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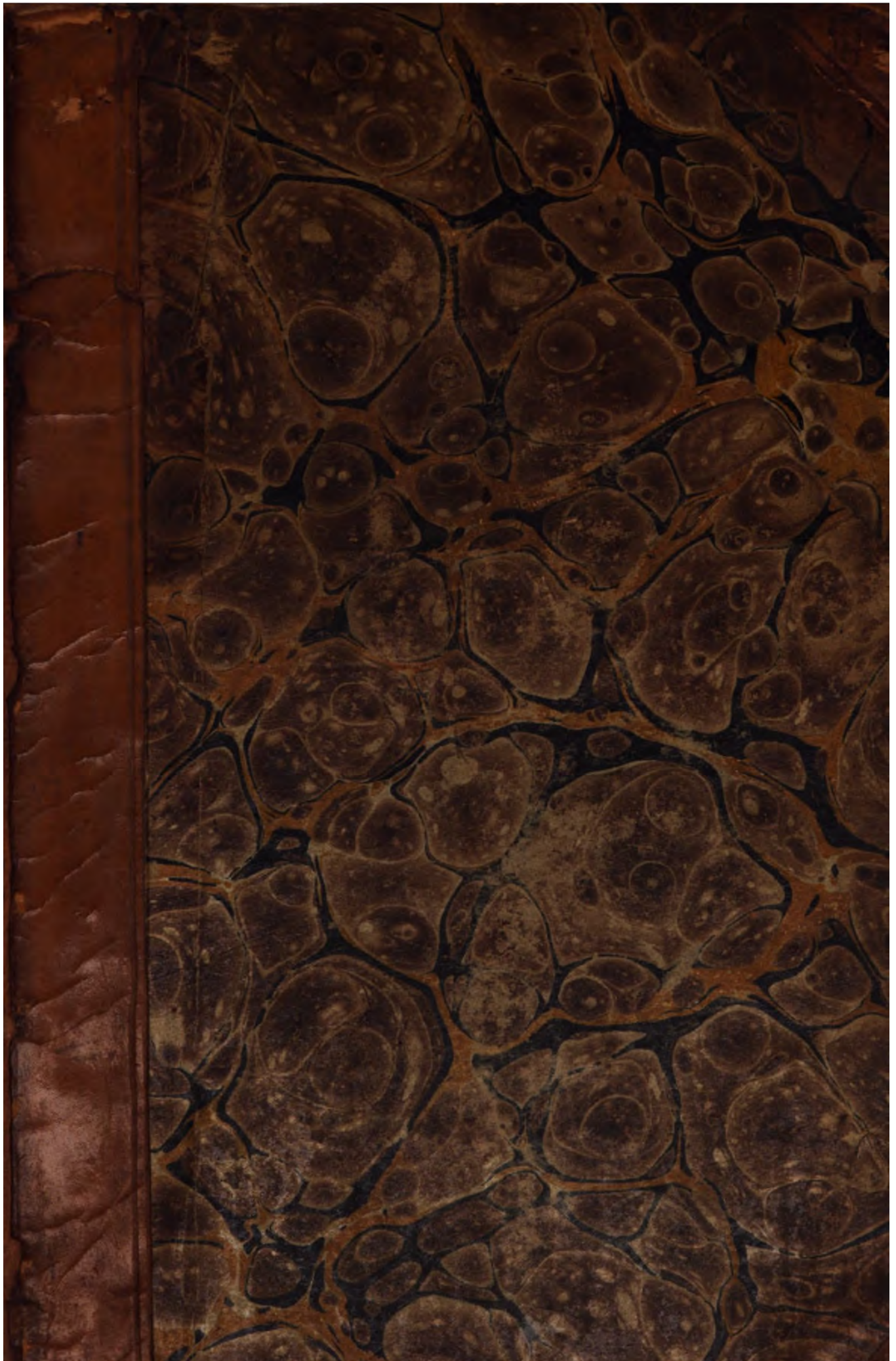
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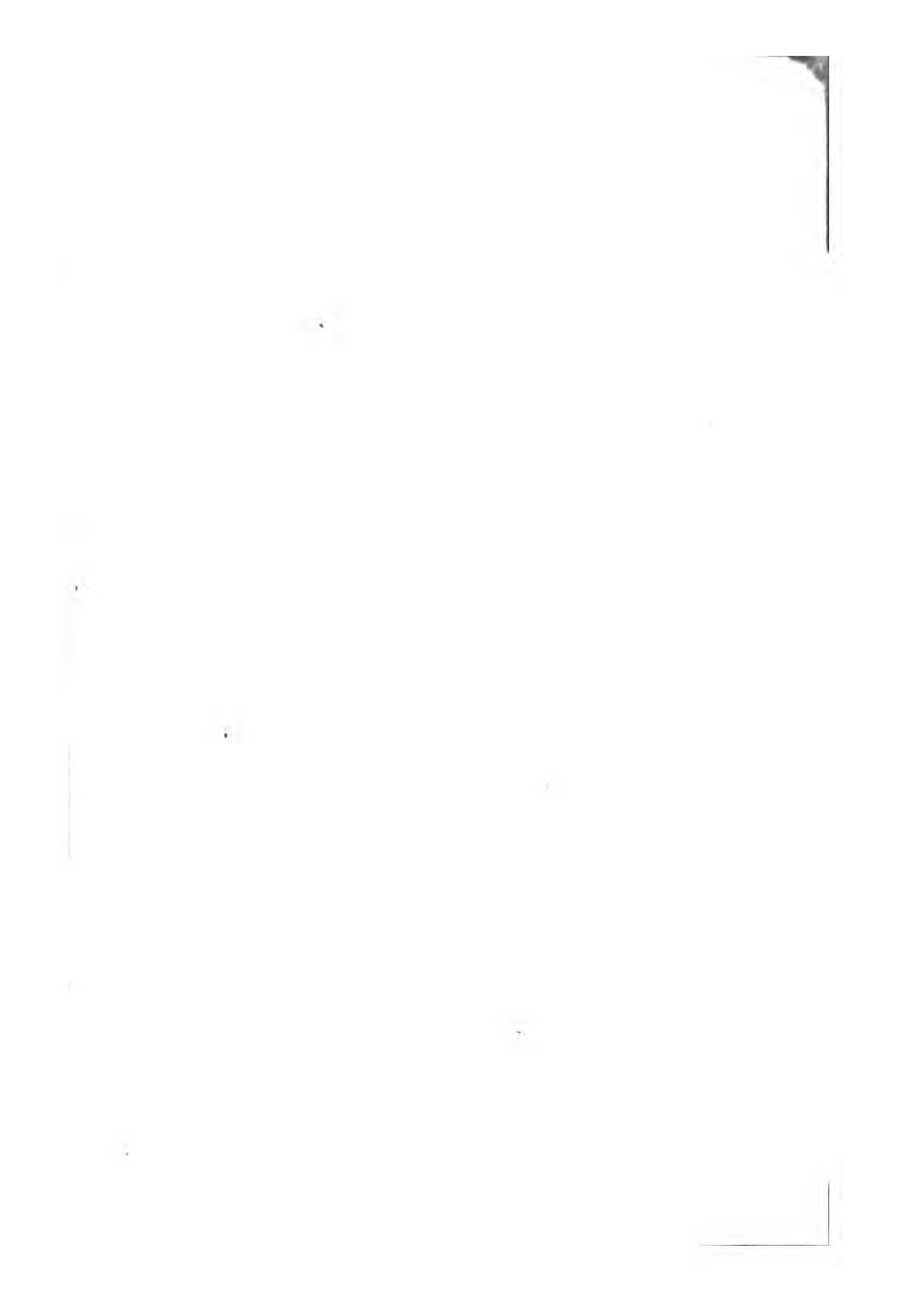
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# E U C L I D E ' S E L E M E N T S ;

The whole Fifteen Books compen-  
diously Demonstrated

By Mr. ISAAC BARROW, Fellow of Tri-  
nity Colledge in CAMBRIDGE.

*And Translated out of the Latin.*

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Καθαροὶ ψυχῆς λογικῆς εἰσὶν αἱ μαθηματικαὶ ὁπισθήμῃ. Hierocl.

---



L O N D O N ,  
Printed by R. DANIEL , for WILLIAM  
NEALAND Bookseller in Cambridge ; and  
are to be sold there , and at the Crown in

Duck-Lane , 1660.

188 0 54



THE  
AUTHORS  
PREFACE.

**I**N order to the Reader's satisfaction concerning the Book put into his hand, I am to advertise him of some few things, and that according to the nature of the Work, briefly; as followeth. My Undertaking aimed principally at two Ends. The first of which was to conjoin the greatest Compendiousness of Demonstration with as much Perspicuity as the quality of the subject would admit; that so the Volume might bear no bigger bulk than would render it conveniently portable. Which I have so farr attained, that though possibly some other person might with greater curiosity, yet (I presume) none could with more concisenesse have demonstrated most propositions; especially, since I have altered nothing in the number and order of the Propositions, nor taken the liberty to leave out any one of *Euclide's* as lesse necessary, or to reduce certain of the easiest into the Classis of Axiomes. Which notwithstanding some have done; as that most accurate Geometrician *Andr. Tacquet*, whom I mention the rather, because I esteem it ingenuous to acknowledge some things taken from him. And, indeed, I should have attempted nothing after his most elegant Edition, had it not pleased that learned Person to publish



## The P R E F A C E.

only Eight of *Euclide's* Books illustrated by his paines, either slighting or undervaluing the other Seven as lesse relating to the Elements of Geometry. But I had a different Purpose from the beginning; not to compose Elements of Geometry any-wise at my discretion, but to demonstrate *Euclide* himself, and all of him, and that with all possible brevity. For as for Foure of his Books, the Seventh, Eighth, Ninth and Tenth, although they do not so neerly pertain to the Elements of Plane and Solid Geometry, as the Six First & the Two subsequent; yet no man that ha's arriv'd to any measure of skill in Geometry is ignorant how exceedingly usefull they are in Geometricall matters, aswell in regard of the very neer alliance between Arithmetick and Geometrie, as for the knowledge of Commensurable and Incommensurable Magnitudes which is highly important to the understanding both of Plane and Solid Figures. And the noble Theory of the Five Regular Bodies, contained in the Three Last Books, could not be omitted without prejudice & injury; since our Author of these Elements, being a Sectator of *Plato's* Schole, is reported to have compiled the whole Systeme only in reference to that Contemplation; which *Proclus* attesteth in these words, Ὁθεν δὴ καὶ τῆς Συμπύσεως στοιχειώσεως τέλος ἀποβήσεται τὸ τῶν καλεμένων Πλατωνικῶν ἑπτάγων ἑξήκων. Moreover, I was easily induc'd to believe, that it would be acceptable to all Lovers of these Sciences to have the Intire work of *Euclide* by them, as it is usually cited and recommended by all men. Wherefore I determin'd

## The P R E F A C E.

min'd to leave out no Book or Proposition of those which are found in *Peter Herigon*, whose footsteps I became necessitated to follow closely by having resolved to make use for the most part of the Schemes of his Book, upon a foresight that my speedy departure out of *England*, would not allow me time to describe New, although I sometimes desired so to doe. Upon the same account also I purposed to use generally no other then *Euclide's* own Demonstrations, contracted into a more succinct form, saving perchance in the Second and Thirteenth, & sparingly in the Seventh, Eighth, and Ninth Books, where it seem'd convenient to vary something from him. So that it may be reasonably hoped that in this Particular our own Design and the Wishes of the Studious are in some manner satisfi'd.

The other End aimed at, was in favour of Their desires who more affect Symbolical then Verbal Demonstrations. In which kind, seeing most of our own Nation are accustomed to the Notes of Mr. *Oughtred*, I esteemed it more convenient to make use of them principally throughout. For no man hitherto that I know of, saving only *Peter Herigon*, ha's attempted to set forth and interpret *Euclide* according to this way. The Method of which most learned Person, though in many other respects very excellent, and exactly accommodated to his peculiar purpose, seem'd to me notwithstanding doubly defective. First, in that, whereas of severall Propositions brought to the proving of some one Theoreme or Probleme the Latter do's

## The P R E F A C E.

do's not alwaies depend on the Former, yet when they do cohere one with another, and when not, cannot readily enough be known; either from their order or any other way; whence it not seldom comes to passe, that through the want of Conjunctions and Adjectives, *Ergo, rursus, &c.* there arises difficulty and occasion of doubting, especially to such as are but little vers'd therein. And in the next place, it oftentimes falls out that the said Method cannot avoid superfluous repetitions, whereby the Demonstrations become sometimes prolix, and sometimes perplex'd and intricate. All which Inconveniencies are easily remedied in our Way by the intermingling of Words and Signes at discretion. And thus much may suffice to be premised concerning the Intent and Method of this Compendium. I shall not alledge in favour of my self the scantnesse of time allotted to this Work, nor the avocations of affairs, nor the scarcity of Helps to this sort of Studies amongst us (as I might not untruly) out of fear lest my Performance should not thoroughly please every body: But I wholly submit to the faire Censure and Judgement of the Ingenuous Reader, what I have undertaken for the advantage of his Studies; to be approved, if he find it serviceable thereunto; or, if otherwise, rejected.

**Ad amicissimum Virum, I. C. de EVCLIDE**  
contracto, Εὐρητισμός.

**F**actum bene! didicit Laconice loqui  
Senex profundus, & aphorismos induit.  
Immensa dudum margo commentarii  
Diagramma circuit minutum; utque Insula  
Problema breve natabat in vasto mari.  
Sed unda jam detumuit; & glossa arctior  
Stringit Theoremata: minoris anguli  
Lateribus ecce totus Euclides jacet,  
Inclusus olim velut Homerus in nuce;  
Pluteoque sarcina modo qui incubuit, levis  
En fit manipulus. Pelle in exigua latet  
Ingens Mathesis, matris utero Hercules,  
In glande quercus, vel Ithaca Eurus in pila.  
Nec mole dum decrescit, usu fit minor;  
Quin auctior jam evadit, & cumulatus  
Contracta prodest erudita pagina.  
Sic ubere magis liquor è presso effluit;  
Sic pleniori vasa inundat sanguinis  
Torrente cordis Systole; sic fusus  
Procurrit æquor ex Abyla angustiis.  
Tantilli operis ars tanta referenda unice est  
**BAROVIANO** nomini, ac solertia,  
Sublimis euge mentis ingenium potens!  
Cui invium nil, arduum esse nil solet;  
Sic usque pergas prospero conamine,  
Radiusque multum debeat ac abacus tibi;  
Sic crescat indies feracior seges,  
Simili colonum germine assiduo beans.  
Specimen futurae messis hic fiet labor,  
Magnaque famæ illustria hæc præludia.  
Juvenis dedit qui tanta, quid dabit senex?

Car. Robotham, **CANTAB.**  
Coll. Trin. Sen. Soc.

# The Explication of the Signes or Characters.

=	}	Equall.	
>		Greater.	
<		Leffer.	
+		More, or to be added.	
-		Leffe, or to be subtracted.	
:		The Difference, or Excesse; Also, that all the quantities which follow, are to be subtracted, the Signes not being changed.	
x		}	Multiplication, or the Drawing one side of a Rectangle into another.
			The same is denoted by the Conjunction of letters; as $AB = A \times B$ .
√		}	The Side or Root of a Square, or Cube, &c.
Q & q			A Square.
C & c	A Cube.		
Q.Q.	The ratio of a square number to a square number.		

*Other Abbreviations of words, where ever they occur, the Reader will without trouble understand of himself; saving some few, which, being of lesse generall use, we referr to be explained in their own places.*



THE FIRST BOOK  
OF  
EUCLIDES  
ELEMENTS.

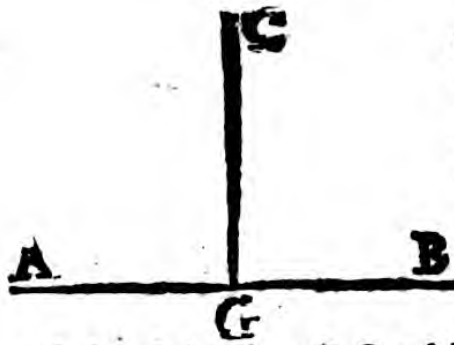
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*Definitions.*

- I. **A** Point is that which hath no part.  
II. A Line is a longitude without latitude.  
III. The ends, or limits, of a line are points.  
IV. A right line is that which lyes equally betwixt it's points.  
V. A Superficies is that which hath only longitude and latitude.  
VI. The extremes, or limits, of a superficies are lines.  
VII. A plaine superficies is that which lyes equally betwixt it's lines.  
VIII. A plaine Angle is the inclination of two lines the one to the other, the one touching the other in the same plain, yet not lying in the same strait line.  
IX. And if the lines which contain the angle be right lines, it is called a right-lined angle.

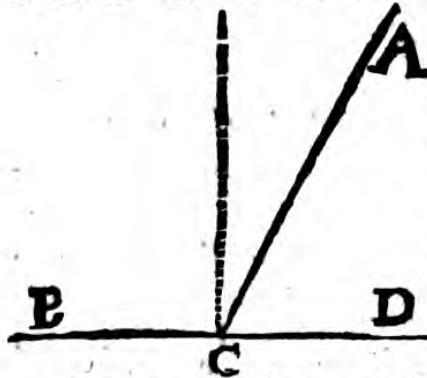
A

X. When



X. When a right line  $CG$  standing upon a right line  $AB$ , makes the angles on either side thereof,  $CGA$ ,  $CGB$ , equall one to the other, then both those equall angles are right angles; and the right line  $CG$ , which it standeth on the other, is termed a Perpendicular to that ( $AB$ ) whereon it standeth.

Note. When severall angles meet at the same point (as at  $G$ ) each particular angle is described by three letters; whereof the middle letter sheweth the angular point, and the two other letters the lines that make that angle: As the angle which the right lines  $CG$ ,  $AG$  make at  $G$ , is called  $CGA$ , or  $AGC$ .



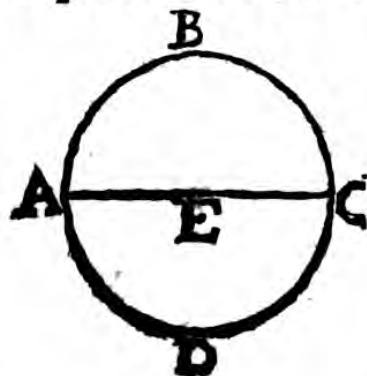
XI. An obtuse angle is that which is greater then a right angle; as  $ACB$ .

XII. An acute angle is that which is lesse then a right angle; as  $ACD$ .

XIII. A Limit, or Term, is the end of any thing.

XIV. A Figure is that which is contained under one or more terms.

XV. A Circle is a plain figure contained under one line, which is called a Circumference; unto which all lines drawn from one point within the figure, and falling upon the circumference thereof, are equall the one to the other.



XVI. And that point is called the Centre of the circle.

XVII. A Diameter of a circle is a right line drawn through the centre thereof, and ending at the circumference on either

# EUCLIDE'S *Elements.*

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ther side, dividing the circle into two equal parts.

**XVIII.** A Semicircle is a figure which is contained under the diameter and under that part of the circumference which is cut off by the diameter.

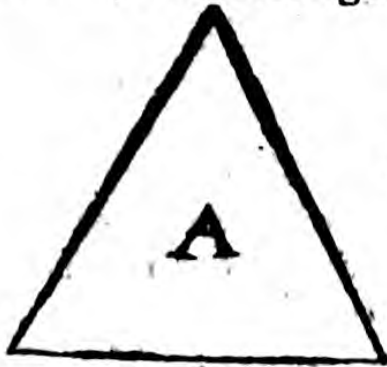
*In the circle E A B C D, E is the centre, A C the diameter, A B C the semicircle.*

**XIX.** Right-lined figures are such as are contained under right lines.

**XX.** Three-sided or Trilateral figures are such as are contained under three right lines.

**XXI.** Four-sided or Quadrilateral figures are such as are contained under four right lines.

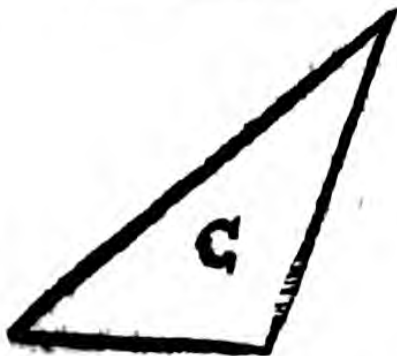
**XXII.** Many-sided figures are such as are contained under more right lines than four.



**XXIII.** Of trilateral figures, that is an Equilateral Triangle, which hath three equal sides; as the Triangle A.

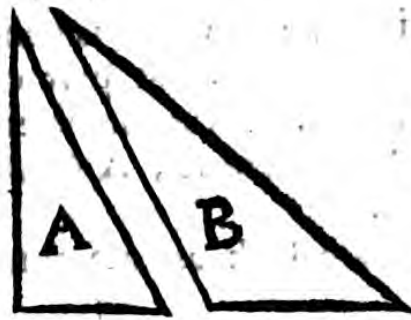


**XXIV.** Isosceles is a Triangle which hath only two sides equal; as the Triangle B.



**XXV.** Scalenum is a Triangle whose three sides are all unequal; as C.

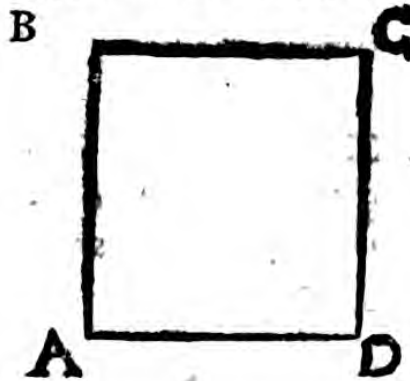




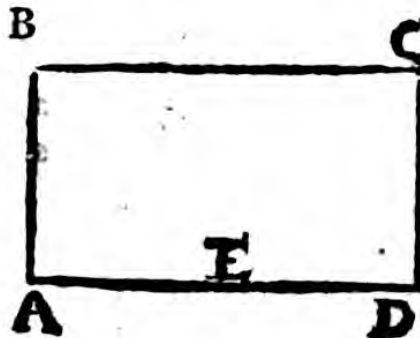
which hath one angle obtuse; as B.



equiangular, if the severall angles of the one figure be equall to the severall angles of the other. The same is to be understood of equilateral figures.

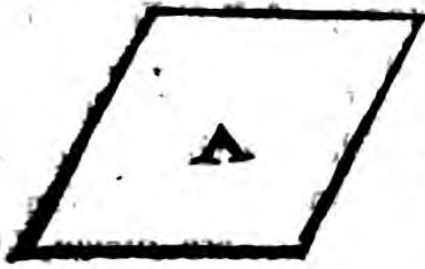


XXIX. Of quadrilateral, or four-sided, figures, a Square is that whose sides are equall, & angles right; as A B C D.

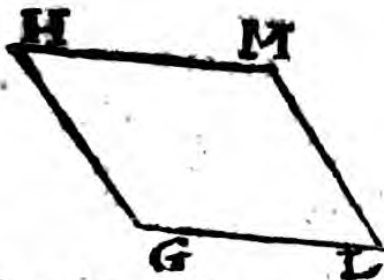


XXX. A figure on the one part longer, or a long square, is that which hath right angles, but not equall sides; as A B C D.

XXXI. A

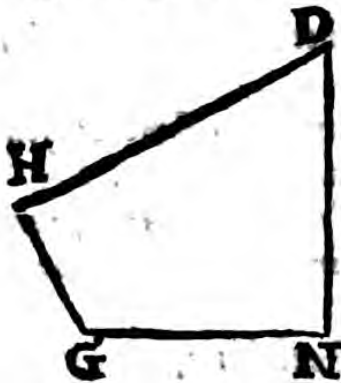


**XXXI.** A Rhombus, or diamond-figure, is that which hath four equall sides , but is not right-angled; as A.

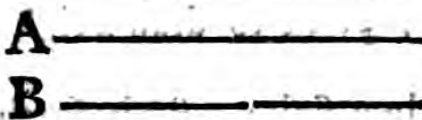


nor right angles; as GL M H.

**XXXII.** A Rhomboides, or diamond-like figure , is that whose opposite sides , and opposite angles , are equall; but hath neither equall sides

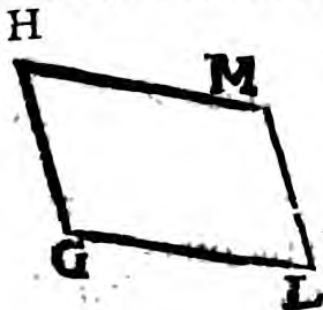


**XXXIII.** All other quadrilateral figures besides these are called *Trapezia*, or *Tables*; as G N D H.



**XXXIV.** Parallel, or equidistant, right lines are such , which being in the same superficies , if infinitely produced , would never meet; as A and B.

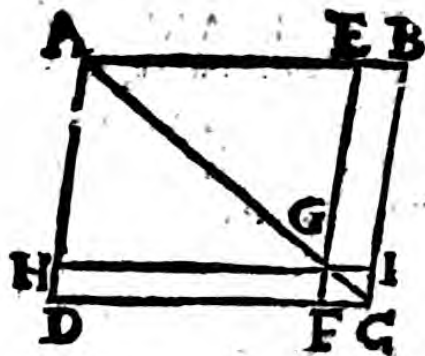
**XXXIV.** Parallel, or equidistant, right lines are such , which being in the



**XXXV.** A Parellelo-gram is a quadrilateral figure, whose opposite sides are parallel, or equidistant; as G L H M.

A 3

**XXXVI.** Lo



XXXVI. In a parallelogram  $ABCD$ , when a diameter  $AC$ , and two lines  $EF$ ,  $HI$  parallel to the sides, cutting the diameter in one and the same point  $G$ , are drawn, so that the parallelogram be divided by them into

four parallelograms; those two,  $DG$ ,  $GB$ , through which the diameter passeth not, are called **Complements**; and the other two,  $HE$ ,  $FI$ , through which the diameter passeth, the **Parallelograms standing about the diameter**.

A **Probleme** is, when something is proposed to be Done or effected.

A **Theoreme** is, when something is proposed to be Demonstrated.

A **Corollary** is a consectary, or some consequent truth gained from a preceding demonstration.

A **Lemma** is the demonstration of some premise, whereby the proof of the thing in hand becomes the shorter.

*Postulates or Petitions.*

1. **F**ROM any point to any point to draw a right line.
2. To produce a right line finite, strait forth continually.
3. Upon any centre, and at any distance, to describe a circle.

*Axiomes.*

1. **T**HINGS equall to the same third, are also equall one to the other;  
As  $A=B=C$ . Therefore  $A=C$ . Or therefore all,  $A, B, C$ , are equall the one to the other.

Note. When severall quantities are joynd the one to the other continually with this mark  $=$ , the first quantity is by vertue of this axiome equall to the last, & every one to every one. In which case we often abstain from

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*from citing the axiome, for brevities sake ; although the force of the consequence depend thereon.*

2. If to equall things you adde equall things, the wholes shall be equall.

3. If from equall things you take away equall things, the things remaining will be equall.

4. If to,unequall things you adde equall things, the wholes will be unequall.

5. If from unequall things you take away equall things, the remainders will be unequall.

6. Things which are double to the same third, or to equall things, are equall one to the other. Understand the same of triple, quadruple, &c.

7. Things which are half of one and the samething, or of things equall , are equall the one to the other. Conceive the same of subtriple, subquadruple, &c.

8. Things which agree together , are equall one to the other.

*The converse of this axiome is true in right lines and angles, but not in figures, unlesse they be like.*

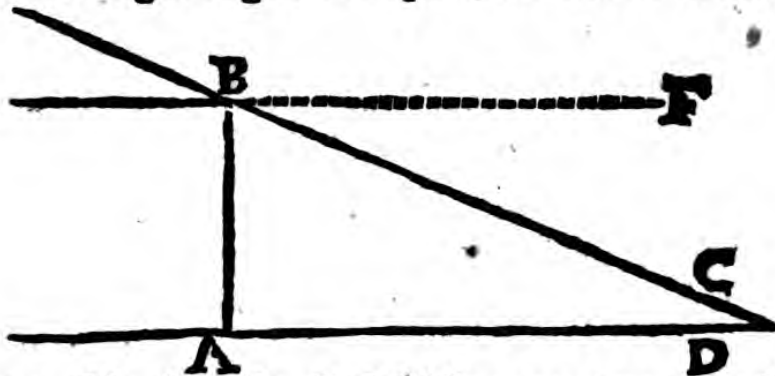
*Moreover , magnitudes are said to agree , when the parts of the one being applyed to the parts of the other, they fill up an equall or the same place.*

9. Every whole is greater then it's part.

10. Two right lines cannot have one and the same segment (or part) common to them both.

11. Two right lines meeting in the same point, if they be both produced , they shall necessarily cut one the other in that point.

12. All right angles are equall the one to the other.



13. If a right line B A falling on two right lines  
A 4
A D,

CB, make the internal angles on the same side,  $\angle BAD$ ,  $\angle ABC$ , lesse then two right angles, those two right lines produced shall meet on that side, where the angles are lesse then two right angles.

14. Two right lines do not contain a space.

15. If to equall things you adde things unequal, the excesse of the wholes shall be equall to the excesse of the additions.

16. If to unequal things equall be added, the excesse of the wholes shall be equall to the excesse of those which were at first.

17. If from equall things unequal things be taken away, the excesse of the remainders shall be equall to the excesse of what was taken away.

18. If from things unequal things equall be taken away, the excesse of the remainders shall be equall to the excesse of the wholes.

19. Every whole is equall to all it's parts taken together.

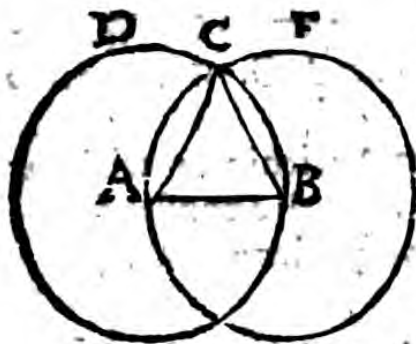
20. If one whole be double to another, and that which is taken away from the first to that which is taken away from the second, the remainder of the first shall be double to the remainder of the second.

*The Citations are to be understood in this manner; When you meet with two numbers, the first shewes the Proposition, the second the Book; as by 4. 1. you are to understand the fourth Proposition of the first Book; and so of the rest. Moreover, ax. denotes Axiome, post. Postulate, def. Definition, sch. Scholium, cor. Corollary.*



PRO.

PROPOSITION I.



**U**pon a finite right line given  $AB$ , to describe an equilateral triangle  $ACB$ .

From the centres  $A$  and  $B$ , at the distance of  $AB$ , or  $BA$ , *a* describe two circles cutting each other in the point  $C$ ; from whence *b* draw two right lines  $CA$ ,

$CB$ . Then is  $AC = AB = BC = CA$ .  
*e* Wherefore the Triangle  $ACB$  is equilateral.  
 Which was to be Done.

*a 3 post*

*b 1. post.*

*c 15. def.*

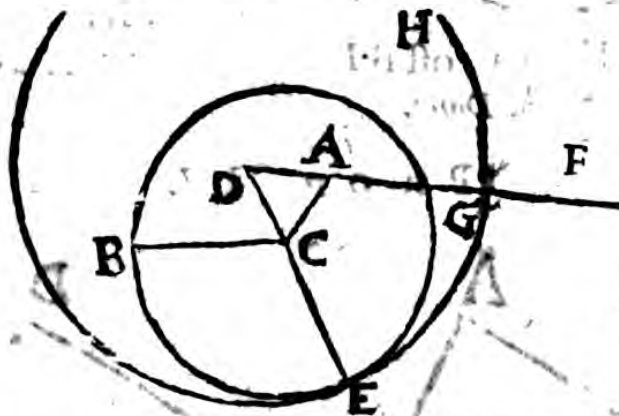
*d 1. ax.*

*e 23. def.*

*Scholium.*

After the same manner upon the line  $AB$  may be described an Isosceles triangle, if the distances of the equall circles be taken greater or lesse then the line  $AB$ .

PROP. II.



At a point given  $A$ , to make a right line  $AG$  equal to a right line given  $BC$ .

From the centre  $C$ , at the distance of  $CB$ , *a* describe the circle  $CBE$ . *b* join  $AC$ ; upon which *c* raise the equilateral triangle  $ADC$ . *d* Produce  $DC$  to  $E$ . From the centre  $D$ , at the distance of  $DE$ , *e*

*a 3 post.*

*b 1. post.*

*c 1. s.*

*d 2. post.*

de-

2. post.

describe the circle DEH; & let DA be produced to the point G in the circumference thereof. Then  $AG = CB$ .

f 15. def.  
g const.  
h 3. ax.  
k 15. def.  
l 1. ax.

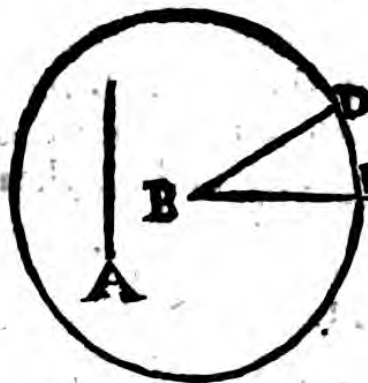
For  $DG = DE$ , and  $DA = DC$ . Wherefore  $AG = CE = BC = AG$ . Which was to be Done.

The putting of the point A within or without the line BC varies the cases; but the construction, and the demonstration, are every where alike.

Schol.

The line AG might be taken with a pair of compasses; but the so doing answers to no postulate, as Proclus well intimates.

PROP. III.



Two right lines, A and BC, being given, from the greater BC to take away the right line BE equal to the lesser A.

At the point B draw the right line  $BD = A$ . The circle described from the center B at the distance

of BD shall cut off  $BE = BD = A = BE$ . Which was to be Done.

2. 1.

b 15. def.  
c const.  
d 1. ax.

PROP. IV.



If two triangles BAC, EDF, have two sides of the one BA, AC equal to two sides of the other ED, DF, each to it's correspondent side (that is,  $BA = ED$ , and

and  $AC = DF$  and have the angle  $A$  equall to the angle  $D$  contained under the equall right lines; they shall have the base  $BC$  equall to the base  $EF$ ; and the triangle  $BAC$  shall be equall to the triangle  $EDF$ ; and the remaining angles  $B, C$ , shall be equall to the remaining angles  $E, F$ , each to each, under which the equall sides are subtended.

If the point  $D$  be applyed to the point  $A$ , and the right line  $DE$  plac'd upon the right line  $AB$ , the point  $E$  shall fall upon  $B$ , because  $DE = AB$ . also the right line  $DF$  shall fall upon  $AC$ , because the angle  $A = D$ . moreover, the point  $F$  shall fall upon the point  $C$ , because  $AC = DF$ . Therefore the right lines  $EF, BC$  shall agree, because they have the same termes, & so consequently are equall. Wherefore the triangles  $BAC, DEF$ , and the angles  $B, E$ , as also the angles  $C, F$ , do agree, and are equall. Which was to be Demonstrated.

PROP. V.



The angles  $ABC, ACB$ , at the base of an Isosceles triangle  $ABC$ , are equall one to the other: And if the equall sides  $AB, AC$  be produc'd, the angles  $CBD, BCE$ , under the base, shall be equall one to the other.

*a* Take  $AE = AD$ ; and *b* join  $CD$ , and  $BE$ .

*a* 3. 1.  
*b* 1. post.

Because, in the triangles  $ACD, ABE$ , are  $AC = AB$ , and  $AE = AD$ , and the angle  $A$  common to them both, therefore is the angle  $ABE = ACD$ , and the angle  $AEB = ADC$ , and the base  $BE = CD$ ; also  $EC = DB$ . Therefore in the triangles  $BEC, BDC$  shall be the angle  $ECB = DBC$ . Which was to be Dem. Also therefore the angle  $EBC = DCB$ . but the angle  $ABE = ACD$ ; therefore the angle  $ABC = ACB$ . Which was to be Dem.

*c* hyp.  
*d* constr.  
*e* 4. 1.

*f* 3. ax.  
*g* 4. 1.

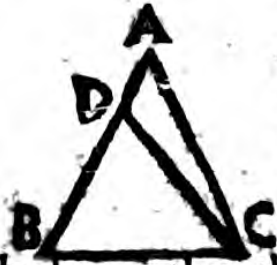
*h* before.  
*k* 3. ax.



Hence, Every equilateral triangle is also equiangular.

P R O P. VI.

If two angles  $A B C, A C B$  of a triangle  $A B C$  be equal the one to the other, the sides  $A B, A C$  subtended under the equal angles, shall also be equal one to the other.



If the sides be not equal, let one be bigger then the other, suppose  $BA > CA$ .<sup>a</sup> Make  $BD = CA$ , and draw the line  $C D$ .

In the triangles  $D B C, A C B$ , because  $B D = CA$ , and the side  $BC$  is common, & the angle  $DBC = ACB$ , the triangles  $DBC, ACB$  shall be equal the one to the other, a part to the whole, *f* Which is impossible.

Coroll.

Hence, Every equiangular triangle is also equilateral.

P R O P. VII.



Vpon the same right line  $A B$  two right lines being drawn  $A C, B C$ , two other right lines equal to the former,  $A D, B D$ , each to each (viz.  $A D = A C$ , and  $B D = B C$ ) cannot be drawn from the same points  $A, B$ , on the same side  $C$ , to severall points, as  $C$  and  $D$ , but onely to  $C$ .

1. Case. If the point  $D$  be set in the line  $A C$ , it is plain that  $A D$  is not equal to  $A C$ .

2. Case. If the point  $D$  be placed within the triangle  $A C B$ , then draw the line  $C D$ , and produce  $B D F$ , and  $B C E$ . Now you would have  $A D = A C$ . then the

a 3. 1.  
b 1. post.

c suppos.  
d hyp.  
e 4. 1.

f 9. ax.

29. ax.

the angle  $ADC = ACD$ ; as also, because  $BD = BC$ , the angle  $FDC = b ACD$ . therefore is the angle  $FDC = d ACD$ , that is, the angle  $FDC = ADC$ . *which is imposs.*

b 5. 1.  
c suppos.  
d 9. ax.

3. case. If  $D$  falls without the triangle  $ACB$ , let  $CD$  be joyned.

Again, the angle  $ACD = ADC$ , and the angle  $BCD = BDC$ . Therefore the angle  $ACD < BDC$ , viz. the angle  $ADC < BDC$ . *Which is impossible.* Therefore, &c.

e 5. 1.  
f 9. ax.

PROP. VIII.



If two triangles  $ABC, DEF$  have two sides  $AB, AC$  equal to two sides  $DE, DF$ , each to each, and the base

$BC$  equal to the base  $EF$ , then the angles contained under the equal right lines shall be equal, viz.  $A$  to  $D$ .

Because  $BC = EF$ , if the base  $BC$  be laid on the base  $EF$ , they will agree: therefore whereas  $AB = DE$ , and  $AC = DF$ , the point  $A$  will fall on  $D$  (for it cannot fall on any other point, by the precedent proposition) and so the sides of the angles  $A$  and  $D$  are coincident; wherefore those angles are equal. *Which was to be Dem.*

a hyp.  
b ax. 8.  
c hyp.

d 8. ax.

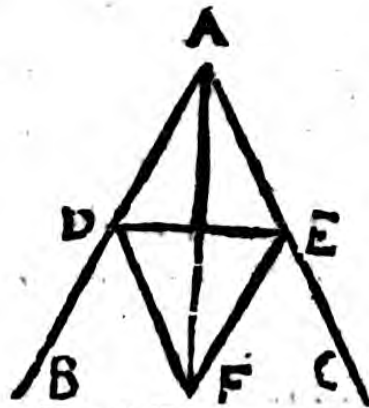
Coroll.

1. Hence, Triangles mutually equilateral are also mutually equiangular.

2. Triangles mutually equilateral are equal one to the other.

PROP.

## P R O P. IX.



a 3. 1.

b 2. 1.

c confr.  
d 8. 1.

To bisect, or divide into two equal parts, a right-lined angle given  $BAC$ .

a Take  $AD = AE$ , and draw the line  $DE$ ; upon which b make an equilateral triangle  $DFE$ . draw the right line  $AF$ ; it shall bisect the angle.

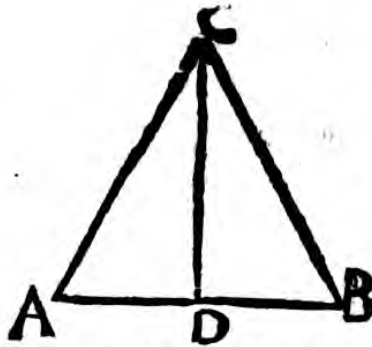
For  $AD = AE$ , and the side  $AF$  is common, & the base  $DF = FE$ . d therefore the angle  $DAF = EAF$ . Which was to be Done.

Coroll.

Hence it appears how an angle may be cut into any equal parts, as 4, 8, 16, &c. to wit, by bisecting each part again.

The method of cutting angles into any equal parts required, by a Rule and Compasse, is as yet unknown to Geometricians.

## P R O P. X.

a 1. 1.  
b 9. 1.c confr.  
d 4. 1.

To bisect a right line given  $AB$ .

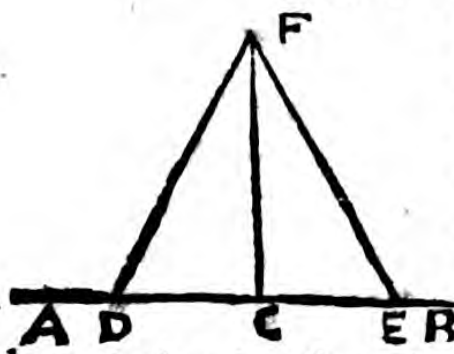
Upon the line given  $AB$  a erect an equilateral triangle  $ABC$ ; and b bisect the angle  $C$  with the right line  $CD$ . That line shall also bisect the line given  $AB$ .

For  $AC = BC$ , & the side  $CD$  is common, & the angle  $ACD = BCD$ . therefore  $AD = BD$ . Which was to be Done.

The practise of this and the precedent proposition is easily shewn by the construction of the 1. prop. of this book.

P R O P.

PROP. XI.



From a point C in a right line given AB to erect a right line CF at right angles.

Take on either side of the point given CD = CE. upon the right line DE erect an equilateral triangle. draw the line FC, and it will be the perpendicular required.

For the triangles DFC, EFC are mutually equilateral; therefore the angle DCF = ECF. therefore FC is perpendicular. Which was to be Done.

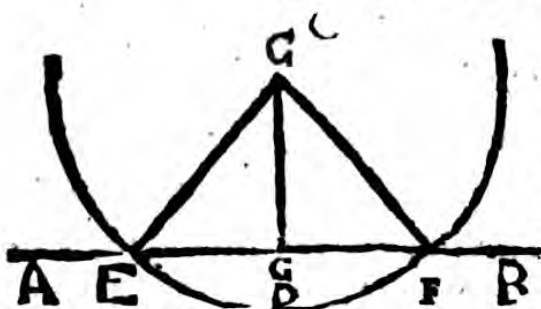
a 3. 1.

b 1. 1.

c constr.  
d 8. 1.  
e 10. def.

The practise of this and the following is easily performed by the help of a square.

PROP. XII.



Upon an infinite right line given AB, from a point given that is not in it, to let fall a perpendicular right line CG.

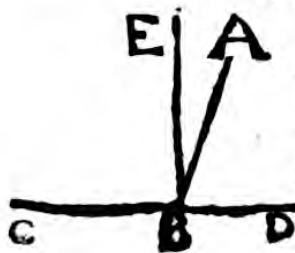
From the centre C describe a circle cutting the right line given AB in the points E & F. Then bisect EF in G, and draw the right line CG, which will be the perpendicular required.

a 3. post.  
b 10. 1.

Let the lines CE, CF be drawn. The triangles EGC, FGC are mutually equilaterall. therefore the angles EGC, FGC are equal, and by consequence right. Wherefore GC is a perpendicular. Which was to be Done.

c constr.  
d 8. 1.  
e 10. def.

PROP. XIII.



When a right line AB standing upon a right line CD maketh angles ABC, ABD; it maketh either two right angles, or two angles equal to two right.

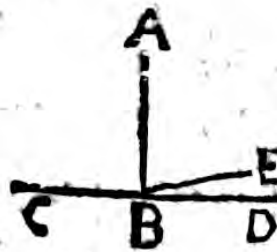
a def. 10.  
 b 11. 1.  
 c 19. ax.  
 d 3. ax.  
 e 2. ax.

If the angles  $A B C$ ,  $A B D$  be equal, <sup>a</sup> then they make two right angles; if unequal, then from the point  $B$  <sup>b</sup> let there be erected a perpendicular  $B E$ . Because the angle  $A B C$  <sup>c</sup>  $\equiv$  to a right  $\rightarrow$   $A B E$ , and the angle  $A B D$  <sup>d</sup>  $\equiv$  to a right  $\rightarrow$   $A B E$ , therefore shall be  $A B C \rightarrow A B D$  <sup>e</sup>  $\equiv$  to two right angles  $\rightarrow A B E \rightarrow A B E \equiv 2$  right angles. Which was to be Dem.

Corollaries.

1. Hence, if one angle  $A B D$  be right, the other  $A B C$  is also right; if one acute, the other is obtuse, and so on the contrary.
2. If more right lines then one stand upon the same right line at the same point, the angles shall be equal to two right.
3. Two right lines cutting each other make angles equal to four right.
4. All the angles made about one point make 4. right; as appears by Coroll. 2.

P R O P. XIV.

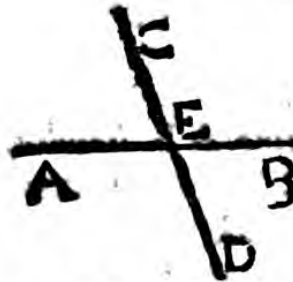


If to any right line  $A B$ , and a point therein  $B$ , two right lines, not drawn from the same side, do make the angles  $A B C$ ,  $A B D$  on each side equal to two right, the lines  $C B$ ,  $B D$  shall make one strait line.

a 13. 1.  
 b hyp.  
 c 9. ax.

If you deny it, let  $C B$ ,  $B E$  make one right line; then shall be the angle  $A B C \rightarrow A B E$  <sup>a</sup>  $\equiv 2$  right angles <sup>b</sup>  $\equiv A B C \rightarrow A B D$ . Which is <sup>c</sup> Absurd.

P R O P. XV.



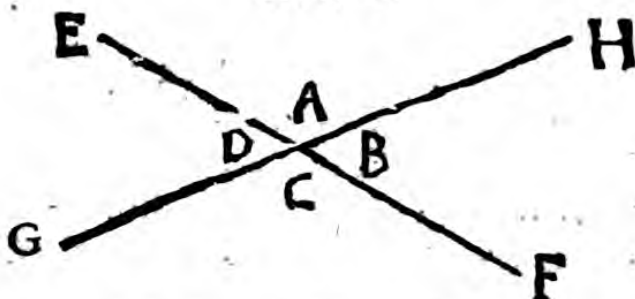
If two right lines  $A B$ ,  $C D$  cut through one another, then are the two angles which are opposite, viz.  $C E B$ ,  $A E D$ , equal one to the other.

a 13. 1.  
 b 3. ax.

For the angle  $A E C \rightarrow C E B$  <sup>a</sup>  $\equiv$  to 2 right angles  $\equiv A E C \rightarrow A E D$ ; <sup>b</sup> therefore  $C E B \equiv A E D$ . Which was to be Done.

Schol.

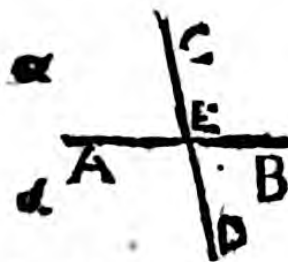
Schol. 1.



If to any right line  $GH$ , and in it a point  $A$ , two right lines being drawn  $EA, AF$ , and not taken on the same side, make the vertical (or opposite) angles  $D$  and  $B$  equal, those right lines  $EA, AF$ , do meet directly and make one strait line.

For 2 right angles are  $a$  equal to the angle  $D + A$  a 13. 1.  
 $a = B + A$  therefore  $EA, AF$  are in a strait b 14. 1.  
 line. Which was to be Dem.

Schol. 2.



If four right lines  $EA, EB, EC, ED$ , proceeding from one point  $E$ , make the angles vertically opposite equal the one to the other, each two lines,  $AE, EB$ , and  $CE, ED$ , are

placed in one strait line.

For because the angle  $AEC + AED + CEB + DEB$   $a =$  to 4 right angles, therefore the angle a 4 cor. 13. 1.  
 $AEC + AED$   $b =$   $CEB + DEB =$  to two b hyp. & 2 ax.  
 right angles. c Therefore  $CED$  &  $AEB$  are strait c 14. 1.  
 lines. Which was to be Dem.

PROP. XVI.



One side  $BC$  of any triangle  $ABC$  being produc'd, the outward angle  $ACD$  will be greater then either of the inward and opposite angles  $CAB, CBA$ .

Let the right lines  $AH, BE$   
 $a$  bisect the sides  $AC, BC$ ; a 10. 1. & 1. post.  
 $b$  produce  $EF = BE$ , and  $HI$  b 3. 1.  
 $B =$

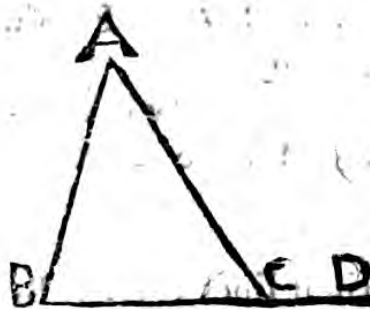
$b = AH$ . and join  $FC$ , and  $IC$ ; and produce  $ACG$ .

*confr.*  
d 15. 1.  
e 4. 1.

f 15. 1.  
g 9. ax.

Because  $CE = EA$ , and  $EF = EB$ , and the angle  $FE C = BEA$ , the angle  $ECF$  shall be equall to  $EAB$ . By the like argument is the angle  $ICH = ABH$ . Therefore the whole angle  $ACD$  ( $fBCG$ )  $g$  is greater then either the angle  $CAB$  or  $ABC$ . Which was to be Dem.

P R O P. X V I I.



Two angles of any triangle  $ABC$ , which way soever they be taken, are lesse then two right angles.

Let the side  $BC$  be produced. Because the angle  $ACD + ACB = 2$  right angles, and the angle  $ACD > A$ , therefore  $A + ACB < 2$  then two right angles. After the same manner is the angle  $B + ACB < 2$  then two right. Lastly, the side  $AB$  being produced, the angle  $A + B$  will be also lesse then two right angles. Which was to be Dem.

a 13. 1.  
b 16. 1.  
c 4. ax.

Coroll.

1. Hence it follows that in every triangle, wherein one angle is either right or obtuse, the two others are acute angles.



2. If a right line  $AE$  make unequal angles with another right line  $CD$ , one acute  $AED$ , the other obtuse  $AEC$ , a perpendicular  $AD$  let fall from any point  $A$  to the other line  $CD$ , shall fall on that side the acute angle is of.

\* 17. 1.

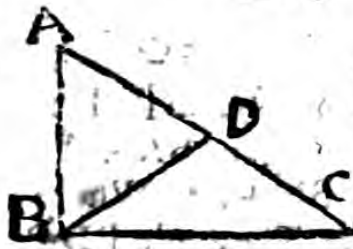
For if  $AC$ , drawn on the side of the obtuse angle, be a perpendicular, then in the triangle  $AEC$  shall  $AEC + ACE$  be greater then two right angles.

\* Which is contrary to the precedent prop.

3. All the angles of an equilateral triangle, and the 2 angles of an Isosceles triangle that are upon the base, are acute.

P R O P.

PROP. XVIII.

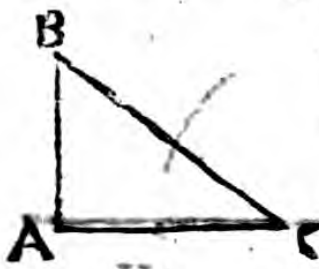


The greatest side AC of every triangle ABC subtends the greatest angle ABC.

From AC take away AD = AB, and join BD. Therefore is the angle ADB = ABD.

But ADB = C; therefore is ABD = C; therefore the whole angle ABC = C. After the same manner, shall be ABC = A. Which was to be Dem.

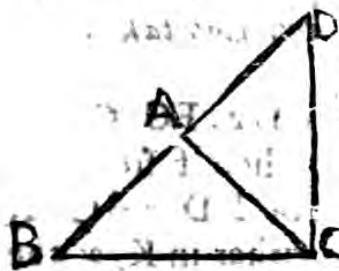
PROP. XIX.



In every triangle ABC, under the greatest angle A is subtended the greatest side BC.

For if AB be supposed equal to BC, then will be the angle A = C, which is contrary to the Hypothesis: and if AB < BC, then shall be the angle C < A, which is against the Hypothesis. Wherefore rather BC < AB; and after the same manner BC < AC. Which was to be Dem.

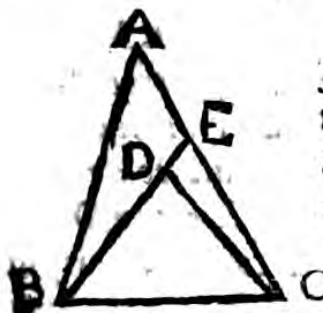
PROP. XX.



Of every triangle ABC two sides BA, AC, any way taken, are greater than the side that remains BC.

Produce the line BA, & take AD = AC, & draw the line DC. then shall the angle D be equal to ACD, therefore is the whole angle BCD = D; therefore BD (e BA + AC) < BC. Which was to be Dem.

PROP. XXI.



If from the utmost points of one side BC of a triangle ABC two right lines BD, CD be drawn to any point within the triangle, then are both those two lines shorter than the two other sides of the



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the triangle, BA, CA; but do contain a greater angle, BDC.

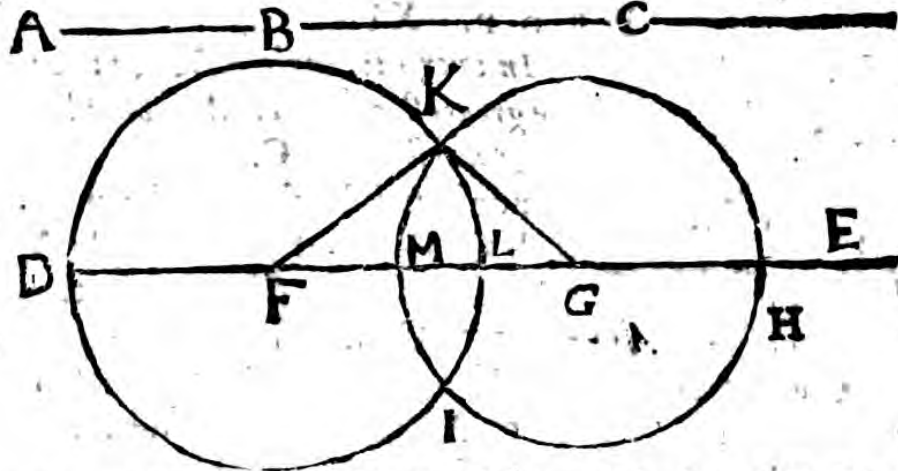
a 20. 1.

b 4. ex.

c 16. 1.

Let BD be produced to E. Then is CE + ED  
 $\square$  CD. adde BD common to both, then shall be  
 $BD + DE + EC \square CD + BD$ . Again, BA + AE  
 $\square$  BE. therefore BA + AC  $\square$  BE + EC. Where-  
 fore 1. BA + AC  $\square$  BD + DC. 2. The angle BDC  
 $\square$  DEC  $\square$  A. Therefore the angle BDC  $\square$  A.  
 Which was to be Dem.

PROP. XXII.



To make a triangle FKG of three right lines FK, FG, GK which shall be equall to three right lines given A, B, C. Of which it is necessary that any two taken together be longer then the third.

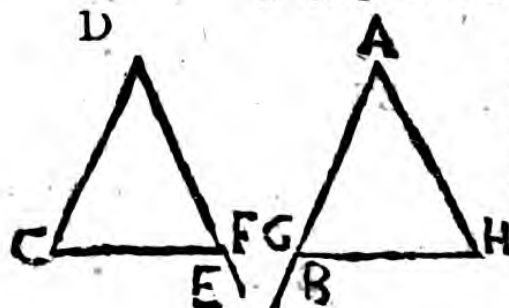
a 3. 1.

b 3. post.

c 15. def.  
d 1. ex.

From the infinite line DE take DF, FG, GH equall to the lines given A, B, C. Then if from the centres F and G by the distances of FD and GH, two circles be drawn cutting each other in K, & the right lines KF, KG be joined, the triangle FKG shall be made, whose sides FK, FG, GK are equall to the three lines DF, FG, GH that is to the three lines given A, B, C. Which was to be Done.

PROP. XXIII.



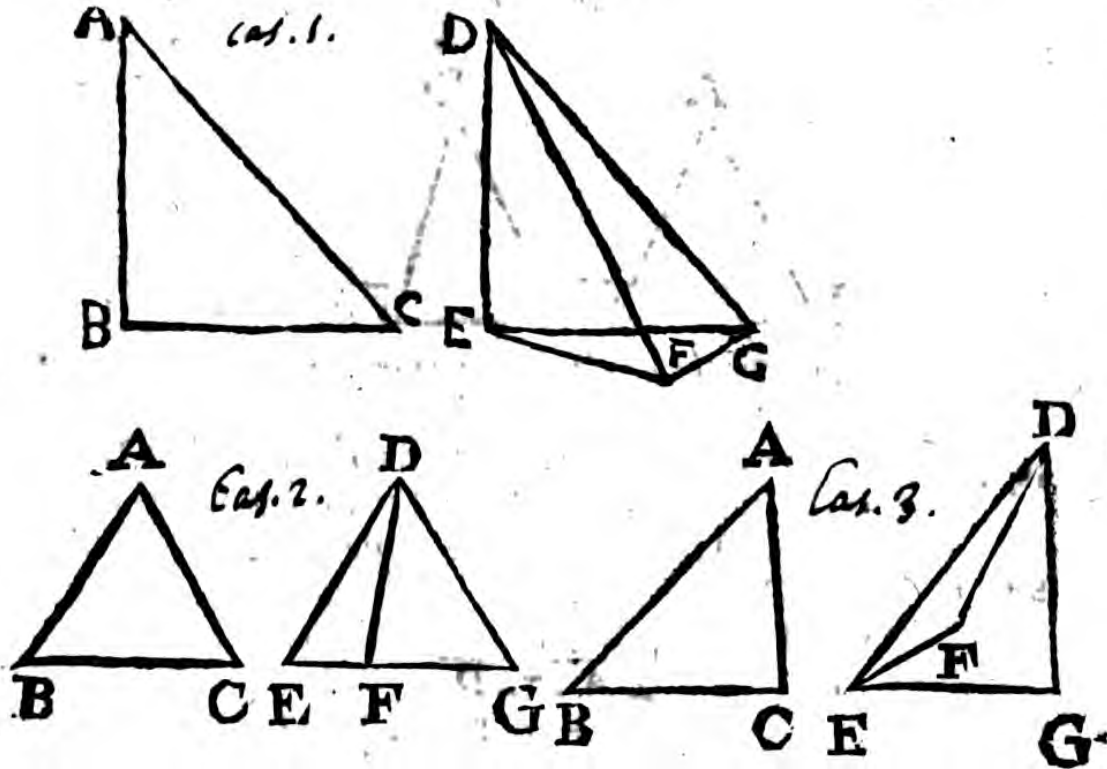
a 1. post.

At a point A in a right line given AB, to make a right-lined angle A equall to a right-lined angle given D.

Draw the right line

line  $CF$  cutting the sides of the angle given any wayes;  $b$  make  $AG = CD$ ; upon  $AG$   $c$  raise a triangle equilateral to the former  $CD F$ , so that  $A H$  be equal to  $D F$ , and  $GH$  to  $CF$ . then shall you have  $d$   $8. 1.$  the angle  $A d = D$ . Which was to be Done.

PROP. XXIV.



If two triangles  $ABC$ ,  $DEF$  have two sides of the one triangle  $AB, AC$  equal to two sides of the other triangle  $DE, DF$ , each to other, and have the angle  $A$  greater then the angle  $EDF$  contained under the equal right lines, they shall also have the base  $BC$  greater then the base  $EF$ .

$a$  Let the angle  $EDG$  be made equal to  $A$ , and the side  $DG$   $b = DF$   $c = AC$ ; & let  $EG$ , and  $FG$  be joyned.

1. Case. If  $EG$  fall above  $EF$ ; Because  $AB$   $d = DE$ , and  $AC$   $e = DG$ , and the angle  $A e = EDG$ ,  $f$  therefore is  $BC = EG$ . But because  $DF e = DG$ ,  $g$  therefore is the angle  $DFG = DGF$ ;  $h$  therefore is the angle  $DFG = EFG$ , and by consequence the angle  $EFG$   $b = EGF$ .  $k$  wherefore  $EG (BC) = EF$ .

B 3

2. Case.

*Handwritten notes:*  $eg \rightarrow efg$

The first Book of

19<sup>ax.</sup> 2. *Case.* If the base EF falls in the same place with EG, it is evident that  $EG(BC) \sqsupset EF$ .  
 3. *Case.* If EG fall below EF, then because  $DG + GE = DF + FE$ , if from both DG, DF be taken away, which are equal, EG(BC) remains  $\sqsupset EF$ , Which was to be Dem.

P R O P. XXV.



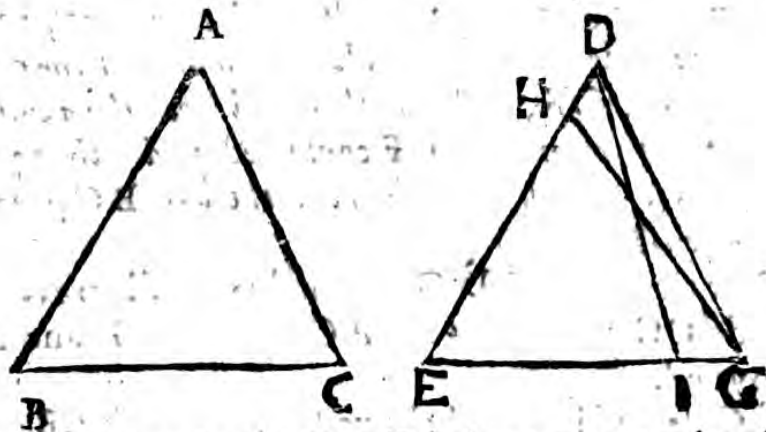
If two triangles ABC, DEF have two sides AB, AC equal to two sides DE, DF, each to other, and have the base BC greater then the base EF, they shall

also have the angle A contained under the equal right lines greater then the angle D.

For if the angle A be said to be equal to D, then is the base  $BC = EF$ , which is against the Hypothesis. If it be said the angle  $A \sqsupset D$ , then  $b$  will be  $BC \sqsupset EF$ , which is also against the Hypothesis. Therefore  $BC \sqsupset EF$ . Which was to be Dem.

$\angle A \sqsupset \angle D$ .

P R O P. XXVI.



If two triangles BAC, EDG have two angles of the one B, C equal to two angles of the other E, DGE, each to his correspondent angle, and have also one side of the one equal to one side of the other, either that side which lyeth betwixt the equal angles, or that which is subtended under one of the equal angles; the other sides also

so

so of the one shall be equal to the other sides of the other<sup>2</sup> each to his correspondent side, and the other angle of the one shall be equal to the other angle of the other.

1. Hypothesis. Let BC be equal to EG, which are the sides that lye between the equal angles. Then I say BA = ED, and AC = DG, & the angle A = EDG. For if it be said that ED < BA, then a let EH be made equal to BA, & let the line GH be drawn.

a 3. 1.

Because AB<sup>b</sup> = HE, and BC<sup>c</sup> = EG, and the angle B<sup>c</sup> = E, therefore shall be the angle EGH = C<sup>e</sup> = DGE. f Which is absurd. After the same manner let AC be equal to DG, then will the angle A be equal to EDG.

b suppos.  
c hyp.  
d 4. 1.  
e hyp.  
f 9 ax.

2. Hyp. Let AB be equal to ED. Then I say BC = EG, and AC = DG, and the angle A = EDG. For if EG be greater then BC, make EI = BC, and joyn the line DI. Now because AB<sup>g</sup> = ED, and BC<sup>b</sup> = EI, and the angle G<sup>g</sup> = E; therefore will be the angle EID<sup>k</sup> = C<sup>l</sup> = EGD. m Which is absurd. Therefore is BC = EG, and so as before AC = DG, and the angle A = EDG. Which was to be Dem.

g hyp.  
h suppos.  
k 4. 1.  
l hyp.  
m 16. 1.

PROP. XXVII.

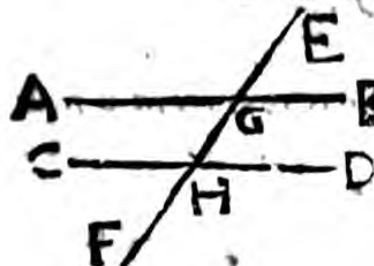


If a right line EF falling upon two right lines AB, CD make the alternate angles AEF, DFE, equal the one to the other, then are the right lines AB, CD parallel.

If AB, CD be said not to be parallel, produce them till they meet in G, which being supposed, the outward angle AEF will be a greater then the inward angle DFE, to which it was equal by Hypothesis. Which is repugnant.

a 16. 1.

PROP. XXVIII.



If a right line EF falling upon two right lines AB, CD make the outward angle AGE of the one line equal to CHG the inward and opposite angle of

B 4

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of the other on the same side, or make the inward angles on the same side AGH, CHG equall to two right angles, then are the right lines AB, CD parallel.

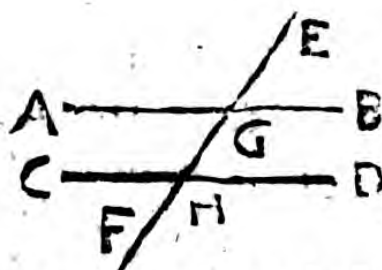
a 15. 1.  
b 27. 1.

Hyp. 1. Because by Hypothesis the angle AGE = CHG, a therefore are BGH, CHG alternate angles and equall: And b therefore are AB and CD parallel.

a 13. 1.  
b 3. ax.  
c 27. 1.

Hyp. 2. Because by Hypothesis the angle AGH + CHG = to two right, a = AGH + BGH, b thence is CHG = BGH; and c therefore AB, CD are parallel. Which was to be Dem.

P R O P. XXIX.

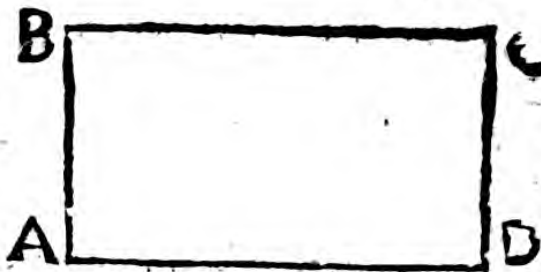


If a right line EF fall upon two parallels, AB, CD, it will both make the alternate angles DHG, AGH equall each to other, and the outward angle BGE equall to the inward and opposite angle on the same side DHE, as also the inward angles on the same side AGH, CHG equall to two right angles.

a 13. ax.  
b 13. 1.  
c 13. ax.  
d 15. 1.

It is evident that AGH + CHG = 2 right angles; a otherwise AB, CD would not be parallel, which is contrary to the Hypothesis: But moreover the angle DHG + CHG b = 2 right; therefore is DHG c = AGH d = BGE. Which was to be Dem.

coroll.



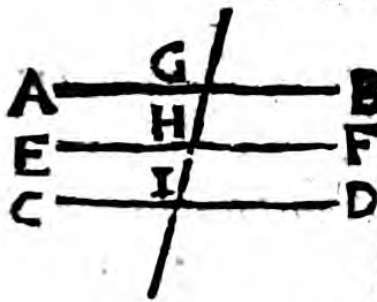
Hence it follows that every parallelogram AC having one angle right A, the rest are also right.

a 29. 1.  
b 3. ax.

For A + B a = 2 right angles. Therefore, whereas A is right, b must B be also right. By the same argument are C and D right angles.

P R O P.

PROP. XXX.

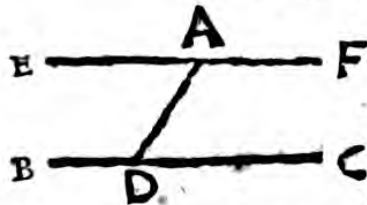


Right lines (AB, CD) parallel to one & the same right line EF, are also parallel the one to the other.

Let GI cut the three right lines given any wayes. then because AB, EF are parallel, will be the angle  $AGI = EHI$ . also because CD and EF are parallel, will be the angle  $EHI = DIG$ . Therefore the angle  $AGI = DIG$  whence AB and CD are parallel. Which was to be Dem.

a 29. 1.  
b 1. ex.  
c 27. 1.

PROP. XXXI.

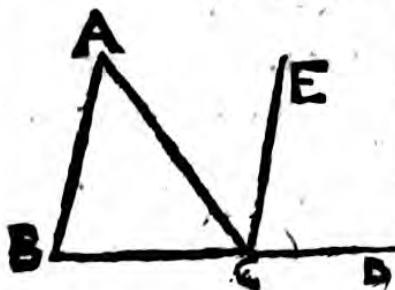


From a point given A to draw a right line AE parallel to a right line given BC.

From the point A draw a right line AD to any point of the given right line; with which at the point thereof A make an angle  $DAE = ADC$ . then will AE and BC be parallel. Which was to be Done.

a 23. 1.  
b 27. 1.

PROP. XXXII.



Of any triangle ABC one side BC being drawn out, the outward angle ACD shall be equal to the two inward opposite angles A, B. and the three inward angles of the triangle, A, B, ACB,

shall be equal to two right angles.

From C draw CE parallel to BA. Then is the angle  $A = ACE$ , and the angle  $B = ECD$ . Therefore  $A + B = ACE + ECD = ACD$ . Which was to be Dem.

a 31. 1.  
b 29. 1.  
c 2. ex.  
d 19. ex.

I affirm  $ACD + ACB = 2$  right angles; therefore  $A + B + ACB = 2$  right angles. Which was to be Dem.

e 13. 1.  
f 1. ex.

Coroll.

1. The three angles of any triangle taken together

are

are equal to three angles of any other triangle taken together. From whence it follows,

2. That if in one triangle, two angles (taken severally, or together) be equal to two angles of another triangle (taken severally, or together) then is the remaining angle of the one equal to the remaining angle of the other. In like manner, if two triangles have one angle of the one equal to one of the other, then is the sum of the remaining angles of the one triangle equal to the summe of the remaining angles of the other.

3. If one angle in a triangle be right, the other two are equal to a right. Likewise, that angle in a triangle which is equal to the other two, is it self a right angle.

4. When in an Isosceles the angle made by the equal sides is right, the other two upon the base are each of them half a right angle.

5. An angle of an equilateral triangle makes two third parts of a right angle. For  $\frac{1}{3}$  of two right angles is equal to  $\frac{2}{3}$  of one.

*Schol.*

By the help of this proposition you may know how many right angles the inward and outward angles of a right-lined figure make; as may appear by these two following Theoremes.

T H E O R E M E I.



*All the angles of a right-lined figure do together make twice as many right angles, bating four, as there are sides of the figure.*

From any point within the figure let right lines  
be

be drawn to all the angles of the figure, which shall resolve the figure into as many triangles as there are sides of the figure. Wherefore, whereas every triangle affords two right angles, all the triangles taken together will make up twice as many right angles as there are sides. But the angles about the said point within the figure make up four right; therefore, if from the angles of all the triangles you take away the angles which are about the said point, the remaining angles, which make up the angles of the figure, will make twice as many right angles, bating four, as there are sides of the figure. *Which was to be Dem.*

*Coroll.*

Hence, All right-lined figures of the same species have the sums of their angles equall.

**T H E O R E M E I I.**

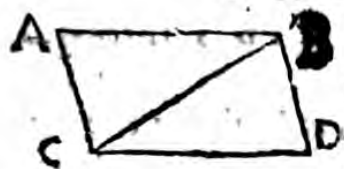
*All the outward angles of any right-lined figure, taken together, make up four right angles.*

For all the severall inward angles of a figure with the severall outward angles of the same make two right angles; therefore all the inward angles together with all the outward, make twice as many right angles as there are sides of the figure; But (as it was now shewn) all the inward angles with four right make twice as many right as there are sides of the figure; therefore the outward angles are equall to 4 right angles. *Which was to be Dem.*

*Coroll.*

All right-lined figures of whatsoever species have the summes of their outward angles equall.

**P R O P. XXXIII.**



*If two equall and parallel lines AB, CD be joynd together with two other right lines AC, BD, then are those lines also equall &*

*parallel.*

Draw a line from C to B. Now because AB and CD are parallel, and the angle ABC = BCD; and also by hypothesis AB = CD, and the side CB

com-

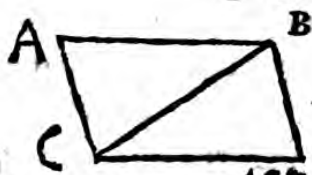


The first Book of

common, therefore is  $AC = BD$ , and the angle  $ACB = DBC$ . whence also  $AC, BD$  are parallel.

b 4. 1.  
c 27. 1.

PROP. XXXIV.



In parallelograms, as  $ABCD$ , the opposite sides  $AB, CD$ , and  $AC, BD$ , are equal each to other; and the opposite angles  $A, D$ , and  $ABC, ADC$  are also equal; and the diameter  $BC$  bisects the same.

a hyp.

Because  $AB, CD$  are parallel, therefore is the angle  $ABC = BCD$ . Also because  $AC, BD$  are parallel, therefore is the angle  $ACB = CBD$ ; therefore the whole angle  $ACD = ABD$ . After the same manner is  $A = D$ . Moreover because the angles  $ABC, ACB$  lye at each end of the side  $CB$ , and are equal to  $BCD, CBD$ , therefore is  $AC = BD$ , and  $AB = CD$ , and so the triangle  $ABC = CBD$ . Which was to be Dem.

b 29. 1.  
c 2. ax.

d 16. 1.

Schol.

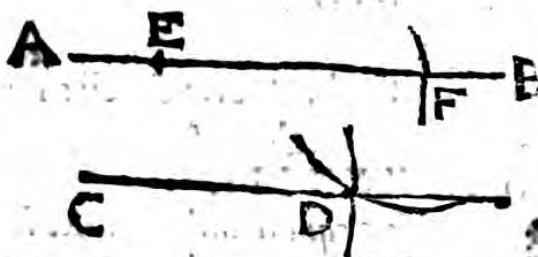
Every four-sided figure  $ABDC$  having the opposite sides equal, is a parallelogram.

a 27. 1.

For by 8. 1. the angle  $ABC = BCD$ ; wherefore  $AB, CD$  are parallel. In like manner is the angle  $BCA = CBD$ ; wherefore  $AC, BD$  are also parallel.

b 35. def. 1.

Therefore  $ABCD$  is a parallelogram. Which was to be Dem.



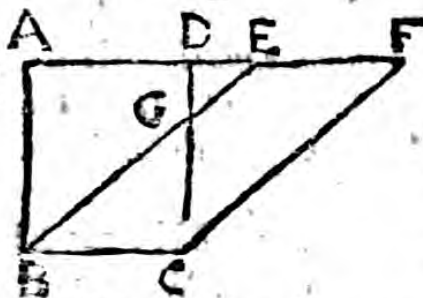
From hence may be learned how to draw a parallel  $CD$  to a right line given  $AB$ , through a

point assigned  $C$ .

Take in the line  $AB$  any point, as  $E$ . From the centres  $E$  and  $C$  at any distance draw two equal circles  $EF, CD$ . From the centre  $F$  by the space of  $EC$  draw a circle  $FD$ , which shall cut the former circle  $CD$  in the point  $D$ . Then shall the line drawn  $CD$  be parallel to  $AB$ . for as it was before demonstrated,  $CEFD$  is a parallelogram.

PROP.

PROP. XXXV.

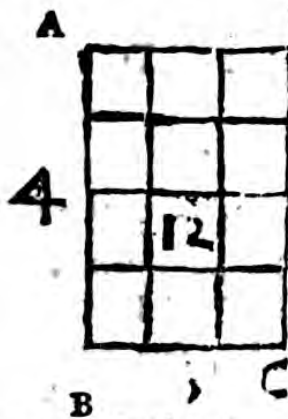


Parallelograms, BC-DA, BC FE, which stand upon the same base BC, and between the same parallels AF, BC, are equal one to the other.

For AD  $a =$  BC  $a =$  a 34 i.  
 EF. adde DE common to both,  $b$  then is AE = DF. b 2 ax.  
 But also AB  $c =$  DC, and the angle A  $c =$  CDF. c 19. s.  
 $d$  Therefore is the triangle ABE = DCF. take away d 4 i.  
 DGE common to both triangles,  $e$  then is the Tra- e 3 ax.  
 pezium ABCD = EGCF. adde BGC common to  
 both,  $f$  then is the parallelogram ABCD = EBCF. f 2. ax.  
 Which was to be Dem.

The demonstration of any other cases is not unlike, but much more plain and easy.

Schol.



If the side AB of a right-angled parallelogram ABCD be conceived to be carried along perpendicularly through the whole line BC, or BC through the whole line AB, the Area or content of the Rectangle ABCD shall be produc'd by that motion. Hence a rectangle is said to be made by the drawing

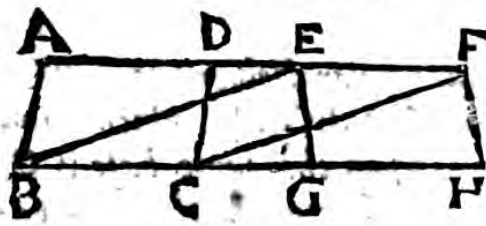
or multiplication of two contiguous sides. For examples sake; let AB be supposed four foot, and BC three: draw 3 into 4, there will be produced 12 square feet for the Area of the Rectangle.

This being supposed, the dimension of any parallelogram (\*EBCF) is found out by this theoreme. For the Area thereof is produced from the altitude BA drawn into the base BC. So the Area of the parallelogram AC = EBCF, is made by the drawing of BA into BC, therefore, &c.

\* See the fig. of prop. 35.

PROP.

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P R O P. XXXVI.

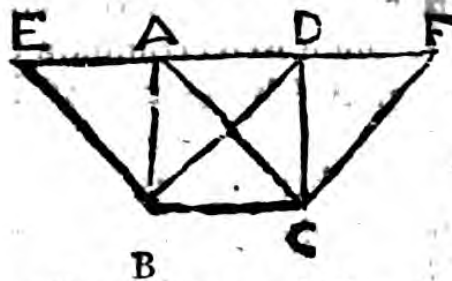


Parallelograms, BC-DA, GHFE, standing upon equall bases BC, GH, and betwixt the same parallels AF, BH, are equall one to the other.

a 2yp.  
b 34. 1.  
c 33. 1.  
d 35. 1.

Draw BE, & CF. Because BC = GH = EF, therefore is BCFE a parallelogram. Whence the parallelogram BCDA = BCFE = GHFE. Which was to be Dem.

P R O P. XXXVII.



Triangles, BCA, BCD, standing upon the same base BC, and between the same parallels BC, EF, are equall one to the other.

a 31. 1.  
b 34. 1.  
c 35. 1. and  
7. ax.

Draw BE parallel to CA, and CF parallel to BD. Then is the triangle BCA = 1/2 of the parallelogram BCAE = 1/2 BDFC = 1/2 BCD. Which was to be Dem.

P R O P. XXXVIII.



Triangles, BCA, EFD, set upon equall bases BC, EF, and between the same parallels GH, BF, are equall the one to the other.

a 34. 1.  
b 36. 1. and  
7. ax.  
c 34. 1.

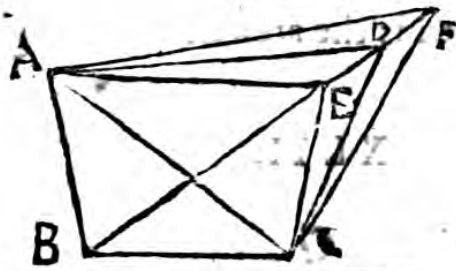
Draw BG parallel to CA, & FH parallel to ED. Then is the triangle BCA = 1/2 Pgr. BCAF = 1/2 EDHF = 1/2 EFD. Which was to be Dem.

Schol.

If the base BC be greater then EF, then is the triangle BAC > EDF, and so on the contrary.

P R O P.

P R O P. XXXIX.

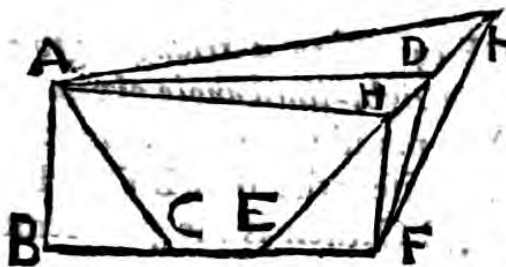


*Equall triangles BCA, BCD, standing on the same base BC, and on the same side, they are also between the same parallels AD, BC.*

If you deny it, let another line AF be parallel to BC; and let CF be drawn. Then is the triangle CBF  $a = CBA$   $b = CBD$ . *c Which is absurd.*

a 37. 1.  
b hyp.  
c 9. ax.

P R O P. XL.

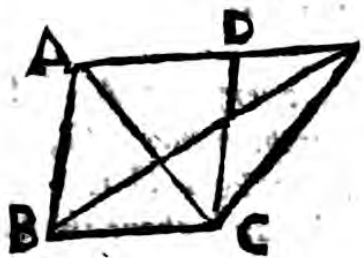


*Equall triangles BCA, EFD, standing upon equall bases BC, EF, and on the same side, they are betwixt the same parallels.*

If you deny it, let another line AH be parallel to BF, and let FH be drawn. Then is the triangle EFH  $a = BCA$   $b = EFD$ . *c Which is absurd.*

a 38. 1.  
b hyp.  
c 9. ax.

P R O P. XLI.



If a Pgr. ABCD have the same base BC with the triangle BCE, and be between the same parallels AE, BC, then is the Pgr. ABCD double to the trian-

gle BCE.

Let the line AC be drawn. Then is the triangle BCA  $a = BCE$ . therefore is the Pgr. ABCD  $b = 2 BCA$   $c = 2 BCE$ . *Which was to be Dem.*

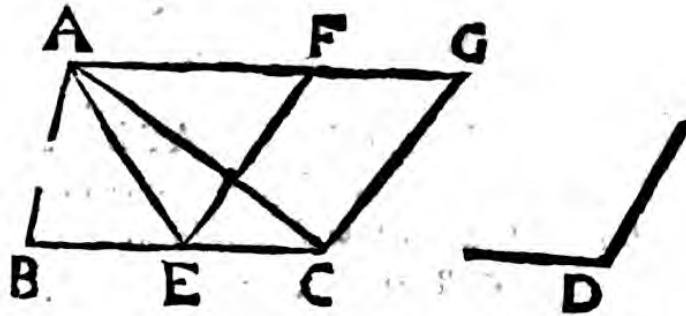
a 37. 1.  
b 34. 1.  
c 6. ax.

*Schol.*

From hence may the Area of any triangle BCE be found, for whereas the Area of the Pgr. ABCD is produced by the altitude drawn into the base, therefore shall the Area of a triangle be produced by half the altitude drawn into the base, or half the base drawn

drawn into the altitude. as if so be the base  $BC$  be 8, and the altitude 7, then is the area of the triangle  $BCE$  28.

P R O P. XLII.



To make a Pgr.  $ECGF$  equall to a triangle given  $ABC$  in an angle equal to a right-lined angle given  $D$ .

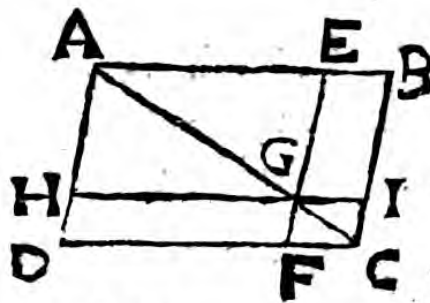
a 31. 1.  
b 23. 1.  
c 10. 2.

Through  $A$  draw  $AG$  parallel to  $BC$ .  $b$  make the angle  $BCG = D$ .  $c$  bisect the base  $BC$  in  $E$ , and draw  $EF$  parallel to  $CG$ . then is the probleme resolved.

d 38. 1.  
e 41. 1.

For  $AE$  being drawn, the angle  $ECG$  is equal to  $D$  by construction, and the triangle  $BAC = 2$   $AEC =$  Pgr.  $ECGF$ . Which was to be Done.

P R O P. XLIII.

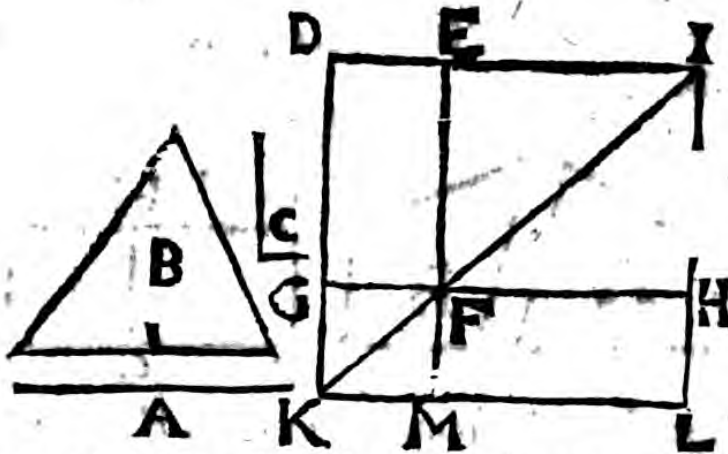


In every Pgr.  $ABCD$ , the complements  $DG$ ,  $GB$  of those Pgrs  $HE$ ,  $FI$ , which stand about the diameter, are equal one to the other.

a 34. 1.  
b 3. 22.

For the triangle  $ACD = ACB$ , and the triangle  $AGH = AGE$ , and the triangle  $GCF = GCI$ . Therefore the Pgr.  $DG = BG$ . Which was to be Dem.

P R O P. XLIV.

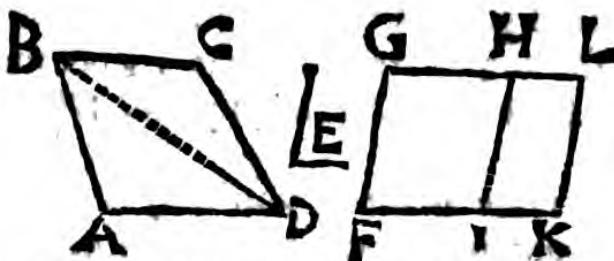


*Upon a right line given A, to make a Parallelogram FL at a right-lined angle given C, equall to a triangle given B.*

*a* Make a Pgr. FD = to the triangle B, so that <sup>a 42. 1.</sup> the angle GFE may be equall to C. Produce GF till FH be equall to the line given A. through H *b* draw IL parallel to EF, which let DE produced meet in the point I. let DG drawn forth meet with <sup>b 31. 1.</sup> a right line drawn from I in the point K. through K *b* draw KL parallel to GH, with which let EF drawn out meet at M, and IH at L. Then shall FL be the Pgr. required.

