their lengths or of their diameters (*g*).

Scholium.

Hence we may find the number of vibrations made in a given time by any musical sound, if we compare it with the sound of a given chord stretched by a given weight.

For example in the experiment above-mentioned (*b*) I found the length of the vibrating chord $AB = 35.55$
inches

(*g*) See *Galileo's* experiments on chords. Dialogo 1^o attenente alla *Mecanica*, towards the end.

(*b*) Prop. XVIII.

Prop. XXIV. HARMONICS. 265

inches and its weight $p = 31$ grains troy; and the sound of it, when stretched by the weight $P = 7$ pounds averdu-
 pois = 49000 grains troy, was two octaves below the sound of the pipe d there mentioned. Hence, by coroll. 2, we have $n = 131.04$, the number of semivibrations made in one second by the wire AB , and $4n = 524.16$, the number of semivibrations made by $\frac{1}{4}AB$, by coroll. 7, or by the pipe d ; which is double the number 262 of its whole vibrations.

The length of the cylindrical part of that pipe was exactly 21.6 inches, and its diameter 1.9, which I mention because the experiment, being accurately made, is of use upon other occasions.

ADVERTISEMENT.

THOUGH the theory of imperfect consonances has been demonstrated pretty clearly, I hope, in the sixth Section, yet as I had considered some parts of it in different lights and searched a little further into some others for my own diversion, I thought it not amiss to print my papers in the form of the following Additions; that if the reader should desire any further information, he may have recourse to them whenever he pleases.

A scholium to prop. viii, explaining an uncommon case of it.

An illustration of prop. xi, with a scholium confirming the theory of the beats of imperfect consonances.

Another demonstration of scholium 4. prop. xi, (concerning the analogy between audible and visible undulations) and of prop. vii.

Another demonstration of prop. xiii and its third corollary, with an illustration of the first, and a scholium or two confirming the theory of the harmony of imperfect consonances, and shewing the absolute times and numbers of their vibrations, short cycles and dislocations of their pulses, contained in the periods between their beats.

A scholium to prop. xx, shewing the methods of altering the pitch of an organ-pipe in order to tune it.

Scho-

Scholium to Prop. viii.

Fig. 63. **W**HEN different multiples as $3AB$ and $2ab$ of the vibrations AB, ab of imperfect unisons, are the single vibrations AD, ac of an imperfect consonance, the multipliers 3 and 2 are in the ratio of the single vibrations $3AB$ and $2AB$, or $3ab$ and $2ab$ of the perfect consonance, and therefore should be irreducible to smaller numbers. The different multiples of the vibrations of imperfect unisons are therefore supposed in the proposition to be the least in the same ratio.

Fig. 63, 64. But if different multiples of the vibrations AB, ab , as $6AB$ and $4ab$, whose multipliers 6 and 4 are reducible by a common divisor, be the single vibrations of an imperfect consonance, (as they may by intermitting 6—1 pulses of AB and 4—1 of ab , so as to leave single pulses at first and between every intermission,) the period of the imperfections of this consonance will not be equal to that of the imperfect unisons AB, ab , but multiple of it by 2, the greatest common divisor of the multipliers 6 and 4.

For those multiple vibrations $6AB$ and $4ab$ are the same as $3 \times 2AB$ and $2 \times 2ab$, or $3AC$ and $2ac$, in which the equimultiples $2AB$ and $2ab$, or AC and ac in fig. 64, are the single vibrations of other imperfect unisons, resulting from an intermission of every second pulse of AB and ab in fig. 63; and the period of their imperfections is equal to that of $3AC$ and $2ac$, or AG and ae by this VIIIth proposition, and is the same multiple of the period of AB and ab , as AC is of AB , or ac of ab by the coroll. to the proposition, that is by 2, the greatest common divisor of the multipliers 6 and 4.

An

An illustration of Prop. xi.

THUS in Fig. 23, after taking away 9 short cycles from each end of the cycle AU of imperfect unisons, there remains $kKlm$, part of two more; and in Fig. 25, after taking away 3 short cycles from each end of the period AX , there remains dDe , part of another; and in Fig. 27, after taking away 4 short cycles of imperfect octaves from each end of the period Aw , there remains $iILm$, part of two more; and lastly in Fig. 34, after taking away 2 short cycles of imperfect v^{th} s from each end of the cycle AZ of the imperfect unisons, there remains $nNqr$, part of another: which though not situated exactly in the middle of AZ , by reason of the part $\Delta \varepsilon Z$, of another short cycle, containing equimultiples of AB and ab , is very near it, when the number of short cycles in the period is very large as usual; in which case the beats will be made very nearly in the middle of every period.

Scholium.

It is very unreasonable to suppose with Mr. *Sauveur* that the beats are made by the united force of the coincident pulses of imperfect unisons (*a*).

For while the imperfect unisons are made to approach gradually to perfection, experience shews that they always beat slower and slower (*b*) and by theory (*c*) the periods of their pulses grow longer and longer. Therefore in consequence of this gentleman's hypothesis, the unisons should also beat at the ends of the periods where the pulses do not coincide: Because

(*a*) See Note pag. 114.

(*b*) See Phænomenon pag. 107.

(*c*) Prop. vii. coroll. 5.

cause it is very improbable that the cycle of the first unisons supposing it simple, while it lengthens gradually, will not sometimes be changed into periods as well as into other simple cycles.

Nor can it be allowed that the unisons will beat only at the ends of their complex cycles. For according as the numeral terms expressing the ratios of the single vibrations of the several successive unisons, happen to be reducible or not, or to be irrational, the cycles of the pulses will sometimes be shortened, sometimes lengthened again, sometimes invariable and sometimes impossible, as shall be explained by and by; which accidents disagree with the gradual and constant retardation of the beats in the present case.

If it be said that the pulses next to the periodical points, fall so close to one another, as to affect the ear in the same manner as if they were quite coincident; it may be so, and most probably is so. And then it will follow that the harmony of the short cycles terminated by such close pulses, will there be the same as that of perfect unisons; at least it will certainly be better about the periodical points and coincident pulses than any where else in the periods. But the sound of a beat has no harmony in it; on the contrary it rather resembles the common sound of a beat or stroke upon any gross, irregular body. And this sound results from pulses of air which rebounding from different parts of the body, disposed to vibrate in different times, will strike the ear one after another at irregular intervals, like those of the pulses in the middle between the periodical points of imperfect unisons. Therefore these are the only pulses in each period which can excite the idea of the beats of imperfect unisons. And the like argument is applicable to any other imperfect consonance by prop. VIII.

Fig.