For the Pgr. FL <sup>c 43. 1.</sup> = FD = B, <sup>d 15. 1.</sup> and the angle MFH = GFE = C. *Which was to be Done.*

P R O P. XLV.



*Upon a right line given FG to make a Pgr. FL equall to a right-lined figure given ABCD, at a right-lined angle given E.*

C

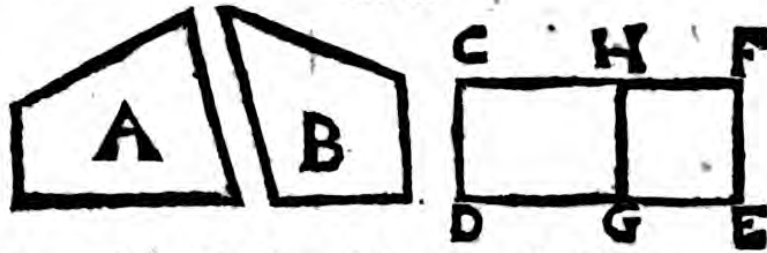
Re-

a 44.1.

b 19. ax.  
c const.

Resolve the right-lined figure given into triangles BAD, BCD. then *a* make a Pgr. FH = BAD, so that the angle F may be equall to E. FI being produced, *a* make on HI the Pgr. IL = BCD. Then is the Pgr. FL *b* = FH + IL *c* = ABCD. Which was to be Done.

Schol.



Hence is easily found the excesse, HE, whereby any right-lined figure, A, exceeds a lesse right-lined figure, B; namely, if to some right line, CD, both be applied, Pgr. DF = A, and DH = B.

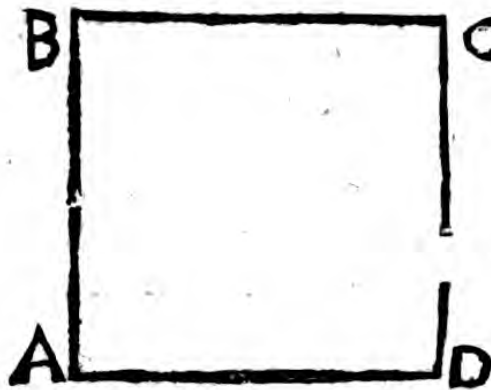
PROP. XLVI.

a 11. 3.

b 3. 1.

c const.  
d 28. 1.  
e const.  
f 34. 1.

g sch. 29. 1.  
h 29. def.



Vpon a right line given AD to describe a square AC.

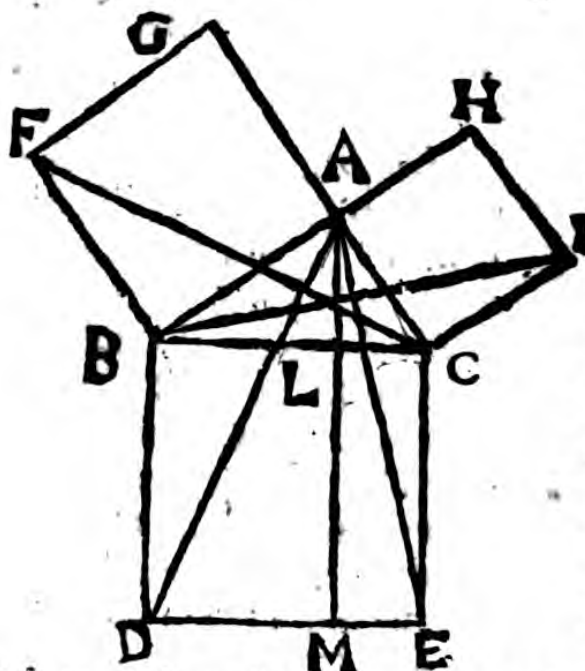
*a* Erect two perpendiculars AB, DC, *b* equall to the line given AD; then joyn BC, & the thing required is done.

For, whereas the angle A + D *c* = 2 right, *d* therefore are AB, DC parallel. But they are also *e* equall; *f* therefore AD, BC are both parallel and equall; therefore the figure AC is a Pgr. and equilateral. Moreover the angles are all right, *g* because one, A, is right; *h* therefore AC is a square. Which was to be Done.

After the same manner you may easily describe a rectangle contained under two right lines given.

PROP.

PROP. XLVII.



In right angled triangles,  $BAC$ , the square  $BE$ , which is made of the side  $BC$  that subtends the right angle  $BAC$ , is equal to both the squares  $BG, CH$ , which are made of the sides  $AB, AC$  containing the right angle.

Join  $AE$ , and

$AD$ ; and draw  $AM$  parallel to  $CE$ .

Because the angle  $DBC$   $a = FBA$ , add the angle  $ABC$  common to them both; then is the angle  $ABD = FBC$ . Moreover  $AB$   $b = FB$ , and  $BD$   $b = BC$ ; therefore is the triangle  $ABD = FBC$ . But the Pgr.  $BM$   $d = 2 ABD$ , and the Pgr.  $BG = 2 FBC$  (for  $GAC$  is one right line by Hypothesis, and 14.1.) therefore is the Pgr.  $BM = BG$ . By the same way of argument is the Pgr.  $CM = CH$ . Therefore is the whole  $BE = BG + CH$ . Which was to be Dem.

Schol.

This most excellent and usefull theoreme hath deserved the title of *Pythagoras* his theoreme, because he was the inventer of it. By the help of which the addition and subtraction of squares are performed; to which purpose serve the 2 following problems.



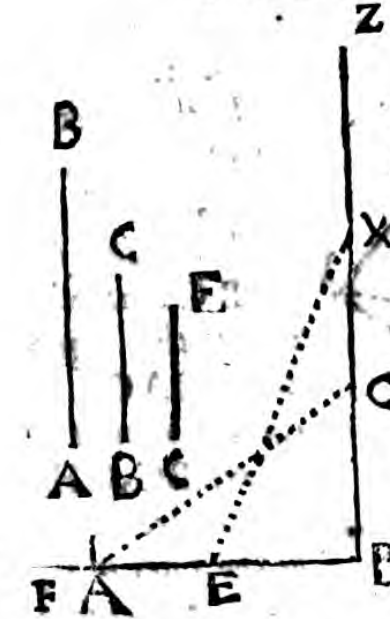
PROBLEME I.

Andr. Tacq.

a 11. 1.

b 47. 1.

c 2. ex.

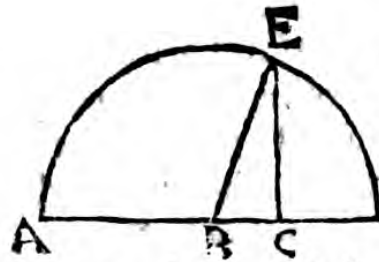


To make one square equal to any number of squares given.

Let three squares be given, whereof the sides are  $AB, BC, CE$ . <sup>a</sup> Make the right angle  $FBZ$  having the sides infinite; and on them transfer  $BA$  and  $BC$ ; join  $AC$ , then is  $ACq = ABq + BCq$ . Then transfer  $AC$  from  $B$  to  $X$ , and  $CE$  the third side given from  $B$  to  $E$ ;

join  $EX$ . <sup>b</sup> Then is  $EXq = EBq (CEq) + BXq (ACq)$  <sup>c</sup>  $= CEq + ABq + BCq$ . Which was to be Done.

PROBLEME II.

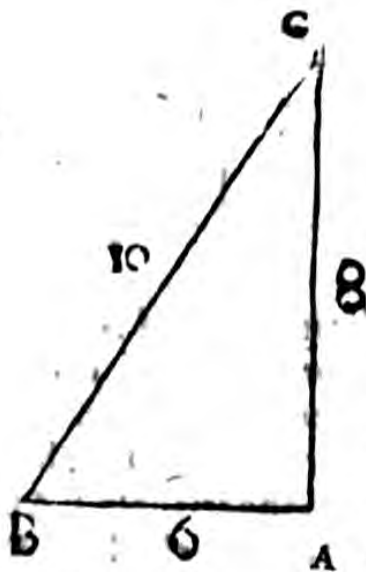


Two unequal right lines being given  $AB, BC$ , to make a square equal to the difference of the two squares of the given lines,  $AB, BC$ .

From the centre  $B$ , at the distance of  $BA$ , describe a circle; and from the point  $C$  erect a perpendicular  $CE$  meeting with the circumference in  $E$ ; and draw  $BE$ . <sup>a</sup> Then is  $BEq (BAq) = BCq + CEq$ . <sup>b</sup> Therefore  $BAq - BCq = CEq$ . Which was to be Done.

a 47. 1.  
b 3. ex.

PROBLEME III.

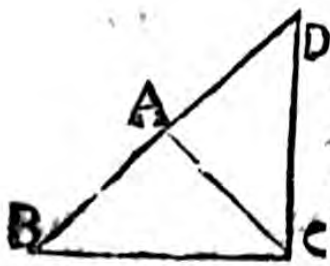


Any two sides of a right-angled triangle ABC being known, to find out the third.

Let the sides AB, AC encompassing the right angle, be, the one 6 foot, the other 8. Therefore, whereas  $ACq + ABq = 64 + 36 = 100 = BCq$ , thence is  $BC = \sqrt{100} = 10$ .

Otherwise, let the sides AB, BC be known, the one 6 foot, the other 10. Therefore since  $BCq - ABq = 100 - 36 = 64 = ACq$ , thence is  $AC = \sqrt{64} = 8$ . Which was to be Done.

PROP. XLVIII.



If the square made upon one side BC of a triangle be equal to the squares made on the other sides of the triangle, AB, AC, then the angle BAC comprehended under those two other sides of the triangle AB, AC, is a right angle.

Draw to the point A in AC a perpendicular line  $DA = AB$ , and join CD.

Now is  $CDq = ADq + ACq = ABq + ACq = BCq$ . \* Therefore is  $CD = BC$ . And therefore the triangles CAB, CAD are mutually equilaterall. Wherefore the angle  $CAB = CAD =$  a right angle. Which was to be Dem.

Schol.

We assumed in the demonstration of the last proposition,  $CD = BC$ , because  $CDq$  was equal to  $BCq$ : our assumption we prove by the following theoreme.

## T H E O R E M E.



The squares AF, CG of equall right lines AB, CD are equall one to the other: And the sides IK, LM of equall squares NK, PM, are equall one to the other.

a 34. 1.  
b 4. 1. &  
c ax.

1. Hypothesis. Draw the diameters EB, HD. Then it is evident that AF is *a* equall to the triangle EAB twice taken, and *b* equall to the triangle HCD twice taken, and equall to *a* CG. Which was to be Dem.

a 46. 1.  
b 1. part.  
c hyp.  
d 9. ax.

2. Hyp. If it may be, let LM be greater then IK. Make LT = IK, and let LS be *a* equall to LTq. Therefore is LS *b* = NK *c* = LQ. *d* Which is Absurd.

Coroll.

After the same manner any rectangles equilateral one to another, are demonstrated to be also equall.

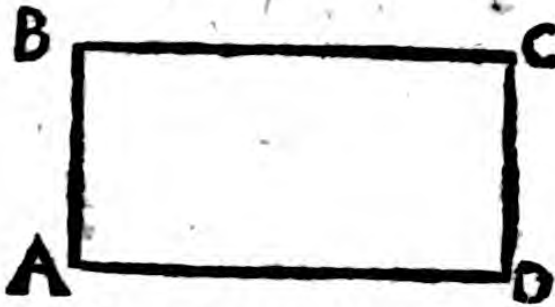
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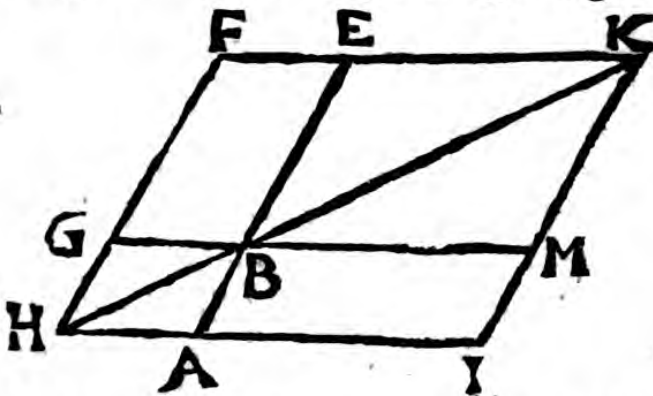
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*Definitions.*



I. **H** Very right-angled Parallelogram  $AB-CD$  is said to be contained under two right lines  $AB, AD$  comprehending a right angle.

*Therefore when you meet with such as these, the rectangle under  $BA, AD$ , or for shortnesse sake the rectangle  $BAD$ , or  $BA \times AD$  (or  $ZA, for Z \times A$ ) the rectangle meant is that which is contained under the right lines  $BA, AD$  set at right angles.*



I I. In every Pgr.  $FHIK$ , any one of those parallelograms which are about the diameter together with its two complements is called a Gnomon. As the Pgr.  $FB + BI + GA$  ( $EHM$ ) is a Gnomon; and likewise the Pgr.  $FB + BI + EM$  ( $GKA$ ) is a Gnomon.



If two right lines  $AF, AB$ , be given, and one of them  $AB$  divided into as many parts or segments as you please; the rectangle comprehended under the two whole right lines  $AB, AF$ , shall be equal to all the rectangles contained under the whole line  $AF$  and the severall segments,  $AD, DE, EB$ .

a 11. 1.

b 19. ex. 1.  
c 34. 1.

Set  $AF$  perpendicular to  $AB$ . Through  $F$  draw an infinite line  $FG$  perpendicular to  $AF$ . From the points  $D, E, B$  erect perpendiculars  $DH, EI, BG$ . Then is  $AG$  a rectangle comprehended under  $AF, AB$ , and is equal to the rectangles  $AH, DI, EG$ , that is (because  $DH, EI, AF$  are equal) to the rectangles under  $AF, AD$ , under  $AF, DE$ , under  $AF, EB$ . Which was to be Dem.

Schol.

If two right lines given be both divided into how many parts soever, the product of the whole multiplied into it self shall be the same with that of the parts multiplied into themselves.

a 1. 2.

b 3. ex.

For let  $Z$  be  $= A + B + C$ , and  $Y = D + E$ ; then, because  $DZ = DA + DB + DC$ , and  $EZ = EA + EB + EC$ , and  $YZ = DZ + EZ$ , shall  $ZY$  be  $= DA + DB + DC + EA + EB + EC$ . Which was to be Dem.

From hence is understood the manner of multiplying compounded right lines into compounded. For you must take all the Rectangles of the parts, and they will present you with the Rectangle of the wholes.

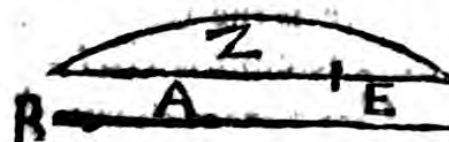
But whensoever in the multiplication of lines into themselves you meet with these signes  $-$  intermingled with these  $+$ , you must also have particular regard to the signes. For of  $+$  multiply'd into  $-$  ariseth  $-$ ; but of  $-$  into  $-$  ariseth  $+$ . ex. gr. let  $+A$  be to be multiply'd into  $B - C$ ; then because  $+A$  is not affirmed of all  $B$ , but only of a part of it, whereby it exceeds  $C$ , therefore  $AC$  must remain de-

denied; so that the product will be  $AB - AC$ . Or thus; because B consists of the parts C and  $B - C$ , \* thence  $AB = AC + A \times B - C$ . take away AC from either, then  $AB - AC = A \times B - C$ . In like manner if A be to be multiply'd into  $B - C$ , then seeing by reason of the figure, that A is not denied of all B, but only of so much as it exceeds C, therefore AC must remain affirmed. whence the product will be  $-AB + AC$ . Or thus; because  $AB = AC + A \times B - C$ ; take away all throughout, and there will be  $-AB = AC - A \times B + C$ ; adde AC to either, and there will be  $-AB + AC = A \times B - C$ .

This being sufficiently understood, the nine following propositions, and innumerable others of that kind, arising from the comparing of lines multiply'd into themselves (which you may find done to your hand in *Vieta* and other Analytical writers) are demonstrated with great facility, by reducing the matter for the most part to almost a simple work.

Furthermore, \* it appears that the product of any magnitude multiply'd into the parts of any number is equal to the product of the same multiply'd into the whole number: As  $5A + 7A = 12A$ , and  $4A \times 5A + 4A \times 7A = 4A \times 12A$ . Wherefore what is here deliver'd of the multiplying of right lines into themselves, the same may be understood of the multiplying of numbers into themselves. so that whatsoever is affirmed concerning lines in the nine following theoremes, holds good also in numbers; seeing they all immediately depend and are deriv'd from this first.

P R O P. II.



If a right line Z be divided any-wise into two parts, the rectangles comprehended under the whole line Z and each of the segments A, E, are equal to the square made of the whole line Z.

a 1. 2. I say that  $ZA + ZE = Zq$ . For take  $B = Z$ ; then is  $BA + BE = BZ$ . that is (because  $B = Z$ )  $ZA + ZE = Zq$ . Which was to be Dem.

P R O P. III.

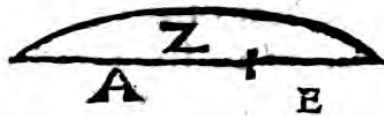


If a right line Z be divided any-wise into two parts, the rectangle comprehended under the whole line Z and one of the segments E is equal to the rectangle made of the segments A, E, and the square described on the said segment E.

a 1. 2.

I say  $ZE = AE + Eq$ . For  $EZ = EA + Eq$ .

P R O P. IV.



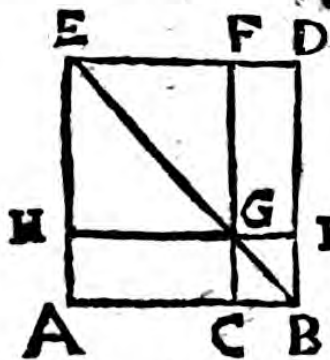
If a right line Z be cut any-wise into two parts, the square made of the whole line Z is equal both to the squares made of the segments A, E, and to twice a rectangle made of the parts A, E.

a 3. 2.

I say that  $Zq = Aq + Eq + 2 AE$ . For  $ZA = Aq + AE$ , and  $ZE = Eq + EA$ . Therefore whereas  $ZA + ZE = Zq$ , thence is  $Zq = Aq + Eq + 2 AE$ . Which was to be Dem.

b 2. 2.

c 1. 2.



Otherwise thus ; Upon the right line AB make the square AD, and draw the diameter EB; through C, the point wherein the line AB is divided, draw the perpendicular CF; and through the point G draw HI parallel to AB.

d 4 cor. 32.

e.

e 32. 1.

f 6 1.

g 34. 1.

h 29. def.

k 19. ex. 1.

Because the angle  $EHG = A$  is a right angle, and  $AEB$  is  $d$  half a right,  $e$  therefore is the remaining angle  $HGE$  half a right angle. Therefore is  $HEf = HGg = EFg = AC$ , so that  $HF$   $b$  is the square of the right line  $AC$ . After the same manner is  $CI$  proved to be  $CBq$ . Therefore  $AG, GD$  are rectangles under  $AC, CB$ . wherefore the whole square  $AD$   $k = ACq + CBq + 2 ACB$ . Which was to be Dem. Co-

Coroll.

1. Hence it appears that the Pgrs which are about the diameter of a square are also squares themselves.

2. That the diameter of any square bisects it's angles.

3. That if  $A = \frac{1}{2} Z$ , then is  $Zq = 4 Aq$ , and  $Aq = \frac{1}{4} Zq$ . As on the contrary, if it be so that  $Zq = 4 Aq$ , then is  $A = \frac{1}{2} Z$ .

*The Squ: of the whole line is 4 times much as the Sq of its half.*

PROP. V.

A ——— | ——— | ——— B  
          C        D

If a right line AB be cut into equal parts AC, CB, and into unequal parts AD, DB, the rectangle comprehended under the unequal parts AD, DB, together with the square that is made of the difference of the parts CD, is equal to the square that is made of the half line CB.

I say that  $CBq = ADB + CDq$ .

For these are all equal ;

$$\left\{ \begin{array}{l} CBq. \\ a CDq + CDB + DBq + CDB. \\ b CDq + c CBD (c AC \times BD) + CDB. \\ d CDq + d ADB. \end{array} \right.$$

a 4. 2.  
b 3. 2.  
c hyp.  
d 1. 2.

This theoreme is somewhat differently express'd and more easily demonstrated thus ; *A Rectangle made of the summe and the difference of two right lines A, E, is equal to the difference out of them.*

For if  $A + E$  be multiply'd into  $A - E$ , \* there ariseth  $Aq - AE + EA - Eq = Aq - Eq$ . Which was to be Dem.

Schol.

A ——— | ——— | ——— B  
          C    E    D

If the line AB be divided otherwise, (viz.) nearer to the point of bisection, in E; Then is  $AEB \sqsupseteq ADB$ .

For  $AEB \sqsupseteq CBq - CEq$ . and  $ADB \sqsupseteq CBq - CDq$ . Therefore, whereas  $CDq \sqsupseteq CEq$ , thence is  $AEB \sqsupseteq ADB$ . W. W. to be Dem.

a 6. 2. &  
3. 2.

Co.



Coroll.

b 4. 2. 1. Hence is  $ADq + DBq \square AEq + EBq$ . For  $ADq + DBq + 2 ADB^b = ABq^b = AEq + EBq + 2 AEB$ . Therefore because  $2 AEB \square 2 ADB$ , thence is  $ADq + DBq \square AEq + EBq$ . *W.W. to be Dem.*

e 3. ex. 2. Hence is  $ADq + DBq - AEq + EBq = 2 AEB - 2 ADB$ .

## P R O P. VI.



If a right line *A* be divided into two equal parts, and another right line *E* added to the same directly in one right line, then the rectangle comprehended under the whole and the line added, (viz.  $A + E$ ), and the line added *E*, together with the square which is made of  $\frac{1}{2}$  the line *A*, is equal to the square of  $\frac{1}{2} A + E$  taken as one line.

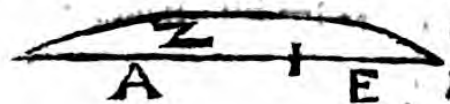
I say that  $\frac{1}{2} Aq + AE + Eq = Q$ .  $\frac{1}{4} A + E$ . For,  $Q^2 = \frac{1}{4} Aq + Eq + AE$ . Which was to be Dem.

a 4. 3.  
Cor. 4. 4.

Coroll.

Hence it follows that if 3 right lines *E*,  $E + \frac{1}{2} A$ ,  $E + A$  be in arithmetical proportion, then the rectangle contained under the extreme termes *E*,  $E + A$ , together with the square of the difference  $\frac{1}{2} A$ , is equal to the square of the middle term  $E + \frac{1}{2} A$ .

## P R O P. VII.



If a right line *Z* be divided any-wise into two parts, the square of the whole line *Z* together with the square made of one of the segments *E*, is equal to a double rectangle comprehended under the whole line *Z* and the said segment *E*, together with the square made of the other segment *A*.

a 4. 2.  
b 3. 2.

I say that  $Zq + Eq = 2 ZE + Aq$ . For  $Zq = Aq + Eq + 2 AE$ . and  $2 ZE = 2 Eq + 2 AE$ . *W. W. to be Dem.*

Co-

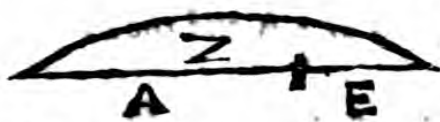
Coroll.

Hence it follows that the square of the difference of any two lines Z, E, is equal to the squares of both the lines less by a double rectangle comprehended under the said lines.

c For  $Zq + Eq - 2 ZE = Aq = Q. Z - E.$

c 7.2. and  
3 ax.

P R O P. VIII.



If a right line Z be divided any-wise into two parts, the rectangle comprehended under the whole line Z and one of the segments E four times, together with the square of the other segment A, is equal to the square of the whole line Z and the segment E taken as one line Z + E.

I say that  $4 ZE + Aq = Q. Z + E.$  For  $2 ZE = Zq + Eq - Aq.$  Therefore  $4 ZE + Aq = Zq + Eq + 2 ZE = Q. Z + E.$  W. W. to be Dem.

a 7.2. and  
3 ax.  
b 4 2.

P R O P. IX.



If a right line AB be divided into equal parts AC, CB, and into unequal parts AD, DB, then are the squares of the unequal parts AD, DB together, double to the square of the half line AC, and to the square of the difference CD.

I say that  $ADq + DBq = 2 ACq + 2 CDq.$  For  $ADq + DBq = ACq + CDq + 2 ACD + DBq.$  But  $2 ACD = 2 BCD = CBq = (ACq) + CDq.$  Therefore  $ADq + DBq = 2 ACq + 2 CDq.$  W. W. to be Dem.

a 4.2.  
b hyp.  
c 7.2.  
d 2. ax.

This may be otherwise deliver'd and more easily demonstrated thus; The aggregate of the squares made of the summe and the difference of two right lines A, E, is equal to the double of the squares made from those lines.

For  $Q: A + E = Aq + Eq + 2 AE.$  and  $Q: A - E = Aq + Eq - 2 AE.$  These added together make  $2 Aq + 2 Eq.$  Which was to be Dem.

P R O P.

The second Book of

P R O P. X.



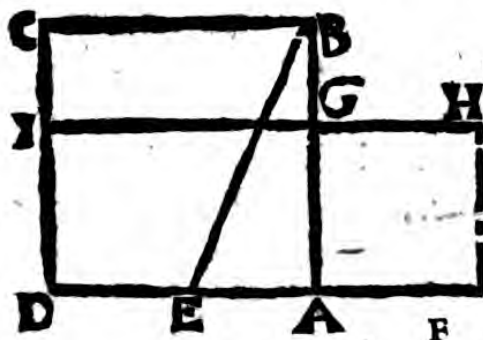
If a right line A be divided into two equal parts, and another line be added in a right line with the same, then

is the square of the whole line together with the added line (as being one line) together with the square of the added line E, double to the square of  $\frac{1}{2}$  A and the added line E, taken as one line.

a 4. 2.  
b cor. 4. 2.  
c 4. 2.

I say that  $Eq + Q. A + E$ , i. e.  $Aq + 2 Eq + 2 AE = 2 Q. \frac{1}{2} A + 2 Q. \frac{1}{2} A + E$ . For  $2 Q. \frac{1}{2} A = \frac{1}{2} Aq$ . And  $2^2 Q. \frac{1}{2} A + E^2 = \frac{1}{2} Aq + 2 Eq + 2 AE$ . W.W. to be <sup>2</sup> Dem.

P R O P. XI.



To cut a right line given AB in a point G, so that the rectangle comprehended under the whole line AB and one of the segments BG shall be equal to the square that is made of the other segment AG.

a 46. 1.  
b 10. 1.

Upon AB<sup>a</sup> describe the square AC. <sup>b</sup> Bisect the side AD in E, & draw the line EB; from the line EA produced take EF = EB. On AF make the square AH. Then is AH = AB x BG.

c 6. 2.  
d constr.  
e 47. 1.  
f 3. ax.

For HG being drawn out to I; the rectangle DH + EAq<sup>c</sup> = EFq<sup>d</sup> = EBq<sup>e</sup> = BAq + EAq; Therefore is DH<sup>f</sup> = BAq = to the square AC. Take away AI common to both, then remains the square AH = GC. that is, AGq = AB x BG. W.W. to be Done.

Schol.

a 6. 13.

This proposition cannot be performed by numbers; for there is no number that can be so divided, that the product of the whole into one part shall be equal to the square of the other part.

P R O P.

PROP. XII.



In obtuse-angled triangles ABC, the square that is made of the side AC subtending the obtuse angle ABC is greater then the squares of the sides BC, AB that contain the obtuse angle ABC, by a double rectangle contained under one of the sides BC, which are about the obtuse angle ABC, on which side produced the perpendicular AD falls, and under the line BD, taken without the triangle from the point on which the perpendicular AD falls to the obtuse angle ABC.

I say that  $AC^2 = CB^2 + AB^2 + 2 CB \times BD$ .

For these are all equal

$$\left\{ \begin{array}{l} a. AC^2 \\ b. CD^2 + AD^2 \\ c. CB^2 + 2 CBD + BD^2 + AD^2 \\ d. CB^2 + 2 CBD + AB^2 \end{array} \right.$$

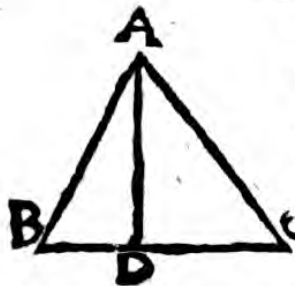
a 47. 1.  
b 47. 1.  
c 47. 1.

Scholium.

Hence, the sides of any obtuse-angled triangle ABC being known, the segment BD intercepted betwixt the perpendicular AD and the obtuse angle ABC, as also the perpendicular it self AD shall be easily found out.

Thus. Let AC be 10, AB 7, CB 5. Then is  $AC^2$  100,  $AB^2$  49,  $CB^2$  25. And  $AB^2 + CB^2 = 74$ . Take that out of 100, then will 26 remain for  $2 CBD$ . Wherefore  $CBD$  shall be 13; divide this by  $CB$  5, there will 2 $\frac{2}{5}$  be found for  $BD$ . Whence  $AD$  will be found out by the 47. 1.

PROP. XIII.



In acute-angled triangles ABC, the square made of the side AB subtending the acute angle ACB, is lesse then the squares made of the sides AC, CB comprehending the acute angle ACB by a double rectangle contained under one of the sides BC, which are about the acute angle ACB, on which the perpendicular AD falls, and under the line DC taken within the triangle from the perpendicular AD to the acute angle ACB.

I

I say that  $ACq + BCq = ABq + 2 BCD.$

a 47. 1.  
b 7. 2.  
c 47. 1.

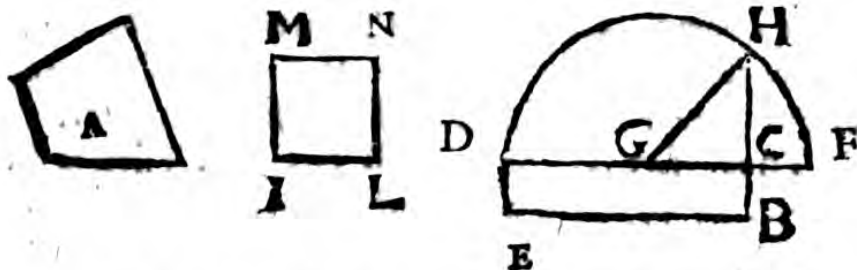
For these are  
equall  $\left\{ \begin{array}{l} a ACq + BCq. \\ b ADq + DCq + BCq. \\ c ADq + BDq + 2 BCD. \\ d ABq + 2 BCD. \end{array} \right.$

Coroll.

Hence, The sides of an acute-angled triangle ABC being known, you may find out the segment DC intercepted betwixt the perpendicular AD and the acute angle ABC, as also the perpendicular it self AD.

Let AB be 13, AC 15, BC 14. Take ABq (169) from ACq + BCq, that is, from 225 + 196 = 421. Then remains 252 for 2 BCD. wherefore BCD will be 126. divide this by BC 14, then will 9 be found out for DC. From whence it follows AD =  $\sqrt{225 - 81} = 12.$

PROP. XIV.



To find a square ML equall to a right-lined figure given A.

a 45. 1.  
b 10. 1.

\* Make the rectangle  $DB = A$ , and produce the greater side thereof DC to F, so that  $CF = CB.$  b bisect DF in G, about which as the centre at the distance of GF describe the circle FHD, and draw out CB till it touch the circumference in H. Then shall be  $CHq = * ML = A.$

\* 46. 1.  
c constr.  
d 5. 2. and  
3. ax.  
e 47. 1. and  
3. ax.

For let GH be drawn. Then is  $A c = DB c = DCF d = GFq - GCq e = HCq c = ML.$  *W.W. to be Done.*

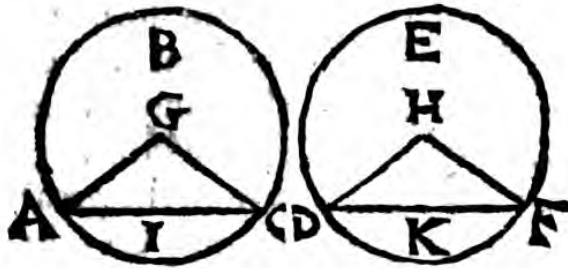
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L I B.

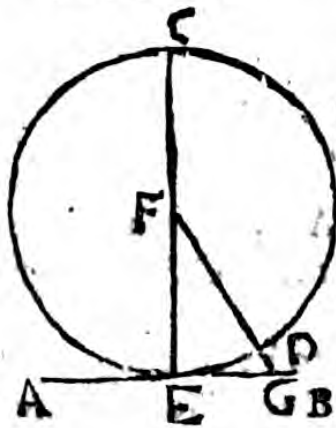
THE THIRD BOOK  
OF  
EUCLIDE'S ELEMENTS.

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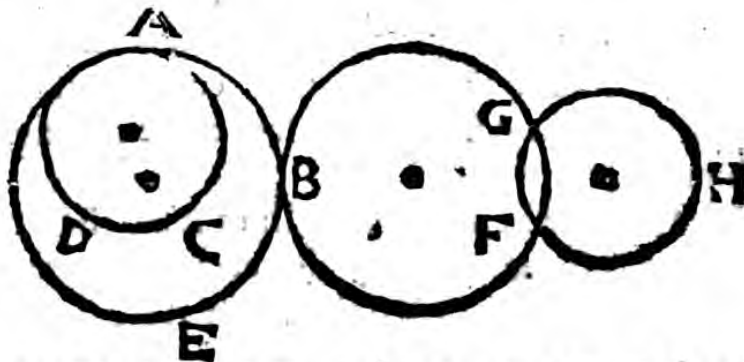
*Definitions.*



I. **E**qual circles (GABC, HDEF) are such whose diameters are equal; or, from whose centres right lines drawn GA, HD, are equal.



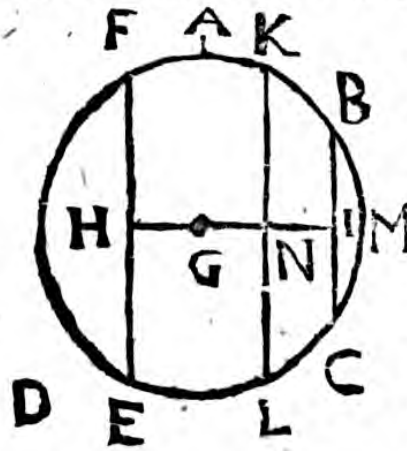
II. A right line AB is said to touch a circle FED, when touching the same, and being produced, it cutteth it not.  
The right line FG cuts the circle FED.



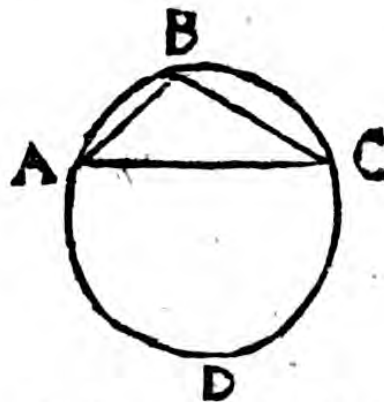
III. Circles DAC, ABE (and also FBG, ABE) are

are said to touch one to the other, which touch, but cut not one the other.

The circle BFG cuts the circle FGH.



IV. In a circle GABD, right lines FE, KL are said to be equally distant from the centre, when perpendiculars GH, GN drawn from the centre G to them are equal. And that line BC is said to be furthest distant from it, on whom the greater perpendicular GI falls.

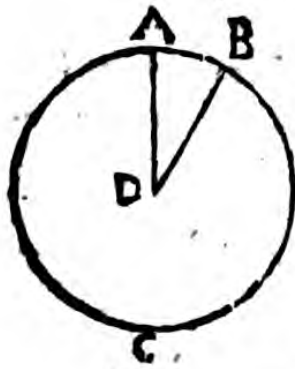


V. A segment of a circle (ABC) is a figure contained under a right line AC, and a portion of the circumference of a circle ABC.

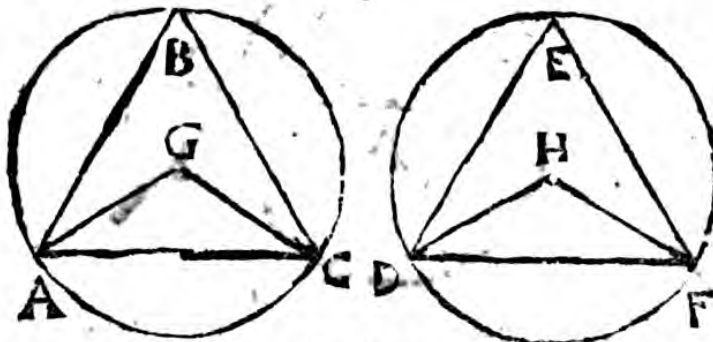
VI. An angle of a segment CAB, is that angle which is contained under a right line CA and an arch of a circle AB.

VII. An angle ABC is said to be in a segment ABC, when in the circumference thereof some point B is taken, and from it right lines AB, CB, drawn to the ends of the right line AC, which is the base of the segment; then the angle ABC contained under the adjoined lines AB, CB, is said to be an angle in a segment.

VIII. But when the right lines AB, BC comprehending the angle ABC, do receive any periphery of the circle ADC, then the angle ABC is said to stand upon that periphery.

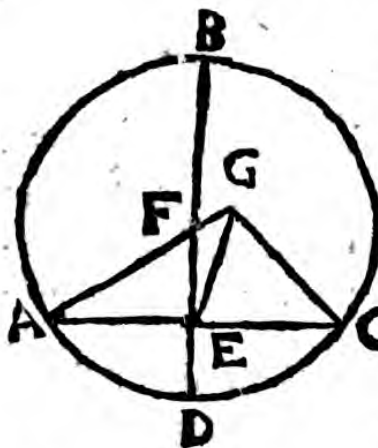


IX. A sector of a circle (ADB) is when an angle ADB is set at the centre D of that circle; namely, that figure ADB, comprehended under the right lines AD, BD containing the angle, and the part of the circumference received by them AB.



X. Like segments of a circle (ABC, DEF) are those which conclude equall angles (ABC, DEF;) or, in whom the angles ABC, DEF are equall.

P R O P. I.



To find the centre F of a circle given ABC.

Draw a right line AC any-wise in the circle, which bisect in E, through E draw a perpendicular DB, and bisect the same in F; the point F shall be the center.

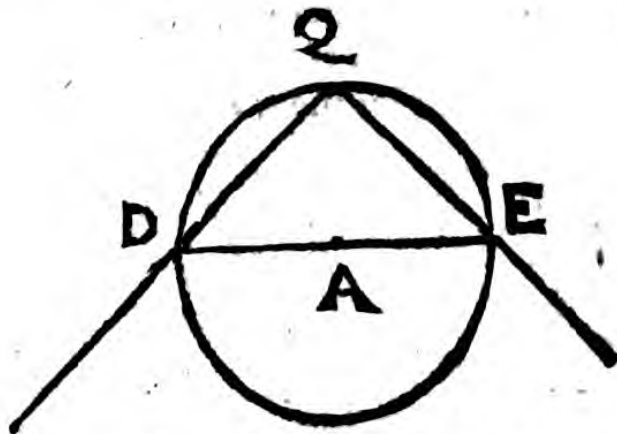
If you deny it, let G a point without the line DB be the centre (for it cannot be in

the line BD, since that cannot be divided equally in any point but F;) let the lines GA, GC, GE be drawn. Now if G be the centre, then is  $GA = GC$ , and  $AE = EC$  by construction, and the side GE common. <sup>b</sup> Therefore are the angles GEA, GEC equall, and <sup>c</sup> consequently right. <sup>d</sup> Therefore the angle  $GEC = FEC$ . <sup>e</sup> Which is absurd.

<sup>a</sup> 15. def. 1.  
<sup>b</sup> 8. 1.  
<sup>c</sup> 10 def. 1.  
<sup>d</sup> 12. ax.  
<sup>e</sup> 9. ax.



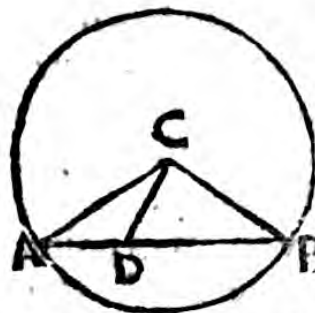
Hence, If a right line  $BD$  bisect any right line  $AC$  in a circle at right angles, the centre shall be in the line  $BD$  that cuts the other.



*Andr. Tacq.*

The centre of a circle is easily found out by applying the top of a Square to the circumference thereof. For if the right line  $DE$  that joins the points  $D, E$ , in which the sides of the Square  $QD, QE$  cut the circumference, be bisected in  $A$ , the point  $A$  shall be the centre. The demonstration whereof depends upon Prop. 31. of this Book.

**P R O P. II.**



If in the circumference of a circle  $CAB$  any two points  $A, B$  be taken, the right line  $AB$  which joins those two points shall fall within the circle.

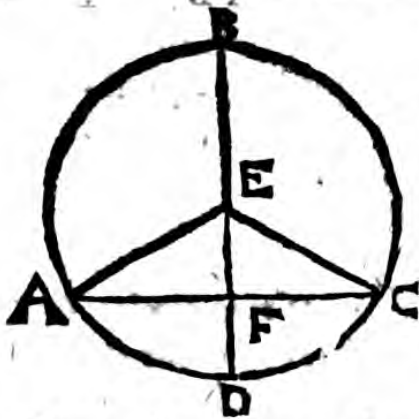
Take in the right line  $AB$  any point  $D$ ; from the center  $C$  draw  $CA, CD, CB$ . Because  $CA = CB$ , therefore is the angle  $A = B$ . But the angle  $CDB < A$ , therefore is  $CD < B$ . therefore  $CB < CD$ . But  $CB$  onely reaches the circumference, therefore  $CD$  comes not so far; wherefore the point  $D$  is within the circle. The same may be proved of any other point in the line  $AB$ . And therefore the whole line  $AB$  falls within the circle. Which was to be Dem.

a 15. d. f. 1.  
b 5. 1.  
c 16. 1.  
d 19. 1.

Coroll.

Hence, If a right line touch a circle, so that it cut it not, it touches but in one point.

P R O P. III.



If in a circle  $EABC$ , a right line  $BD$  drawn through the centre, bisect any other line  $AG$  not drawn through the centre, it shall also cut it at right angles: And if it cuts it at right angles, it shall also bisect the same.

From the centre  $E$  let the lines  $EA$ ,  $EC$  be drawn.

1. Hyp. Because  $AF^a = FC$ , and  $EA^b = EC$ , and the side  $EF$  common; the angles  $EFA$ ,  $EFC$  shall be equal, and  $d$  consequently right. Which was to be Dem.

*a hyp.*  
*b 15. def. 1.*  
*c 8. 1.*  
*d 10. def. 1.*

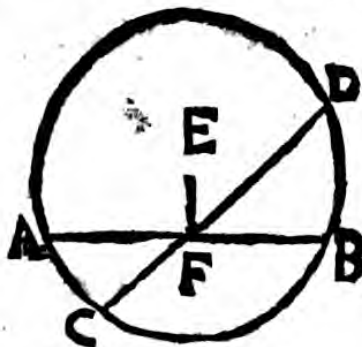
Hyp. 2. Because  $EFA^e = EFC$ , and the angle  $EAF^f = ECF$ , and the side  $EF$  common; therefore is  $AF = FC$ . Therefore  $AC$  is cut into two equal parts. Which was to be Dem.

*e hyp and*  
*12. ax.*  
*f 5. 1.*  
*g 16. 1.*

Coroll.

Hence, In any equilateral or Isoceles triangle, if a line drawn from the verticall angle bisect the base, that line is perpendicular to it. And on the contrary, a perpendicular drawn from the verticall angle bisects the base.

P R O P. IV.



If in a circle  $ACD$  two right lines  $AB, CD$  cut through one another, yet neither of them passe through the centre  $E$ , then neither of those lines are divided into equal parts.

For if one line passe through the

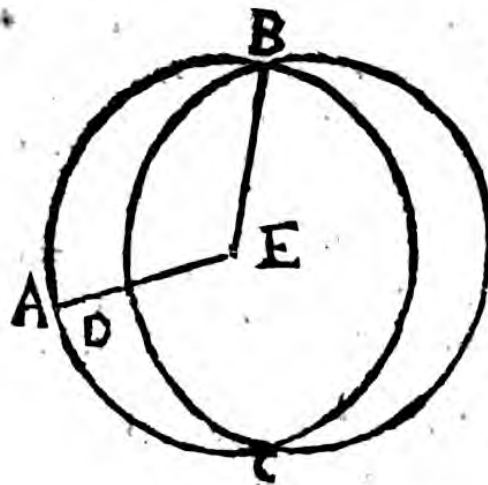
*then both of the lines are not bisected.*

the centre, it appears that it cannot be bisected by the other; because by Hypothesis, the other does not pass through the centre.

If neither of them pass through the centre, then from the centre E draw EF: now if AB, CD were both bisected in F, then  $\angle$  EFB, EFD be both right, and consequently equal. *b* Which is absurd.

a 3.3.  
b 9. ax.

PROP. V.

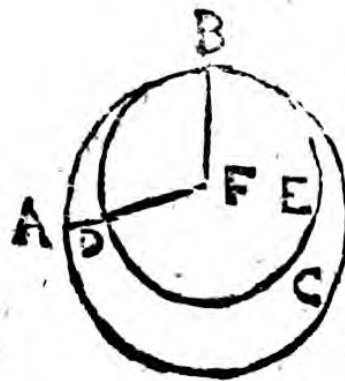


If two circles BAC, BDC cut one the other, they shall not have the same centre E.

For otherwise the lines EB, EDA drawn from E the common centre, would DE be  $\angle$  EFB  $\angle$  EFA. *b* Which is absurd.

a 15. def. 1.  
b 9. ax.

PROP. VI.

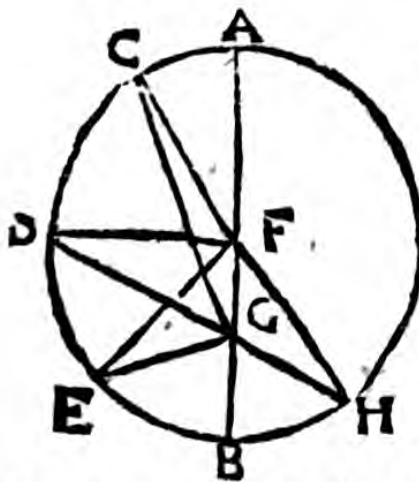


If two circles BAC, BDC, inwardly touch one the other (in B) they have not one and the same centre F.

For otherwise the right lines FB, FDA drawn from the centre F, would be  $\angle$  FDB  $\angle$  FBA. *b* Which is absurd.

a 15. def. 1.  
b 9. ax.

PROP. VII.



If in  $AB$  the diameter of a circle some point  $G$  be taken, which is not the centre of the circle, and from that point certain right lines  $GC, GD, GE$  fall on the circle, the greatest line shall be that ( $GA$ ) in which is the centre  $F$ ; the least, the remainder of the same line ( $GB$ .) And of

all the other lines, the line  $GC$  nearest to that which was drawn through the centre is always greater than any line farther removed  $GD$ ; and only two lines are equal  $GE, GH$ , which fall upon the circle from the same point, on each side of the least  $GB$  or of the greatest  $GA$ .

From the centre  $F$  draw the right lines  $FC, FD, FE$ , and  $a$  make the angle  $BFH = BFE$ .

$a$  23. 1.

1.  $GF + FC$  (that is  $GA$ )  $a$   $\square$   $GC$ . Which was to be Dem.

$a$  20. 1.

2. The side  $FG$  is common, and  $FC b = FD$ , and the angle  $GFC c \square GFD$ ;  $d$  wherefore the base  $GC \square GD$ .

$b$  15. def. 1.

$c$  9. ax.

$d$  24. 1.

3.  $FB (FE) e \supset GE + GF$ . Therefore  $FG$ , which is common, being taken away from both, there remains  $BG \supset EG$ .

$e$  10.

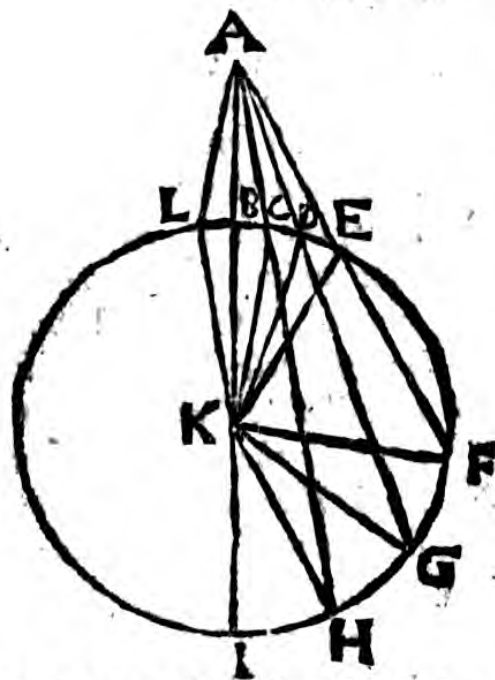
$f$  5. ax.

4. The side  $FG$  is common, and  $FE = FH$ , and the angle  $BFH g = BFE$ ;  $b$  Therefore is  $GE = GH$ . But that no other line  $GD$  from the point  $G$  can be equal to  $GE$ , or  $GH$ , is already proved. Which was to be Demonstrated.

$g$  conft

$h$  4. 1.

The third Book of  
P R O P. V I I I.



If some point A be taken without a circle, and from that point be drawn certain right lines AI, AH, AG, AF to the circle, & of those one AI be drawn through the centre K, and the others anywise; of all those lines that fall on the concave of the circumference, that is the greatest AI which is drawn through the

centre; and of the others, that which is nearest (AH) to the line that passes through the centre is greater than that which is more distant AG. But of all those lines that fall on the convexe part of the circle, the least is that AB which is drawn from the point A to the diameter IB; and of the others, that (AC) which is nearest to the least, is lesse than that which is farther distant AD. And from that point there can be onely two equall right lines AC, AL drawn, which shall fall on the circumference on each side of the least line AB or of the greatest AI.

From the centre K draw the right lines KH, KG, KF, KC, KD, KE. and make the angle AKL = AKC.

b 20. 1.

1.  $AI (AK + KH) = AH.$

b 24. 1.

2. The side AK is common, and  $KH = KG$ , and the angle  $AKH = AKG$ ; therefore the base  $AH = AG.$

c 20. 1.

d 5. ax.

$AB = AC.$

3.  $KA = KC + CA.$  From hence take away KC, KB that are equall; then will remain  $AB = AC.$

e 21. 1.

f 5. ax.

4.  $AC + CK = AD + DK.$  From thence take away CK, DK that are equall; then remains  $AC = AD.$

5. The

5. The side  $KA$  is common, and  $KL = KC$ , and the angle  $AKL = AKC$ ; therefore  $KA = CA$ . <sup>g const. h 4. 1.</sup> But that no other line could be drawn equal to these, was proved above. Therefore, &c.

P R O P. IX.

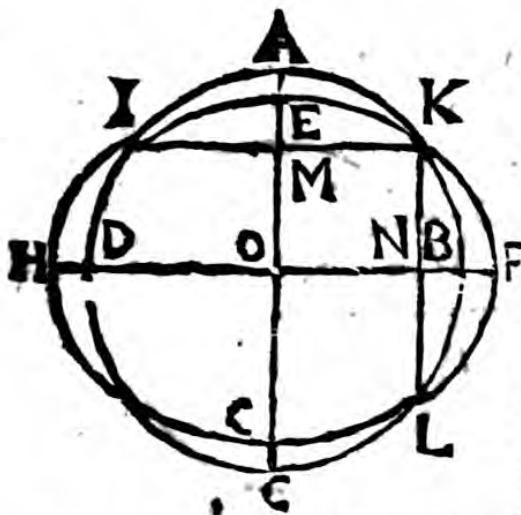


If in a circle  $BCK$  a point  $A$  be taken, and from that point more than two equal right lines  $AB$ ,  $AC$ ,  $AK$ , drawn to the circumference, then is that point  $A$  the centre of the circle.

For a from no point without the centre can more than two right lines equal be drawn to the circumference. <sup>a 7. 3.</sup> Therefore  $A$

is the centre. Which was to be Dem.

P R O P. X.



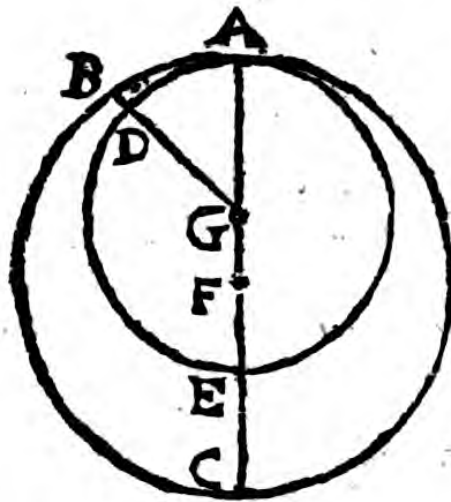
A circle  $IAKBL$  cannot cut another circle  $IEKFL$  in more than two points.

Let one circle, if it may be, cut the other in three points  $I, K, L$ . and  $IK, KL$  being join'd, let them be bisected in  $M$  and  $N$ .

<sup>a cor. 1. 3.</sup> Both circles have their centres in their

perpendiculars  $MC, NH$ , and in the intersection of those perpendiculars which is  $O$ . Therefore the circles that cut each other have the same centre. Which is false, by *Prop. 5. 3.*

PROP. XI.



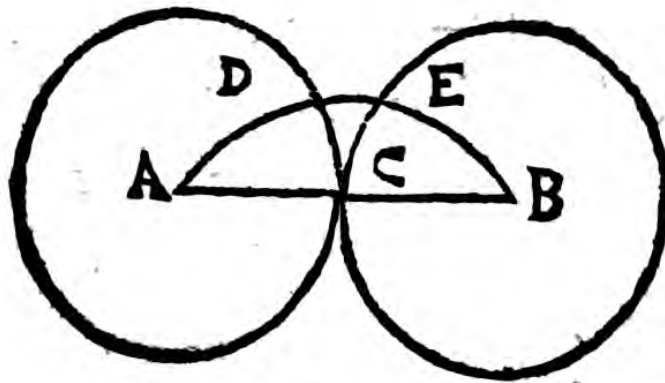
If two circles  $G A-DE$ ,  $F A B C$  touch one the other inwardly, and their centres be taken  $G, F$ ; a right line  $F G$  joyning their centres, and produced, shall cut the circumference in  $A$  the point of contact of the circles.

If it can be, let the right line  $F G$  produced cut the circles in some other point then  $A$ . so that not  $F G A$ , but  $F G D B$  shall be a right line. Let the line  $G A$  be drawn. Now, because  $G D = G A$ , &  $G B \perp G A$  (since the right line  $F G B$  passes through  $F$  the centre of the greater circle) therefore is  $G B \perp G D$ . *c Which is absurd.*

a 15. def 1.  
b 7. 3.

c 9. ex. 1

PROP. XII.



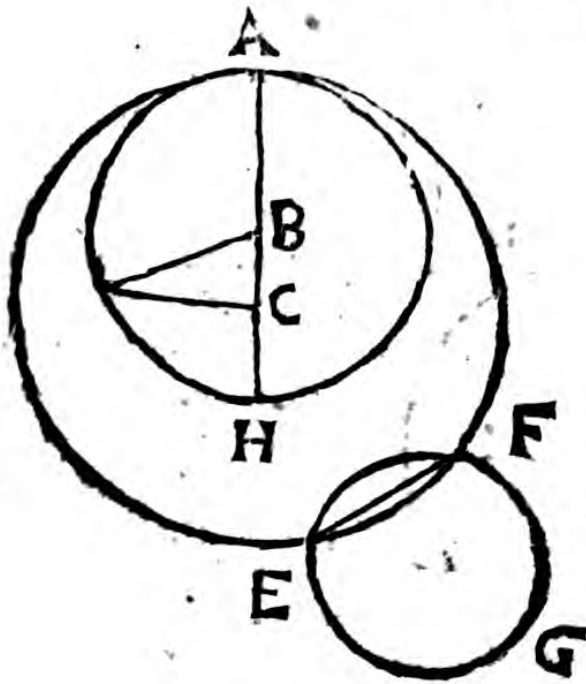
If two circles  $A C D$ ,  $B C E$  touch one the other outwardly, the right line  $A B$  which joins their centres  $A, B$ , shall passe through the point of contact  $C$ .

If it may be, let  $A D E B$  be a right line cutting the circles not in the point of contact  $C$ , but in the points  $D, E$ ; draw  $A C, C B$ . then is  $A D + E B (A C + C B) < A D E B$ . *b Which is absurd.*

a 20. 1.  
b 9. ex.

PROP.

P R O P. XIII.



A circle CAF cannot touch a circle BAH in more points than one A, whether it be inwardly or outwardly.

1. Let one circle (if it can be) touch another in two points A, H. Then will the right line CB that joins the centres, if

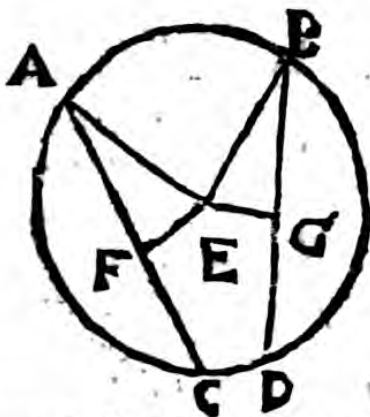
it be produced, fall as well in A, as H. Now because  $CH = CA$ , and  $BH = CH$ , therefore is  $BA = BH = CA$ . Which is absurd.

b 15. def. 1.  
c 15. def. 1.  
d 9. ax.

2. If it be said to touch outwardly in the points E and F, then draw the line EF, which will be in both circles. Therefore those circles cut one the other; Which is against the Hyp.

e 2. 3.

P R O P. XIV.



In a circle EABC equal right lines AC, BD are equally distant from the centre E: and right lines AC, BD which are equally distant from the centre, are equal among themselves.

From the centre E draw the perpendiculars EF, EG. which will bisect the lines AC, BD. join EA, EB.

a 3. 3.

1. Hyp.  $AC = BD$ . therefore  $AF = BG$ . But also

b 7. ax.  
so



c 47. 1. and 3. ax. so  $EA = EB$ . therefore  $FEq c = EAq - AFq = EBq - BGq c = EGq$ . d Therefore  $FE = EG$ .  
 d schol 48. 1. 2. Hyp.  $EF = EG$ . Therefore  $AFq e = EAq - EFq = EBq - EGq = BGq$ . Therefore  $AF d = GB$ , and  
 e 6. ax. e consequently  $AC = BD$ . Which was to be Dem.

PROP. XV.



In a circle  $GABC$  the greatest line is  $AD$  the diameter; and of all other lines, that line  $FE$ , which is nearest to the centre  $G$  is greater than any line  $BC$  farther distant from it.

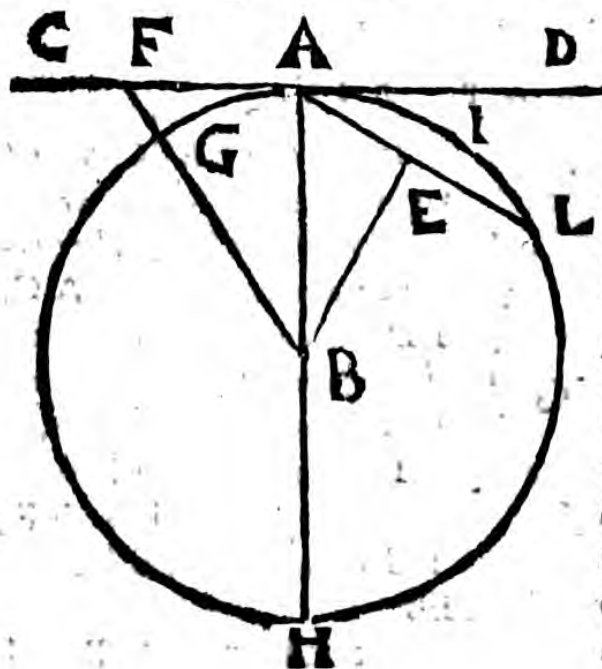
1. Draw  $GB$ , and  $GC$ . The diameter  $AD$  ( $a GB + GC$ )  $b \perp BC$ .

2. Let the distance  $GI$  be  $\perp GH$ . Take  $GN = GH$ . Through the point  $N$  draw  $KL$  perpendicularly to  $GI$ ; join  $GK, GL$ . Because  $GK = GB$ , and  $GL = GC$ , and the angle  $KGL \perp BGC$ ; c therefore is  $KL (FE) \perp BC$ . Which was to be Dem.

a 15. def. 1.  
b 20. 1.

c 14. 1.

PROP. XVI.



A line  $CD$  drawn from the extreme point of the diameter  $HA$  of a circle  $BALH$ , perpendicular to the said diameter, shall fall without the circle; and between the same right line and the circumference cannot be drawn another line  $AL$ . And the angle of the

the semicircle  $BAI$ , is greater then any right-lined acute angle  $BAL$ ; and the remaining angle without the circumference  $DAI$  is lesse then any right-lined angle.

1. From the centre  $B$  to any point  $F$  in the right line  $AC$ , draw the right line  $BF$ . The side  $BF$  subtending the right angle  $B A F$  is <sup>a</sup> greater then the side  $BA$  which is opposite to the acute angle  $BFA$ . Therefore whereas  $BA$  ( $BG$ ) reaches to the circumference,  $BF$  shall reach further; and so the point  $F$ , and for the same reason is any other point of the line  $AC$  placed without the circle.

2. Draw  $BE$  perpendicular to  $AL$ . The side  $BA$  opposite to the right angle  $BEA$  is <sup>b</sup> greater then the side  $BE$  which subtends the acute angle  $BAE$ ; therefore the point  $E$ , and so the whole line  $EA$ , falls within the circle.

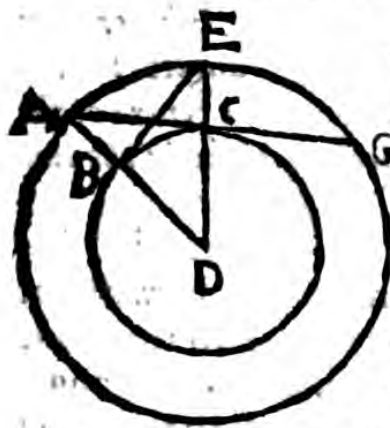
3. Hence it follows that any acute angle, to wit,  $EAD$ , is greater then the angle of contact  $DAI$ , and that any acute angle  $BAL$  is lesse then the angle of a semicircle  $BAI$ . Which was to be Dem.

Coroll.

Hence, A right line drawn from the extremity of the diameter of a circle, and at right angles, is a tangent to the said circle.

From this proposition are gathered many paradox and wonderfull consecratives, which you may meet with in the interpreters.

P R O P. XVII.



From a point given  $A$  to draw a right line  $AG$  which shall touch a circle given  $DBC$ .

From  $D$  the centre of the circle given let a line  $DA$  cutting the circumference in  $B$ , be drawn to the point given  $A$ ; from the centre  $D$  describe another circle through the point  $A$ ; and from  $B$  draw a perpendicular to  $AD$ , which shall meet with the circle

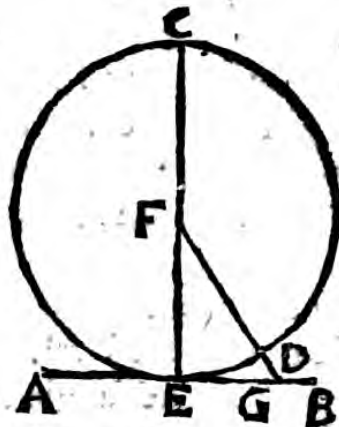
circle AE in the point E; and draw ED meeting with the circle BC in the point C. Then the line drawn from A to C shall touch the circle DBC.

a 15 def. 1.  
b 4. 1.

c cor. 16. 3.

For  $DB^a = DC$ , and  $DE^a = DA$ , and the angle D is common; *b* therefore the angle  $ACD = EBD$  and right. *c* Therefore AC touches the circle in C. Which was to be Done.

PROP. XVIII.



If any right line AB touch a circle FEDC, and from the centre to the point of contact E a right line FE be drawn; that line FE shall be perpendicular to the tangent AB.

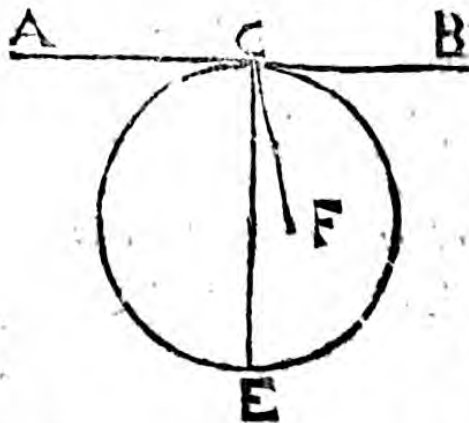
If you deny it, let some other line FG be drawn from the centre F perpendicular to the tangent, and *a* cutting the circle in D.

Therefore, whereas the angle FGE is said to be right, *b* thence is the angle FEG acute; *c* so that FE (FD)  $\sphericalangle$  FG. *d* Which is absurd.

a 1. def. 3.

b cor. 17. 1.  
c 19. 1.  
d 9. ax. 1

PROP. XIX.



If any right line AB touch a circle, and from the point of contact C a right line CE be erected at right angles to the tangent, the centre of the circle shall be in the line CE so erected.

If you deny it, let the centre be without

the line CE in the point F; and from F to the point of contact let FC be drawn. Therefore the angle FCB is right, and *a* consequently equal to the angle ECB, which was right by Hypothesis. *b* Which is absurd.

a 12. ax.  
b 9. ax.

PROP.

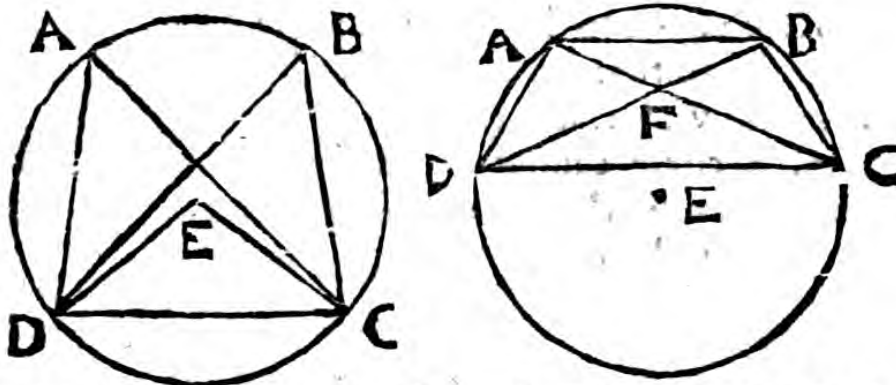
PROP. XX.



In a circle  $DABC$ , the angle  $BDC$  at the centre is double of the angle  $BAC$  at the circumference, when the same arch of the circle  $BC$  is the base of the angles.

Draw the diameter  $ADE$ . The outward angle  $BDE$   $a = DAB + DBA$   $b = 2 DAB$ . Likewise the angle  $EDC = 2 DAC$ . Therefore in the first case the whole angle  $BDC = 2 BAC$ . and in the third case the  $c$  remaining angle  $BDC = 2 BAC$ . Which  $c 10. ax.$  was to be Dem.

PROP. XXI.



In a circle  $EDAC$ , the angles  $DAC$  and  $DBC$  which are in the same segment, are equal one to the other.

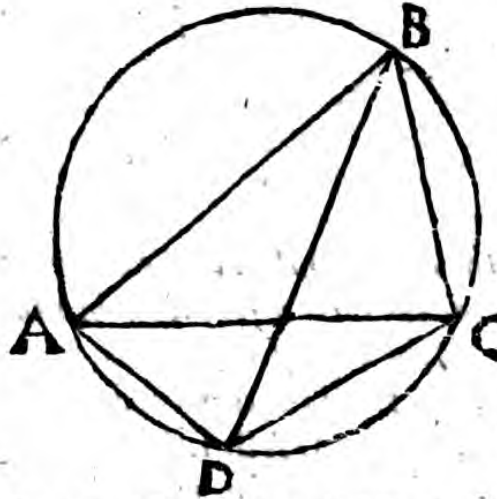
1. Case. If the segment  $DABC$  be greater than a semicircle, from the centre  $E$  draw  $ED, EC$ . Then is twice the angle  $A$   $a = E a = 2 B$ . *W. W. to be Dem.*

2. Case. If the segment be less than a semicircle, then is the sum of the angles of the triangle  $ADF$  equal to the sum of the angles of the triangle  $BCF$ . from each let  $AFD$  be taken away  $b$  equal to  $b 15. 1.$   $BFC,$

e by the 1. case.

BFC, and ADB  $\hat{=}$  ACB be likewise taken away; then remains DAC  $\hat{=}$  DBC. *W.W. to be Dem.*

P R O P. XXII.



The angles ADC, ABC of a quadrilateral figure ABCD described in a circle, which are opposite one to the other, are equal to two right angles.

Draw AC, BD. The angle ABC + BCA + BAC  $\hat{=}$  2 right. But BDA  $\hat{=}$  BCA, and BDC  $\hat{=}$

a 32. 1.  
b 21. 3.

c 1. ax.

BAC.  $\hat{c}$  Therefore ABC + ADC  $\hat{=}$  2 right angles. Which was to be Dem.

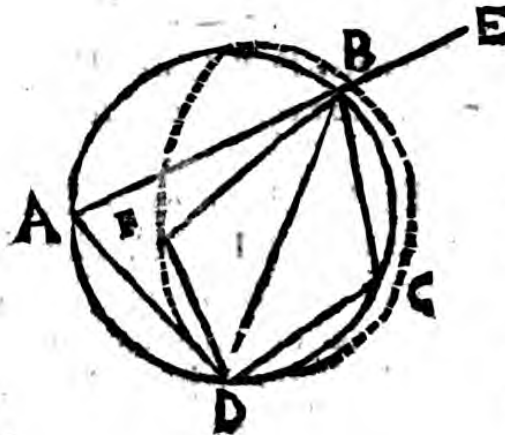
Coroll.

\* See the following Diagr.

1. Hence, If one side \* AB of a quadrilateral described in a circle be produced, the external angle EBC is equal to the internal angle ADC, which is opposite to that ABC which is adjacent to EBC. as appears by 13. 1. and 3. ax.

2. A circle cannot be described about a Rhombus; because it's opposite angles are greater or less than two right angles.

Schol.



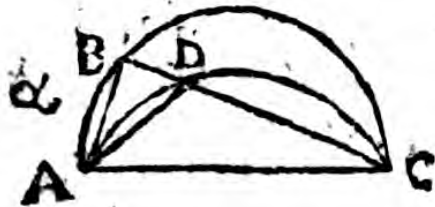
If in a quadrilateral ABCD the angles A and C, which are opposite, be equal to two right, then a circle may be described about that quadrilateral.

For a circle will pass through any 3 angles (as shall appear by 5. 4.) I say that shall pass through A the fourth also of such

a quadrilaterall: For if you deny it, let the circle passe through F. Therefore the right lines BF, FD, BD being drawn, the angle  $C + F = 2 \text{ right} = C + A$  wherefore,  $A = F$ . *Which is absurd.*

a 22. 3.  
b 27.  
c 3. ex.  
d 21. 1.

PROP. XXIII.



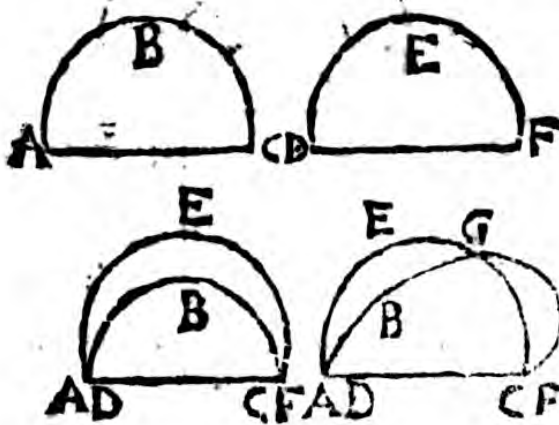
Two like and unequal segments of circles  $AB$ ,  $C$ ,  $ADC$ , cannot be set on the same right line  $AC$ , and the same side thereof.

thereof.

For if they are said to be like, draw the line  $CB$  cutting the circumferences in  $D$  and  $B$ ; join  $AB$  and  $AD$ . Because the segments are supposed like, therefore is the angle  $ADC = ABC$ . *Which is absurd.*

a 10. def. 3.  
b 16. 1.

PROP. XXIV.



Like segments of circles  $ABC$ ,  $DEF$  upon equal right lines  $AC$ ,  $DF$ , are equal one to the other.

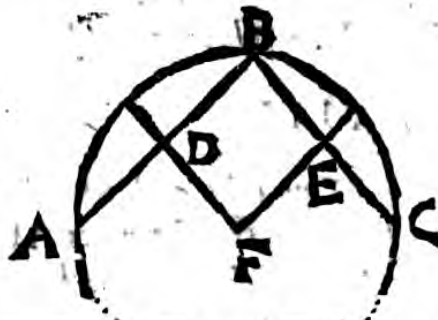
The base  $AC$  being laid on the base  $DF$  will agree with it, because  $AC = DF$ . Therefore the segment  $ABC$

shall agree with the segment  $DEF$  (for otherwise it shall fall either within or without, and if so then the segments are not like, which is contrary to the Hypothesis, and at least it shall fall partly within and partly without, and so cut in 3 points, which is absurd. *c* Therefore the segment  $ABC = DEF$ . *Which was to be Dem.*

a 13. 3.

b 10. 3.  
c 8. ex.

P R O P. XXV.



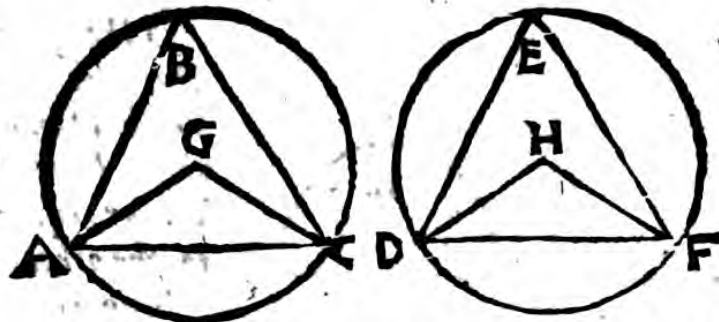
A segment of a circle ABC being given, to describe the whole circle whereof that is a segment.

Let two right lines be drawn AB, BC, which bisect in the points D and E. From D and E draw the perpendiculars DF, EF meeting in the point F. I say this point shall be the centre of the circle.

a cor. 1. 3.

For the centre shall be as well in a DF as; EF, therefore it must be in the common point F. Which was to be Done.

P R O P. XXVI.



In equall circles GABC, HDEF, equall angles stand upon equall parts of the circumference, AC, DF; whether those angles be made at the centres, G, H, or at the circumferences, B, E.

a 4. 1.  
b 10. 3.  
c h. p.  
d 10. def. 3.  
e 24. 3.  
f 3. ex.

Because the circles are equall, therefore is GA = HD, and GC = HF; also by Hypothesis the angle G = H; a therefore AC = DF. Moreover the angle B = E. b Therefore the segments ABC, DEF are like, and c consequently equall. d whence the remaining segments also AC, DF are equall. Which was to be Dem.

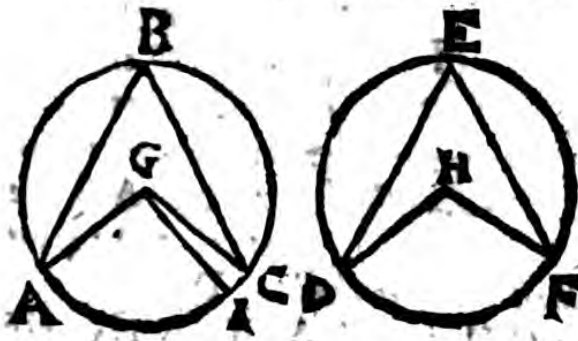
Schol.

Schol.



In a circle ABCD let an arch AB be equal to DC; then shall AD be parallel to BC. For the right line AC being drawn, the angle  $ACB^a = CAD$ ; wherefore by 27. 1. the said sides are parallel. a 16. 3.

P R O P. XXVII.

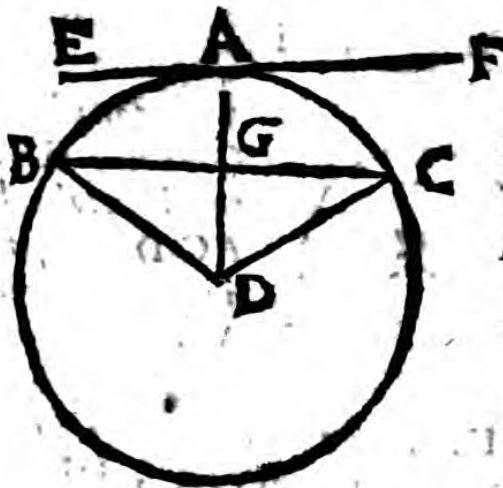


In equall circles GABC, HD-  
EF, the angles standing upon equall parts of the circumference A-  
C, DF, are equall between themselves, whether

they be made at the centres G, H, or at the circumferences, B, E.

For if it be possible, let one of the angles AGC be  $\neq$  DHF, and make  $AGI = DFH$ . thence is the arch AI  $^a = DF^b = AC$ . *c Which is absurd.* a 16. 3.  
b by p.  
c 9. ax.

Schol.



A right line EF, which being drawn from A the middle point of any periphery BC, toucheth the circle, is parallel to the right line BC subtending the said periphery.

From the centre D draw a right line DA to the point of contact A, and

join DB, DC.

The side DG is common, and  $DB = DC$ , and the angle BDA  $^a = CDA$ , (because the arches BA, a 27. 3

E 2

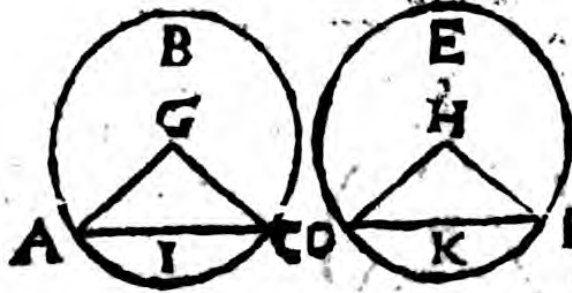
CA



b hyp.  
c 4 1.  
d 10. def. 1.  
e hyp.  
f 28. 1.

CA are *b* equal) therefore the angles at the base DGB, DGC are *c* equal, and *d* consequently right; But the inward angles GAE, GAF are also *e* right, *f* therefore BC, EF are parallel. Which was to be Dem.

P R O P. XXVIII.







In equal circles GABC, HDEF, equal right lines AC, DF cut off equal parts of the circumference, the greatest ABC equal to the great-

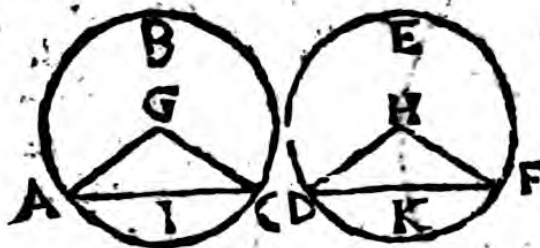
est DEF, and the least AIC to the least DKF.

a hyp.  
b 28 1.  
c 6. 3.  
d 1. cor.

From the centres G, H draw GA, GC, & HD, HF. Because GA = HD, and GC = HF, and AC = DF, *b* therefore is the angle G = H; *c* whence the arch AIC = DKF; *d* and so the remaining arch ABC = DEF. Which was to be Dem.

But if the subtended line AC be  or  then DF, then in like manner will the arch AC be  or  then DF.

P R O P. XXIX.



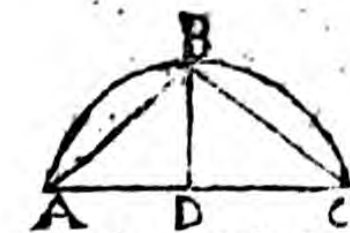
In equal circles GABC, HDEF, equal right lines AC, DF subtend equal peripheries ABC, DEF.

a hyp.  
b 27 3.  
c 4. 1.

Draw the lines GA, GC, and HD, HF. Because GA = HD, and GC = HF, and (because the arches AC, DF are *a* equal) the angle G = H. *c* therefore is the base AC = DF. Which was to be Dem.

This and the three precedent propositions may be understood also of the same circle.

PROP. XXX.



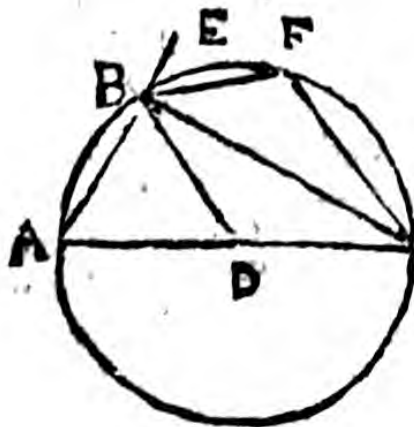
To cut a periphery given ABC into two equal parts.

Draw the right line AC, & bisect it in D; from D draw a perpendicular DB, meeting with the arch in B, it shall bisect the same.

For join AB, and CB. The side DB is common, and AD = DC, and the angle ADB = CDB. therefore AB = BC; whence the arch AB = BC. Which was to be Done.

a conf.  
b 12 ex.  
c 4. 1.  
d 28 3.

PROP. XXXI.



In a circle the angle ABC, which is in the semi-circle, is a right angle; but the angle  $\angle BAC$ , which is in the greater segment BAC is less than a right angle, and the angle which is in the lesser segment BFC is greater than a right angle. Moreover, the angle of the greater segment is greater

than a right angle, and the angle of the lesser segment is less than a right angle.

From the centre D draw DB. Because DB = DA, therefore is the angle A = DBA, and the angle DCB = DBC, therefore the angle  $\angle ABC = \angle A + \angle ACB = \angle EBC$ , so that ABC and EBC are right angles. W. W. to be Dem. Therefore BAC is an acute angle. f. W. W. to be Dem. And further, whereas  $\angle BAC + \angle BFC = 2$  right, therefore BFC is an obtuse angle. Lastly the angle contained under the right line CB, and the arch BAC is greater than the right angle ABC; but the angle, made by the right line CB and the periphery of the lesser segment BFC is less than the right angle ABC. Which was to be Dem.

a 5. 1.  
b 2. ex.  
c 32. 1.  
d 10. def. 1.  $\angle ABC$   
e cor. 17. 1.

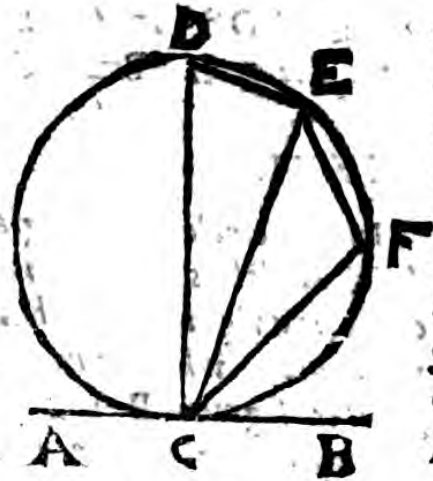
f 28. 3.  
; because the 2 of angles in quadrilateral fig: insc: in circle are = to one an

g 9. ar.

In a right-angled triangle  $ABC$ , if the hypotenuse (or subtended line)  $AC$  be bisected in  $D$ , a circle drawn from the centre  $D$  through the point  $A$  shall also passe through the point  $B$ : As you may easily demonstrate from this prop. and 21. 1.

P R O P. XXXII.

If a right line  $AB$  touch a circle, and from the point of contact be drawn a right line  $CE$  cutting the circle, the angles  $ECB, ECA$  which it makes with the tangent line are equal to those angles  $EDC, EFC$  which are made in the alternate segments of the circle.



Let  $CD$ , the side of

the angle  $EDC$ , be perpendicular to  $AB$  (a for it's to the same purpose) b therefore  $CD$  is the diameter. c therefore the angle  $CED$  in a semicircle is a right angle, d and therefore the angle  $D + DCE =$  to a right angle e  $= ECB + DCE$ . f Therefore the angle  $D = ECB$ . Which was to be Dem.

Now whereas the angle  $ECB + ECA$  g  $= 2$  right b  $= D + F$ , from both of these take away  $ECB$  and  $D$ , which are equal, k then remains  $ECA = F$ . Which was to be Dem.

P R O P. XXXIII.

Vpon a right line  $AB$  to describe a segment of a circle  $AIEB$  which shall contain an angle  $AIB$  equal to a right-lined angle given  $C$ .



a Make the angle

a 16 3.  
b 19. 3.  
c 31. 3.  
d 32. 1.  
e confr.  
f 3 ax.

g 13. 1.  
h 12 3.  
k 3. ax.

a 23. 1.

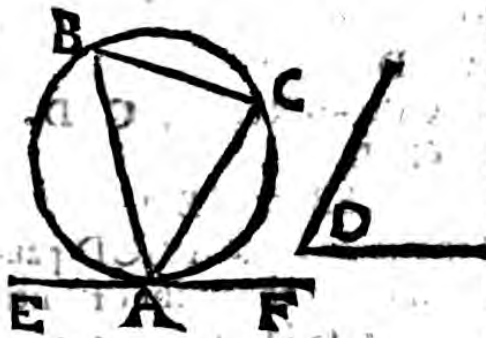
angle  $BAD = C$ . Through the point  $A$  draw the line  $AE$  perpendicular to  $HD$ . At one end of the line given  $AB$  make an angle  $ABF = BAF$ , one side whereof let it cut the line  $AE$  in  $F$ ; from the centre  $F$  through the point  $A$ , describe a circle which shall passe through  $B$  (because the angles  $FBA = FAB$ , and therefore  $FB = FA$ .)  $AIB$  is the segment sought. For because  $HD$  is perpendicular to the diameter  $AE$ , it therefore touches the circle  $HD$  which  $AB$  cuts. And therefore the angle  $AIB = BAD = C$ . Which was to be done.

b constr. c 6. 1.

d cor 16. 3. e 32. 3. f constr.

PROP. XXXIV.

From a circle given  $ABC$  to cut off a segment  $ABC$  containing an angle  $B$  equal to a right-lined angle given  $D$ .



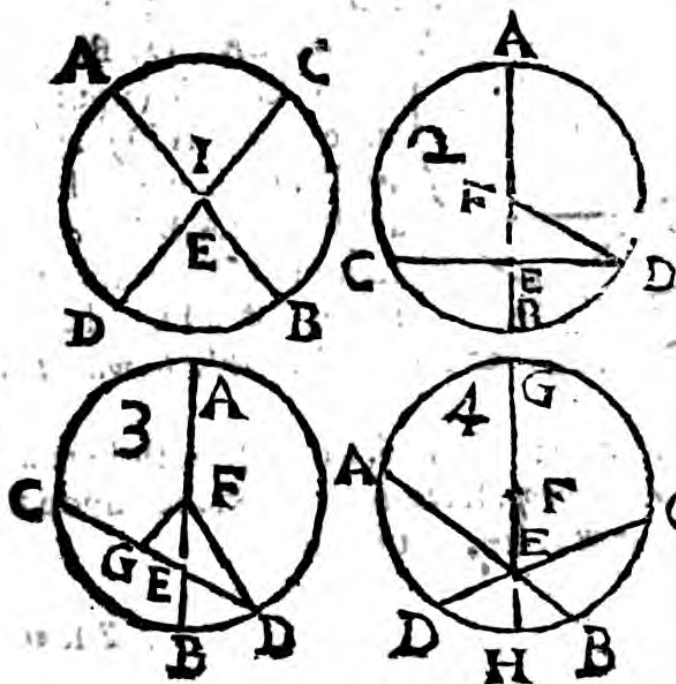
Draw a right line  $EF$  which shall touch the circle given in  $A$ .

Let  $AC$  be drawn also making an angle  $FAC = D$ . This line shall cut off  $ABC$  containing an angle  $B = CAF = D$ . Which was to be Done.

a 17. 3. b 23. 1. c 32. 3. d constr.

PROP. XXXV.

If in a circle  $FBCA$  two right lines  $AB, DC$  cut each other, the rectangle comprehended under the segments  $AE, EB$  of the



circ.

The third Book of

one, shall be equal to the rectangle comprehended under the segments **CE, ED** of the other.

1. *Case.* If the right lines cut one the other in the centre, the thing is evident.

2. *Case.* If one line **AB** passe through the centre **F**, and bisect the other line **CD**, then draw **FD**. Now the rectangle **AEB + FEq**  $\doteq$  **FBq**  $\doteq$  **FDq**  $\doteq$  **EDq + FEq**  $\doteq$  **CED + FEq**. Therefore the rectangle **AEB = CED**. Which was to be Dem.

3. *Case.* If one of the lines **AB** be the diameter, and cut the other line **CD** unequally, bisect **CD** by **FG** a perpendicular from the centre.

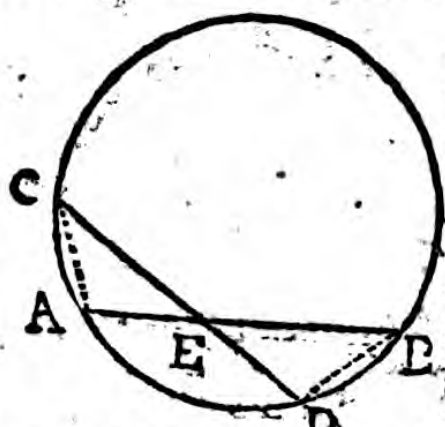
f. 2.  
b 5b. 48. 1.  
c 47. 1.  
d 47.  
e 3. ex.

All These are equal.  $\left\{ \begin{array}{l} \text{The rectangle } AEB + FEq. \\ f \text{ FBq (FDq.)} \\ g \text{ FGq} + GDq. \\ h \text{ FGq} + b \text{ GEq} + \text{Rectang. CED,} \\ k \text{ FEq} + CED. \end{array} \right.$

f 5. 2.  
g 47. 1.  
h 5. 2.  
k 47. 1.

l 3. ex.

Therefore the rectangle **AEB = CED**.  
4. *Case.* If neither of the right lines **AB, CD** passe through the centre, then through the point of intersection **E**, draw the diameter **GH**. By that which hath been already demonstrated, it appears that the rectangle **AEB = GEH = CED**. *W.W. to be Dem.*



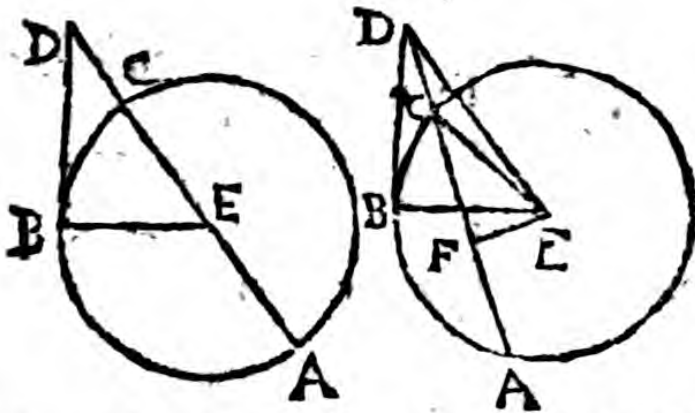
More easily, and generally, thus; Join **AC** and **BD**, Then because the angles  $\sphericalangle$  **CEA, DEB**, and  $\sphericalangle$  **C, B** (upon the same arch **AD**) are equal, thence are the triangles **CEA, BED** equiangular. Therefore **CE, EA :: EB, ED**, and consequently

a 15. 1.  
b 21. 3.  
c cor. 32. 1.  
d 4. 6.  
e 16. 6.

**CE x ED = AE x EB**. Which was to be Dem.

The citations out of the 6. Book, both here and in the following *prop.* have no dependance upon the same; so that it was free to use them.

PROP. XXXVI.



If any point *D* be taken without a circle *EBC*, and from that point two right lines *DA*, *DB* fall upon the circle, whereof one *DA* cut the circle, the other *DB* touches it, the rectangle comprehended under the whole line *DA* that cuts the circle, and under *DC* that part which is taken from the point given *D* to the convexe of the periphery, shall be equal to the square made of the tangent line.

1. *Case.* If the secant *AD* passe through the centre, then join *EB*, this *a* will make a right angle with the line *DB*, wherefore  $DBq + EBq (ECq) b = EDq$   
 $c = AD \times DC + ECq$ . Therefore  $AD \times DC = DBq$ . Which was to be Dem.

a 18. 3.  
 b 47. 1.  
 c 6. 2.  
 d 3. ex.

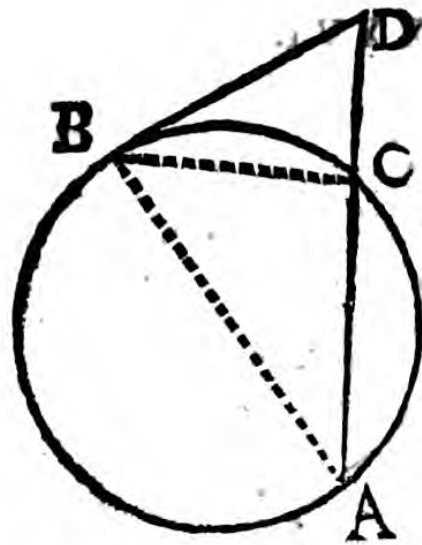
2. *Case.* But if *AD* passe not through the centre then draw *EC*, *EB*, *ED*, and *EF* perpendicular to *AD*, wherefore *AC* is bisected in *F*.

Because  $BDq + EBq b = DEq b = EFq + FDq c = EFq + ADC + FCq d = ADC + CEq (EBq) e$ . Therefore is  $BDq = ADC$ . Which was to be Dem.

a 3. 3.  
 b 47. 1.  
 c 6. 2.  
 d 47. 1.  
 e 3. ex.

a 31. 3.

b 31. 1.  
c 4 6  
d 17. 6.



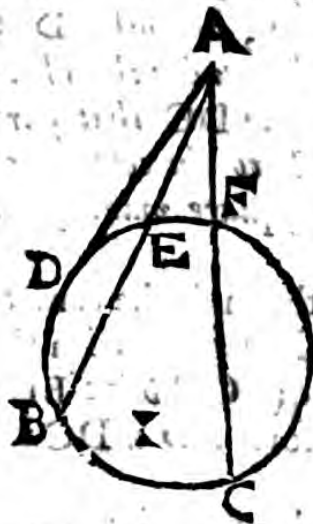
More easily, and generally, thus; Draw  $AB$  and  $BC$ . Then because the angles  $A$ , and  $DBC$  <sup>a</sup> are equal, and the angle  $D$  common to both, thence are the triangles  $BDC$ ,  $\sphericalangle$   $DB$  <sup>b</sup> equiangular. <sup>c</sup> Wherefore  $AD \cdot DB :: DB \cdot CD$ . and <sup>d</sup> consequently  $AD \times DC = DB^2$ . Which was to be Dem.

Coroll.

1. Hence, If from any point  $A$  taken without a circle, there be severall lines  $AB, AC$  drawn which cut the circle, the rectangles comprehended under the whole lines  $AB, AC$ , and the outward parts  $AE, AF$  are equal between themselves.

For if the tangent  $AD$  be drawn, then is  $CAF = AD^2 = BAE$ .

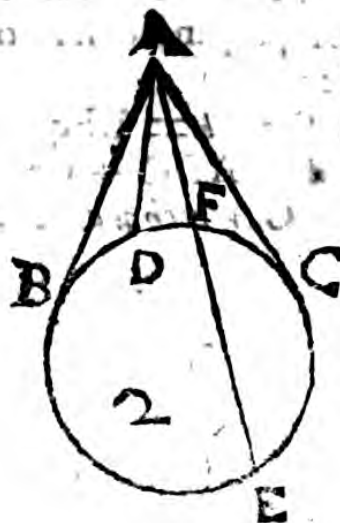
a 36. 1.



2. It appears also from hence, that if two lines  $AB, AC$  drawn from the same point do touch a circle, those two lines are equal one to the other.

For if  $AE$  be drawn cutting the circle, then is  $AB^2 = EAF = AC^2$ .

a 36. 1.  
b 36. 30



3. It

3. It is also evident that from a point A taken without a circle, there can be drawn but two lines AB, AC that shall touch the circle.

For if a third line AD be said to touch the circle, thence is  $AD \text{ c} = AB \text{ c} = AC$ . *d* which is absurd.

c 2. cor.  
d 8. 3.

4. And on the contrary, it is plain, that if two equal right lines AB, AC fall from any point A upon the convex periphery of a circle, and that if one of these equal lines AB touch the circle, then the other AC touches the circle also.

For if possible, let not AC, but another line AD, touch the circle; therefore is  $AD \text{ e} = AC \text{ f} = AB$ . *g* which is absurd.

e 2. cor.  
f hyp.  
g 8. 3.

PROP. XXXVII



If without a circle EBF any point D be taken, and from that point two right lines DA, DB fall on the circle, whereof one line DA cuts the circle, the other DB falls upon it; and if also the rectangle comprehended under the whole line that cuts the circle, and under that part of it DC which is taken betwixt the point D and the convex periphery, be equal to that square

which is made of the line DB falling on the circle, I say that that line DB so falling shall touch the circle given.

From the point D let a tangent DF be drawn, and from the centre E draw ED, EB, EF. Now because  $DB \text{ b} = AD \text{ c} = DF \text{ d}$ , therefore is  $DB \text{ a} = DF$ : But  $EB = EF$ , and the side ED common; therefore the angle  $EBD = EFD$ . but EFD is a right angle, and therefore EBD is right also. and therefore DB touches the circle. *W. W. to be Dem.*

a 17. 3.  
b hyp.  
c 36. 3.  
d 1. ax. &  
sel. 41. 1.  
e 8. 1.  
f 12. ax.  
g cor. 16. 3.

Coroll.

From hence it follows that the angle  $EDB = BDF$ .

The End of the third Book.


THE

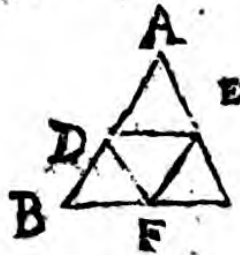


THE FOURTH BOOK  
OF  
EUCLIDE'S ELEMENTS.

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*Definitions.*

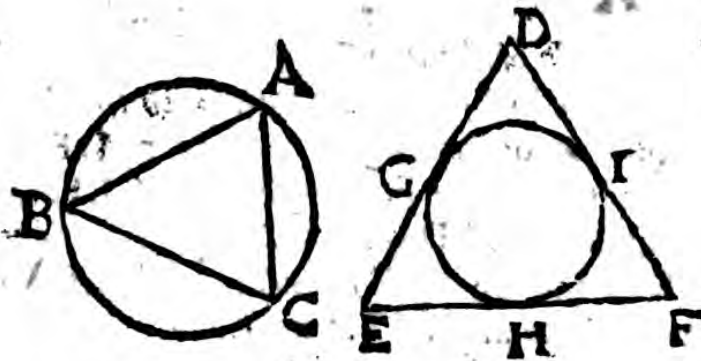
**I.**  Right-lined figure is said to be inscribed in a right-lined figure, when every one of the angles of the inscribed figure touch every one of the sides of the figure wherein it is inscribed.



*So the triangle DEF is inscribed in the triangle ABC.*

**II.** In like manner a figure is said to be described about a figure, when every one of the sides of the figure circumscribed touch every one of the angles of the figure about which it is circumscribed.

*So the triangle ABC is described about the triangle DEF.*



**III.** A right-lined figure is said to be inscribed in a circle, when all the angles of that figure which is inscribed do touch the circumference of the circle.

**IV.** A right-lined figure is said to be described about a circle, when all the sides of the figure which

which is circumscribed touch the periphery of the circle.

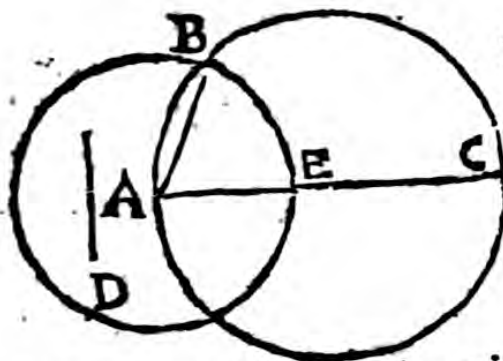
V. After the like manner a circle is said to be inscribed in a right-lined figure, when the periphery of the circle touches all the sides of the figure in which it is inscribed.

VI. A circle is said to be described about a figure when the periphery of the circle touches all the angles of the figure, which it circumscribes.



VII. A right line is said to be coapted or applied in a circle when the extremes thereof fall upon the circumference; as the right line AB.

PROF. I. Probl. 1.



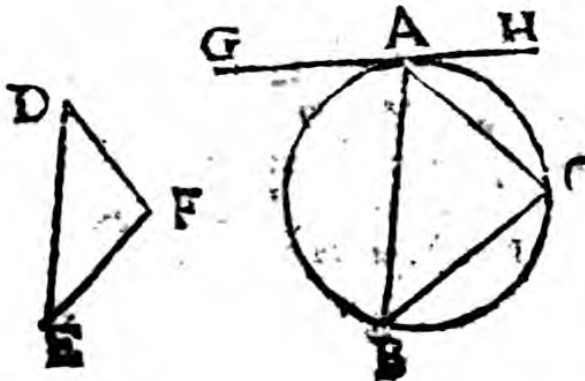
In a circle given ABC to apply a right line AB equall to a right line given D, which doth not exceed AC the diameter of the circle.

From the centre A by the space  $AE = D$

describe a circle meeting with the circle given in B. draw AB. Then is  $AB = AE = D$ . W. W. to be Done.

a 370<sup>th</sup>. and 3. 1.  
b 19. def. 1.  
c const.

PROF. II. Probl. 2.



In a circle given ABC to describe a triangle ABC, equiangular to a triangle given DEF.

Let the right

The fourth Book of

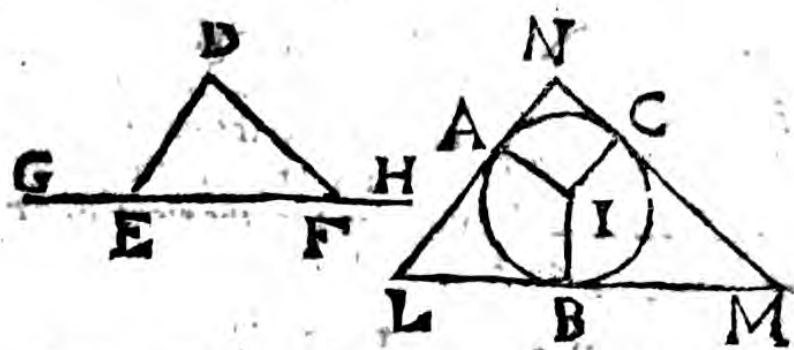
a 17. 3.  
b 23. 1.

c 32. 3.  
d *confr.*  
e 32. 1.

right line GH <sup>a</sup> touch the circle given in A; <sup>b</sup> make the angle HAC = E, and the angle GAB = F. then join BC; and the thing is done.

For the angle B <sup>c</sup> = HAC <sup>d</sup> = E, and the angle C <sup>e</sup> = GAB <sup>d</sup> = F; whence also the angle BAC = D. Therefore the triangle BAC inscribed in the circle is equiangular to DEF. *Which was to be done.*

P R O P. III.



About a circle given I A B C to describe a triangle LNM equiangular to a triangle given DEF.

a 23. 1.

b 17. 3.

Produce the side EF on both sides; at the centre I <sup>a</sup> make an angle AIB = DEG, and an angle BIC = DFH. Then in the points A, B, C let three right lines LN, LM, NM <sup>b</sup> touch the circle, and the thing is done.

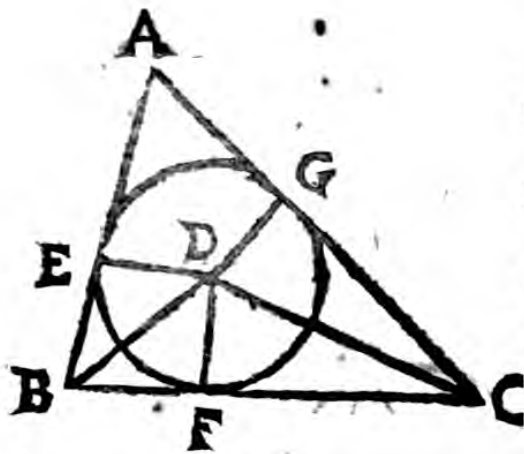
c 13. 2.  
d 11. 3.

For it's evident that the right lines LN, LM, MN will meet and make a triangle, <sup>c</sup> because the angles LAI, LBI are right; so that the <sup>d</sup> right line AB produced will make the angles LAB, LBA, lesse then 2 right angles.

e *schol.* 32. 1.  
f 13. 1.  
g *confr.*  
h 3. ax.  
k 32. 1.

Since, therefore the angle AIB + L <sup>e</sup> = 2 right angles <sup>f</sup> = DEG + DEF, and AIB <sup>g</sup> = DEG; <sup>b</sup> therefore is the angle L = DEF. By the like way of argument the angle M = DEF. <sup>k</sup> therefore also the angle N = D. And therefore the triangle LNM described about the circle is equiangular to EDF the triangle given. *Which was to be done.*

PROP. IV.



In a triangle given ABC, to describe a circle EFG.

a Bisect the angles s 9. 1.  
 B and C with the  
 right lines BD, CD  
 meeting in the point  
 D, <sup>b</sup> & draw the per- b 12. 1.  
 pendiculars DE,  
 DF, DG. A circle de-  
 scribed from the cen-  
 tre D through E, will pass through G and F, and  
 touch the three sides of the triangle.

For the angle DBE <sup>c</sup> = DBF; and the angle c const.  
 DEB <sup>d</sup> = DFB; and the side DB common. <sup>d 12. ax.</sup> e there- e 16. 1.  
 fore DE = DF. By the like argument DG = DF.  
 The circle therefore described from the centre D  
 passes through the 3 points E, F, G. and whereas the  
 angles at E, F, G are right, therefore it touches all  
 the sides of the triangle. Which was to be done.

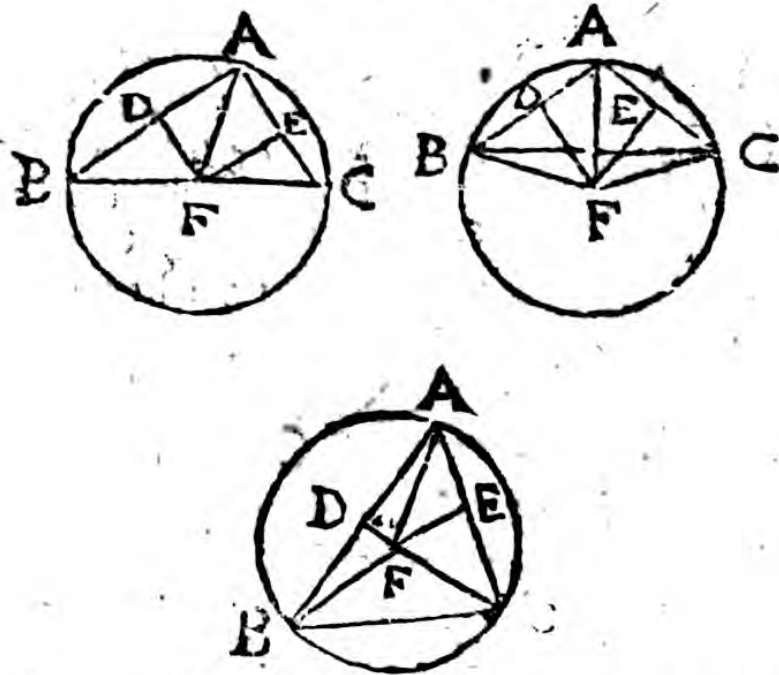
Schol.

Hence, The sides of a triangle being known, their Prop. Metric.  
 segments which are made by the touchings of the circle  
 inscribed shall be found, Thus;

Let AB be 12, AC 18, BC 16. then is AB + BC  
 = 28. Out of which subduct 18 = AC = AE +  
 FC, then remains 10 = BE + BF. Therefore BE,  
 or BF = 5; and consequently FC, or CG, = 11.  
 Wherefore GA, or AE, = 7.

PROP.

The fourth Book of  
P R O P. V.



About a triangle given  $A B C$  to describe a circle  $F A B C$ .

10. & 11. 1.

Bisect any two sides  $B A$ ,  $C A$  with perpendiculars  $D F$ ,  $E F$  meeting in the point  $F$ . I say this shall be the centre of the circle.

b const.  
c const. and  
12. ax.  
d 4. 1.

For, let the right lines  $F A$ ,  $F B$ ,  $F C$  be drawn. Now because  $A D = D B$  and the side  $D F$  common, and the angles  $F D A = F D B$ , therefore is  $F B = F A$ . After the same manner is  $F C = F A$ . Therefore a circle described from the centre  $F$  shall pass through the angles of the triangle given (*viz.*)  $B$ ,  $A$ ,  $C$ . Which was to be done.

Coroll.

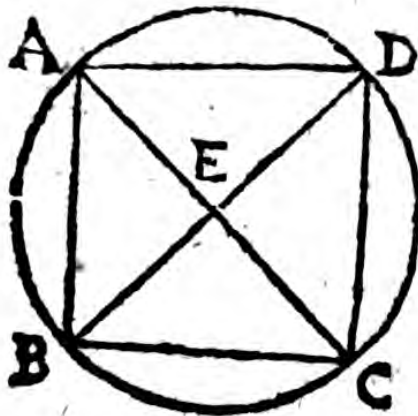
11. 3.

Hence, if a triangle be acute-angled, the centre shall fall within the triangle; if right-angled, in the side opposite to the right angle, & if obtuse-angled, without the triangle.

Schol.

By the same method may a circle be described, that shall pass through 3 points given, not being in the same strait line.

PROP. VI.



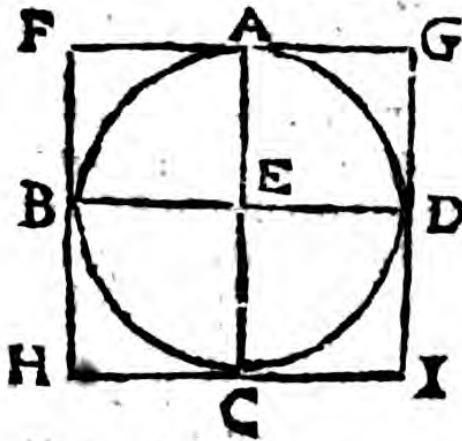
In a circle given EABCD to inscribe a square ABCD.

• Draw the diameters AC, BD cutting each other at right angles in the centre E. Join the extremes of these diameters with the right lines

AB, BC, CD, DA. And the thing is done.

Now because the 4 angles at E are right, the b arches and c subtended lines AB, BC, CD, DA are equal; therefore is the figure ABCD equilateral, and all the angles in semicircles, and so d right. Therefore ABCD is a square inscribed in a circle given. Which was to be done.

PROP. VII.



About a circle given EABCD to describe a square FHIG.

Draw the diameters AC, BD cutting one the other at right angles; through the extremes of these diameters a draw tangents meeting in F, H, I, G

then I say it's done.

For because b the angles A and C are right, c therefore is FG parallel to HI. After the same manner is FH parallel to GI, and therefore FHIG is a Pgr. and also right-angled. It is equilaterall because FG d = HI d = DB e = CA d = FH d = GI. Wherefore FHIG is a f square described about the circle given. Which was to be done.

E.

Schol.

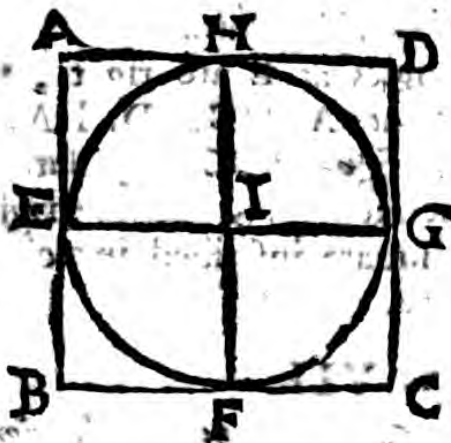
Schol.



A square ABCD described about a circle is double of the square EFGH inscribed in the same circle.

For the rectangle HB = 2 HEF, and HD = 2 HGF by the 4L. 1.

PROP. VIII.



In a square given ABCD to inscribe a circle IEFHG.

Bisect the sides of the square in the points H, E, F, G cutting one the other in I. a circle drawn from the centre I through H shall be inscribed in

the square.

For because AH and BF are <sup>a</sup> equal and <sup>b</sup> parallel, <sup>c</sup> therefore is AB parallel to HF parallel to DC. After the same manner is AD parallel to EG, parallel to BC; therefore IA, ID, IB, IC are parallelograms. Therefore AH<sup>d</sup> = AE<sup>e</sup> = HI = EI = FI = IG. The circle therefore described from the centre I through H shall pass through H, E, F, G and touch the sides of the square being the angles H, E, F, G are right. Which was to be done.

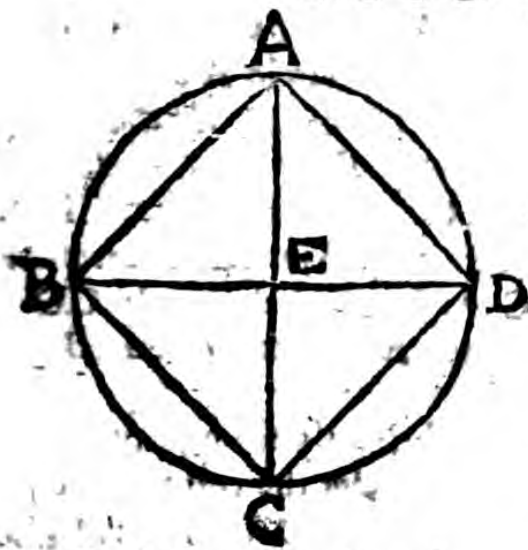
<sup>a</sup> 7. ax.  
<sup>b</sup> hyp.  
<sup>c</sup> 33. 1.

<sup>d</sup> 7. ax.  
<sup>e</sup> 34. 1.

PROP.

# EUCLIDE'S *Elements.*

## PROP. IX.



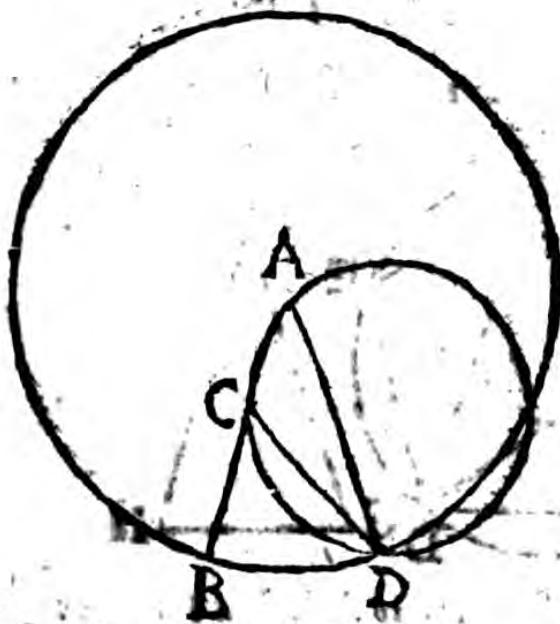
About a square given  $ABCD$  to describe a circle  $EABCD$ .

Draw the diameters  $AC$ ,  $BD$  cutting one the other in  $E$ . From the centre  $E$  through  $A$  describe a circle, then I say that circle is

described about the square.

For the angles  $ABD$  and  $BAC$  are a half of right angles,  $b$  therefore  $EA = EB$ . After the same manner is  $EA = ED = EC$ . The circle therefore described from the centre  $E$  passes through  $A, B, C, D$  the angles of the square given. Which was to be done.

## PROP. X.



To make an Isosceles triangle  $BD$ , having each angle at the base  $B$  and  $D$  double to the remaining angle  $A$ .

Take any right line  $AB$ , and divide it in  $C$ ,  $a$  so that  $AB \times BC$  may be equal to

$AC^2$ . From the centre  $A$  through  $B$ , describe the circle  $ABD$ ; and in this circle  $b$  apply  $BD = AC$ , and  $b. 1.$  join  $AD$ , I say  $ABD$  is the triangle required.

F 2

For



e 5. 4.  
d 37. 3.  
e 32. 3.  
f 2. ax.  
g 32. 1.  
h 5. 1.  
k 1. ax.  
l 6. 1.  
m constr.  
n 5. 1.

For, draw DC, and through the points C, D, A. draw a circle. Now because  $AB \times BC = AC^2$ , it is evident that D touches the circle ACD which CD cutteth; therefore is the angle  $BDC = A$ , and therefore the angle  $BDC + CDA = A + CDA = BCD$ . But  $BDC + CDA = BDA = CBD$ . therefore the angle  $BCD = CBD$ ; and therefore  $DC = DB = AC$ , wherefore the angle  $CDA = A = BDC$ . therefore  $ADB = 2A = ABD$ . Which was to be done.

a 3. 6.  
b constr.  
c hyp.  
d 6. 1.  
e 31. 1.  
f 2. ax.  
g 17. 6.

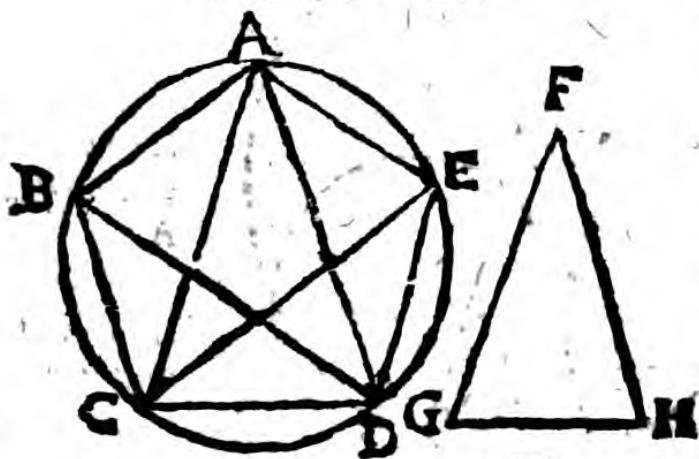
This construction is Analytically found out thus; Take the thing for done, and let the right line DC bisect the angle BDA; therefore  $DA, DB :: CA, CB$ . also because the angle  $CDA = \frac{1}{2} ADB = A$ , therefore  $CA = DC$ . and because the angle  $DCB = A + CDA = 2A = B$ , thence will be  $DB = DC$ . from whence also  $DB = CA$ . and so  $DA (BA.) CA :: CA, CB$ . whence  $BA \times CB = CA^2$ .

Coroll.

d 32. 1.

Whereas all the angles A, B, D make up two right angles, it's evident that A is  $\frac{1}{5}$  of two right angles.

PROP. XI.



In a circle given ABCDE to describe a Pentagone figure ABCDE equilateral and equiangular.

a 10. 4.  
b 2. 4.

Describe an Isosceles triangle FGH, having each angle at the base double to the other; inscribe a triangle

triangle CAD equiangular to the said triangle FGH. Bisect the angles at the base ACD & ADC with the right lines DB, CE meeting with the circumference in B and E. join the right lines CB, BA, AE, ED. Then I say it is done. e 9. 1.

For it is evident by construction that the angles CAD, CDB, BDA, DCE, ECA are equal; wherefore the <sup>d</sup> arches and <sup>e</sup> subtended lines DC, CB, BA, AE, DE are equal. Therefore the Pentagone is equilateral, and equiangular <sup>f</sup> because the angles of it BAE, AED, &c. stand on equal <sup>g</sup> arches BCDE, ABCD, &c. d 26. 1.  
e 29. 3.  
f 27. 3.  
g 2. 27.

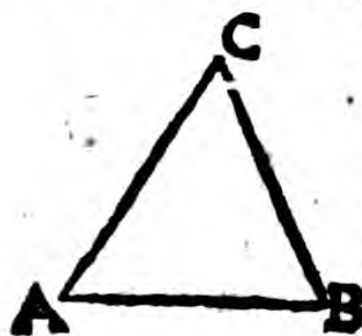
A more easy practise of this probleme shall be deliver'd at 10. 13.

*Coroll.*

Hence, Each angle of an equilateral and equiangular Pentagone is equal to  $\frac{2}{5}$  of two right angles, or  $\frac{6}{5}$  of one right angle.

*Schol.*

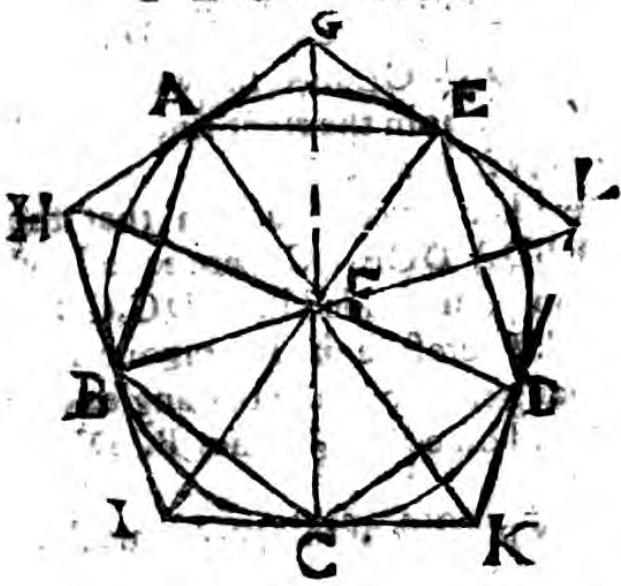
Generally all figures of odd number of sides are inscribed in circles by the help of Isosceles triangles, whose angles at the base are multiples of those at the top: and figures of even number of sides are inscribed in a circle by the help of Isosceles triangles, whose angles at the base are multiples sesquialter of those at the top. Prop. 11. 1.



As in the Isosceles triangle CAB if the angle  $A = 3 C = B$ , then will AB be the side of a Heptagone. If  $A = 4 C$ ; then is AB the side of an Enneagone. But if  $A = \frac{1}{2} C$  then is AB the side of a  $\frac{1}{2}$  square. And if  $A = 2 \frac{1}{2} C$  AB will

subtend the sixth part of a circumference and likewise if  $A = 3 \frac{1}{2} C$  then will AB be the side of an Octagone.

The fourth Book of  
PROP. XII.



About a circle given *FABCDE*, to describe an equilateral and an equiangular pentagone *HIKLG*.

11. 4.

be ar. 16. 13.  
c 2 cor. 36 3.  
d 8. 1.

e 27. 3.  
f 7. ax.  
g 12. ax.  
h 16. 1.  
i 3. ax.

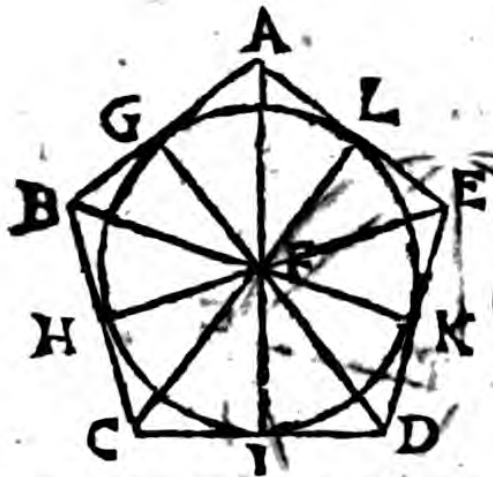
Inscribe a pentagone *ABCDE* in the circle given; and from the centre draw the right lines *FA*, *FB*, *FC*, *FD*, *FE*; and to those lines draw so many perpendiculars *GAH*, *HBI*, *ICK*, *KDL*, *LEG* meeting in the points *H*, *I*, *K*, *L*, *G*. then I say it is done. For because *GA*, *GE* from the same point *G* touch the circle, therefore is  $GA = GE$ , and therefore the angle  $GFA = GFE$ , therefore the angle  $AFE = 2 GFA$ . After the same manner is the angle  $AFH = FHB$ , and consequently the angle  $AFB = 2 AFH$ . But the angle  $AFE = AFB$ , therefore the angle  $GFA = AFH$ . But also the angle  $FAH = FAG$ , and the side *FA* is common, therefore  $HA = AG = GE = EL$ , &c. Therefore *HG*, *GL*, *LK*, *KI*, *IH* the sides of the pentagone are equal, the angles also are, because double of the equal angles *AGF*, *AHF*. therefore, &c.

Coroll.

After the same manner, if any equilateral and equiangular figure be described in a circle, and at the extreme points of the semi-diameters drawn from the centre at angles, be drawn perpendicular lines to the said diameters, I say that these perpendiculars shall

shall make another figure of as many equall sides and equall angles, described about the circle.

P R O P. XIII.



In an equilateral and equiangular pentagone given ABCDE to inscribe a circle FGHK.

*a* Bisect two angles <sup>9.1.</sup> of the pentagone A and B with the right lines AF, BF meeting in the point F. From F draw the perpendi-

culars FG, FH, FI, FK, FL. Then a circle described from the centre F through G will touch all the sides of the pentagone.

Draw FC, FD, FE. Because BA <sup>b</sup> = BC and the side BF common, and the angle FBA <sup>c</sup> = FBC, <sup>d</sup> therefore is AF = FC and the angle FAB = FCB, but the angle FAB <sup>e</sup> =  $\frac{1}{2}$  BAE =  $\frac{1}{2}$  BCD. Therefore the angle FCB =  $\frac{1}{2}$  BCD. After the same manner are all the whole <sup>2</sup> angles C, D, E bisected. Now whereas the angle FGB <sup>f</sup> = FHB, and the ang. FBH = FBG & the side FB is common, <sup>g</sup> therefore is FG = FH. In like manner are all the right lines FH, FI, FK, FL, FG equal. Therefore a circle described from the centre F through G passes through the points H, I, K, L and <sup>h</sup> touches the sides of the pentagone because the angles at those points are right. Which was to be done. <sup>h cor. 16.1.</sup>

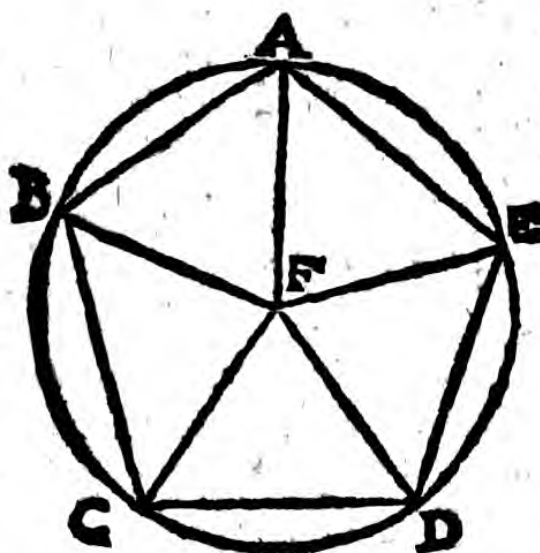
Coroll.

Hence, If any two nearest angles of an equilateral and equiangular figure, and from that point in which the lines meet that bisect the angles be drawn right lines to the remaining angles of the figure, all the angles of the figure shall be bisected.

Schol.

By the same method shall a circle be inscribed in any equilateral and equiangular figure.

P R O P. XIV.



About a pentagone given *ABCDE* equilateral and equiangular to describe a circle *FABCDE*.

Bisect any two angles of the pentagone with the right lines *AF*, *BF* meeting in the point *F*; the circle described from the centre *F* through *A* shall be described about the pentagone.

For 13. 4.  
b 6. 1.

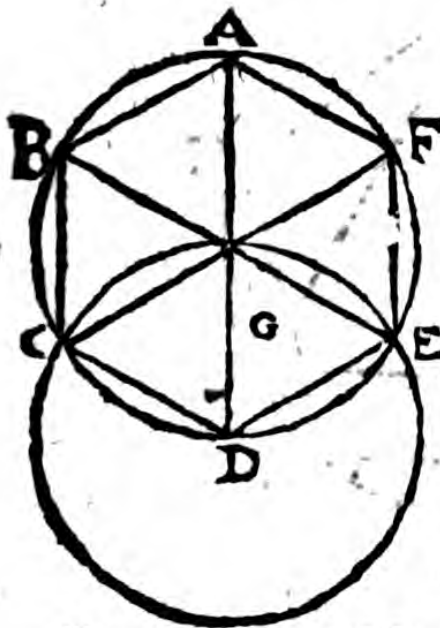
For let *FC*, *FD*, *FE* be drawn. <sup>a</sup> Then the angles *C*, *D*, *E* are bisected; & therefore *FA*, *FB*, *FC*, *FD*, *FE* are equall; therefore the circle described from the centre *F* passes through *A*, *B*, *C*, *D*, *E* all the angles of the pentagone. Which was to be done.

Schol.

By the same art is a circle described about any figure which is equilateral and equiangular.

P R O P.

P R O P. XV.



In a circle given GABCDEF to inscribe an Hexagone (or six-sided figure) equilateral and equiangular ABCDEF.

Draw the diameter AD; from the centre D through the centre G describe a circle cutting the circle given in the points C and E. Draw the diameters CF, EB; and join AB, BC, CD, DE, EF, FA. Then I say it's done.

For the angle CGD  $\hat{c} = \frac{1}{2}$  of 2 right  $\hat{c} = \text{DGE}$  a 32. 2.  
 $\hat{b} = \text{AGF}$   $\hat{b} = \text{AGB}$ . b 15. 2. Therefore BGC  $\hat{c} =$  c cor. 13. 2.  
 $\frac{1}{2}$  of 2 right  $\hat{c} = \text{FGE}$ ; therefore the  $\hat{d}$  arches and  $\hat{e}$  subtenses AB, BC, CD, DE, EF are equal. d 26. 3. Therefore e 29. 3.  
 the Hexagone is equilateral; but it is equiangular also,  $\hat{f}$  because all the angles of it stand upon equal f 27. 3.  
 arches.

Coroll.

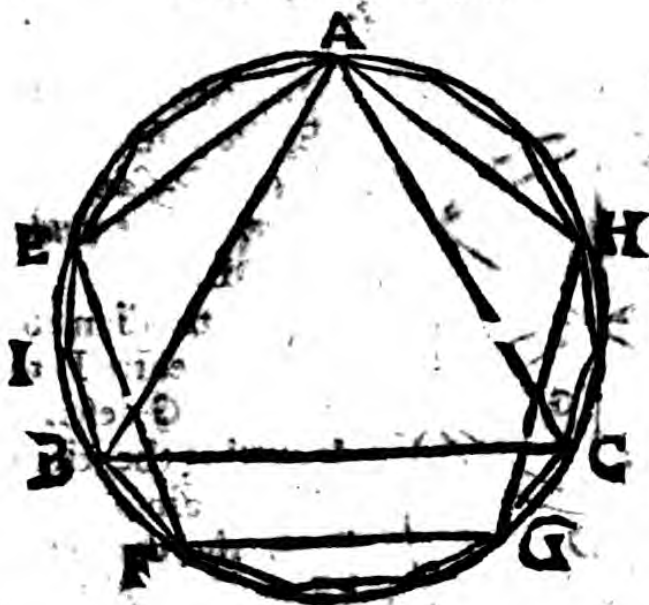
1. Hence, The side of an Hexagone inscribed in a circle is equal to the semidiameter.

2. Hereby an equilateral triangle ACE may very easily be described in a circle given.

Schol.

To make a true Hexagon upon a right line given CD. An dr. Traq.  
 $\hat{a}$  Make an equilateral triangle CGD upon the a 1. 1.  
 line given CD; from the centre G through C and D describe a circle. That circle shall contain the Hexagone made upon the given line CD.

P R O P.



In a circle given  $AEB C$  to inscribe a quindecagone (or fifteen-sided figure) equilateral and equiangular.

a 11. 4.  
b 3. 4.

Inscribe an equilateral pentagone  $A E F G H$  in the circle given, and also an equilateral triangle  $A B C$ . then I say  $B F$  is the side of the quindecagone required.

c 10. 17.

For the arch  $A B$  is  $\frac{1}{3}$  or  $\frac{5}{15}$  of that periphery whereof  $A F$  is  $\frac{2}{3}$  or  $\frac{6}{15}$ . therefore the remaining part  $B F$  is  $\frac{1}{15}$  of the periphery; and therefore the quindecagone, whose side is  $B F$ , is equilateral; but it is equiangular also because all the angles insit on equall arches of a circle, whereof every one  $\frac{13}{15}$  of the whole circumference. Therefore, &c.

d 27. 3.

Schol.

A circle is geometrically divided into parts

$\left\{ \begin{array}{l} 4, 8, 16, \&c. \text{ by } 6, 4, \text{ and } 9, 1. \\ 3, 6, 12, \&c. \text{ by } 15, 4, \text{ and } 9, 1. \\ 5, 10, 20, \&c. \text{ by } 11, 4, \text{ and } 9, 1. \\ 15, 30, 60, \&c. \text{ by } 16, 4, \text{ and } 9, 1. \end{array} \right.$	4, 8, 16, &c. by 6, 4, and 9, 1.
	3, 6, 12, &c. by 15, 4, and 9, 1.
	5, 10, 20, &c. by 11, 4, and 9, 1.
	15, 30, 60, &c. by 16, 4, and 9, 1.

Any other way of dividing the circumference into any parts given is as yet unknown; wherefore in the construction of ordinate figures, we are forced to have recourse to mechanick artifices, concerning which you may consult the writers of practical Geometry.

THE

21

THE FIFTH BOOK  
OF  
EUCLIDES' ELEMENTS.

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*Definitions.*

I. **A** Part, is a magnitude of a magnitude, a lesse of a greater, when the lesse measureth the greater.

II. **Multiplex** is a greater magnitude in respect of a lesse, when the lesse measureth the greater.

III. **Ratio** (or rate) is the mutuall habitude or respect of two magnitudes of the same kind each to other, according to quantity.

*In every ratio that quantity which is referred to another quantity is called the antecedent of the ratio, and that to which the other is referd is called the consequent of the ratio. as in the ratio of 6 to 4, 6 is the antecedent and 4 the consequent.*

*Note: The quantity of any ratio is known by dividing the antecedent by the consequent; as the ratio of 12 to 5 is expressed by  $\frac{12}{5}$ ; or the quantity of the ratio of A to B is  $\frac{A}{B}$ . Wherefore often for brevity sake we denote*

*the quantities of rations thus;  $\frac{A}{B}$ , or  $\frac{C}{D}$  or  $\frac{C}{D}$*

*that is, the ratio of A to B is greater, equall, or lesse then the ratio of C to D. And this note must be diligently observed in the understanding of the following Book.*

*Concerning the diverse species of ratio's, you may please to consult interpreters.*

IV. **Proportion** is a similitude of ratio's.

*That which is here termed proportion, is more rightly called proportionality or analogy; for proportion*



The fifth Book of

tion commonly denotes no more then the ratio betwixt two magnitudes.

V. Those numbers are said to have a ratio betwixt them, which being multiplyed may exceed one the other.

E, 12. | A, 4. B, 6. | G, 24. VI. Magnitudes are F, 30. | C, 10. D, 15. | H, 60. said to be in the same ratio, the first A to the

second B, and the third C to the fourth D, when the equimultiples E and F of the first A, and the third C compared with the equimultiples G, H of the second B and the fourth D, according to any multiplication whatsoever, either both together E, F are lesse then GH both together, or equall taken together, or exceed one the other together, if those be taken E, G and F, H, which answer one to the other.

The note hereof is :: ; as A.B :: C.D. That is, as A is to B, so is C to D. which signifyes that A to B, and C to D are in the same ratio. We sometimes thus expresse it;  $A = \frac{C}{D}$  that is A.B :: C.D.

$\frac{A}{B} = \frac{C}{D}$

VII. Magnitudes that have the same ratio (A.B :: C.D. are called proportional.

E, 30. | A, 6. B, 4. | G, 28. VIII. When of equimultiples, E the multiplex of the first

magnitude A exceeds the G the multiplex of the second B, but F the multiplex of the third C exceeds not H the multiplex of the fourth D, then the first A to the second B has a greater ratio then the third C to the fourth D.

If  $A < \frac{C}{D}$ , it is not necessary from this definition that E should alwayes exceed G, when F is lesse then H; but it is granted that this may be.

IX. Proportionality consist in three termes at least. whereof the second supplyes the place of two.

X. When 3 magnitudes A, B, C are proportional, the

the first A shall have a duplicate ratio to the 3 C of that it hath to the second B; But when four magnitudes A, B, C, D are proportional, the first A shall have a triplicate ratio to the fourth D of what it had to the second B; and so alwayes in order one more, as the proportion shall be extended.

*Duplicate ratio is thus expressed*  $A = A \text{ twice, that is, } \frac{A}{C} = \frac{A}{B}$

*is, the ratio of A to C is double of the ratio of A to B.*

*Triple ratio is thus expressed; A = A thrice. That is,  $\frac{A}{D} = \frac{A}{B}$*

*the ratio of A to D is triple of the ratio of A to B.*

*∴ denotes continued proportionals; as A, B, C, D; or 2, 6, 18, 64. are ∴*

**X I.** Magnitudes of a like ratio, are antecedents to antecedents, and consequents to consequents; *As if A. B :: C. D. A and C; and B and D are homologous or magnitudes of a like ratio.*

**X I I.** Alternate proportion is the comparing of antecedent to antecedent, and consequent to consequent. *As if A. B :: C. D. therefore alternately, or by permutation, A. C :: B. D. by the 16. of 5.*

*In this definition, and the 5 following, names are given to the six wayes of arguing which are often used by Mathematicians: the force of which inferences depends on the propositions of this book, which are named in their explications.*

**X I I I.** Inverse ratio is when the consequent is taken as the antecedent, and so compared to the antecedent as the consequent; *as A. B :: C. D. therefore generally B. A :: D. C. by cor. 4. 5.*

**X I V.** Compounded ratio is when the antecedent and consequent taken both as one are compared to the consequent it self. *As A. B :: C. D. therefore by composition A + B. B :: C + D. D by 18. 5.*

**X V.** Divided ratio is when the excess where in the antecedent exceedeth the consequent, is compared to the consequent. *As A. B :: C. D. therefore by division A - B. B :: C - D. D. by 17. 5.*

**X V I.**

*The fifth Book of*

XVI. **Converse ratio** is when the antecedent is compared to the excess wherein the antecedent exceeds the consequent. *As*  $A.B :: C.D.$  therefore by *converse ratio.*  $A.A - B :: C.C - D.$  by the coroll. of the 19. of the 5.

XVII. **Proportion of equality** is where there are taken more magnitudes than two in one order, and also as many magnitudes in another order, comparing two to two being in the same ratio; it cometh to passe that as in the first order of magnitudes, the first is to the last, so in the second order of magnitudes is the first to the last. Or otherwise: it is a comparison of the extremes together, the mean magnitudes being taken away.

XVIII. **Ordinate proportionality** is, when, as the antecedent is to the consequent, so is the antecedent to the consequent, and as the consequent is to any other, so is the consequent to any other. *As*  $A.B :: D.E.$  also  $B.C :: E.F.$  it shall be true also  $A.C :: D.F.$  by the 22. of the 5.

XIX. **Inordinate proportion** is, when three magnitudes being put, and others also, which are equal to these in multitude, as in the first magnitudes the antecedent is to the consequent, so in the second magnitudes is the antecedent to the consequent; and as in the first magnitudes the consequent is to any other, so in the second magnitudes any other thing to the antecedent. *As*  $A.B :: F.G.$  also  $B.C :: E.F.$  it shall be true in inordinate proportion:  $A.C :: E.G.$  by the 23. of the 5.

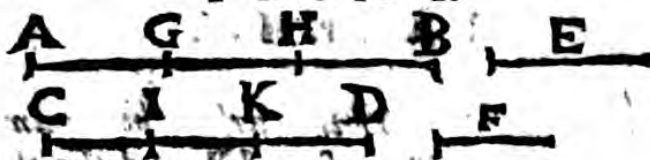
XX. Any number of magnitudes being put; the proportion of the first to the last is compounded out of the proportions of the first to the second, the second to the third, and the third to the fourth, and so forwards till the proportion arise.

Let there be any number of magnitudes  $A, B, C, D.$  by this definition  $\frac{A}{D} = \frac{A}{B} + \frac{B}{C} + \frac{C}{D}.$

Axiome.

Magnitudes equimultiples to the same multiplex, are also equimultiples betwixt themselves.

P R O P. I.

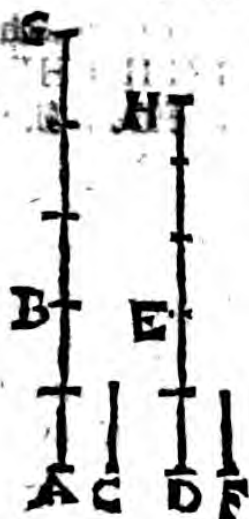


If there be a number of magnitudes how many soever,  $AB, CD$  equimultiples to a like number of magnitudes  $E, F$  each to other; how multiplex one magnitude  $AB$  is to one  $E$ , so multiplices are all the magnitudes  $AB + CD$  to all the other magnitudes  $E + F$ .

Let  $AG, GH, HB$  the parts of the quantity  $AB$ , be equal to  $E$ , and also let  $CI, IK, KD$  the parts of the quantity  $CD$  be equal to  $F$ . The number of these are put equal to those. Now whereas  $AG + CI = E + F$ ; and  $GH + IK = E + F$ ; and  $HB + KD = E + F$ , it is evident that  $AB + CD$  doth so often contain  $E + F$  as one  $AB$  contains  $E$ . Which was to be done.

P R O P. II.

If the first  $AB$  be equimultiplex to the second  $C$ , as the third  $DE$  is to the fourth  $F$ , and if the fifth  $BG$  be equimultiplex to the second  $C$  as the sixth  $EH$  is to the fourth  $F$ ; then shall the first compounded with the fifth ( $AG$ ) be equimultiplex to the second  $C$ , as the third compounded with the sixth ( $DH$ ) is to the fourth  $F$ .



The number of parts in  $AB$  equal each to  $C$  is put equal to the numbers of parts in  $DE$ , whereof each part is equal to  $F$ . Likewise the number of parts in  $BG$  is put equal to the number of parts in  $EH$ . Therefore the number of parts in  $AB + BG$  is equal to the number of parts in  $DE + EH$ .

o 2. 47

**EH.** That is, the whole line AG is as equimultiple of C, as the whole line DH is of F. *Which was to be Dem.*

**PROP. III.**

*If the first A be equimultiple of the second B, and the third C of the fourth D, and there be taken EI, FM equimultiples of the first and third, then will each of the magnitudes taken be alike equimultiple of both, the one EI to the second B, the other FM to the fourth D.*

Let EG, GH, HI the parts of the multiplex EI be equal to A, also let FK, KL, LM the parts of the multiplex FM be equal to C, the number of these is equal to the number of those. Moreover A (that is) EG or GH or HI is put as equimultiple of B, as C,

or FK, &c. of D. <sup>b</sup> Therefore EG + GH is equimultiple of the second B, as FK + KL is of the fourth D. <sup>c</sup> By the same way of argument is EI (EH + HI) as multiplex of B, as FM (FL + LN) is of D. *Which was to be done.*



o 27.

b 25.

c 25.

**PROP.**

P R O P. IV.



If the first **A** have the same ratio to the second **B**, as the third **C** to the fourth **D**; then also **E** and **F** the equimultiples of the first **A** and the third **C**, shall have the same ratio to **G** and **H** the equimultiples of the second **B** and the fourth **D**, according to any multiplication, if so taken as they answer each to other (**E, G :: F, H.**)

Take **I** and **K** the equimultiples of **E** and **F**; and also **L** and **M** the equimultiples of **G** & **H**.  
 Then is **I** as multiplex of **A**, as **K** of **C**; and also **L** is as multiplex of **B**, as **M** of **D**. Therefore whereas it is **A. B :: C. D**; according to the sixth definition, if **I** be  $\square, =, \sqsupset$  **L**, then consequently after the same manner is **K**  $\square, =, \sqsupset$  **M**. Therefore when **I** and **K** are taken as multiples of **E** and **F**, as **L** and **M** of **G**, & **H**, then

<sup>c</sup> 3. 5.

<sup>b</sup> 5. p. 6

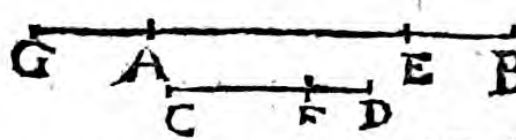
will it be by the 7 definition **E. G :: F. H.** Which was to be Dem.

Coroll.

From hence is wont to be demonstrated the proofe of inverse ratio.

For because **A. B :: C. D**, therefore if **E**  $\square, =, \sqsupset$  **G**, then is <sup>c</sup> likewise **F**  $\square, =, \sqsupset$  **H**. therefore it is <sup>c</sup> 2. def. 5. evident that if **G**  $\square, =, \sqsupset$  **E**, then is **H**  $\square, =, \sqsupset$  **F**; <sup>d</sup> therefore **B. A :: D. C.** Which was to be <sup>d</sup> 6. def. 5. Dem.

P R O P. V.



If a magnitude  $AB$  be as multiplex of a magnitude  $CD$ , as a part taken from the one  $AE$  of a part taken from the other  $CF$ ; the residue of the one  $EB$  shall be as multiplex of the residue of the other  $FD$  as the whole  $AB$  is of the whole  $CD$ .

Take any other  $GA$ , which shall be as multiplex to  $FD$  the residue, as  $AB$  is of the whole  $CD$ , or as the part taken away  $AE$  is of the part taken away  $CF$ . Therefore the whole  $GA + AE$  is as multiplex of the whole  $CF + FD$ , as the one  $AE$  of the one  $CF$ , that is as  $AB$  of  $CD$ . therefore  $GE = AB$ ; and so  $AE$  that was common being taken away, there remains  $GA = EB$ .

a 1. 5.

b 6. ax.  
c 3. ax.

P R O P. VI.



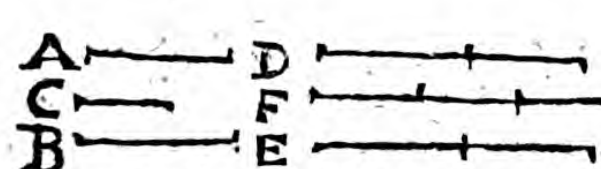
If two magnitudes  $AB$ ,  $CD$  be equimultiples of two magnitudes  $E$ ,  $F$ ; and some magnitudes  $AG$  and  $CH$  equimultiples of the same  $E$ ,  $F$ , be taken away; then the residues  $GB$ ,  $HD$  are either equal to these magnitudes  $E$ ,  $F$ , or else equimultiples of them.

For because the number of parts in  $AB$ , whereof each is equal to  $E$ , is put equal to the number of parts in  $CD$ , whereof each is equal to  $F$ . and also the number of parts in  $AG$  equal to the number of parts in  $CH$ ; If from one you take  $AG$ , and from the other  $CH$ , then remains the number of parts in the remainder  $GB$  equal to the number of parts in  $HD$ . therefore if  $GB$  be once  $E$ , then is  $HD$  once  $F$ . if  $GB$  be many times  $E$ , then is  $HD$  so of  $F$ . Which was to be Dem.

a 3. ax.

P R O P.

P R O P. VII.



*Equall magnitudes A and B have to the same magnitude C the same proportion or ratio. And one and the same magnitude C hath the same ratio to equall magnitudes A and B.*

Take D and E equimultiples of the equall magnitudes A and B, and F any-wise multiplex of C; then is  $D = E$ . Wherefore if  $D \square, =, \supset F$ , then also E will be  $\square, =, \supset F$ . *b* therefore  $A. C :: B. C.$  & *c* by inversion  $C. A :: c C. B.$  *W.W. to be Dem.*

*a 6 ax.  
 b 6 def. 5.  
 c cor. 4. 5.*

*Schol.*

If in stead of the multiplex F, two equimultiplicés be taken, it shall be the same way proved that equall magnitudes have the same ratio to other magnitudes that are equall between themselves.

P R O P. VIII.



Of unequall magnitudes AB, AC; the greater AB hath a greater ratio to the same third line D, then the lesser AC; and the same third line D hath a greater ratio to the lesser AC, then to the greater AB.

Take EF, EG equimultiplicés of the said AB, AC, so that EH being multiplex of D be greater then EG, but lesser then EF. (which will easily happen, if both EG and GF be taken greater then D.) It is manifest from 8 def. 5. that  $AB \square AC$ , and  $D \supset D$  which was  $\overline{D} \quad \overline{D} \quad \overline{AB} \quad \overline{AC}$  to be Dem.



The fifth Book of

PROP. IX.

Magnitudes which to one and the same magnitude have the same ratio, are equal to one to the other. And if a magnitude have the same ratio to other magnitudes, those magnitudes are equal one to the other.

a 8. s.

A B C

1. Hyp. If  $A.C :: B.C$ ; I say that  $A = B$ . For let A be greater or less than C. <sup>a</sup> then is  $A \sqsubset \bar{C}$  or  $\sqsupset B \bar{C}$  which is contrary to the

Hypothesis.

b 8. s.

2. Hyp. If  $C.B :: C.A$ . I say that  $A = B$ . For let A be  $\sqsubset B$ , <sup>b</sup> then  $C \sqsubset \bar{B} \bar{A}$  which is against the Hypothesis.

pothesis.

PROP. X.

Of magnitudes having ratio to the same magnitude, that which has the greater ratio, is the greater magnitude: and that magnitude to which the same carries a greater ratio, is the lesser magnitude.

a 7. s.

A B C

1. Hyp. If  $A \sqsubset \bar{B} \bar{C}$  I say that  $A \sqsubset B$ .

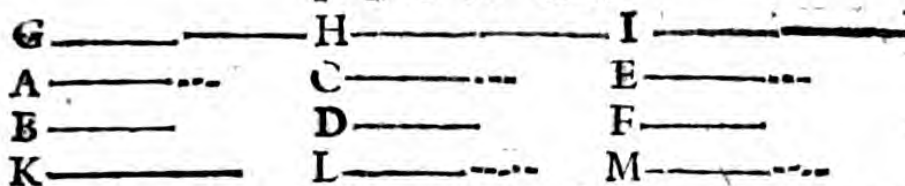
b 8. s.

For if it be said that  $A = B$ , <sup>a</sup> then  $A.C :: B.C$  which is contrary to the Hyp. If  $A \sqsupset B$ , <sup>b</sup> then is  $A \sqsupset \bar{B} \bar{C}$  which is also against the Hyp.

c 7. s.  
d 8. s.

2. Hyp. If  $C \sqsubset \bar{B} \bar{A}$  I say that  $B \sqsupset A$ . for if you say  $B = A$ , it's against the Hypothesis, for it will follow that  $C.B :: C.A$ . If you say  $B \sqsubset A$ , <sup>d</sup> then is  $C \sqsubset \bar{A} \bar{B}$  which is also against the Hyp.

P R O P. XI.



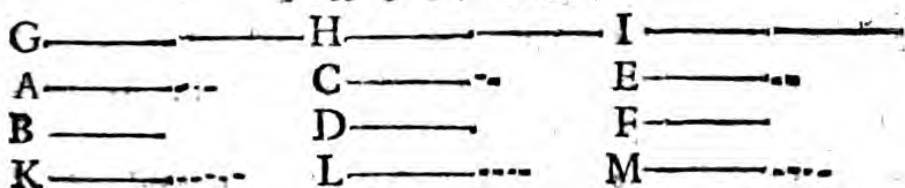
*Proportions which are one and the same to any third, are also the same one to another.*

Let  $A. B :: E. F$ , and  $C. D :: E. F$ . I say that  $A. B :: C. D$ . Take  $G, H, I$ , the equimultiples of  $A, C, E$ ; and  $K, L, M$  the equimultiples of  $B, D, F$ . Now <sup>a</sup> because  $A. B :: E. F$ ; if  $G \square, =, \sqsupset K$ , <sup>b</sup> then after the same manner  $I \square, =, \sqsupset M$ . And likewise <sup>a</sup> because  $E. F :: C. D$ . if  $I \square, =, \sqsupset M$ , <sup>b</sup> then is  $H$  likewise  $\square, =, \sqsupset L$ . <sup>c</sup> Wherefore  $A. B :: C. D$ , which was to be Dem.

Schol.

Proportions that are one and the same to the same proportions, are the same betwixt themselves.

P R O P. XII.

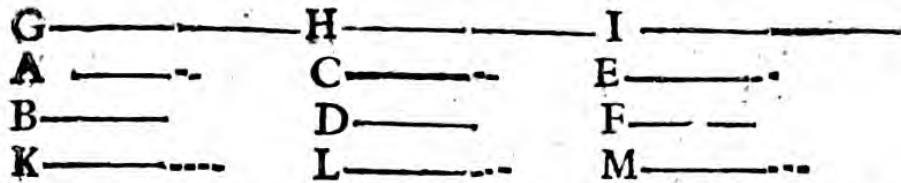


*If any number of magnitudes  $A, B; C, D; E$  and  $F$  be proportionall; as one of the Antecedents  $A$  is to one of the Consequents  $B$ , so are all the antecedents  $A, C, E$  to all the consequents  $B, D, F$ .*

Take the equimultiples of the antecedents  $G, H, I$ , and of the consequents  $K, L, M$ . Because that as multiplex as one  $G$  is of one  $A$ , <sup>a</sup> so multiplices are all  $G, H, I$ , of all  $A, C, E$ ; & likewise as multiplex as one  $K$  is of one  $B$ , so multiplices are all  $K, L, M$ , of all  $B, D, F$ . moreover because  $A. B :: C. D :: E. F$  <sup>b</sup> if  $G$  be  $\square, =, \sqsupset K$ , then will  $H$  likewise be  $\square, =, \sqsupset L$ , and  $I \square, =, \sqsupset M$ . and so if  $G \square, =, \sqsupset K$ . in like manner will  $G + H + I$  be  $\square, =, \sqsupset K + L + M$ . <sup>c</sup> wherefore  $A. B :: A + C + E. B + D + F$ . which was to be Dem.

From hence, if like proportionals be added to like proportionals, the wholes shall be proportionall.

P R O P. XIII.



If the first A have the same ratio to the second B, that the third C hath to the fourth D; and if the third C have a greater proportion to the fourth D, then the fifth E to the sixth F; then also shall the first A have a greater proportion to the second B, then the fifth E to the sixth F.

Take G, H, I equimultiples of A, C, E, and K, L, M, equimultiples of B, D, F. Now because that  $A : B :: C : D$ , if  $H \sqsubset L$ , a then is  $G \sqsubset K$ . but because  $C \sqsubset E$ , b it may be that  $H \sqsubset L$ , and

a 6. def. 3.  
b 8. def. 5.  
c 8. def. 5.

yet I not  $\sqsubset M$ . c Therefore  $\frac{A}{B} \sqsubset \frac{E}{F}$ . Which was to be Dem.

Schol.

But if  $\frac{C}{D} \sqsupset \frac{E}{F}$ , then also is  $\frac{A}{B} \sqsupset \frac{E}{F}$ . Also, if  $\frac{A}{B} \sqsubset \frac{C}{D} \sqsubset \frac{E}{F}$ , then is  $\frac{A}{B} \sqsubset \frac{E}{F}$ . And if  $\frac{A}{B} \sqsupset \frac{C}{D} \sqsupset \frac{E}{F}$ , then is  $\frac{A}{B} \sqsupset \frac{E}{F}$ .

PROP. XIV.



If the first A have the same ratio to the second B, that the third C hath to the fourth D; and if the first A be greater then the third C, then shall the second B be greater then the fourth D. But if the first A be equall to the third C, then the second B shall be equall to the fourth D. but if A be lesser, then is B also lesser.

Let  $A \sqsupset C$ . <sup>a</sup> then  $A \sqsupset C$  <sup>b</sup> but <sup>a</sup> 8. 5. <sup>b</sup> hyp.

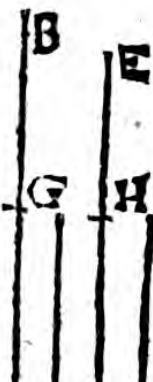
$A \sqsupset C$ . therefore  $C \sqsupset C$ . <sup>c</sup> therefore <sup>c</sup> 13. 5.

$B \sqsupset D$ . By the like way of argument, if  $A \sqsupset C$ , <sup>d</sup> then is  $B \sqsupset D$ . But if A be put equall to C, then <sup>d</sup> 10. 5.  $C. B :: e A. B$  <sup>e</sup> 7. 5.  $f :: C. D.$  <sup>f</sup> hyp.  $g$  therefore  $B = D$ . <sup>g</sup> 11. 5. & 9. <sup>5.</sup> Which was to be Dem.

Schol.

By an argument *a fortiori*, if  $A \sqsupset C$ , and  $A \sqsupset B$ , then is  $B \sqsupset D$ . Likewise if  $A = B$ , then is  $C = D$ . and if  $A \sqsubset$ , or  $\sqsupset B$ , then also is  $C \sqsupset$  or  $\sqsupset D$ .

PROP. XV.



Parts C and F are in the same ratio, with their like multiples AB and DE, if taken correspondently. ( $AB. DE :: C.F.$ )

Let AG, GB parts of the multiplex AB be equall to C; and let DH, HE parts of the multiplex DE be equall to F.

<sup>a</sup> The number of these parts is equall to <sup>a</sup> hyp. <sup>b</sup> 7. 5. the number of those. Therefore whereas

<sup>b</sup>  $AG. C :: DH. F$ , and  $GB. C :: HE.$  <sup>c</sup> 12. 5.  $ACDF$ , therefore is <sup>c</sup>  $AG + GB (AB.)$   $DH + HE (DE) :: C.F.$  Which was to be Dem.

The fifth Book of  
P R O P. XVI.



If four magnitudes A, B, C, D be proportionall, they also shall be alternately proportionall (A. C :: B. D.)

Take E and F equimultiples of A and B ; take also G and H equimultiples of C and D. Therefore E. F  $a$  :: A. B  $b$  :: C. D  $a$  :: G. H. Wherefore if E  $\square$ , =,  $\square$  G,  $c$  then likewise is F  $\square$ , =,  $\square$  H.  $d$  Therefore A. C :: B. D. Which was to be Dem.

Schol.

Alternate ratio has place onely then when the quantities are of the same kind. For heterogeneous quantities are not compared together.

P R O P. XVII.

If magnitudes compounded be proportionall (A B. C B :: D E. F E,) they shall be proportionall also when divided. (A C. C B :: D F. F E.)

Take G H, H L, I K, K M, in order the equimultiples of A C, C B, D F, F E ; and also L N, M O, the equimultiples of C B, F E. The whole G L is  $a$  as multiplex of the whole A B, as one G H of one A C,  $b$  that is as I K of D F,  $c$  or as the whole I M of the whole D E. Also H N (H L + L N) is as  $d$  multiplex of C B, as K O (K M + M O) is of F E. Therefore, whereas by Hyp. A B. B C :: D E. E F. I if G L be  $\square$ , =,  $\square$  H N, then likewise  $e$  will I M  $\square$ , =,  $\square$  K O



a 15. 5.  
b 17. 5.  
c 11. 5. &  
14. 5.  
d 6. def. 5.

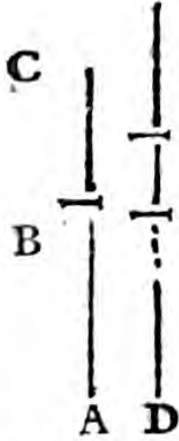
a 1. 9.  
b const.  
d 1. 9.  
e 2. 5.

K O. Take from these H L, K M that are equal; and if the remainder GH be  $\square, =, \sqsupset$  L N, then will I K  $\square, =, \sqsupset$  M O. *g* whence A C. C B :: D F. F E. Which was to be Dem. e 6. def. 5.  
f 5. ax.  
g 6. def. 5.

P R O P. XVIII.

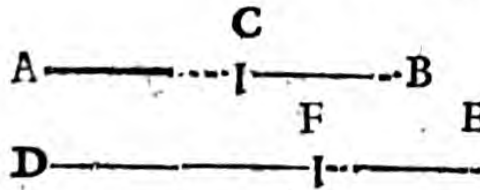
**F** If magnitudes divided be proportionall (A B. B C :: D E. E F.) the same also being compounded shall be proportionall (A C. C B :: D F. F E.)

**E** For if it can be, let A B. C B :: D F. F G  $\sqsupset$  F E. *a* Then by division will A B. B C :: D G. G F. *b* that is, D G. G F :: D E. E F. and being D G  $\square$  D E, *c* therefore is G F  $\square$  E F. *d* Which is Absurd. The like absurdity will follow if it be said A B. C B :: D E. G F  $\square$  F E. a 17. 5.  
b hyp. & 11.  
c 14. 5.  
d 9. ax.



P R O P. XIX.

**C** If the whole A B be to the whole D E as the part taken away A C is to the part taken away D F, then shall the residue C B be to the residue F E as the whole A B is to the whole D E.



Because *a* A B. D E :: A C. D F, *b* therefore by permutation A B. A C :: D E. D F. *c* and thence by division A C. C B :: D F. F E. *b* wherefore again by permutation A C. D F :: C B. F E. *d* that is, A B. D E :: C B. F E. *W. W.* to be Dem. a hyp.  
b 10. 5.  
c 17. 5.  
d hyp. & 11. 5.

Coroll.

Hence, If like proportionals be subtracted from like proportionals, the residues shall be proportionall.

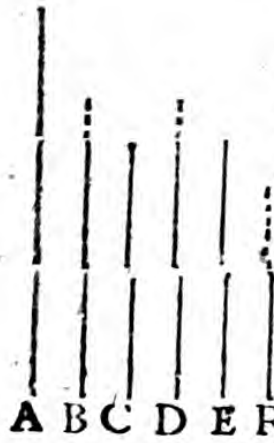
2. Hence is converse ratio demonstrated.

Let A B. C B :: D E. F E. I say that A B. A C :: D E. D F. For by *a* permutation A B. D E :: C B. F E, *b* therefore A B. D E :: A C. D F. whence again by permutation A B. A C :: D E. D F. *W. W.* to be Dem. a 16. 5.  
b 19. 5.

P R O P.

The fifth Book of

PROP. XX.



If there be three magnitudes A, B, C, and others D, E, F equall to those in number, which being tak'n two & two in each order are in the same ratio, (A. B :: D. E; and B. C :: E. F,) and if of equality the first A be greater then the third C; then shal the fourth D be greater then the sixth F. But if the first A be equall to the third C, then the fourth D is so to the sixth F; & if A be lesse then C, so D is lesse then F.

a hyp.  
b cor 4. 5.  
c hyp & 8. 5.  
d schol. 13. 5.

1. Hyp. Let  $A \sqsubset C$ . Because  $E F :: B. C.$  by  $b$  inversion shall be  $F. E :: C. B.$   $c$  But  $C \sqsupset A$  therefore  $\frac{C}{B} \sqsupset \frac{A}{B}$ ,

e 10. 5.

$\frac{F}{E} \sqsupset \frac{A}{B}$  or  $D \sqsupset F$  therefore  $D \sqsupset F$ . W.W. to be Dem.

f 7. 5.  
g 11. 5. and  
9. 5.

2. Hyp. By the same way of argument, if  $A \sqsupset C$ , it will appear that  $D \sqsupset F$ .

3. Hyp. If  $A = C$ . Because  $F. E :: C. B :: f A. B :: D. E.$  g therefore is  $D = F$ . W.W. to be Dem.

PROP. XXI.



If there be three magnitudes A, B, C, and others also D, E, F equall to them in number, which taken two and two are in the same ratio; and their proportion inordinate (A. B :: E. F. and B. C :: D. E.) & if of equality, the first A be greater then the third C; then is the fourth D greater then the sixth F; but if the first be equall to the third, then is the fourth equall to the sixth; if lesse, so is the other likewise.

a hyp.  
b 8. 5.

1. Hyp. If  $A \sqsubset C$ ; then because  $D. E :: B. C$ , therefore inversely  $E. D :: C. B.$  but  $C \sqsupset A$  there-

c schol. 13. 5.  
d 10. 5.

fore  $\frac{E}{D} \sqsupset \frac{A}{B}$  that is  $\frac{E}{D} \sqsupset \frac{A}{B}$  therefore  $D \sqsupset F$ .

2. Hyp.

2. Hyp. By the like argument, if  $A \sim C$ , then is  $D \sim F$ .

3. Hyp. If  $A = C$ ; then because  $E, D :: e C, B :: e A, B :: f E, F$ ,  $g$  therefore is  $D = F$ . *W.W. to be Dem.*

e 7. 9.  
f hyp.  
g 9. 5.

PROP. XXII.



If there be any number of magnitudes  $A, B, C$ , and others equal to them in number  $D, E, F$ , which taken two and two are in the same ratio ( $A, B :: D, E$ . and  $B, C :: E, F$ .) they shall be in the same ratio also by equality, ( $A, C :: D, F$ .)

Take  $G, H$  equimultiples of  $A, D$ ; &  $I, K$  of  $B, E$ ; and also  $L, M$  of  $E, F$ .

Because  $a A, B :: D, E$ ,  $b$  therefore  $G, I :: H, K$ . and in like manner  $I, L :: K, M$ . therefore, if  $G \sim, =, \sim, L$ ,  $c$  then is  $H \sim, =, \sim$

a hyp.  
b 4. 5.

$M$ , therefore  $A, C :: D, F$ . By the same way of demonstration if further  $C, N :: F, O$ , then by equality  $A, N :: D, O$ . *W.W. to be Dem.*

c 20. 9.  
d 6. def. 5.



The fifth Book of  
P R O P. XXIII.



If there be three magnitudes A, B, C, and others D, E, F, equal to them in number, which taken two and two are in the same ratio, and their proportionality inordinate ( $A.B :: E.F.$  and  $B.C :: D.E.$ ) they shall be in the same ratio also by equality.

Take G, H, I equimultiples of A, B, D; and also K, L, M equimultiples of C, E, F. Then  $G.H :: a.A.B :: b.E.F$   $a :: L.M.$  Moreover because  $b.B.C :: D.E$ , thence is  $*H.K :: I.L$ ; therefore G, H, K, and I, L, M are according to 21. 5. Wherefore if  $G$  be  $\square, =, \sqsupset K$ , then is likewise  $I \square, =, \sqsupset M$ .

a 15. 5.  
b hyp.

c 4. 5.

d 6. def. 5.  $\square, \sqsupset M$ . and so  $d$  consequently  $A.C :: D.F.$  *W.W.* to be Dem.

If there be more magnitudes then three, this way of demonstration holds good in them also.

Coroll.

\* 22. and 23.  
5. & 20.  
def. 1.

From hence \* it follow's that ratio's compounded of the same ratio's, are among themselves the same; as also that the same parts of the same ratio's, are among themselves the same.

P R O P. XXIV.

A \_\_\_\_\_ | \_\_\_\_\_ If the first magnitude A B  
C \_\_\_\_\_ B G have the same ratio to the se-  
D \_\_\_\_\_ | \_\_\_\_\_ cond C, which the third D E  
F \_\_\_\_\_ E H hath to the fourth F; & if the  
fifth B G have the same ratio  
to the second C, which the sixth E H hath to the fourth F,  
then shall the first compounded with the fifth (AG) have  
the same ratio to the second C, which the third com-  
pounded with the sixth (D H) hath to the fourth F.

a hyp.

For because  $a.A.B.C :: D.E.F$ , and by the Hyp. and

and inversion  $C. BG :: F. EH$ ; therefore by *b* equality  $AB. BG :: DE. EH$ . whence by compounding,  $AG. BG :: DH. EH$ . Also *c*  $BG. C :: EH. F$ . Therefore again by *b* equality  $AG. C :: DH. F$ . *W.W. to be Dem.*

P R O P. XXV.



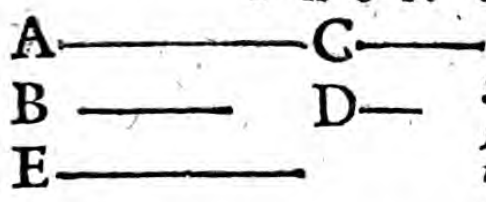
If four magnitudes be proportionall ( $AB. CD :: E. F$ ) the greatest  $AB$  and the least  $F$  shall be greater then the rest  $CD$ , and  $E$ .

Make  $AG = E$ , and  $CH = F$ . Because  $AB. CD :: a E. F :: b AG. CH$ , *c* thence is  $AB. CD :: GB. HD$ . *d* but  $AB > CD$ . *e* therefore  $GB < HD$ . But  $AG + F = E + CH$ , therefore  $AG + F + GB < E + CH + HD$ , that is,  $AB + F < E + CD$ . *W.W. to be Dem.*

*b* 22. 5.  
*c* hyp.  
*a* hyp.  
*b* 7. 5.  
*c* 19. 5.  
*d* hyp.  
*e* schol. 14. 5.

These propositions which follow are not Euclid's, but taken out of other Authors, and here subjoined because of their frequent use.

P R O P. XXVI.



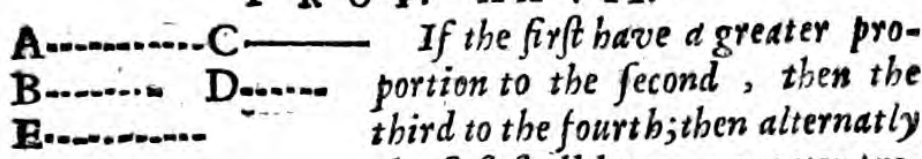
If the first have a greater proportion to the second; then the third to the fourth, then contrarywise, by conversion, the second shall have a lesse proportion to the first, then the fourth to the third.

Let  $\frac{A}{B} > \frac{C}{D}$  I say that  $\frac{B}{A} < \frac{D}{C}$ . For conceive

$\frac{C}{D} = \frac{E}{B}$  *a* therefore  $\frac{A}{B} > \frac{E}{B}$  *b* whence  $A > E$ . *c* therefore  $\frac{B}{A} < \frac{B}{E}$  *d* or  $\frac{B}{A} < \frac{D}{C}$ . *W.W. to be Dem.*

*a* 13. 5.  
*b* 10. 5.  
*c* 8. 5.  
*d* cor. 4. 5.

P R O P. XXVII.



If the first have a greater proportion to the second, then the third to the fourth; then alternatly the first shall have a greater proportion to the third, then the second to the fourth.

Let

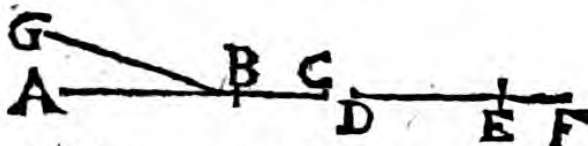
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Let  $\frac{A}{B} \square \frac{C}{D}$  then I say  $\frac{A}{C} \square \frac{B}{D}$  For conceive

a 10. 5.  
b 1. 5.  
c 16. 5.

$\frac{E}{B} \square \frac{C}{D}$  a therefore  $A \square E$ , b and therefore  $\frac{A}{C} \square \frac{E}{C}$   
c or B *W.W. to be Dem.*  
 $\frac{D}{D}$ .

P R O P. XXVIII.



If the first have a greater proportion to the second than the third to the fourth, then the first compounded with the second shall have a greater proportion to the second, then the third compounded with the fourth to the fourth.

a 10. 5.  
b 4. ax.  
c 8. 5.  
d 18. 5.

Let  $\frac{AB}{BC} \square \frac{DE}{EF}$  I say that  $\frac{AC}{BC} \square \frac{DE}{EF}$  For conceive  $\frac{GB}{BC} \square \frac{DE}{EF}$  a therefore is  $AB \square GB$ . adde BC to each part, then b will  $AC \square GC$ . c therefore  $\frac{AC}{BC} \square \frac{GC}{BC}$  d that is  $\frac{DE}{FE}$  *W.W. to be Dem.*

P R O P. XXIX.



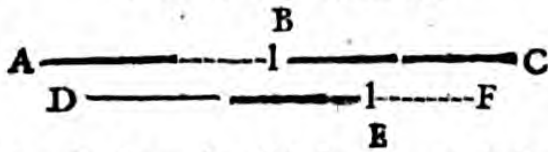
If the first compounded with the second have a greater proportion to the second, then the third compounded with the fourth hath to the fourth; then by division the first shall have a greater proportion to the second, then the third to the fourth.

a 10. 5.  
b 5. ax.  
c 8. 4.  
d 17. 5.

Let  $\frac{AC}{BC} \square \frac{DE}{EF}$  then I say  $\frac{AB}{BC} \square \frac{DE}{EF}$  For conceive  $\frac{GC}{BC} \square \frac{DE}{EF}$  a therefore  $AC \square GC$ . Take away BC, that is common; then b remains  $AB \square GB$ . c Therefore  $\frac{AB}{BC} \square \frac{GB}{BC}$  d or  $\frac{DE}{EF}$  *W.W. to be Dem.*

P R O P.

PROP. XXX.



If the first compounded with the second have a greater proportion to the second, then the third compounded with the fourth hath to the fourth, then by converse ratio shall the first compounded with the second have a lesser ratio to the first, then the third compounded with the fourth shall have to the third.

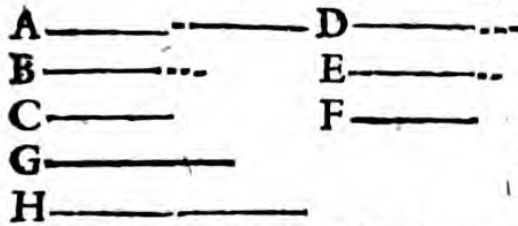
Let  $\frac{AC}{BC} > \frac{DF}{EF}$ . Then I say that  $\frac{AC}{AB} < \frac{DF}{DE}$ . For

because that  $\frac{AC}{BC} > \frac{DF}{EF}$  therefore by division  $\frac{AB}{BC} < \frac{DE}{EF}$ .

a hyp.  
b 19. 5.  
c 16. 5.  
d 18. 5.