Fig. 55. As to those uncertain lengths abovementioned of the simple and complex cycles of the pulses of imperfect unisons, while their interval is continually diminished or increased; let one of the sounds be fixt and the time of its single vibration be represented by any given line V and those of the variable sound by the successive lines $A, B, C, \&c$, all which lines may constitute any increasing or decreasing progression; and supposing n to represent any large given number, let $A:V::n:a$, $B:V::n:b$, $C:V::n:c$, $\&c$.

Then will the cycles of the pulses of V and A, V and B, V and $C, \&c$, be $nV=aA$, $nV=bB$, $nV=cC$, $\&c$, provided every one of the numbers $a, b, c, \&c$, be integers and primes with respect to the assumed number n . In which case the several cycles are equal to one another and to nV .

But if the terms of all or any of those ratios have a common divisor, the corresponding cycles will be shortened in proportion as the greatest common divisors are larger; and therefore their lengths cannot increase or decrease successively in regular order while the successive intervals of the unisons continually decrease or increase, unless the greatest common divisors decrease or increase in regular order too; which can happen but very rarely.

And when the terms of the ratios of any of the vibrations happen to be incommensurable, a second coincidence of their pulses will be impossible: because no multiple of one vibration can be equal to any multiple of the other.

But in all cases whatever, the periods of the pulses of V and A, V and B, V and $C, \&c$, which are $\frac{nV}{n-a}$, $\frac{nV}{n-b}$, $\frac{nV}{n-c}$, $\&c$ (a), will decrease continually

ly

(a) Def. 111. Sect. vi.

ly in the same proportions with the fractions $\frac{n}{n-a}$, $\frac{n}{n-b}$, $\frac{n}{n-c}$, whose magnitudes can never be altered by any common divisors of their terms, whether integers, fractions or surds.

Coroll. 1. Hence in all cases, the periods of V and A , V and B , V and C , &c, are inversely in the same proportions as $\frac{n-a}{n}$, $\frac{n-b}{n}$, $\frac{n-c}{n}$, &c, or as $\frac{A-V}{A}$, $\frac{B-V}{B}$, $\frac{C-V}{C}$, &c, or as the magnitudes of the ratios of the vibrations, A to V , B to V , C to V , &c (a), or as the intervals of their sounds (b); as we proved before in coroll. 5. prop. VII.

Coroll. 2. If the vibrations V , A , B , C , &c, be a geometrical progression, and consequently the successive intervals of their sounds be equal, the periods of the pulses of V and A , V and B , V and C , &c, will be also a geometrical progression.

For since $nV = aA = bB = cC = \&c$, the numbers n , a , b , c , &c, are the reciprocals of V , A , B , C , &c, and consequently are also a geometrical progression.

Whence $n-a$, $n-b$, $n-c$, &c, and $\frac{nV}{n-a}$, $\frac{nV}{n-b}$, $\frac{nV}{n-c}$, &c, are also geometrical progressions.

Coroll. 3. Supposing n , a , b , c , &c, to be an arithmetical progression, the vibrations V , A , B , C , &c, and the periods of V and A , V and B , V and C , &c, will be musical progressions. Because $n-a$, $n-b$, $n-c$, &c, is also an arithmetical progression, and the pro-

(a) See pag. 89. Note

(b) Sect. 1. Art. 10.

progressions of the reciprocals $V, A, B, C, \&c$, and $\frac{nV}{n-a}, \frac{nV}{n-b}, \frac{nV}{n-c}, \&c$, of arithmetical progressions are called musical progressions.

For further illustration I will add an example or two of this last corollary.

$n = 319$	Simp. & comp. cycles	Simple cycle & periods.
$a = 320$	$319V = 320A$	$\frac{1}{1} \times 319V = \frac{1}{1} \times 320A$
$b = 321$	$319V = 321B$	$\frac{1}{2} \times 319V = \frac{1}{2} \times 321B$
$c = 322$	$319V = 322C$	$\frac{1}{3} \times 319V = \frac{1}{3} \times 322C$
$d = 323$	$319V = 323D$	$\frac{1}{4} \times 319V = \frac{1}{4} \times 323D$

$n = 320$	Simp. & comp. cycles	Simple cycles & periods.
$a = 321$	$320V = 321A$	$320V = 321A$
$b = 322$	$320V = 322B$	$160V = 161B$
$c = 323$	$320V = 323C$	$106\frac{2}{3}V = 107\frac{2}{3}C$
$d = 324$	$320V = 324D$	$80V = 81D$

It is well known that the terms of an arithmetical progression as $n, a, b, c, \&c$, whose common difference is but a small part of any of them, are very nearly in geometrical progression. Consequently since $80V = 81D$, these 4 simple cycles and periods are equal respectively to the times between the successive beats of imperfect unisons when their interval is about

1, 2, 3, 4 quarters of a comma (a), or to the times between the successive beats of any imperfect consonance whose vibrations are equal to any different multiples of those of the unisons (b) and whose temperament is respectively about 1, 2, 3, 4 quarters of a comma.

I have lengthened this scholium not so much to confute Mr. *Sauveur's* conjecture, which I had done before, as to illustrate the theory of the periods and beats of imperfect consonances: and with the same design I will add

*Another demonstration of scholium 4. Prop. xi,
and of Prop. vii.*

THE breadth of the apparent Undulations of the lights and shades seen at a distance upon two rows of parallel objects, may be also found by the following construction.

Fig. 56. Let a plane passing through a distant eye at z , cut the axes of the parallel objects at right angles in the points $a, b, c, \&c, \alpha, \beta, \gamma, \&c$, which are supposed equidistant in both the parallel lines $abc, \alpha\beta\gamma$. From any object in one of these lines to any successive objects in the other, draw the lines $\alpha a, \alpha b, \alpha c, \&c$, and the lines $zvV, zxX, zyY, \&c$, drawn parallel to them, will intercept the equal breadths of the apparent undulations.

Because while the eye is gradually directed from the middle of any of the breadths $VX, XY, \&c$ towards either of its extremities, the objects will appear closer together in couples, in proportion to their smaller distances

(a) Sect. II, Art. 4.

(a) Prop. VIII.

distances from the next extremity, (which was shewn in the scholium to be the cause of the undulations).