$\frac{AB}{BC} < \frac{DE}{EF}$  by conversion c therefore  $\frac{BC}{AB} > \frac{EF}{DE}$  and therefore by composition  $\frac{AC}{AB} > \frac{DF}{DE}$  W. W. to be Dem.

PROP. XXXI.



If there be three magnitudes A, B, C, & others also D, E, F equal to them in number; & if there be a greater proportion of the first of the former to the second, then there is of the first of the last to their second ( $\frac{A}{B} > \frac{D}{E}$ ) and

and there, be also a greater proportion of the second of the first magnitudes to the third, then there is of the second of the last magnitudes to their third ( $\frac{B}{C} > \frac{E}{F}$ ).

Then by equality also shall the ratio of the first of the former magnitudes to the third be greater then the ratio of the first of the latter magnitudes to the third ( $\frac{A}{C} > \frac{D}{F}$ ).

$\frac{A}{C} > \frac{D}{F}$   $G = E$

Conceive  $\frac{G}{F}$  a therefore is  $\frac{B}{G} > \frac{E}{F}$ ; & b therefore  $\frac{A}{G} > \frac{A}{B}$ . Again conceive  $\frac{H}{G} = \frac{D}{F}$  c therefore

a 10. 5.  
b 8. 5.  
c 13. 5.  
d 10. 5.  
e 8. 5.  
f 12. 5.

$\frac{H}{G} > \frac{A}{B}$  therefore much more  $\frac{H}{G} > \frac{A}{G}$  d wherefore  $\frac{A}{G} < \frac{H}{G}$ .

$\frac{A}{G} < \frac{H}{G}$  & consequently  $\frac{A}{C} > \frac{H}{C}$  for D  $\frac{H}{F}$ .

PROP.

P R O P. XXXII.

A ————— D —————  
 B ————— E —————  
 C ————— F —————  
 G —————  
 H —————

If there be three magnitudes A, B, C, and others D, E, F, equal to them in number ; and there be a greater proportion of the first of the former magnitudes to the second , then there is of the second of the latter to the third  $\left(\frac{A}{B} > \frac{E}{F}\right)$

and also the ratio of the second of the former to the third be greater then the ratio of the first of the latter to the second  $\left(\frac{B}{C} > \frac{D}{E}\right)$  then by equality also shall the proportion of the first of the former to the third , be greater then that of the first of the latter to the third  $\left(\frac{A}{C} > \frac{D}{E}\right)$

The demonstration of this proposition is altogether like that of the precedent.

P R O P. XXXIII.

E  
 A-----|-----B  
 C-----|-----D  
 F

If the proportion of the whole A B to the whole C D be greater then the proportion of the part taken away A E to the part taken away C F ; then shall also the ratio of the remainder E B to the remainder F D be greater then that of the whole A B to the whole C D.

a 17. 5.  
 b 27. 5.  
 c 30. 5.

Because that  $\frac{AB}{CD} > \frac{AE}{CF}$  therefore by permutation  $\frac{AB}{AE} > \frac{CD}{CF}$  therefore by converse ratio  $\frac{AB}{EB} > \frac{CD}{FD}$  and by permutation again  $\frac{AB}{EB} > \frac{EB}{FD}$   
 W. W. to be Dem.

P R O P. XXXIV.

A	—————	D	—————	If there be any number of magni- tudes, and others al- so equall to them in number ; and the proportion of the first of the former to the first of the latter be greater then that of the second to the second , and that greater then the proportion of the third to the third , and so forward: all the former magni- tudes together shall have a greater ratio to all the latter together , then all the former , leaving out the first, shall have to all the latter, leaving out the first ; but lesse then that of the first of the former to the first of the latter ; and lastly greater then that of the last of the former to the last of the latter.
B	—————	E	—————	
C	—————	F	—————	
G	—————	H	—————	

You may please to consult Interpreters for the demonstration hereof, we having for brevities sake omitted it , and because 'tis of no use in these Elements.

The End of the fifth Book.

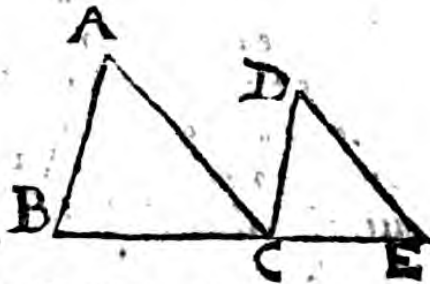
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THE

THE SIXTH BOOK  
OF  
EUCLIDE'S ELEMENTS.

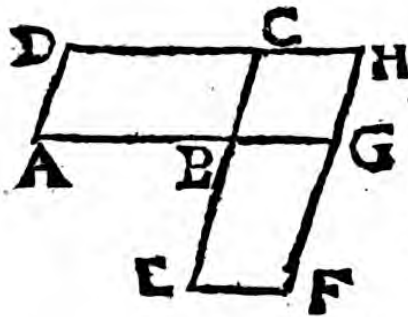
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*Definitions.*

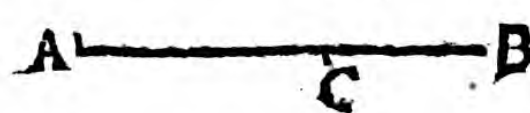


I. Like right-lined figures ( $ABC, DCE$ ) are such whose severall angles are equall one to the other, and also their sides about the equall angles, proportional.

*The angle  $B = DCE$ , and  $AB. BC :: DC. CE$ . Also the angle  $A = D$ , and  $BA. AC :: CD. DE$ . Lastly the angle  $ACB = E$ , and  $BC. CA :: CE. ED$ .*

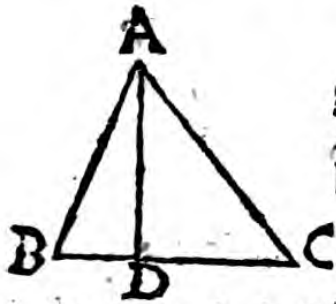


II. Reciprocal figures are ( $BD, BF$ ) when in either figure are the terms antecedents and consequents of ratios (that is,  $AB. BG :: EB. BC$ .)



III. A right line  $AB$  is said to be cut according to mean and extreme proportion, when as the whole  $AB$  is to the greater segment  $AC$ , so is the greater segment  $AC$  to the lesse  $CB$  ( $AB, AC :: AC. CB$ .)

IV.

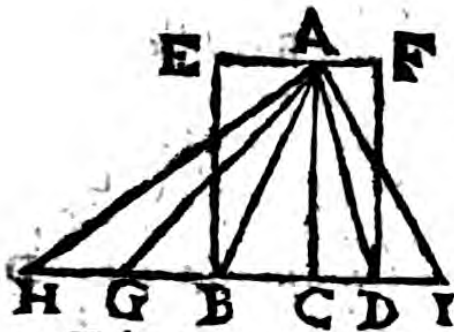


IV. The altitude of any figure ABC, is a perpendicular line AD drawn from the top A to the base BC.

V. A ratio is said to be compounded of two ratios, when the quantities of the ratios being multiplied the one into the other, do produce any ratio. As the ratio of A to C is compounded of the ratios of A to B and B to C. For  $\frac{A}{B} \times \frac{B}{C} = \frac{A}{C} = \frac{AB}{BC}$ .

a 10. def. 5.  
b 15. 4.

P R O P. I.



Triangles ABC, ACD, and parallelograms BCAG, CDAF, which have the same height, are in proportion one to the other, as their bases, BC, CD are.

a Take as many as you please, BG, GH, equal to BC, and also DI = CD. and join AG, AH, AI. a 3. 4

b The triangles ACB, ABG, AGH are equal, and also the triangle ACD = ADI. Therefore the b 38. 1.

the triangle ACH is as multiplex of the triangle ACB, as the base HC is of the base BC; and the triangle ACI as multiplex of the triangle ACD, as the base CI is of CD. But if HC =, =, = CI,

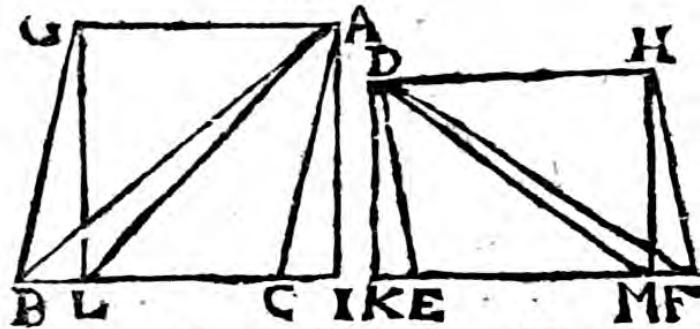
e then is likewise the triangle AHC =, =, = ACI; and therefore BC. CD :: the triangle ABC. ACD

c 28. 38 1.  
d 6 def. 5.  
e 41. 1. and  
15. 5.

:: e Pgr. CE. CF. W. W. to be Dem.



The sixth Book of  
Scholium.

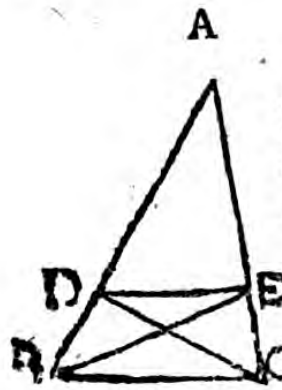


Hence, Triangles ABC, DEF, and Pgrs. AGBC, DEFH, whose bases BC, EF are equal, are in such proportion as their altitudes, AI, DK are.

a 3. 1.  
b 7. 1.  
c 1. 6  
d 41. 1. and  
15. 5.

a Take  $IL = CB$ , and  $KM = EF$ ; and join LA, LG, MD, MH. then is it evident that the triang. ABC. DEF :: b ALI. DKM :: c AI. DK :: d pgr. AGBC. DEFH. *W.W. to be Dem.*

PROP. II.



If to one side BC of a triangle ABC be drawn a parallel right line DE, the same shall cut the sides of the triangle proportionally ( $AD. BD :: AE. EC$ .) And if the sides of the triangle be proportionally cut ( $AD. BD :: AE. EC$ ) then a right line DE joined at the sections D, E, shall be parallel to the remaining side of the triangle BC. Draw CD and BE.

a 37. 1.  
b 7. 5.  
c 2. 6.  
d 11. 5.

1. *Hyp.* Because the Triangle DEB a = DEC, therefore shall be the triangle ADE. DBE :: ADE. ECD. But the triang. ADE. DBE :: c AD. DB, and the triangle ADE. DEC :: AE. EC, d therefore  $AD. DB :: AE. EC$ .

e 1. 6.  
f 9. 5.  
g 39. 1

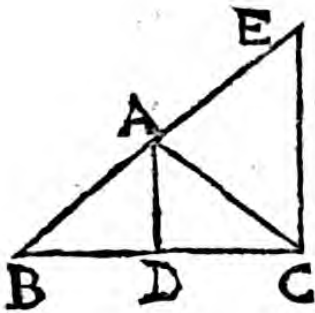
2. *Hyp.* Because  $AD. DB :: AE. EC$ , e that is the triangle ADE. DBE :: ADE. ECD; f therefore is the triangle DBE = ECD; and g therefore DE, BC are parallels. *W.W. to be Dem.*

Scholium.

If there be drawne many paralleles to one side of any triangle, then all the segments of the sides shall

shall be proportionall ; as is easily deducible from the precedent.

PROP. III.



If an angle BAC of a triangle BAC be bisected, and the right line AD, that bisects the angle, cut the base also; then shall the segments of the base have the same ratio that the other sides of the triangle have,

(BD. DC :: AB. AC.) And if the segments of the base have the same ratio, that the other sides of the triangle have (BD. DC :: AB. AC) then a right line AD drawn from the top A to the section D, shall bisect that angle BAC of the triangle.

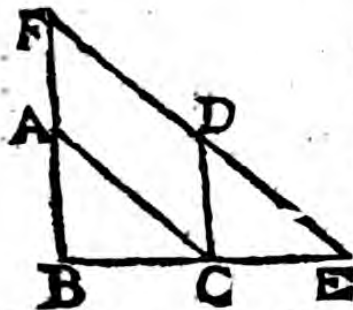
Produce BA, and make AE = AC, and join CE.

1. Hyp. Because AE = AC, therefore is the angle ACE = E = 1/2 BAC = DAC; therefore DA, CE are parallels. Wherefore BA. AE (AC) :: BD. DC.

2. Hyp. Because BA.AC (AE) :: BD.DC, therefore are DA, CE parallels; and therefore is the angle BAD = E; and the angle DAC = ACE = E. therefore the angle BAD = DAC. Wherefore the angle BAC is bisected. W.W. to be Dem.

a 5. 1.  
b 32. 1.  
c hyp.  
d 27. 1.  
e 2. 6.  
f 2. 6.  
g 29. 1.  
h 5. 1.  
k 1. ax.

PROP. IV.



Of equiangular triangles ABC, DCE, the sides are proportionall which are about the equall angles, B, DCE, (AB. BC :: DC. CE, &c.) and the sides AB, DC, & C which are subtended under the equall angles ACB, E, &c. are homologous, or of like ratio.

Set the side BC in a direct line to the side CE, and produce BA and ED till they meet.

a 32. 1.

b hyp.  
c 18. 1.  
d 14. 1.  
e 2. 6.  
f 16. 5.  
g 22. 5.

Because the angle  $\angle B = \angle ECD$ , therefore  $BF, CD$  are parallel: Also because the angle  $\angle BCA = \angle CED$ , therefore are  $CA, EF$  parallel. Therefore the figure  $CAFD$  is a Pgr. therefore  $AF = CD$ , and  $AC = FD$ . Whence it is evident that  $AB.AF (CD) :: BC.CE$ . *f* by permutation therefore  $AB.BC :: CD.CE$ . also  $BC.CE :: FD (AC).DE$ . *f* and thence by permutation  $BC.AC :: CE.DE$ . *g* Wherefore also by equality  $AB.AC :: CD.DE$ . Therefore, &c.

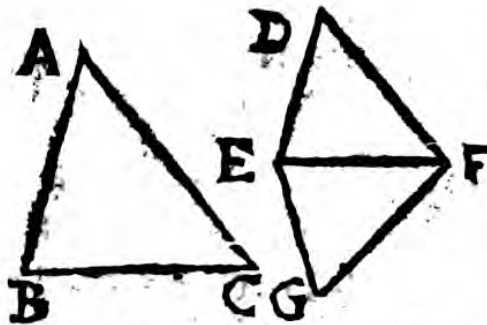
Coroll.

Hence  $AB.DC :: BC.CE :: AC.DE$ .

Schol.

Hence, If in a triangle  $FBE$  there be drawn  $AC$  a parallel to one side  $FE$ , the triangle  $ABC$  shall be like to the whole  $FBE$ .

PROP. V.



If two triangles  $ABC, DEF$  have their sides proportionall ( $AB.BC :: DE.EF$ , and  $AC.BC :: DF.EF$ , & also  $AB.AC :: DE.DF$ ) those triangles are equian-

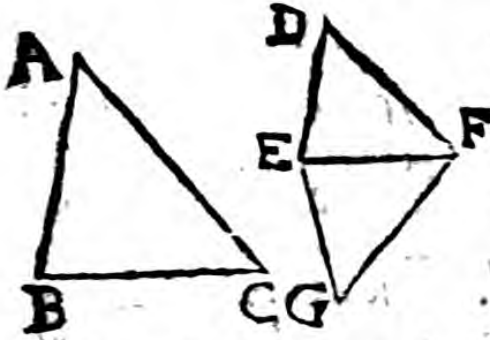
gular, and those angles equall under which are subtended the homologous sides.

At the side  $EF$  *a* make the angle  $\angle FEG = B$ , and the angle  $\angle EFG = C$ ; *b* whence the angle  $\angle G = A$ . Therefore  $GE.EF$  *c*  $:: AB.BC$  *d*  $:: DE.EF$ . *e* and therefore  $GE = DE$ . Likewise  $GF.FE$  *c*  $:: AC.CB$  *d*  $:: DF.FE$ . *e* therefore  $GF = DF$ . Therefore the triangles  $DEF, GEF$  are mutually equilaterall. *f* Therefore the angle  $\angle D = \angle G = A$ , and the angle  $\angle FED$  *f*  $= \angle FEG = B$ , and *g* consequently the angle  $\angle DFE = C$ . Therefore, &c.

a 23. 1.  
b 32. 1.  
c 4. 6.  
d hyp.  
e 11. 5. and  
f 8. 1.  
g 32. 1.

PROP.

PROP. VI.



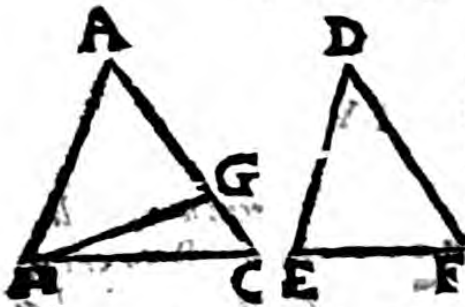
If two triangles  $ABC, DEF$  have one angle  $B$  equal to one angle  $DEF$ , and the sides about the equal angles  $B, DEF$  proportionall ( $AB. BC :: DE. EF$ ) then those triangles

$ABC, DEF$  are equiangular, and have those angles equal, under which are subtended the homologous sides.

At the side  $EF$  make the angle  $FEG = B$ , and the angle  $EFG = C$ ; then will the angle  $G = A$ . Therefore  $GE. EF :: AB. BC :: DE. EF$ , and therefore  $DE = GE$ . But the angle  $DEF = B = GEF$ ; therefore the angle  $D = G = A$ , and consequently the angle  $EFD = C$ . *W. W. to be Dem.*

a 32. 1.  
b 4. 6.  
c hyp.  
d 9. 5.  
e hyp.  
f const.  
g 4. 1.  
h 32. 1.

PROP. VII.



If two triangles  $ABC, DEF$  have one angle  $A$  equal to one angle  $D$ , and the sides about the other angles  $ABC, E$ , proportionall ( $AB. BC :: DE. EF$ ) and if they have both

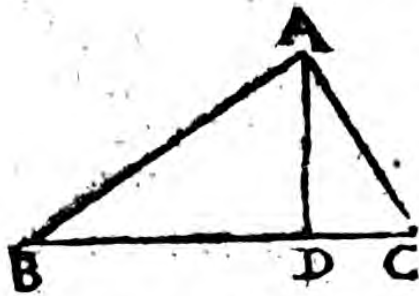
of the remaining angles  $C, F$  either lesse or not lesse then a right angle; then shall the triangles  $ABC, DEF$  be equiangular, and have those angles equal about which the proportionall sides are.

For, if it can be, let the angle  $ABC < E$ , and make the angle  $ABG = E$ . Therefore, whereas the angle  $A = D$ , thence is the angle  $AGB = F$ . Therefore  $AB. BG :: DE. EF :: AB. BC$ . therefore  $BG = BC$ . therefore the angle  $BGC = BCG$ . Therefore  $BGC$  or  $C$  is lesse then a right angle, & consequently  $AGB$  or  $F$  is greater then a right. Therefore the angles  $C$  and  $F$  are not of the same species or kind, which is against the Hypothesis.

a hyp.  
b 32. 1.  
c 4. 6.  
d hyp.  
e 9. 5.  
f 5. 1.  
g cor. 17. 1.  
h cor. 13. 1.

The sixth Book of

PROP. VIII.



If in a right-angled triangle ABC, from the right-angle BAC there be drawn AD a perpendicular to the base BC; then the triangles about the perpendicular (ADB, ADC) are like both to the whole triangle

ABC, and also one to the other.

a hyp.  
b 12. ax.  
c 32. & 4. 6.  
d 21. 6.

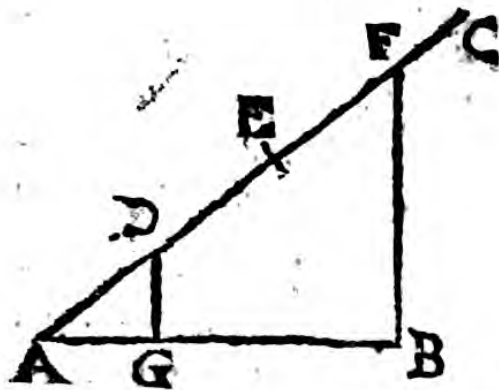
For because BAC, ADB are <sup>a</sup> right angles, <sup>b</sup> and so equall, and B common; the triangles BAC, ADB are like. By the same argum. BAC, ADC are like <sup>c</sup> whence also ADB, ADC will be like. *W.W. to be Dem.*

Coroll.

e 1. def. 6.

Hence, 1.  $BD \cdot DA \text{ } \therefore \text{ } DA \cdot DC$ .  
2.  $BC \cdot AC \text{ } \therefore \text{ } AC \cdot DC$  and  $CB \cdot BA \text{ } \therefore \text{ } BA \cdot BD$ .

PROP. IX.



From a right line given AB to cut off any part required, as  $\frac{1}{5}$  (AG.)

From the point A draw an infinite line AC any-wise, in which <sup>2</sup> take any three equall parts AD, DE, EF. join

f 3. 1.

g 31. 4.

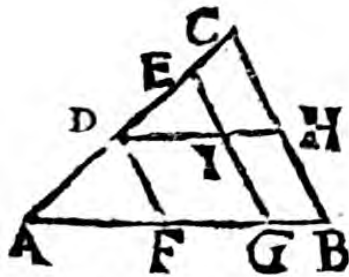
<sup>1</sup> FB, to which from D <sup>b</sup> draw the parallel DG; and the thing is done.

e 2. 6.  
d 18. 5.

For  $GB \cdot AG \text{ } \therefore \text{ } FD \cdot AD$ ; whence by <sup>a</sup> composition  $AB \cdot AG \text{ } \therefore \text{ } AF \cdot AD$ . therefore, whereas AD =  $\frac{1}{5}$  of AF, therefore is AG =  $\frac{1}{5}$  of AB. *W.W. to be Done.*

PROP

PROP. X.



To divide a right line given AB not divided (in F and G) as another line given AC was cut (in D and E.)

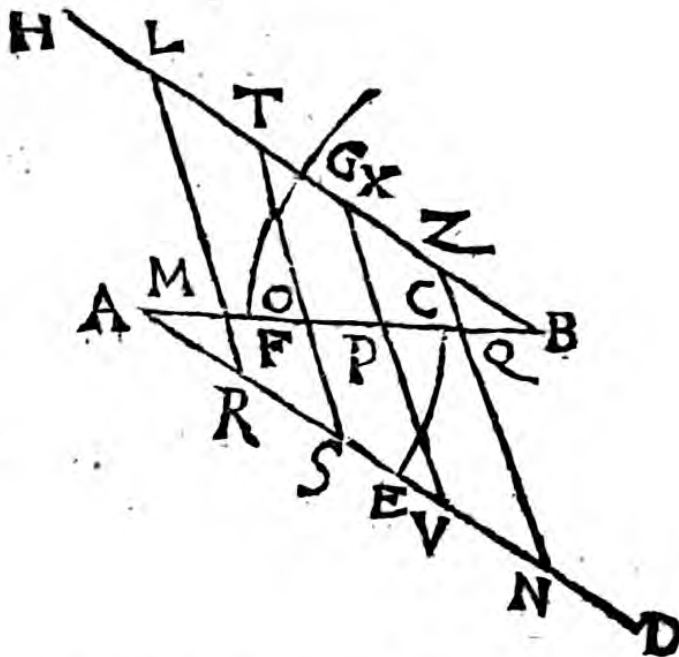
Let a right line BC join the extremities of the line divided, and of the line not divided; and to that line from the points E, D draw the parallels EG, DF meeting with the right line that is to be cut, in G & F; Then the thing is Done.

For let DH be drawn parallel to AB. Then AB, DE :: AF, FG. and DE, EC :: DI, IH :: FG, GB. W. W. to be Done.

a 31. 1.

b 2. 6.  
c 34. 1. and  
7. 5.

Schol.



Hence is learnt to cut a right line given AB into as many equal parts as you please (suppose 5;) which will be more easily performed thus.

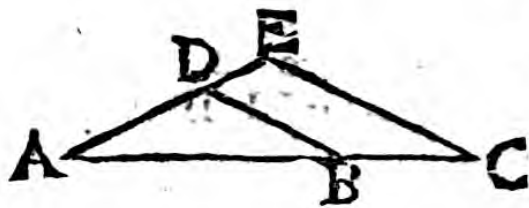
Draw an infinite line AD, and another BH parallel to it and infinite also. Of those take equal parts, AR, RS, SV, VN; and BZ, ZX, XT, TL; in

in each line leſſe parts by one, then are required in AB; then let the right lines LR, TS, XV, ZN be drawn; these lines ſo drawn ſhall cut the right line given AB into five parts.

a 33. 1.  
b conſtr.  
c 2. 6.

For RL, ST, VX, NZ are *a* parallels; therefore, whereas AR, RS, SV, VN are *b* equall; *c* thence AM, MO, OP, PQ are equall alſo. Likewise, becauſe that BZ = ZX, therefore is BQ = PQ. and therefore AB is cut into five parts. *W. W. to be Done.*

P R O P. XI.

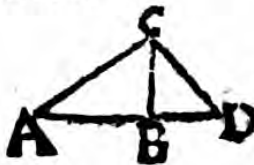


Two right lines being given, AB, AD, to find out a third in proportion to them (DE.)

Join BD, and from AB being produced take BC = AD. Through C draw CE parallel to BD; with which let AD produced meet in E. then is DE the proportionall required.

b 2. 6.

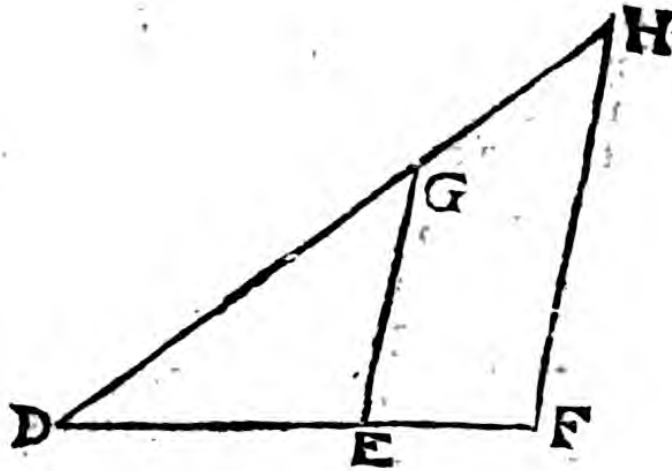
For AB. BC (AD) *a* :: AD. DE. *W. W. to be Done.*



b cor. 9.

Or thus: make the angle ABC right, and alſo the angle ACD right. then *b* AB. BC :: BC. BD.

PROP. XII.



Three right lines being given DE, EF, DG, to find out a fourth proportionall GH.

Join EG, and thorough F draw FH parallel to EG; with which let DG produced to H meet. Then it is evident that DE. EF  $\therefore$  DG. GH. *W. W. to be Done.* a 2. 6.

PROP. XIII.



Two right lines being given AE, EB, to find out a mean proportionall E-F.

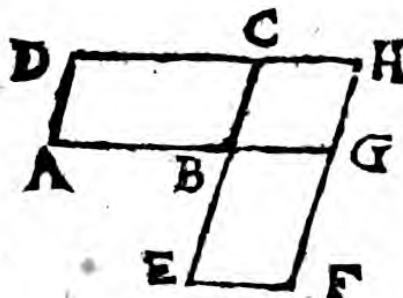
Upon the whole line AB as a diameter describe a semicircle AFB, & from E erect a perpendicular EF meeting with the periphery in F. then AE. EF  $\therefore$  EF. EB. For let AF & FB be drawn; <sup>a</sup> then from the right angle of the right-angled triangle AFB is drawn a right line FE perpendicular to the base. <sup>b</sup> Therefore AE. FE  $\therefore$  FE. EB. *W. W. to be Done.* a 31. 3. b cor. 8. 6.



Coroll.

Hence, A right line drawn in a circle from any point of the diameter perpendicularly, and extended to the circumference, is a mean proportion all betwixt the two segments of the diameter.

P R O P. XIV.



Equall Parallelograms having one angle ABC equall to one EBG, have the sides BD, BF which are about the equall angles reiprocall (A B. B G :: E B. B C;) and those Parallelograms B D, B F

which have one angle ABC equall to one EBG, and the sides which are about the equall angles reciprocally, are equall.

For let the sides AB, BG about the equall angles make one right line; a wherefore EB, BC shall doe the same. Let FG, DC be produced till they meet.

1. Hyp. AB.BG *b* :: BD. BH :: *c* BF. BH :: *d* BE. BC. *e* therefore, &c.

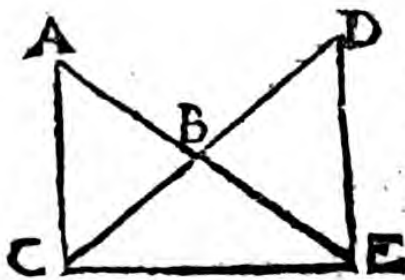
2. Hyp. BD. BH :: *f* AB. BG :: *g* BE. BC :: *h* BF. BH. *k* Therefore the Pgr. BD = BF. Which was to be

a 7. 1. 1.  
b 1. 6.  
c 7. 5.  
d 1. 6.  
e 11. 5.  
f 1. 6.  
g Hyp.  
h 1. 6.

k 11. 5. 5. Dem.

P R O P.

PROP. XV.



Equall triangles having one angle ABC equall to one DBE, their sides which are about the equall angles are reciprocall (AB. BE :: DB. BC.) And those triangles that have one angle

ABC equall to one DBE, and have also the sides that are about the equall angles reciprocall (AB. BE :: DB. BC.) are equall.

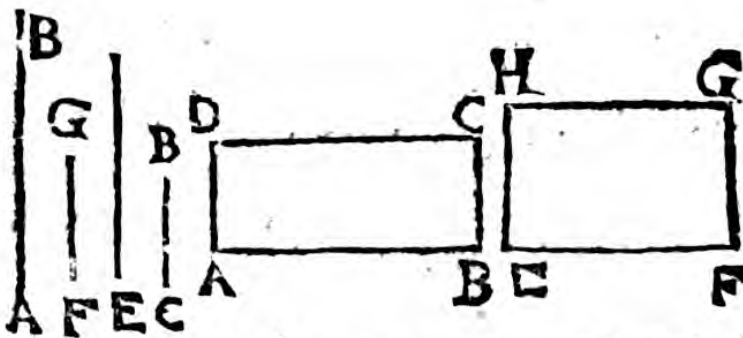
Let the sides CB, BE, which are about the equall angles be set in a strait line; therefore ABE is a right line. Let CE be drawn.

1. Hyp. AB. BE :: b the triangle ABC. CBE :: the triangle DBE. CBE :: d DB. BC. e therefore, &c.

2. Hyp. The triangle ABC. CBE :: f AB. BE :: g DB. BC :: the triangle DBC. CBE. k Therefore the triangle ABC = DBE. w.w. to be Dem.

b 1. 6.  
c 7. 5.  
d 1. 6.  
e 11. 5.  
f 1. 6.  
g hyp.  
h 1. 6.  
k 11. & 9. 5.

PROP. XVI.



If four right lines be proportionall (AB. FG :: EF. CB) the rectangle AC comprehended under the extremes AD, CB, is equall to the rectangle EG comprehended under the means FG, EF. And if the rectangle AC comprehended under the extremes AB, CB be equall to the rectangle EG comprehended under the means FG, EF, then are the four right lines proportionall. (AB. FG :: EF. CB)

I. Hy

The sixth Book of

b 12. cor.

b 14. 6.

c hyp.

d 14. 6.

1. Hyp. The angles B and F are right, and  $\therefore$  consequently equal, & by hypothesis  $AB. FG :: EF. CB.$

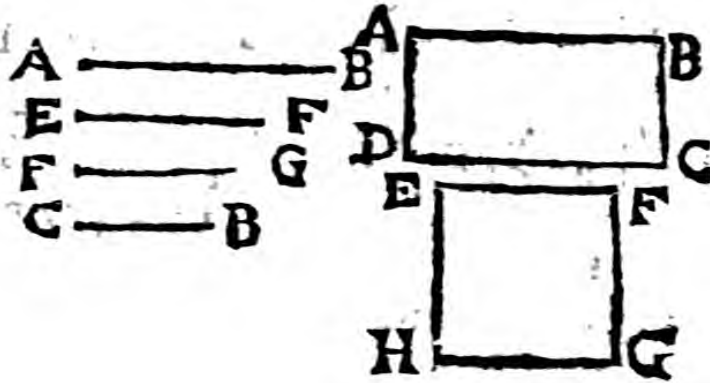
$\therefore$  therefore the rectangle  $AC = EG.$   
 2. Hyp. The rectangle  $AC = EG,$  and the angle  $B = F;$   $\therefore$  therefore  $AB. FG :: EF. CB. W.W. to be Dem.$

Coroll.

e 12. 6.

Hence, it is easy to apply a rectangle given  $EG$  to a right line given  $AB;$  (*viz.*) by making  $AB. EF :: FG. BC.$

PROP. XVII.



If three right lines be proportionall ( $AB. EF :: EF. CB.$ ) the rectangle  $AC$  made under the extremes  $AB, CB$  is equal to the square  $EG$  made of the middle  $EF.$  And if the rectangle  $AC$  comprehended under the extremes  $AB, CB$  be equal to the square  $EG$  made of the middle  $EF,$  then the three lines are proportionall, ( $AB: EF :: EF. CB$ ).

Take  $FG = EF.$

a hyp.  
 b 16. 6.  
 c 19. def. 1.  
 d hyp.  
 e 16. 6.

1. Hyp.  $AB. EF :: EF (FG.) CB.$  therefore the rectangle  $AC = EG = EFq.$

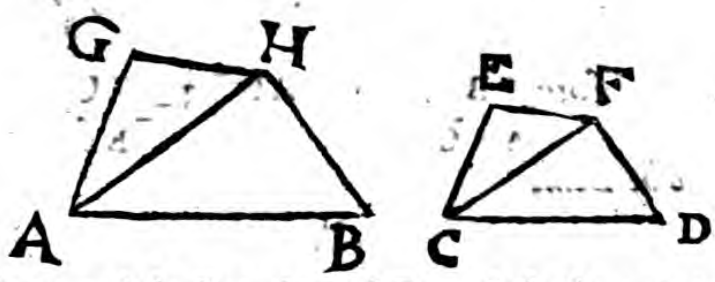
2 Hyp. The rectangle  $AC =$  to the square  $EG = EFq.$  therefore  $AB. EF :: FG (EF.) BC. W.W. to be Dem.$

Coroll.

Let  $A \times B = Cq.$  therefore  $A.C :: C.B.$

PROP.

EUCLIDE'S *Elements.*  
 P R O P. XVIII.



From a right line given  $AB$  to describe a right-lined figure  $AGHB$  like and alike situate to a right-lined figure given  $CEFD$ .

Resolve the right-lined figure given into triangles;  
 \* Make the angle  $ABH = D$ , <sup>a</sup> & the angle  $BAH = DCF$ , <sup>a</sup> and the angle  $AHG = CFE$ , <sup>a</sup> and the angle  $HAG = FCE$ . then  $AGHB$  shall be the right-lined figure sought.

For the angle  $B = D$ , and the angle  $BAH = DCF$ , <sup>b</sup> wherefore the angle  $AHB = CFD$ . <sup>b</sup> also the angle  $HAG = FCE$ , and the angle  $AHG = CFE$ . <sup>c</sup> wherefore the angle  $G = E$ , and the whole angle  $GAB = ECD$ , and the whole angle  $GHB = EFD$ . The polygons therefore are mutually equiangular. Moreover because the triangles are equiangular, therefore  $AB.BH :: CD.DF$ ; and  $AG.GH :: CE.EF$ . Likewise  $AG.AH :: CE.CF$ . and  $AH.AB :: CF.CD$ . <sup>f</sup> From whence by equality  $AG.AB :: CF.CD$ . After the same manner  $GH.HB :: EF.FD$ . Therefore the Polygons  $ABHG$ ,  $CDFE$  are like and alike situate. *W.W. to be Done.*

P R O P. XIX.



Like triangles  $ABC$ ,  $DEF$  are in duplicate ratio of their homologous sides,  $BC$ ,  $EF$ .

<sup>a</sup> Let there be made  $BC.EF :: EF.BG$ , and let  $AG$  be drawn. Because that  $AB.DE$

<sup>a</sup> 11. 6.

b cor. 4. 6)  
 c constr.  
 d 15. 6.  
 e 1. 6.  
 f 10. def. 5.  
 g 11. 5.

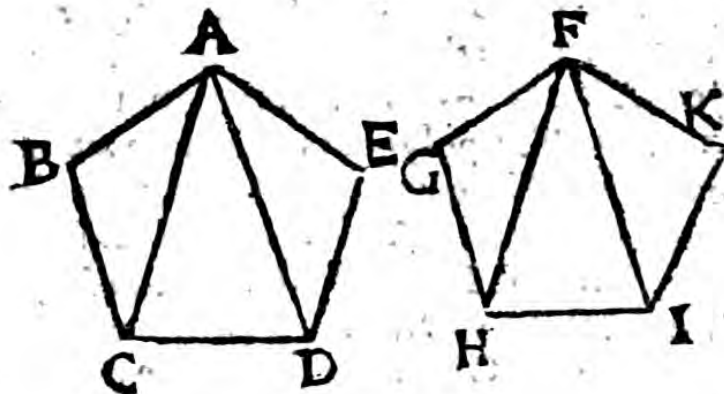
DE  $b :: BC$ . EF  $c :: EF$ . BG, and the angle  $B = E$ .  
 therefore is the triangle  $ABG = DEF$ . But the  
 triangle ABC.  $ABG :: e BC$ . BG, and  $f \frac{BC}{BG} = \frac{BC}{EF}$   
 twice; therefore ABC that is  $\frac{ABC}{ABG} g = \frac{BC}{EF}$  twice.

W.W. to be Dem.

Coroll.

Hence, If three right lines (BC, EF, BG) be proportionall, then as the first is to the third, so is a triangle made upon the first BC, to a triangle like and alike described upon the second EF : or so is a triangle described upon the second EF to a triangle like and alike described upon the third.

PROP. XX.



Like polygones ABCDE, FGHIK are divided into equall triangles ABC, FGH, and ACD, FHI, and ADE, FIK ; both equall in number and homologous to the wholes (ABC. FGH :: ABCDE. FGHIK :: ACD. FHI :: ADE. FIK.) And the polygones ABCDE, FGHIK have a double ratio one to the other of what one homologous side BC hath to the other homologous side GH.

a hyp.  
 b 6. 6.

1. For the angle  $B = G$ , and  $AB \cdot BC :: FG \cdot GH$ . therefore the triangles ABC, FGH are

are equiangular. After the same manner are the triangles AED, FKI like. Whereas therefore the angle BCA = GHF, and the angle ADE = FIK, and the whole angles BCD, GHI, and the whole angles CDE, HIK are equal, there remains the angle ACD = FHI, and the angle ADC = FIH; from whence also the angle CAD = HFI. therefore the triangles ACD, FHI are like: Therefore, &c.

2. Because that the triangles BCA, GHF are like, therefore is  $BCA \sim BC$  twice. For the same reason  $GHF \sim GH$

is  $CAD \sim CD$  twice; lastly  $DEA \sim DE$  twice.  $HFI \sim HI$   $IKF \sim IK$

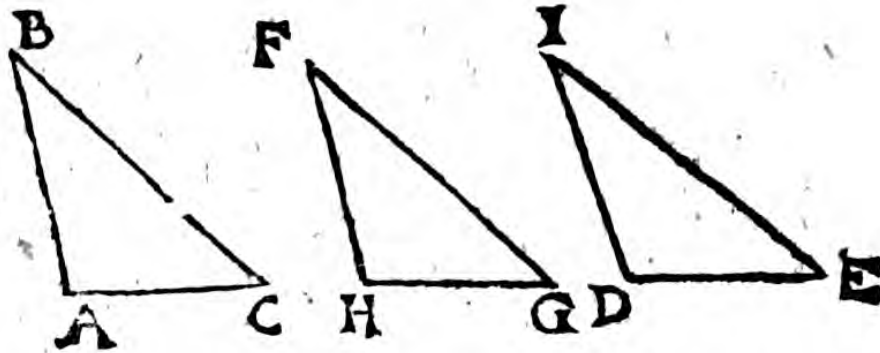
now whereas that  $BC.GH :: CD.HI :: DE.IK$ , therefore is the triangle  $BCA.GHF :: CAD.HFI :: DEA.IKF ::$  the polygon  $ABCDE.FGHIK :: BC$  twice.  $GH$

Coroll.

1. Hence, If there be three right lines proportionall; then as the first is to the third, so is a polygone made upon the first to a polygone made on the second like and alike described; or so is a polygone described upon the second to a polygone made on the third like and alike described.

By which is found out a method of inlarging or diminishing any right-lined figure in a ratio given: As if you would make a pentagone quintuple of that pentagone whereof CD is the side, then betwixt AB and 5 AB find out a mean proportional, upon this raise a pentagone like to that given, and it shall be quintuple of the pentagone given. \* 18 6.

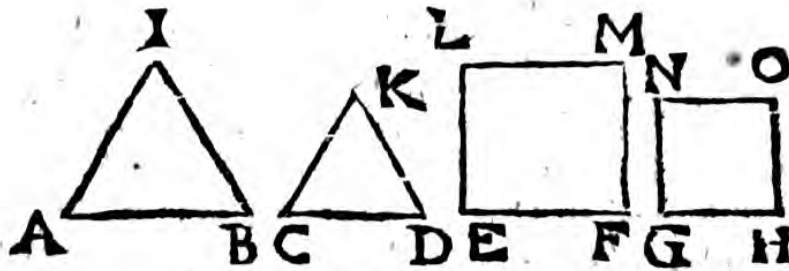
2. Hence also, If the homologous sides of like figures be known, then will the proportion of the figures be evident, viz. by finding out a third proportional.



Right-lined figures ABC, DIE which are like to the same right-lined figure HFG, are also like one to the other.

<sup>a</sup> 1. def. 6. For the angle A = H = D; and the angle C = G = E; and the angle B = F = I. Also <sup>a</sup> AB.AC :: HF.HG :: DI.DE. & <sup>a</sup> AC.CB :: HG.GF :: DE.EI. And AB.BC :: HF.FG :: DI.IE. Therefore <sup>a</sup> ABC, DIE are like. W. W. to be Dem.

P R O P. XXII.



If four right lines be proportionall (AB. CD :: EF. GH) the right-lined figures also described upon them being like and in like sort situate, shall be proportionall (ABI.CDK :: EM. GO.) And if the right-lined figures described upon the lines, like and alike situate, be proportionall (ABI.CDK :: EM. GO) then the right lines also shall be proportionall (AB. CD :: EF. GH.)

<sup>a</sup> 19. 6.

1. Hyp.  $\frac{ABI}{CDK} = \frac{AB \text{ twice}}{CD} = \frac{EF \text{ twice}}{GH} = \frac{EM}{GO}$

<sup>b</sup> therefore ABI.CDK :: EM. GO.

<sup>b</sup> hyp.

<sup>c</sup> 10. 6.

<sup>d</sup> cor. 23. 5.

2. Hyp.  $\frac{AB \text{ twice}}{CD} = \frac{ABI}{CDK} = \frac{EM}{GO} = \frac{EF \text{ twice}}{GH}$

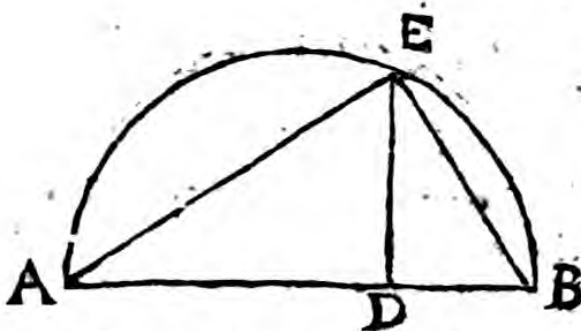
Therefore AB. CD :: EF. GH. W. W. to be Dem.

Schol.

Schol.

Hence is deduced the manner and reason of multiplying surd quantities. Ex.g. Let  $\sqrt{5}$  be to be multiplied into  $\sqrt{3}$ . I say that the product will be  $\sqrt{15}$ . For by the definition of multiplication it ought to be, as 1.  $\sqrt{3} :: \sqrt{5}$ . to the product. Therefore by this q.1. q.  $\sqrt{3} :: q. \sqrt{5}$ . q. of the product. That is 1.3. :: 5. to the square of the product. therefore the square of the product is 15. Wherefore  $\sqrt{15}$  is the product of  $\sqrt{3}$  into  $\sqrt{5}$ . W.W. to be Dem.

THEOREME.



If a right line AB be cut any-wise in D, the rectangle *comprehended under the parts AD, DB is a mean proportionall betwixt their squares. Likewise the rectangle comprehended under the whole AB and one part AD, or DB, is a mean proportionall betwixt the square of the whole AB and the square of the said part AD, or DB.*

Upon the diameter AB describe a semicircle; from D erect a perpendicular DE meeting with the periphery in E. join AE, BE.

It's evident that  $AD \cdot DE :: DE \cdot DB$  therefore  $ADq \cdot DEq :: DEq \cdot DBq$  that is,  $ADq \cdot ADB :: A \cdot B \cdot DBq$ . W.W. to be Dem.

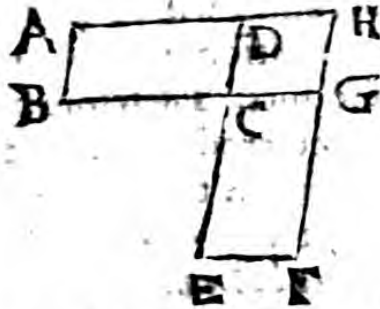
*Prop. 12.6.*  
*b 22.6.*  
*c 17.6.*

Moreover  $BA \cdot AE :: AE \cdot AD$  therefore  $BAq \cdot AEq :: AEq \cdot ADq$  that is  $BAq \cdot BAD :: BAD \cdot ADq$ . After the same manner  $ABq \cdot ABD :: ABD \cdot BDq$ . W.W. to be Dem.

*d cor. 8.6.*  
*e 22.6.*  
*f 17.6.*

Or thus: suppose  $Z = A + E$ . It is manifest that  $Aq \cdot AE :: A \cdot E :: AE \cdot Eq$  also  $Zq \cdot ZA :: Z \cdot A :: ZA \cdot Aq$  and  $Zq \cdot ZE :: Z \cdot E :: ZE \cdot Eq$ .





Equiangular parallelograms AC, CF, have the ratio one to the other, which is compounded of their sides.

$$\left(\frac{AC}{CF} = \frac{BC}{CG} + \frac{DC}{CE}\right)$$

a. 15.

Let the sides about the equal angles C be set in a direct line, and let the pgr. CH be completed, Then is the ratio of

b. 20 def. 5.  
c. 1. 6.

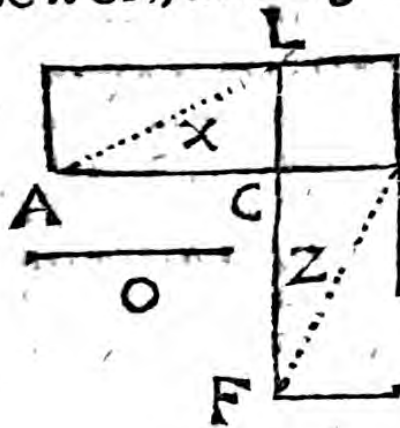
$$\frac{AC}{CF} = \frac{AC}{CH} + \frac{CH}{CF} = \frac{BC}{CG} + \frac{DC}{CE} \text{ W.W. to be Dem.}$$

Coroll.

Andr. Taeg.  
15. 5.

Hence, and from 34. 1. it appears 1. That triangles which have one angle equal (at C) have a ratio compounded of the ratio's of the right lines, (AC to CB, and LC to CF,) containing the equal angle.

35. 1.

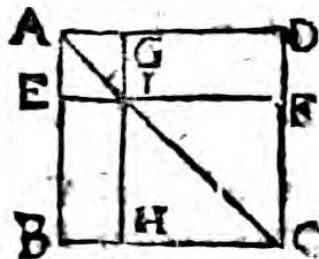


2. That all rectangles, & consequently all parallelograms, have their ratio one to the other compounded of the ratio's of base to base, and altitude to altitude. After the like manner you may argue in triangles.

3. From hence is apparent how to give the proportion of triangles and parallelograms. Let there be two pgrs. X, and Z, whose bases are AC, CB, and altitudes CL, CF. Make CL. CF :: CB. O. \* then will it be X, Z :: AC. O.

\* 14 6 and  
1. 6

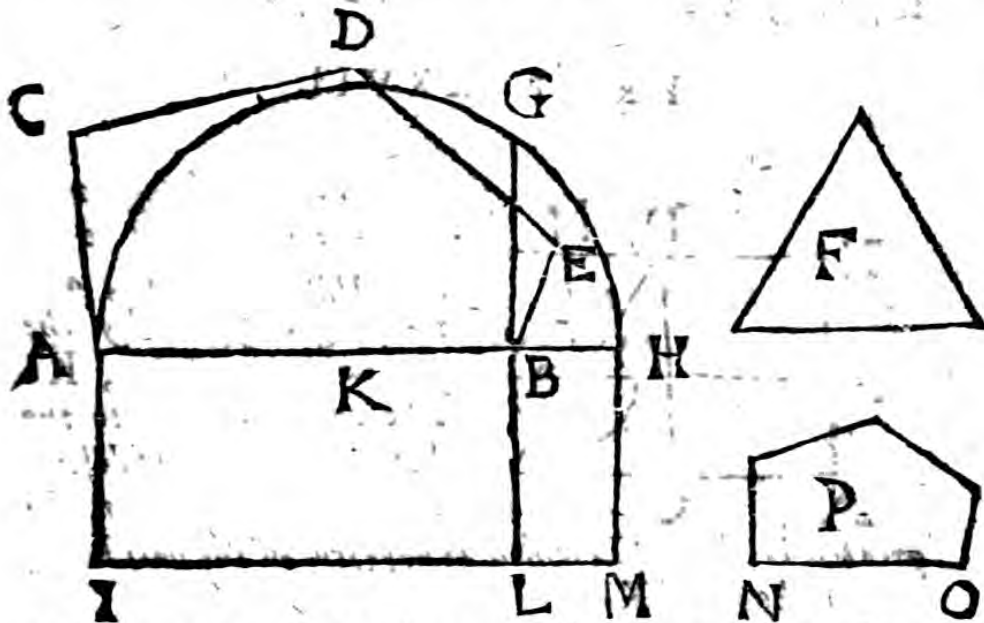
P. R O P. XXIV.



In every parallelogram ABCD, the parallelograms EG, HF which are about the diameter AC are like to the whole, and also one to the other.

For the Pgrs. EG, HF have each of them one angle common with the whole; therefore they are equiangular to the whole, & also one to the other. Also both the triangles ABC, AEI, IHC and the triangles ADC, AGI, IFC are equiangular mutually; therefore  $AE.EI :: AB.BC$ , and  $AE.AI :: AB.AC$ , and  $AI.AG :: AC.AD$ . Therefore by equality,  $AE.AG :: AB.AD$ . Therefore the Pgrs. EG, BD are like. After the same manner are HF, BD like also. Therefore, &c.

P R O P. XXV.



Unto a right-lined figure given ABEDC to describe another figure P like and alike situate, which also shall be equall to another right-lined figure given F.

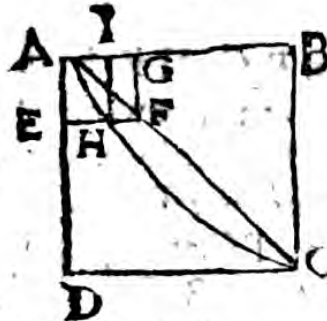
Make the rectangle AL = ABEDC; also upon BL make the rectangle BM = F; Betwixt AB and BH find out a mean proportionall NO; Upon NO, make the polygone B like to the right-lined figure given ABEDC. P say the polygone P so made shall be equall to F that was given. 13 For

a 45. 1.  
b 44. 1.  
c 13. 6.  
d 18. 6.

Cor. 20.6.  
 f 1.6.  
 g 14.5.  
 h confr.

For ABEDC (AL.) P :: <sup>e</sup> AB. BH :: <sup>f</sup> AL. BM.  
 Therefore P<sub>g</sub> = BM = F.W.W. to be Done.

P R O P. XXVI.



If from a Parallelogram AB-  
 CD, be taken away another Pa-  
 rallelogram AGFE, like unto the  
 whole, and in like sort set, having  
 also an angle common with it  
 EAG; then is that parallelogram  
 about the same diagonall AC

with the whole.

If you denye AG to be the common diagonall,  
 then let AHC be it, cutting EF in H, and let HI be  
 drawn parallel to AE. Then are Pgrs EI, DB <sup>a</sup> like,  
<sup>b</sup> therefore AE.EH :: AD.DC :: <sup>c</sup> AE.EF. & <sup>d</sup> con-  
 sequently EH = EF. <sup>f</sup> Which is Absurd.

g 24.6.  
 b 1. def.6.  
 c hyp.  
 d 9.5.  
 f 9. ex.

P R O P. XXVII.

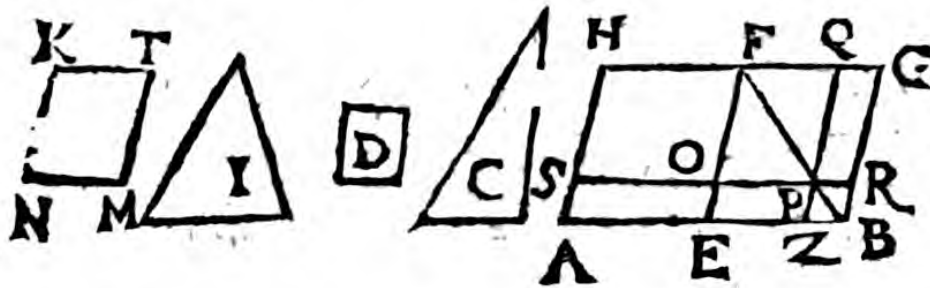


Of all parallelograms,  
 AD, AG applied to the  
 same right line AB, and  
 wanting in figure by the  
 parallelograms CE, KI  
 like and alike set to the  
 Pgr. AD which is descri-  
 taken away bed upon the  
 greatest is that AD which is applied to the half being  
 like to the defect KI.

For because that GE = GC, and KI added in  
 common, <sup>b</sup> thence is KE = CI = AM. adde CG in  
 common, <sup>d</sup> then is AG = to the Gnomon MBL. But  
 the Gnomon MBL = <sup>e</sup> CE (AD.) Therefore AG  
 = AD. W.W. to be Dem.

g 43. 1.  
 h 2. ex.  
 i 36. 1.  
 k 2. ex.  
 l 9. ex.

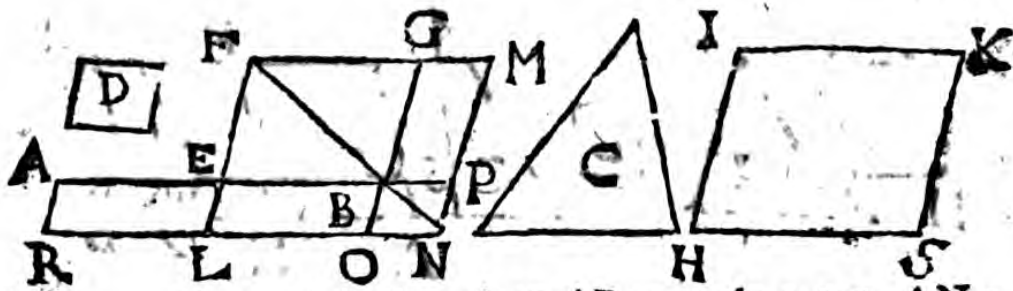
PROP. XXVIII.



Upon a right line given AB, to apply a parallelogram AP equal to a right-lined figure given C, deficient by a parallelogram ZR which is like to another parallelogram given D. \* Now it is requisite that the right-lined figure given C, whereunto the Pgr. to be applied AP must be equal, be not greater than the Pgr. AF which is applied upon the half line, the defects being like, namely the defect of the Pgr. AF, which is applied to the half line, and the defect of the Pgr. D to be applied whose defect is to be like to the Pgr. given. \* 27.6.

Bisect AB in E; upon EB make the Pgr. EG like to the Pgr. D; and let  $EG = C + I$ . Make the Pgr. NT = I, and like to the Pgr. given D, or EG; Draw the diameter FB; Make  $FO = KN$ , and  $FQ = KT$ ; thorough O and Q, draw the parallels SR, QZ. Then is the Pgr. AP that which was sought. a 18.6.  
b 45.1.  
c 15.6.

For the Pgrs D, EG, OQ, NT, ZR are all like one to the other, and the Pgr.  $EG = NT + C = OQ + C$ . wherefore  $C =$  to the Gnomon  $O B Q = A O + P G = A O + E P = A P$ . d constr. &  
14.6.  
e constr.  
f 3. ax.  
g 2. ax.  
h 43. 1.  
Which was to be Done.



Vpon a right line given AB to apply a pgr. AN equal to a right-lined figure given C, exceeding by a pgr. OP, which shall be like to another pgr. given D.

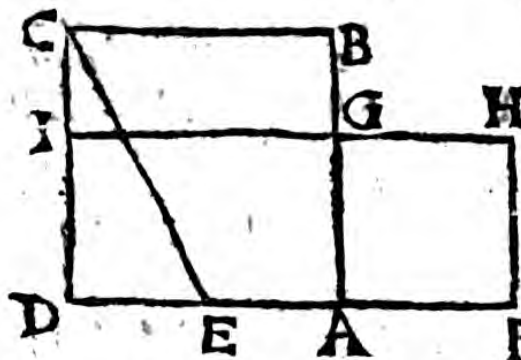
Bisect AB in E. Upon EB make a Pgr. EG like to D, which was given. And let the pgr. HK = EG + C, and like to D given, or EG. Make FEL = IH; and FG = HK. Thorough L, M draw the parallels MN and RN. and AR parallel to NM. produce ABP, GBO. Draw the diameter FBN. Then is AN the parallelogram required.

For the pgrs. D, HK, LM, EG are like. therefore the pgr. OP is like to the pgr. LM, or D. Also LM = HK = EG + C. Therefore C = to the Gnomon ENG. But AL = LB = BM. therefore C = AN. *W.W. to be Done.*

a 18. 6.  
b 25. 6.  
c 3. 1.

d anfr.  
e 24. 6.  
f constr.  
g 3. ax.  
h 16. 1.  
k 41. 1.  
l 2. & 1. ax.

PROP. XXX.



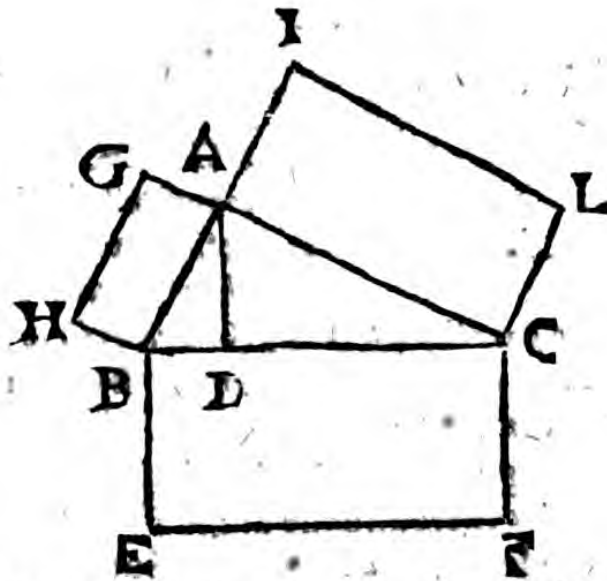
To cut a finite right line given AB according to extreme and mean ratio (AB. AG :: AG. GB.)

Cut AB in G, in such wise that AB x BG = AG<sup>2</sup>.

Then BA. AG :: AG. GB. Which was to be Done.

a 11. 2.  
b 17. 6.

PROP. XXXI.



In right-angled triangles  $BAC$ , any figure  $BF$  described upon the side  $BC$  subtending the right-angle  $BAC$ , is equal to the figures  $BG$ ,  $AL$ , described upon the sides  $BA$ ,  $AC$ , containing the right-angle, like and alike situate to the former  $BF$ .

From the right angle  $BAC$  let down a perpendicular  $AD$ . Because that  $DC \cdot CA :: CA \cdot CB$ , therefore  $AL \cdot BF :: DC \cdot CB$ . Also, because  $DB \cdot BA :: BA \cdot BC$ , therefore  $BG \cdot BF :: DB \cdot BC$ , therefore  $AL + BG \cdot BF :: DC + DB \cdot BC$ . Therefore  $AL + BG = BF$ . *W.W. to be Dem.*

a cor. 8. 6.  
b cor. 20 6.

c 24 5.  
d schol. 14. 5.

Or thus:  $BG \cdot BF :: BA^2 \cdot BC$ . And  $AL \cdot BF :: AC^2 \cdot BC$ . Therefore  $BG + AL \cdot BF :: BA^2 + AC^2 \cdot BC$ . Therefore whereas  $BA^2 + AC^2 = BC^2$ , thence is  $BG + AL = BF$ . *W.W. to be Dem.*

e 22. 6.

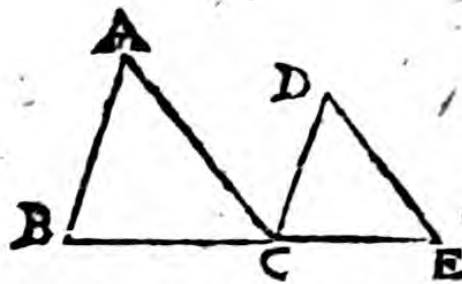
f 24. 5.  
g schol. 14. 5.  
h 47. 1.

Coroll.

From this proposition you may learn how to add or subtract any like figures, by the same method that is used in adding and subtracting of squares, in Schol. 47. 1.

PROP.

PROP. XXXII.



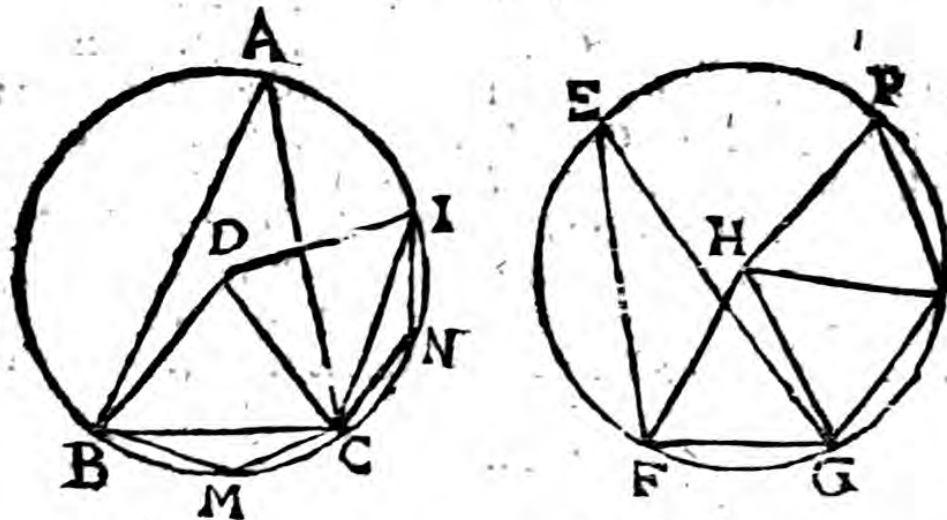
If two triangles ABC, DCE having two sides proportional to two (AB. AC:: DC. DE.) be so compounded or set together at one angle ACD that their homologous

sides be also parallel (AB to DC, and AC to DE,) then the remaining sides of those triangles (BC, CE) shall be found placed in one strait line.

For the angle  $A = ACD = D$ . and  $AB. AC :: DC. DE$ . therefore the angle  $B = DCE$ . Therefore the angle  $B + A = ACE$ . But the angle  $B + A + ACB = 2$  right. therefore the angle  $ACE + ACB = 2$  right. therefore BCE is a right line. *W.W. to be Dem.*

a 29. 1.  
b hyp.  
c 6. 6.  
d 2. ax.  
e 32. 1.  
f 1. ax.  
g 14. 1.

PROP. XXXIII.



In equall circles DBCA, HFGP, the angles BDC, FHG have the same ratio with their peripheries BC, FG on which they insist; whether the angles be set at the centers (as BDC, FHG) or at the circumferences, A, H: And in like sort are the Sectors BDC, FHG, because described upon the centers.

Draw

Draw the right lines BC, FG. Make  $CI = CB$ , &  $GL = FG = LP$ . and join DI, HL, HP.

The arch BC  $a = CI$ ,  $a$  also the arch FG, GL, LP are a 28. 3. b 27. 3. equall;  $b$  therefore the angle  $BDC = CDI$ ,  $b$  and the angle  $FGH = GHL = LHP$ . Therefore the arch BI is as multiplex of the arch BC, as the angle BDI is of the angle BDC. And in like manner is the arch FP as multiplex of the arch FG, as the angle FHP is of the ang. FHG. But if the arch BI  $c = FP$ , c 27. 3. d 6. def. 5. e 15. 5. f 20. 3. then likewise is the angle BDI  $c = FHP$ . Therefore is the arch BC. FG ::  $d$  the angle BDC. FHG  $e :: BDC. FHG f :: A. E. W.W. to be Dem.$

Moreover, the angle BMC  $g = CNI$ ;  $g$  and therefore the segment BCM  $h = CIN$ . g 27. 3. h 24. 3. k 4. 1. l 2. ar. Also the triangle BDC = CDI;  $l$  wherefore the Sector BDCM = CD-IN. After the same manner are the sectors FHG, GHL, LHP equall one to the other. Therefore since accordingly as the arch BI  $m = FGP$ , so is likewise the sector BDI  $m = FHP$ ; m 6. def. 5. thence shall be the sector BDC. FHG :: the arch BC. FG. *W.W. to be Dem.*

Coroll.

1. Hence, *As sector is to sector, so is angle to angle.* 11. 8.

2. *The angle BDC in the center is to four right angles, as the arch BC, on which it insists, to the whole circumference.*

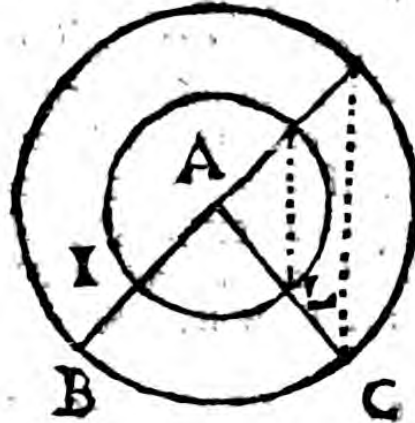
For as the angle BDC is to a right angle, so is the arch BC to a quadrant. Therefore BDC is so to four right angles as the arch BC is to four quadrants, that is, the whole circumference. Also, the angle A. 2 right :: the arch BC. periphery.

3. Hence, *The arches IL, BC of unequall circles which subtend equall angles, whether at the centers, as IAL and BAC, or at the periphery, are like.*

For



For  $IL$ . periph  $::$  angle  $IAL$  ( $BAC$ .)  $4$  right. Also Arch  $BC$ . Periph  $::$  angle  $BAC$ .  $4$  right. Therefore  $IL$ . periph  $::$   $BC$  periph. And consequently the arches  $IL$  and  $BC$  are like. Whence



**FIG. 4.** Two semidiameters  $AB$ ,  $AC$  cut off like arches  $IL$ ,  $BC$  from concentricall peripheries.


*The End of the sixth Book of Euclide's Elements.*

**T H E**

THE SEVENTH BOOK  
OF  
EUCLIDE'S ELEMENTS.

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*Definitions.*

I. nity is that, by which every thing that is, is called One.

II. Number is a multitude composed of unities.

III. One number is a Part of another, the lesser of the greater, when the lesser measureth the greater.

*Every part is denominated from that number, by which it measures the number whereof it is a part; as 4 is called the third part of 12, because it measures 12 by 3.*

IV. But the lesser number is termed Parts, when it measureth not the greater.

*All parts whatsoever are denominated from those two numbers, by which the greatest common measure of the two numbers measures each of them; as 10 is said to be  $\frac{2}{3}$  of the number 15; because the greatest common measure, which is 5, measures 10 by 2, and 15 by 3.*

V. A number is Multiplex (or Manifold) a greater in comparison of a lesser, when the lesser measureth the greater.

VI. An Even number is that which may be divided into two equall parts.

VII. But an Odde number is that which cannot be divided into two equall parts; or, that which differeth from an even number by an unitie.

VIII. A number Evenly Even is that which an even number measureth by an even number.

IX. But a number Evenly Odde is that which an even number measureth by an odde number.

X. A

X. A number Oddly Odd is that which an odd number measureth by an odd number.

XI. A Prime (or first) number is that, which is measured only by an unities.

XII. Numbers Prime the one to the other, are such as onely an unities doth measure, being their common measure.

XIII. A Composed number is that which some certain number measureth.

XIV. Numbers Composed the one to the other, are they, which some number, being a common measure to them both, doth measure.

*In this, and the preceding definition, unities is not a number.*

XV. One number is said to Multiply another, when the number multiplied is so often added to it self, as there are unities in the number multiplying, and another number is produced.

*Hence in every multiplication a unities is to the multiplier, as the multiplied is to the product.*

*Obs. That many times, when any numbers are to be multiplied (as A into B) the conjunction of the letters denotes the Product; So  $AB = A \times B$ , and  $CDE = C \times D \times E$ .*

XVI. When two numbers multiplying themselves produce another, the number produced is called a Plane number; and the numbers which multiplied one another, are called the Sides of that number: So  $2 (C) \times 3 (D) = 6 = CD$  is a plane number.

XVII. But when three numbers multiplying one another produce any number, the number produced is termed a Solid number; and the numbers multiplying one another, are the sides thereof: So  $2 (C) \times 3 (D) \times 5 (E) = 30 = CDE$  is a solid number.

XVIII. A Square number is that which is equally equal; or, which is contained under two equal numbers. Let A be the side of a square; the square is thus noted, AA, or Aq.

XIX. A Cube is that number which is equally equal

quall equally; or, which is contained under three equal numbers. *Let A be the side of a Cube; the Cube is thus noted, AAA, or Ac.*

*In this definition, and the three foregoing, unitie is a number.*

**XX.** Numbers are proportionall, when the first is as multiplex of the second as the third is of the fourth; or, the same part; or, when a part of the first number measures the second, and the same part of the third measures the fourth, equally: and on the contrary. *So A.B :: C. D. that is, 3.9 :: 5.15.*

**XXI.** Like Plane, and solid numbers are they, which have their sides proportionall: *Namely, not all the sides, but some.*

**XXII.** A Perfect number is that which is equal to its own parts.

*As 6. & 28. But a number that is lesse then it's parts is called an Abounding number; and a greater a Diminutive: so 12 is an abounding, 15 a diminutive number.*

**XXIII.** One number is said to measure another, by that number, which, when it multiplies, or is multiplied by it, it produceth.

*In Division, a unitie is to the quotient as the divisor is to the dividend. Note, that a number placed under another with a line between them, signifies division: So*  

$$\frac{A}{B} = A \text{ divided by } B, \text{ \& } \frac{CA}{B} = C \times A \text{ divided by } B.$$

Two numbers are called Termes or Roots of Proportion, lesser then which cannot be found in the same proportion.

*Postulates, or Petitions.*

1. **T**hat numbers equal or manifold to any number may be taken at pleasure.
2. That a greater number may be taken then any number whatsoever.
3. That Addition, Substraction, Multiplication, Di-

**Division**, and the Extractions of roots or sides of square and cube numbers, be also granted as possible.

*Axiomes.*

1. **W**Hatsoever agrees with one of many equall numbers, agrees likewise with the rest.

2. Those parts that are the same to the same part, or parts, are the same amongst themselves.

3. Numbers that are the same parts of equall numbers, or of the same number, are equall amongst themselves.

4. Those numbers, of whom the same number, or equall numbers, are the same parts, are equall amongst themselves.

5. Unitie measures every number by the unities that are in it; that is, by the same number.

6. Every number measures it self by a unitie.

7. If one number, multiplying another, produce a third, the multiplier shall measure the product by the multiplied; and the multiplied shall measure the same by the multiplier.

*Hence, No prime number is either a plane, solid, square, or cube number.*

8. If one number measure another, that number by which it measureth shall measure the same by the unities that are in the number measuring, that is, by the number it self that measures.

9. If a number measuring another, multiply that by which it measureth, or be multiplied by it, it produceth the number which it measureth.

10. How many numbers so ever any number measureth, it likewise measures the number composed of them.

11. If a number measure any number, it also measureth every number which the said number measureth.

12. A number that measures the whole & a part taken away, doth also measure the residue.

**P R O-**

PROPOSITION I.

A .....	E ..	G . B	8 5 3	If two unequal numbers AB, CD, being given, the lesser CD be continually taken from the greater AB (and the residue EB from CD, &c.) by an alternate subtraction, and the number remaining do never measure the precedent, till the unitie GB be taken; then are the numbers which were given AB, CD, prime the one to the other.
C ...	F ..	D	5 3 2	
H ...			5 2 1	

If you deny it, let AB, CD have a common measure, namely the number H. Therefore H measuring CD, doth *a* also measure AE; and consequently the remainder EB; *a* therefore it likewise measures CF, and *b* so the remainder FD; *a* wherefore it also measures EG. But it measured the whole EB, and *b* therefore it must measure that which remaineth GB, a unitie, it self being a number. *c* Which is Absurd. a 11. ax. 7. b 12. ax. 7. c 9. ax. 1.

PROP. II.

A .....	E .....	B	15 9 6	Two numbers AB, CD being given, not prime the one to the other, to find out their greatest common measure FD.
C .....	F ...	D	9 6 3	
	G . . .		6 3 0	

Take the lesser number CD from the greater AB as often as you can. If nothing remains, *a* it is manifest that CD is the greatest common measure. But if there remains something (as EB) then take it out of CD, and the residue FD out of EB, and so forward till some number (FD) measure the said EB (*b* for this will be, before you come to a unitie.) FD shall be the greatest common measure. a 6. ax. 7. b 1. 7.

For FD *c* measures EB, and *d* therefore also CF; and *e* consequently the whole CD; *d* therefore likewise AE; and so measures the whole AB. Wherefore

K it

d 11. ax. 7.  
e 22. ax. 7.  
g suppos.  
h 9. ax. 1.

it is evident that FD is a common measure. If you deny it to be the greatest, let there be a greater (G;) then whereas G measureth CD, it must likewise measure AE, e & the residue EB, d as also CF, e and by consequence the residue FD, g the greater the lesse. <sup>b</sup> Which is absurd.

Coroll.

Hence, A number that measures two numbers, does also measure their greatest common measure.

P R O P. III.

A.....	12	Three numbers being given, A, B, C, not prime to one another, to find out their greatest common measure
B.....	8	
	D....	4
C.....	6	E.
	E..	2
F...-		

Find out D the greatest common measure of the two numbers A, B. If D measures C the third, it is clear that D is the greatest common measure of all the three numbers. If D does not measure C, at least D and C will be composed the one to the other, by the Coroll. of the Prop. preceding. Therefore let E be the greatest common measure of the said numbers D and C, and it appears to be the number required.

a constr.  
b 11. ax. 7.

c cor. 1. 7.

d suppos.  
e 9. ax. 1.

For E a measures C and D, and D measures A and B; therefore b E measures each of the numbers A, B, C: neither shall any greater (F) measure them; for if you affirm that, c then F measuring A and B, does likewise measure D their greatest common measure; and in like manner, F measuring D and C, does also measure E c their greatest common measure, d the greater the lesse. e Which is absurd.

Coroll.

Hence, A number that measures three numbers, does also measure their greatest common measure.

P R O P. I V.

A ..... 6                      *Every lesse number A is of every*  
 B ..... 7                      *greater B either a part or parts.*  
 B ..... 18                      *If A and B be prime to one*  
 B ..... 9                      *another, a A shall be as many* <sup>a 4.def.7.</sup>  
    *parts of the number B, as there*  
*are unities in A (as  $6 = \frac{6}{7} 7$ .) But if A measures B,*  
*it is b plain that A is a part of B (as  $6 = \frac{1}{3} 18$ .)* <sup>b 3.def.7.</sup>  
*Lastly, if A and B be otherwise composd to one*  
*another, c the greatest common measure shall de-* <sup>c 4.def.7.</sup>  
*termine how many parts A does contain of B; as  $6$*   
 *$= \frac{2}{3} 9$ .*

P R O P. V.

A ..... 6	D .... 4
6      6	4      4
B ..... G ..... C 12.	E .... H .... F 8.

*If a number A be a part of a number BC, and another number D the same part of another number EF; then both the numbers together (A + D) shall be the same part of both the numbers together (BC + EF,) which one number A is of one number BC.*

*For if BC be resolvd into it's parts BG, GC, equall to A; and EF also into its parts EH, HF equall to D; a the number of parts in BC shall be equall* <sup>a hyp.</sup>  
*to the number of parts in EF. Therefore since  $A + D$*   
 *$= BG + EH = GC + HF$ , thence  $A + D$  shall* <sup>b const. and</sup>  
*be as often in  $BC + EF$ , as A is in BC. Which was to* <sup>c 2.ax.1.</sup>  
*be Demonstrated.*

*Or thus. Let  $a = \frac{x}{2}$ , and  $b = \frac{y}{2}$ . then  $2 a = x$ ,* <sup>c 2.ax.1.</sup>  
*and  $2 b = y$ . wherefore  $2 a + 2 b = x + y$ . there-*  
*fore  $a + b = \frac{x + y}{2}$ . W. W. to be Dem.*



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PROP. VI.

$$\begin{array}{ccccccc} & 3 & & 3 & & 4 & & 4 \\ A \dots G \dots B & 6 & D \dots H \dots E & 8 & & & & \\ C \dots \dots \dots & 9 & F \dots \dots \dots & 12. & & & & \end{array}$$

If a number AB be parts of a number C, and another number DE the same parts of another number F; then both numbers together AB+DE shall be of both numbers together C+F the same parts, that one number AB is of one number C.

a hyp.  
b 5.7.

c 2. ax. 7.

Divide AB into its parts AG, GB; and DE into its parts DH, HE. The multitude of parts in both AB, DE is equal by supposition; Wherefore since AG<sup>a</sup> is the same part of the number C, that DH is of the number F; therefore AG+DH<sup>b</sup> shall be the same part of the compounded number C+F, that one number AG is of one number C.<sup>b</sup> In like manner GB+HE is the same part of the said C+F, that one number GB is of one number C.<sup>c</sup> Therefore AB+DE is the same parts of C+F, that AB is of C. *W.W. to be Dem.*

Or thus. Let  $a = \frac{2}{3}x$ , and  $b = \frac{2}{3}y$ , and  $x+y = g$ . then, because  $3a = 2x$ , and  $3b = 2y$ , is  $3a + 3b = 2x + 2y = 2g$ . therefore  $a + b = \frac{2}{3}g = \frac{2}{3}(x+y)$ .

PROP. VII.

$$\begin{array}{ccccccc} & 5 & & 3 & & & & \\ A \dots \dots E \dots B & 8 & & & & & & \\ & 6 & & 10 & & 6 & & \\ G \dots \dots C \dots \dots F \dots \dots D & 16 & & & & & & \end{array}$$

If a number AB be the same part of a number CD, that a part taken away AE is of a part taken away CF; then shall the residue EB be the same part of the residue FD that the whole AB is of the whole CD.

a 1. post. 7.  
b 5. 7.

c 6 ax. 1.  
d 3. ax. 1.  
e 2. ax. 7.

Let EB be the same part of the number GC that AB is of CD, or AE of CF, <sup>b</sup> therefore AE+EB is the same part of CF+GC that AE is of CF, or AB of CD. <sup>c</sup> therefore GF=CD. Take away CF common to both, & <sup>d</sup> there remains GC=FD. <sup>e</sup> Wherefore EB is the same part of the residue FD (GC) that the whole AB is of the whole CB. *Which was to be Dem.*

PROP.

Or thus. Let  $a + b = x$ ; and  $c + d = y$ ; and  $x = 3y$ , in like manner as  $a = 3c$ ; I say  $b = 3d$ . For  $3c + 3d = 3y = x = a + b$ . take away from both  $3c = a$ , and  $b$  there remains  $3d = b$ . *W.W. to be Dem.*

f 1. 2.  
g hyp.

PROP. VIII.

$\begin{array}{ccccccc} & 6 & & 2 & & 4 & & 2 & & 2 \\ \text{A} & \dots\dots & \text{H} & \dots & \text{G} & \dots & \text{E} & \dots & \text{L} & \dots & \text{B} \\ & & & 18 & & & & 6 & & & \end{array}$	<p>If a number AB be the same parts of a number CD, that a part taken away AE is of a part taken away CF; the residue also EB shall be of the residue FD the same parts, that the whole AB is of the whole CD.</p>
$\text{C} \dots\dots\dots \text{F} \dots\dots, \text{D} 24$	<p>16 24</p>

Divide AB into AG, GB, parts of the number CD; also AE into AH, HE, parts of the number CF; and take  $GL = AH = HE$ . wherefore  $HG = EL$ . And because  $AG = GB$ , therefore  $HG = LB$ . Now whereas the whole AG is the same part of the whole CD that the part taken away AH is of the part taken away CF, the residue HG or EL shall be the same part also of the residue FD that AG is of CD. In like manner, because GB is the same part of the whole CD, that HE or GL are of CF, therefore the residue LB shall be the same part of the residue FD that GB is of the whole CD. Therefore EL + LB (EB) is the same parts of the residue FD, that the whole AB is of the whole CD. *W.W. to be Dem.*

a 3. ex. 1.  
b constr.  
c 3 ex. 1.  
d 7 7.

Or thus more easily. Let  $a + b = x$ , and  $c + d = y$ . Also  $y = \frac{2}{3}x$  as well as  $c = \frac{2}{3}a$ ; or, which is the same,  $3y = 2x$ ; and  $3c = 2a$ . I say  $d = \frac{2}{3}b$ . For  $3c + 3d = 3y = 2x = 2a + 2b$ . Therefore  $3c + 3d = 2a + 2b$ . take away from each  $3c = 2a$ , and there remains  $3d = 2b$ . therefore  $d = \frac{2}{3}b$ . *Which was to be Dem.*

e 9 ex. 7.  
f 1. 2.  
g 1. ex. 1.  
h hyp  
k 3. ex. 1.  
l 8 ex. 7.

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PROP. IX.

A .... 4  
 4 4  
 B ... G ... C 8  
 5 D ..... 5  
 5 5  
 E ..... H ..... F 10

If a number A be a part of a number BC, and another number D the same part of another number EF; then alternately what part or parts the first A is of the third D, the same part or parts shall the second BC

be of the fourth EF.

A is supposed  $\supset$  D. therefore let BG, GC, and EH, HF, parts of the numbers BC, EF be equal; BG and GC to A; and EH, HF to D. The multitude of parts is put equal in both. But it is clear that BG is the same part or parts of EH, that GC is of HF; wherefore BC (BG + GC) is the same part or parts of EF (EH + HF) that BG alone (A) is of EH alone (D.) Which was to be Dem.

a 1. ex. 7.  
 and 4. 7.  
 b 5. & 6. 7.

Or thus. Let  $a = \frac{b}{3}$ , and  $c = \frac{d}{3}$ ; or  $3a = b$ ,

\* 15. 5.

and  $\frac{3c}{a} = d$ , then is  $\frac{c}{a} = \frac{3c}{3a} = \frac{d}{b}$

PROP. X.

A .. G .. B 4  
 C ..... 6  
 5 5  
 D ..... H ..... E 10  
 F ..... I 15

If a number AB be parts of a number C, and another number DE the same parts of another number F; then alternately, what parts or part the first AB is of the third DE. the same parts or part

shall the second C be of the fourth F.

AB is taken  $\supset$  DE, and C  $\supset$  F. Let AG, GB, and DH, HE be parts of the numbers C and F, viz. as many in AB, as in DE. It is manifest that AG is the same part of C, that DH is of F. whence alternately AG is of DH, and likewise GB of HE, & so conjointly AB of DE the same part, or parts, that C is of F. Which was to be Dem.

a 9. 7.  
 b 5. and 9. 7.

Or thus. Let  $a = 2b$ , &  $c = 2d$ ; or  $3a = 2b$ , &  $3c = 2d$ .

Then is  $\frac{c}{a} = \frac{3c}{3a} = \frac{2d}{2b} = \frac{d}{b}$

PROP.

P R O P. XI.

$A \dots E \dots B$  <sup>4 3</sup>  
 $C \dots F \dots D$  <sup>8 6</sup>

*If a part taken away AE be to a part taken away CF, as the whole AB is to the whole CD, the residue also EB shall be to the residue FD, as the whole AB is to the*

*whole CD.*

First, let AB be  $\sqsupset$  CD; *a* then AB is either a part or parts of the number CD; and likewise AE is *b* the same part or parts of CF; *c* therefore the residue EB is the same part or parts of the residue FD that the whole AB is of the whole CD. *b* & so  $AB, CD :: EB, FD$ . But if AB be  $\sqsubset$  CD, then according to what is already shewn, will  $CD, AB :: FD, EB$ . therefore by inversion  $AB, CD :: EB, FD$ .

P R O P. XII.

$A, 4. C, 2. E, 3.$  *If there be divers numbers,*  
 $B, 8. D, 4. F, 6.$  *how many soever, proportionall*  
*( $A, B :: C, D :: E, F$ ;) then as*  
*one of the antecedents A is to one of the consequents B,*  
*so shall all the antecedents ( $A + C + E$ ) be to all the*  
*consequents ( $B + D + F$ .)*

First, let A, C, E, be  $\sqsupset$  B, D, F; then (because of the same proportions) *a* shall A be the same part or parts of B that C is of D; *b* and likewise conjointly  $A + C$  shall be the same part or parts of  $B + D$  that A alone is of B alone. In the like manner  $A + C + E$  is the same part or parts of  $B + D + F$  that A is of B. *c* Therefore  $A + C + E, B + D + F :: A, B$ . But if A, C, E, be put greater then B, D, F, the same may be shewn by inversion.

## P R O P. XIII.

A, 3. C, 4. If there be four numbers proportionall  
B, 5. D, 12. ( $A. B :: C. D.$ ) then alternately they  
shall also be proportionall, ( $A. C :: B. D.$ )

a 10. def. 7.  
b 9. and 10.  
7.

First let A and C be  $\square$  B and D, and A  $\square$  C. By reason of the same proportion  $a$  shall A be the same part or parts of B, that C is of D.  $b$  Therefore alternately A is the same part or parts of C that B is of D. and so  $A. C :: B. D$  But if A be  $\square$  C, and A and C supposed  $\square$  B and D, the matter will be the same by inverting the proportions.