For the lines $d\delta, e\epsilon, f\zeta, g\eta, \&c.$, being parallel to $a\alpha$, are parallel to Vv by construction; and the lines $i\theta, k\iota, l\kappa, m\lambda, \&c.$, being parallel to $b\alpha$, are parallel to Xx ; and so on. Let lines drawn from z through the objects $\delta, \epsilon, \zeta, \&c.$, of one row, cut the line of the other in $D, E, F, \&c.$ Then because the rows are parallel, the ratio of $D\delta$ to δz , $E\epsilon$ to ϵz , $F\zeta$ to ζz , $\&c.$, is the same as Vv to vz , or Xx to xz , $\&c.$ (a). Whence also, because of the parallels between the rows, we have

$$\left. \begin{array}{l} D\delta : dV :: D\delta : \delta z \\ E\epsilon : eV :: E\epsilon : \epsilon z \\ F\zeta : fV :: F\zeta : \zeta z \\ G\eta : gV :: G\eta : \eta z \\ Hi : iX :: H\theta : \theta z \\ Ik : kX :: I\iota : \iota z \\ Kl : lX :: K\kappa : \kappa z \\ Lm : mX :: L\lambda : \lambda z \\ \&c. \quad \quad \quad \&c. \end{array} \right\} :: Vv : vz;$$

That is, all those ratios are equal, and, alternately, the lesser apparent intervals $D\delta, E\epsilon, F\zeta, G\eta$, are proportional to their distance dV, eV, fV, gV , from the next extremity V of the breadth VX ; and also Hi, Ik, Kl, Lm , proportional to iX, kX, lX, mX , their distances from the next extremity X of the same breadth VX . And the breadths $VX, XY, \&c.$, are equal, because $ab, bc, \&c.$ are so, and the triangles VzX and $a\alpha b$, XzY and $b\alpha c$, $\&c.$ are similar by construction. *Q. E. D.*

Coroll.

(a) 2. VI. Euclid.

Coroll. 1. The projections DE , EF , &c of the equal intervals $\delta\varepsilon$, $\varepsilon\zeta$, &c, are to these intervals in the constant ratio of Dz to δz , or $E\varepsilon$ to εz , or Vv to vz , and consequently are equal to one another. Therefore supposing the lines DE and $\delta\varepsilon$ or de to represent the times of the single vibrations of imperfect unisons, the periods of the nearest approaches of their pulses D, E , &c, d, e , &c, are VX, XY , &c; And in going from their extremities V, X to the middle, the alternate lesser intervals between the successive pulses, are proportional to their distances from the next extremity, as we shewed just now: which is another proof of prop. VII.

Coroll. 2. If the eye be moved in a line parallel to the rows, the breadths of the apparent undulations will be constantly the same, and if it be moved uniformly in any other right line, their breadths will vary uniformly, and be constantly proportional to the distance of the eye from the rows. Because the triangles VzX , VzY , &c, are constantly similar to aab , aac , &c. And this conclusion seems to agree with what I have transiently observed of these undulations.

But it is easy to collect from the construction of the figure, and the different ratios of zV to zv expressed by numbers, that the intervals between the apparent conjunctions of the objects will increase and decrease very irregularly; and that no conjunctions can happen except when the eye arrives at certain points of its course, and none at all, mathematically speaking, when its distances from the two rows, measured upon any right line, happen to be incommensurable. Which conclusions being contrary to the continual appearance of the undulations to the eye in all places, and to the regular increase or decrease of their breadth, shew, that the breadth is not equal

equal to the interval between the apparent conjunctions, no more than the interval between the beats of imperfect unisons is equal to the interval between their coincident pulses.

L E M M A.

In any period between the successive beats of any imperfect consonance, any given number of short cycles next to one side of the least dislocation of the pulses, is more harmonious, and the same number of them next to the other side is less harmonious than the same number of them next to either side of the coincident pulses: and these degrees of harmony differ more in those periods where the two least dislocations differ less, and most of all in the periods where these dislocations are equal when possible.

Fig. 59. Let AB and ab represent the times of the single vibrations of imperfect unisons, A and a their coincident pulses, $B, C, D, \&c.$, $b, c, d, \&c.$, their successive pulses on each side of A, a ; Rr their least dislocation in any given period, and consequently the nearest to the periodical point z , which is here placed under A , for the convenience of seeing at one view, the short cycles next to both sides of Rr and Aa .

First I say, the short cycles $RS, ST, \&c.$, which include z , are more harmonious, and $RQ, QP, \&c.$, less harmonious than $AB, BC, \&c.$, the numbers of them being the same: and that the degrees of their harmony differ more in the periods where the two least dislocations Rr, sS differ less, and most of all where $Rr = sS$, when possible (a).

$$\text{For } bB = (AB - Ab = RS - rs =) Rr + sS (b).$$

$$\text{And } cC = (AC - Ac = RT - rt =) Rr + tT.$$

$\&c.$

$\&c.$

Hence

(a) See prop. VII. coroll. 2. (b) Prop. VII. coroll. 1.

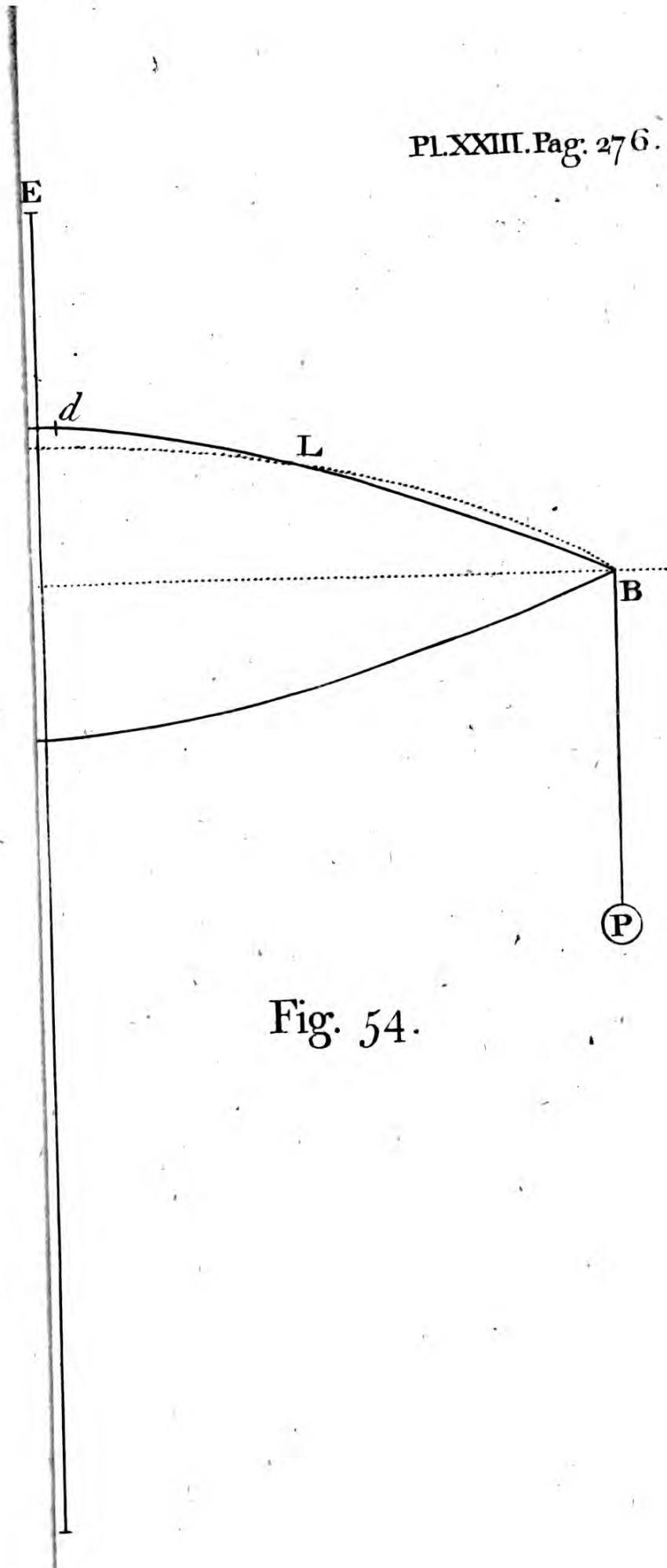


Fig. 54.



Hence the successive dislocations $sS, tT, \&c,$ are respectively smaller than $bB, cC, \&c,$ by Rr , as appears also by their smaller distances from z (*a*). But on the other side of Rr , the dislocations $Qq, Pp, \&c,$ are respectively greater than $bB, cC, \&c,$ by the same Rr , for the like reasons.

Now the short cycle RS which includes z , is more harmonious than AB next to A . For though the dislocations $Rr + sS$ are $= bB$, yet those parts of bB , as being smaller than bB , will give less offence to the ear than the whole: the whole may be perceived and give some offence even when one or both its parts are imperceptible. And for the same reasons the short cycle RS will be still more harmonious than AB in other periods where Rr, sS are less unequal, and the most harmonious where they are equal when possible (*b*): their sum being every where the same.

The next short cycle ST is also more harmonious than BC ; the dislocations sS, tT being respectively smaller than bB, cC . Therefore the short cycles RS, ST , taken together, are more harmonious than AB, BC taken together; and still more harmonious in other periods where sS, tT are smaller, till sS be equal to Rr .

But on the other side of Rr and Aa , the short cycle RQ is less harmonious than AB , the dislocations Qq, Rr being larger than Bb and o . The next short cycle QP is also less harmonious than BC ; the dislocations Pp, Qq being respectively larger than Cc, Bb . Therefore RQ, QP together are less harmonious than AB, BC together; and still less harmonious in other periods where Rr, Qq, Pp are larger, till Rr be equal to sS . And the same is evident in any larger

(a) Prop. VII.

(b) Prop. VII. coroll. 2.

larger equal numbers of short cycles, throughout the period between the successive pulses.

Secondly, any imperfect unisons will be changed into imperfect octaves whose single vibrations are AC and Ab , or Ac and AB , by conceiving every second pulse of the series $A, B, C, \&c$, or $a, b, c, \&c$, to be intermitted, which would depress one of the unisons an octave lower.

Now if that intermission should take away the alternate pulses $S, U, \&c$, or $s, u, \&c$, the short cycles of the octaves, next to one side of Rr , will be $RT, TW, \&c$, and on the other, $RP, PN, \&c$: I say the former as including z are more, and the latter less harmonious than $AC, CE, \&c$, the numbers of them being equal.

For we had $Rr + tT = cC$, consequently the short cycle RT is more harmonious than AC , for the same reason as in unisons, and because the intermediate dislocations sS, bB are vanished, one of their constituent pulses in each being taken away. And RT is still more harmonious than AC in other periods where Rr and tT are less unequal.

The next short cycle TW is also more harmonious than CE , the dislocations tT, wW being respectively smaller than cC, eE , as in unisons; and is still more harmonious in other periods where tT, wW are smaller, that is where tT and Rr are less unequal.

But on the other side of Rr and Aa , the short cycle RP is less harmonious than AC , and PN than CE , the dislocations Rr, Pp being respectively bigger than o and Cc ; and Pp, Nn bigger than Cc, Ee , respectively: and is still less harmonious in other periods where Rr, Pp, Nn are larger, that is where Rr, tT are less unequal.

There-

Therefore the short cycles $RT, TW, \&c$ are more, and $RP, PN, \&c$ are less harmonious than $AC, CE, \&c$.

Likewise if that alternate intermission should take away the pulses $R, T, W, \&c$, or $r, t, w, \&c$, then the least dislocation is sS , and the short cycles $SQ, QO, \&c$, as including z , will be more, and $SU, UX, \&c$, less harmonious than $AC, CE, \&c$, for the very same reasons as before.

Thirdly, any imperfect unisons will be changed into imperfect v^{th} s, whose vibrations are Ac and AD , (or AC and Ad) that is, $2ab$ and $3AB$, by intermitting $2-1$ pulses of the series $a, b, c, d, \&c$, which depresses the acuter unison an $viii^{th}$ lower, and $3-1$ pulses of the series $A, B, C, D, \&c$ leaving single ones between, which depresses the graver unison a xii^{th} or $viii+v^{th}$ lower; and thus the interval of the new sounds is an imperfect v^{th} , as represented in the uppermost parallel in the figure.

Now in the period where those intermissions leave the pulses $r, t, w, y, \&c$, $R, U, Y, \&c$, (as in the 4^{th} parallel) the intermediate ones will be taken away, and then Rr being the lesser of the two dislocations in the short cycle RY which includes z , is the least of all in this period. And the short cycles $RY, \&c$, on this side of Rr , will be more harmonious than $AG, \&c$ (in the first parallel); and on the other side, the short cycles $RL, \&c$, will be less harmonious than $AG, \&c$: For the same reasons as above.

Likewise in the period where the pulses $q, s, u, x, \&c$, $Q, T, X, \&c$ are left (in the 5^{th} parallel), the intermediate ones will be absent, and then Qq is the least dislocation in this period, and a greater difference than before will be found in the harmony of the short cycles on each side of Qq and Aa ; the difference

$Xx - Qq$ being less than $Yy - Rr$ in the former case.

Lastly in the period where $p, r, t, w, \&c, P, S, W, \&c,$ are left (in the lowest parallel), the intermediate ones are intermitted, and then Pp is the least dislocation in this period, and a difference still greater will be found in the harmony of the short cycles on each side of Pp and Aa , for the like reason. And the greatest difference will be found where these dislocations are equal when possible; that is, when a periodical point z bisects a short cycle of any consonance, which consists of any odd number of those of the unisons; and also when either of the coincident pulses at the ends of the complex or simple cycles of the unisons, bisects a short cycle of any consonance, consisting of any even number of those of the unisons as in Fig. 35. Plate XII. The like proof is plainly applicable to the vibrations AC, Ad , or to those of any other consonance. $Q, E. D.$

Coroll. Hence any two imperfect consonances will be as equally harmonious as they possibly can be, when the periods (between their successive beats) which are bisected by their coincident pulses, are made equally harmonious; these periods having a mean degree of harmony among those of all the other periods in each consonance.