## P R O P. XIV.

A, 9. D, 6. If there be numbers, how many soever,  
B, 6. E, 4. A, B, C, and as many more equal to  
C, 3. F, 2. them in multitude, which may be com-  
pared two and two in the same propor-  
tion ( $A. B :: D. E.$  and  $B. C :: E. F.$ ;) they shall also, of  
equality, be in the same proportion ( $A. C :: D. F.$ )

a 13. 7.

For because  $A. B :: D. E.$   $a$  therefore alternately is  
 $A. D :: B. E :: a C. F.$   $a$  and so again by changing  
 $A. C :: D. F.$  *W.W. to be Dem.*

## P R O P. XV.

I. D.. If a unite measure any num-  
B ... 3. E ..... 6. ber B, and another number D do  
equally measure some other num-  
ber E; alternately also shall a unite measure the third  
number D, as often as the second B doth the fourth E.

a 9. 7.

For seeing I is the same part of B, that D is of E;  
 $a$  therefore alternately shall I be the same part of D,  
that B is of E. *W.W. to be Dem.*

P R O P.

P R O P. XVI.

B, 4. A, 3. If two numbers A, B, multiplying themselves the one into the other, produce any numbers AB, BA; the numbers produced AB and BA shall be equal the one to the other.

For because  $AB = A \times B$ , <sup>a</sup> therefore shall I be as often in A, as B in AB, <sup>b</sup> and by consequence alternately I shall be as often in B as A in AB. But for that  $BA = B \times A$ , <sup>a</sup> therefore shall I be as often in B, as A in BA. therefore as often as I is in AB, so often is I in BA. and <sup>c</sup> so  $AB = BA$ . *W.W. to be Dem.* <sup>a 15. def. 7. b 15. 7. c 4. ax. 7.</sup>

P R O P. XVII.

A, 3. If a number A multiplying two numbers B, C, produce other numbers AB, AC; the numbers produc'd of them shall be in the same proportion that the numbers multiplied are. (AB.AC :: B.C.)

For being  $AB = A \times B$ , <sup>a</sup> therefore shall I be as often in A, as B in AB. <sup>a</sup> Likewise because  $AC = A \times C$ , therefore shall I be as often in A, as C in AC. and so also B as often in AB as C in AC. <sup>b</sup> wherefore  $B.AB :: C.AC$ . <sup>c</sup> and therefore also alternately  $B.C :: AB.AC$ . *W.W. to be Dem.* <sup>a 15. def. 7. b 20. def. 7. c 13. 7.</sup>

P R O P. XVIII.

C, 5. C, 5. If two numbers A, B, multiplying any number C, produce other numbers AC, BC; the numbers produced of them shall be in the same proportion that the numbers multiplying are (A.B :: AC.BC.)

For <sup>a</sup>  $AC = CA$ , and  $BC = CB$ ; so the same C multiplying A and B produceth AC and BC. <sup>b</sup> therefore  $A.B :: AC.BC$ . *W.W. to be Dem.* <sup>a 16. 7. b 17. 7.</sup>

Schol.

Schol.

Hence is deduced the vulgar manner of reducing fractions ( $\frac{3}{7}, \frac{7}{9}$ ) to the same denomination. For multiply 9 both by 3 and 5, and they produce  $\frac{27}{45} = \frac{3}{5}$ ; because by this, 3. 5 :: 27. 45. Likewise multiply 5 by 7 and 9; there arises  $\frac{35}{45} = \frac{7}{9}$ ; because 7. 9 :: 35. 45.

PROP. XIX.

A, 4. B, 6. C, 8. D, 12. If there be four numbers in proportion (A. B :: C. D) the number produced of the first and fourth (AD) is equal to the number which is produced of the second and third (BC.)

And if the number which is produced of the first and fourth (AD) be equal to that produced of the second & third (BC) those four numbers shall be in proportion (A. B :: C. D.)

a 17. 7.  
b hyp.  
c 18. 7.  
d 9. 5.  
e hyp.  
f 7. 5.  
g 17. 7.  
h 18. 7.  
k 11. 5.

1. Hyp. For AC. AD <sup>a</sup> :: C. D <sup>b</sup> :: A. B <sup>c</sup> :: AC. BC. <sup>d</sup> therefore AD = BC. W.W. to be Dem.

2. Hyp. Because <sup>e</sup> AD = BC, therefore AC. AD <sup>f</sup> :: AC. BC. But AC. AD <sup>g</sup> :: C. D. and AC. BC <sup>h</sup> :: A. B. <sup>k</sup> therefore C. D :: A. B. W.W. to be Dem.

PROP. XX.

A. B. C. If there be three numbers in proportion (A. B :: B. C.) the number contained under the extremes (AC) is equal to the square made of the middle (BB.)

And if the number contained under the extremes be equal to that (Bq,) produced of the middle, those three numbers shall be in proportion ( $\frac{A}{B} :: \frac{B}{C}$ )

a 1. 42. 7.  
b 19. 7.

1. Hyp. For take D = B. <sup>a</sup> therefore A. B :: D (B.) C. <sup>b</sup> wherefore AC = BD <sup>c</sup> or BB. Which was to be Dem.

2. Hyp.

2. Hyp. Because  $AC^c = BD^d$  therefore  $A. B :: D^c$  *hyp.*  
 (B.) C. *W.W. to be Dem.* d 19. 7.

P R O P. XXI.

A ... G .. B 5.    E ..... 10.    Numbers A B,  
 C .. H . D 3.    F ..... 6.    CD, being the least  
 of all that have the  
 same proportion with them (E, F) doe equally measure  
 the numbers E, F, having the same proportion with them;  
 the greater AB the greater E, and the lesser CD the  
 lesser F.

For AB.CD *a* :: E.F. *b* therefore alternately AB. *a hyp.*  
 E :: CD.F. *c* therefore AB is the same part or parts *b 13 7.*  
 of E that CD is of F. Not parts; for if so, then let *c 20. def. 7.*  
 AG, GB be parts of the number E; and CH, HD,  
 parts of the number F. *c* therefore AG. E :: CH. F.  
 and by inversion AG. CH *d* :: E. F *e* :: AB. CD. *d 13. 7.*  
 therefore AB, CD are not the least in their propor- *e hyp.*  
 tion; which is contrary to the hypothesis. There-  
 fore, &c.

P R O P. XXII.

A, 4.    D, 12.    If there be three numbers A,  
 B, 3.    E, 8.    B, C; and other numbers equall  
 C, 2.    F, 6.    to them in multitude, D, E, F;  
 which may be compared two &  
 two in the same proportion: and if also the proportion of  
 them be perturbed (A.B :: E. F. and B.C :: D.E.) then  
 of equality they shall be in the same proportion (A.C ::  
 D. F.)

For because A.B *a* :: E.F, therefore shall AF = BE; *a hyp.*  
 and because B. C :: *a* D. E, *b* therefore BE = CD. *b 19. 7.*  
*c* and consequently AF = CD. *d* Wherefore A. C :: *c 1. ax 6.*  
 D.F. *W.W. to be Dem.* d 19. 7.

P R O P.



## P R O P. XXIII.

A, 9. B, 4.      Numbers prime the one to the other,  
 C ---- D ---      A, B, are the least of all numbers  
 E . -              that have the same proportion  
                          with them.

a 21. 7.      If it be possible, let C and D be lesse then A and B, and in the same proportion; a therefore C measures A equally as D measures B, namely by the same number F; and so C shall be b as often in A as 1 is in E; c likewise alternately, E as often in A as 1 in C. By the like inference as many times as 1 is in D, so many times shall E be in B. Therefore E measures both A and B; which consequently are not prime the one to the other, *contrary to the Hypoth.*

b 13 def. 7.  
c 15. 7.

## P R O P. XXIV.

A, 9. B, 4.      Numbers A, B, being the least of  
 C ---              all that have the same proportion  
 D --- E ---      with them, are prime the one to the  
                          others.

If it be possible, let A and B have a common measure C; and let the same measure A by D, and B by E; a therefore  $CD = A$ , b and  $CE = B$ . b Wherefore  $A.B :: D.E$ . But D and E are lesser then A and B, as being but parts of them. Therefore A and B are not the least in their proportion, *against the Hypoth.*

a 9. ex. 7.        
b 17. 7.

## P R O P. XXV.

A, 9. B, 4.      If two numbers A, B, be prime the  
 C, 3. D --      one to the other, the number C measuring one of them A, shall be prime to the other number B.

For if you affirme any other D to measure the numbers B and C, a then D measuring C does also measure A; and consequently A and B are not prime the one to the other: *Which is against the Hypothesis.*

a 11. ex. 7.

P R O P.

PROP. XXVI.

A, 5. C, 8. If two numbers, A, B, be  
 B, 3. prime to any number C, the  
 AB, 15. E ---- number also produced of them  
 F ---- AB shall be prime to the  
 same C.

If it be possible, let the number E be a common  
 measure to AB, and C; and let  $\frac{AB}{E} = F$ ; thence a 9. 27.

$AB = EF$ ; b 19. 7.  
 wherefore also  $E.A :: B.F$ . But because c 25. 7.  
 A is prime to C, which E measures, d 23. 7.  
 therefore E and A are prime to one another, e 21. 7.  
 and so least in their  
 own proportion, and consequently they must mea-  
 sure B and F; namely F shall measure B, and A shall  
 measure F. Therefore seeing E measures both B  
 and C, they shall not be prime to one another: con-  
 trary to the Hypoth.

PROP. XXVII.

A, 4. B, 5. If two numbers A, B, be prime to  
 Aq, 16. one another, that also which is pro-  
 D, 4. duc'd of one of them (Aq) shall be  
 prime to the other B.

Take  $D = A$ ; a 1. 27.  
 therefore each of D, and A are b 16. 7.  
 prime to B. wherefore AD or Aq is prime to B.  
*W.W. to be Dem.*

PROP. XXVIII.

A, 5. C, 4. If two numbers A, B be prime to  
 B, 3. D, 2. two numbers C, D each to either of  
 AB, 15. CD, 8. both, the numbers also produced of  
 them AB, CD shall be prime to one  
 another.

For being A and B are prime to C, a 16. 7.  
 therefore shall  
 AB also be prime to the same. And by the same rea-  
 son

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son shall AB be prime to D. b Therefore AB is prime to CD. W.W. to be Dem.

P R O P. XXIX.

A, 3. B, 2. If two numbers A, B, be prime  
Aq, 9. Bq, 4. to one another, and each multi-  
Ac, 27. Bc, 8. plying himself produce another  
number (Aq, and Bq;) then the  
numbers produced of them (Aq, Bq) shall be prime to  
one another. And if the numbers given at first, A, B, mul-  
tiplying the said produced numbers (Aq, Bq) produce o-  
thers (Ac, Bc) those numbers also shall be prime to one  
another: and this shall ever happen about the extremes.

a 27. 7.

b 28. 7.

For because A is prime to B, a therefore Aq shall  
be prime to B. and Aq being prime to B, a therefore  
Aq shall also be prime to Bq. Again, because A is as  
well prime to B and Bq, as Aq is to the said B and  
Bq, b therefore shall A x Aq, that is, Ac, be prime to  
B x Bq, that is, to Bc: And so forth of the rest.

P R O P. XXX.

3 5 If two numbers AB,  
A ..... B ..... C 13. D - - - - BC, be prime the one to  
the other; then both ad-  
ded together (AC) shall be prime to either of them AB,  
BC. And if both added together AC be prime to any one  
of them AB, the numbers also given in the beginning  
AB, BC, shall be prime to one another.

a 12. ex 7.

1. Hyp. For if you would have AC, AB to be com-  
posed, let D be the common measure: a this shall  
measure the residue BC: and therefore AB, BC, are  
not prime to one another; which is against the  
hypoth.

b 10. ex 7.

2. Hyp. AC, AB being taken for prime to one an-  
other, let D be the common measure of AB, BC.  
b But seeing that measures the whole AC, therefore  
AC, AB, are not prime to one another; contrary to the  
hypoth.

*Coroll.*

Hence, A number, which being compounded of two, is prime to one of them, is also prime to the other.

P R O P. XXXI.

A, 5. B, 8. *Every prime number A is prime to every number B, which it measureth not.*

For if any common measure doth measure both, A, B, <sup>a 11. def. 7.</sup> then A will not be a prime number; *contrary to the Hyp.*

P R O P. XXXII.

A, 4. D, 3. *If two numbers A, B, multiplying*  
 B, 6. E, 8. *one another produce another AB, and*  
 $\overline{AB}, 24.$  *some prime number D measure the*  
*number produced of them AB; then*  
*shall it also measure one of those numbers, A or B, which*  
*were given at the beginning.*

Suppose the number D not to measure the number A, and let  $\overline{AB} = E.$  <sup>a 9. ex. 7. (</sup> then  $\overline{AB} = DE;$  <sup>b 19. 7.</sup> whence <sup>c hyp. and</sup>

$\overline{D.A} :: \overline{B.E.}$  <sup>d 31. 7.</sup> But D is prime to A; <sup>d 23. 7.</sup> therefore D and A are the least in their proportion; <sup>e 21. 7.</sup> and consequently D measures B as often as A measures E. *Which was to be Dem.*

P R O P. XXXIII.

A, 12. *Every composed number A is measured*  
 B, 2. *by some prime number B.*

Let one or more numbers <sup>a 13. def. 7.</sup> measure A, of which let the least be B; that shall be a prime number: for if it be said to be composed, then some <sup>b 11. ex. 7.</sup> lesser number shall measure it, which shall also consequently measure A. Wherefore B is not the least of them which measure A. *contrary to the Hyp.*

P R O P.

## P R O P. XXXIV.

A, 9. Every number A, is either a prime, or measured by some prime number.

For A is necessarily either a prime or a composed number. If it be a prime, 'tis that we affirm. If composed, 'tis then some prime number measureth it. *W.W. to be Dem.*

a 33. 7.

## P R O P. XXXV.

A, 6. B, 4. C, 8. H -- I -- K ----  
 D, 2. L ---  
 E, 3. F, 2. G, 4.

How many numbers soever A, B, C, being given, to find the least numbers E, F, G, that have the same proportion with them.

a 23. 7.

b 3. 7.

If A, B, C, be prime to one another, 'tis they shall be the least in their proportion. If they be composed, b let their greatest common measure be D, which let measure them by E, F, G. These are then least in the proportion A, B, C.

c 9. ex. 7.

d 17. 7.

e 21. 7.

f 9. ex. 7.

g 1. ex. 1.

h 19. 7.

k suppos.

l 20. def. 7.

For  $D \times E, F, G,$  produceth ABC, 'tis therefore they are all in the same proportion. But allow other numbers H, I, K to be the least in the same proportion; 'tis which shall therefore equally measure A, B, C, namely by the number L, f therefore  $L \times H, I, K,$  shall produce A, B, C, g and consequently  $ED = A = HL.$  b from whence  $E.H :: L. D.$  But  $E \nmid K;$  therefore  $L \nmid D,$  and so D is not the greatest common measure of A, B, C. Which is against the Hypoth.

Coroll.

Hence, The greatest common measure of how many numbers soever do's measure them by the numbers which are least of all that have the same proportion with them. Whereby appears the vulgar method of reducing fractions to the least termes.

P R O P.

PROP. XXXVI.

"Two numbers being given A, B, to find out the least number which they measure."

A, 5. B, 4.  
AB, 20.  
D-----  
E--- F---

1. Case. If A and B be prime the one to the other, AB is the number required. For it is manifest that A and B measure AB. If it be possible, let A and B measure some

other number D  $\supset$  AB, if you please by E, and F. therefore AE = D = BF, and so A.B :: F.E. But because A and B are prime the one to the other, and so least in their proportion, A shall equally measure F as B does E. But B.E f :: B. AE (D) g Therefore AB shall also measure D, which is lesse then it self. Which is Absurd.

a 9 ax. 7. and  
1. ax. 1.  
b 19. 7.  
c hyp.  
d 23. 7.  
e 21. 7.  
f 17. 7.  
g 10. def. 7.

A, 6. B, 4. F----  
C, 3. D, 2. G---H---  
AD, 12.

2. Case. But if A and B be composed one to another, let there be found C and D the

least in the same proportion. therefore AD = BC; and AD or BC shall be the number sought for.

For it is plain that B and D doe measure AD or BC. Conceive A and B to measure F  $\supset$  AD, namely A by G, and B by H. therefore AG = F = BH. whence A. B :: H. G :: C. D. and consequently C equally measures H as D does G. But D. G :: AD. AG (F.) therefore AD measures F, the greater the lesse. Which is Absurd.

h 35. 7.

k 19. 7.

l 7 ax. 7.

m 9 ax. 7.

n 19. 7.

o const.

p 21. 7.

q 17. 7.

r 20. def. 7.

Coroll.

Hence, If two numbers multiply the least that are in the same proportion, the greater the lesse, and the lesse the greater, the least number which they measure shall be produced.

## PROP. XXXVII.

A, 2. B, 3.      If two numbers A, B, measure any  
 E ..... 6      number CD, the least number which  
 C --- F --- D they measure E shall also measure  
 the same CD.

If you deny it, take E from CD as often as you  
 can, and leave FD  $\square$  E. therefore seeing A and B  
 measure E, a and E measures CF, c likewise A and  
 B will measure CF. But a they measure the whole  
 CD; d therefore also they measure the residue FD;  
 and consequently E is not the least which A and B  
 measure; *Contrary to the Hypoth.*

a hyp.  
 b conf.  
 c 11. ex. 7.  
 d 12. ex 7.

## PROP. XXXVIII.

A, 3. B, 4. C, 6.      Three numbers being given  
 D, 12.      A, B, C, to find out the least which  
 they measure.

a Find D to be the least that two of them A and  
 B do measure; which if the third C do also measure,  
 it is manifest that D is the number sought for. But  
 if C do not measure D, let E be the least that C and  
 D do measure. E shall be the number required.

A, 2. B, 3. C, 4.      For it appears by the 11. ax. 7.  
 D, 6. E, 12.      that A, B, C measure E; and it is  
 F ---      easily shewn that they measure  
 no other lesse F. For if you af-

firm they do, b then D measures F, b and consequent-  
 ly E measures the same F, the greater the lesse. Which  
 is *Absurd.*

*Coroll.*

Hence it appears that, If three numbers measure  
 any number, the least also, which they measure, shall  
 measure the same.

PROP.

a 16. 7.

b 17. 7.

PROP. XXXIX.

A, 12. If any number B measure a number  
 B, 4. C, 3. A, the number measured A, shall have a  
 part C denominated of the number mea-  
 suring B.

For because  $A \overset{a}{=} C \overset{b}{=} B$ , shall  $A \overset{c}{=} BC$ . therefore  $A \overset{a}{=} B$ . *W.W. to be Dem.*

PROP. XXXX.

A, 15. If a number A have any part whatsoever  
 B, 3. C, 5. B, the number C, from which the part B  
 is denominated, shall measure the same.

For being  $BC \overset{a}{=} A$ , thence  $A \overset{b}{=} B$ . *W.W. to be Dem.*

PROP. XXXXI.

G, 12. To find out a number G, which being  
 H, the least, containeth the parts given  
 $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ .

Let G be found the least which the denomina-  
 tors 2, 3, 4, measure; b it is evident that G ha's the  
 parts  $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ . If it be possible let H  $\supset$  G have the  
 same parts; c therefore 2, 3, 4, measure H; and so G is  
 not the least which 2, 3, 4 measure: against the constr.

The End of the seventh Book.



THE EIGHTH BOOK  
OF  
EUCLIDE'S ELEMENTS.

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P R O P. I.

A, 8. B, 12. C, 18. D, 27.  
E-F--G---H----

**I**F there be divers numbers how many soever in continuall proportion, A, B, C, D, and their extremes A, D, prime to one another; then those numbers A, B, C, D, are the least of all numbers that have the same proportion with them.

For, if it be possible, let there be as many others E, F, G, H, lesse then A, B, C, D, and in the same proportion with them. Therefore of equality  $A \cdot D :: E \cdot H$  and consequently A & D are prime numbers, *b* and so the least in their proportion, *a* equally measuring E and H, which are lesse then themselves. Which is Absurd.

*a* 14. 7.

*b* 23. 7.

*c* 21. 7.

P R O P. II.

I.

A, 2. B, 3.

Aq, 4. AB, 6. Bq, 9.

Ac, 8. AqB, 12. ABq, 18. Bc, 27.

To find out the least numbers continually proportionali, as many as shall be required, in the proportion given of A to B.

Let A and B be the least in the proportion given; Then Aq, AB, Bq, shall be the three least in the same continuall proportion that A is to B.

For

For  $AA. AB \cdot a :: A. B \cdot a :: AB. BB.$  Likewise because  $A$  and  $B$  are  $b$  prime to one another,  $c$  shall  $Aq, Bq,$  be also prime to one another,  $d$  and so  $Aq, AB, Bq,$  are  $\therefore$  the least in the proportion of  $A$  to  $B.$

Moreover, I say  $Ac, AqB, ABq, Bc,$  are the four least in the proportion of  $A$  to  $B.$  For  $AqA. AqB \cdot e :: A. B. e :: ABA (AqB.) ABB. e$  and  $A. B :: ABq. BBq (Bc.)$  Therefore since  $Ac,$  and  $Bc,$  are  $f$  prime to one another, likewise  $g$  shall  $Ac, AqB, ABq, Bc$  be the four least  $\therefore$  in the proportion of  $A$  to  $B.$  In the same manner may you find out as many proportionall numbers as you please. *W. W. to be Done.*

*Coroll.*

1. Hence, If three numbers, being the least, are proportionall, their extremes shall be squares; if four, cubes.

2. How many extremes proportionall soever there be, being by this propos. found to be the least in the given proportion, they are prime to one another.

3. Two numbers, being the least in the given proportion, doe measure all the mean numbers whatsoever of the least in the same proportion; because they arise from the multiplication of them into certain other numbers.

4. Hence also it appears by the construction, that the series of numbers  $1, A, Aq, AC; 1, B, Bq, Bc; Ac, AqB, ABq, BC,$  consists of an equall multitude of numbers; and consequently, the extreme numbers of how many soever the least continually proportionals are the last of as many other continually proportionalls from a unite. as the extremes  $Ac, Bc,$  of the continually proportionals  $Ac, AqB, ABq, Bc,$  are the last of as many proportionals from a unite.  $1, A, Aq, Ac;$  and  $1, B, Bq, Bc,$

5.  $1, A, Aq, Ac;$  and  $B, BA, BAq;$  and  $Bq, ABq,$  are  $\therefore$  in the proportion of  $1$  to  $A.$  Also  $B, Bq, Bc;$  and  $A, AB, ABq;$  and  $Aq, AqB$  are  $\therefore$  in the proportion of  $1$  to  $B.$

## PROP. III.

A, 8. B, 12. C, 18. D, 28. If there be numbers continually proportionall, how many soever, A, B, C, D, being also the least of all that have the same proportion with them; their extremes A, D, are prime to one another.

For if there be found as many numbers the least in the proportion of A to B, they shall be no other then A, B, C, D; therefore, by the second Coroll. of the precedent prop. the extremes A and D are prime to one another. *W. W. to be Dem.*

## PROP. IV.

A, 6. B, 5. C, 4. D, 3.  
H, 4. F, 24. E, 20. G, 15.  
I, K, L, ...

Proportions how many soever being given in the least numbers (A, to B, and C to D) to

find out the least numbers continually proportionall in the proportions given.

Find out E the least number which B and C do measure; and let B measure E *b* as often as A does another F, viz. by the same number H. Also let C measure the said E as often as D measures another G. then F, E, G, shall be the least in the proportions given. For  $AH \epsilon = F$ , and  $BC \epsilon = E$ ; therefore  $A.B :: AH. BH \epsilon :: F. E$ . In like manner  $C, D :: E.G$ ; therefore F, E, G, are continually proportionall in the proportions given. And they are moreover the least in the said proportions; for conceive other numbers I, K, L, to be the least; then A and B must equally measure I and K, and C and D likewise K and L; and so B and C measure the same K. Wherefore also E measures the same number K, which is lesse then it self. *Which is Absurd.*

A, 6. B, 5. C, 4. D, 3. E, 5. F, 7.  
H, 24. G, 20. I, 15. K, 21.

But three proportions being given, A to B, C to D, and E to F; find out as before three numbers H, G, I, the least continually in the proportions of A to B, and C to D. Then if E measures I, take another number K which may be equally measur'd by F; and those four numbers H, G, I, K, shall be continually the least in the given proportions; which we need go no other way to prove then we did in the first part. h 3. post. 7.

A, 6. B, 5. C, 4. D, 3. E, 2. F, 7.  
H, 24. G, 29. I, 25.  
M, 48. L, 60. K, 30. N, 105.

If E doe not measure I, let K be the least which E and I doe measure; and as often as I measures K, let G as often measure L, and H also M. so likewise let F measure N as often as E measures K. The four numbers M, L, K, N shall be least continually in the given proportions; which we may demonstrate as before.

P R O P. V.

C, 4. E, 3. *Plane numbers CD,*  
D, 6. F, 16. ED, 18. EF, *are in that propor-*  
 $\frac{CD}{EF} = \frac{ED}{EF}$  *tion to one another which*  
*is composed of their sides.*

For because  $CD \cdot ED :: C \cdot E$ ; and  $ED \cdot EF :: D \cdot F$ ; &  $\frac{CD}{EF} = \frac{CD}{ED} + \frac{ED}{EF}$  then shall be the proportion  $\frac{CD}{EF} = \frac{C}{E} + \frac{D}{F}$  *W.W. to be Dem.*

a 17. 7.  
b 20. def. 3.  
c 11. 5.

P R O P. VI.

A, 16. B, 24. C, 36. D, 54. E, 81. *If there be num-*  
F, 4. G, 6. H, 9. *bers continual-*  
*ly proportional*  
*how many soever, A, B, C, D, E, and the first A doe not*  
*measure the second B, neither shall any of the other mea-*  
*sure any one of the rest.*

a 10. def. 7.

b 35. 7.

c 4. ar 7.

d 3. 3.

e 14. 7.

Because A does not measure B, <sup>a</sup> neither shall any one measure that which next followes; being A. B :: B. C :: C. D, &c. <sup>b</sup> Take three numbers, F, G, H the least in the proportion of A to B. therefore since A does not measure B, <sup>c</sup> neither shall F measure G. <sup>c</sup> therefore F is not a unite. But F and H are prime to one another; and so, <sup>d</sup> being of equality A. C :: F. H. and F does not measure H, <sup>e</sup> neither shall A measure C; and consequently neither shall B measure D, nor C measure E, &c. because A. C :: B. D :: C. E, &c. In like manner four or five numbers being taken the least in the proportion of A to B, it will appear that A does not measure D and E; nor does B measure E and F, &c. Wherefore none of them shall measure any other. *W. W. to be Dem.*

## P R O P. VII.

A, 3. B, 6. C, 12. D, 24. E, 48.

*If there be numbers continually proportionall how many soever A, B, C, D, E, and the first A measure the last E, it shall also measure the second B.*

a 6. 7.

If you deny that A measures B, <sup>a</sup> then neither shall it measure E; which is contrary to the Hypoth.

## P R O P. VIII.

A, 24. C, 36. D, 54. B, 81. G, 8. H, 12. I, 18. K, 27. E, 32. L, 48. M, 72. F, 108.

*If between two numbers A, B, there fall mean proportionall numbers in continuall*

*proportion C, D; as many mean continually proportionall numbers as fall between them, so many also mean continually proportionall numbers shall fall between two other numbers E, F, which have the same proportion with them. (L, M.)*

a 35. 7.

b 14. 7.

c hyp.

d 3. 8.

e 21. 7.

f constr.

<sup>a</sup> Take G, H, I, K, the least :: in the proportion of A to C; <sup>b</sup> of equality shall G. K :: A. B <sup>c</sup> :: E. F. But G, and K <sup>d</sup> are prime to one another. <sup>e</sup> Wherefore G measures E as often as K does F. Let H measure L, and I likewise M by the same number, <sup>f</sup> therefore E, L, M, F are in such proportion as G, H, I, K, that is as A, B, C, D. *W. W. to be Dem.*

## P R O P.

PROP. IX.

**I.**  
 E, 2. F, 3.  
 G, 4. H, 6. I, 9.  
 A, 8. C, 12. D, 18. B, 17.

If two numbers A, B be prime to one another, and mean numbers in continual proportion C, D fall between them; as many mean numbers in continual proportion as fall between them, so many means also (E, G; and F, I) shall fall in continual proportion between either of them and a unitie.

It is evident, that I, E, G, A, and I, F, I, B, are  $\therefore$ , and as many as A, C, D, B, namely by the 4. Coroll. 2. 8. W.W. to be Dem.

PROP. X.

A, 8. I, 12. K, 18. B, 27.  
 E, 4. DF, 6. G, 9.  
 D, 2. F, 3.  
**I.**

If between two numbers A, B, and a unite, numbers continually proportionall (E, D, and F, G,) do fall, how many mean numbers in continual proportion fall between either of them and a unite, so many means also shall fall in continual proportion between them, I, K.

Nam E, DF, G, and A, DqF (I) DG (K) B are  $\therefore$  by 2. 8. therefore, &c.

PROP. XI.

A, 2. B, 3.  
 Aq, 4. AB, 6. Bq, 9.

Between two square numbers Aq, Bq, there is one mean proportionall number A B: and the square Aq to the square Bq is in double proportion of that of the side A to the side B.

It is manifest that Aq, AB, Bq, are  $\therefore$ ; and consequently also  $\frac{Aq}{Bq} = \frac{A}{B}$  doubly. W.W. to be Dem. a 17. 7.  
b 10. def. 5.

PROP.

PROP. XII

Ac, 27. AqB, 36. ABq, 48. Bc, 64. Between two  
 A, 3. B, 4. cube numbers,  
 Aq, 9. AB, 12. Bq, 16. Ac, Bc, there  
 are two mean  
 proportionall numbers AqB, ABq: and the cube Ac is to  
 the cube Bc in treble proportion of that in which the side  
 A is to the side B.

a 2. B.  
 b 10. def 5.

For Ac, AqB, ABq, Bc, are  $\therefore$  in the proportion  
 of A to B; and therefore  $Ac = A$  treblely.  $W.W.$  to be  
 Dem.  $\frac{Ac}{Bc} = \frac{A}{B}$

PROP. XIII

A, 2. B, 4. C, 8.

Aq 4. AB, 8. Bq, 16. BC, 32. Cq, 64.  
 Ac, 8. AqB, 16. ABq, 32. Bc, 64. BqC, 128. BCq, 256. Cc, 512.

If there be numbers in continuall proportion how many  
 soever A, B, C; and every of them multiplying it self pro-  
 duce certain numbers; the numbers produced of them Aq,  
 Bq, Cq, shall be proportionall: And if the numbers first  
 given A, B, C, multiplying their products Aq, Bq, Cq,  
 produce other numbers, Ac, Bc, Cc, they also shall be  
 proportionall; and this shall ever happen to the extremes.

a 2. B.  
 b 14. 7.

For Aq, AB, Bq, BC, Cq are  $\therefore$ ; therefore of e-  
 quality Aq.Bq :: Bq.Cq.  $W.W.$  to be Dem.

Also Ac, AqB, ABq, Bc, BqC, BCq, Cc, are  $\therefore$ ;  
 therefore likewise of equality Ac. Bc :: Bc. Cc.  
 $W.W.$  to be Dem.

PROP. XIV.

Aq, 4. AB, 12. Bq, 36. If a square number Aq  
 A, 2. B, 6. measure a square number  
 Bq, the side also of the one

(A) shall measure the side of the other (B:) & if the side  
 of one square A measure the side of another B, the square  
 Aq shall likewise measure the square Bq.

I. Hyp.

1. Hyp. For  $Aq \cdot AB \cdot a :: AB \cdot Bq$ . therefore seeing by the hypothesis  $Aq$  measures  $Bq$ ,  $b$  it shall measure also  $AB$ . But  $Aq \cdot AB :: A \cdot B \cdot c$  therefore also  $A$  measures  $B$ . *W.W. to be Dem.*

a 2. & 11. 2.  
b 7. 8.  
c 10. def. 7.

2. Hyp.  $A$  measures  $B$ .  $c$  therefore  $Aq$  shall as well measure  $AB$ ,  $e$  as  $AB$  measures  $Bq$ ,  $d$  and consequently  $Aq$  measures  $Bq$ . *W.W. to be Dem.*

d 11. def. 7.

PROP. XV.

$A, 2. B, 6.$  If a cube number  $Ac, 8. AqB, 24. ABq, 72. Bc, 216.$   $Ac$  measure a cube number  $Bc$ , then the side of the one ( $A$ ) shall measure the side of the other ( $B$ ). And if the side  $A$  of one cube  $Ac$  measure the side  $B$  of the other  $Bc$ , also the cube  $Ac$  shall measure the cube  $Bc$ .

1. Hyp. For  $Ac, AqB, ABq, Bc \cdot a$  are  $:::$ , therefore  $Ac, b$  measuring the extreme  $Bc$ , shall also  $c$  measure the second  $AqB$ . But  $Ac \cdot AqB :: A \cdot B \cdot d$  therefore  $A$  shall also measure  $B$ .

a 2. & 12. 2.  
b hyp.  
c 7. 8.  
d 10. def. 7.

2. Hyp.  $A$  measures  $B$ ;  $d$  therefore  $Ac$  measures  $AqB$ , which also measures  $ABq$ , and that  $Bc$ ;  $e$  therefore  $Ac$  shall measure  $Bc$ . *W.W. to be Dem.*

e 11. def. 7.

PROP. XVI.

$A, 4. B, 9.$  If a square number  $Aq$  doe not measure a square number  $Bq$ , neither shall the side of the one  $A$  measure the side of the other  $B$ : And if  $A$  the side of the one square  $Aq$  doe not measure  $B$  the side of the other  $Bq$ , neither shall the square  $Aq$  measure the square  $Bq$ .

1. Hyp. For if you affirm that  $A$  measures  $B$ , then  $Aq$  also shall measure  $Bq$ . against the Hyp.

a 14. 8.

2. Hyp. If you maintain  $Aq$  to measure  $Bq$ ; then likewise  $A$  shall measure  $B$ . contrary to the Hypoth.

PROP.



## P R O P. XVII.

A, 2. B, 3. If a cube number  $Ac$  doe not  
 Ac, 8. Bc, 27. measure a cube number  $Bc$ , nei-  
 ther shall the side of one  $A$ , mea-  
 sure the side of the other  $B$ : And if  $A$  the side of one cube  
 $Ac$  do not measure  $B$  the side of the other  $Bc$ , neither  
 shall the cube  $Ac$  measure the cube  $Bc$ .

a 15. 8.

1. Hyp. Let  $A$  measure  $B$ ;  $a$  then  $Ac$  shall measure  
 $Bc$ . against the Hyp.

2. Hyp. Let  $Ac$  measure  $Bc$ ; then  $A$  shall measure  
 $B$ ; which is also against the Hyp.

## P R O P. XVIII.

C, 6. D, 2. Between two like plane num-  
 CD, 12. bers  $CD$  and  $EF$  there is one  
 E, 9. F, 3. DE, 18. mean proportional number  $DE$ .  
 EF, 27. And the plane  $CD$  is to the  
 plane  $EF$  in double proportion  
 of that which the side  $C$  hath to the homologous side (or  
 of like proportion)  $E$ .

a 21. def. 7.

a 17. 7.

b 11. 5.

a 10. def. 5.

Being  $*$  by the Hypoth.  $C.D :: E.F.$  therefore by  
 inversion  $C.E :: D.F.$  But  $C.E a :: CD. DE$ ;  $a$  and  
 $D.F. b :: DE. EF.$   $b$  therefore  $CD. DE :: DE. EF.$   
 $c$  Wherefore the proportion of  $CD$  to  $EF$  is double  
 to that of  $CD$  to  $DE$ , that is, to the proportion of  
 $C$  to  $E$ , or  $D$  to  $F$ .

Coroll.

Hence it is apparent, That between two like plane  
 numbers there falls one mean proportionall in the  
 proportion of the homologous sides.

P R O P.

PROP. XIX.

CDE, 30. DEF, 60. FGE, 120. FGH, 240.

CD, 6. DF, 12. FG, 24.

C, 2. D, 3. E, 5. F, 4. G, 6. H, 10.

Between two like solid numbers CDE, FGH, there are two mean proportionall numbers DFE, FGE. And the solid CDE is to the solid FGH, in treble proportion of that which the homologous side C ha's to the homologous side F.

Whereas by the \* hyp. C. D :: F. G, & D. E :: G. H. therefore <sup>a</sup> by inversion shall C. F :: D. G :: <sup>a</sup> E. H. But CD.DF<sup>b</sup> :: C. F, & DF.FG<sup>b</sup> :: D. G; <sup>c</sup> wherefore CD.DF :: DF.FG :: E. H. <sup>d</sup> and accordingly CDE. DFE :: DFE.FGE :: E. H :: FGE.FGH. Therefore between CDE, FGH, fall two mean proportionals DFE, FGE. And so it is plain that the proportion of CDE to FGH is treble to that of CDE to DFE, or C to F. *W.W. to be Dem.*

\* 21. def. 7.  
a 13. 7.  
b 17. 7.  
c 11. 5.  
d 17. 7.

e 10. def. 5.

Coroll.

Hereby it is manifest, That between two like solid numbers there fall two mean proportionals in the proportion of the homologous sides.

PROP. XX.

A, 12, C, 18. B, 27. If between two numbers

D, 2. E, 3. F, 6. G, 9. A, B, there fall one mean proportionall number C;

those numbers A, B, are like plane numbers.

<sup>a</sup> Take D and E the least in the proportion of A <sup>a 35. 7.</sup> to C, or C to B. then D measures A equally as E does C, viz. by the same number F; <sup>b</sup> also D equally measures C, as E does B, viz. by the same number G. <sup>b 21. 7.</sup>

<sup>c</sup> Therefore DF = A, and EG = B. <sup>d</sup> and consequently A and B are plane numbers. But because <sup>e 9. ax. 7.</sup> <sup>d 6 def. 7.</sup> <sup>e 19. 7.</sup> EF = C = DG, shall D. E :: F. G. and alternately

nately

21. def. 7.

nately D.F :: E. G. Therefore the plane numbers A and B are also like. *W.W. to be Dem.*

PROP. XXI.

A, 16. C, 24. D, 36. B, 54. If between two numbers A, B there E, 4. F, 6. G, 9. numbers A, B there H, 2. P, 2. M, 4. K, 3. L, 3. N, 6. fall two mean proportionall numbers C, D; those numbers A, B are like solid numbers.

a 1. 8.  
b 10. 8.

e 11. def. 7.  
d cor. 18. 8.  
e 11. 7.

f 9. ax. 7.  
g 17. def. 7.  
h 17. 7.  
k 7. 5.  
l confr.  
m 21. def. 7.

Take E, F, G, the least :: in the proportion of A to C. then E and G are like plane numbers: let the sides of this be H & P, & of that K and L. therefore H.K :: P.L :: E.F. But E, F, G, doe equally measure A, C, D. viz. by the same number M. and likewise the said numbers E, F, G, doe equally measure the numbers C, D, B, viz. by the same number N. Therefore A = EM = HPM, and B = GN = KLN; and so A and B are solid numbers. But for that C = FM, and D = FN, therefore shall M.N :: FM.FN :: C.D :: E.F :: H.K :: P.L. wherefore A and B are like solid numbers. *W.W. to be Dem.*

Lemma.

AE, BF, CG, DH. If proportionall numbers A, A, B, C, D, B, C, D measure proportionall numbers AE, BF, CG, DH by E, F, G, H. the numbers (E, F, G, H) shall be proportionall.

a 19. 7.

b 1. ax. 7.  
c 9. ax. 7.

For being AEDH = BFCG, and AD = BC, thence will  $\frac{AEDH}{AD} = \frac{BFCG}{BC}$  that is, EH = FG.

Therefore E.F :: G.H. *W.W. to be Dem.*

Coroll.

d 15. def. 7.

e lem. prop.

Hence  $\frac{Bq}{Aq} = \frac{B}{A} \times \frac{B}{A}$  For 1.B :: B.Bq and r. A :: A.Aq. therefore 1. B :: B.Bq therefore  $\frac{Bq}{Aq} = \frac{B}{A} \times \frac{B}{A}$ . In like manner  $\frac{B}{Ac} \times \frac{B}{Ac} = \frac{Bc}{Acc}$  and so of the rest.

PROP.

PROP. XXII.

Aq, B, C. If three numbers Aq, B, C be continually proportionall, and the first Aq a square, the third C shall also be a square.

For because  $Aq \cdot C = Bq$ ,<sup>b</sup> thence is  $C = \frac{Bq}{Aq} = \frac{B}{A}$  a 20 7.  
b 7. ex. 7.  
c cor. of the  
lem. prece  
d hyp. and  
14. 8.  
 Q. B But it is plain that B is a number,<sup>d</sup> because Rq  
 or C is a number. Therefore if three, &c.

PROP. XXIII.

Ac, B, C, D. If four numbers A, B, C, D be continually proportionall, and the first of them Ac a cube, the fourth also D shall be a cube.

For because  $Ac \cdot D = BC$ ,<sup>b</sup> therefore  $D = \frac{BC}{Ac}$  a 19 7.  
b 7. ex. 1.  
c cor. of the  
prece. lem.  
20. 7.  
 $= \frac{B}{Ac} \times C$ ; that is (because  $Ac \cdot C = Bq$ , and  
<sup>b</sup> thence  $C = \frac{Bq}{Ac}$ )  $D = \frac{B}{Ac} \times \frac{Bq}{Ac} = \frac{Bc}{Ac} = C : B$   
 But it is evident<sup>c</sup> that  $\frac{Bc}{Ac}$  or  $\frac{B}{Ac}$  is supposed a number. Therefore if four numbers, &c. e 15. 8.

PROP. XXIV.

A, 16. 24. B, 36. If two numbers A, B, be in the same proportion one to another, that a square number C is to a square number D, and the first A be a square number, the second also B shall be a square number.

Between C and D being square numbers, <sup>a</sup> and so between A and B having the same proportion, <sup>b</sup> falls one mean proportionall. Therefore <sup>c</sup> being A is a 8. 8.  
b 11. 8.  
c hyp.

e 22. 2.

is a square number,  $e$  B also shall be a square number. *W.W. to be Dem.*

## Coroll.

a 10, & 18.  
2.

1. Hence, If there be two like numbers AB, CD ( $A.B :: C.D$ ) and the first AB be a square, the second also CD shall be a square.

\* For  $AB.CD :: Aq.Cq$ .

2. From hence it appears, That the proportion of any square number to any other not square cannot possibly be declared in two square numbers. Whence it cannot be  $Q.Q :: 1.2$ . nor  $1.5 :: Q.Q$ , &c.

## P R O P. XXV.

C, 64. 96. 144. D, 216. If two numbers A, B, be  
A, 8, 12. 18. B, 27. in the same proportion one  
to another, that a cube number C is to a cube number D, the first of them A being a cube number; the second B shall likewise be a cube number.

a 12. 2.  
b 2. 2.  
c hyp.  
d 12. 2.

\* Between the cube numbers C and D,  $b$  and so between A and B having the same proportion, fall two mean proportionals; therefore  $c$  because A is a cube,  $d$  shall B be a cube also. *W.W. to be Dem.*

## Coroll.

a 12, & 19.  
2.

1. Hence, If there be two numbers ABC, DEF ( $A.B :: D.E$ , and  $B.C :: E.F$ ;) and the first ABC be a cube, the second DEF shall be a cube also.

\* For  $ABC.DEF :: Ac.Dc$ .

2. It is perspicuous from hence, That the proportion of any cube number to any other number not a cube cannot be found in two cube numbers.

PROP. XXVI.

A, 20. C, 30. B, 45. Like plane numbers A, B,  
D, 4. E, 6. F, 9. are in the same proportion one  
to another, that a square num-  
ber is in to a square number.

Between A and B  $a$  falls one mean proportionall  $a$  18. 8.  
number C;  $b$  take three numbers D, E, F the least  $b$  2. 8.  
 $\frac{a}{b}$  in the proportion of A to C. the extremes D,  
F,  $b$  shall be square numbers. But of equality A. B  
 $c$  :: D.F. therefore A.B :: Q. Q. *W.W. to be Dem.*

$c$  14. 7.

PROP. XXVII.

A, 16. C, 24. D, 36. B, 54. Like solid numbers  
E, 8. F, 12. G, 18. H, 27. A, B, are in the same  
proportion one to an-  
other, that a cube number is in to a cube number.

Between A and B fall two mean proportionall  $a$  19. 8.  
numbers, namely C and D:  $b$  take four numbers E,  $b$  2. 8.  
F, G, H the least  $\frac{a}{b}$  in the proportion of A to C;  
 $b$  the extremes E, H, are cube numbers. But A.B  $c$  ::  $c$  14. 7.  
E.H :: C.C. *W.W. to be Dem.*

Schol.

1. From hence is inferred, that no numbers in pro-  
portion superparticular, superbipartiate, or double, *See Clavius.*  
or any other manifold proportion not denominated  
from a square number, are like plane numbers.

2. Likewise, that neither any two prime numbers,  
nor any two numbers prime one to another, nor  
being squares, can be like plane numbers.

The End of the eighth Book.

# THE NINTH BOOK OF EUCLIDES ELEMENTS.

## PROPOSITION I.

A, 6. B, 54.  
Aq, 36. AB, 324.

**I**f two like plane numbers A, B multiplying one another, produce a number AB, the number produced AB shall be a square number.

a 17. 7.  
b 18. 8.  
c 8. 8.

d 22. 8.

For  $A : B :: Aq : AB$ ; wherefore since one mean proportionall  $b$  falls between A and B, likewise one mean proportionall number shall fall between Aq and AB: therefore being the first Aq is a square number, the third AB shall be a square number too. *W.W. to be Dem.*

e 19. 7.  
f 1. 2 x 7.

Or thus. Let  $a, b, c, d$ , be like plane numbers; namely  $a, b :: c, d$ . & therefore  $ad = bc$ . and so likewise  $a, b, c, d$ , or  $a, d, b, c$ ,  $= ad = Q; ad$ .

## PROP. II.

A, 6. B, 54.  
Aq, 36. AB, 324.

If two numbers A, B, multiplying one another, produce a square number AB, those numbers A, B are like plane numbers.

a 17. 7.  
b 11. 8.  
c 8. 8.  
d 20. 8.

For  $A : B :: Aq : AB$ ; wherefore being between Aq, AB,  $b$  there falls one mean proportionall number, likewise one mean shall fall between A and B. therefore A and B are like planes. *W.W. to be Dem.*

PROP.

P R O P. III.

A, 2. Ac, 3. Acc, 64. If a cube number Ac multiplying it self produce a number Acc, the number produced Acc shall be a cube number.

For  $1.A :: A.Aq^2 :: Aq.Ac$ . therefore between 1 and Ac fall two mean proportionalls. But  $1.Ac :: Ac.Acc$ . therefore between Ac and Acc, fall also two mean proportionalls: and so by consequence seeing Ac is a cube,  $\therefore$  Acc shall be a cube also. Which was to be Dem.

Or thus,  $aaa (Ac)$  multiplied into it self makes  $aaaaaa (Acc)$ ; this is a cube, whose side is  $aa$ .

P R O P. IV.

Ac, 8. Bc, 27. If a cube number Ac multiplying a cube number Bc produce a number AcBc, the produced number AcBc shall be a cube.

For  $Ac.Bc :: Acc.AcBc$ . But between Ac and Bc, two mean proportionall numbers fall;  $\therefore$  therefore there fall as many between Acc and AcBc. So that whereas Acc is a cube number,  $\therefore$  AcBc shall be such also. W.W. to be Dem.

Or thus.  $AcBc = aaabbb (ababab) = C: ab$ .

P R O P. V.

Ac, 8. B, 27. If a cube number Ac multiplying a number B produce a cube number AcB, the number multiplied B shall also be a cube.

For  $Acc.AcB :: Ac.B$ . But between Acc and AcB, two mean proportionalls;  $\therefore$  therefore also as many shall fall between Ac and B. whence Ac being a cube number,  $\therefore$  B shall be a cube number too. W.W. to be Dem.



## The ninth Book of

## PROP. VI.

A, 8. Aq, 64. Ac, 512. If a number A multiplying it self produce a cube Aq, that number A it self is a cube.

a hyp.  
b 19. def. 7.  
c 5. 9.

For because Aq is a cube, and AqA (Ac) also a cube; therefore c shall A be a cube. W.W. to be Dem.

## PROP. VII.

A, 6. B, 11. AB, 66. If a composed number A multiplying any number B, produce a number AB, the number produced AB shall be a solid number.

a 13. def. 7.  
b 9. ax 7.  
c 17. def. 7.

Being A is a composed number, & some other number D measures it, conceive by E. therefore A = DE: c whence DEB = AB is a solid number. W.W. to be Dem.

## PROP. VIII.

1. a, 3. a<sup>2</sup>, 9. a<sup>3</sup>, 27. a<sup>4</sup>, 81. a<sup>5</sup>, 243. a<sup>6</sup>, 729.

If from a unite there be numbers continually proportionall how many soever (1, a, a<sup>2</sup>, a<sup>3</sup>, a<sup>4</sup>, &c.) the third number from a unite a<sup>2</sup> is a square number; and so are all forward, leaving one between (a<sup>4</sup>, a<sup>6</sup>, a<sup>8</sup>, &c.) But the fourth a<sup>3</sup> is a cube number; and so are all forward, leaving two between (a<sup>6</sup>, a<sup>9</sup>, &c.) The seventh also a<sup>6</sup> is both a cube number and a square; and so are all forward, leaving five between (a<sup>12</sup>, a<sup>18</sup>, &c.)

For 1. a<sup>2</sup> = Q.a. and a<sup>4</sup> = aaaa = Q. aa. and a<sup>6</sup> = aaaaaa = Q. aaa, &c.

2. a<sup>3</sup> = aaa = C.a. and a<sup>6</sup> = aaaaaa = C.aa. and aaaaaaaa = C.aaa, &c.

3. a<sup>6</sup> = aaaaaa = C.aa = Q.aaa. therefore, &c.

a hyp.  
b 10. 7.  
c 12. 8.

Or according to Euclide; Because 1. a<sup>0</sup> :: a. a<sup>2</sup>. b shall a<sup>2</sup> = Q:a. therefore seeing a<sup>2</sup>, a<sup>3</sup>, a<sup>4</sup>, are ::, c the third a<sup>4</sup> shall be a square number; and so likewise a<sup>6</sup>, a<sup>8</sup>, &c. Also because 1. a<sup>0</sup> :: a<sup>2</sup>. a<sup>3</sup>. therefore shall a<sup>3</sup>, b = a<sup>2</sup> x a = C: a. d therefore the fourth from a<sup>3</sup>, namely a<sup>6</sup>, shall be likewise a cube, &c. and consequently a<sup>6</sup> is both a cube and a square number, &c.

d 13. 8.

PROP.

P R O P. IX.

1, 2, 4, 8, 16, 32, 64, 128, 256, &c. If from a unite there be numbers how many soever continually proportionall (1, a, a<sup>2</sup>, a<sup>3</sup>, &c.) and the number following the unite (a) be a square; then all the rest, a<sup>2</sup>, a<sup>3</sup>, a<sup>4</sup>, &c. shall be squares too. But if the number next the unite (a) be a cube, then all the following numbers a<sup>2</sup>, a<sup>3</sup>, a<sup>4</sup>, &c. shall be cube numbers.

1. Hyp. For a<sup>2</sup>, a<sup>4</sup>, a<sup>6</sup>, &c. are square numbers by the prec. prop. also being a is taken to be a square, therefore the third a<sup>3</sup> shall be a square, and likewise a<sup>5</sup>, a<sup>7</sup>, &c. and so all.

2. Hyp. a is taken to be a cube, therefore a<sup>4</sup>, a<sup>7</sup>, a<sup>10</sup> are cubes; but by the prec. a<sup>3</sup>, a<sup>6</sup>, a<sup>9</sup>, &c. are cubes; lastly because 1.a :: a.aa. therefore shall a<sup>2</sup> = Q: a. but a cube multiplied into it self produces a cube; therefore a<sup>2</sup> is a cube, and consequently the fourth from it a<sup>5</sup>, and in like manner a<sup>8</sup>, a<sup>11</sup>, &c. are cubes, therefore all. W. W. to be Dem.

Peradventure more clearly thus. Let b be the side of the square number a, and so the series a, a<sup>2</sup>, a<sup>3</sup>, a<sup>4</sup>, &c. will be otherwise expressed, thus, bb, b<sup>4</sup>, b<sup>6</sup>, b<sup>8</sup>, &c. It is evident that all these numbers are squares, and may be thus expressed, Q: b, Q: bb: Q, bbb, Q: bbbb, &c.

In like manner, if b be the side of the cube a, the series may be expressed thus, b<sup>3</sup>, b<sup>6</sup>, b<sup>9</sup>, b<sup>12</sup>, &c. or C: b, C: b<sup>2</sup>, C: b<sup>3</sup>, C: b<sup>4</sup>, &c.

P R O P. X.

1, 2, 4, 8, 16, 32, 64, 128, 256, &c. If from a unite there be numbers how many soever continually proportionall (1, a, a<sup>2</sup>, a<sup>3</sup>, &c.) and the number next the unite (a) be not a square number; then is none of the rest following a square number, excepting a<sup>2</sup> the third from the unite, and so all forward, leaving one between (a<sup>4</sup>, a<sup>6</sup>, a<sup>8</sup>, &c.)

M 3 But

But if that (a) which is next after the unite, be not a cube number, neither is any other of the following numbers a cube, saving a; the fourth from the unite, and so all forward, leaving two between, 26, 29, a<sup>12</sup>, &c.

a Hyp. b suppos and 8 9. c 24. 8. 1. Hyp. For it it be possible, let a, be a square number; therefore because a. a<sup>2</sup> :: a<sup>4</sup>. a<sup>4</sup>, and by inversion a<sup>5</sup>. a<sup>4</sup> :: a<sup>2</sup>. a; and also a<sup>5</sup> and a<sup>4</sup> b square numbers, and the first a<sup>2</sup> a square, therefore a shall be likewise a square; contrary to the Hyp.

d 24. 7. 2. Hyp. If it may be, let a<sup>4</sup> be a cube; being of equality a<sup>4</sup>. a<sup>6</sup> :: a. a<sup>3</sup>, and inversely a<sup>6</sup>. a<sup>4</sup> :: a<sup>3</sup>. a; and also being a<sup>6</sup> and a<sup>4</sup> are cubes, and the first a<sup>3</sup> a cube, therefore a shall be a cube also; against the Hyp.

e 25. 8.

PROP. XI.

1. 1, a, a<sup>2</sup>, a<sup>3</sup>, a<sup>4</sup>, a<sup>5</sup>, a<sup>6</sup>. If there be numbers 1, 3, 9, 27, 81, 243, 729. how many soever in continuall proportion from a unite (1, a, a<sup>2</sup>, a<sup>3</sup>, &c.) the lesse measureth the greater by some one of them that are amongst the proportionall numbers.

Because 1. a :: a. aa. therefore  $\frac{aa}{a} = a = \frac{aaa}{aa}$

a 5. ax. 7. b 20 def. 7. b 14. 7.

Also because 1. aa b :: a. aaa. therefore  $\frac{aaa}{a} = aa = \frac{aaaa}{aa}$

$\frac{a^4}{aa} = a^5$ , &c. Lastly because 1. a<sup>3</sup> b :: a. a<sup>4</sup>. therefore

$\frac{a^4}{a} = a^3 = \frac{a^6}{a}$ , &c.

Coroll.

Hence, If a number that measures any one of proportionall numbers, be not one of the said numbers, neither shall the number by which it measures the said proportionall numbers, be one of them.

PROP.

P R O P. XII.

$1, a, a^2, a^3, a^4.$  If there be numbers how many  
 $1, 6, 36, 216, 1296.$  soever in continuall proportion  
 $B, 3$  from a unite ( $1, a, a^2, a^3, a^4.$ )  
 whatsoever prime numbers B  
 measure the last  $a^4$ , the same (B) shall also measure  
 the number (a) which follows next after the unite.

If you say B does not measure  $a^4$ , then B is prime <sup>a 31.7.</sup>  
 to  $a^4$ , and also B is prime to  $a^3$ ; <sup>b 27.7.</sup> and so conse-  
 quently to  $a^2$ ; <sup>c 16.7.</sup> and so conse-  
 quently to  $a$ , which it is supposed to measure. which  
 is Absurd.

Coroll.

1. Therefore every prime number that measures  
 the last, does also measure all those other numbers  
 that precede the last.

2. If any number not measuring that next to the  
 unite, does yet measure the last, it is a composed  
 number.

3. If the number next to the unite be a prime, no  
 other prime number shall measure the last.

P R O P. XIII.

$1, a, a^2, a^3, a^4,$  If from a unite be num-  
 $1, 5, 25, 125, 625.$  bers in continuall propor-  
 $H - - G - - F - - E - -$  tion how many soever ( $a,$   
 $a^2, a^3, \&c.$ ) and that  
 after the unite (a) a prime; then shall no other measure  
 the greatest number, but those which are amongst the said  
 proportionall numbers.

If it be possible, let some other B measure  $a^4$ , viz.  
 by F. then F shall be some other beside  $a, a^2, a^3$ . But <sup>a cor. 11.9.</sup>  
 because E measuring  $a^4$ , does not measure  $a$ , <sup>b 2. cor. 12.</sup> there-  
 fore E shall be a composed number, and some <sup>c 33.7.</sup>  
 prime number measure it, which does consequent- <sup>d 11. ex. 7.</sup>  
 ly measure  $a^4$ , and so is no other then a. therefore <sup>e 3. cor. 12.9.</sup>  
 a measures B. So also may F be shewn to be a com-  
 posed

f 9. ex 7.  
 g 19. 7.  
 h 20. def. 7.  
 k cor. 11. 9

l 23. def. 7.  
 m 20. def. 7.

posed number, measuring a 4, and so that a measures F. Therefore seeing  $EF = a^4 = a \times a^3$ , g shall a.  $E :: F. a^3$ . Consequently, whereas a measures B, likewise F shall equally measure a 3, viz. by the same number G. Nor shall G be a, or a 2, therefore, as before, G is a composed number, and a measures it. Wherefore being that  $FG = a^3 = a^2 \times a$ , g shall  $a.F :: G. a^2$ . and so because A measures F, b G shall equally measure a 2. viz. by the same number H, which is not a. Therefore being  $GH = a^2 = aa$ , l thence  $H. a :: a.G$ . & because a measures G (as before) m H also shall measure a, which is a prime number. Which is impossible.

PROP. XIV.

A, 30. If certain prime numbers B, C,  
 B, 2. C, 3. D, 5. D, do measure the least number A,  
 E, F. no other prime number E shall measure the same, besides those that measured it at first.

a 9. ex 7.  
 b 31. 7.

If it is possible, let A be  $= F. \frac{A}{B}$  then  $A = EF$ .  
 therefore every of the prime numbers B, C, D measures one of those E, F. not E, which is taken to be a prime; therefore F, which is lesse then it self A; contrary to the Hyp.

PROP. XV.

A, 9. B, 12. C, 16. If three numbers continually proportionall, B, C, be  
 D, 3. E, 4. the least of all that have the same proportion with them; any two of them added together shall be prime to the third,

a 37. 7.  
 b 2. 2.

Take D and E the least in the proportion of A to B; then  $A = Dq$ , &  $C = Eq$ , and  $B = DE$ .  
 But

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But because  $D^c$  is prime to  $E$ ,  $d$  therefore shall  $D + E$  be prime to both  $D$  and  $E$ .  $^*$  therefore  $D \times D + E = Dq + DE$  ( $f A + B$ ) is prime to  $E$ , and so to  $C$  or  $Eq$ . *W. W. to be Dem.*

c 14. 7.  
d 30. 7.  
e 16. 7.  
e 3. 2.  
f before.

In like manner  $DE + Eq$  ( $B + C$ ) is prime to  $D$ , and consequently to  $A = Dq$ . *W. W. to be Dem.*

Lastly, because  $B^b$  is prime to  $D + E$ , it shall also be prime to the square of it  $Dq + 2 DE + Eq$  ( $A + 2 B + C$ );  $^l$  wherefore the said  $B$  shall be prime to  $A + B + C$ ,  $^l$  and so likewise to  $A + C$ . *which was to be Dem.*

g 27. 7.  
h 26. 7.  
k 4. 2.  
l 30. 7.

## P R O P. XVI.

$A, 3. B, 5. C - \dots$  *If two numbers A, B, be prime to one another, the second B shall not be to any other C, as the first A is to the second B.*

If you affirm  $A.B :: B.C$ . then whereas  $A$  and  $B$  are the least in their proportion,  $b$  shall measure  $B$  as many times as  $B$  does  $C$ ; but  $A^c$  measures it self also; therefore  $A$  and  $B$  are not prime to one another. *against the Hyp.*

a 23. 7.  
b 21. 7.  
c 6. ex. 7.

## P R O P. XVII.

$A, 8. B, 12. C, 18, D, 27. E - \dots$  *If there be numbers how many soever in continuall proportion A, B, C, D, and the extremes of them A, D be prime one to another, the last D shall not be to any other number E, as the first A is to the second B.*

Suppose  $A.B :: D.E$ . then alternately  $A.D :: B.E$ . therefore seeing  $A$  and  $B$  are the least in their proportion,  $A^b$  shall measure  $B$ ,  $c$  and  $B$  likewise  $C$ , and  $C$  the following number  $D$ .  $d$  and so  $A$  shall measure the said number  $D$ . Wherefore  $A$  and

a 23. 7.  
b 21. 7.  
c 20 def 7.  
d 11. ex. 7.

D

are not prime to one another, *contrary to the Hypoth.*

## P R O P. XVIII.

A, 4. B, 6. C, 9. *Two numbers being given A, B, to consider if there may be a third number found proportionall to them C.*  
Bq, 36.

a 9. ex 7.  
b 10. 7.

If A measure Bq (by any number C, a then  $AC = Bq$ . from whence b it is manifest that  $A.B :: B.C$ . *W.W. to be Dem.*

c 7. ex 7.

A, 6. B, 4. Bq, 16. *But if A doe not measure Bq, there will not be any third proportionall. For suppose  $A.B :: B.C$ . then  $AC = Bq$ . = a and consequently  $Bq = C$ . namely A measures Bq. *Which is against the Hypoth.**

## P R O P. XIX.

A, 8. B, 12. C, 18. D, 27. *Three numbers being given A, B, C, to consider if a fourth proportionall to them D may be found.*  
BC, 216.

a 9. ex 7.  
b ex. 19. 7.

If A measures BC by any number D, a then  $AD = BC$ ; b therefore it appears that  $A.B :: C.D$ . which was required.

But if A do not measure BC, then there can no fourth proportionall be found; which may be shewn as in the prec. prop.

## P R O P. XX.

A, 2. B, 3. C, 5. *More prime numbers may be given D, 30. G ---- then any multitude whatsoever of prime numbers A, B, C, propounded.*

a 38. 7.  
b 33. 7.

a Let D be the least which A, B, C, measure; If  $D + 1$  be a prime, the case is plain; if composed, b then some prime number, conceive G, measures  $D + 1$ ; which

which is none of the three A, B, C; For if it be, seeing it *c* measures the whole  $D + 1$ , *d* and the part taken away *D*, *e* it shall also measure the remaining unite. *c* *suppos.*  
*d* *confr.*  
*e* 12. *ax.* 7.  
 Which is *abs.* Therefore the propounded number of prime numbers is increased by  $D + 1$ , or at least by *G*.

PROP. XXI.

A <sup>5</sup>..... E <sup>5</sup>..... B <sup>3</sup>... F <sup>3</sup>... C <sup>2</sup>.. G <sup>2</sup>.. D 20.

If even numbers, how many soever, AB, BC, CD, be added together, the whole AD shall be even.

Take EB =  $\frac{1}{2}$  AB, and FC =  $\frac{1}{2}$  BC, and GD =  $\frac{1}{2}$  CD. *a* 6. *def.* 7.  
*b* 12. 7.  
 It is plain that EB + FC + GD =  $\frac{1}{2}$  AD. *c* 6. *def.* 7.  
 therefore AD is an even number. Which was to be Dem.

PROP. XXII.

A <sup>9</sup>..... F <sup>7</sup>..... B <sup>5</sup>..... G <sup>5</sup>..... C <sup>3</sup>..... H <sup>3</sup>..... D <sup>3</sup>..... L <sup>3</sup>..... E 22.

If odd numbers, how many soever, AB, BC, CD, DE, be added together, and their multitudes even, the whole also AE shall be even.

A unite being taken from each odde number, there will remain AF, BG, CH, DL, even numbers, *a* 7. *def.* 7.  
*b* 21. 9.  
*c* *hyp.*  
 and thence the number compounded of them will be even. adde to them the even number made of the remaining unites, and the whole AE will *d* 21. 9.  
 thereby be even. *W.W.* to be Dem.

PROP.



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PROP. XXIII.

$\begin{matrix} 7 & 5 & 1 \\ A & \dots & B & \dots & C & \dots & E & \dots & D & 15. \end{matrix}$

$\begin{matrix} 3 \\ \end{matrix}$

If odd numbers how many soever AB, BC, CD, be added together, and the multitude of them be odde, the whole AD shall be odde.

a 12. 9.  
b 21. 3.  
c 7. def. 7.

For CD one of the odde numbers being taken away, the number compounded of the others AC is even. Whereto adde CD - 1, the whole AE is also even; wherefore the unite being restored the whole AD will be odd. *W. W. to be Dem.*

PROP. XXIV.

$\begin{matrix} 4 & 5 & 1 \\ A & \dots & B & \dots & D & \dots & C & 10. \end{matrix}$

$\begin{matrix} 6 \\ \end{matrix}$

If an even number AB be taken away from an even number AC, that which remains BC shall be even.

c 7. def. 7.

b hyp.  
c 21. 9.

For if BD (BC - 1) be odde, BC (BD + 1) will be even. *W. W. to be Dem.* But if you say BD is even, because AB is even, thence AD will be so; & consequently AC (AD + 1) will be odde, contrary to the Hypoth. therefore BC is even. *W. W. to be Dem.*

PROP. XXV.

$\begin{matrix} 6 & 1 & 3 \\ A & \dots & D & \dots & C & \dots & B & 10. \end{matrix}$

$\begin{matrix} 7 \\ \end{matrix}$

If from an even number AB, an odde number AC be taken away, the remaining number CB shall be odde.

a 7. def. 7.  
b 24. 9.  
c 7. def. 7.

For AC - 1 (AD) is even. b therefore DB is even; c and consequently CB (DB - 1) is odd. *W. W. to be Dem.*

PROP. XXVI.

$\begin{matrix} 4 & 6 & 1 \\ A & \dots & C & \dots & D & \dots & B & 11. \end{matrix}$

If from an odde number AB be taken away an odde number CB, that which remaineth AC shall be even.

For

For  $AB - 1$  ( $AD$ ) and  $CB - 1$  ( $CD$ )<sup>a</sup> are even; <sup>a 7. def. 7.</sup>  
<sup>b</sup> therefore  $AD - CD$  ( $AC$ ) is even. *W.W. to be Dem.* <sup>b 24. 9.</sup>

PROP. XXVII.

*If from an odde number  
 A, D ... C ... B II AB be taken away an even  
 number CB, the residue AC  
 shall be odde.*

For  $AB - 1$  ( $DB$ )<sup>a</sup> is even, and  $CB$  is supposed <sup>a 7. def. 7.</sup>  
 to be even; <sup>b</sup> therefore the residue  $CD$  is even; <sup>b 24. 9.</sup> there-  
 fore  $CD + 1$  ( $CA$ ) is odde. *W.W. to be Dem.* <sup>c 7. def. 7.</sup>

PROP. XXVIII.

*If an odde number A multiplying an even  
 number B produce a number AB, the num-  
 ber produced AB shall be even.*

For  $AB$ <sup>a</sup> is compounded of the odde <sup>a hyp. and</sup>  
 number  $A$  taken as many times as a unite is con- <sup>b 15. def. 7.</sup>  
 tained in  $B$  an even number. <sup>b</sup> Therefore  $AB$  is an <sup>b 21. 9.</sup>  
 even number.

Schol.

In like manner, if  $A$  be an even number,  $AB$  shall  
 be an even number also.

PROP. XXIX.

*If an odde number A multiplying an odde  
 number B, produce a number AB, the number  
 produced AB shall be odde.*

For  $AB$ <sup>a</sup> is compounded of the odde <sup>a 15. def. 7.</sup>  
 number  $B$  taken as often as a unite is included in  $A$   
 likewise an odde number, <sup>b</sup> Therefore  $AB$  is an odd <sup>b 23. 9.</sup>  
 number. *W.W. to be Dem.*

Schol.

Schol.

$B, 12$  (C, 4.  
 $A, 3$

1. An odd number A measuring an even number B, measures the same by an even number C.

a 9 ax. 7.  
b 19. 9.

For if C be affirmed to be odde, then because  $a B = AC, b$  therefore B shall be odde, against the Hyp.

$B, 15$  (C, 5  
 $A, 3$

2. An odde number A measuring an odde number B, measures the same by an odde number C.

a 18. 9.

For if C be said to be even, then AC, or B will be even, contrary to the Hypoth.

$B, 15$  (C, 5.  
 $A, 3$

3. Every number (A and C) that measures an odde number B, is itself an odde number.

a 18. 9.

For if either A or C be affirmed to be even, B shall be an even number, against the Hypoth.

P R O P. XXX.

$B, 24$ .  
 $A, 3$

(C, 8.

$D, 12$ .  
 $A, 3$

(E, 4.

If an odde number A measure an even number B, it shall also measure the half of it D.

Let B be  $= C. b$  then C is an even number.

a hyp.  
b 1. schol.  
29. 9.  
c 9 ax. 7.  
d 1. 2  
e 9 P.  
f 7. ax. 1.  
g 7. ax. 7.

Therefore let E be  $= \frac{1}{2} C$ , then B  $= CA = 2 EA = 2 D. f$  therefore  $EA = D$ ; g and consequently  $D = E. W.W. to be Dem.$

P R O P. XXXI.

A, 5. B, 8. C, 16. D. . . . If an odde number A be prime to any number B, it shall also be prime to the double thereof C.

a 1. schol. 29.  
9.  
b 30. 9.

If it be possible, let some number D measure A and C, then D measuring the odde number A shall be odde it self, b & so shall measure B the half of the even number C. therefore A & B are not prime one to another. Which is against the Hyp. Coroll.

Coroll.

It follows from hence that an odde number which is prime to any number of double progression, is also prime to all the numbers of that progression.

P R O P. XXXII.

A, 2. B, 4. C, 8. D, 16. All numbers A, B, C, D, &c. in double progression from the binarie are evenly even only.

It is evident that all these numbers 1, A, B, C, D, are even, and  $b \div a$ , namely in a double proportion, and so every lesse measures the greater by some one of them. Wherefore all are evenly even. But for that A is a prime number, no number beside these shall measure any of them. Therefore they are evenly even only. *W.W. to be Dem.*

a 6 def. 7.  
b 20 def. 7.  
c 12. 9.  
d 8 def. 7.  
e 13 9.

P R O P. XXXIII.

A, 30. B, 15. If of a number A, the half B be odd, the same A is evenly odd only.

Being an odde number B measures A by two an even number, therefore B is evenly odde. If you affirm it to be evenly even, then some even number D measures it by an even number E. whence  $2 B \div A \div DE$ . wherefore  $2 E \div D \cdot B$ . and therefore as 2f measures the even number E, so D an even number measures B an odde. Which is impossible.

a hyp.  
b 9 def. 7.  
c 8. def. 7.  
d 9 ex. 7.  
e 19 7.  
f 6 def. 7.  
g 20. def. 7.

P R O P. XXXIV.

A, 24. If an even number A be neither doubled from two, nor have it's half part odde, it is both evenly even and evenly odde.

It is undoubtable, that A is evenly even, because the half of it is not odde. But because if A be divided into two equal parts, and so continuing the bipartition

tition

a 7. def. 7.

b 1. sch. 199.

tion, we shall at length light upon some a odde number (not upon the number two ; because A is not supposed to be doubled upward from two) which shall measure A by an even number. (for b otherwise A it self should be odde , against the Hyp.) Therefore A is also evenly odde. *W.W. to be Dem.*

P R O P. XXXV.



If there be numbers in continuall proportion how many soever A . BG, C, DN , and the number FG be taken from the second, and KN from the last , equal to the first A; as the excessse of the second BF is to the first A, so shall the excessse of the last DK be to all the numbers that precede it, A, BG, C.

a hyp. b 17. §.

c 12. §. d 3. cor. 1.

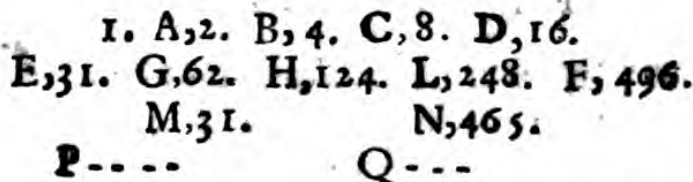
e 18. §.

From DN take NL = BG, and NH = C. Because DN. C (HN) e :: HN. BG (LN) e :: LN (BG.) A (KN.) b therefore by dividing each , shall DH.HN :: HL.LN :: LK. KN. c wherefore DK. C + BG + A :: LK (d BF.) KN (A.) *W.W. to be Dem.*

Coroll.

Hence e by compounding , DN + BG + C. A + BG + C :: BG. A.

P R O P. XXXVI.



If from a unite be taken how many numbers soever 1, A, B, C, D, in double proportion continually, untill the whole added together E be a prime number ; and if this whole

whole E multiplying the last produce a number F, that which is produced F shall be a perfect number.

Take as many numbers E, G, H, L, likewise in double proportion continually; then <sup>a</sup> of equality A.D :: E.L. <sup>b</sup> therefore AL = DE <sup>c</sup> = F. <sup>d</sup> whence L = F. Wherefore E, G, H, L, F, are :: in double <sub>2</sub>

proportion. Let G - E be = M, and F - E = N; <sup>e</sup> then M.E :: N.E + G + H + L. <sup>f</sup> But M = E. <sup>g</sup> therefore N = E + G + H + L. <sup>h</sup> therefore F = I + B + C + D + E + G + H + L = E +

N. Moreover because D <sup>k</sup> measures DE (F); therefore every one, I, A, B, C, <sup>l</sup> measuring D, as <sup>m</sup> also E, G, H, L, does measure F. And further, no other number measures the said F. For if there do, let it be P, which measures F by Q. <sup>n</sup> therefore PQ = F = D.

F. <sup>o</sup> therefore E. Q :: P. D. therefore seeing A a prime number measures D, <sup>p</sup> and so no other P measures the same, <sup>q</sup> consequently E does not measure Q. Wherefore E being supposed a prime number,

it shall be prime to Q. <sup>r</sup> wherefore E and Q are the least in their proportion; <sup>s</sup> and so E measures P as many times as Q does D; <sup>t</sup> therefore Q is one of them A, B, C. Let it be B. seeing then of equality B. D :: E. H. <sup>x</sup> and so BH = DE = F = PQ. <sup>y</sup> and so also Q. B :: H. P. <sup>z</sup> therefore H = P. therefore P is also one of them A, B, C, &c. against the Hypoth.

Wherefore no other beside the foresaid numbers measures F, and <sup>z</sup> consequently F is a perfect number. Which was to be Demonstrated.


The End of the ninth Book.


N THE

THE TENTH BOOK  
OF  
EUCLIDE'S ELEMENTS.

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Definitions.

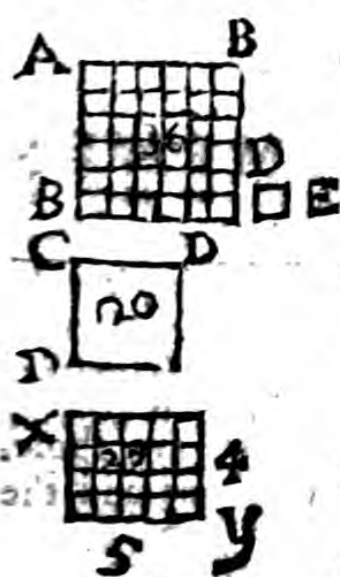
I.  Commensurable magnitudes are those, which are measured by one and the same measure.

 The note of commensurability is  $\square$ , as  $\square$  A  $\square$  B; that is, the line A of 8 foot is commensurable to the line B of 13 foot; because  $\square$  a line of one foot measures both A & B. Also  $\sqrt{18} \square \sqrt{50}$ ; because  $\sqrt{2}$  measures both  $\sqrt{18}$  and  $\sqrt{50}$ . For  $\sqrt{\frac{18}{2}} = \sqrt{9} = 3$ . and  $\sqrt{\frac{50}{2}} = \sqrt{25} = 5$ . wherefore  $\sqrt{18} \cdot \sqrt{50} :: 3 \cdot 5$ .

II. Incommensurable magnitudes are such, of which no common measure can be found.

Incommensurability is denoted by this mark  $\square$ ; as  $\sqrt{6} \square \sqrt{25}$  (5;) that is,  $\sqrt{6}$  is incommensurable to the number 5, or to a magnitude designed by that number; because there is no common measure of them, as shall appear hereafter.

III. Right lines are commensurable in power, when the same space does measure their squares.



The mark of this commensurability is  $\square$ ; as  $AB \square CD$ . i.e. the line  $AB$  of 6 foot is in power commensurable to the line  $CD$ , which is expressed by  $\sqrt{20}$ . because  $E$  the space of one foot square does as well measure  $ABq$  (36) as the rectangle  $XY$  (20) to which the square of the line  $CD$  ( $\sqrt{20}$ ) is equal. The same note  $\square$  sometimes signifies commensurable in power only.

IV. Lines incommensurable in power are such, to whose squares no space can be found to be a

common measure.

This incommensurability is denoted thus;  $5 \square \sqrt{8}$ . i.e. the numbers or lines 5, and  $\sqrt{8}$  are incommensurable in power, because their squares 25 and  $\sqrt{8}$  are incommensurable.

V. From which it is manifest, that to any right line given right lines infinite in multitude are both commensurable and incommensurable; some in length and power, others in power only. The right line given is called a Rationall line.

The note of which is  $\rho$ .

VI. And lines commensurable to this line, whether in length and power, or in power only, are also called Rationall,  $\rho$ .

VII. But such as are incommensurable to it, are called Irrationall,

And denoted thus  $\rho$ .

VIII. Also the square which is made of the said given right line is called Rationall,  $\rho$ .

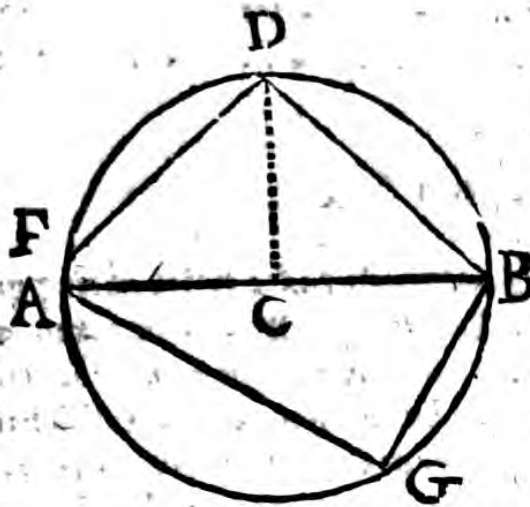
IX. And likewise such figures as are commensurable to it, are Rationall  $\rho$ .

X. But such as are incommensurable, Irrationall  $\rho$ .

XI. And those right lines also, which contain them in power, are Irrationall  $\rho$ .



Schol.



That the last seven definitions may be rendered more clear by an example, let there be a circle AD-BP, whose semidiameter is CB inscribe therein the sides of the ordinate figures, as of a Hexagone BP, of a Triangle AP, of a

square BD, of a Pentagone FD. Therefore, if according to the 5. def. the semidiameter CB be the Rationall line given, expressed by the number 2, to which the other lines BP, AP, BD, FD, are to be compared, then  $Pa = BC = 2$ , wherefore BP is  $\dot{p} \sqcup BC$ , according to the 6. def. Also  $AP^b = \sqrt{12}$  (for  $AB^q (16) - BP^q (4) = 12$ ) therefore AB is  $\dot{p} \sqcup BC$  likewise accord. to the 6. def. and  $AP^q (12)$  is  $\dot{p} \nu$  by the 9. def. Moreover  $BD^b = \sqrt{DC^q + BC^q} = \sqrt{8}$ ; whence BD is  $\dot{p} \sqcup BC$ ; and  $BD^q \dot{p} \nu$  Lastly,  $FD^q = 10 - \sqrt{20}$  (as shall appear by the praxis to be delivered at the 10. 13.) shall be  $\dot{p} \nu$ , according to the 10. def. & consequently  $FD = \sqrt{10 - \sqrt{20}}$  is  $\dot{p}$ , according to the 11. def.

a cor. 15 4

b 47. 1.

A Postulate.

That any magnitude may be so often multiplied, till it exceed any magnitude whatsoever of the same kind.

Axiomes.

1. A magnitude measuring how many magnitudes soever, does also measure that which is composed of them.

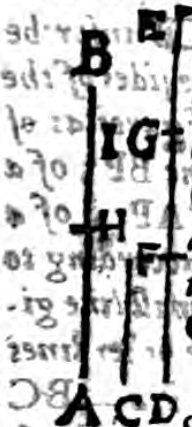
2. A

2. A magnitude measuring any magnitude whatsoever, does likewise measure every magnitude which that measures.

3. A magnitude measuring a whole magnitude and a part of it taken away, does also measure the residue.

P R O P. I.

*Two unequal magnitudes AB, C, being given, if from the greater AB there be taken away more then half (AH) and from the residue (HB) be again taken away more then half (HI) and this be done continually, there shall at length be left a certain magnitude IB, lesse then the lesse of the magnitudes first given*



Take C so often, till it's multiplex DE post. 10. doe somewhat exceed AB, and there be DF = FG = GE = C. Take from AB more then half HA, and from the remainder HB more then half HI, and so continually, till the parts AH, HI, IB, be equall in multitudine to the parts DF, FG, GE. Now it is plain, that FE, which is not lesse then  $\frac{1}{2}$  DE, is greater then HB, which is lesse then  $\frac{1}{2}$  AB  $\supset$  DE. And in like manner GE, which is not lesse then  $\frac{1}{2}$  FE, is greater then IB  $\supset$   $\frac{1}{2}$  HB. therefore C, or GE  $\supset$  IB. *W. W. to be Dem.*

The same may also be demonstrated, if from AB the half AH be taken away, and again from the residue HB the half HI, and so forward.

PROP. II.



Two unequal magnitudes being given (AB, CD) if the lesse AB be continually taken from the greater CD, by an interchangeable subtraction, and the residue doe not measure the magnitude going before, then are the magnitudes given incommensurable.

If it be possible, let some magnitude E be the common measure. Then because AB taken from CD, as often as it can be, leaves a magnitude FD lesse then it self, and FD taken from AB leaves GB, and so forward; therefore at length some magnitude GB shall be left. therefore E measuring AB, and so CF, and the whole CD, shall also measure the residue FD. consequently also AG; wherefore it shall likewise measure the remainder GB; lesse then it self. Which is Absurd.

a 1. 10.  
b hyp.  
c 2. ex. 10.

d 3. ex. 10.

PROP. III.



Two commensurable magnitudes being given AB, CD, to find out their greatest common measure FB.

Take AB from CD, and the residue ED from AB, and FB from ED, till FB measure ED (which will come to passe at length, because by the Hyp. AB  $\square$  CD) FB shall be the magnitude required.

For FB measures ED, and so also AF; but it measures it self too, therefore likewise AB, and consequently CE, and so the whole CD. Wherefore FB is the common measure of AB, CD. If you affirm G to be a common measure greater then that, then G measuring AB and CD, measures also CE, and the remainder ED, and so AF; and consequently the remainder FB, the greater the lesse. Which is Absurd.

a 1. 10.

b consr.  
c 2. ex. 10.  
d 1. ex. 10.

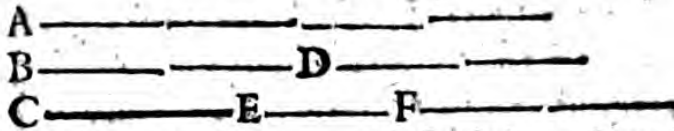
e 2. ex. 10.  
f 3. ex. 10.

Co-

Coroll.

Hence, A magnitude that measures two magnitudes, does also measure their greatest common measure.

P R O P. I V.



Three commensurable magnitudes being given A, B, C, to find out their greatest common measure.

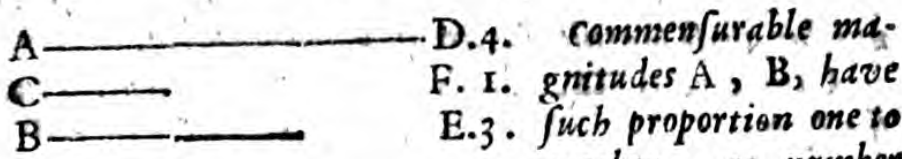
Find out D the greatest common measure of any two A, B ; also E the greatest common measure of D and C. therefore E is the magnitude sought for.

For it is clear, E measuring D and C, does measure the three, A, B, C. Conceive another magnitude F greater then that to measure them ; then F measures D, and consequently E the greatest common measure of D, and C, the greater the lesse. Which is Absurd.

Coroll.

Hence also it appears, that if a magnitude measure three magnitudes, it shall likewise measure their greatest common measure.

P R O P. V.



both to number.

C being found the greatest common measure of A, B ; as often as C is contained in A and B, so often is 1 contained in the numbers D and E ; therefore C. A :: 1. D ; wherefore inversely A. C :: D. 1. but likewise C. B :: 1. E. therefore of equality A. B :: D. E :: N. N. *W.W. to be Dem.*

N 4

P R O P.

PROP. VI.

E \_\_\_\_\_ F, 1. If two magnitudes A,  
 A \_\_\_\_\_ C, 4. B, have such proportion  
 B \_\_\_\_\_ D, 3. one to another, as num-  
 ber C hath to number  
 D, those magnitudes A, B shall be commensurable.

a. 6. 10. 6.  
 b. constr.  
 c. hyp.  
 d. 2. 5.  
 e. 5. 27. 7.  
 f. 20. def. 7.  
 g. constr.  
 h. 1. def. 10.

What part 1 is of the number C, that let E be of A. Therefore because E.A b :: 1.C. and A.B c :: C.D. therefore of equality shall E.B :: 1. D. Wherefore seeing 1 e measures the number D, f likewise E measures B; but it g also measures A. therefore A  $\square$  B. W.W. to be Dem.

PROP. VII.

A \_\_\_\_\_ Incommensurable magni-  
 B \_\_\_\_\_ des A, B, have not that propor-  
 tion one to another, which num-  
 ber hath to number.

a. 6. 10.

If you affirm A.B :: N.N. then A  $\square$  B. against the Hyp.

PROP. VIII.

A \_\_\_\_\_ If two magnitudes A, B, have  
 B \_\_\_\_\_ not that proportion one to an-  
 other, which number hath to  
 number, those magnitudes are incommensurable.

a. 9. 10.