All those degrees of harmony occur in practical music, and whether sensibly different or not (*a*), must be used as if they were equal, and in theory we must take the medium among them.

As the proof of this conclusion has been pretty long, I avoided it in the Book by a paragraph (*b*) in the demonstration of prop. XIII, which may now be proved somewhat differently.

Another

(*a*) See pag. 126. (*b*) Pag. 137.

*Another demonstration of Prop. xiii and its
third corollary.*

Fig. 60, 61. Let op and OP represent the times of the single vibrations of imperfect unisons; ab and AB those of other imperfect unisons; o and O , a and A their coincident pulses; and if $ab = op$, the period of the pulses of the former unisons, will be to that of the latter, as bB to pP (a). Because the inverse ratio of the intervals of the imperfect unisons op and OP , ab and AB , or of the magnitudes of the ratios between op and OP , ab and AB (b), is bB to pP (c).

Taking bB to pP as 1 to 2, this is now the ratio of the lengths of the periods of the unisons op and OP , ab and AB ; and the latter is of the same length as the period of the dislocations of imperfect octaves, whose single vibrations are ab and AC or $2AB$, by intermitting 2—1 pulses of the series $A, B, C, D, \&c$, by prop. VIII.

Now the shorter length of the short cycles of the unisons op, OP , is $op = ab$, and that of the short cycles of the imperfect octaves is ac or $2ab$, and the ratio of their lengths is 1 to 2, which being the same as that of the periods of the unisons and octaves, shews that their short cycles are equally numerous in them.

The longer length of the short cycle of the octaves is AC or $2AB$, and the difference of the lengths is $2AB - 2ab = 2bB = cC$, the dislocation of the pulses at the end of the first short cycle, and is equal to pP , because we took $bB : pP :: 1 : 2$; therefore the several dislocations $eE, \&c, qQ, \&c$, at the ends of the subsequent short cycles of the octaves and unisons, are equal respectively throughout their half periods, which are therefore equally harmonious.

Be-

(a) Prop. VII. coroll. 5. (b) Sect. I. 10. (c) Note in pag. 89.

Because those dislocations are the causes that spoil the harmony, more or less according as they are greater or smaller; and causes constantly equal must have equal effects: And because the harmony of these half periods is the medium among the degrees of harmony of all the rest, by the coroll. to the lemma.

Fig. 60, 62. Again, taking bB to pP as 1 to 3, this is now the ratio of the periods of the imperfect unisons op and OP , ab and AB ; and the latter period is equal to that of imperfect x_{11}^{th} , whose vibrations will be ab and AD or $3AB$ by intermitting 3—1 pulses of the series $A, B, C, D, \&c$, so as to leave single pulses between every intermission. And since $Pp = (3bB =) dD$, it appears that the several subsequent dislocations $qQ, \&c, gG, \&c$, of the unisons and x_{11}^{th} , are equally numerous and equal respectively throughout the half periods on each side of oO and aA ; which render the consonances as equally harmonious as they possibly can be, for the reasons above.

Fig. 60, 63. Lastly, taking bB to pP as 1 to 2×3 , this is now the ratio of the lengths of the periods of the dislocations of the imperfect unisons op and OP , ab and AB , for the reason above. And the latter period is of the same length as that of imperfect v^{th} , whose single vibrations $2ab$ and $3AB$ (or $2AB$ and $3ab$) result from intermitting 2—1 pulses of the series $a, b, c, d, e, f, g, \&c$, and 3—1 pulses of the series $A, B, C, D, E, F, G, \&c$, so as to leave single pulses at the beginning and between every intermission, by prop. VIII.

Now the shorter length of the short cycle of the unisons op, OP is $op = ab$, and that of the short cycle of the imperfect v^{th} is $2 \times 3ab$ (because $2ab : 3ab :: 2 : 3$) and the ratio of these lengths is 1 to 2×3 , the same as that of the periods of the imperfect unisons

sons op , OP and the v^{th} s, whose short cycles op and ag are therefore equally numerous in them.

The longer length of the imperfect short cycle of the v^{th} s is $2 \times 3AB$ (because $2AB : 3AB :: 2 : 3$) and the difference of the longer and shorter lengths is $2 \times 3AB - 2 \times 3ab = 2 \times 3, \overline{AB - Ab} = 2 \times 3bB = gG$, the dislocation of the pulses at the end of that short cycle, and is equal to pP , because we took $bB : pP :: 1 : 2 \times 3$. Therefore the several dislocations nN , &c, qQ , &c, at the ends of all the subsequent short cycles of the v^{th} s and unisons, are respectively equal in magnitude and number too, throughout the half periods on each side of the coincident pulses aA, oO ; which equalities make these consonances as equally harmonious as they possibly can be, for the reasons above.

Instead of the terms 2 and 3 of the ratio of the vibrations of perfect v^{th} s, if we substitute those of any other perfect consonance, or r and s indeterminately for them, the method of demonstration will be evidently the same as in the last example.

Now those imperfect consonances of $viii^{\text{th}}$ s, xii^{th} s, v^{th} s, &c are not only equally harmonious with the same imperfect unisons op, OP , but also with one another; the dislocations pP, cC, dD, gG , at the ends of their first and subsequent short cycles, being equal and equally numerous in their periods. And since any one of them is equally harmonious to another of the same name, at any other pitch, when their short cycles are equally numerous in their periods (a), it appears that all sorts of imperfect consonances are as equally harmonious as possible, when their short cycles are equally numerous in the periods of their imperfections. *Q. E. D.*

Coroll.

(a) Prop. xii .

Coroll. Hence imperfect consonances are equally harmonious, when their temperaments have the inverse ratio of the products of the terms expressing the ratios of the single vibrations of the perfect consonances.

This is the third corollary to prop. XIII and may be demonstrated in this other manner.

The interval of the sounds of imperfect unisons is the temperament of the interval of any consonance, whose single vibrations are different multiples of the vibrations of those unisons. Thus in Fig. 63 the interval of the imperfect unisons ab and AB is the temperament of the interval of the v^{th} s whose perfect vibrations are $2ab$ and $3ab$, and whose tempered ones are $2ab$ and $3AB$, or ac and AD ; because the difference of their two ratios is $3ab$ to $3AB$, or ab to AB , the ratio of the vibrations which terminate the temperament.

Now in all the examples of tempered consonances we made the term ab constant and AB variable. Consequently the several magnitudes of the ratios of ab to AB were proportional to the corresponding magnitudes of the differences $bB(a)$, which in the

$viii^{\text{th}}$ s and v^{th} s were made equal to $\frac{1}{1 \times 2} pP$ and

$\frac{1}{2 \times 3} pP$ respectively. Therefore when these consonances are equally harmonious, the ratio of their

temperaments is $\frac{1}{1 \times 2}$ to $\frac{1}{2 \times 3}$.

Now when either of them is equally harmonious to another of the same name at a different pitch, their temperaments are equal (b), and the terms of the ratio of the vibrations of the perfect consonances of that name are the same.

Con-

(a) Note in pag. 89.

(b) Prop. XII. coroll.

Consequently the direct ratio of the temperaments and the inverse ratio of the products of those terms, are the same in all equally harmonious consonances.

An illustration of coroll. 1. Prop. xiii.

Fig. 57. In the line AP let the intervals of the equidistant points $A, I, II, III, \&c$ be the longer or the shorter lengths of the imperfect short cycles of any given consonance; A and a its coincident pulses; ab the lesser of the vibrations of the imperfect unisons whose half period is also AP . Make the perpendicular $PQ = \frac{1}{2} ab$, and draw AQ cutting the perpendiculars at $I, II, III, \&c$, in $D, D, D, \&c$. Then are these perpendiculars equal to the dislocations of the pulses between the successive short cycles of the imperfect consonance, by prop. VII and VIII.

Fig. 58. Make the like construction denoted by the greek letters, for any other imperfect consonance of the same or a different name. And if it be equally harmonious to the former, its half period $\alpha\pi$ will contain the same number of short cycles as AP does (a); suppose 6 in each. By lessening its temperament, let its half period be lengthened to $\alpha\rho$, where erecting the perpendicular $pq = \pi\kappa$ join αq cutting all the intermediate perpendiculars in $e, e, \&c$. Then the several new dislocations $1e, 2e, 3e, \&c$ will be smaller than $1\delta, 2\delta, 3\delta, \&c$ respectively. Therefore the short cycles $\alpha 6e$, contained in part of the new half period $\alpha\rho$, are not only more harmonious than the short cycles $\alpha 6\delta$, contained in the old half period $\alpha\pi\kappa$, or than $AVID$, but those in the remaining part $e678e$ continue the harmony in the new half period

(a) Prop. XII, XIII.

period αp , when that of the former half periods is quite extinguished by the beats at the ends of them.

Coroll. Since only the corresponding short cycles of imperfect consonances can admit of a just comparison, one by one, in the order of their succession, beginning from the coincident pulses, or from their least dislocations next to the periodical points; (as explained in the demonstration of the lemma;) if the periods of two consonances contain unequal numbers of their short cycles, the comparison will be imperfect: which is another argument *à priori* for the truth of the XII and XIII propositions.

Scholium I.

In any pure consonance (*a*) the short cycle contains but one vibration of the base, as in Fig. 61, 62, and the equal times between the pulses of the treble are never subdivided by any pulses of the base, except at the ends of the short cycles; and here the dislocations *cC*, *dD* are considered and adjusted with the analogous ones in other pure consonances, by the XIII proposition.

But in any other consonance whose short cycle contains several vibrations of the base, the equal times between the pulses of the treble are subdivided by the pulses of the base, not only at the ends of the short cycles, but between them, as at *D*, *K*, &c, Fig. 63; where the consideration of the inequalities of the intervals *cD* and *De*, *iK* and *Kl*, &c, seems to have been neglected in the said proposition, but in reality is implied in it.

Fig. 63. For supposing the alternate pulses *D*, *K*, &c to be intermitted or taken away, those v^{th} will be changed into XIIth an octave lower than the XIIth in Fig. 62; but will not be equally harmonious with them,

(*a*) Sect. III. Art. 8.

them, as the v^{th} s were supposed to be before that intermission, till the dislocations $gG, nN, \&c$, in Fig. 63 be doubled; that the temperaments of both the xii^{th} s may be equal and their periods proportional to their vibrations and short cycles (*a*).

While the dislocations $gG, nN, \&c$, remain doubled, restore the pulses $D, K, \&c$, to their places, and now the intermediate inequalities $Dc - De, Ki - Kl, \&c$ are also the doubles of their former magnitudes and the new v^{th} s are less harmonious than the xii^{th} s in Fig. 62, and will not be equally harmonious with them till the dislocation, $gG, nN, \&c$, and consequently the inequalities $Dc - De, Ki - Kl, \&c$ be contracted to their former magnitudes.

Therefore these interrupted consonances are not considered as pure ones in the xii and $xiii$ propositions, but allowance is made on course for the effect of the intermediate pulses of the base.

Scholium 2.

Fig. 63. Supposing the letters d, k to be restored to the places of the absent pulses of the imperfect unisons, that fall next to D and K , I call the lines or times dD, kK the Aberrations of the interior pulses D, K , from the places d, k which they have in the perfect short cycles. Likewise in the upper part of the same figure, if AE and ad be the single vibrations of an imperfect 4^{th} , then Ee is an aberration of one of the interior pulses of the base in the first short cycle.

Now if the ratio of the times of the single vibrations of any perfect consonance be m to n in the least integers, and when it is tempered, if $2D$ be the sum of the exterior dislocations in any given short cycle, the aberration or sum of the aberrations of the interior

(*a*) Prop. xii . coroll.

rior pulses of the base, from the places they have when the consonance is perfect, will be $\overline{n-1} \times D$.

The reason of the theorem will soon appear by drawing a short cycle or two of a 4th, III^d, &c, and by observing, that as n is the number of the vibrations of the base contained in any short cycle, so $n-1$ is the number of its pulses exclusive of the extremes; and that the sum of the exterior dislocations is equal to the sum of any two interior aberrations equidistant from them, or to double the aberration in the middle; as is plain from the arithmetical progression of the alternate lesser intervals of the imperfect unisons, from which the given consonance is derived.

Therefore in two equally harmonious consonances, as the sum of the exterior dislocations in any short cycle of the one, is to the sum of them in the corresponding short cycle of the other, in a certain constant ratio (a), so the interior aberration or the sum of the interior aberrations in the former short cycle, is to the sum of them in the latter in another constant ratio; and *componendo*, the totals of the exterior dislocations and interior aberrations are also in another constant ratio.

But the temperaments and periods of the two consonances must be adjusted by the first given ratio alone, without any regard to the second or third.