Conceive A  $\square$  B. then A.B :: N.N. contrary to the Hyp.

PROP. IX.

A \_\_\_\_\_ The squares described of right  
 B \_\_\_\_\_ lines commensurable in length,  
 E, 4. have that proportion one to an-  
 F, 3. other, that a square number hath  
 to a square number. And squares, which have that pro-  
 portion

portion one to another, that a square number hath to a square number, shall also have their sides commensurable in length. But such squares as are made of right lines incommensurable in length, have not that proportion one to another, which a square number hath to a square number. And squares which have not such proportion one to another as a square number hath to a square number, have not their sides commensurable in length.

1. Hyp. A  $\perp$  B. I say Aq.Bq :: Q. Q.

For let A. B :: number E. number F. therefore  
 $\frac{Aq}{Bq} = \frac{E^2}{F^2}$   $\frac{E}{F} = \frac{Eq}{Fq}$  therefore

a 5. 10.  
 b 20. 6.  
 c 11. 23. 5.  
 d 11. 8.  
 e 11. 1.

Aq.Bq :: Eq.Fq :: Q. Q. *W.W. to be Dem.*

2. Hyp. Aq.Bq :: Eq.Fq :: Q. Q. I say A  $\perp$  B. For A

twice  $\frac{Aq}{Bq} = \frac{Eq}{Fq} = \frac{E}{F}$  therefore A.

f 20 6.  
 g hyp.  
 h 11. 8.  
 i 11. 23 5.  
 k 6. 10.

B :: E. F :: N. N. & wherefore A  $\perp$  B. Which was to be Dem.

3. Hyp. A  $\perp$  B. I deny that Aq.Bq :: Q. Q. For suppose Aq. Bq :: Q. Q. then A  $\perp$  B, as is shewn before, against the Hyp.

4. Hyp. Not Aq. Bq :: Q. Q. I say that A  $\perp$  B. For conceive A  $\perp$  B. then Aq Bq :: Q. Q. as above, against the Hyp.

Coroll.

Lines  $\perp$  are also  $\perp$ . but not on the contrary. And lines  $\perp$  are not therefore  $\perp$ . but  $\perp$  are also  $\perp$ .

PROP.

PROP. X.

If four magnitudes be proportionall (C A :: B. D) and the first C be commensurable to the second A, the third B shall be commensurable to the fourth D. And if the first C be incommensurable to the second A, also the third B shall be incommensurable to the fourth D.

a 9. 10.  
b 6. 10.  
c 7. 10.  
d 8. 10.

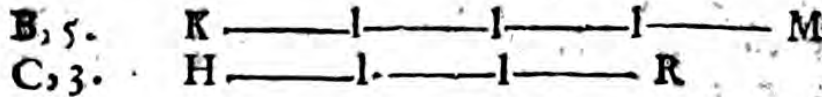
C A B D. If C  $\square$  A, then C.A :: N. N<sup>b</sup> :: B. D.<sup>b</sup> therefore B  $\square$  D. But if  $\square$  A, then shall not C.A :: N. N :: B. D.<sup>d</sup> wherefore B  $\square$  D. *W.W. to be Dem.*

Lemma 1.

To find out two plane numbers, not having the proportion which a square number hath to a square.

Any two plane numbers not like will satisfy this Lemma, as those numbers which have superparticular, superbipartient, or double proportion; or any two prime numbers. See schol. 27. 8.

Lemma 2.



To find out a line HR, to which a right line given KM hath the proportion of two numbers given B, C.

a f. 10. 6.

Divide KM into as many equal parts as there are unites in the number B. and let as many of these, as there are unites in the number C, make the right line HR. it is manifest that KM.HR :: B. C.

b 3. 1.

Lemma 3.

To find out a line D, to the square of which the square of a right line given KM hath the proportion of two numbers given B, C.

a 1. lem. 10.  
b 13. 6.  
c 10. 6.  
d con. fr.

Allow B. C<sup>a</sup> :: KM. HR. and between KM and HR, find a mean proportionall D. Therefore KMq. Dq. :: KM.HR<sup>d</sup> :: B. C.

PROP. XI.

A ————— B, 20. To find two right lines in-  
 E - - - - - C, 16. commensurable to a right  
 D - - - - - line given A, one D in  
 length only, the other E in  
 power also.

1. Take the numbers B, C, & so that there be not  
 B. C :: Q. Q. b and let B. C :: Aq. Dq. c it is plane  
 that A  $\square$  D. But Aq d  $\square$  Dq. W.W. to be Done. a 1.lem.10.  
 10.  
 b 3.lem.10.  
 10.

2. Make A.E. & E.D. I say Aq  $\square$  Eq. For A.D  
 e :: Aq. Eq. therefore since A  $\square$  D, as before; there-  
 fore Aq  $\square$  Eq. W.W. to be Done. c 9. 10.  
 d 6. 10.  
 d 13. 6.  
 e 10. 6.  
 f 10. 10.

PROP. XII.

Magnitudes (A, B) commensurable to the  
 same magnitude C, are also commensurable one  
 to the other.

Because A  $\square$  C, and C  $\square$  B, let A. B  
 D, 18. E, 8, C :: N. N :: D. E, & C. B ::  
 F, 2. G, 3, N. N :: F. G. b take three num- b 4. 8.  
 H, 5. I, 4. K, 6, bers H, I, K, the least :: in  
 A B C the proportions of D to E, & F to G. Now  
 because A. C e :: D. E e :: H. I. and C. B e :: F. G ::  
 I. K. d therefore of equality A. B :: H. K :: N. N. c constr.  
 d 22. 5.  
 e 6. 10.  
 e therefore A  $\square$  B. W.W. to be Dem.

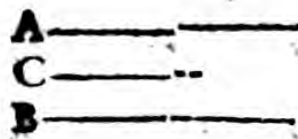
Schol.

Hence, Every right line commensurable to a ra-  
 tionall line is also it self rationall. And all right  
 lines rationall are commensurable to one another, at  
 least in power. Also, every space commensurable to a  
 rationall space is rationall too: and all rationall spa-  
 ces are commensurable one to another. But magni-  
 tudes whereof one is rationall, the other irrational,  
 are incommensurable amongst themselves. 12. 10. and  
 def. 6.  
 def. 9.  
 def. 7. & 10.

PROP.



PROP. XIII.



If there be two magnitudes A, B, and one of them A commensurable to a third C, but the other B incommensurable, those magnitudes A, B are incommensurable.

a hyp. b 12. 10.

Conceive B  $\perp$  A. then being C  $\perp$  A, therefore C  $\perp$  B, against the Hyp.

PROP. XIV.

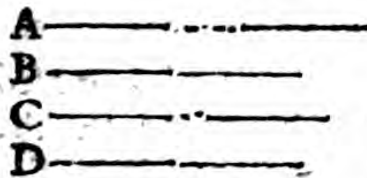


If there be two magnitudes commensurable A, B; and one of them A incommensurable to any other magnitud: C, the other also B shall be incommensurable to the same C.

a hyp. b 12. 10.

Imagine B  $\perp$  C. then for that A  $\perp$  B, therefore A  $\perp$  C, against the Hyp.

PROP. XV.



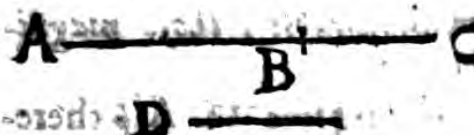
If four right lines be proportionall (A. B. :: C. D) and the first A be in power more then the second B by the square of a right line commensurable to it self in length, then also the third C shall be more in power then the fourth D by the square of a right line commensurable to it self in length. But if the first A be more in power then the second B by the square of a right line incommensurable to it self in length, then shall the third C be more in power then the fourth D by the square of a right line incommensurable to it self in length.

a hyp. b 22. 6. c 17. 5. d 22. 6. e cor 4. 5. f 21. 5.

For because A. B  $\perp$  :: C. D. therefore Aq. Bq :: Cq. Dq. therefore by division Aq - Bq. B  $\perp$  :: Cq - Dq. Dq. therefore  $\sqrt{\text{Aq} - \text{Bq}} : B :: \sqrt{\text{Cq} - \text{Dq}} : D$ , and so inversely B  $\sqrt{\text{Aq} - \text{Bq}} : A :: D \sqrt{\text{Cq} - \text{Dq}} : C$ . therefore of equality A.  $\sqrt{\text{Aq} - \text{Bq}} :: C. \sqrt{\text{Cq} - \text{Dq}}$ . consequently if A  $\perp$ , or  $\perp \sqrt{\text{Aq} - \text{Bq}}$

— Bq. & then likewise  $C \sqsupseteq$ , or  $\sqsupseteq \sqrt{Cq - Dq}$ . § 10. 10.  
*W.W. to be Dem.*

PROP. XVI.

 If two magnitudes commensurable AB, BC, be composed, the whole magnitude AC shall be commensurable to each of the parts AB, BC. And if the whole magnitude AC be commensurable to either of the parts B, or BC, those two magnitudes given at first AB, BC, shall be commensurable.

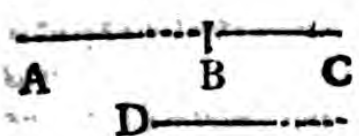
1. Hyp. • Let D be the common measure of AB, BC; & so D measures AC. and therefore  $AC \sqsupseteq$  AB, and BC. *Which was to be Dem.* § 3. 10. b 1. ax. 10. c 1. def. 10.

2. Hyp. • Let D be the common measure of AC, AB. • therefore D measures  $AC - AB$  (BC) and consequently  $AB \sqsupseteq BC$ . *W.W. to be Dem.* d 3. ax. 10.

Coroll.

Hence it follows, if a whole magnitude composed of two be commensurable to any one of them, the same shall be commensurable to the other also.

PROP. XVII.

 If two incommensurable magnitudes AB, BC, be composed, the whole magnitude also AC shall be incommensurable to either of the two parts AB, BC. And if the whole magnitude AC be incommensurable to one of them AB, the magnitudes first given AB, BC, shall be incommensurable.

1. Hyp. If it can be, let D be the common measure of AC, AB. • therefore D measures  $AC - AB$  (BC) & and therefore also  $AB \sqsupseteq BC$ , against the Hypoth.

2. Hyp.

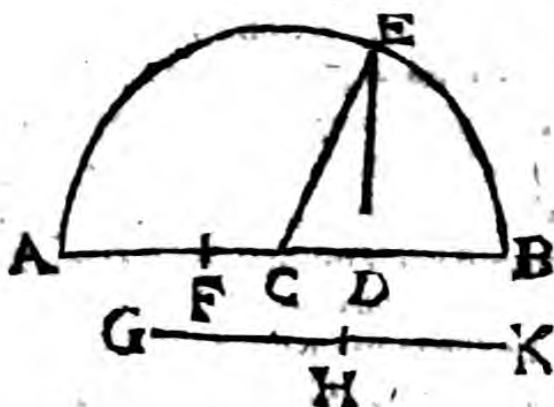
c 16. 10

2. Hyp. Conceive  $AB \perp BC$ , & therefore  $AC \perp AB$ . Against the Hyp.

Coroll.

Hence also, If one magnitude, composed of two, be incommensurable to any one of them, the same also shall be incommensurable to the other.

PROP. XVIII.



If there be two unequal right lines AB, GK, and upon the greater AB a parallelogram ADB equal to the fourth part of a square made of the lesse line GK, and wanting in figure

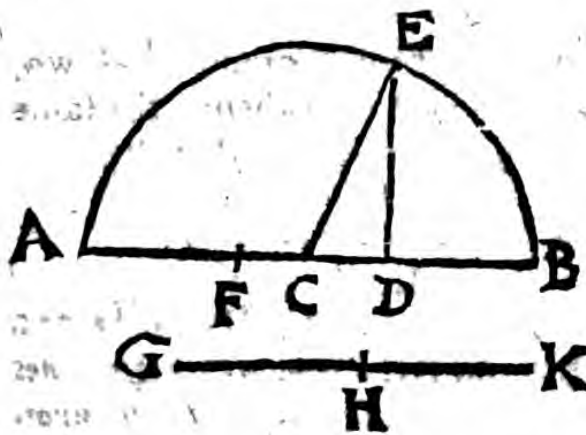
by a square, be applyed, and divide the said AB into parts commensurable in length AD, DB; then shall the greater line AB be more in power then the lesse GK by the square of a right line FD commensurable in length to the greater. And if the greater AB be in power more then the lesse GK by the square of the right line FD commensurable unto it self in length, and a parallelogram ADB equal to the fourth part of the square made of the lesse line GK, & wanting in figure by a square, be applyed to the greater AB, then shall it divide the same into parts AD, DB commensurable in length.

a Divide GK equally in H, and b make the rectangle  $ADB = GHq$ . Cut off  $AF = DB$ , then is  $ABq = 4 ADB^d (4 GHq \text{ or } GKq) + FDq$ . Now in the first place, if  $AD \perp DB$ , then shall  $AB \perp BD$  &  $\perp 2 DB^f (AF + DB, \text{ or } AB - FD)$  b therefore  $AB \perp FD$ . *W. W. to be Dem.* But secondly, if  $AB \perp FD$ , b then shall  $AB \perp AB - FD (2 DB)$  & therefore  $AB \perp DB$ . l wherefore  $AD \perp DB$ . *W. W. to be Dem.*

a 10. 11  
b 28. 6.  
c 8. 2.  
d const. and  
4. 1.  
e 16. 10.  
f cor. 16. 10.  
h cor. 16. 10.  
k 12. 10.  
l 16. 10.

PROP.

PROP. XIX.



If there be two right lines unequal AB, GK, and to the greater AB be applied a parallelogram ADB equall to the fourth part of a square made upon the lesse GK, and wanting in figure

by a square, and also thus applied divide the said AB into parts AD, DB incommensurable in length; the greater line AB shall be in power more then the lesse GK by the square of the right line FD incommensurable to the greater in length. And if the greater line AB be more in power then the lesse GK by the square of a right line FD incommensurable unto it self in length, and if also upon the greater AB be applied a parallelogram ADB equall to the fourth part of the square of the lesse GK and wanting in figure by a square, then shall it divide the said greater line AB into parts incommensurable in length AD, DB.

Suppose all the same that was done and said in the prec. prop. Therefore first, If  $AD \perp DB$ , <sup>a</sup> then shall  $AB \perp DB$ . <sup>b</sup> Wherefore  $AB \perp 2 DB$  ( $AB = FD$ ) <sup>c</sup> therefore  $AB \perp FD$ . *W.W. to be Dem.*

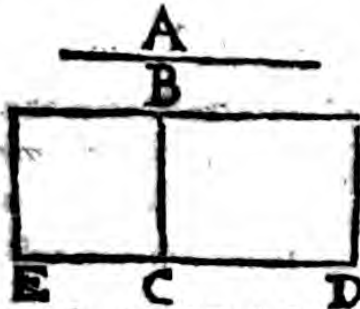
<sup>a</sup> 17. 10.  
<sup>b</sup> 13. 10.

Secondly, If  $AB \perp FD$ , then  $AB \perp AB - FD$  ( $= DB$ ); <sup>d</sup> wherefore  $AB \perp DB$ , <sup>e</sup> and consequently  $AD \perp DB$ . *W.W. to be Dem.*

<sup>c</sup> cor 17. 10.  
<sup>d</sup> 13. 30.  
<sup>e</sup> 17. 10.

PROP.

PROP. XX.



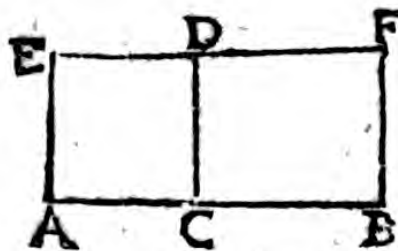
A rectangle BD comprehended under right lines BC, CD, rational and commensurable in length, according to one of the foresaid wayes, is rational.

Let A be given  $\rho$ , and the square BE described upon BC. Because DC. CE (BC)  $b :: BD.BE$ , and DC  $e \perp BC$ ,<sup>d</sup> therefore shall the rectangle BD be  $\perp$  square BE. wherefore seeing the square BE  $e \perp Aq$ , shall also BD be  $\perp Aq$ . and so the rectangle BD is  $\rho v. W. W.$  to be Dem.

Note. There are three kinds of lines rational commensurable one to another. For either, of two lines rational commensurable in length one to the other, one is equal to the rational line propounded, or neither of them is equal to it, notwithstanding both of them are commensurable to it in length; or lastly both of them are commensurable to the rational line given only in power. And these are the wayes which the present theoreme speaks of.

In numbers, let there be BC,  $\sqrt{8} (2\sqrt{2})$  and CD  $\sqrt{18} (3\sqrt{2})$  then shall the rectangle BD =  $\sqrt{144} = 12$ .

PROP. XXI:



If a rational rectangle DB be applied to a rational line DC, it makes the breadth thereof CB rational, and commensurable in length to that line DC,

whereto DB is applied.

Let G be propounded  $\rho$ , and the square DA described on BC. because BD. DA  $:: BC. CA$ ; and BD, DA  $b$  are  $\rho a c$  & so  $\perp$ .<sup>d</sup> therefore BC  $\perp CA$ . but CD (CA) is  $f. e$  therefore BC is  $\rho v. W. W.$  to be Dem.

In

a 1. 6.  
b 1. 6.  
c hyp.  
d 10. 10.  
e hyp. and 9.  
def. 10.  
f 12. 12.

a 1. 6.  
b hyp.  
c fob 12. 10.  
d 10. 10.  
e fob 12. 10.

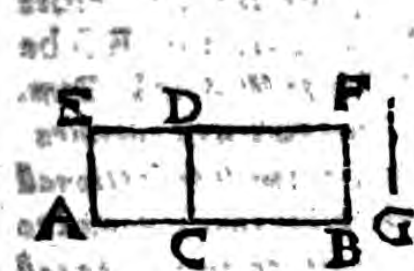
In numbers, let there be the rectangle DB, 12. and DC,  $\sqrt{8}$ . then shall CB,  $\sqrt{18}$ . but  $\sqrt{18} = 3\sqrt{2}$ . and  $\sqrt{8} = 2 \times \sqrt{2}$ .

Lemma.

To find out two right lines rati-  
 onall commensurable only in power.  
 Let A be propounded  $\rho$ . Take  
 B  $\perp$  A, and C  $\perp$  B. b it is clear that B and C  
 are the lines required.

a 11. 10.  
 b 5. 12. 10.

PROP. XXII.



A rectangle DB com-  
 prehended under right  
 lines rati-  
 onall DC, CB  
 commensurable in power  
 only, is irrati-  
 onall: & the  
 right line H, which con-  
 taineth that rectangle in power is irrati-  
 onall, and called  
 a Medial line.

Let G be the propounded  $\rho$ , and the square DA  
 described on DC, and let Hq = DB. Because AC.  
 CB  $\rho$  :: DA.DB. b & AC  $\perp$  CB, c shall be DA  $\perp$   
 DB (Hq.) d but Gq  $\perp$  DA. e therefore Hq  $\perp$   
 Gq. f wherefore H is  $\rho$ . Which was to be Dem. and let  
 it be called a Medial line, because AC. H :: H.  
 CB.

a 1. 6.  
 b hyp.  
 c 10. 10.  
 d hyp. and 9.  
 e 13. 10.  
 f 11. 10.

In numbers, let there be DC, 3. and CB,  $\sqrt{6}$ . then  
 shall the rectangle be DB (Hq)  $\sqrt{54}$ . wherefore H  
 is  $\sqrt{54}$ .

The note of a medial line is  $\mu$ , of a medial rect-  
 angle  $\mu\nu$ , of more together  $\mu\alpha$ .

Schol.

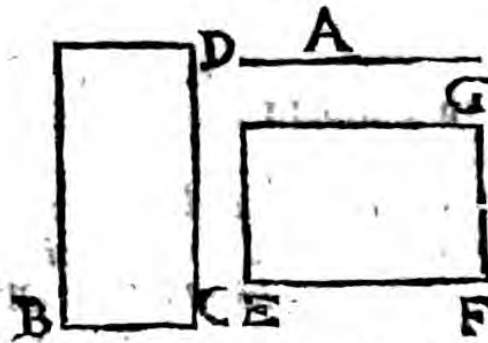
Every rectangle that can be contained under two  
 right lines rati-  
 onall commensurable only in power,  
 is medial, although it be contained under two right  
 lines irrati-  
 onall: and every medial rectangle may  
 be contained under two right lines rati-  
 onall, com-  
 mensurable only in power. as for example, the



The tenth Book of

$\sqrt{24}$  is  $\mu\nu$ , because it is contained under  $\sqrt{3}$ , and  $\sqrt{8}$ , which are  $\rho$ ,  $\sigma$ . although it may be contained under  $\nu\sqrt{6}$ , and  $\nu\sqrt{96}$  irrationalls; for  $\sqrt{24} = \nu\sqrt{576} = \nu\sqrt{6} \times \nu\sqrt{96}$ .

P R O P. XXIII.



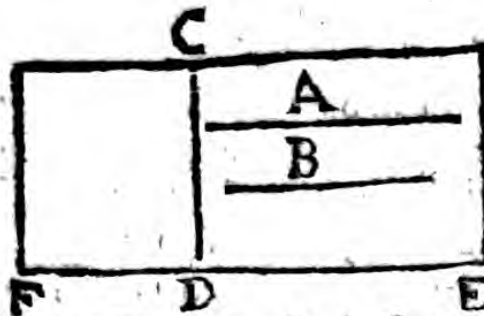
If the rectangle BD made of a medially line A, be applied on a rational line BC, it makes the breadth CD rational, and incommensurable in length to the line BC, whereunto

the rectangle BD is applied.

a sch. 22. 10.  
 b 1. ax. 1.  
 c 14. 6.  
 d 22. 6.  
 e hyp.  
 f sch. 12. 10.  
 g 10. 10.  
 h sch. 12. 10.  
 k 1. 6.  
 l 10. 10.  
 m sch. 12. 10.  
 n 13. 10.  
 o 1. 6.  
 p 10. 10.

Because A is  $\mu$ , therefore shall Aq be equal to some rectangle (EG) contained under EF and FG  $\rho$   $\sigma$ . b therefore  $D = EG$ . c whence  $BC. EF :: FG. CD$ . d therefore  $BCq. EFq :: FGq. CDq$ . But  $BCq$  and  $EFq$  are  $\rho\alpha$ , f and so  $\square$ . g therefore  $FGq \square CDq$ . Wherefore being FG is  $\rho$ , b therefore CD shall be  $\rho$ . Moreover, because  $EF.FG^k :: EFq.EG$  (BD;) for that  $EF \square FG$ , e shall  $EFq$  be  $\square$  BD. But  $EFq \square CDq$ . n therefore the rectangle  $BD \square CDq$ . Whence being  $CDq.BD^o :: CD.BC$ . p shall CD be  $\square$  BC. therefore, &c.

P R O P. XXIV.



A right line B commensurable to a medially line A is also a medially line.

Upon CD  $\rho$  make the rectangle  $CE = Aq$ ; a and the rectangle

CF = Bq. Because Aq (CE) is  $\mu\nu$ , b and CD  $\rho\sigma$  therefore shall the latitude DE be  $\rho$   $\sigma$  CD. But for

a 11. 6.  
 b by p.  
 c 23. 10.

# EUCLIDE'S Elements.

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for that  $CE \cdot CF \propto ED \cdot DF$ . and  $CE \perp CF$ ,  
 therefore  $ED \perp DF$ . therefore  $DF$  is  $\perp$  to  $CD$ .  
 whence the rectangle  $CF$  ( $Bq$ ) is  $\mu\nu$ , and so  $B$  is  
 $\mu$ . *W.W. to be Dem.*

d 1. 6.  
 e by p.  
 f 10. 10.  
 g 12. and 13.  
 10.  
 h 21. 10.

Obf. that the note  $\perp$  for the most part signifies  
 commensurable in power only, as in this and the prece-  
 dent demonstrations, &c.

*Coroll.*

Hereby it is manifest that a space commensurable  
 to a medially space, is also medially.

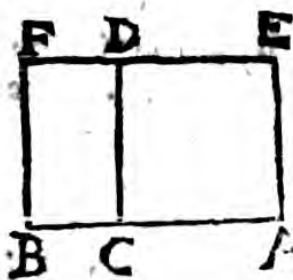
*Lemma.*

A ————— To find out two right lines medially  
 B ————— A, B, commensurable in length, and  
 C ————— also two, A, C, commensurable on-  
 ly in power,

Let  $A$  be any  $\mu$ , take  $B \perp A$ , and  $C \perp$   
 $A$ . it appears to be done.

a lem. 22. 10.  
 and 13. 6.  
 h 2. lem. 10.  
 10.  
 c 3. l. m. 10.  
 10.  
 d constr. and  
 24. 10.

## P R O P. XXV.

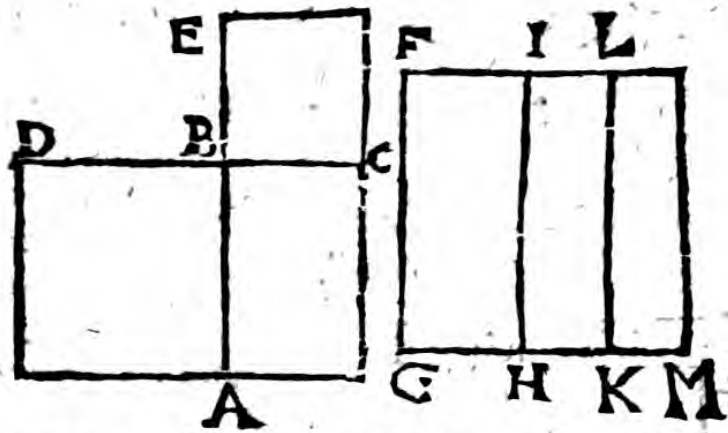


A rectangle  $DB$  contained un-  
 der  $DC$ ,  $CB$  medially right lines  
 commensurable in length, is me-  
 dially.

Upon  $DC$  describe the square  
 $DA$ . Being  $AC$  ( $DC$ )  $CB$   $a ::$   
 $DA$ .  $DB$ , &  $DC \perp CB$ ;  $b$  shall  
 $DA \perp DB$ . therefore  $DB$  is  $\mu\nu$ . Which was to be  
*Dem.*

a 1. 6.  
 b 10. 10.  
 c 24. 10.





A rectangle AC comprehended under medially right lines AB, BC commensurable only in power, is either rational or medially.

a 46. 1.  
b cor. 16.  
6.

Upon the lines AB, BC, a describe the squares AD, CE; and upon FG, b make the rectangles FH, = AD, b and IK = AC, b and LM = CE.

c hyp. & 24.  
10.  
d 23. 10.  
e 10. 10.  
f 30. 10.  
g sch. 22. 6.  
h 1. 6.  
k 17. 6.  
l 12. 10.  
m 20. 10.  
n 22. 10.

The squares AD, CE, that is, the rectangles FH, LM, c are  $\mu$  and  $\square$ . therefore GH, KM, having the same proportion d are  $\rho$ , e and  $\square$ . f therefore GH x KM is  $\rho$ v. But because AD, AC, CE, that is, FH, IK, LM, g are  $\div$ ; b and so GH, HK, KM also  $\div$ ; k thence HKq = GH x KM. l therefore HK is  $\rho$ , or  $\square$ , or  $\square$  IH (GF;) if  $\square$ , m then the rectangle IK or AC is  $\rho$ v. but if  $\square$ , n then AC is  $\mu$ v. Which was to be Dem.

Lemma.



If A and E be  $\square$  only, Then first, shall Aq, Eq,

a hyp. and  
16. 10.  
b 1. 6.  
c hyp.  
d 10. 10.  
e 14. 10.

Aq + Eq, Aq - Eq a  $\square$ . And secondly, Aq, Eq, Aq + Eq, Aq - Eq  $\square$  AE and 2 E. For A.E b :: Aq. AE b :: AE. Eq. therefore seeing A c  $\square$  E, d shall Aq  $\square$  AE, e and 2 AE. also Eq d  $\square$  AE, e and 2 AE. wherefore because Aq + Eq  $\square$  Aq and Eq; and Aq - Eq  $\square$  Aq and Eq. f therefore shall Aq + Eq, f and Aq - Eq be  $\square$  AE, and 2 AE.

Hence



## PROP. XXVIII.

To find out medially lines (C and D,) which contain a rationally rectangle CD.

Take A and B  $\rho$   $\perp$  b make A. C :: C. B. c and A. B :: C. D. I say the thing required is done. For AB (Cq) d is  $\mu\nu$ ,

whence C is  $\mu$ . but being that A. B  $\rho$  :: C. D. f therefore C  $\perp$  D. g and conse-

quently D is  $\mu$ . Moreover by inversion A. C :: B. D. i. e. C. B :: B. D. b therefore Bq = CD. But Bq is  $\rho\nu$ . b therefore CD is  $\rho\nu$ . W. W. to be done.

In numbers, let A be  $\sqrt{2}$ ; and B  $\sqrt{6}$ . therefore C is  $\sqrt{12}$ . make  $\sqrt{2} \cdot \sqrt{6} :: \sqrt{12} \cdot D$ . or  $\sqrt{4} \cdot \sqrt{36} :: \sqrt{12} \cdot D$ . then shall D be  $\sqrt{108}$ . but  $\sqrt{12} \times \sqrt{108} = \sqrt{1296} = \sqrt{36} = 6$ . therefore CD is 6. likewise C. D ::  $1 \cdot \sqrt{3}$ . wherefore C  $\perp$  D.

## PROP. XXIX.

To find out medially right lines commensurable in power only, D and E, containing a medially rectangle DE.

Take A, B, C  $\rho$   $\perp$ . make A. D b :: D. B. c and B. C :: D. E. I say the thing desired is performed.

For AB d = Dq. and AB e is  $\mu\nu$ . A D B C E therefore D is  $\mu$ ; and Bf  $\perp$  C, g whence D  $\perp$  E. therefore b E is  $\mu$ . Moreover B. C f :: D. E. and by inversion B. D :: C. E. i. e. D. A :: C. E. l therefore DE = AC. But AC m is  $\mu\nu$ . therefore DE is  $\mu\nu$ . W. W. to be Done.

In numbers, let A be 20. and B,  $\sqrt{200}$ , and C,  $\sqrt{80}$ . Therefore D is  $\sqrt{80000}$ ; and E  $\sqrt{12800}$ . Therefore DE =  $\sqrt{1024000000} = 32000$ . and D. E ::  $\sqrt{10} \cdot 2$ . wherefore D  $\perp$  E.

Schol.

a lem. 21. 10.  
b 13. 6.  
c 12. 6.  
d 22. 10. 1  
e const.  
f 10. 10.  
g 14. 10.  
h 17. 6.  
i sch. 12. 10.

a lem. 21. 10.  
b 13. 6.  
c 12. 6.  
d 17. 6.  
e 22. 10.  
f const.  
g 10. 10.  
h 24. 10.  
i const. and  
cor 45.  
l 16. 6.  
m 22. 6.

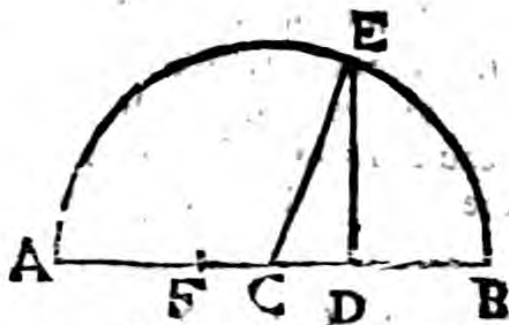
*Schol.*

A,6. C,12. *To find out two plane numbers,*  
 B,4. D,8. *like or unlike.*

$\overline{AB}, 24.$   $\overline{CD}, 96.$  Take any four numbers pro-  
 portionall A.B :: C.D. it is ma-  
 nifest that AB and CD are like  
 plane numbers. And you may  
 find out as many unlike plane  
 numbers, as you please, by help

of *Schol.* 27.8.

*Lemma.*



*To find out two square numbers (DEq and CDq) so  
 that the number composed of them (CEq) be square  
 also.*

Take AD, DB like plane numbers (of which let  
 both be equal, or both odde) viz. AD, 24. and DB,  
 6. The totall of these (AB) is 30; the difference  
 (FD) 18. half of which (CD) is 9. Now the like  
 plane numbers AD, DB, have one mean number  
 proportionall, namely DE. therefore it is evident  
 that every of those numbers CE, CD, DE, are ratio-  
 nall, and by consequence CEq (6 CDq + DEq) is  
 the square number required. b 18. 8.

Whereby it will be easy to find out two square  
 numbers, the exceffe of which is a square or not a  
 square number. namely by the same construction  
 shall CEq - CDq be = DEq. b 47. 1.

But if AD, DB be plane numbers unlike, the me-  
 diall c 3. 4. 1.

diall proportionall line (DE) shall not be a rati-  
onall number, and so neither shall the excesse (DEq)  
of the square numbers, CEq, CDq, be a square num-  
ber.

Lemma 2.

2. To find out two such square numbers B, C, as the  
number compounded of them D is not square. Also to  
divide a square number A into two numbers B, C, not  
squares.

A, 3. B, 9. C, 36. D, 45.

1. Take any square number B, and let C be  $\equiv$   
 $4B$ , and  $D \equiv B + C$ . I say the matter is done.

a 14. 8.

b cor, 24. 8.

For B is Q. by the constr. likewise because B.C ::  
1.4 :: Q.Q. therefore C also shall be a square num-  
ber. But because B + C. (D) C :: 5. 4 :: not Q.  
Q. b therefore shall not D be a square number. *W. W.*  
to be done.

A, 36. B, 24. C, 12. D, 3. E, 2. F, 1.

2. Let A be some square number. Take D, E, F,  
plane numbers unlike, and let D be  $\equiv E + F$ . make  
D.E :: A. B. and D. F :: A. C. I say the thing re-  
quired is done.

a 14. 5.  
b 21 def. 7.  
c 16. 8.

For because D.E + F :: A, B + C. and  $D \equiv E$   
+ F, therefore shall  $A \equiv B + C$ . Now suppose  
B to be square, b then A and B, c and consequently  
D and E are like plane numbers. *Which is contrary*  
*to the Hyp.*

The same absurdity will follow if C be supposed  
a square number. Therefore, &c.

PROP.

PROP. XXX.



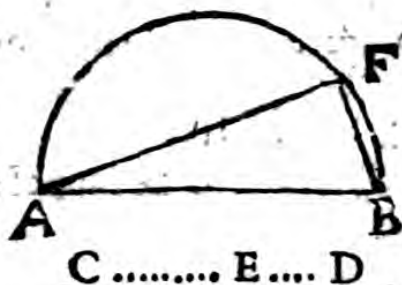
To find out two such rational right lines AB, AF, commensurable only in power, as the greater AB shall be in power more then the lesse AF by the square of a right line BF commensurable in length to the greater.

Let AB be the line given  $\rho$ . Take the square numbers CD, CE, so that CD - CE (ED) be not Q<sup>b</sup> and let there be CD. ED :: ABq. AFq. In a circle described upon the diameter AB draw AF, and also BF. Then I say AB, AF, are the lines required.

For ABq. AFq<sup>d</sup> :: CD. ED. e therefore ABq AFq. but AB is  $\rho$ . f therefore AF is also  $\rho$ . But because CD is Q: and ED not Q: g therefore shall AB be  $\perp$  AF. Moreover by reason of the <sup>b</sup> right angle AFB, is ABq<sup>k</sup> = AFq + BFq; therefore seeing ABq. AFq :: CD. ED. by conversion of proportion shall ABq. BFq :: CD. CE :: Q. Q. l therefore AB  $\perp$  BF. W. W. to be Done.

In numbers, let there be AB, 6; CD, 9; CE, 4; wherefore ED, 5. Make 9.5 :: 36. (Q:6) AFq. then AFq shall be 20. and consequently AF  $\sqrt{20}$ . therefore BFq = 36 - 20 = 16. wherefore BF is 4.

PROP. XXXI.



To find out two rational lines AB, AF commensurable only in power, so that the greater AB shall be in power more then the lesse AF by the square of a right line BF incommensurable in length to the greater.

Let AB be the line given  $\rho$ . Take the square numbers CE, ED, so that CD = CE + ED be not Q. and in the rest follow the construction of the preced. prop. I say then the thing required is done.

For

a 1.lem. 29.  
10.  
b 3.lem. 10.  
10.  
c 1. 4.  
d constr.  
e 6. 10.  
f sch. 12. 10.  
g 9. 10.  
h 34. 9.  
k 47. 1.  
19. 10.

a 2.lem. 29.  
10.

b9. 10.

For, as above,  $AB, AF$ , are  $\rho \sqcup$ . also  $ABq, BFq :: CD, ED$ . therefore being  $CD$  is not  $Q$ .  $AB, BFb$  shall be  $\sqcup$ . Which was to be Done.

In numbers, let there be  $AB, 5$ .  $CD, 45$ .  $CE = 36$ .  $ED = 9$ . Make  $45, 9 :: 25 (ABq.) 5 (AFq.)$  therefore  $AF = \sqrt{5}$ . consequently  $BFq = 45 - 25 = 20$ . wherefore  $BF = \sqrt{20}$ .

## P R O P. XXXII.

A \_\_\_\_\_ To find out two medially  
 B \_\_\_\_\_ lines C, D, commensurable  
 C \_\_\_\_\_ only in power, compre-  
 D \_\_\_\_\_ hending a rational rectan-  
 gle CD, so that the greater  
 C be more in power than the lesser D by the square of a  
 right line commensurable in length to the greater.

Take A and B  $\rho \sqcup$ ; so as  $\sqrt{Aq - Bq} \sqcup A$ .  
 b and make  $A.C :: C.B$ . c and  $A.B :: C.D$ . I say the  
 thing is done.

For because A and  $^d B$  are  $\rho \sqcup$ . e therefore shall  
 C ( $f \sqrt{AB}$ ) be  $\mu$ . g & thence also  $C \sqcup D$ . therefore  
 D is likewise  $\mu$ . Furthermore, whereas  $A, B d ::$   
 $C, D$ ; and inversely  $A.C :: B.D :: C.B$ ; and  $Bq$  is  $\rho v$ .  
 therefore shall  $CD$  ( $^k Bq$ ) be  $\rho v$ . Lastly, because  $\sqrt{Aq - Bq} \sqcup A$ ,  
 l shall  $\sqrt{Cq - Dq}$  be  $\sqcup C$ . therefore, &c. But if  $\sqrt{Aq - Bq} \sqcup Aq$ , then shall  $\sqrt{Cq - Dq}$  be  $\sqcup C$ .

In numbers; let there be  $A 8, B \sqrt{48}$  ( $\sqrt{64 - 16}$ )  
 therefore  $C = \sqrt{AB} = \sqrt{3072}$ . and  $D = \sqrt{1728}$ . wherefore  $CD = \sqrt{5308416} = \sqrt{2304}$ .

## P R O P. XXXIII.

A \_\_\_\_\_ To find out two medially  
 D \_\_\_\_\_ lines D, E, commensurable in  
 B \_\_\_\_\_ power only, comprehending a  
 C \_\_\_\_\_ medially rectangle DE, so that  
 D \_\_\_\_\_ the greater D shall be more in  
 power than the lesse E, by the  
 square of a right line commensurable to the greater in  
 length.

P R O P.

a 30. 10.  
 b 13. 6.  
 c 12. 6.  
 d const.  
 e 22. 10.  
 f 17. 6.  
 g 10. 10.  
 h 24. 10.

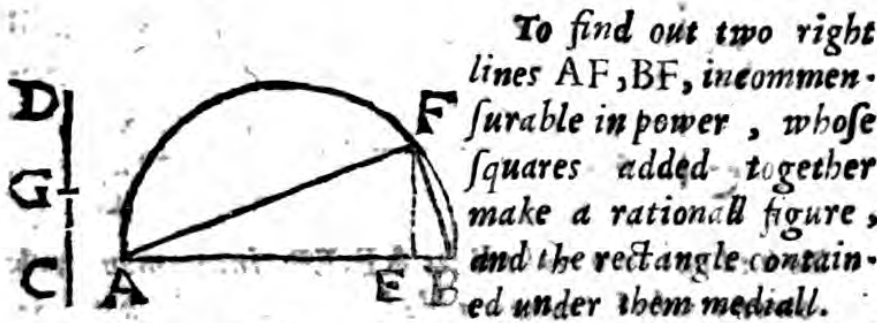
k 17. 6.  
 l 15. 10.

*a* Take A and C  $\rho' \sqsupset$ , so that  $\sqrt{Aq - Cq} \sqsupset A$ .  
*b* take also B  $\sqsupset A$  and C, and make A.D  $c :: D. B$   
*d*  $:: C.E.$  then D, and E are the lines sought for.

For because A and C  $e$  are  $\rho', e$  and B  $\sqsupset A$  and C,  
*f* therefore shall B be  $\rho'$ , and D ( $\sqrt{AB}$ )  $g$  shall be  $\mu$ .  
 But because A.D  $:: C.E.$ , therefore inversely A.C  $::$   
 D.E. wherefore seeing A  $\sqsupset C$ , therefore D shall be  
 $\sqsupset E$ . therefore E is  $\mu$ . Furthermore,  $l$  being D.B  $::$   
 C.E. and BC is  $\mu\nu$ . also DE, equall to it, is  $\mu\nu$ . Lastly  
 because A.C  $:: D.E.$  seeing  $\sqrt{Aq - Cq} \sqsupset A$ .  
 therefore  $\sqrt{Dq - Eq} \sqsupset D$ . therefore, &c. But if  
 $\sqrt{Aq - Cq} \sqsupset A$ . then  $\sqrt{Dq - Eq} \sqsupset Eq$ .

In numbers, let there be A 8, C  $\sqrt{48}$ . B  $\sqrt{28}$ .  
 then D  $\sqrt{3072}$ . and E  $\sqrt{588}$ . wherefore D.E  $::$   
 $2.\sqrt{3}$ . and  $DE = \sqrt{1344}$ .

PROP. XXXIV.



To find out two right  
 lines AF, BF, incommen-  
 surable in power, whose  
 squares added together  
 make a ratiōnall figure,  
 and the rectangle contain-  
 ed under them mediall.

*a* Let there be found AB, CD,  $\rho' \sqsupset$ ; so that  $\sqrt{ABq - CDq} \sqsupset AB$ . divide CD equally in G. *e* make  
 the rectangle AEB = GCq. Upon AB the dia-  
 meter draw the semicircle AFB, erect the perpendi-  
 cular EF, and draw AF, BF. These are the lines re-  
 quired.

For AE.BE  $a :: BA \times AE. AB \times BE$ . But  $BA \times AE$   
*e* = AFq; and  $AB \times BE = Fq$ . *f* therefore AE.EB  
 $:: AFq. FBq$ . therefore being  $\sqsupset EG \sqsupset EB$ , *b* AFq  
 shall be  $\sqsupset FBq$ . Moreover ABq ( $k$  AFq + FBq) *l* is  
 $\rho\nu$ . Lastly EFq *l* = AEB *l* = CGq. *m* therefore EF =  
 CG. therefore CD x AB = 2 EF x AB. But CD x  
 AB *n* is  $\mu\nu$ . therefore AB x EF, *p* or AF x FB, is  $\mu\nu$ .  
*W.W. to be Dem.*

*a* 30. 10.  
*b* lem. 21. 10.  
*c* 13. 6.  
*d* 12. 6.  
*e* constr.  
*f* sch. 12. 10.  
*g* 22. 10.  
*h* 10. 10.  
*k* 24. 10.  
*l* 22. 10.  
*m* 16. 6.  
*n* 15. 5.

*a* 31. 10.  
*b* 10. 1.  
*c* 28. 6.  
*d* 12. 6.  
*e* cor. 8. 6. &  
 17. 6.  
*f* 7. 5.  
*g* 19. 10.  
*h* 10. 10.

*k* 31. 3. and  
 47. 1.  
*l* constr.  
*m* 1. ax. 1.  
*n* 21. 10.  
*o* 24. 10.  
*p* sch. 22. 6.

The

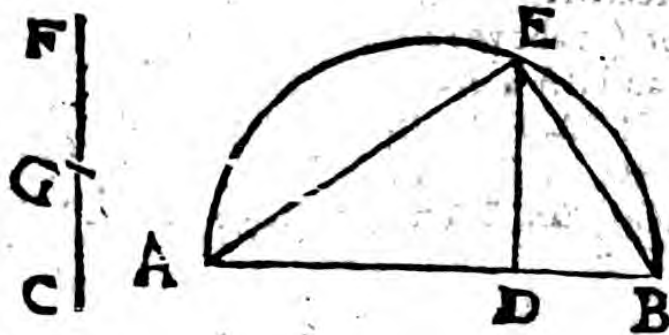


The Explication of the same by numbers.

Let  $AB$  be 6.  $CD \sqrt{12}$ . then  $CG = \sqrt{\frac{14}{4}} = \sqrt{3\frac{1}{2}}$   
 3. But  $AE = 3 + \sqrt{6}$ . and  $EB = 3 - \sqrt{6}$ . whence  
 $AF$  shall be  $\sqrt{18 + 216}$ . and  $FB \sqrt{18 - 216}$ .  
 Also  $AFq + FBq$  is 36, and  $AF \times FB = \sqrt{108}$ .

But  $AE$  is found in this manner. Because  $B \Delta (6.)$   
 $AF :: AF \cdot AE$ . therefore  $6 \cdot AE = AFq = AEq + 3$   
 $(EFq.)$  therefore  $6 \cdot AE - AEq = 3$ . Put  $3 + e =$   
 $AE$ . then  $18 + 6e - 9 - 6e - ee$ , that is  $9 - ee =$   
 $3$ . or  $ee = 6$ . wherefore  $e = \sqrt{6}$ . & so  $AE = 3 +$   
 $\sqrt{6}$ .

PROP. XXXV.



To find out two right lines  $AE, EB$ , incommensurable  
 in power, whose squares added together make a mediall  
 figure, and the rectangle contained under them ratio-  
 nall.

a 31. 10.

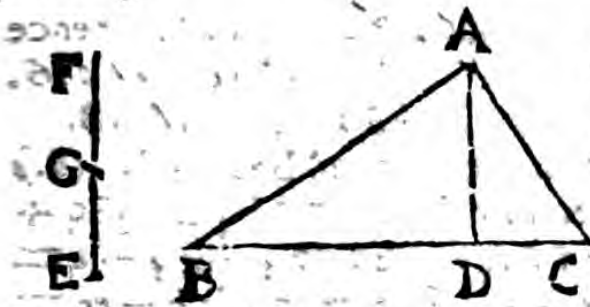
Take  $AB$  and  $CF \mu \sqcup$ , so that  $AB \times CF$  be  
 $\rho \nu$ , and  $\sqrt{ABq - CFq} \sqcup AB$ . and let the rest be  
 done as in the prec. prop.  $AE, EB$  are the lines re-  
 quired.

Dempr.  
 e 31. 12. 10.  
 d feb. 22. 6.

For, as it is shewn there,  $AEq \sqcup EBq$ . also  $ABq$   
 $(AEq + EBq)$  is  $\mu \nu$ . and lastly  $AB \times CF$  is  $\rho \nu$ .  
 therefore also  $AB \times DE$ , that is,  $AE \times EB$ , is  $\rho \nu$ .  
 therefore, &c.

PROP.

PROP. XXXVI.



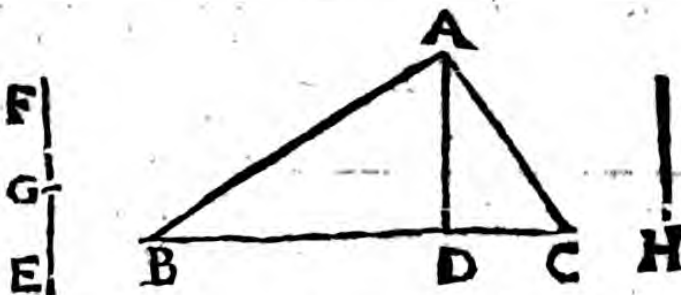
To find out two right lines BA, AC, incommensurable in power, whose squares added together make a medially figure, & the rectangle also contained under

them medially, and incommensurable to the figure composed of the squares.

a Take BC and EF  $\mu$   $\square$ , so that BC x EF be  $\mu\nu$ . a 33. 10.  
and  $\sqrt{BCq - EFq} \square BC$ . and so forward, as in the prec. BA, AC, shall be the lines sought for.

For (as above) BAq  $\square$  ACq. also BAq + ACq is  $\mu\nu$ . and BA x AC is  $\mu\nu$ . Lastly BC  $b$   $\square$  EF, and b confr. c 13. 10. d 1. 6. e 14. 10.  
so BC  $\square$  EG; likewise BC. EG  $d$  :: BCq. BC x EG (BC x AD, or BA x AC)  $e$  therefore BCq (ABq + ACq)  $\square$  BA x AC. therefore, &c.

Schol.




To find out two medially lines incommensurable both in length and power.

a Take BC  $\mu$ . and let BA x AC be  $\mu\nu$ , and  $\square$  BCq (BAq + ACq)  $b$  make BA. H :: H. AC. then I say BC & H are  $\mu$   $\square$ . For BC is  $\mu$ ,  $a$  and BA x AC (c Hq) is  $\mu\nu$ . wherefore H is also  $\mu$ .  $d$  Likewise BA x AC  $\square$  BCq; therefore Hq  $\square$  BCq. therefore, &c. a 36. 10. b 33. 6. c 17. 6. d 14. 10.

Here

Here begin the senaries of lines irrationall by composition.


PROP. XXXVII.

*If two ratioll lines*  

*CA, BC, commensurable on'y in power, be added together, the whole line AC is irrationall, and is called a binomiall line, or of two names.*

a hyp.  
 b lem. 26. 10.  
 c 11. def. 10.

For because  $AB \text{ is } \sqrt{\square} BC$ , thence  $b$  shall  $ACq$  be  $\sqrt{\square} ABq$ . But  $AB \text{ is } \rho \cdot c$  therefore  $AC$  is  $\rho$ . Which was to be Dem.

PROP. XXXVIII.

*If two mediall lines*  

*AB, BC, only in power commensurable, be compounded, and contain a ratioll rectangle, the whole line AC is irrationall, and called a first bimediall line.*

a hyp.  
 b lem. 26. 10.  
 c 11. def. 10.

For being that  $AB \text{ is } \sqrt{\square} BC$ ,  $b$  shall  $ACq$  be  $\sqrt{\square} AB \times BC$ ,  $\rho \cdot v$ . therefore  $AC$  is  $\rho \cdot W$ . to be Dem.

Lemma.



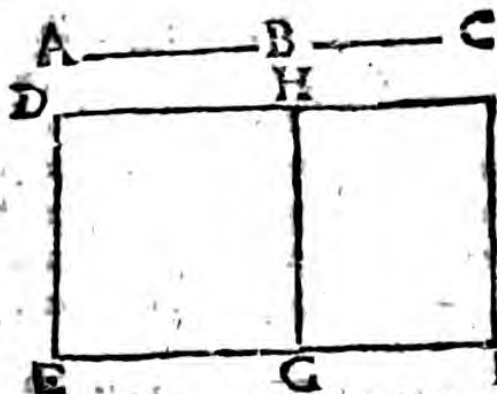
*A rectangle AC, contained under a ratioll line AB and an irrationall line BC, is irrationall.*

a hyp.  
 b 21. 10.

For if the rectangle  $AC$  be affirmed  $\rho \cdot v$ , then being  $AB$  is  $\rho$ ,  $b$  the breadth  $BC$  shall be also  $\rho$ . against the Hyp.

PROP.

PROP. XXXIX.



If two medially lines AB, BC, commensurable only in power, containing a medially rectangle, be compounded, the whole line AC shall be irrational, and is called a Second bimedial line.

Upon the propounded line DE  $\rho$  make the rectangle DF = ACq;  $b$  and DG = ABq + BCq.

Because ABq  $\square$  BCq,  $d$  therefore ABq + BCq, i.e. DG,  $\square$  ABq: but ABq  $\rho$  is  $\mu\nu$ ,  $e$  therefore DG is  $\mu\nu$ . But the rectangle ABC is taken  $\mu\nu$ ;  $e$  and consequently 2 ABC (f HF) is  $\mu\nu$ .  $g$  therefore EG and GF are  $\rho$ . Being also that DG  $b$   $\square$  HF; and DG. HF ::  $k$  EG.GF;  $l$  therefore EG  $\square$  GF.  $m$  therefore the whole EF is  $\rho$ .  $n$  wherefore the rectangle DF is  $\rho\nu$ .  $o$  therefore  $\sqrt{DF}$ , i.e. AC, is  $\rho$ . W.W. to be Dem.

a cor 16.6.  
b 47.1. and  
11.6.  
c hyp. 1  
d 16.10.  
e 24.10.  
f 4.2.  
g 27.10.  
h lem. 26.  
10.  
k 1.6.  
l 10.10.  
m 37.10.  
n lem. 38.  
10.  
o 11. def. 10.

PROP. XXXX.

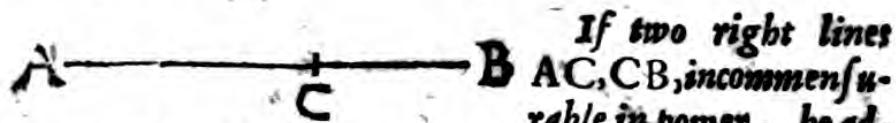
If two right lines AB, BC commensurable only in power, be added together, making that which is composed of their squares rational, and the rectangle contained under them medially, the whole right line AC is irrational and is called a Major line.

For whereas ABq + BCq  $\rho$  is  $\rho\nu$ . and  $b$   $\square$  2 ABC  $\rho$   $\mu\nu$ ; and so ACq ( $d$  ABq + BCq + 2 ABC)  $e$   $\square$  ABq + BCq  $\rho\nu$ .  $f$  therefore shall AC be  $\rho$ . Which was to be Dem.

a hyp.  
b feb. 12. 10.  
c hyp. & 24.  
10.  
d 4. 2.  
e 17. 10.  
f 11. def. 10.

PROP.

PROP. XLI.

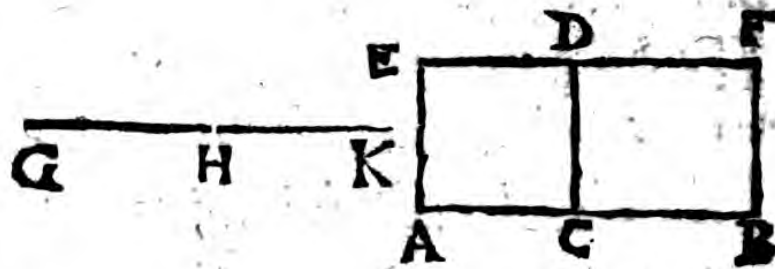


If two right lines AC, CB, incommensurable in power, be added together, having that which is made of their squares added together mediall, and the rectangle contained under them rationall, the whole right line AB shall be irrationall, and is called A line containing in power a rationall and a mediall rectangle.

a hyp. and  
 feb 12. 10.  
 b feb. 12. 10.  
 c hyp.  
 d 17. 10.  
 o 11. def. 10.

For 2 rectangles ACB a p v, b  $\square$  ACq + CBq e  $\mu$  v. d therefore 2 ACB d  $\square$  ABq. wherefore e AB is p. W.W. to be Dem.

PROP. XLII.



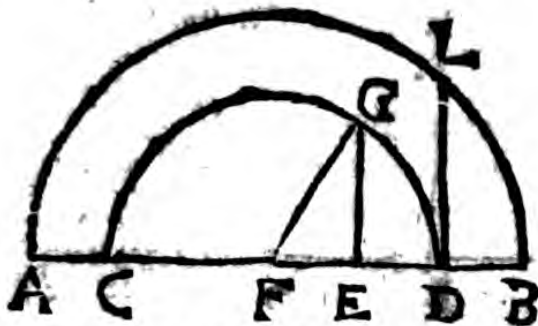
If two right lines GH, HK, incommensurable in power be added together, having both that which is composed of their squares mediall, and the rectangle contained under them mediall, and incommensurable to that which is composed of their squares, the whole right line GK is irrationall, and is called A line containing in power two mediall figures.

a hyp.  
 b 21. 10.  
 c 4. 2.  
 d 2. 6.  
 e 20. 10.  
 f 37. 10.  
 g lem. 38. 10.  
 h 21. def. 10.

Upon the propounded line FB p make the rectangles AF = GKq, & CF = GHq + HKq. Being GHq + HKq (CF) a is  $\mu$  v, the breadth CB b shall be p. Also because 2 rectangles GHK (c AD) a is  $\mu$  v, therefore AC b shall be p. Moreover because the rectangle AD a  $\square$  CF, d and AD. CF :: AC. CB. e thence shall e AC be  $\square$  CB. f wherefore A is g p. therefore the rectangle AF. i.e. GKq is p v; b and consequently GK is p. W.W. to be Dem.

PROP.

PROP. XLIII.

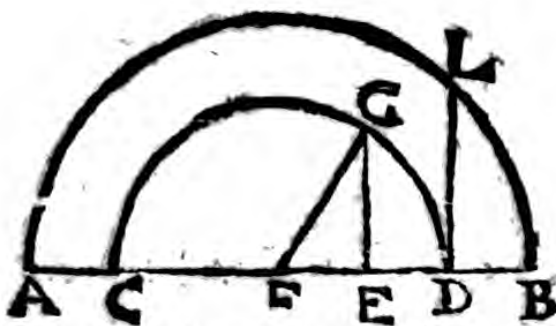


A line of two names, or binomiall, AB, can at one point only D be divided into its names AD, DB.

If it be possible, let the binomiall line AB be divided at the point E, into other names AE, EB. It is manifest that the line AB is in both cases divided unequally, since AD  $\neq$  DB, and AE  $\neq$  EB.

Because the rectangles ADB, AEB are  $\mu\alpha$ ;  $a$  and each of ADq, DBq, AEq, EBq is  $\rho^2 a \cdot b$  and so ADq + DBq  $\neq$  AEq + EBq are also  $\rho^2 a \cdot b$  therefore ADq + DBq  $\neq$  AEq + EBq. i. e.  $2 \text{ AEB} - 2 \text{ ADB}$  is  $\rho^2 v$ . therefore AEB - ADB is  $\rho^2 v$ . therefore  $\mu v$  exceeds  $\mu v$  by  $\rho^2 v$ . which is Absurd.

PROP. XLIV.



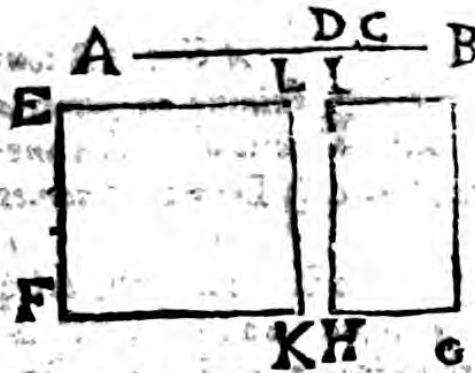
A first binomiall line AB, is in one point only D divided into its names AD, DB.

Conceive AB to be divided into other names AE, EB. whereupon every one ADq, DBq, EBq, will be  $\mu\alpha$ . and the rectangles ADB, AEB, and the doubles of them,  $\rho^2 a \cdot b$  therefore  $2 \text{ AEB} - 2 \text{ ADB}$ . i. e. ADq + DBq  $\neq$  AEq + EBq is  $\rho^2 v$ . which is Absurd.

P

PROP.

PROP. XLV.



A second bimediall line AB, is divided into its names AC, CB, only at one point C.

Suppose there were other names AD, DB. Upon the propounded line EF, make the

rectangles  $EG = ABq$ , and  $EH = ACq + CBq$ , as also  $EK = ADq + DBq$ .

- a 39. 10.
- b 16 and 24. 10.
- c 23. 10.
- d 24. 10.
- e 4. 2.
- f 15. 16. 10.
- g 1. 6.
- h 10. 10.
- k 37. 10.

Because  $ACq, BCq$  are  $\mu\alpha$ ;  $ACq + CBq$  (EH) shall be  $\mu\gamma$ . therefore the breadth FH is  $\rho$ . moreover the rectangle ACB, and so  $2 ACB$  (IG) is  $\mu\gamma$ , therefore HG is also  $\rho$ . And since EH is  $f$  IG, &  $EH.IG :: FH.HG$ , therefore FH, HG shall be  $\square$ . & therefore FG is a binomial, whose names are FH, HG. By the same reason FG is binomial, and the names of it FK, KG: contrary to the 43. of this Book.

PROP. XLVI.



A Major line AB is at one point only D divided into its names AD, DB.

Imagine other names AE, EB. whereupon the rectangles ADB, AEB, are  $\mu\alpha$ . and as well  $ADq + DBq$ , as  $AEq + EBq$  are  $\rho\alpha$ . therefore  $ADq + DBq = AEq + EBq$ , i.e.  $2 AEB = 2 ADB$  is  $\rho\gamma$ . which is impossible.

- a 40. 10.
- b 37. 10.
- c 5. 2.
- d 17. 10.

PROP.

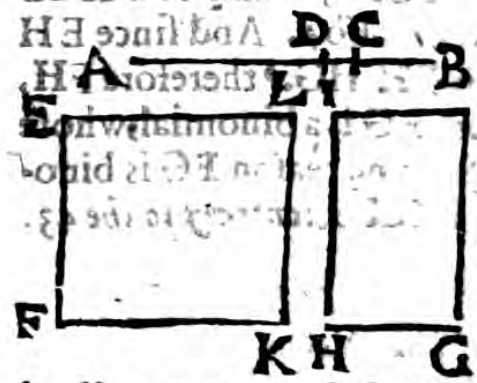
PROP. XLVII.

A line AB containing in power a rationall and a mediall figure is divided at one point only D into its names AD, DB.

Conceive other names AE, EB. then both AEq + EBq, and ADq + DBq are  $\mu a$ . and the rectangles AEB, ADB are  $\rho a \cdot b$  therefore  $2 AEB = 2 ADB$ ,  $e. e. ADq + DBq = AEq + EBq$  is  $\rho r$ . Which is Absurd.

a 41. 10.  
b 51. 17. 10.  
c 51. 5. 2.  
d 27. 10.

PROP. XLVIII.



A line AB containing in power two mediall rectangles, is at one point only C divided into its names AC, CB.

If you would divide AB into other names AD, DB. draw upon the line propounded EF, the rectangles EG = ABq, and EH = ACq + CBq, and EK = ADq + DBq. then because ACq + CBq, namely EH,  $a$  is  $\mu r$ ,  $b$  the breadth FH shall be  $\rho$ . Also because  $2 ACB$ ,  $e$  that is, IG, is  $a \mu r$ , HG  $b$  shall be likewise  $\rho$ . Therefore, whereas EH  $e \perp$  IG. and EH. IG  $d ::$  FH. HG, thence FH  $e$  shall be  $\perp$  HG.  $f$  therefore FG is a binomiall, and the names of it FH, HG. In like manner FK, KG shall be the names of it, against the 43. of this Book.

a 41. 10.  
b 23. 10.  
c 4. 2.  
d 1. 6.  
e 10. 10.  
f 37. 10.

Second Definitions.

A Rationall line being propounded, and the binomiall divided into its names, the greatest of whose names is more in power then the lesse by a square of a right line commensurable to the greater in length; then



I. If the greater name be commensurable in length to the ratioll line propounded, the whole line is called a first binomial line.

II. But if the lesser name be commensurable in length to the ratioll line propounded, the whole line is called a second binomial.

III. If neither of the names be commensurable in length to the ratioll line propounded, it is called a third binomial.

Furthermore, if the greater name be more in power than the lesse by the square of a right line incommensurable to the greater in length, then

IV. If the greater name be commensurable to the propounded ratioll line in length, it is called a fourth binomial.

V. If the lesser name be so, a fifth.

VI. If neither, a sixth.

P R O P. XLIX.

A ... 4 C ... 5 B

D \_\_\_\_\_

E \_\_\_\_\_ G

F

H \_\_\_\_\_

To find out a first binomial line, EG.

Take AB, AC, square numbers, whose excess CB is not Q. Let

D be propounded. Take EF  $\perp$  D, and make AB.CB :: EFq.FGq. then EG shall be a 1 bin.

For EF  $\perp$  D, therefore EF is  $\rho$ . f also EFq  $\perp$  FGq. g therefore FG is also  $\rho$ . likewise  $\Delta$  because EFq.FGq :: AB.CB :: Q. not Q. therefore EF  $\perp$  FG. Lastly because by conversion of proportion, EFq. EFq - FGq :: AB. AC :: Q. Q. thence EF shall be  $\perp \sqrt{EFq - FGq}$ . therefore EG is a 1 binomial. *W.W. to be Done.*

In numbers thus; let there be D 8. EF 6. AB 9. CB 5. wherefore because 9.5 :: 36. 20, therefore FG is  $\sqrt{20}$ . and consequently EG is  $6 + \sqrt{20}$ .

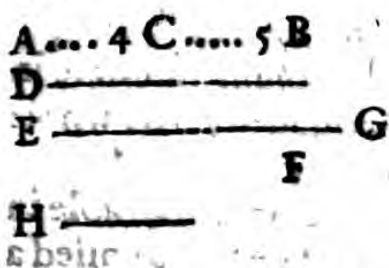
P R O P.

a feb. 10. 10.  
b 2. lem. 10. 10.  
c 3. lem. 10. 10.

d const. 10.  
e 6. def. 10. 10.  
f 6. 10.  
g feb. 12. 10. 10.  
h 9. 10.

k 9. 10.  
l 1. def. 48. 10.

PROP. L.



To find out a second binomial line, EG.

Take AB and AC square numbers, the excess of which is CB not Q. Let D be the line propounded  $\rho$ .

Proof is as the prop.

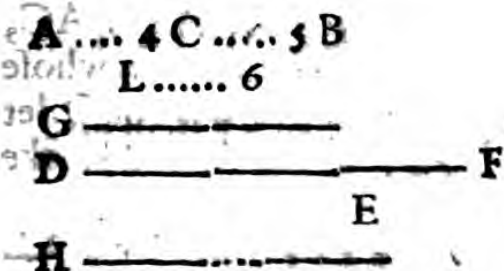
take FG  $\perp$  D, and make CB. AB :: FGq. EFq. then EG will be the line desired.

For FG  $\perp$  D. wherefore FG is  $\rho$ . Also EFq  $\perp$  FGq. therefore EF is  $\rho$ . Likewise because FGq. EFq :: CB. AB :: not Q. Q. thence FG is  $\perp$  EF. Lastly seeing CB. AB :: FGq. EFq, and inversely AB. CB :: EFq. FGq. therefore as in in the foregoing prop. EF  $\perp$   $\sqrt{EFq - FGq}$ . whereby EG is a 2. binomial. *W. W. to be Done.*

as def. 48. 10.

In numbers; let there be D 8, FG 10, AB 9, CB 5. then EF is  $\sqrt{180}$ . wherefore EG is  $10 + \sqrt{180}$ .

PROP. LI.



To find out a third binomial line, DF.

Take AB, AC, square numbers, the excess of which CB is not Q. and let L be a

as feb. 19. 10.

number not Q next greater then CB, viz. by a unite or two. Let G be the line propounded  $\rho$ . Make L. AB :: Gq. DEq. b and AB. CB :: DEq. EFq. then DF shall be a 3. bin.

b 3. lem. 10. 10.

c constr. 6. 10.

d feb. 12. 10. e 6. 10.

For because DEq c  $\perp$  Gq, d DE is  $\rho$ . also Gq. DEq :: L. AB :: not Q. Q. e therefore G  $\perp$  DE. Likewise being that DEq e  $\perp$  EFq, d also EF is  $\rho$ . Moreover because DEq. EFq :: AB. CB :: Q not

P 3

Q

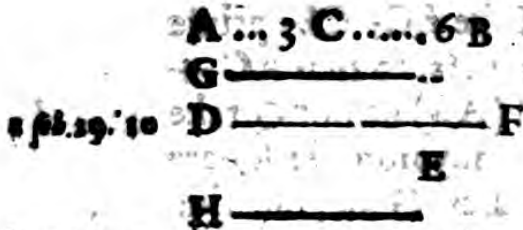
19. 10.  
 2/1. 27. 2.  
 19. 10.

13. def. 48.  
 19.

Q<sup>is</sup> DE  $\perp$  EF. and being that by constr. and of equality Gq. EFq :: L. CB :: not Q. Q. (for L and CB are not like plane numbers.) <sup>b</sup> therefore shall G be also  $\perp$  EF. Lastly, as in the prec. prop.  $\sqrt{DEq} - EFq \perp DE$ . <sup>a</sup> therefore DF is a 3 bin. *which was to be Done.*

In numbers ; let there be AB, 9. CB, 5. L, 6. G, 8. then shall be DE  $\sqrt{96}$ . & EF  $\sqrt{45}$ . wherefore DF =  $\sqrt{96} + \sqrt{45}$ .

PROP. LII.



19. 10.

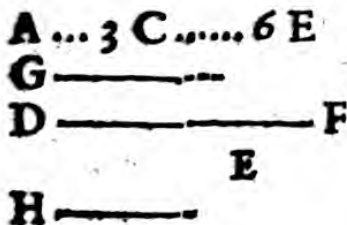
12. lem. 10.  
 10.  
 13. lem. 10.  
 10.  
 19. 10.  
 14. def. 48.  
 19.

line propounded <sup>a</sup> take DE  $\perp$  LG. <sup>c</sup> and make AB. CB :: DEq. EFq. then DF shall be a 4. bin.

For, as in the 49 of this Book, DF may be shewn to be a binom. and also because by constr. and conversion of propotion DEq. DEq - EFq :: AB.AC :: Q not Q. <sup>a</sup> shall DE be  $\perp \sqrt{DEq} - EFq$ . therefore DF is a 4. bin.

In numbers, let G be 8, DE, 6. then EF shall be  $\sqrt{24}$ . therefore DF is  $6 + \sqrt{24}$ .

PROP. LIII.



To find out a fifth binomial line DF.

Take any square number AB, whose segments AC, CB are not Q. Let G be the line propounded.

<sup>a</sup> take EF  $\perp$  LG. and make CB. AB :: EFq. DEq. then shall DF be a 5. bin.

For

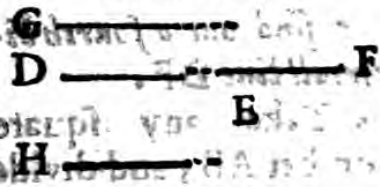
For DF shall be a bin. as in the 50. of this book. and because by construction, and inversion,  $DEq. EFq. :: AB. CB$  and so by conversion of proportion,  $DEq. DEq - EFq. :: AB. AC :: Q. not Q.$  therefore shall DE be  $\sqrt{DEq - EFq}$ . therefore DF is 5 bin. *W.W. to be Done.*

19. 10.  
b 5. def. 48.  
10.

In numbers, let there be G, 7. EF, 6. then DE shall be  $\sqrt{54}$ . wherefore DF is  $6 + \sqrt{54}$ .

P R O P. L I V.

A ..... 5 C ..... 7 B  
L ..... 9



To find out a sixth binomial line.

Take AC, CB, prime numbers, so that AC + CB (AB) be not Q. take also any number square L. Let G be the line

propounded  $\rho$ . and make L.  $AB :: Gq. DEq.$  and  $AB. CB :: DEq. EFq.$  then DF shall be a 6. binomial.

a 3. sm. 10.  
10.

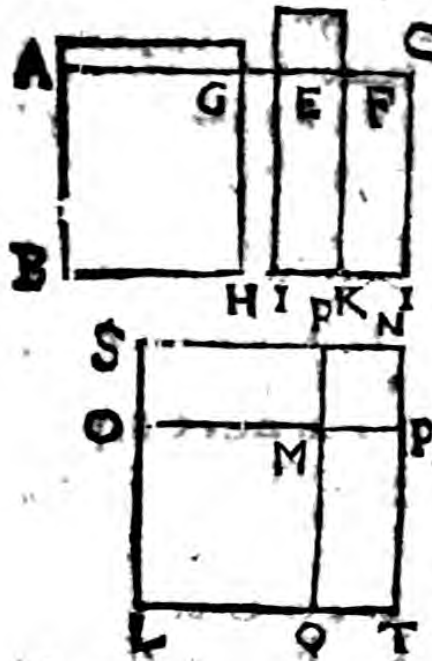
For DF may be demonstrated bin. as in the 51. of this Book. and also by reason that DE and EF  $\perp$  G. lastly likewise because by constr. and conversion of proportion  $DEq. DEq - EFq. :: AB. AC :: not Q. Q.$  (For AB is prime to AC, b and so unlike to it) therefore DE  $\perp \sqrt{DEq - EFq}$ . therefore DF is a 6. bin. *Which was required.*

b 6. 27. 8.  
c 9. 10  
e 6. def. 48.  
10.

In numbers, let there be G 6. DE  $\sqrt{48}$ . then EF shall be  $\sqrt{28}$ . wherefore DF is  $\sqrt{48} + \sqrt{28}$ .

Lemma.

Let AD be a rectangle, and the side thereof AC divided unequally in E; also let the lesser portion EC be equally divided in F, upon the line AE<sup>a</sup> make the rectangle AGE = EFq. and from the points G, E, F<sup>b</sup> draw GH, EI, FK, parallel to AB.<sup>c</sup> Let the square LM be made, equal to the rectangle AH, and upon OMP produced the square MN = GI<sup>d</sup> & let the right lines LOS, LQT, NRS, NPT be produced.



a 18. 6.

b 31. 1.  
c 14. 2.

d 15. 11. 1.  
e 13. 2.

e 12. 11.

d hyp.  
e 17. 6.  
f 1. 6.  
g 16. 22. 6.  
h 9. 5.  
k 36. 1.  
l 43. 1.  
m 1. 22. 1.  
n 16. 10.

o 18. and 16.  
p 10.

q 10. 10.

I say 1. MS, MT, are rectangles. For by reason of the right angles of the squares OMQ, RMP, shall QMR be a right line. b therefore RMO, QEP, are right angles. wherefore the parallelograms MS, MT are rectangles.

2. Hence it is plain that LS = LT, and consequently that LN is a square.

3. The rectangles SM, MT, EK, FD are equal. For because the rectangle AGE = EFq. e thence shall AE.EF :: EF.GE, f and so AH.EK :: EK.GI. that is by constr. LM.EK :: EK.MN. g but LM. SM :: SM.MN, therefore EK = SM = FD = MT.

4. Hence LN = AD.

5. Being that EC is equally divided in F, it is plain that EF, FC, EC are  $\square$ .

6. If AE  $\square$  EC, and AE  $\square$   $\sqrt{AEq} - ECq$ , then shall AG, GE, AE, be  $\square$ . also, because AG, GE :: AH, GI, therefore shall AH, GI, i. e. LM, MN, be  $\square$ . Likewise thereupon,

7. OM

7. OM  $\perp$  MP. For by the Hyp. AE  $\perp$  EC.  
 therefore EC  $\perp$  GE. therefore EF  $\perp$  GE. q 14. 10. r 10. 10.  
 but EF. GE :: EK. GI. therefore EK  $\perp$  GI, that  
 is, SM  $\perp$  MN. but SM. MN :: OM. MP. therefore  
 OM  $\perp$  MP.

8. If AB be supposed  $\perp \sqrt{AEq - ECq}$ , it fig. and 172 10.  
 is apparent that AG, GE, AE, are  $\perp$ . whence  
 LM  $\perp$  MN, for AG. GE :: AH. GI :: LM.  
 MN.

*These being well considered, we shall easily dispatch  
 the six following Propositions.*

P R O P. L V.

*If a space AD be contained under a ratioll line  
 AB, and a first binomiall line AC (AE + EC) the  
 right line OP which containeth that space in power is  
 irratioll, and called a binomiall line.*

All that being supposed which is described and  
 demonstrated in the next foregoing Lemma, it is  
 manifest that the right line OP containeth in power  
 the space AD. Likewise AG, GE, AE, are  $\perp$ . a hyp. and lem. 54. 10.  
 therefore seeing AE<sup>b</sup> is  $\perp$  AB. shall also AG b hyp.  
 and GE be  $\perp$  AB. c 12. 10.  
 therefore the rectangles AH, d 20. 10.  
 GI, that is, the squares LM, MN are  $\perp$ . e lem. 54. 10.  
 therefore OM, MP are  $\perp$ . f 37. 10.  
 and consequently OP is a bi-  
 nomiall. *W. W. to be Dem.*

In numbers, let there be AB 5. AC  $4 + \sqrt{12}$ .  
 wherefore the rectangle AD =  $20 + \sqrt{300}$  = to  
 the square LN. therefore OP is  $\sqrt{15} + \sqrt{5}$ , na-  
 mely a 6 binomial.

P R O P. L V I.

*If a space AD be comprehended under a ratioll  
 line AB, and a second binomiall AC (AE + EC) the  
 right line OP, which containeth that space AD in power,  
 is irratioll, and called a first mediall line.*

The

a hyp. and  
lem. 54. 10.  
b hyp.  
c sch. 12. 10.  
elem. 54. 10.

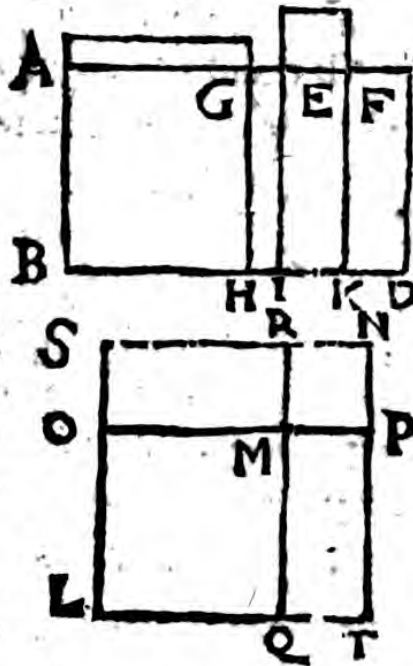
1 hyp. 12. 10.  
2 10. 10.  
b 38. 10.

The foresaid Lemma of the 54. of this Book being again supposed, then shall  $OP$  be  $= \sqrt{AD}$ . also  $AE, G, GE$  are  $\perp$ . therefore since  $AE$  is  $\perp$   $AB$ , likewise  $AG, GE$  shall be  $\perp$   $AB$ . therefore the rectangles  $AH, GI$ , i.e.  $OMq, MPq$  are  $\mu\alpha$ . Moreover  $OM \perp MP$ . Lastly  $EF \perp EC$ , and  $EC \perp AB$  wherefore  $EK$ , i.e.  $SM$ , or  $OMP$ , is  $\mu\nu$ . Consequently  $OP$  is a first bimediall. Which was to be Dem.

In numbers, let there  $AB, 5$ ; and  $AC, \sqrt{48} + 6$ . then the rectangle  $AD = \sqrt{1200} + 30 = OPq$ . therefore  $OP$  is  $\sqrt{675} + \sqrt{75}$ . viz. a first bimediall.

See Scheme 57. M

PROP. LVII.



If a space  $AD$  be contained under a rationally line  $AB$  and a third binomial line  $AC$  ( $AE + EC$ ) the right line  $OP$  which containeth in power the space  $AD$ , is irrational, and called a second bimediall line.

As above,  $OPq = AD$ . also the rectangles  $AH, GI$ , that is  $OMP, MPq$  are  $\mu\alpha$ . Likewise  $EK$  or  $OMP$  is  $\mu\nu$ . therefore  $OP$  is a second bimediall.

a hyp. and  
22. 10.  
b 39. 10.

In numbers; let there be  $AB, 5$ .  $AC, \sqrt{32} + \sqrt{34}$ . wherefore  $AD$  is  $\sqrt{800} + \sqrt{600} = OPq$ . and so  $OP$  is  $\sqrt{450} + \sqrt{50}$ . that is a 2. bimed.

PROP.

PROP. LVIII.



If a space AD be comprehended under a rati-  
onall line B and a fourth  
binomial AC (AE +  
EC) the right line OP  
containing the space AD in  
power, is that irrati-  
onall line which is called a Major  
line.

For again, OMq  $\square$  MPq; & the rectangle AI,  
i.e. OMq + MPq is  $\rho\nu$ .  
c also EK or OMP is  $\mu\nu$ .  
d therefore OP ( $\sqrt{AD}$ )  
is a Major line. W.W. to be  
Dem.

a lem. 54.  
10.

b hyp. and  
20. 10.

c hyp. and  
22. 10.

d 40. 10.

In numbers; let there be AB 5, and AC  $4 + \sqrt{3}$ .  
then the rectangle AD is  $20 + \sqrt{200}$ . wherefore  
OP is  $\sqrt{20 + \sqrt{200}}$ .

PROP. LIX.

If a space AD be contained under a rati-  
onall line AB, and a first binomial AC, the right line OP which  
containeth the space AD in power, is that irrati-  
onall line, which is a line containing a rati-  
onall and a medi-  
al rectangle in power.

Again OMP  $\square$  MPq. and the rectangle AI or  
OMq + MPq is  $\rho\nu$ . Likewise the rectangle EK or  
OMP is  $\rho\nu$ . b therefore OP ( $\sqrt{AD}$ ) contains in  
power  $\rho\nu$  and  $\mu\nu$ . W.W. to be Dem.

a as in the  
prac.  
b 41. 10.

In numbers, let there be AB 5, and AC  $2 + \sqrt{3}$ .  
then the rectangle AD =  $10 + \sqrt{200}$  = OPq.  
Wherefore OP is  $\sqrt{10 + \sqrt{200}}$ .

PROP.



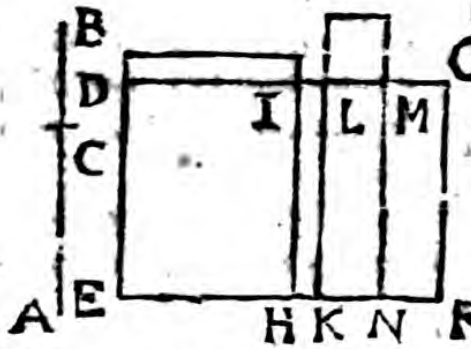
If a space AD be contained under a ratioll line AB and a six binomial AC (AE + EC) the line OP containing the space AD in power is irratioll, which containeth in power two mediall rectangles.

241. 10.

As often before, OMq  $\square$  MPq, and OMq + MPq is  $\mu$ r. and also the rectangle (EK) OMP is  $\mu$ r. therefore OP =  $\sqrt{AD}$  contains in power 2  $\mu$ r. W.W. to be Dem.

In numbers, let there be AB 5. AC  $\sqrt{12} + \sqrt{8}$ . therefore the rectangle AD or OPq is  $\sqrt{300} + \sqrt{200}$ . and so OP is  $\sqrt{\sqrt{300} + \sqrt{200}}$ .

Lemma.



Let a right line AB be unequally divided in C, and let AC be the greater portion, and upon some line DE apply the rectangles DF = ABq, and DH = ACq, and IK = CBq. and let LG be divided equally in M,

and also MN drawn parallel to GF.

I say 1. The rectangle ACB is = LN or MF. For

242. and 3.

241. b 7. 2.

c 1. 6. d 16. 10.

2. ACB = LF.

3. DL  $\square$  LG. for DK (ACq + CBq)  $\square$  LF (2 ACB) therefore being DK, LF are of equal altitude, DL shall be  $\square$  LG.

4. If AC  $\square$  CB, then shall the rectangle DK be  $\square$  ACq and CBq.

e 14. 16. 10. f 10. 10.

5. Also DL  $\square$  LG. for ACq + CBq  $\square$  2 ACB, i. e. DK  $\square$  LF. but DK. LF  $\square$  DL. LG. therefore DL  $\square$  LG.

g 1. 6. h 17. 6. i 10. 10. m 18. 10.

6. Moreover DL  $\square$   $\sqrt{DLq - LGq}$ . For ACq. ACB  $\square$  ACB. CBq. that is, DH. LN  $\square$  LN. IK. therefore DI. LM  $\square$  LM. IL. therefore DI x IL = LMq. therefore seeing ACq  $\square$  CBq, that is DH  $\square$  IK. and so DI  $\square$  IL. shall DL be  $\square$   $\sqrt{DLq - LGq}$ . W.W. to be Dem. 6. But

6. But if ACq be put  $\square$  CBq, then shall DL be  $\square$   $\sqrt{DLq} - LGq$ .

This Lemma is preparatory to the 6 following Propositions.

PROP. LXI.

The square of a binomial line (AC + CB) applied unto a rational line DE, makes the latitude DG a first binomial line.

Those things being supposed, which are described and demonstrated in the next preceding Lemma; because AC, CB are  $\rho$   $\square$ , the rectangle DK shall be  $\square$  ACq, and so DK is  $\rho v$ . therefore DL  $\square$  DE; but the rectangle ACB, and so 2 ACB (LF) is  $\mu v$ . therefore the latitude LG is  $\rho$   $\square$  DE. therefore also DL  $\square$  LG also DL  $\square$   $\sqrt{DLq} - LGq$ . from whence it follows that DG is a first binomial. *W.W. to be Dem.*

a hyp.  
b lem. 60.  
10  
c feb. 12. 10.  
d 21. 10.  
e 22. &  
24. 10.  
f 23. 10.  
g 13. 10.  
n lem. 60.  
10.  
k 1 def. 45.  
10.

PROP. LXII.

The square of a first binomial line (AC + CB) being applied to a rational line DE, makes the latitude DG a second binomial line.

The aforesaid Lemma being again supposed; The rectangle DK  $\square$  ACq, therefore DK is  $\mu v$ . therefore the latitude DL is  $\rho$   $\square$  DE. But because the rectangle ACB, and so LF (2 ACB) is  $\rho v$ , shall LG be  $\rho$   $\square$  DE. therefore DL, LG are  $\square$ . f also DL  $\square$   $\sqrt{DLq} - LGq$ . g from whence it is clear that DG is a second binomial. *W.W. to be Dem.*

a 24. 10.  
b 23. 10.  
c hyp. and  
feb. 22. 10.  
d 21. 10.  
e 13. 10.  
f lem 60. 10.  
g 2 def. 45.  
10.

## PROP. LXIII.

The square of a second binomial line  $(AC + CB)$  applied to a rational line  $DE$  makes the breadth  $DG$  a third binomial line.

a hyp. and  
24.10.  
b 23. 10.  
elem. 60. 10  
d 3 def. 48.  
10.

As in the prec.  $DL$  is  $\rho \sqsupset DE$ . Furthermore because the rectangle  $ACB$ , and so  $LF$  ( $2 ACB$ ) is  $\mu \nu$ . b therefore shall  $LG$  be  $\rho \sqsupset DE$ . c Moreover  $DL \sqsupset LG$ . and also  $DL \sqsupset \sqrt{DLq - LGq}$ . d therefore  $DG$  is a third binomial. *W.W. to be Dem.*

## PROP. LXIV.

The square of a Major line  $(AC + CB)$  applied to a rational line  $DE$ , makes the breadth  $DG$  a fourth binomial line.

a hyp. and  
feb 12. 10.  
b 21. 10.  
c hyp and  
24. 10.  
d 13. 10.  
e 13. 10.  
f lem. 60. 10.  
g 4. def. 48.  
10.