1. Because the exterior dislocations are of a different kind from the interior aberrations. For as in seeing so in hearing, it is more difficult for the sense to perceive the quantity of a small inequality in the larger successive intervals of the points or pulses c, D, e , Fig. 63, than to perceive the same or a different small quantity when bounded by two visible points or pulses g, G . And the difficulty is greater
in

(a) By the foregoing illustration.

in more complex short cycles of imperfect 4ths, III^{ds}, VI^{ths}, &c, where the successive intervals between the points analogous to *c*, *D*, *e*, do not err from the simplest ratio of 1 to 1, but from the more complex ones of 1 to 2, 1 to 3, &c; as will easily appear from the disposition of the pulses in such cycles in Fig. 5, Plate I. For which reason the ratio of the sum of the interior aberrations ought not to be compared and compounded with that of the exterior dislocations.

2. Because it appears from the last corollary, that no other regard can be had to the interior aberrations, than what follows on course from the given ratio of the exterior dislocations, determined by the equality of their numbers in the periods of the two consonances, as in prop. XII and XIII.

Scholium 3.

To give the reader more determinate ideas of the numbers of vibrations, short cycles and dislocations contained in the periods of imperfect consonances, and of their absolute duration in practical music, I will conclude with a computation of them in a consonance of v^{ths} tempered by $\frac{1}{4}$ comma, as it usually is, more or less, in organs and harpsichords.

Fig. 63. If $AB : ab :: 322 : 321$, the interval of the sounds of these vibrations is $\frac{1}{4}$ comma very nearly (*a*). Whence $321 AB, = 322 ab = AZ$ in Fig. 34 or 35 Plate XII, is the length of the simple cycle of the dislocations of the pulses of the vibrations *AB*, *ab*, or of the period of the imperfections of any consonances whose vibrations are different multiples of *AB* and λb (*b*) and whose temperament is the interval of the sounds of *AB* and *ab* (*c*).

Now

(*a*) Pag. 127. (*b*) Prop. VIII. (*c*) See dem. coroll. pag. 284.

Now the vibrations of imperfect v^{th} s are AD and ac , or $3AB$ and $2ab$, and the two constant lengths of their short cycles are $AG=2AD=2 \times 3AB$ and $ag=3ac=3 \times 2ab$.

$$\text{Hence } AZ = 321AB = \frac{321}{3} \times 3AB = 107AD = \frac{321}{6} \times 6AB = 53\frac{3}{6}AG;$$

$$\text{Likewise } AZ = 322ab = \frac{322}{2} \times 2ab = 161ac = \frac{322}{6} \times 6ab = 53\frac{4}{6}ag.$$

And after the coincidence of the pulses, their first dislocation is $gG = \frac{1}{161}AD$: and the limit of the

greatest dislocation is $\frac{1}{2}ab = \frac{107}{644}AD$.

For $AB : AB - Ab$, or $bB :: 322 : 1$, whence $bB = \frac{1}{322}AB$, and $gG = 6bB = \frac{6AB}{322} = \frac{3AB}{161} = \frac{1}{161}AD$: and $\frac{1}{2}ab = \frac{1}{2} \times \frac{321}{322}AB = \frac{1}{2} \times \frac{321}{322} \times \frac{1}{3}AD = \frac{107}{644}AD$, and is less than the greatest dislocation or alternate lesser interval of the imperfect unisons AB, ab in any half period by prop. VII.

Now by an experiment mentioned in prop. XVIII, I found that the particles of air in an organ-pipe called *d* or *d-la-sol-re*, in the middle of the scale of the open diapason, made 262 complete vibrations or returns to the places they went from, and consequently propagated 262 pulses of air to the ear (*a*), in one second of time, though the pitch of the organ was above

(a) Newt. Princip. prop. 43. lib. 2.

above half a tone lower than the present pitch at the Opera. And taking that found for the base of our v^{ths} , whose vibration AD represents a certain quantity of time, we have $262AD = 1$ second, and hence

the absolute times AZ , AG , Gg and $\frac{1}{2} ab$.

$$\text{For, } 262AD : AZ \text{ or } 107AD :: 1'' : \frac{107}{262} \times 1''.$$

$$\text{and } 262AD : AG \text{ or } 2AD :: 1'' : \frac{1}{131} \times 1''.$$

$$\begin{aligned} \text{and } 262AD : Gg \text{ or } \frac{1}{161}AD :: 1'' : \frac{1}{262 \times 161} \times 1'' \\ = \frac{1}{42182} \times 1''. \end{aligned}$$

$$\begin{aligned} \text{and } 262AD : \frac{1}{2} ab \text{ or } \frac{107}{644}AD :: 1'' : \frac{107}{262 \times 644} \times 1'' \\ = \frac{1}{1568} \times 1''. \end{aligned}$$

And the reciprocal of the periodical time $AZ = \frac{107}{262} \times 1''$ is the number of periods and also of beats in $1''$ (a), namely $\frac{262}{107} = 2.45$ nearly in $1''$, or 245 in $100''$ nearly.

And the least dislocations in the short cycles which include the successive periodical points Z , are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$ of the dislocation Gg next to the coincident pulses.

And these measures are to those in any other v^{ths} in the scale of that organ, in the given ratio of the times of the single vibrations of their bases. And the like

(a) Prop. XI.

like measures in any other given consonance, whose temperament is given, may be computed in the like manner, or derived from these by the corollaries to prop. ix.

In this example the v^{th} s were tempered sharp, and when they are equally tempered flat, by taking Ad and AC for the single vibrations, the computation and the measures will be but very little different (*a*).

Scholium 4. to prop. xx.

The sound of an open metalline pipe will be flattened or sharpened a little, by bending a small part of the metal at the open end, a little inwards or outwards, respectively.

The sound of a stopt pipe, made of metal, will be flattened or sharpened a little, by bending the ears, at the sides of the mouth, a little inwards or outwards, respectively.

The sound of an open wooden pipe will be flattened or sharpened a little, by depressing or raising the leaden plate that hangs over the open end, respectively.

The sound of a stopt wooden pipe will be flattened or sharpened by drawing the plugg outwards or forcing it inwards, respectively.

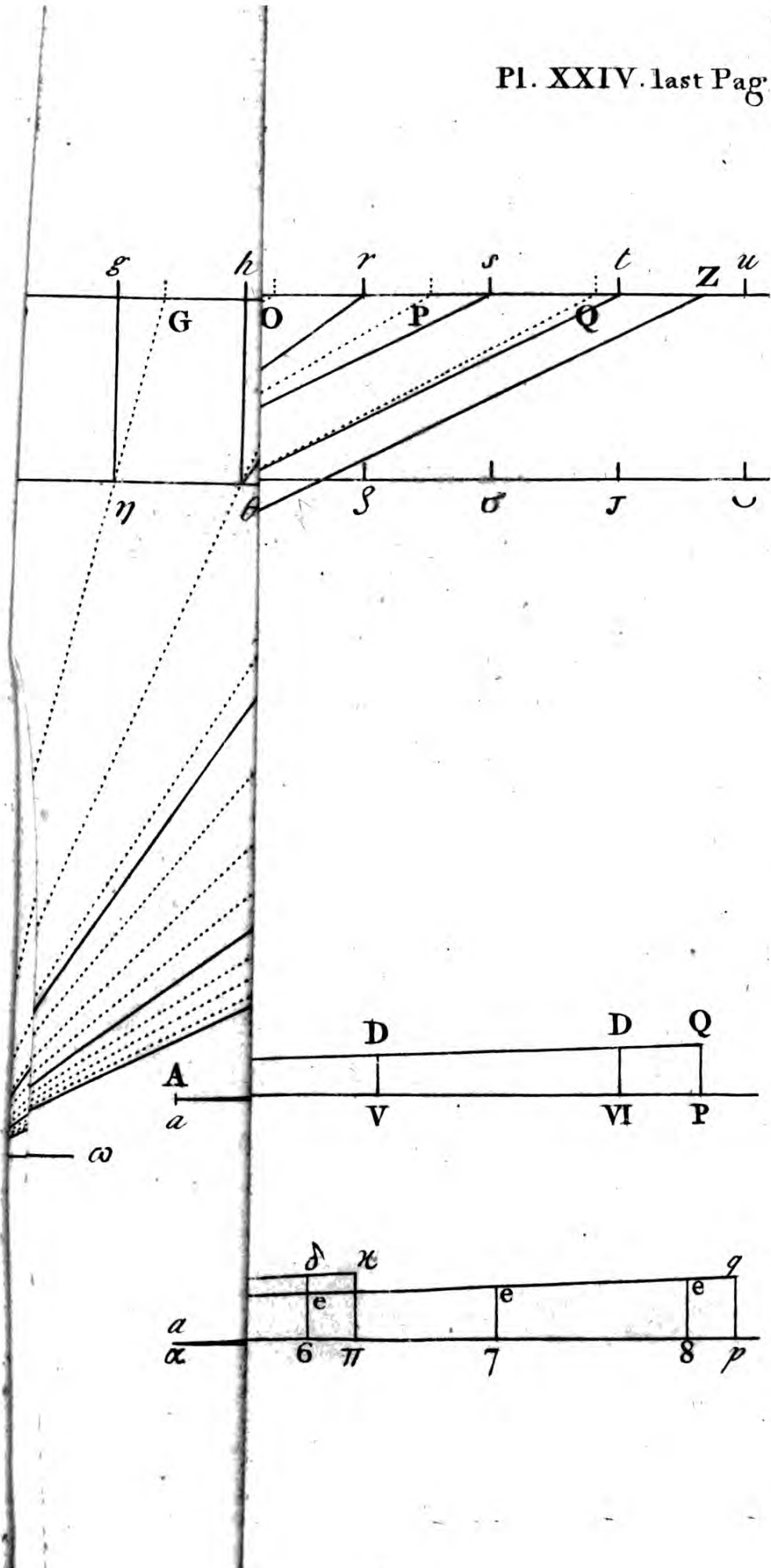
The sound of a reed-pipe will be flattened or sharpened by causing the brazen tongue to be lengthened or shortened, respectively.

There is something curious in the reasons of these effects, but as they cannot be well explained in few words, I must omit them.

(*a*) Prop. x.



AN





The diagram consists of seven horizontal lines. Above the first line are the letters D and G. Below the first line are the lowercase letters c, e, and g. Above the second line are the uppercase letters B, C, D, E, F, and G. Below the second line are the lowercase letters b, c, d, e, f, and g. Above the third line are the uppercase letters T, U, W, X, and Y. Below the third line are the lowercase letters t, u, n, x, and y. Above the fourth line is the uppercase letter U. Below the fourth line are the lowercase letters t, n, and y. Above the fifth line is the uppercase letter X. Below the fifth line are the lowercase letters u and x. Above the sixth line is the uppercase letter W. Below the sixth line are the lowercase letters t, n, and y. On the left side, there are vertical lines and some letters: 'E' is next to the third line, 'f' is next to the fifth line, and 'F' is next to the sixth line.



A N

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CORRECTIONS & ADDITIONS.

PAG. 15. line 3. *read* in the table, except 81 to 80, or in any one of them compounded once or oftener with the ratio of 2 to 1, the consonance &c. P. 33. l. 4. *r.* to *E*, *E* to *B*; and &c. P. 41. l. 17. *r.* In this notation then &c. P. 43. l. 16. *r.* the third *df* and sixth *fd* &c. P. 48. *note* l. 4. *for True read* Be it so. P. 64. l. 15. *r.* or *AN* is a ratio &c. P. 65. l. 1. *r.* is less than *AS*, and l. 2. *r.* less than *ET*.

P. 67. l. 22. *r.* first condition. P. 72. l. 3. $r. = \frac{-t}{4r+t}$.

P. 77. Def. II. l. 3. *r.* the least terms &c. P. 79. l. 8. *r.* which physically speaking must &c. P. 80. l. 18.

r. $\frac{n}{2} V$: *To note (a) add* For if 2 measured *n*, it would

measure $m = n + 2$, and thus the terms m, n of the ratio of *V* to *v* would not be the least. P. 85. l. 1. *r.* *Kl.* P. 97. l. 3. *r.* of *AD* and *ac.* P. 98. l. 17. *r.* *AB* and *ab* whose pulses &c. P. 103. prop. x. l. 3. *r.* period of the least imperfections of the tempered consonance is &c. P. 115. l. 9. *r.* notice of the cause of the beats &c. P. 116. *note (a)* l. 1. *r.* Les battements &c. P. 171. l. 14. *r.* the parts in any assigned places of &c. P. 189. Tab. I. l. 4. *r.* 369 : 370. P. 199. l. 4. *r.* would be 16 as before, and l. 19. *r.* the 16 false &c. P. 216. l. 8 & 20. *r.* the perfect v^{ths} &c. P. 217. l. 15. *r.* answers one part of the design &c. P. 222. l. 9. *r.* III^{ds} and x^{ths} . P. 224. l. 7. *r.* made by two given pipes &c.—l. 8. *r.* and these pipes &c,—l. 10. *r.* alter their pitch. P. 242. l. *penult.* *r.* though one of &c.—*note (b)* *r.* prop. xx. Schol. 2. Art. 6. P. 259. l. 8. *r.* of the harmonical curve. P. 260. *note (d)* *r.* corol. 8. Def. P. 268, l. 16. *r.* is comparatively very near it.

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Hydrostatical and Pneumatical Lectures by Mr. *Cotes*, published by Dr. *Smith*: Cambridge 1747, 2^d Edition, 8^{vo}.