Again  $ACq + CBq$ . i.e.  $DK$  is  $\rho \nu$ . b therefore  $DL$  is  $\rho \sqsupset DE$ . also  $ACB$ , and so  $LF$  ( $2 ACB$ ) is  $\mu \nu$ . d therefore  $LG$  is  $\rho \sqsupset DE$ . e and consequently  $DL \sqsupset LG$ . Lastly because  $AC \sqsupset BC$ . f shall  $DL$  be  $\sqsupset DLq - LGq$ . g whence  $DG$  is a fourth binomial. *W.W. to be Dem.*

## PROP. LXV.

The square of a line containing in power a rational and a medial rectangle  $(AC + CB)$  applied to a rational line  $DE$  makes the latitude  $DG$  a fifth binomial.

a 23. 10.  
b 21. 10.  
c 13. 10.  
d lem. 60.  
10.  
e 5. def. 48.  
10.

Again,  $DK$  is  $\mu \nu$ . b therefore  $DL$  is  $\rho \sqsupset DE$ . also  $LF$  is  $\rho \nu$ . b therefore  $LG$  is  $\rho \sqsupset DE$ . c therefore  $DL \sqsupset LG$ . d likewise  $DL \sqsupset \sqrt{DLq - LGq}$ . e and so by consequence  $DG$  is a fifth binomial. *Which was to be Dem.*

## PROP. LXVI.

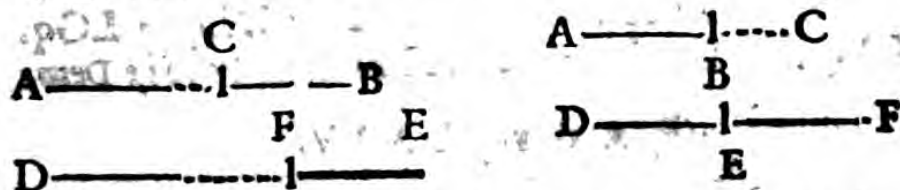
The square of a line containing in power two medial rectangles  $(AC + CB)$  applied to a rational line  $DE$ , makes the latitude  $DG$  a sixth binomial line.

As

As before, DL and LG and LG are  $\square$  DE.  
 But for that ACq + CBq (DK)  $\square$  ACB,  
 b and so DK  $\square$  LF (2 ACB) and also DK. LF  $\epsilon$  ::  
 DL. LG.  $\therefore$  therefore shall DL be  $\square$  LG. Lastly  
 DL  $\square$   $\sqrt{DLq - LGq}$ . f by which it appears that  
 DG is a fixt binomiall.

a hyp.  
 b 14. 10.  
 c 1. 6.  
 d 10. 10.  
 e lem. 60.  
 10.  
 f 6 def. 48.  
 10.

Lemma.



Let AB, DE be  $\square$  and make AB. DE :: AC. DF.  
 I say 1. AC  $\square$  DF. as appears by 10. 10. also 19. 5.  
 CB  $\square$  FE.  $\because$  because AB. DE :: CB. FE.  
 2. AC. CB :: DF. FE. For AC. DF :: AB. DE  
 :: CB. FE. therefore inversely AC. CB :: DF.  
 FE.

3. The Rectangle ACB  $\square$  DFE. For ACq. ACB  
 $\epsilon$  :: AC. CB  $\epsilon$  :: DF. EF :: DFq. DFE. wherefore by  
 inversion ACq. DFq :: ACB. DFE. therefore being  
 ACq  $\square$  DFq.  $\therefore$  shall ACB be  $\square$  DFE.

b 1. 6.  
 c before.  
 d 10. 10.

4. ACq + CBq  $\square$  DFq + FEq. For because  
 ACq. CBq  $\epsilon$  :: DFq. FEq. therefore by addition  
 ACq + CBq. CBq :: DFq + FEq. FEq. therefore  
 being CBq  $\square$  FEq, f shall also ACq + CBq be  
 $\square$  DFq + FEq.

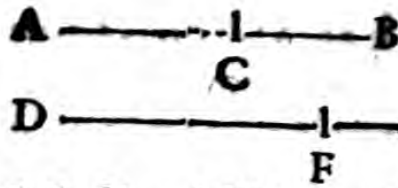
e 22. 6.  
 f 10. 10.

5. Hence, if AC  $\square$  or  $\square$  CB, g then likewise  
 shall DE be  $\square$  or  $\square$  EF.

g 10. 10.

PROP.

PROP. LXVII.



A line DE, commensurable in length to a binomial line (AC + CB) is itself a binomial line, and of the same order.

Make AB. DE :: AC. DF. then are AC, DF  $\square$ . and CB, FE  $\square$ . whence being that AC and CB are  $\rho$   $\square$ , c thence DF, FE  $\rho$   $\square$ . therefore DE is a binomial. But for that AC. CB :: DF. FE. if AC  $\square$  or  $\square \sqrt{ACq - BCq}$ , then in like manner DF  $\square$  or  $\square \sqrt{DFq - FEq}$ . also if AC  $\square$  or  $\square \rho$  propounded, e then shall DF be  $\square$  or  $\square \rho$  propounded. But if CB  $\square$  or  $\square \rho$ . likewise FE  $\square$  or  $\square \rho$ . If both AC, CB,  $\square \rho$ . f then also both DF, FE,  $\square \rho$ . g That is, whatsoever binomial AB is, DE shall be of the same order. W.W. to be Dem.

a lem. 66. 10. a  
b hyp.  
c lem. 66. 10.  
and feb. 11.  
10.  
d 19. 10.  
e 12. 10. and  
14. 10.  
g 14. 10.  
f by def. 48.  
10.

PROP. LXVIII.

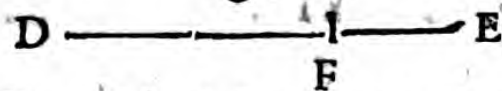
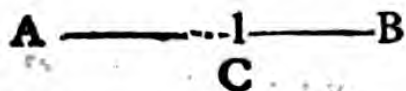
A line DE commensurable in length to a binomial line (AC + CB) is also a binomial line, and of the same order.

Make AB. DE :: AC. DF. b therefore AC  $\square$  DF. and CB  $\square$  FE. therefore seeing AC and CB are  $\mu$ , d also DF and FE shall be  $\mu$ . and for that AC  $\rho$   $\square$  CB, e therefore DF  $\rho$   $\square$  FE. f therefore DE is  $2 \mu$ . Wherefore if the rectangle ACB be  $\rho \nu$ . because DFE  $\rho$   $\square$  ACB, g likewise DFE. is  $\rho \nu$ . and if that be  $\mu \nu$ , b this shall be  $\mu \nu$  too. k That is, whether AB be 1 bimed. or 2 bimed. DF shall be of the same order. W.W. to be Dem.

a 12. 6.  
b lem. 66.  
10.  
c hyp.  
d 24. 10.  
e 10. 10.  
f 38. 10.  
g feb. 12. 10.  
h 24. 10.  
k 38. or 39.  
10.

PROP.

PROP. LXIX.



A line DE commensurable to a Major line (AC+CB) is itself a Major line.

Make AB.DE :: AC.DF. Because AC  $\perp$  CB,  $\therefore$  thence DF  $\perp$  FE. also ACq + CBq is  $\rho\nu$ . and so being DFq + FEq  $\perp$  ACq + CBq,  $\therefore$  also DFq + FEq is  $\rho\nu$ . lastly, the rectangle ACB  $\perp$  is  $\mu\nu$ .  $\therefore$  therefore the rectangle DFE is  $\mu\nu$ . (because DFE is  $\perp$  ACB)  $\therefore$  wherefore DE is a Major line. W.W. to be Dem.

a hyp. b lem. 66. 10.

c sch. 12. 10. d 24. 10. e 40. 10.

PROP. LXX.

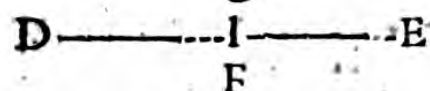
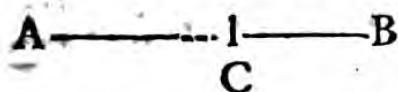
A line DE commensurable to a line containing in power a rationally and a medially rectangle (AC+CB) is a line containing in power a rationally and a medially rectangle.

Again make AB.DE :: AC.DF. Because AC  $\perp$  CB,  $\therefore$  also DF  $\perp$  FE. likewise because ACq + CBq is  $\mu\nu$ ,  $\therefore$  therefore DFq + FEq shall be  $\mu\nu$ . lastly, because the rectangle ACB  $\perp$  is  $\rho\nu$ ,  $\therefore$  also DFE is  $\rho\nu$ . Therefore DE contains in power  $\rho\nu$  and  $\mu\nu$ . W.W. to be Dem.

a hyp. b lem. 66. 10.

c 24. 10. d sch. 12. 10. e 41. 10.

PROP. LXXI.



A line DE commensurable to a line containing two medially rectangles in power (AC + C) is also a line containing in

power two medially rectangles.

Divide DE, as in the prec. Because ACq  $\perp$  CBq,  $\therefore$  thence shall DFq be  $\perp$  FEq. also for that ACq + CBq is  $\mu\nu$ ,  $\therefore$  shall DFq + FEq be also  $\mu\nu$ . And in like manner because ACB  $\perp$  is  $\rho\nu$ ,  $\therefore$  also DFE is  $\rho\nu$ . Lastly, because ACq + CBq  $\perp$  ACB,  $\therefore$  shall

a hyp. b lem. 66. 10.

c 24. 10. d 24. 10.

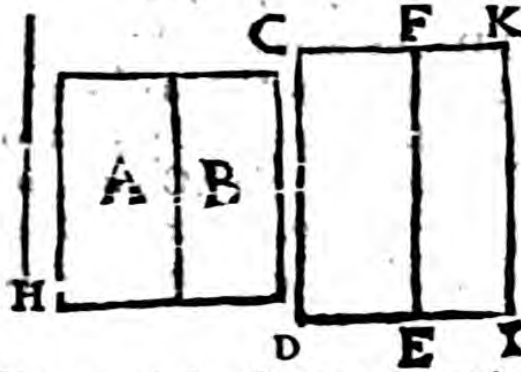
Q

e 14. 10.  
f 42. 10.

The tenth Book of

e shall DFq + FEq be  $\perp$  DFE. f From whence it follows that DE contains in power 2  $\mu$ . Which was to be Dem.

PROP. LXXII.



If a rationally rectangle A and a medially B, be composed together, these four irrational lines will be made; either a binomial, or a first binomial, or a major, or a

line containing in power a rationally and a medially rectangle.

Namely, if  $Hq = A + B$ , then H shall be one of the four lines which the Theoreme mentions. For upon CD the propounded  $\rho$ , make the rectangle  $CE = A$ , and  $FI = B$ , and so  $CI = Hq$ . Whereas then is  $A \rho$ , likewise CE is  $\rho$ , therefore the latitude CF is  $\rho \perp CD$ . and because B is  $\mu$ , also FI shall be  $\mu$ , therefore FK is  $\rho \perp CD$ . therefore CF, FK are  $\rho \perp$ . and so the whole CK is binom. wherefore if  $A \sqsubset B$ , i.e.  $CE \sqsubset FI$ , then  $CF \sqsubset FK$ . therefore if  $CF \perp \sqrt{CFq - FKq}$ , likewise CK shall be a 1. bin. and consequently  $H = \sqrt{CI}$  k is a bin. If CF be supposed  $\perp \sqrt{CFq - FKq}$ , then shall CK be a 4. bin. wherefore  $H (\sqrt{CI})$  is a major line. But if  $A \supset B$ , then shall CF be  $\supset FK$ . consequently if  $FK \perp \sqrt{FKq - CFq}$ , then shall CK be a 2. bin. wherefore H is a first  $2 \mu$ . lastly if  $FK \perp \sqrt{FKq - CFq}$ , then CK shall be a first binom. whence H shall contain in power  $\rho$  and  $\mu$ . W.W. to be Dem.

a cor. 16. 6.

b 2. ex. 1.  
c 21. 10.

d 23. 10.

e 13. 10.

f 37. 10.

g 2. 6.

h 1. def. 48.

i 10.

k 55. 10.

l 4 def. 48.

m 58. 10.

n 2. def. 48.

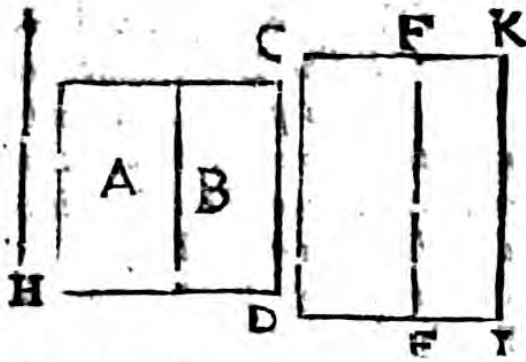
o 56. 10.

p 5. def. 48.

q 10.

r 59. 10.

PROP. LXXIII.



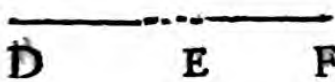
If two medially rectangles A, B, incommensurable to one another be composed together, the two remaining irrational lines are made, either a second binomial, or

a line containing in power two medially rectangles.

As H containing in power  $A + B$  is one of the said irrational lines. For upon CD propounded  $\rho$  draw the rectangle  $CE = A$ , and  $FI = B$ . whence  $Hq = CI$ . Therefore because CE and FI are  $\mu a$ . a hyp. b the latitudes CF, FK, shall be  $\rho \sqrt{\square} CD$ . also because CE  $\mu \sqrt{\square} FI$ , and  $CE.FI c :: CF.FK$ , b 23. 10. c 1. 6. d 10. 10. e 3. def. 48. f 57. 10. g 6. def. 48. h 60. 10. therefore  $CF \sqrt{\square} FK$ . e therefore CK is a 3. bin. namely, if  $CF \sqrt{\square} \sqrt{CFq - FKq}$ . whence  $H = \sqrt{CI}$  shall be a second  $2 \mu$ . But if  $CF \sqrt{\square} \sqrt{CFq - FKq}$ , g then CK shall be a 6 binom. b and consequently H contains in power  $2 \mu a$ . *W. W. to be Dem.*

Here begin the Senaries of lines irrational by Subtraction.

PROP. LXXIV.



If from a rational line DF a rational line DE, commensurable in power only to the

whole DF, be taken away, the residue EF is irrational, and is called an Apotome or residual line.

For  $EFq \mu \sqrt{\square} DEq$ ; b but  $DEq$  is  $\rho^2$ ; c therefore EF is  $\rho$ . *W. W. to be Dem.*

a lem. 16. 10. b hyp. c 10. & 11. def. 10.

In numbers; let there be DF, 2. DE,  $\sqrt{3}$ . then EF shall be  $2 - \sqrt{3}$ .






$\rightarrow BCq$ ) <sup>d</sup> therefore  $ACq \rightarrow ABq \sqsupset BCq$ . <sup>e</sup> therefore  $BC$  is  $\rho$ . *W.W. to be Dem.*

<sup>d</sup> 17. 10.  
<sup>e</sup> 11. def. 10

In numbers, let  $AC$  be  $\sqrt{18} + \sqrt{108}$ ;  $AB \sqrt{18} - \sqrt{108}$ . then  $BC$  is  $\sqrt{18} - \sqrt{108} - \sqrt{18} - \sqrt{108}$ .

PROP. LXXVIII.

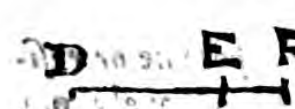
 If from a right line  $DF$  be taken away a right line  $DE$ , being incommensurable in power to the whole line  $DF$ , and with the whole  $DF$  making that which is composed of their squares medially, and the rectangle contained under the same lines rational, the line remaining  $EF$  is irrational, and is called a line making a whole space medially with a rational space.

For  $2 FDE$  <sup>a</sup> is  $\mu$ . <sup>b</sup> and  $DFq + DEq$  is  $\nu$ . <sup>c</sup> therefore  $2 FDE \sqsupset DFq + DEq$  <sup>d</sup> ( $2 FDE + EFq$ ) <sup>e</sup> therefore  $EF$  is  $\rho$ . *W.W. to be Dem.*

<sup>a</sup> hyp. & sch. 12. 10.  
<sup>b</sup> hyp.  
<sup>c</sup> sch. 12. 10.  
<sup>d</sup> 7. 2.  
<sup>e</sup> sch. 12. 10. and 11. def. 10.

In numbers, let  $DF$  be  $\sqrt{216} + \sqrt{72}$ ;  $DE \sqrt{216} - \sqrt{72}$ . therefore  $EF$  is  $\sqrt{216} + \sqrt{72} - \sqrt{216} - \sqrt{72}$ .

PROP. LXXIX.

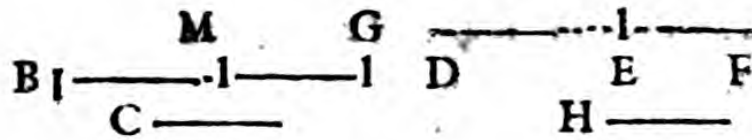
 If from a right line  $DF$  be taken away a right line  $DE$ , incommensurable in power to the whole  $DF$ , and which together with the whole makes that which is composed of their squares medially, and the rectangle contained under them also medially and incommensurable to that which is composed of their squares, the remainder is irrational, and is called a line making a whole space medially with a medial space.

For  $2 FDE$ , and  $FDq + DEq$  <sup>a</sup> are  $\mu$ ; <sup>b</sup> therefore  $EFq$  ( $DFq + DEq - 2 FDE$ ) is  $\rho$ . <sup>d</sup> and so consequently  $EF$  is  $\rho$ . *W.W. to be Dem.*

<sup>a</sup> hyp. & sch. 10.  
<sup>b</sup> 27. 10.  
<sup>c</sup> cor. 7. 2.  
<sup>d</sup> 11. def. 10.

In numbers; let  $DF$  be  $\sqrt{180} + \sqrt{60}$ .  $DE \sqrt{180} - \sqrt{60}$ . then  $EF$  shall be  $\sqrt{180} + \sqrt{60} - \sqrt{180} - \sqrt{60}$ .

Lemma.



If there be the same excessse between the first magnitude BG and the second C (MG) as is between the third magnitude DF and the fourth H(EF;) then alternately, the same excessse shall be between the first magnitude BG and the third DF, as is between the second C and the fourth H.

a hyp.

b 15. ar. 1.

For because that *a* to the equalls BM, DE, are added the equalls MG, EF, that is, C, H; the excessse of the wholes BG, DF, *b* shall be equall to the excessse of the parts added C, H. *W.W. to be Dem.*

Coroll.

Hence, Four magnitudes Arithmetically proportionall, are alternately also Arithmetically proportionall.

P R O P. LXXX.

*To an Apotome or residuall line AB onely one rationally right line BC, being commensurable in power onely to the whole AB, is congruent, or can be joynd.*

If it be possible, let some other line BD be added to it; *a* then the rectangles ACB, ADB, *b* and so consequently the doubles of them are *μz*. wherefore seeing ACq + BCq = 2 ACB *c* = ABq *c* = ADq + DBq = 2 ADB. therefore alternately ACq + BCq = ADq + BDq *d* = 2 ACB = 2 ADB. But ACq + BCq = ADq + BDq *e* is *p r f* therefore 2 ACB = 2 ADB is *p r f*. Which is Absurd.

a 22. 10.  
b 22. 10.

c cor. 7 2.  
d lem. 79.  
10.  
e hyp. and  
29. 10.  
f sch. 12. 10.  
g 27. 10.

P R O P.

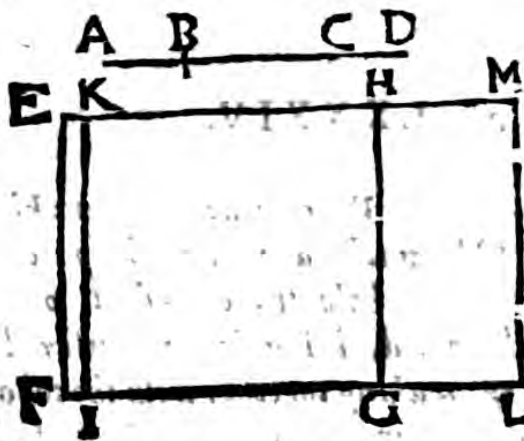
PROP. LXXXI.

*To a first medially residual line AB only one medially right line BC, being commensurable only in power to the whole, and comprehending with the whole line a rational rectangle, can be joined.*

Conceive BD to be such a line as may be joined to it; then because ACq and BCq, as well as ADq and BDq are  $\mu\alpha$   $\square$ . *b* also ACq + BCq, & ADq + BDq shall be  $\mu\alpha$ . *c* but the rectangles ACB, ADB, *d* and so 2 ACB and 2 ADB are  $\rho\alpha$ . *e* therefore 2 ACB = 2 ADB, *f* that is ACq + BCq = ADq + BDq is  $\rho\gamma$ . *g* which is Absurd.

*a* hyp. *b* 10 and 24. 10. *c* hyp. *d* sch. 12. 10. *e* sch. 17. 10. *f* 7. 2. and lem. 79. 10. *g* 17. 10.

PROP. LXXXII.



*Unto a second medially residual line AB only one medially right line BC, commensurable only in power to the whole, and with it containing a medially rectangle, can be joined.*

If it be possible, let some other line BD be added to it; and upon EF  $\rho$  make the rectangle EG = ACq + BCq; as also the rectangle EL = ADq + BDq. likewise EI = ABq. Now 2 ACB + ABq = ACq + BCq = EG. therefore seeing EI = ABq, *a* also KG shall be = 2 ACB. moreover ACq and BCq *b* are  $\mu\alpha$   $\square$ . *c* therefore EG (ACq + BCq) is  $\mu\gamma$ . *d* therefore the breadth EH is  $\rho$   $\square$  EF. *e* Further, the rectangle ACB *f* and so 2 ACB (KG) is  $\mu\gamma$ . *d* therefore KH is also B,  $\rho$   $\square$  EF. lastly, because ACq + BCq (EG) *g*  $\square$  2 ACB (KG) and EG. KG  $\rho$  :: EH. KH. *k* therefore

*a* 4. 2. and 3. *b* hyp. *c* 24. 10. *d* 13. 10. *e* hyp. *f* 24. 10. *g* lem 26. 10. *h* 1. 6. *k* 10. 10.

74. 10.

fore EH  $\square$  HK. <sup>l</sup> therefore EK is a residuall line, whereto KH is congruent, by the same reason also shall KM be congruent to the said EK. which is repugnant to the 80. prop. of this Book.

P R O P. L X X X I I I.

-----  
 A      B      D      C      To a Minor line AB onely one right line BC can be joined being incommensurable in power to the whole, and making together with the whole line that which is composed of their squares rational, and the rectangle which is contained under them mediall.

a Hyp.  
 b lem. 97.  
 10.  
 c feb. 27. 10.  
 d 27. 10.

Conceive any other BD to be congruent to it; Therefore whereas ACq + BCq, and ADq + BDq are p<sup>a</sup>, their excesse (2 b ACB - : 2 ADB) c is p<sup>r</sup>. Which is Absurd; because ACB and ADB are  $\mu\alpha$  by the Hyp.

P R O P. L X X X I V.

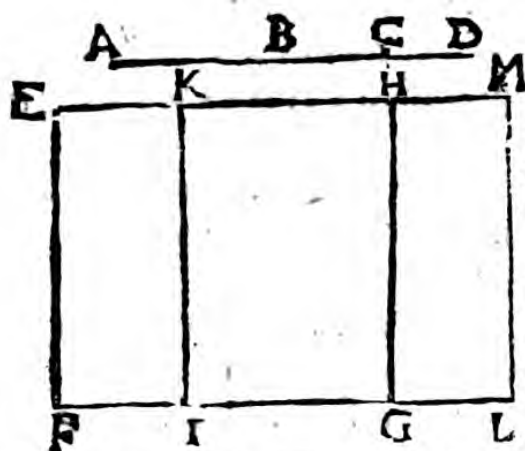
-----  
 A      B      D      C      Vnto a line (AB) making with a rationall space a whole space mediall onely one right line BC can be joined, being incommensurable in power to the whole, and making together with the whole that which is composed of their squares mediall, and the rectangle which is contained under them rational.

a Hyp.  
 b feb. 12. 10.  
 c lem. 79.  
 10.  
 d feb. 17. 10.

Suppose some other BD to be congruent also to it; then the rectangles ACB, ADB, b and so 2 ACB and 2 ADB are p<sup>a</sup>. therefore 2 ACB - : 2 ADB, that is, ACq + BCq - : ADq + BDq d is p<sup>r</sup>. Which is Absurd. since ACq + BCq, and ADq + BDq are  $\mu\alpha$  by the Hyp.

P R O P.

PROP. LXXXV.



To a line AB, which with a medially space makes a whole space medially, can be joined only one right line BC, incommensurable in power to the whole, and making with the whole both that which is composed of their

squares medially, and the rectangle which is contained under them medially and incommensurable to that which is composed of their squares.

Those things being supposed which are done and shewn in the 82. prop. of this Book; it is clear that EH and KH are  $\square$  EF. Besides, being that ACq + CBq, that is, the rectangle EG, <sup>a</sup> is  $\square$  ACB. <sup>b</sup> & so EG  $\square$  2 ACB (KG;) & EG.KG <sup>c</sup> :: EH.KH; shall EH be  $\square$  KH. therefore EK is a residuall line, & the line congruent to it is KH. In like manner may KM be shewn to be congruent to the said residuall EK, against the 80. prop. of this Book.

<sup>a</sup> 2yp.  
<sup>b</sup> 14. 10.  
<sup>c</sup> 1. 6.

Third Definitions.

**A** Rationall line & a residuall being propounded, if the whole be more in power than the line joined to the residuall, by the square of a right line commensurable unto it in length; then

I. If the whole be commensurable in length to the rationall line propounded, it is called a first residuall line.

II. But if the line adjoined be commensurable in length to the rationall line propounded, it is called a second residuall line.

III. If neither the whole nor the line adjoined be commensurable in length to the rationall line propounded, it is called a third residuall line.

Moreover

The tenth Book of

Moreover, If the whole be more in power then the line adjoined by the square of a right line incommensurable to it in length, then

I V. If the whole be commensurable in length to the rationall line propounded, it is called a fourth residuall line.

V. But if the line adjoined be commensurable in length to the rationall line propounded, it is a fift residuall.

VI. If neither the whole nor the line adjoined be commensurable in length to the rationall line propounded, it is termed a sixt residuall line.

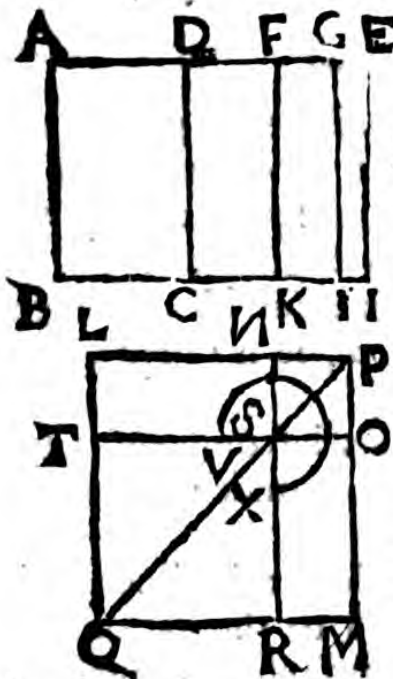
PROP. LXXXVI, 87, 88, 89, 90, 91.

A.... 4 C..... 5 B To find out a first, second, third, fourth, fifth, and sixth residuall line.

D.....  
E.....F  
G  
H.....

Residuall lines are found out by subducting the lesse names or parts of binomials from the greater. Ex. gr. Let  $6 + \sqrt{20}$  be a first binom. then shall  $6 - \sqrt{20}$  be a first residuall. So that it is not necessary to repeat more concerning the finding of them out.

Lemma.



Let AC be a rectangle contained under the right lines AB, AD. Let AD be drawn forth to E, and DE equally divided in F. and let the rectangle AGE be = FEq. & the rectangles AI, DK, FH, finished. Then let the square LM = AH be made, and the square NO = GI; and the lines NSR, OST, produced.

I say 1. the rectangle AI = LM + NO = TQq + SOq. which appears by the constr.

2. The

2. The rectangle  $DK = LO$ . For because the rectangle  $AGE = FEq$ .  $b$  then  $e$  are  $AG, FE, GE, \therefore c$  and so  $AH, FI, GI \therefore, d$  that is  $LM, FI, NO, \therefore; e$  but  $LM, LO, NO d$  are  $\therefore; f$  therefore  $FI = e LO f = DK = g NM$ .

a const.  
b 17. 6.  
c 1. 6.  
d sch. 22. 6.  
e 9. 5.  
f 16. 1.  
g 13. 2.

3. Hence,  $AC = AI - DK - FI = LM + NO - LO - NM = TR$ .

4.  $b$  It is manifest that  $DF, FE, DE$ , are  $\perp$ .

h 16. 10.

5. If  $AE \perp DE$ , and  $AE \perp \surd AEq - DEq$ , & then shall  $AG, GE, AE$  be  $\perp$ .

k 18. and 10.

6. Also, because  $AE \perp DE$ ,  $m$  thence shall  $AE, FE$  be  $\perp$ .  $n$  and so  $AI, FI$ , that is,  $LM + NO$  and  $LO$  are  $\perp$ .

l hyp.

m 13. 10.

n 1. 6. and

10. 10.

7. Because  $AG \perp GE$ ,  $n$  shall  $AH, GI$ , that is  $LM, NO$  be  $\perp$ .

o before,

8. But because  $AE \perp DE$ ,  $o$  therefore shall  $FE, GE$  be  $\perp$ ,  $n$  and so the rectangle  $FI \perp GI$ , that is  $LO \perp NO$ . wherefore seeing  $LO, NO \perp \therefore TS, SO$ .  $p$  therefore shall  $TS, SO$  be  $\perp$ .

o 14. 10.

p 2. 6.

q 10. 10.

9. If  $AE$  be put  $\perp \surd AEq - DEq$ , then shall  $AG, GE, AE$  be  $\perp$ .

r 19. 10. and

17. 10.

s 1. 6. and

10. 10.

10.  $f$  Wherefore the rectangles  $AH, GI$ , that is  $TOq, SOq$  shall be  $\perp$ .

PROP.



PROP. XCII.



If a space AC be contained under a ratioll line AB, and a first residuall line AD (AE - DE) the right line TS, which containeth the space AC in power, is a residuall line.

Use the foregoing Lemma for a preparatory to the demonstration of this prop. Therefore  $TS = \sqrt{AC}$ . Also AG, GE, AE, are  $\perp$ ; therefore since AE  $\perp$  AB  $\rho$ , b also AG and GE shall be  $\perp$  AB. c therefore the rectangles AH & GI, that is,

a hyp.  
b 12. 10.  
c 10. 10.

d lem. 91. 10.  
e 74. 10.

TOq and SOq are  $\rho\alpha$ . d Likewise TO, SO, are  $\perp$  and consequently TS is a residuall line. W.W. to be Dem.

PROP. XCIII.

See the prec. Scheme.

If a space AC be contained under a ratioll line AB and a second residuall AD (AE - DE) the right line TS, containing the space AC in power, is a first medially residuall line.

Again, by the foregoing Lemma, AG, GE, AE are  $\perp$ . therefore a since AE is  $\rho$   $\perp$  AB, b also AG, GE, shall be  $\rho$   $\perp$  AB. c therefore the rectangles AH, GI, that is, TOq, SOq are  $\mu\alpha$ . d likewise TO  $\perp$  SO. Lastly, because DE e  $\perp$  AB  $\rho$ , f the right angle DI, and the half thereof DK or LO, that is TOS shall be  $\rho\gamma$ . g from whence it follows that TS ( $\sqrt{AC}$ ) is a first medially residuall. W. W. to be Dem.

a hyp.  
b 13. 10.  
c 22. 10.  
d lem. 74. 10.

e hyp.  
f 20. 10.

g 75. 10.

PROP. XCIV.

See Scheme. 92.

If a space AC be contained under a rationally line AB and a third residuall AD (AE - DE) the right line TS containing in power the space AC is a second mediall residuall line.

As in the former, TO and SO are  $\mu$ . Therefore because DE <sup>a hyp.</sup> is  $\rho$   $\perp$  AB. <sup>b 22.10.</sup> the rectangle DI, <sup>c 24.10.</sup> and <sup>d 76.10.</sup> so DK, or TOS, shall be  $\mu\nu$ . therefore  $TS = \sqrt{AC}$  is a second mediall residuall. *W.W. to be Dem.*

PROP. XCV.

See Scheme 92.

If a space AC be contained under a rationally line AB and a fourth residuall AD (AE - DE) the right line TS containing the space AC in power, is a Minor line.

As before, TO  $\perp$  SO. Therefore because AE <sup>a lem. 91.10.</sup> is  $\rho$   $\perp$  AB, <sup>b hyp.</sup> c shall AI (TOq + SOq) be  $\rho\nu$ . but, as <sup>c 20.10.</sup> before, the rectangle TOS is  $\mu\nu$ . <sup>d 77.10.</sup> therefore  $TS = \sqrt{AC}$  is a Minor line. *W.W. to be Dem.*

PROP. XCVI.

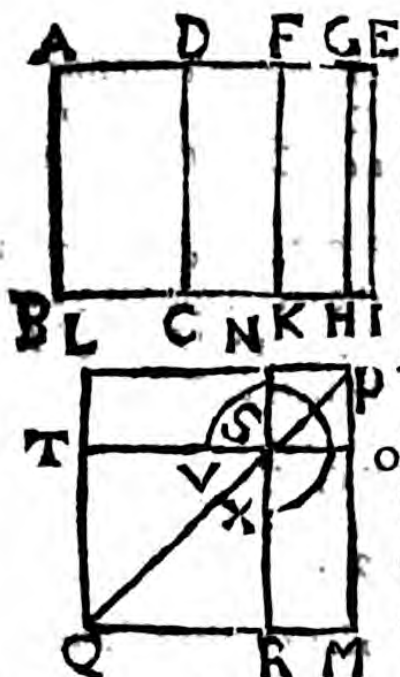
See Scheme 92.

If a space AC be contained under a rationally line AB and a fifth residuall AD (AE - DE) the right line TS containing in power the space AC, is a line which maketh with a rationally space the whole space mediall.

For again TO  $\perp$  SO. therefore since AE <sup>a hyp.</sup> is  $\rho$   $\perp$  AB, <sup>b 22.10.</sup> also AI, that is TOq + SOq shall be <sup>c 78.10.</sup>  $\mu\nu$ . But, as in the 93. the rectangle TOS is  $\rho\nu$ . whence  $TS = \sqrt{AC}$  is a line which with  $\rho\nu$  makes a whole  $\mu\nu$ . *W.W. to be Dem.*

PROP.

PROP. XCVII.



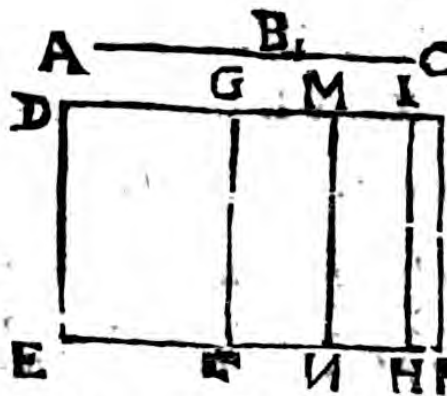
If a space AC be contained under a rational line AB, and a fixt residuall AD (AE - DE) the right line TS containing in power the space AC, is a line making with a mediall rectangle, a whole space mediall.

As often above,  $TO \sqsupseteq SO$ . also, as in 96,  $TOq + SOq$  is  $\mu v$ . but the rectangle TOS is  $\rho v$ , as in 94. <sup>a</sup> Lastly  $TOq + SOq \sqsupseteq TOS$ . <sup>b</sup> therefore  $TS = \sqrt{AC}$  is a line which with  $\mu v$  makes

a whole  $\mu v$ . W.W. to be Dem.

a lem 91.  
20.  
b 79. 10.

Lemma.



Upon a right line DE \* apply the rectangles  $LDf = ABq$ , and  $DH = ACq$ , and  $IK = BCq$ . and let GL be bisected in M, and the line MN drawn parallel to GF.

Then 1. the rectangle DK is  $= ACq + BCq$ . as the construction manifests.

2. The rectangle ACB  $= GN$  or  $MK$ . For  $DK = ACq + BCq$  <sup>b</sup>  $= 2 ACB + ABq$ . but  $ABq = DF$ . therefore  $GK$  <sup>c</sup>  $= 2 ACB$ . and <sup>d</sup> consequently  $GN$  or  $MK = ACB$ .

3. The rectangle DIL  $= MLq$ . For because  $ACq = ACB \cdot AC$  <sup>e</sup>  $ACB \cdot BCq$ , that is  $DH \cdot MK :: MK \cdot IK$ .

a constr.  
b 7. 2.  
c 3. ax. 1.  
d 7. ax. 1.  
e 1. 6.

IK. e thence is DI. ML :: ML. IL. f therefore  $DI \cdot IL = ML^2$ .

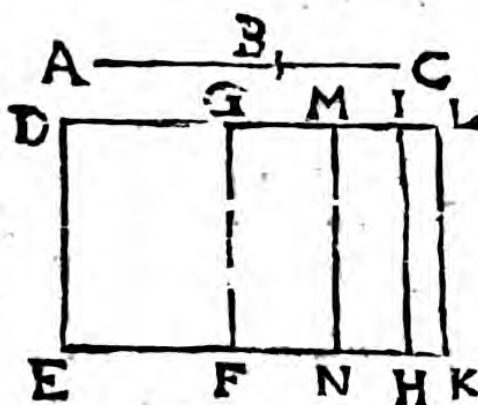
4. If AC be taken  $\perp$  BC, then DK shall be  $\perp$  ACq. For  $ACq + BCq = DK^2$  g  $\perp$  ACq. g 16. 10.

5. Likewise  $DL^2 = DLq - GLq$ . For because DH (ACq)  $\perp$  IK (BCq) b thence shall DI be  $\perp$  IL. k therefore  $\sqrt{DLq - GLq} = DL$ . h 10. 10. k 18. 10.

6. Also  $DL \perp GL$ . For  $ACq + BCq = DK^2$  l 2 ACB. that is,  $DK \perp GK$ . m therefore  $DL \perp GL$ . l lem. 26. 10. m 10. 10.

7. But if AC be taken  $\perp$  BC, n then DL shall be  $\perp \sqrt{DLq - GLq}$ . n 19. 10.

PROP. XCVIII.



The square of a residual line AB (AC - BC) applied to rational line DE, makes the breadth DG a first residual line.

Doe as is enjoined in the Lemma next preceding. Then because

AC, BC, a are  $\rho'$   $\perp$ . b also DK ( $ACq + BCq$ ) shall be  $\perp$  ACq. c therefore DK is  $\rho'$ . d wherefore DL is  $\rho'$   $\perp$  DE. e Likewise the rectangle GK (2 ACB) is  $\mu$ . f therefore GL is  $\rho'$   $\perp$  DE. g and consequently  $DL \perp GL$ . h But  $DLq = GLq$ . k therefore DG is a residual, l and that of the first order (because m AC  $\perp$  BC, and therefore  $DL \perp \sqrt{DLq - GLq}$ ) w.w. to be Dem.

a hyp. b lem. 97. 10. c sch. 12. 10. d 21. 10. e 22. and 24. 10. f 23. 10. g 13. 10. h sch. 12. 10. k 74. 10. l 1. def. 85. 10. m lem. 97. 10.

PROP.

PROP. XCIX.

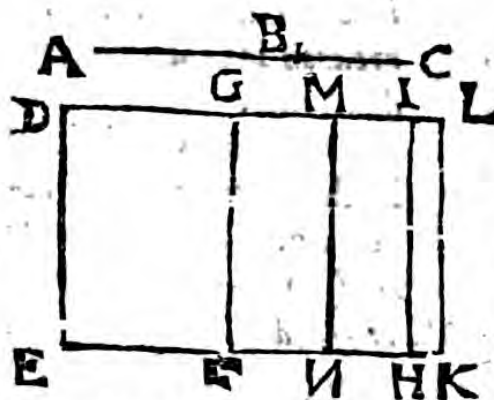
See the following Scheme.

The square of a first medially residual line AB (AC - BC) applied to a rational line DE, makes the breadth DG a second residual line.

a hyp.  
b lem. 97.  
10.  
c 24. 10.  
d 23. 10.  
e hyp. & feb.  
12. 10.  
f 21. 19.  
g 13. 10.  
h feb. 12. 10.  
k 74. 10.  
l lem. 97. 10.  
m 2. def. 85.  
10.

Supposing the foregoing Lemma; because AC and BC are  $\mu$   $\perp$ , b thence shall DK (ACq + BCq) be  $\perp$  ACq. c wherefore DK is  $\mu\nu$ . d therefore DL is  $\rho$   $\perp$  DE. e also GK (2 ACB) is  $\rho\nu$ . f therefore GL is  $\rho$   $\perp$  DE; g wherefore DL  $\perp$  GL. b But DLq  $\perp$  GLq. k therefore DG is a residual line; and because DL is  $\perp$   $\sqrt{DLq - GLq}$ . m therefore shall DG be a second residual. W. W. to be Dem.

PROP. C.



The square of a second medially residual line AB (AC - BC) applied to a rational line DE, makes the breadth DG a third residual line.

a 23. 10.  
b lem. 26.  
10.  
c 1. 6. and  
10. 10.  
d feb. 12. 10.  
e 74. 10.  
f 3. def. 85.  
10.  
g lem. 97. 10.

Again DK is  $\mu\nu$ . a wherefore DL is  $\rho$   $\perp$  DE. also DGK is  $\mu\nu$  a whence GL is  $\rho$   $\perp$  DE. b likewise DK  $\perp$  GK. e wherefore DL  $\perp$  GL. d but DLq  $\perp$  GLq. e therefore DG is a residual line, and that of the third order, g because DL  $\perp$   $\sqrt{DLq - GLq}$ . W. W. to be Dem.

PROP. CI.

See the foregoing Scheme.

The square of a Minor line AB (AC - BC) applied

to a ratioll line DE, makes the breadth DG a fourth residuall.

As before, ACq + BCq, that is DK, is  $\rho v$ . a therefore DL is  $\rho \sqsupset$  DE. but the rectangle ACB, and so GK (2 ACB) \* is  $\mu v$ . b wherefore GL is  $\rho \sqsupset$  DE. c therefore DL  $\sqsupset$  GL. d but DLq  $\sqsupset$  GLq. and because + ACq  $\sqsupset$  BCq, e thence shall DL be  $\sqsupset \sqrt{DLq - GLq}$ . f therefore DG ha's the conditions required to a fourth residuall. *W.W. to be Dem.*

a 21.10.  
\* hyp.  
b 12.10.  
c 13.10.  
d sch. 11. 10.  
e lem. 97.  
f 4 def. 85.  
10.

PROP. CII.

See Scheme 100.

The square of a line AB (AC - BC) which makes with a ratioll space the whole space medially, applied to a ratioll line DE, makes the breadth DG a fifth residuall line.

For, as above, DK is  $\mu v$ . a wherefore DL is  $\rho \sqsupset$  DE. also GK is  $\rho v$ . b whence GL is  $\rho \sqsupset$  DE. c therefore DL  $\sqsupset$  GL. d but DLq  $\sqsupset$  GLq. Moreover DL  $\sqsupset \sqrt{DLq - GLq}$ . wherefore DG f is a fifth residuall. *W.W. to be Dem.*

a 13.10.  
b 21.10.  
c 12.10.  
d sch. 12. 10.  
e lem. 97.  
f 5. def. 85.  
10.

PROP. CIII.

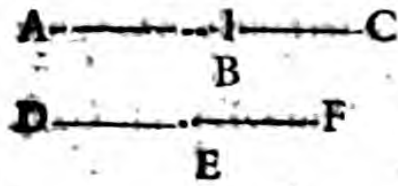
See the last Scheme.

The square of a line AB (AC - BC) making with a medially space the whole space medially, applied to a ratioll line DE, makes the breadth DG a sixth residuall line.

As above DK and GK are  $\mu a$ ; a wherefore DL and GL are  $\rho \sqsupset$  DE, also DK b  $\sqsupset$  GK. c whence DL  $\sqsupset$  GL. d therefore DG is a residuall. b And whereas ACq  $\sqsupset$  BCq. and so DL  $\sqsupset \sqrt{DLq - GLq}$ , e therefore DG shall be a sixth residuall. *W.W. to be Dem.*

a 23.10.  
b hyp. and lem. 97. 10.  
c 10.10.  
d 74.10.  
e 6. def. 85.  
10.

PROP. CIV.



A right line DE commensurable in length to a residuall AB (AC - BC) is it self also a residuall, and of the same order.

Lemma.

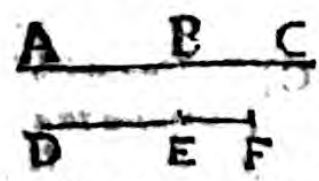
Let AB.DE :: AC.DF. and AB  $\perp$  DE.

I say AC + BC  $\perp$  DF + EF. For AC.BC :: DF.EF. therefore by addition AC + BC.BC :: DF + EF.EF. therefore by inversion AC + BC.DF + EF :: BC.EF. but BC  $\perp$  EF. therefore AC + BC  $\perp$  DF + EF. *W.W. to be Dem.*

a lem. 66. 10.  
b 10. 10.  
a 12. 6.  
b lem. 103.  
10.  
c hyp.  
d 67. 10.  
e by 47. 85.  
10.

Make AB.DE :: AC.DF. therefore AC + BC  $\perp$  DF + EF. therefore seeing AC + BC is a binomial, and DF + EF shall be a binomial too, and of the same order. therefore DF - EF is a residuall of the same order with AC - BC. *W.W. to be Dem.*

PROP. CV.



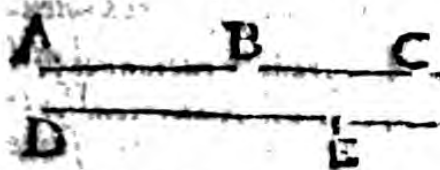
A right line DE commensurable to a medially residuall line AB (AC - BC) is it self a medially residuall, and of the same order.

Again make AB.DE :: AC.DF. whence AC + BC  $\perp$  DF + EF. therefore DF + EF is a binomial of the same order with AC + BC, and consequently DF - EF shall be a medially residuall of the same order with AC - BC. which was to be Demonstrated.

a 12. 6.  
b lem. 103.  
10.  
c 68. 10.  
d 75. and 76.  
10.

PROP.

PROP. CVI.

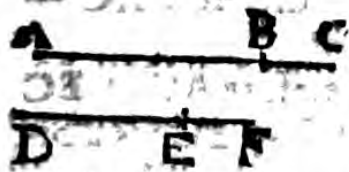


A right line DE commensurable to a Minor line AB (AC - BC) is it self also

a Minor line.

Make  $AB:DE :: AC:DF$ , then is  $AC + BC$  al. 101.  
 $DF + EF$ . but  $AC + BC$  10. is a Major line; b hyp. therefore  $DF + EF$  is also a Major line; c 69. 10. and consequently  $DF - EF$  is a Minor line. d 77. 10. *W. W. to be Dem.*

PROP. CVII.

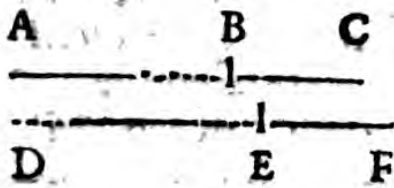


A right line DE commensurable to a line AB (AC - BC) which makes with a ratioll space the whole space mediall,

is it self also a line making with a ratioll space the whole space mediall.

For, accordingly as in the former, we may show  $DF + EF$  to contain in power  $\rho$  and  $\mu$ . a 78. 10. whence  $DF - EF$  is a line making, &c.

PROP. CVIII.



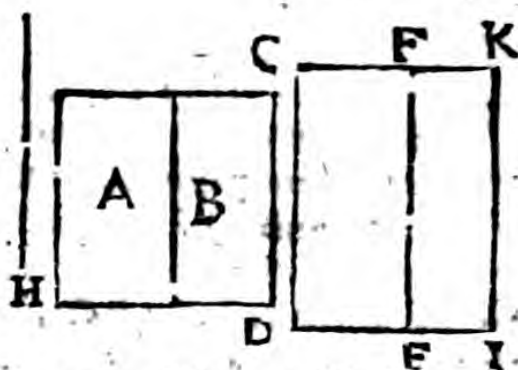
A right line DE commensurable to a line AB (AC - BC) which with a mediall space makes the whole space mediall, is it

self a line making with a mediall space the whole space mediall.

For according to the preced.  $DF + EF$  shall contain in power  $2\mu$ . a therefore  $DF - EF$  shall be, as a 79. 10. in the prop.



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P R O P. C I X.



A medially rectan-  
gle B being taken  
from a rationally rect-  
angle A+B, the right  
line H which contain-  
eth in power the space  
remaining A, is one  
of these two irrational

lines. viz. either a residuall line, or a Minor line.

a 3. ax. 1.  
b hyp. and  
constr.  
c 21. 10.  
d 23. 10.  
e 13. 10.  
f 74. 10.  
g 1. def. 85.  
10.  
h 92. 10.  
k 4 def. 85.  
10.  
l 95. 10.

Upon CD  $\rho'$  make the rectangles  $CI = A + B$ ,  
and  $FI = B$ . whence  $CE = A = Hq$ . wherefore be-  
cause  $CI$  is  $\rho'v$ , c therefore  $CK$  is  $\rho' \perp CD$ . but being  
 $FI$  is  $\mu v$ , d shall  $FK$  be  $\rho' \perp CD$ . e whence  $CK$   
 $\perp FK$ . f therefore  $CF$  is a residuall line. Wherefore  
if  $CK$  be  $\perp \sqrt{CKq - FKq}$ , g then  $CF$  shall be a  
first residuall. h therefore  $\sqrt{CE}$  (H) is a residuall  
line. But if  $CK \perp \sqrt{CKq - FKq}$ , k then  $CF$  shall  
be a first residuall; and consequently H ( $\sqrt{CE}$ )  
l shall be a Minor line. W.W. to be Dem.

P R O P. C X.

See the prec. Scheme.

A rationally rectangle B being taken away from a me-  
dially rectangle A + B, other two irrational lines are  
made, namely either a first medially residuall line, or a line  
making with a rationally space the whole space me-  
dially.

a 3. ax. 1.  
b hyp. and  
constr.  
c 23. 10.  
d 21. 10.  
e 13. 10.  
f 74. 10.  
g 2. def. 85.  
10.  
h 93. 10.  
k 5 def. 85.  
10.  
l 96. 10.

Upon CD the propounded  $\rho'$  make the rectangles  
 $CI = A + B$ , and  $FI = B$ . a whence  $CE = A =$   
 $Hq$ . Therefore because  $CI$  is  $\mu v$ ; c shall  $CK$  be  $\rho' \perp$   
 $CD$ . but because  $FI$  is  $\rho'v$ . d thence  $FK \rho' \perp CD$ .  
e whence  $CK \perp FK$ . f therefore  $CF$  is a residuall,  
g and that a second. If  $CK \perp \sqrt{CKq - FKq}$ ,  
h then H ( $\sqrt{CE}$ ) is a first medially residuall. B ut if  
 $CK \perp + CKq - FKq$ , k then shall  $CF$  be a first re-  
siduall; and l consequently H ( $\sqrt{CE}$ ) shall be a line  
making  $\mu v$  with  $\rho'v$ . W.W. to be Dem.

P R O P.

PROP. CXI.

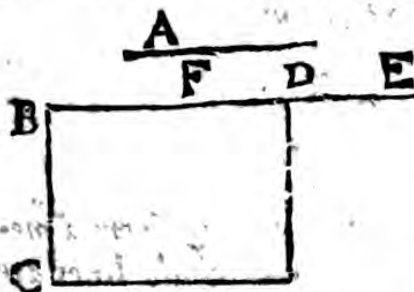
See the same Scheme.

A medially space B being taken away from a medially space A + B, which is incommensurable to the whole A + B, the other two irrational lines are made; viz. either a second medially residual line, or a line making with a medially space the whole space medially.

Upon CD  $\rho$  make the rectangles CI = A + B, and FI = B.  $a$  wherefore CE = A = Hq. Because therefore CI is  $\mu\nu$ ,  $b$  thence CK is  $\rho$   $\perp$  CD, and in like manner FK  $\rho$   $\perp$  CD. Likewise because CI  $\perp$  FI,  $d$  therefore CK  $\perp$  FK.  $e$  wherefore CF is a residual,  $f$  namely a third. If CK  $\perp$   $\sqrt{CKq}$  = FK  $g$  whence H ( $\sqrt{CE}$ ) shall be a second medially residual. but if CK  $\perp$   $\sqrt{CKq}$  = FK  $h$  then shall CF be a first residual.  $k$  wherefore A shall be a line making  $\mu\nu$  with  $\mu\nu$ . *W.W. to be Dem.*

a 3. ex. 1.  
b 23. 10.  
c hyp.  
d 10. 10.  
e 74. 10.  
f 3. def. 85.  
10  
g 94. 10.  
h 6. def. 85.  
10.  
k 97. 10.

PROP. CXII.



A residual line A is not the same with a binomial line.

Upon BC propounded  $e$  make the rectangle CD = Aq. Therefore seeing A is a residual,  $a$  BD shall

be a first residual, to which let DE be the line congruent or that may be adjoined.  $b$  wherefore BE, DE, are  $\rho$   $\perp$ .  $c$  and BE  $\perp$  EC. If you conceive A to be a binomial, then BD is a first bin. whose names let be BF, FD; and let BF be  $\perp$  FD.  $d$  therefore BF, FD are  $\rho$   $\perp$ ; and BF  $\perp$  BC. therefore since BC  $\perp$  BE,  $f$  shall BE be  $\perp$  BF.  $g$  and thence BE  $\perp$  FE.  $h$  therefore FE is  $\rho$ . Likewise because BE  $\perp$  DE,  $k$  shall FE be  $\perp$  DE.  $l$  wherefore FD is a residual, & so FD is  $\rho$ . but it was shewn  $\rho$ . which are repugnant. Therefore A is falsely conceived to be a binomial. *W.W. to be Dem.*

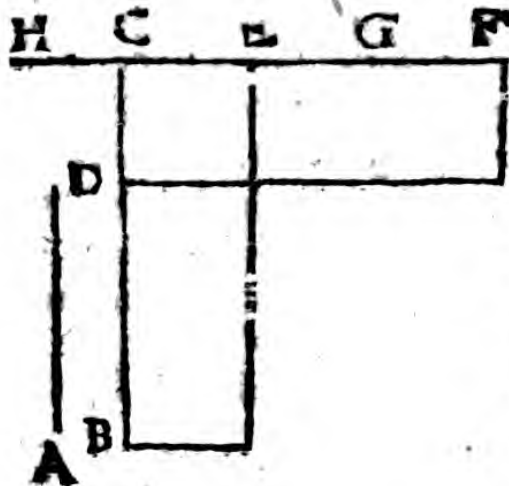
a 98. 10.  
b 74. 10.  
c 1. def. 85.  
10.  
d 37. 10.  
e 1. def. 48.  
10.  
f 12. 10.  
g cor. 16. 10.  
h feb 12. 10.  
k 14. 10.  
l 74. 10.

The names of the 13. irrationall lines differing one from another.

1. A Mediall line.
2. A binomial line, of which there are six species.
3. A first bimediall line.
4. A second bimediall.
5. A Major line.
6. A line containing in power a rationally superficies and a mediall superficies.
7. A line containing in power two mediall superficies.
8. A residuall line; of which there also six kinds.
9. A first mediall residuall line.
10. A second mediall residuall line.
11. A Minor line.
12. A line making with a rationally superficies the whole superficies mediall.
13. A line making with a mediall superficies the whole superficies mediall.

Being the differences of breadths do argue differences of right lines, whose squares are applied to some rationally line, and it is demonstrated in the preced. Propositions that the breadths which arise from applying of the squares of these 13. lines do differ one from another, it evidently follows that these 13. lines do also differ one from another.

PROP. CXIII.

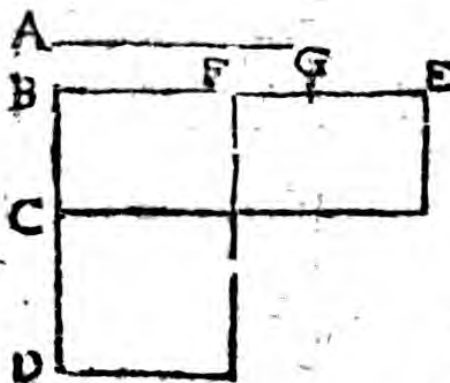


The square of a rationally line A applied to a binomial BC (BD + DC) makes the breadth EC a residuall line, whose names EH, CH, are commensurable to the names BD, DO, of the binomial line, and in

in the same proportion (EH. BD :: CH. DC:) and moreover, the residuall line EC which is made, is of the same order with BC the binomiall.

Upon DC the lesse name  $a$  make the rectangle DF a cor. 16 6.  
 $= Aq = BE$ . whence  $BC. CD b :: FC. CE$ . there- b 14 6.  
 fore by division,  $BD DC :: FE. EC$ . And whereas c hyp. d 14 5.  
 $BD c = DC$ , thence  $FE$  shall be  $= EC$ . Take  $EG$   
 $= EC$ , & make  $FG. GE :: BC. CH$ . Then  $EH$ , &  $CH$   
 shall be the names of the residuall  $EC$ , whereunto  
 all is agreeable that is propounded in the theoreme.  
 For being that by addition  $FE. GE (EC) :: EH.$  e 12. 5. f before. g hyp. h 10. 10.  
 $CH$ . therefore  $FH. EH e :: EH. CH f :: FE.$   
 $EC f :: BD. DC$ . wherefore since  $BD g =$   
 $DC$ , thence shall  $EH$  be  $= CH$ ,  $^b$  and  $FHq =$   
 $EHq$ . Therefore because  $FHq. EHq k :: FH. CH.$  k cor. 10. 6. l 16. 10.  
 $^l$  shall  $FH$  be  $= CH$ ,  $^l$  and so  $FC = CH$ . More-  
 over  $CD g$  is  $p$ , and  $DF (Aq) g$  is  $p v$ , therefore  $FC$  m 11. 10. n sch. 12. 10.  
 is  $p$ ,  $= CD$ . whence also  $CH$  is  $p = CD$ . therefore  $EH, CH$  are  $p$  and  $=$ , as before. therefore  $EC$  o 74. 10.  
 is a residuall line, to which  $CH$  may be joined.  
 Furthermore  $EH. CH f :: BD. DC$ . and so by inver-  
 sion  $EH. BD :: CH. DC$ . whence because  $CH f =$   
 $DC$ ,  $^p$  shall  $EH$  be  $= BD$ . But suppose  $BD =$  p 10. 10. q 15. 10.  
 $BDq = DCq$ .  $^q$  then shall  $EH$  be  $= \sqrt{EHq} =$   
 $CHq$ . Also if  $BD = p$  propounded, then shall  $EH$   
 be  $=$  to the same  $p$ .  $^s$  that is, if  $BC$  be a first bino-  
 miall,  $^s$   $EC$  shall be a first residuall. In like manner, if r 12. 10. s 1. def. 48. 10.  
 $DC$  be to the  $=$  propounded  $p$ , then is  $CH =$  to  
 the same  $p$ .  $^u$  that is, if  $BC$  be a second binomial,  $^u$   $EC$  t 1. def. 85. 10.  
 shall be a second residuall: and if this be a third bi-  
 nom. then that shall be a third residuall, &c. But if u 2. def. 48. 10. x 2. def. 85. 10. y 15. 10.  
 $BD$  be  $= \sqrt{BDq} = DCq$ ,  $^y$  then shall  $EH$  be  $=$   
 $\sqrt{EHq} = CHq$ . therefore if  $BC$  be a 4, 5, or 6 bino-  
 miall,  $EG$  shall be likewise a 4, 5, or 6 residuall.  
*W.W. to be Dem.*

PROP. CXIV.



The square of a rational line A applied to a residuall line BC (BD - CD) makes the breadth BE a binomiall; whose names BE, GE are commensurable to the names BD, BC of the residuall line BC, &

in the same proportion. and moreover, the binomiall line which is made (BE) is of the same order with the residuall line (BC.)

Make the rectangle DF = Aq. and BF.FE :: EG.GF. whence for that DF = Aq = CE, therefore BD. BC :: BE. BF. therefore by conversion of proportion BD. CD :: BE. FE :: EG. GF :: BG. EG. but BD e  $\perp$  CD. f therefore BG  $\perp$  GE. therefore because BGq. GEqg :: BG. GF. shall BG be  $\perp$  GF. & and so BG  $\perp$  BF. moreover BD e is  $\perp$ , and the rectangle DF (Aq) e is  $\perp$  therefore BF is  $\perp$  BD. m therefore also BG is  $\perp$  BD. n therefore BG, GE are  $\perp$ . o wherefore BE is a binomiall. Lastly, because BD.CD :: BG. GE. and inversely BD.BG :: CD. GE. and BD  $\perp$  BG. p thence shall CD be  $\perp$  GE. therefore if CB be a first residuall, BE shall be a first binomiall, &c. as in the prec. therefore, &c.

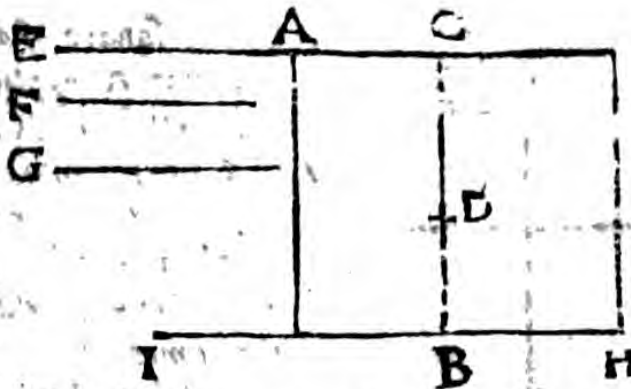
a cor. 16. 6.  
b 12. 6.  
c 14 6.

d 19 5.  
e hyp.  
f 10. 10.  
g cor. 20. 6.  
h 10. 10.  
k cor. 16. 10.  
l 21. 10.  
m 12. 10.  
n sch. 12. 10.  
o 37. 10.

p 10. 10.

PROP.

PROP. CXV.



If a space AB be contained under a residuall line AC (CE - AE) and a binomial CB, whose names CD, DB are commensurable to the names CE, AE, of the residuall line, and in the same proportion (CE. AE :: CD. DB) then the right line F which containeth in power that space AB, is irrationall.

Let G be  $\rho$ , and make the rectangle CH = Gq; then shall BH (HI - IB) be a residuall line, and HI  $\perp$  CD  $\perp$  CE, and BI  $\perp$  DB, and HI. BI :: CD. DB  $b$  :: CE. EA. therefore by inversion HI. CE :: BI. EA.  $c$  therefore BH. AC :: HI. CE :: BI. EA. wherefore since  $d$  HI  $\perp$  CE,  $e$  thence BH  $\perp$  AC  $f$  therefore the rectangle HC  $\perp$  BA. But HC (Gq)  $b$  is  $\rho v$ .  $g$  therefore BA (Fq) is  $\rho v$ ; and consequently F is  $\rho$ . W.W. to be Dem.

a 113. 10.  
b hyp.  
c 19. 5.  
d 12. 10.  
e 10. 10.  
f 1. 6 and  
g feb 12. 10.

Coroll.

Hereby it appears that a rationall superficies may be contained under two irrationall right lines.

PROP. CXVI.



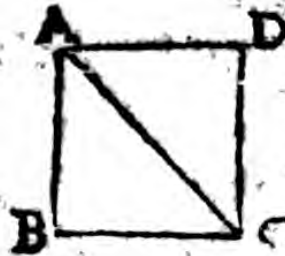
Of a medially line AB are produced infinite irrationall lines BE, EF, &c. whereof none is of the same kind with any of the precedent.

Let AC be propounded  $\rho$ . & AD a rectangle contained

a 17. 10.  
b 11. 10.

tained under AC, AB.  $a$  therefore AD is  $\rho$ . Take BE  $= \sqrt{AD}$ .  $b$  then BE is  $\rho$ , and the same with none of the former. For no square of any of the former being applied to  $\rho$ , makes the breadth mediall. Let the rectangle DE be finished,  $a$  then DE shall be  $\rho$ , and  $b$  consequently EF ( $\sqrt{DE}$ ) shall be  $\rho$ , and not the same with any of the former. for no square of the former being applied to  $\rho$ , makes the latitude BE. therefore, &c.

PROP. CXVII.



a 47. 1.  
b cor. 24. 2.  
c 9. 10.

Let it be required to shew that in square figures BD, the diameter AC is incommensurable in length to the side AB.

For ACq. ABq  $a :: 2. 1 b ::$  not Q. Q.  $a$  therefore AC  $\nparallel$  AB. *W.W. to be Dem.* This theoreme was

of great note with the ancient philosophers; so that he that understood it not was esteemed by Plato undeserving the name of a man, but rather to be reckoned among brutes.

The End of the tenth Book.

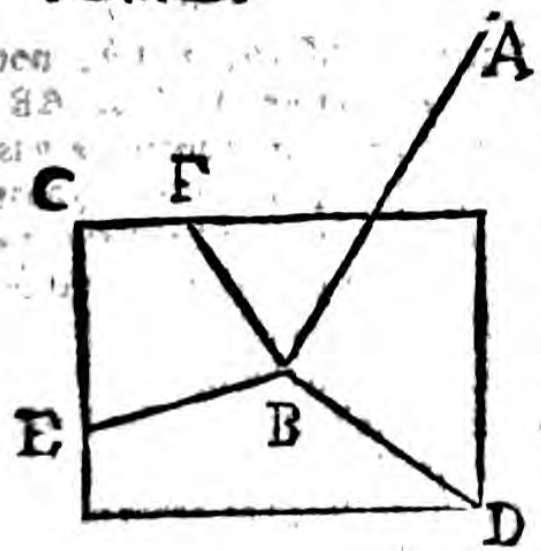
THE

# THE ELEVENTH BOOK OF EUCLIDE'S ELEMENTS.

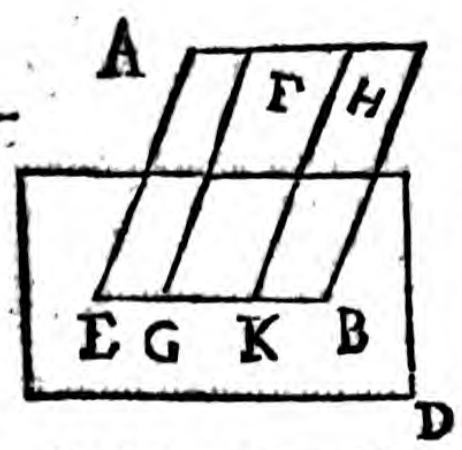
## Definitions.

I. **A Solid** is that which hath length, breadth, and thicknesse.

II. The terme, or extreme, of a Solid is a Superficies.



III. A right line **AB** is perpendicular to a Plane **CD**, when it makes right angles **ABD**, **ABE**, **ABF** with all the right lines **BD**, **BE**, **BF**, that touch it, and are drawn in the said Plane.

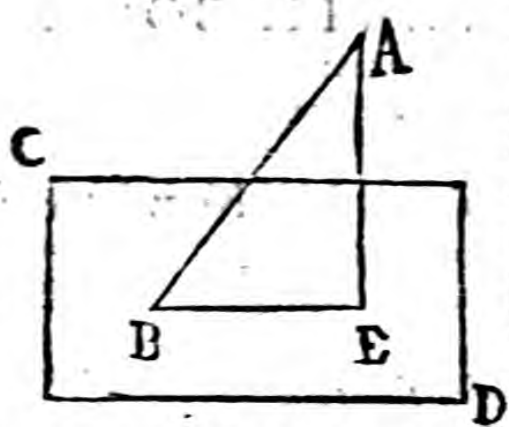


IV. A Plane **AB**, is perpendicular to a Plane **CD**, when the right lines **FG**, **HK**, drawn in one Plane **AB** to the line of common section of the two Planes **EB**, and making right angles therewith, do also make right angles with the other Plane **CD**.

so make right angles with the

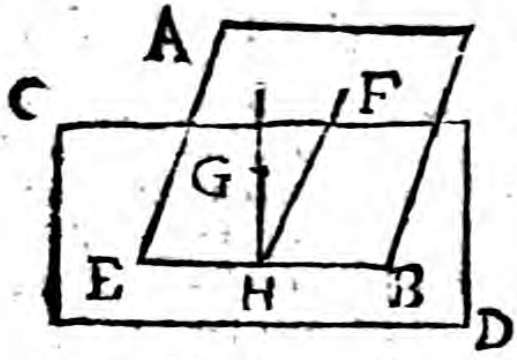
V. The





V. The inclination of a right line AB to a Plane CD, is, when a perpendicular AE is drawn from A the highest point of that line AB to the plane CD, and another line EB

drawn from the point E, which the perpendicular AE makes in the Plane CD, to the end B of the said line AB which is in the same Plane, whereby the angle is acute ABE which is contained under the insisting line AB, and the line drawn in the plane EB.



VI. The inclination of a Plane AB to a Plane CD, is an acute angle FHG contained under the right lines FH, GH, which being drawn

in either of the Planes AB, CD to the same point H of the common section BE, make right angles FHB, GHB, with the common section BE.

VII. Planes are said to be inclined to other planes in the same manner, when the said angles of inclination are equal one to another.

VIII. Parallel Planes are those which being prolonged never meet.

IX. Like solid figures are such as are contained under like planes equal in number.

X. Equal and like solid figures are such as are contained under like planes equal both in multitude and magnitude.

XI. A solid angle is the inclination of more then two right lines which touch one another, and are not in the same superficies.

*Or thus;*

A solid angle is that which is contained under more then two plane angles not being in the same superficies, but consisting all at one point.

XII. A Pyramide is a solid figure comprehended under divers planes set upon one plane, (which is the base of the pyramide,) and gathered together to one point.

XIII. A Prisme is a solid figure contained under planes, whereof the two opposite are equall, like, and parallel; but the others are parallelograms.

XIV. A Sphere is a solid figure made when the diameter of a semicircle abiding unmoved, the semicircle is turned round about, till it return to the same place from whence it began to be moved.

*Coroll.*

Hence, all the rayes drawn from the center to the superficies of a sphere, are equall amongst themselves.

XV. The Axis of a sphere, is that fixed right line, about which the semicircle is moved.

XVI. The Centre of a Sphere, is the same point with that of the semicircle.

XVII. The Diameter of a Sphere, is a right line drawn though the centre, and terminated on either side in the superficies of the sphere.

XVIII. A Cone is a figure made, when one side of a rectangled triangle (*viz.* one of those that contain the right angle) remaining fixed, the triangle is turned round about till it return to the place from whence it first moved. And if the fixed right line be equall to the other which containeth the right angle, then the Cone is a rectangled cone; but if it be lesse, it is an obtuse-angled Cone; if greater, an acute-angled Cone.

XIX. The Axis of a Cone is that fix'd line about which the triangle is moved.

XX. The

**XX.** The Base of a Cone is the circle, which is described by the right line moved about.

**XXI.** A Cylinder is a figure made by the moving round of a right-angled parallelogram, one of the sides thereof, (namely which contain the right angle) abiding fix'd, till the parallelogram be turned about to the same place, where it began to move.

**XXII.** The Axis of a Cylinder is that quiescent right line, about which the parallelogram is turned.

**XXIII.** And the Bases of a Cylinder are the circles which are described by the two opposite sides in their motion.

**XXIV.** Like Cones and Cylinders, are they, both whose Axes and Diameters of their Bases are proportionall.

**XXV.** A Cube is a solid figure contained under six equall squares.

**XXVI.** A Tetraedron is a solid figure contained under four equall and equilaterall triangles.

**XXVII.** An Octaedron is a solid figure contained under eight equall and equilaterall triangles.

**XXVIII.** A Dodecaedron is a solid figure contained under twelve equall, equilaterall and equiangular Pentagones.

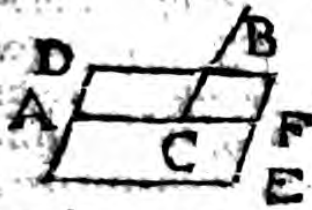
**XXIX.** An Icofaedron is a solid figure contained under twenty equall and equilaterall triangles.

**XXX.** A Parallelipedon is a solid figure contained under six quadrilaterall figures, whereof those which are opposite are parallell.

**XXXI.** A solid figure is said to be inscribed in a solid figure, when all the angles of the figure inscribed are comprehended either within the angles, or in the sides, or in the planes of the figure wherein it is inscribed.

**XXXII.** Likewise a solid figure is then said to be circumscribed about a solid figure, when either the angles, or sides, or planes of the circumscribed figure touch all the angles of the figure which it contains.

PROPOSITION I.



One part AC of a right line cannot be in a plane superficies, and another part CB elevated upward.

Produce AC in the plane directly to F. If you conceive CB to be drawn straight from AC, then two right lines AB, AF, have one common segment AC. *a* Which is impossible. *a* 10 ax. 1.

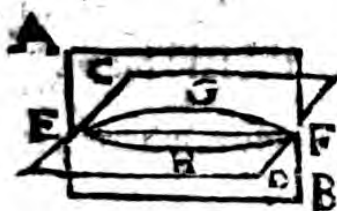
PROP. II.



If two right lines AB, CD, cut one another, they are in the same plane: And every Triangle DEB is in one and the same plane.

For imagine EFG, part of the triangle DEB, to be in one plane, and the part FD-G to be in another. then EF part of the right line ED is in a plane, and the other part elevated upwards. *a* Which is Absurd. Therefore the triangle EDB is in one and the same plane; and so also are the right lines ED, EB; *a* wherefore the whole lines *a* 1. 11. AB, DC, are in one plane. Which was to be Demonstrated.

PROP. III.

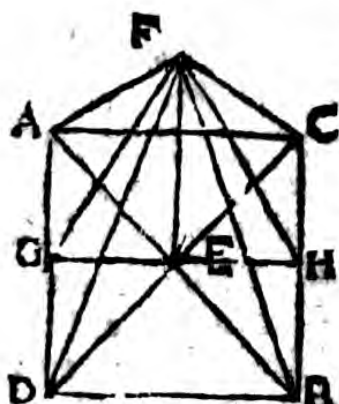


If two planes AB, CD, cut one the other, their common section EF is a right line.

If EF the common section be not a right line, *a* then in *a* 1. post 1. the plane AB draw the right line EGF, *a* and in the plane CD the right line EHF. therefore two right lines EGF, EHF include a superficies. *b* Which is *b* 14 ax. 1. Absurd.

PROP.

## P R O P. I V.



If a right line  $EF$  be at right angles erected upon two lines  $AB$ ,  $CD$ , cutting one the other, at the common section  $E$ ; it shall also be at right angles to the plane  $ACBD$  drawn by the said lines.

Take  $EA, EC, EB, ED$ , equall one to the other, and join the right lines  $AC, CB, BD, AD$ . draw any right line  $GH$  through  $E$ , & join  $FA, FC, FD, FB, FG, FH$ . Because  $AE = EB$ , and  $DE = EC$ , and the angle  $AED = CEB$ , therefore  $AD = CB$ , & likewise  $AC = DB$ . therefore  $AD$  is parallel to  $CB$ , and  $AC$  to  $DB$ . wherefore the angle  $GAE = EBH$ , and the angle  $AGE = EHB$ . But also  $AE = EB$ . therefore  $GE = EH$ , and  $AG = BH$ . whence by reason of the right angles, by the hyp. and so equall, at  $E$ , the bases  $FA, FC, FB, FD$ , are equall. Therefore the triangles  $ADF, FBC$ , are equilaterall one to another. and thence the angle  $DAF = BCF$ . Therefore in the triangles  $AGF, FBH$ , the sides  $FG, FH$  are equall; and so by consequence the triangles  $FEG$  and  $FEH$  are mutually equilaterall. therefore the angles  $FEG, FEH$  are equall, and so right angles. In like manner,  $FE$  makes right angles with all the lines drawn through  $E$  in the plane  $ADBC$ , and is therefore perpendicular to the said plane.

*a* constr.  
b 15. 1.  
c 4. 1.  
d f. 6. 34. 1.  
e 29. 1.

*f* constr.

*g* 16. 1.  
*h* 4. 1.

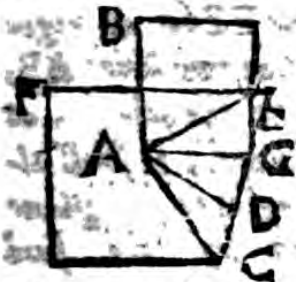
*k* 8. 1.

*l* 4. 1.  
*m* 8. 1.  
*n* 10. def. 1.

*o* 3. def. 11.

P R O P.

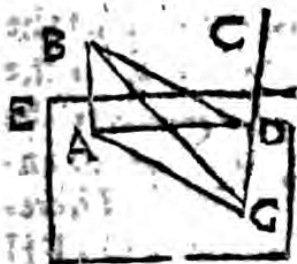
P R O P. V.



If a right line  $AB$  be erected perpendicular to three right lines  $AC$ ,  $AD$ , touching one the other at the common section, those three lines are in the same plane.

For  $AC, AD$ ,  $a$  are in one plane  $a$  2. 11.  
 $FC$ ;  $a$  and  $AD, AE$ , are in one plane  $BE$ . which if you conceive to be severall planes, then let their intersection  $b$  be the right line  $AG$ ; therefore because  $BA$  by the Hypoth. is perpendicular to the right lines  $AC, AD$ .  $c$  and so to the plane  $FC$ ,  $d$  it is also perpendicular to the right line  $AG$ . therefore (since  $a$  that  $AB$  is in the same plane with  $AC, AE$ ) the angles  $BAG, BAE$ , are right angles, and consequently equal, the part & the whole. Which is Absurd.  $b$  3. 11.  $c$  4. 11.  $d$  3. def. 11.

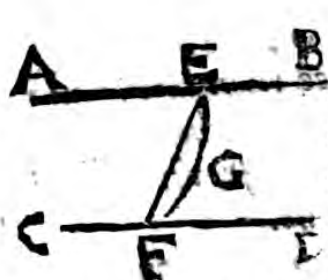
P R O P. VI.



If two right lines  $AB, DC$ , be erected perpendicular to one and the same plane  $EF$ , those right lines  $AB, DC$  are parallel one to the other.

Draw  $AD$ , whereunto let  $DG$   $\perp$   $AB$  be perpendicular in the plane  $EF$ , and join  $BD, BG, AG$ . Being in the triangles  $BAD, ADG$ , the angles  $DAB, ADG$   $a$  are right angles, and  $AB$   $b$   $\perp$   $DG$ , and  $AD$  is common,  $c$  therefore  $BD$  is  $\perp$   $AG$ . whence in the triangles  $AGB, BGD$ , equilaterall one to the other, the angle  $BAG$  is  $d$   $\perp$   $BDG$ ; of which being  $BAG$  is a right angle,  $BDG$  shall be so also. but the angle  $GDC$  is supposed right, therefore the right line  $GD$  is perpendicular to the three lines  $DA, DB, CD$ .  $e$  which are therefore in the same plane  $f$  wherein  $AB$  is. Wherefore since  $AB$  and  $CD$  are in the same plane, and the internall angles  $BAD, CDA$ , are right angles,  $g$   $AB$  and  $CD$  shall be parallels. *W. W. to be Dem.*  $a$  hyp.  $b$  const.  $c$  4. 1.  $d$  8. 1.  $e$  5. 11.  $f$  2. 11.  $g$  8. 16.

P R O P. VII.

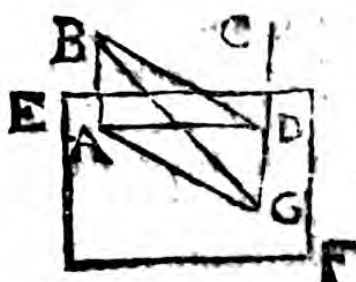


If there be two parallel right lines AB, CD, and any points E, F, be taken in both of them, the line EF which is joined at these points, is in the same plane with the parallels ABCD.

Let the plane in which AB, CD, are, be cut by another plane at the points E, F. then if EF is not in the plane ABCD, it shall not be the common section. Therefore let EGF be the common section; which then is a right line. therefore two right lines EF, EGF, include a superficies. *b* Which is absurd.

a 3. 11.  
b 14. ax. 7.

P R O P. VIII.

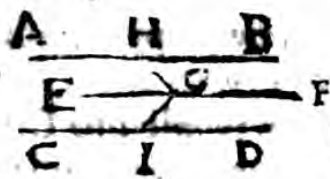


If there be two parallel right lines AB, CD, whereof one AB is perpendicular to a plane EF. then the other CD shall be perpendicular to the same plane EF.

The preparation and demonstration of the sixth of this Book being transferr'd hither; the angles GDA, and GDB are right angles: *a* therefore GD is perpendicular to the plane, wherein are AD, DB (*b* in which also AB, CD, are.) *c* therefore GD is perpendicular to CD. but the angle CDA is also *d* a right angle. *e* therefore CD is perpendicular to the plane EF. *W.W. to be Dem.*

a 4. 11.  
b 7. 11.  
c 3 def. 11.  
d 29. 1.  
e 4. 11.

PROP. IX.



Right lines (AB, CD) which are parallel to the same right line EF, but not in the same plane with it, are also parallel one to

the other.

In the plane of the parallels AB, EF, draw HG perpendicular to EF; also in the plane of the parallels EF, CD, draw IG perpendicular to EF. <sup>a 4. 11.</sup> therefore EG is perpendicular to the plane wherein HG, GI are; and AH, CI are perpendicular to the same plane. <sup>b 8. 11.</sup> therefore AH and CI are parallels. <sup>c 6. 11.</sup> W.W. to be Dem.

PROP. X.



If two right lines AB, AC, touching one another be parallel to two other right lines ED, DF, touching one another, and not being in the same plane, those right lines contain equal angles, BAC, EDF.

Let AB, AC, DE, DF, be equal one to the other, and draw AD, BC, EF, BE, CF. Being <sup>a hyp. and const.</sup> B, DE, <sup>b 33. 1.</sup> are parallels and equal, <sup>c 2. ex 1. & 30. 1.</sup> also BE, AD, are parallels and equal. In like manner CF, AD, are parallels and equal; <sup>d 33. 1.</sup> therefore also BE, FC, are parallels and equal. <sup>e 8. 1.</sup> Therefore BC, EF are equal. Wherefore since the triangles BAC, EDF, are of equal sides one to the other, the angles BAC, EDF shall be equal. W.W. to be Dem.

PROP. XI.



From a point given on high A to draw a right line AI perpendicular to a plane below BC.

In the plane BC draw any line



a 12. 1.  
b 11. 1.

line DE ; to which from the point A *a* draw the perpendicular AF , and *b* likewise FH in the plane BC cutting the said line DE at F ; *a* then let fall AI perpendicular to FH. Which AI shall be perpendicular to the plane BC.

c 31. 1.  
d *confr.*  
e 4. 11.  
f 8. 11.  
g 3. *def.* 11  
h *confr.*  
i 4. 11.

For through I *c* let KIL be drawn parallel to DE. Because DE *d* is perpendicular to AF , and FH, *e* therefore DE shall be perpendicular to the plane IFA. and so also KL *f* is perpendicular to the same plane, *g* therefore the angle KIA is a right angle. but the angle AIF is also *b* a right angle. *l* therefore AI is perpendicular to the plane BC. *W.W. to be Done.*

P R O P. XII.



In a plane given BC , at a point given therein A, to erect a perpendicular line AF.

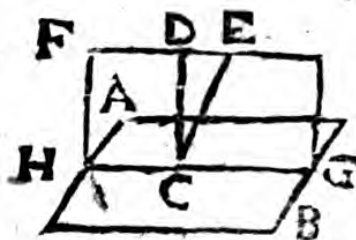
a 11. 11.

b 31. 1.  
c 8. 11.

From some point without the plane, D, *a* draw DE perpendicular to the said plane BC. and joining the points A, E, by a line AE, *b* draw AF parallel to DE. *c* it is apparent that AF is perpendicular to the plane BC. *W.W. to be Done.*

This and the preceding probleme are practically performed by applying two Squires to the point given; as appears by 4. 11.

P R O P. XIII.



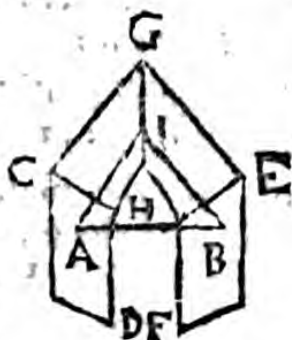
a 6. 11.

At a point given C in a plane given AB, two right lines CD, CE, cannot be erected perpendicular on the same side.

For both CD , and CE , *a* should then be perpendicular to the plane AB , and consequently parallels ; which is repugnant to the definition of parallel lines.

P R O P.

PROP. XIV.



Planes CD, FE, to which the same right line AB is perpendicular, are parallel.

If you deny this; then let the planes CD, FE, meet, so that their common section be the right line GH. in which take any point I, draw to it the right lines IA, IB, in the said planes. where-

by in the triangle IAB, two angles IAB, IBA are right angles. *b* Which is Absurd.

*a* hyp. and 3.  
*c* def 11.  
*b* 17. 1.

PROP. XV.

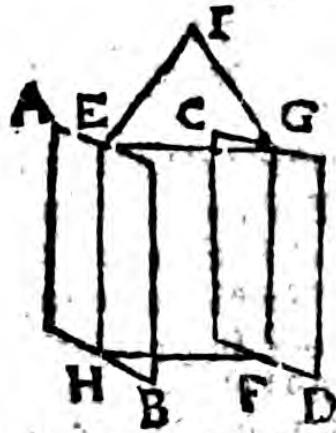


If two right lines AB, AC, touching one the other, be parallel to two other right lines DE, DF, touching one the other, and not being in the same plane with them, the planes BAC, EDF, drawn by those right lines are parallel one to the other.

From A draw AG perpendicular to the plane EF. *b* and let GH, GI be parallel to DE, DF. *c* these also shall be parallel to AB, AC. Therefore since the angles IGA, HGA, *d* are right angles, also CAG, BAG, *e* shall be right angles. *f* therefore GA is perpendicular to the plane BC; but the same is perpendicular to the plane EF. *b* therefore the planes BC, EF, are parallel. *w. w.* to be Dem.

*a* 11. 11.  
*b* 31. 1.  
*c* 30. 1.  
*d* 3. def. 11.  
*e* 19. 1.  
*f* 4. 11.  
*g* constr.  
*h* 14. 11.

P R O P. XVI.



If two parallel planes AB, CD, be cut by some other plane HEIGF, their common sections EH, GF are parallel one to the other.

For if they be conceived to be otherwise; being in the same plane that cuts them, they will meet some where, if produced; suppose in I;

11. 11.

wherefore since the whole lines HEI, FGI are in the planes AD, CD, being produced, the planes also shall meet. *contrary to the Hyp.*

P R O P. XVII.



If two right lines ALB, CMD, be cut by parallel planes EF, GH, IK; they shall be cut proportionally, (AL. LB. :: CM. MD.)

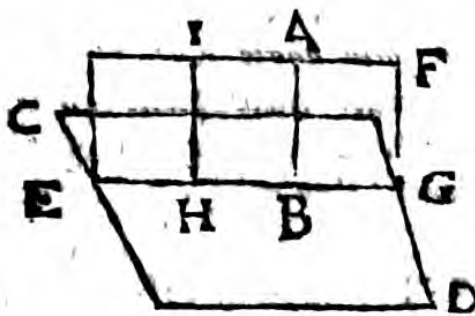
Let the right lines AC, BD, be drawn in the planes EF, IK; as also AD passing through the plane GH in the point N. and join NL, LM. the planes of the triangles ADC,

16. 11.  
2. 6

ADB, make the sections BD, LN, and AC, NM a parallels. Therefore AL. LB :: AN. ND :: CM. MD. *W.W. to be Dem.*

P R O P.

PROP. XVIII.

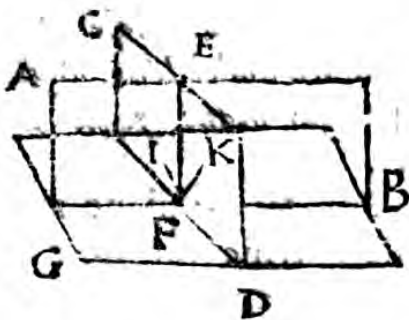


If a right line  $AB$  be perpendicular to some plane  $CD$ , all the planes extended by that right line  $AB$  ( $EF$ , &c.) shall be perpendicular to the same plane  $CD$ .

Let there be some plane  $EF$  drawn by  $AB$ , making the section  $EG$  with the plane  $CD$ ; from some point whereof  $H$ ,<sup>a</sup> draw  $HI$  parallel to  $AB$  in the plain  $EF$ ; <sup>b</sup> then shall  $HI$  be perpendicular to the plane  $CD$ , and so likewise any other lines, that are perpendicular to  $EG$ .<sup>c</sup> therefore the plane  $EF$  is perpendicular to the plane  $CD$ ; and by the same reason any other planes drawn by  $AB$  shall be perpendicular to  $EF$ . *W.W. to be Dem.*

<sup>a</sup> 31. 1.  
<sup>b</sup> 8. 11.  
<sup>c</sup> 4. def. 11.

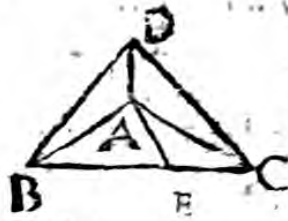
PROP. XIX.



If two planes  $AB, CD$ , cutting one the other, be perpendicular to some plane  $GH$ , their line of common section  $EF$  shall be perpendicular to the same plane ( $GH$ .)

Because the planes  $AB, CD$ , are taken perpendicular to the plane  $GH$ , it appears by 4. def. 11. that out of the point  $F$  there may be drawn in both planes  $AB, CD$ , a perpendicular to the plane  $GH$ . which shall be <sup>a</sup> but one; and therefore the common section of the said planes. <sup>a</sup> 13. 11. *W.W. to be Dem.*

PROP. XX.



If a solid angle ABCD be contained under three plane angles, BAD, DAC, BAC, any two of them howsoever taken are greater then the third.

If the three angles are equal, the assertion is evident; if unequal, then let the greatest be BAC; from whence take away BAE = BAD, and make AD = AE; and also draw BE, EC, BD, DC.

Because the side BA is common, and AD = AE; and the angle BAE = BAD, thence is BE = BD. but BD + DC is < BC. therefore DC < EC. Wherefore since AD = AE, and the side AC is common, and DC < EC. the angle CAD shall be < EAC. g therefore the angle BAD + CAD < BAC. *W.W. to be Dem.*

u 23. 1.

b constr.

c 4. 1.

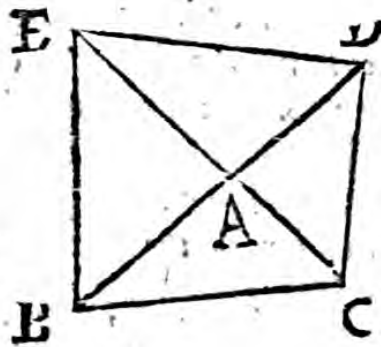
d 20. 1.

e 5. ax. 1.

f 25. 1.

g 4. ax. 1.

PROP. XXI.



Every solid angle A is contained under lesse angles then four plane right angles.

For let a plane anywise cutting the sides of the solid angle A make a many-sided figure BCDE, and as many triangles ABC, ACD, ADE, AEB. I denote all the angles of the polygone by X; and I term the summe of the angles at the bases of the triangles Y. wherefore X + 4 Right ang. = Y + A. but being that, (of the angles at B) b the angle ABE + ABC is < CBE, and the same is true also of the angles at C, at D, and at E, c it is manifest that Y is < X. and

a 32. 1. and

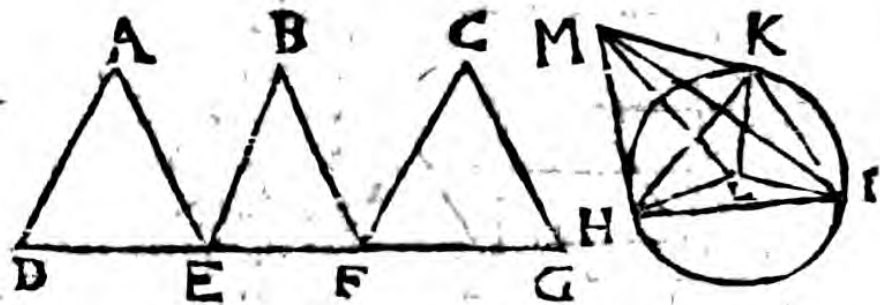
feb. 32. 1.

b 20. 1.

c 5. ax. 1.



PROP. XXIII.



\* 21. 11.

To make a solid angle MHIK of three plane angles A, B, C, whereof two howsoever taken are greater than the third. \* But it is necessary that those three angles be lesse then four right angles.

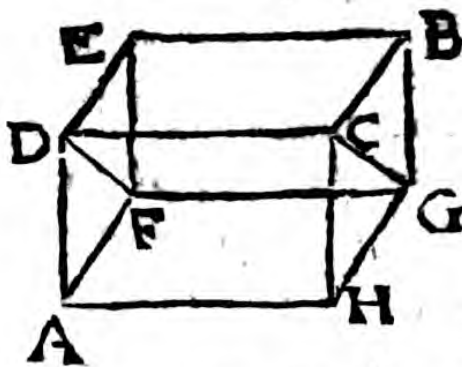
a 23 11. and  
21. 1.  
b 5. 4.  
\* See Cla-  
vius.  
c sch. 47. 1.  
d 12. 11.  
e 3. def 11.  
f 47. 1.  
g confir.

Make AD, AE, BE, BF, CF, CG, equal one to the other; and of the subtended lines DE, EF, FG (that is, of the equal lines HI, IK, KH) <sup>a</sup> make the triangle HKI; about which <sup>b</sup> describe the circle LHKI. \* But because AD is  $\perp$  HL. <sup>c</sup> let ADq be  $\equiv$  HLq + LMq. <sup>d</sup> and let LM be perpendicular to the plane of the circle HKI. and draw HM, KM, IM. wherefore since the angle HLM <sup>e</sup> is a right angle, <sup>f</sup> thence is MHq  $\equiv$  HLq + LMq <sup>g</sup>  $\equiv$  ADq. therefore MH  $\equiv$  AD. By the same reason MK, MI, AD (that is AE, EB, &c.) are equal; therefore since HM  $\equiv$  AD, and MI  $\equiv$  AE, and DE <sup>h</sup>  $\equiv$  HI, <sup>k</sup> the angle A shall be  $\equiv$  HMI, <sup>k</sup> as likewise the angle IMK  $\equiv$  B, <sup>k</sup> and the angle HMK  $\equiv$  C. wherefore a solid angle is made at M of the three given plane angles. *W.W. to be Done.* AD is assumed to be  $\perp$  HL. But this is manifest. For if AD be  $\equiv$  or  $\perp$  HL, then is the angle A  $\equiv$  <sup>m</sup> or  $\perp$  HLI. In like manner shall B be equal or  $\perp$  HLK, and C  $\equiv$  or  $\perp$  KLI. wherefore A + B + C \* shall either equal or exceed four right angles. *contrary to the Hypoth.* therefore rather let AD be  $\perp$  HL. *W.W. to be Dem.*

h confir.  
k 8. 1.

l confir. and  
8. 1.  
m 21. 1.  
\* 4 cor. 13.  
1.

PROP. XXIV.



If a solid AB be contained under parallel planes, the opposite planes thereof (AG, DB, &c.) are like and equal parallelograms.

The plane AC cutting the parallel planes AG, BB, & makes the

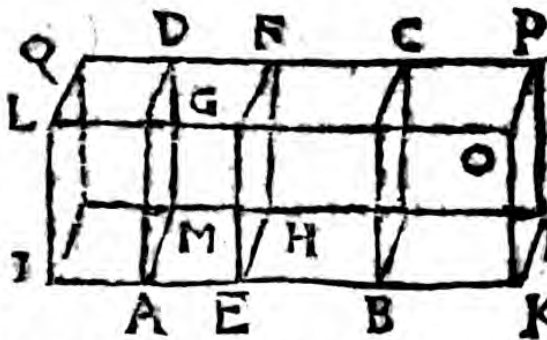
sections AH, DC, parallels. and by the same reason AD, HC are parallels. Therefore ADCH is a pgr. By the like argument the other planes of the parallelepipedon are pgrs. wherefore being AF is parallel to HG, and AD to HC, the angle FAD shall be = CGH. therefore because AF = HG, and AD = HC, and so AF . AD :: HG . HC. the triangles FAD, GHC, are like and equal; and consequently the pgrs. AE, HB are like and equal. and the same may be shewn of the rest opposite planes. therefore, &c.

a 16. 11.

b 35. def. 11.  
c 10. 11.  
d 34. 1.  
e 7. 5.  
g 6. 6.  
h 4. 1.

k 6. ex. 11.

PROP. XXV.



If a solid Parallelepipedon ABCD be cut by a plane EF parallel to the opposite planes AD, BC; then as the base AH is to the base BH, so shall

solid AHD be to solid BHC.

Conceive the Parallelepipedon to be extended on either side. and take AI = AE, and BK = EB, and put the planes IQ, KP, parallel to the planes AD, BC; then the pgrs. IM, AH, and DL, DG, and IQ, AD, EF, &c. are like and equal. wherefore the Parallelepipedon AQ is = AF; and by the same reason

a 36. 1. and  
1 def. 6.  
b 24. 11. 1.  
c 10. def. 11.



d 24. 11. and  
9 def. 11.  
e 6. def. 5.

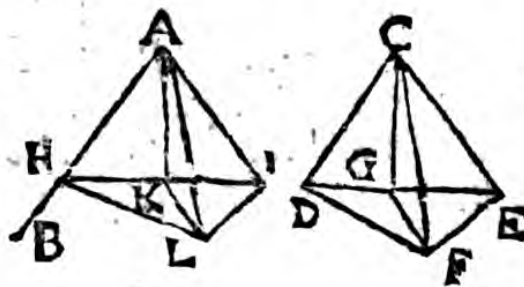
reason the Parallelepipedon  $BP = BF$ . therefore the solids  $IF, EP$  are as multiplex of the solids  $F, EC$ , as the bases  $IH, KH$ , are of the bases  $AH, BH$ . And if the basis  $IH$  be  $\square, =, \triangleright KH$ , & likewise shall the solid  $IE$  be  $\square, =, \triangleright EP$ . consequently  $AH.BH :: AF.EC$ . *W.W. to be Dem.*

The same may be accommodated to all sort of prisms, whence

Coroll.

If any prisme whatsoever be cut by a plane parallel to the opposite planes, the section shall be a figure equall and like to the opposite planes.

PROP. XXVI.



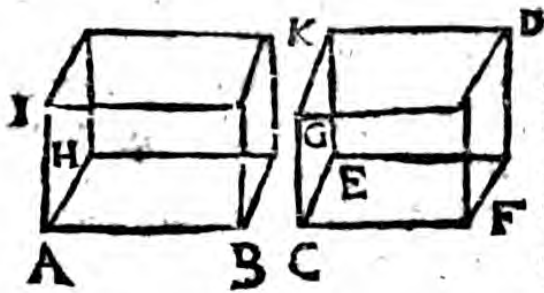
Upon a right line given  $AB$ , and at a point given in it  $A$ , to make a solid angle  $A-HIL$  equall to a solid angle given  $CDEF$ .

e 11. 11.

From some point  $F$  draw  $FG$  perpendicular to the plane  $DCE$ , and draw the right lines  $DF, FE, EG, GD, CG$ . Make  $AH = CD$ , and the angle  $HAI = DCE$ , and  $AI = CE$ ; and in the plane  $HAI$  make the angle  $HAK = DCG$ , and  $AK = CG$ . then erect  $KL$  perpendicular to the plane  $HAI$ , and let  $KL = GF$ . & draw  $AL$ : then  $AHIL$  shall be a solid angle equall to that given  $CDEF$ . For the construction of this do's wholly resemble the framing of that, as may easily appear to any that examine it.

PROP.

PROP. XXVII.

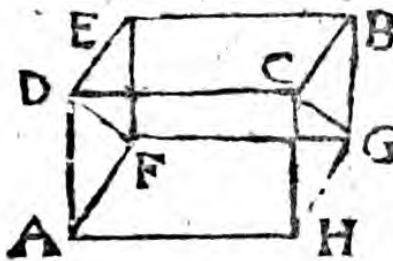


Upon a right line given AB to describe a parallelepipedon AK, like, and in like manner situate, with a solid parallelepipedon given CD.

Of the plane angles, BAH, HAI, BAI, which are equal to FCE, ECG, FCG, *a* make the solid angle A equal to the solid angle C. also *b* make FC. CE :: BA. AH. *b* and CE. CG :: AH. AI (*c* whence of equality FC. CG :: BA. AI) and finish the parallelepipedon AK, which shall be like to that which is given.

For by the construction, the Pgr. *d* BH is like to FE, and *d* HI to EG, and *d* BI to FG, & *e* so the opposites of these to the opposites of them: therefore the six planes of the solid AK are like to the six planes of the solid CD, *f* and consequently AK, CD, are like solids. *W.W. to be Done.*

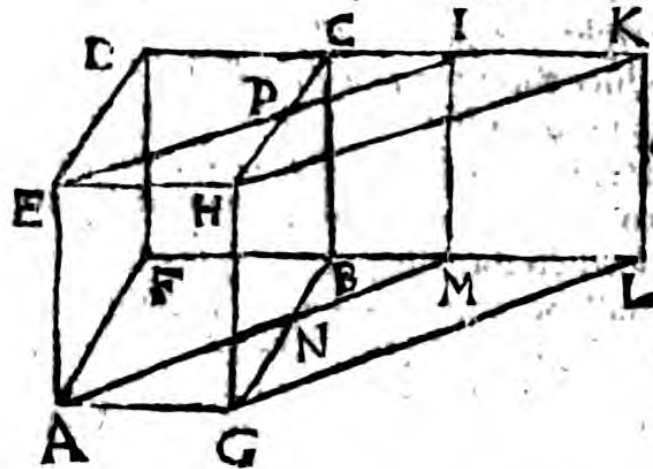
PROP. XXVIII.



If a solid parallelepipedon AB be cut by a plane FGCD drawn by the diagonall lines DF, CG, of the opposite planes AE, HB, that solid AB shall be equally bisected by the plane FGCD.

For because DC, FG, are *a* equal and parallels, *b* the plane FGCD is a Pgr. and being *a* the Pgrs. AB, HB, are equal and like, *b* also the triangles AFD, HGC, CGB, DFE are equal and like. But the Pgrs. AC, AG, are equal and like to FB and FD. therefore all the planes of the prisme FGCD AH are equal and like to all the planes of the prisme FG-CDEB, and *c* consequently this prisme is equal to that. *W.W. to be Dem.*

PROP.

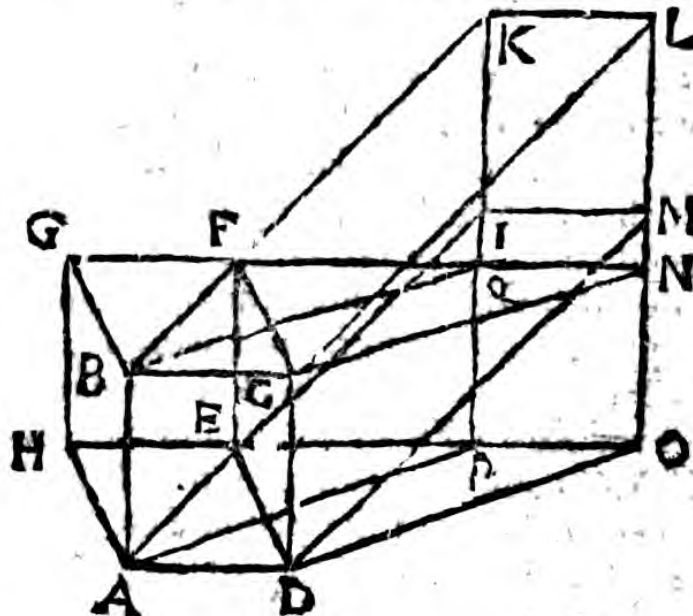


*solid parallelepipeds AGHEFBCE, AGHEMLKI, being constituted upon the same base AGHE, and in the same height, whose insisting lines AF, AM, are placed in the same right lines AG, FL, are equal one to the other.*

*For if from the equal prisms AFMEDI, GB-LHCK, the common prisme NBMPCI be taken away, and the solid AGNEHR be added, the Parallelepiped. AGHEFBCE shall be = AGHEMLKI. W.W. so be Dem.*

*a. i. e. be-  
tween the pa-  
rallel planes  
AGHE,  
FLKD. and  
so understand  
it in the foll.  
a 10. def. 11.  
and 35. 1.  
b 3. and 22  
or 11.*

PROP. XXX.

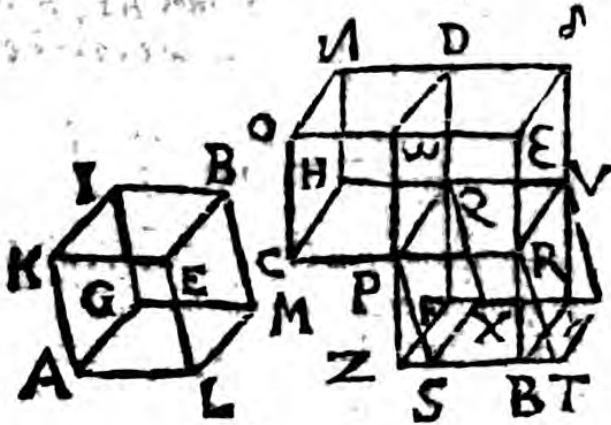


*Solid parallelepipeds ADBCHEFG, ADCBIMLK being*

being constituted upon the same base  $ADFC$ , and in the same height, whose insisting lines  $AH, AI$  are not placed in the same right lines, are equal one to the other.

For produce the right lines  $HEO, GFN,$  and  $LMO, KIP$ ; and draw  $AP, DO, BQ, CN$ . <sup>a 34. 1.</sup> then shall  $DC, AB, HG, EF, PQ, ON$  be as well equal and parallel one to the other as  $AD, HE, GF, BC, KL, IM, QN, PO$ . <sup>b 29. 11.</sup> wherefore the parallelepipedon  $ADCB-PONQ$  shall be equal to either parallelepipedon  $ADCBHEFG, ADCBIMLK$ ; and <sup>c 1. ex. 1.</sup> consequently these two are equal one to the other. *W. W. to be Dem.*

PROP. XXXI.



Solid parallelepipedons,  $ALEKGMBI, CPwOHQ-DN$ , being constituted upon equal bases  $ALEK, CPwO$ , and <sup>\*</sup> in the same height are equal, one to the other.

First, let the parallelepipedons  $AB, CD$ , have the sides perpendicular to the bases. and at the side  $CP$  being produced, <sup>a</sup> make the Pgr.  $PRTS$  equal and like to the pgr.  $KELA$ . <sup>b</sup> and so the parallelepipedon  $PRTSQVYX$  equal and like to the parallelepipedon  $AB$ . Produce  $OwE, NDd, wPZ, DQF, ERB, dVz, TSZ, YXF$ ; and draw  $Ez, Bz, ZF$ .

The planes  $OwN, CRVH, ZTYF$ . <sup>c</sup> are parallels one to the other; <sup>d</sup> and the Pgrs.  $ALEK, CDwO, PRTS, PRBZ$  are equal. Therefore since the parallelepipedon  $CD.PVw$  :: Pgr.  $Cw (PRBZ) Pw$  ::

<sup>\*</sup> by height understand the perpendicular drawn from the plane of the base to the opposite plane.

<sup>a</sup> 18. 6.  
<sup>b</sup> 27. 11. and 10 def. 11.  
<sup>c</sup> 30. def. 11.  
<sup>d</sup> hyp. and 35. 1.

<sup>e</sup> 25. 11.

pa-

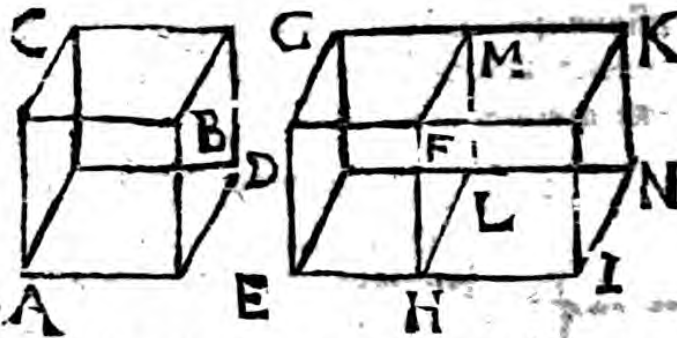
f 9. 5.  
g 19. 11.  
h 100. fr.

parallelepipedon PRBZQV, F.PV do; the parallelepipedon CDf shall be = PRBZQV, Fg = PRVQSTYX b = AB. W.W. to be Dem.

k 19. 11.  
m 1. ax. 1.

But if the parallelepipedons AB, CD, have sides oblique to the base, then on the same bases and in the same height place parallelepipedons whose sides are perpendicular to the base. & They shall be equal to one another, and those that are oblique whence also the oblique parallelepipedons AB, CD are equal. W.W. to be Dem.

P R O P. XXXII.

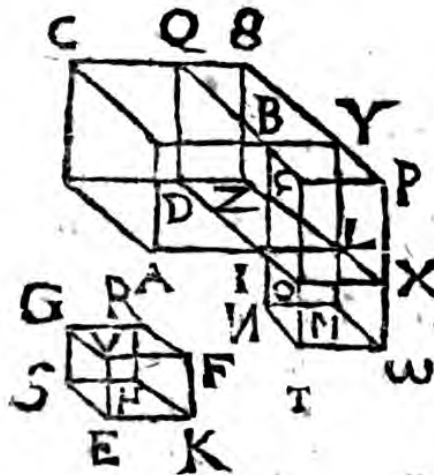


Solid parallelepipedons ABCD, EFGH, of the same height, are one to the other, as their bases, AB, EF.

45. 1.  
b 31. 1.  
c 31. 11.  
d 25. 11.

Produce EHI, a and make the pgr. FI = AB, and b complet the parallepp. FINM. It is clear that the parallepp. FINM. (c ABCD.) EFGH d :: FI (AB) EF. W.W. to be Dem.

P R O P. XXXIII.



23. 8.

Like solid parallelepipedons, ABCD, EFGH, are in tripled proportion one to the other, of that in which their homologous sides or of like proportion AI, EK, are.

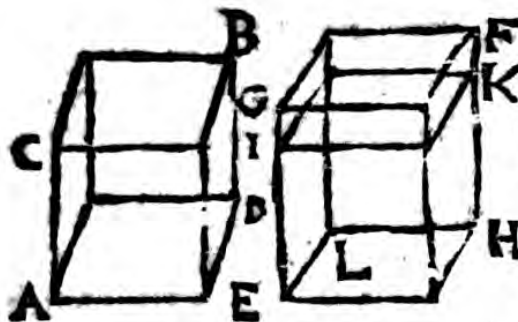
Produce the right lines AIL, DIO, BIN, and a make IL, IO, IN, equal to

to EK, KH, KF, <sup>b</sup> and so the parallepp. IXMT equall <sup>b 27. 11.</sup>  
 and like to the parallepp. EFGH. <sup>c</sup> Let the pa- <sup>c 31. 1.</sup>  
 rallepps. IXPB, DLYQ be finished. <sup>d</sup> Then shall be <sup>d 17. 9.</sup>  
**AL:IL (EK) :: DI:IO (HK) :: BI:IN.KF.** <sup>e</sup> that is <sup>e 1. 6.</sup>  
 the Pgr. AD.DL :: DL.IX :: BO.IT. <sup>f</sup> *fj.e.* the paral- <sup>f 32. 11.</sup>  
 lepp. ABCD. DLQY :: DLQY. IXBP :: IXBP.IX-  
 MT. (<sup>g</sup> EFGH.) <sup>g</sup> therefore the proportion of ABCD <sup>g const.</sup>  
 to EFGH is triple of the proportion of ABCD <sup>h 10. def. 5.</sup>  
 to DLQY, <sup>k</sup> or of AI to EK. *W.W. to be Dem.* <sup>k 1. 6.</sup>

Coroll.

Hence it appears that if four right lines be conti-  
 nually proportionall, as the first is to the fourth, so  
 is a parallelepipedon described on the first to a pa-  
 rallelepipedon described on the second, being like  
 and in like manner described.

P R O P. XXXIV.



In equall solid pa-  
 rallelepipedons AD-  
 CB, EHGK, the bases  
 and altitudes are reci-  
 procall (AD. EH ::  
 EG. AC.) And solid  
 parallelepipedons, AD-  
 CB, EHGK, whose

bases and altitudes are reciprocally, are equall.

First, let the sides CB, GE be perpendicular to the  
 bases; then if the altitudes of the solids are equall,  
 the bases also shall be equall. and the thing is clear.  
 But if the altitudes are unequal, from the greater  
 EG, take EI = AC, and at I draw the plane IK  
 parallel to the base EH. then

1. Hyp. AD.EH <sup>c</sup> :: parallepp. ADCB.EHIK <sup>d</sup> ::  
 parallepp. EHGK. EHIK <sup>e</sup> :: GL. IL <sup>e</sup> :: GE. IE.  
 (fAC.) <sup>g</sup> it is plain therefore that AD.EH :: GE.AC.  
*W.W. to be Dem.*

2. Hyp. ADCB.EHIK <sup>b</sup> :: AD.EH <sup>k</sup> :: EG.EI <sup>l</sup> ::  
 GL. IL <sup>m</sup> :: parallepp. EHGK. EHIK, where

T

fore

<sup>a</sup> 27. 11.  
<sup>b</sup> 31. 1.  
<sup>c</sup> 32. 11.  
<sup>d</sup> 17. 9.  
<sup>e</sup> 1. 6.  
<sup>f</sup> const.  
<sup>g</sup> 11. 5.  
<sup>h</sup> 32. 11.  
<sup>i</sup> hyp.  
<sup>j</sup> 1. 6.  
<sup>k</sup> 32. 11.  
<sup>l</sup> 9. 5.

fore the parallelepipedon  $ADCB = EHGF$ . *W.W. to be Dem.*

Moreover, let the sides be oblique to the bases, and erect right parallelepipedons upon the same bases in the same altitude; the oblique parallepps shall be equall to them. Wherefore since by the first part, the bases and altitudes of those be recipocall, the bases and altitudes of these also shall be recipocall. *W.W. to be Dem.*

*Coroll.*

*All that hath been demonstrated of parallelepipedons in the 29, 30, 31, 32, 33, 34. Prop. does also agree to triangular prismes, which are half parallelepipedons, as appears by Prop. 28. Therefore,*

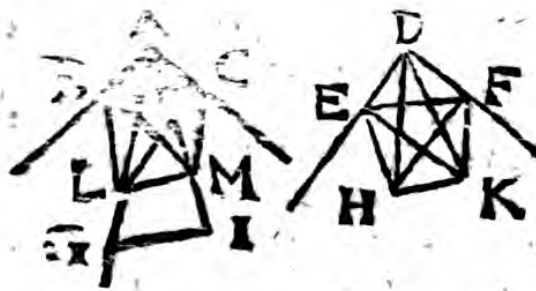
1. Triangular prismes are of equal height with their bases.

2. If they have the same or equall bases and the same altitude, they are equall.

3. If they be like, their proportion is treble to that of their sides of like proportion.

4. If they be equall, their bases and altitudes are recipocall; and if their bases and altitudes be recipocall, they are also equall.

P R O P. XXXV.



If there be two plane angles  $BAC$ ,  $EDF$ , equall, and from the points of those angles two right lines  $AG$ ,  $DH$ , be elevated on high, con-

teining equall angles with the lines first given, each to his correspondent angle (the angle  $GAB = HDE$ , and  $GAC = HDF$ .) and if in those elevated lines  $AG$ ,  $DH$ , some points be taken,  $G$ ,  $H$ ; and from these points perpendicular lines  $GI$ ,  $HK$ , drawn to the planes  $BAC$ ,  $EDF$ , in which the angles first given are, and right lines

lines  $AI, DK$ , be drawn to the angles first given from the points  $I, K$ , which are made by the perpendiculars in the planes; those right lines with the elevated lines  $AG, DH$  shall contain equall angles  $GAM, HDK$ .

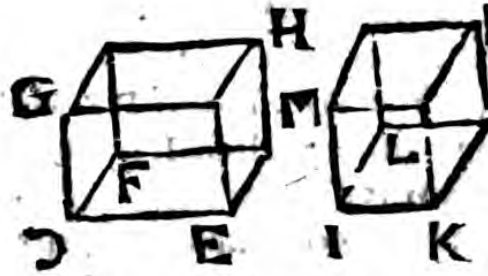
Make  $DH, AL$ , equall; and  $GI, LM$  parallels. and  $MC$  to  $AC$ ,  $MB$  to  $AB$ ,  $KF$  to  $DF$ ,  $KE$  to  $DE$  perpendicular; and draw the right lines  $BC, LB, LC$ , and  $EF, HF, HE$ ; <sup>a 8. 11.</sup> and  $LM$  is perpendicular to the plane  $BAC$ ; <sup>b 3. def. 11.</sup> wherefore the angles  $LMC, LMA, LMB$ ; and by the same reason the angles  $HKF, HKD, HKE$  are right angles. Therefore  $ALq \text{ e} = LMq + AMq \text{ e} = LMq + CMq + ACq \text{ e} = LCq + ACq$ . <sup>c 47. 11.</sup> <sup>d 48. 1.</sup> therefore the angle  $ACL$  is a right angle. Again  $ALq \text{ e} = LMq + MAq \text{ e} = LMq + BMq + BAq \text{ e} = BLq + BAq$ . <sup>e 47. 11.</sup> therefore the angle  $ABL$  is also a right angle. By the like inference the angles  $DFH, DEH$  are right angles; <sup>f 16. 11.</sup> therefore  $AB = DE$ , <sup>g 4. 1.</sup> and  $BL = EH$ , <sup>h 1. ax. 1.</sup> and  $AC = DF$ , and <sup>k 16. 1.</sup>  $CL = FH$ . <sup>l 47. 1.</sup> therefore also  $BC = EF$ ; <sup>m const.</sup> and the angle  $ABC = DEF$ , <sup>n 47. 1 and 3. ax.</sup> and the angle  $ACB = DFE$ . <sup>o 8. 11.</sup> whence the other right angles  $CBM, BCM$ , are equall to the other  $FEK, EFK$ . <sup>1</sup> therefore  $CM = FK$ , and so also  $AM = DK$ . therefore if from  $LAq \text{ m} = HDq$  be taken away  $AMq = DKq$ , <sup>2</sup> there remains  $LMq = HKq$ . wherefore the triangles  $LAM, HDK$  are equilaterall one to the other; <sup>3</sup> therefore the angle  $LAM = HDK$ . *W.W. to be Dem.*

*Coroll.*

Therefore, if there be two plane angles equal, from whose points equal right lines be elevated on high containing equall angles with the lines first given, each to each; perpendiculars drawn from the extreme points of those elevated lines to the planes of the angles first given, are equall one to the other; *viz.*  $LM = HK$ .



PROP. XXXVI.



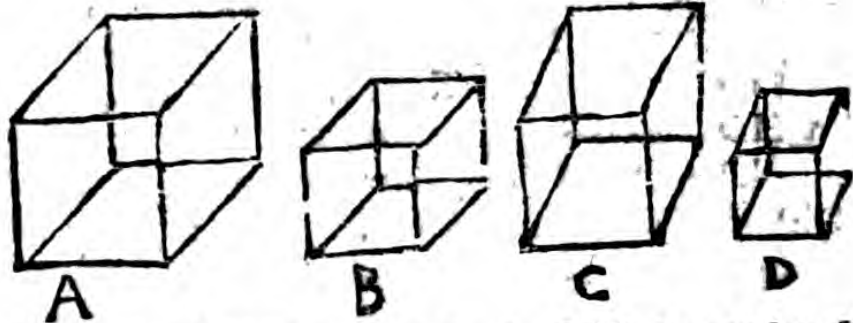
If there be three right lines DE, DG, DF proportionall, the solid parallelepipedon DH made of them, is equall to the solid parallelepipedon IN made of the middle line DG (IL) which is also equilaterall, and equiangular to the said parallelepipedon DH.

a hyp. b 14. 6.

c 31. 11.

Because DE.IK  $a ::$  IL.DF.  $b$  the pgr. LK shall be  $=$  FE. and by reason of the equality of the plane angles at E and I, and of the lines GD, IM, also the altitudes of the parallelepipedons are equall by the prec. Coroll.  $c$  therefore the parallepp. are equall one to the other. *W.W. to be Dem.*

PROP. XXXVII.



If there be four right lines A, B, C, D, proportionall, the solid parallelepipedons A, B, C, D being like, and in like sort described from them, shall be proportionall. And if the solid parallelepipedons, being like and in like sort described, be proportionall (A. B  $::$  C. D.) then those right lines A, B, C, D, shall be proportionall.

a 33. 12. b sch. 23. 5.

For the proportions of the parallepp.  $a$  are triple of those of the lines; therefore if A. B  $::$  C. D.  $b$  then shall the parallepp. A. parallepp. B  $::$  parallepp. C. parallepp. D. and so also contrarily.

P R O P . X X X V I I I .



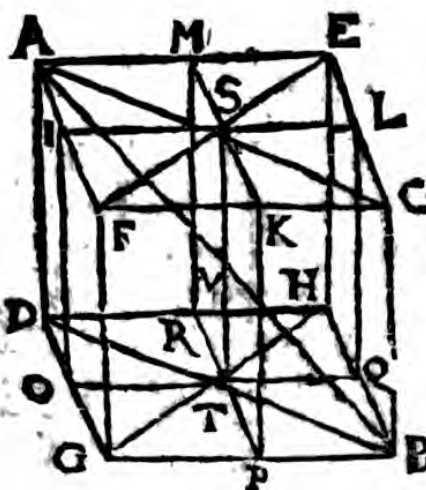
If a plane AB be perpendicular to a plane AC, and a perpendicular line EF

be drawn from a point E in one of the planes (AB) to the other plane AC, that perpendicular EF shall fall upon the common section of the planes AD.

If it be possible, let F fall without the intersection AD. and in the plane AC draw FG perpendicular to AD, and join EG. The angle FGE is a right angle, and EFG is supposed to be such also; therefore two right angles are in the triangle EFG. which is Absurd.

a 12. 1.  
b 4. and 3.  
def. 11.  
c 17. 1.

P R O P . X X X I X .



If the sides (AE, FC, AF, EC, and DH, GB, DG, HB) of the opposite planes AC, DB, of a solid parallelepiped AB, be divided into two equal parts, & planes ILQO, PKMR, be drawn through their sections, the common section of the planes ST, and the diameter of the solid parallelepiped AB shall divide one the other into two equal parts.

Draw the right lines SA, SC, TD, TB. Because the sides DO, OT are equal to the sides BQ, QT, and the alternate angles TOD, TQB equal also, the bases DT, TB, & the angles DTO, BTQ are equal. & therefore DTB is a right line. & so in like manner is ASC. Moreover AD is parallel & equal to FG, & FG to CB, & thence AD is parallel & equal to CB; & consequently AC to DB. wherefore AB

a 34. 1.  
b 29. 1.  
c 4. 1.  
d sch. 15. 1.  
e 34. 1.  
f 9. 11. and  
1. ax.  
g 33. 1.  
h 7. 11.

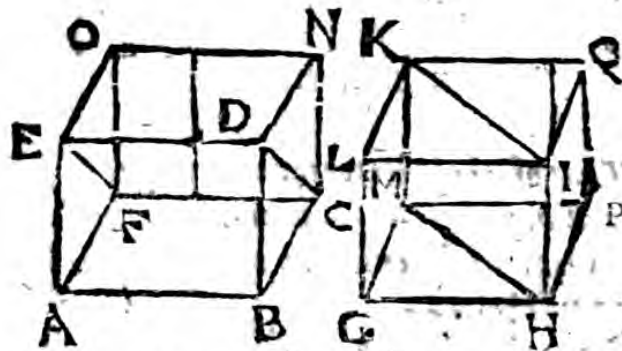
and  $ST$  are in the same plane  $ABCD$ . Therefore since the vertical angles  $AVS, BVT$ , & the alternate angles  $ASV, BTV$  are equal;  $\therefore$  and  $AS = BT$ ; therefore shall  $AV = BV$ , and  $SV = VT$ . *W.W. to be Dem.*

by ax. 1.  
16. 1.

Coroll.

Hence in every parallelepipedon all the diameters bisect one another in one point,  $V$ .

PROP. XL.



If two prisms  $ABCFED, GHMLIK$ , be of equal altitude, whereof one hath its base  $ABCF$  a parallelogram, and the other  $GHM$  a triangle; and if the parallelogram  $ABCF$  be double to the triangle  $GHM$ ; those prisms  $ABCFED, GHMLIK$  are equal.

a 31. 11.  
b 34. 1. and  
7. ax.  
c hyp.  
d 18. 11.  
e 7. ax. 1.

For if the parallelepps.  $AN, GQ$ , be completed, they shall be equal, because of the equality of the bases  $AC, GP$ , & of the altitudes.  $\therefore$  therefore also the prisms, the halves thereof shall be equal. *W.W. to be Dem.*

Schol.

Andr. Tacq.

From the preceding demonstrations, the dimension of triangular prisms, and quadrangular, or parallelepipedons, is learnt; viz. by multiplying the altitude into the base.

As if the altitude be 10 foot, and the base 100 square foot (the base may be measured by *feb. 35. 1.* or by *41. 1.*) then multiply 100 by 10. and 1000 cubic

cubic foot shall be produced for the solidity of the prisme given.

For as a rectangle, so also is a right parallelepipedon produced of the altitude multiplied into the base. Therefore every parallelepipedon is produced of the altitude multiplied into the base, as appears by 31. of this Book.

Moreover, since the whole parallelepipedon is produced of the altitude drawn into the base, the half thereof (that is, a triangular prisme) shall be produced of the altitude drawn into half the base, namely the triangle.

*An Advertisement.*

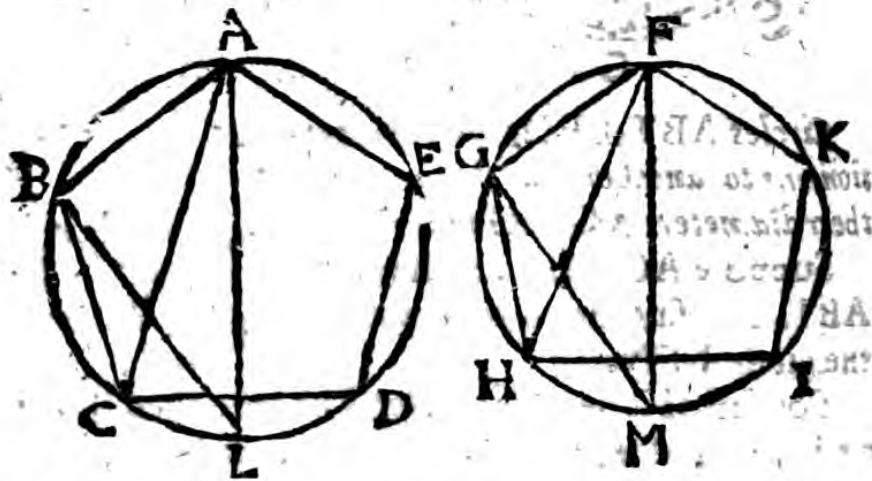
*Obs.* That of those letters which denote a solid angle, the first is alwayes at the point in which the angle is; but of those letters which denote a pyramide, the last is at the supreme point thereof.

Ex. gr. the solid angle ABCD is at the point A; and the supreme point of the pyramide BCDA is at the point A. and the base is the triangle BCD.

*The End of the eleventh Book.*

THE TWELFTH BOOK  
OF  
EUCLIDE'S ELEMENTS.

PROPOSITION I.



*I*ke polygonous figures  $ABCDE, FGHK,$  described in circles  $ABD, FGI,$  are one to another, as the squares described of the diameters of the circles  $AL, FM.$

Draw  $AC, BL, FH, GM.$  Because  $a$  the angle  $ABC = FGH,$  and  $AB \cdot BC :: FG \cdot GH,$   $b$  therefore shall the angle  $ACB$  ( $c$   $ALB$ ) be  $= FHG$  ( $c$   $FMG.$ ) but the angles  $ABL, FGM$   $d$  are right and so equal;  $e$  therefore the triangles  $ABL, FGM$  are equiangular.  $f$  wherefore  $AB \cdot FG :: AL \cdot FM,$   $g$  therefore  $ABCDE \cdot FGHK :: AL \cdot FM.$

*Coroll.*

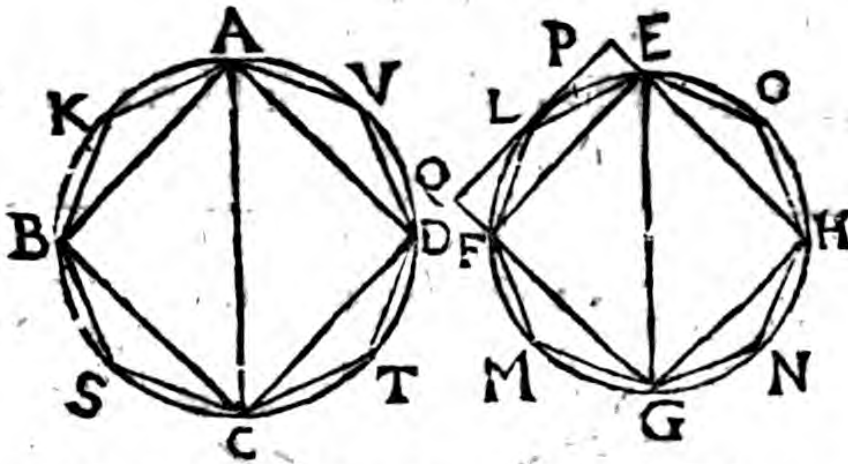
Hence (because  $AB \cdot FG :: AL \cdot FM :: BC \cdot GH,$  &c.) the contents of like polygonous figures described in a circle are in  $b$  proportion as the diameters.

**PROP.**

$a$  1 def. 6.  
 $b$  6. 6.  
 $c$  21. 3.  
 $d$  31. 3.  
 $e$  32. 3.  
 $f$  cor. 4. 6.  
 $g$  22. 6.

$b$  1. 12. and  
13. 7.

PROP. II.



Circles ABT, EFN, are in proportion one to another, as the squares of their diameters AC, EG are.

Suppose  $AC \cdot EG ::$  the circle ABT. I. I say then I is equal to the circle EFN.



For first, if it be possible, let I be lesse then the circle EFN, and let K be the excess or difference. Inscribe the square EFGH in the circle EFN, *a* it being the half of a circumscribed square, and so greater then the semicircle. *b* Divide equally in two the arches EF, FG, GH, HE, and at the points of the divisions join the right lines EL, LF, &c. at L draw the tangent PQ (*c* which is parallel to EF) and produce HEP, GFQ. then is the triangle ELF *d* the half of the pgr. EPQF, and so greater then the half of the segment ELF; and in like sort the rest of those triangles exceed the halves of the rest of the segments. And if the arches EL, LF, FM, &c. be again bisected, and the right lines joined, the triangles will likewise exceed the half of the segments. Wherefore if the square EFGH be taken from the circle EFN, and the triangles from the other segments, and this be done continually, at length *e* there will remain some magnitude lesse then K. Let us have gone so farre, namely to the segments EL, LF, FM, &c. taken together

*a* *schol.* 7. 4.

*b* 30. 3.

*c* *schol.* 27. 3.  
*d* 41. 1.

*e* 1. 10.

f hyp. and 32  
ax  
g 30. 3. and  
i. post. 1.

h 1. 12.  
k hyp.  
l 9. ax. 1.  
m 14 5.

n hyp.

o 14 5.  
p 11 5.

gether lesse then K. Therefore I (f the circle EFN — K)  $\supset$  the polyg. ELFMNHO (the circle EFN — the segm. EL + LF, &c.) In the circle ABT g conceive a like polygon AKBSCTDV inscribed. therefore since AKBSCTDV. ELFMGNHO b :: ACq. EGq k :: the circle ABT. I. and the polyg. AKBSCTDV l  $\supset$  the circle ABT. the polyg. ELFMGNHO m shall be  $\supset$  I. but before, I was  $\supset$  ELFMGNHO. which is repugnant.

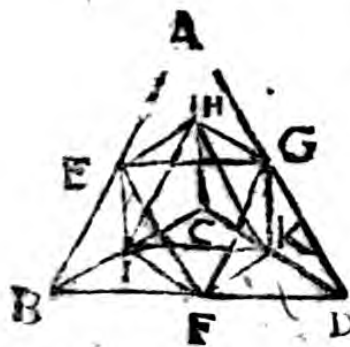
Again, if it be possible, let I be  $\supset$  the circle EFN. Therefore because ACq. EGq n :: the circle ABT. I; and inversely I. the circle ABT :: EGq. ACq. suppose I. the circle ABT :: the circle EFN. K. therefore the circle ABT  $\supset$  K. p and EGq. ACq :: the circle EFN. K. which is shewn to be repugnant.

Therefore it must be concluded, that I is = to the circle EFN. *W.W. to be Dem.*

Coroll.

Hence it follows, that as a circle is to a circle, so is a polygon described in one to a like polygon described in the other.

PROP. III.

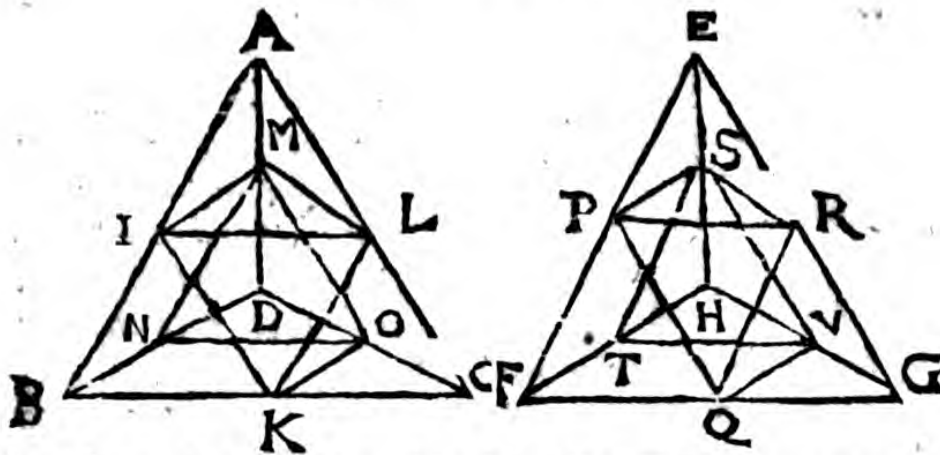


Every Pyramide ABDC having a triangular base, may be divided into two pyramides AEGH, HIKC, equall, and like one to the other, having bases triangular, and like to the whole ABDC; and into two equall prismes, BFGI, EGDH; which two prismes are greater then the half of the whole pyramide ABDC.

Divide the sides of the pyramide into two parts at the points E, F, G, H, I, K. and join the right lines EF, FG, GE, EI, IF, FK, KG, GH, HE. Because the sides of the pyramide are proportionally cut,

cut, <sup>a</sup> thence  $HI, AB$ ; and  $GF, AB$ ; and  $IF, DC$ ; and  $a 2. 6.$   
 $HG, DC$ , &c. are parallels. and consequently  $HI$ ,  
 $FG$ ; and  $GH, FI$  are also parallels. therefore it is ap-  
 parent that the triangles  $ABD, AEG, EBF, FDG,$   
 $HIK$ , <sup>b</sup> are equiangular. and that the four last are <sup>b 29. 1.</sup>  
<sup>c 26. 10</sup> equal: in like manner the triangles  $ACB, AHE,$   
 $EIB, HIC, FGK$  are equiangular; and the four last  
 are equal one to the other. Also the triangles  
 $BFI, FDK, IKC, EGH$ ; & lastly the triangles  $AHG,$   
 $GDK, HKC, EFI$  are like and equal. Moreover the  
 triangles,  $HIK$  to  $ADB$ , and  $EGH$  to  $BDC$ , and  
 $EFI$  to  $ADC$ , and  $FGK$  to  $ABC$ , <sup>d</sup> are parallel. <sup>d 15. 11.</sup>  
 From whence it evidently follows, first that the py-  
 ramids  $AEGH, HIKC$  are equal, and <sup>e</sup> like to the <sup>e 10. def. 11.</sup>  
 whole  $ABDC$ , and to one another. Next, that the  
 solids  $BFGEIH, FGDIHK$  are prismes, and that of  
 equal height, as being placed between the parallel  
 planes  $ABD, HIK$ . but the base  $BFGE$  is <sup>f</sup> double <sup>f 2. 42. 1.</sup>  
 of the base  $FDG$ . wherefore the said prismes are <sup>g 49. 11.</sup>  
 equal; whereof the one  $BFGEIH$  is greater than the  
 pyramide  $BEFI$ , that is, than  $AEGH$ , the whole then  
 its part; & consequently the two prismes are greater  
 than the two pyramids and so exceed the half of the  
 whole pyramide  $ABDC$ . *W.W. to be Dem.*

P R O P. I V.



If there be two pyramids  $ABCD, EFGH$ , of the  
 same altitude, having triangular bases  $ABC, EFG$ ;  
 and



and either of them be divided into two pyramids (AILM, MNOD; and EPRS, STVH) equall one to the other and like to the whole; and into two equall prismes (IBKLMN, KLCNMO; and PFQRST, QRGTSV;) and if in like manner either of those pyramids made by the former division be divided, and this be done continually; then as the base of one pyramide is to the base of the other pyramide, so are all the prismes which are in one pyramide, to all the prismes which are in the other pyramide, being equall in multitude.

a 15. 5.

b 22. 6.

c 2. 6. &amp; 1.

d 16. 5.

e 12. 34. 11.

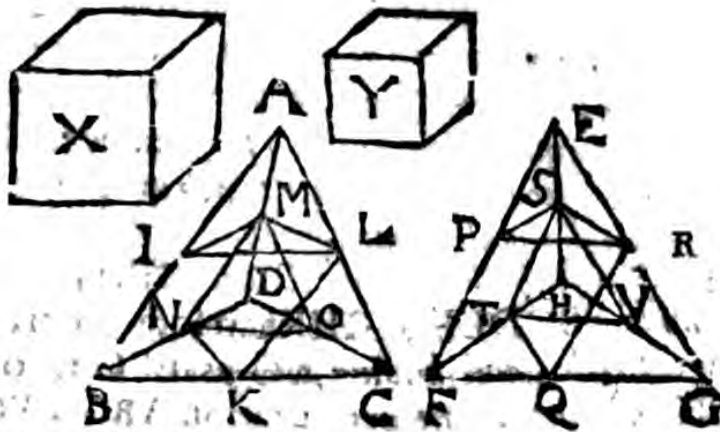
f 7. 5.

g 12. 5.

For (applying the construction of the precedent prop.) BC. KC  $a ::$  FG. QG.  $b$  therefore the triangle ABC is to the like triangle LKC as EFG is to  $c$  the like RQG. therefore by permutation ABC. EFG  $d ::$  LKC. RQG  $e ::$  the prisme KLCNMO. QRGTSV (for these are of equall altitude)  $f ::$  IBKLMN. PFQRST.  $g$  wherefore the triang. ABC. EFG  $::$  the prism. KLCMNO + IBKLMN. the prisme QRGTSV + PFQRST. *W.W. to be Dem.*

But if the pyramids MNOD, AILM; and EPRS, STVH; be further divided in like manner the four new prismes made hereby shall be to the four produced before as the bases MNO and AIL are to the bases STV, and EPR; that is, as LKC to RQG, or as ABC to EFG.  $b$  wherefore all the prismes of the pyramide ABCD are to all the prismes of the pyramide EFGH as the base ABC is to the base EFG. *W.W. to be Dem.*

PROP. V.

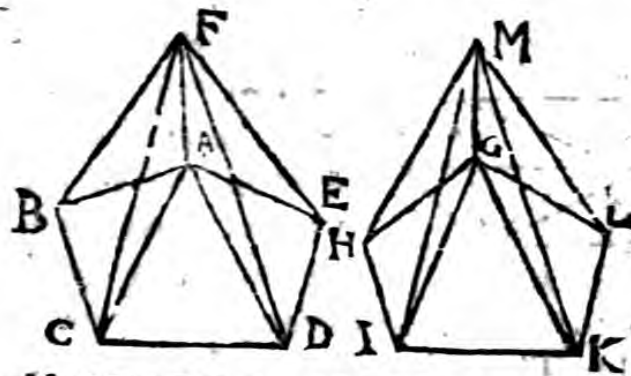


Pyramids  $ABCD$ ,  $EFGH$ , being under the same altitude, having triangular bases  $ABC$ ,  $EFG$ , are one to another as their bases  $ABC$ ,  $EFG$ , are.

Let the triangle  $ABC$ .  $EFG$  ::  $ABCD$ .  $X$ . I say  $X$  is equal to the pyramide  $EFGH$ . For if it be possible, let  $X$  be  $\sqsupset EFGH$ . and let the excess be  $Y$ . Divide the pyramide  $EFGH$  into prismes and pyramids, and the other pyramids in like manner, <sup>a</sup> till  $a$  11 10. the pyramids left  $EPRS$ ,  $STVH$ , be lesse then the solid  $Y$ . Therefore since the pyramide  $EFGH = X + Y$ , it is manifest that the remaining prismes  $PFQRST$ ,  $QRGTSV$  are greater then the solid  $X$ . Conceive the pyramide  $ABCD$  divided after the same manner; <sup>b</sup> then will be the prisme  $IBKLMN + KLCNMO$ .  $PFQRST + QRGTSV$  ::  $ABC$ .  $EFG$  <sup>b 4 12.</sup>  $c$  :: the pyr.  $ABCD$ .  $X$ . <sup>c hyp.</sup> <sup>d 4 5.</sup> <sup>d</sup> therefore  $X \sqsubset$  the prisme  $PFQRST + QRGTSV$ ; which is contrary to that which was affirmed before.

Again, conceive  $X \sqsubset$  the pyr.  $EFGH$ . and make the pyr.  $EFGH$ .  $Y$  ::  $X$ . the pyr.  $ABCD$   $e$  ::  $EFG$ .  $ABC$ . Because  $EFGH$   $f$   $\sqsupset X$ , <sup>e hyp. and</sup> <sup>cor. 4 5.</sup> <sup>f suppos.</sup> <sup>g 4 5.</sup> <sup>g</sup> thence  $Y \sqsupset$  the pyr.  $ABCD$ . which is shewn before to be impossible. Therefore I conclude, that  $X$  is equal to the pyr.  $EFGH$ . *W. W. to be Dem.*

PROP. VI.



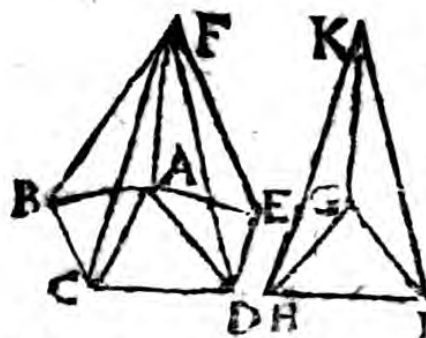
Pyramids  $ABCDEF$ ,  $GHIKLM$ , consisting under the same altitude, and having polygonous bases  $ABCDE$ ,  $GHIKL$ , are to one another as their bases  $ABCDE$ ,  $FGHIKL$  are.

Draw the right lines  $AC, AD, GI, GK$ , then is the base  $ABC.ACD$  *a* :: the pyr.  $ABCF.ACDF$ . *b* therefore by composition,  $ABCD.ACD$  :: the pyr.  $AB.CDF.ACDF$ . *c* but also  $ACD.ADE$  :: the pyr.  $AC.DF.ADEF$ . *c* therefore of equality  $ABCD.ADE$  ::  $ABCDF.ADEF$ . and *b* thence by composition  $ABCDE.ADE$  :: the pyr.  $ABCDEF.ADEF$ . moreover  $ADE.GKL$  *d* :: the pyr.  $ADEF.GKLM$ ; and, as before, and inversely  $GKL.GHIKL$  :: the pyr.  $GKLM.GHIKLM$ . *e* therefore again of equality  $ABCDE.GHIKL$  :: the pyr.  $ABCDEF.GHIKLM$ . *W. W. to be Dem.*

*a* 5. 12.  
*b* 18. 5.

*c* 22. 5.

*d* 5. 12.



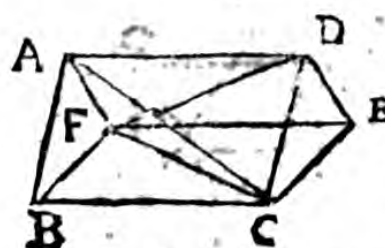
If the bases have not sides of equall multitude, the demonstration will proceed thus. The base  $ABC.GHI$  *a* :: the pyr.  $ABCF.GHIK$ . *e* &  $ACD.GHI$  :: the pyr.  $ACDF.GHIK$ . *f* therefore the base  $ABCD.GHI$  :: the pyr.  $ABCDEF.GHIK$ . *e* Moreover the base  $ADE.GHI$  :: the pyr.  $ADEF.GHIK$ . *f* therefore the base  $ABCDE.GHI$  :: the pyr.  $ABCDEF.GHIK$ .

*e* 5. 12.

*f* 4. 5.

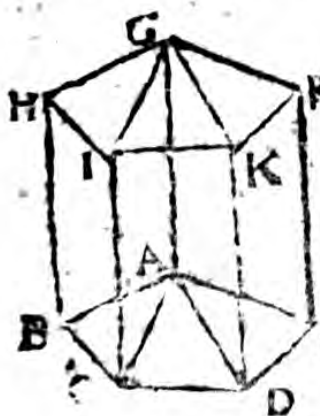
*a*  $ABC.GHI$  :: the pyr.  $ABCF.GHIK$ . *e* Moreover the base  $ADE.GHI$  :: the pyr.  $ADEF.GHIK$ . *f* therefore the base  $ABCDE.GHI$  :: the pyr.  $ABCDEF.GHIK$ .

PROP. VII.



Every *prisme*,  $ABCDEF$ , having a triangular base, may be divided into three *pyramids*  $ACBF$ ,  $ACDF$ ,  $CDFE$ , equall one to the other, & having triangular bases.

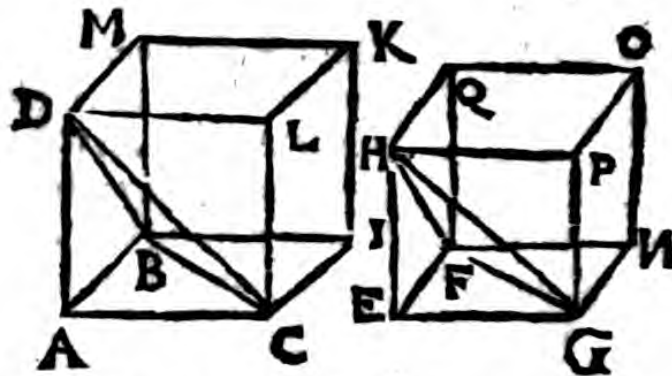
Draw the diameters of the parallelograms,  $AC$ ,  $CF$ ,  $FD$ . Then the triangle  $ACB$  is  $a = ACD$ . <sup>a 34. 1.</sup> <sup>b 5. 12.</sup> therefore the *pyramids* of equall height  $ACBF$ ,  $ACDF$ . are equall. In like manner the *pyr.*  $DFAC =$  the *pyr.*  $DFEC$ . but  $ACDF$  and  $DFAC$  are one & the same *pyramide*. <sup>c 1. 22. 1.</sup> therefore the three *pyramids*  $ACBF$ ,  $ACDF$ ,  $DFEC$ , into which the *prisme* is divided, are equall one to the other. *W. W. to be Dem.*



Hence, every *pyramide* is the third part of the *prisme* that ha's the same base and height with it; or every *prisme* is treble of the *pyramide* that ha's the same base and height with it.

For resolve the polygonous *prisme*  $ABCDEGHKIF$  into triangular *prismes*; and the *pyramide*  $ABCDEH$  into triangular *pyramides*; <sup>a 7. 12.</sup> then all the parts of the *prisme* shall be treble to all the parts of the *pyramide*; <sup>b 1. 5.</sup> consequently the whole *prisme*  $ABCDEGHKIF$  is treble to the whole *pyramide*  $ABCDEH$ . *W. W. to be Dem.*

PROP. VIII.



Like pyramids  $ABCD, EFGH$ , which have triangular bases  $ABC, EFG$  are in triple proportion of that in which their sides of like proportion  $AC, EG$ , are.

a 27. 11.  
 b 9 def. 11.  
 c 28. 11. and  
 7. 12.  
 d 15. 5.  
 e 33. 11.

Complete the parallelepipedons  $ABICDMKL, EFNGHQOP$ , which  $b$  are like, and  $c$  sextuple of the pyramids  $ABCD, EFGH$ .  $d$  and therefore in the same proportion with them one to another,  $e$  that is, triple of that of the sides of like proportion, &c.

Coroll.

Hence, also like polygonous pyramids have proportion tripled to that of the sides of like proportion; as may easily be proved by resolving the same into triangular pyramids.

PROP. IX.

See the prec. Scheme.

In equall pyramids  $ABCD, EFGH$ , having triangular bases  $ABC, EFG$ , the bases and altitudes are recipocall; And pyramids having triangular bases, whose altitudes and bases are recipocall, are equall.

a 28. 11. and  
 7. 12.

I. Hyp. The completed parallelepipedons  $ABICDMKL, EFNGHQOP$  are  $a$  sextuple of the equall pyramids  $ABCD, EFGH$  (either of either) and so equall

equall one to the other. therefore the altitude (H.)  
 the alt. (D)  $b :: ABIC. EFNG c :: ABC.EFG.W.W.$  b 34. 11.  
c 15. 5.  
 to be Dem.

2. Hyp. The altitude (H.) the alt. (D)  $d :: ABC. d hyp. 3$   
 $EFGe :: ABIC.EFNG. f$  therefore the parallelepi- e 15. 5.  
 pedons  $ABICDMKL, EFNGHQOP$  are equall. f 34. 11.  
g 6. ax. 1.  
 g consequently also the pyramids  $ABCD, EFGH$   
 being subsextuple of the same, are equall. *W.W.* to  
 be Dem.

The same is applicable to polygonous pyramids;  
 for they may also in like manner be reduced to trian-  
 gulars.

Coroll.

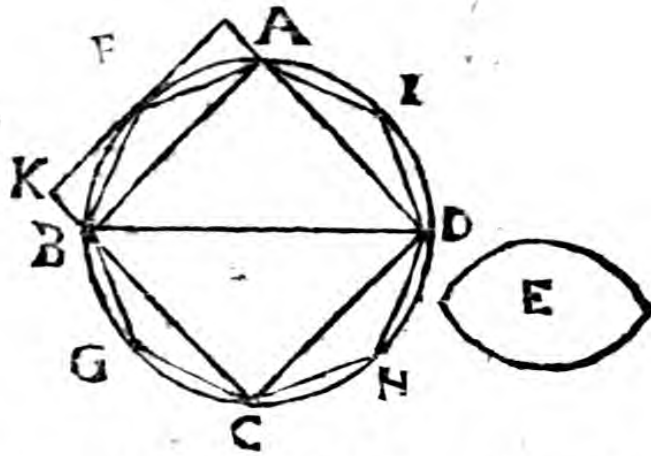
Whatsoever is demonstrated of pyramids in prop. 6,  
 8, 9. do's likewise agree to any sort of prismes; seeing  
 they are triple of the pyramids that have the same base  
 and altitude with them. Therefore.

1. The proportion of prismes of equall altitude  
 is the same with that of their bases.
2. The proportion of like prismes is triple of  
 that of the sides of like proportion.
3. Equall prismes have their bases and altitudes  
 reciprocal; and prismes which are so reciprocal,  
 are equall.

Schol.

From what is hitherto demonstrated the dimen-  
 sion of any prismes and pyramids may be col-  
 lected.

The soliditie of a prisme is produced of the alti- a 1. cor 12.  
and sch. 40.  
 tude multiplied into the base; *b* and therefore like- 11.  
 wise that of a pyramide, of the third part of the alti- b 7. 12.  
 tude multiplied into the base.



Every Cone is the the third part of a cylinder having the same base with it  $ABCD$ , and the altitude equall.

See the se-  
cond fig. of  
this Book.  
a feb 7 4. and  
cor 9. 12.

b feb. 27. 3.  
and cor. 9.  
12.

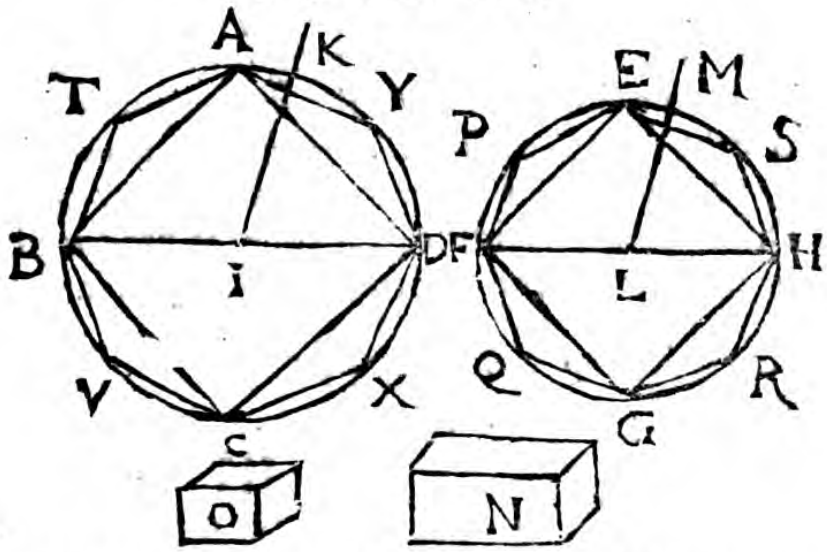
c s. ax. 1.  
d hyp.  
e cor. 7. 12.

If you deny it, then first let such cylinder be more then triple to the cone, and let the exceſſe be  $E$ . A priſme deſcribed on a ſquare in the circle  $ABCD$  is ſubduple of a priſme deſcribed upon a ſquare about the circle, being equall to it and the cylinder in height. Therefore a priſme upon the ſquare  $ABCD$  exceeds the half of the cylinder. and likewise a priſme upon the baſe  $AFB$ , of equall height to the cylinder,  $b$  is greater then the half of the ſegment of the cylinder  $AFB$ . continue an equall biſection of the arches, & ſubſtract the priſmes till the remaining ſegments of the cylinder, namely at  $AF, FB$ , &c. become leſs then the ſolid  $E$ . Therefore the cyl. — ſegm.  $AF, FB$ , &c. (the priſme on the baſe  $AFBGCHDI$ )  $c$  is greater then the cylinder —  $E$  ( $d$  the triple of the cone.) therefore the pyramide,  $e$  a third part of the ſaid priſme (being placed on the ſame baſe, and of the ſame height) is greater then the cone of equall height on the baſe  $ABCD$  a circle, *i. e.* the part greater then the whole. *Which is Abſurd.*

But if the cone be affirmed to be greater then the third part of the cylinder, then let the exceſſe be  $E$ . Detract the pyramids from the cone, as you did in the firſt part the priſmes from the cylinder, till ſome ſegments

segments of the cone remain, conceive at AF, FB, BG, &c. lesse then the solid E: therefore the cone — E ( $\frac{1}{3}$  of the cylin.)  $\supset$  the pyr. AFBGCHDI (the cone  $\frac{1}{3}$  — segm. AF, FB, &c.) therefore the prisme tri-  
 ple to the pyramide (*viz.* of equall height, and on the same base) is greater then the cylinder on the base ABCD, the part then the whole. *Which is Abs.* Wherefore it must be granted, that the cylinder is equall to triple of the cone. *W.W. to be Dem.*

PROP. XI.



Cylinders and Cones ABCDK, EFGHM, being under the same altitude, are to one another as their bases ABCD, EFGH are.

Let the circle ABCD. the cir. EFGH :: the cone ABCDK. N. I say N is equall to the cone EFGHM.

For if it be possible, let N be  $\supset$  the cone EFGHM, and let the excess be O. The preparation and argumentation of the prec. prop. being supposed; then shal O be greater then the segments of the cone EP, PF, FQ, &c. and so the solid N  $\supset$  the pyr. EPFQGRHSM. In the circle ABCD <sup>a</sup> make a like polygonous fig. ATBVCXDY. Because the pyr. ABVYK. the pyr. EFQSM <sup>b</sup> :: the polyg. ATBVY. the polyg. EPFQS <sup>c</sup> :: the cir. ABCD. the cir. EFGH <sup>d</sup> :: the cone ABCDK. N. <sup>e</sup> thence the pyr. EPFQGRHSM shall be  $\supset$  N. contrary to what was affirmed before. Again conceive N  $\supset$  the cone EFGHM.

a 30 3 and  
 1. post.  
 b 6. 12.  
 c cor. 2. 12.  
 d hyp.  
 e 14 5.



f hyp. and by  
inversion,  
B 14. 5.

& make the cone EFGHM. O :: N. the cone ABCDKf :: the circ. EFGH. ABCD. g therefore O  $\supset$  the cone ABCDK; which is absurd, as appears by what is shewn in the first part.

Therefore rather admit ABCD.EFGH :: the cone ABCDK.EFGHM. *W.W. to be Dem.*

The same may be demonstrated of cylinders, if cylinders and prismes be conceived in the place of cones and pyramids. therefore, &c.

Schol.

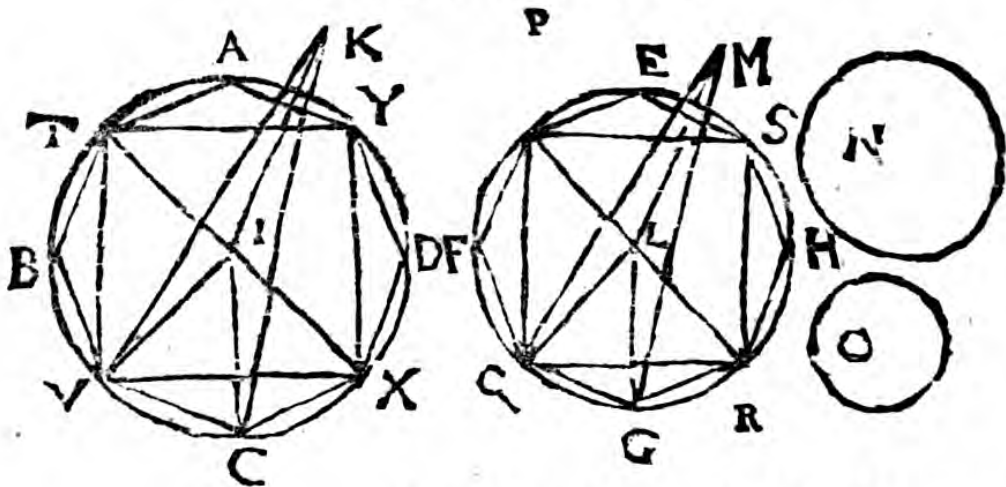
a 1. Prop. de  
dimens. circ

b 11. 12.

Hence, is gathered the dimension of all sorts of cylinders and cones. The soliditie of a right cylinder is produced of the circular base (a the dimension whereof is to be learnt out of Archimedes) multiplied into the height; b whence in like manner that of every cylind.

Therefore the soliditie of a cone is produced of the third part of the altitude multiplied into the base.

P R O P. XII.



Like cones and cylinders ABCDK, EFGHM, are in triple proportion of that of the diameters TX, PR, of their bases ABCD, EFGH.

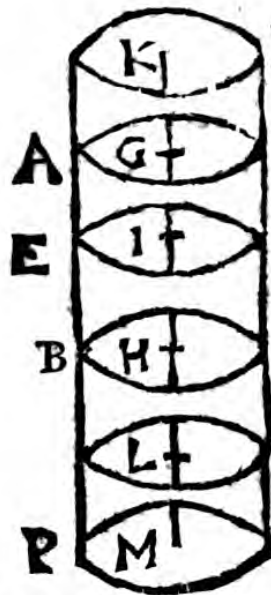
Let the cone A have to N triple proportion of TX to PR. I say N is = the cone EFGHM. For if it be possible let N be  $\supset$  EFGHM. and let the excess be O. therefore, N  $\supset$  the pyr. EPFQGRHSM. Let the axes of the cones be IK, LM, and join the right lines VK, CK, VI, CI, and QM,

QM, GM, QL, GL. Because the cones are like,  
<sup>a</sup> thence VI. IK :: QL. LM. but the angles VIK, <sup>a 14 def. 11.</sup>  
 QLM <sup>b</sup> are right angles. <sup>b 18. def. 11.</sup> <sup>c</sup> therefore the triangles <sup>c 6. 6.</sup>  
 VIK, QLM are equiangul. <sup>d</sup> whence VC. VI :: QG. <sup>d 4. 6.</sup>  
 QL. also VI. VK :: QL. QM. therefore of equality  
 VC. VK :: QG. QM. <sup>e</sup> moreover VK. CK :: QM. <sup>e 7. 5.</sup>  
 MG. therefore again of equality VC. CK :: QG. GM.  
<sup>f</sup> therefore the triangles VKC, QMG are like: and <sup>f 5. 6.</sup>  
 by the same reason the other triangles of this pyra-  
 mids themselves are like. <sup>g</sup> wherefore the py- <sup>g 9. def. 11.</sup>  
 ramids themselves are like. <sup>h</sup> But they are in triple <sup>h cor. 8. 12.</sup>  
 proportion of that of VC to QG, <sup>k</sup> that is, of VI to <sup>k 4. 6.</sup>  
 RL, <sup>l</sup> or TX to PR. <sup>m</sup> therefore the pyr. AIBVCXD- <sup>l 15. 5.</sup>  
 YK. the pyr. EPFQGRHSM :: the cone ABCDK. <sup>m hyp. and</sup>  
 N. <sup>n</sup> whence the pyr. EPFQGRHSM  $\sqsupset$  N. which <sup>n 14. 5.</sup>  
 is repugnant to what was affirmed before.

Again, take N  $\sqsubset$  the cone EFGHM. make the  
 cone EFGHM. O :: N. the cone ABCDK <sup>o</sup> :: the <sup>o before. &</sup>  
 pyr. EPRM. ATCK <sup>p</sup> :: GQ. VC thrice :: <sup>q</sup> PR. TX <sup>inversely.</sup>  
 thrice. but O <sup>r</sup> is  $\sqsupset$  ABCDK. which was before <sup>p cor. 8. 12.</sup>  
 shewn to be repugnant. Wherefore N = the cone <sup>q 4 6.</sup>  
 EFGHM. *W. W. to be Dem.* <sup>r 14. 5.</sup>

But forasmuch as what proportion soever cones  
 have, also cylinders, being triple of them, have the  
 same; therefore cylinder to cylinder shall have pro-  
 portion triple of the diameters of the bases.

PROP. XIII.



If a cylinder ABCD be divided by a plane EF parallel to the opposite planes BC, AD, then as one cylinder AEFD is to the other cylinder EBCF, so is the axis GI to the axis IH.

The axis being produced, take  $GK = GI$ , and  $HL = IH = LM$ . and conceive planes drawn at the points K, L, M, parallel to the circles AD, BC, therefore the cylinder  $FD =$  the cyl. AN. and the cyl.  $EC =$  BO  $=$  OP. therefore the cy-

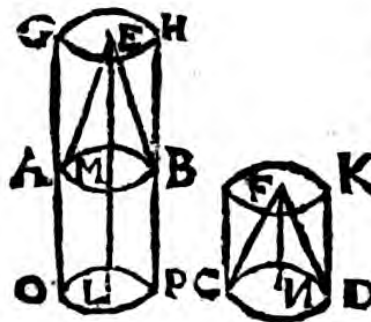
linder EN is as multiplex of the cylinder ED as the axis IK is of the axis IG. and in like manner the cylinder FP is as multiplex of the cylinder BF, as the axis IM is of the axis IH. but as  $IK =$   $IG$ ,  $IM =$   $IH$ , so is the cylinder  $EN =$   $FP$ . therefore the cylinder AEFD. the cyl. EBCF :: GI. IH. *W.W. to be Dem.*

a 3. 1.

b 11. 12.

c 11. 12.  
d 6. def. 5.

PROP. XIV.



Cones AEB, CFD, and cylinders AH, CK, consisting upon equal bases AB, CD, are to one another as their altitudes ME, NF are.

The cylinder HA, and the axis EM being produced, take  $ML = FN$ ; and at the point L draw a plane parallel to the

base AB. then shall the cyl. AP be  $=$  CK. but the cyl. AH. AP (CK) :: ME. ML (NF.) *W.W. to be Dem.*

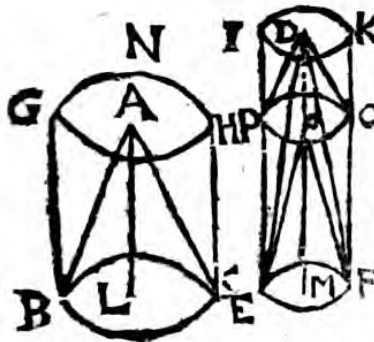
The same may be affirmed of cones subtriple of cylinders; \* as also of prismes and pyramids.

PROP.

a 11. 12.  
b 13. 12.

\* apply 9,  
and 7. 12.

PROP. XV.



In equall cones BAC, EDF, and cylinders BH, EK, the bases and altitudes are reciprocal ( $BC \cdot EF :: MD \cdot LA$ ) And cones and cylinders, whose bases and altitudes are reciprocal, are equall one to the other.

If the altitudes be equall then the bases are equall too, and the thing is evident. If unequal, then take away  $MO = LA$ .

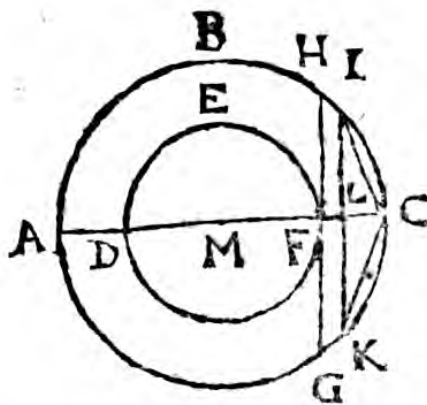
1. Hyp. Then is  $MD \cdot MO$  ( $a$   $LA$ )  $b ::$  the cyl. EK. ( $c$   $BH$ )  $EQ$   $d ::$  the cir. BC. EF. Which was to be Dem.

a 14. 12.  
b const.  
c hyp.  
d 11. 12.  
e hyp.  
f 11. 12.  
g 11. 5.  
h 11. 12.  
k 9. 5.

2. Hyp.  $BC \cdot EF$   $e ::$   $DM \cdot OM$  ( $LA$ )  $f ::$  the cyl. EK.  $EQ$   $g ::$   $BC \cdot EF$   $b ::$   $BH \cdot EQ$   $h$ . Therefore the cylind. EK = BH. Which was to be Dem.

The same argument may be used for cones.

PROP. XVI.



Two unequal circles ABCG, DEF, having the same centre M, to inscribe in the greater circle ABCG a polygonous figure of equal and even sides, which shall not touch the lesser circle DEF.

Through the center M draw the line AC cutting the circle DEF in F, from whence raise a perpendicular FH.  $a$  divide the semicircle ABC into two equal parts; and the half thereof BC also; and so do continually,  $b$  till the arch IC become lesse then the arch HC. from I let fall the perpendicular IL. It

c sch. 16.4.

d cor. 16.3.

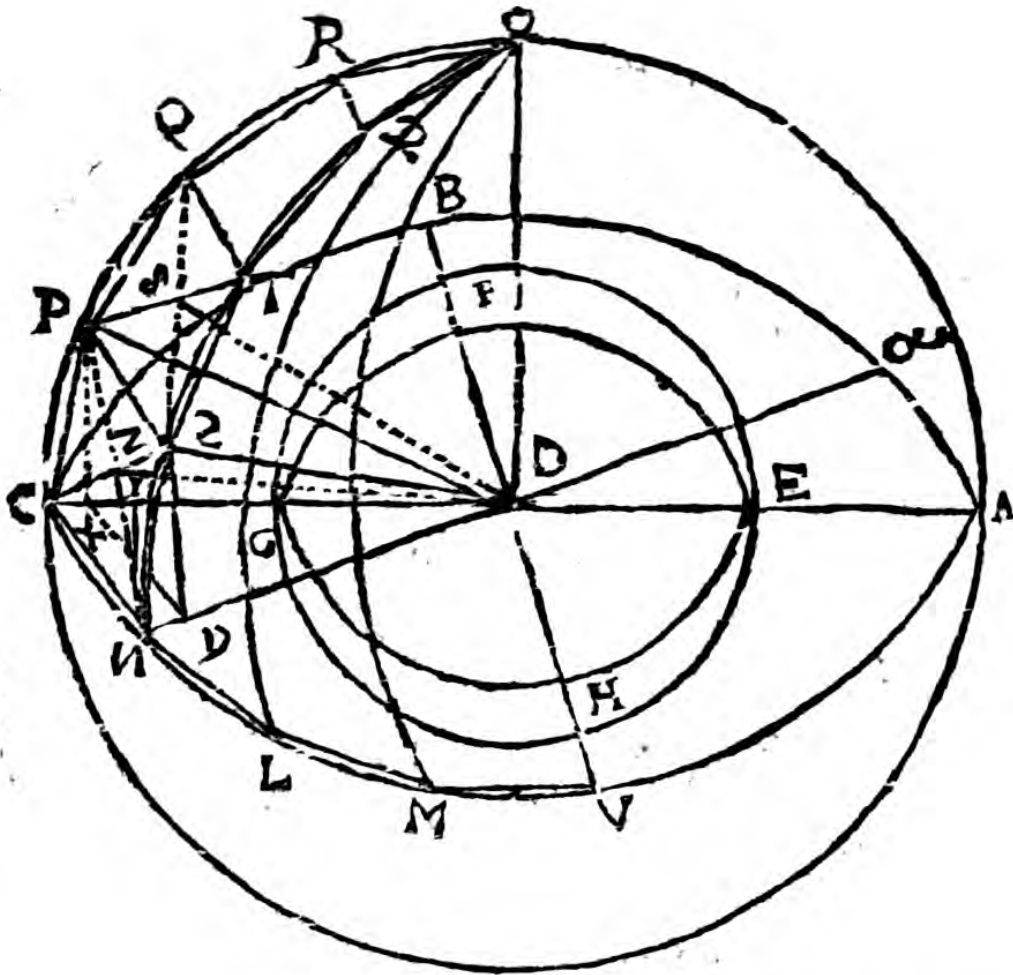
e 28. 1.

f 34. def. 1.

is manifest that the arch IC measures the whole circle, and that the number of arches is even, and so that the subtended line IC is the side of the polygon that may be inscribed without touching the lesser circle DEF. For HG touches the circle DEF, to which IK is parallel, and placed outwardly; wherefore IK does not touch the circle DEF; much less do CI, CK, and the other sides of the polygon more remote from the center. *W. W. to be Done.*

Coroll. Observe that IK touches not the circle DEF.

P R O P. XVII.



Two spheres ABCV, EFGH, consisting about the same center D, being given, to inscribe a solide of many sides (or Polyedron) in the greater sphere ABCV, which shall

shall not touch the superficies of the lesser sphere EFGH.

Let both the spheres be cut by a plane passing by the center making the circles EFGH, ABCV; and the diameters AC, BV drawn, cutting perpendicularly. In the circle ABCV, <sup>a</sup> inscribe the equilaterall polygone VMLNC, &c. not touching the circle EFGH: then draw the diameter Na, and erect DO perpendicular to the plane ABC. by DO, and by the diameters AC, Na, conceive planes DOC, DON erected, which shall be <sup>b</sup> perpendicular to the circle ABCV, and so in the superficies of the sphere make <sup>c</sup> the quadrants DOC, DON. In which let the right lines CP, PQ, QR, RO, NS, ST, Tγ, γO <sup>d</sup> be fitted, <sup>e</sup> quall, and of equall multitude with CN, NL, &c. make the same construction in the other quadrants OL, OM, &c. and in the whole sphere. Then I say the thing required is done.

From the points P, S, to the plane ABCV draw the perpendiculars PX, SY, <sup>e</sup> which shall fall on the sections AC Na. Therefore because both <sup>f</sup> the right angles PXC, SYN, <sup>g</sup> and PCX, SNY insisting on <sup>b</sup> equall circumferences, <sup>f</sup> are equall, the triangles also PCX, SNY <sup>h</sup> are equiangular. Wherefore being PC <sup>k</sup> = SN, <sup>l</sup> also is PX = SY, <sup>l</sup> and XC = YN; <sup>m</sup> whence DX = DY, <sup>n</sup> and therefore DX. XC :: DY. YN. <sup>o</sup> therefore YX, NC are parallels. but because PX, SY are equall, and since being perpendicular to the same plane ABCV, they are also <sup>p</sup> parallels, <sup>q</sup> therefore YX, SP shall be equall and parallels. <sup>r</sup> whence SP, NC, are parallel one to the other; and so the <sup>s</sup> quadrilaterall NCPS, and by the same reason on SPQT, TQRG, as also the <sup>t</sup> triangle γRO are so many planes. In like manner the whole sphere may be shewn full of such quadrilateralls and triangles. wherefore the figure inscribed is a polyedron.

From the center D, draw DZ perpendicular to the plane NCPS; and join ZN, ZC, ZS, ZP. Because DN. NC \* :: DY. YX. thence NC is  $\perp$  YX (SP)

a 16. 12.

b 18. 11.

c cor. 33. 6.

d 4. 1.

e 38. 11.

f 12. ax.

g 27. 3.

h 32. 1.

k const.

l 26. 1.

m 3. ax. 2.

n 7. 5.

o 2. 6.

p 6. 11.

q 33. 1.

r 9. 12.

s 7. 11.

t 2. 11.

u 11. 11.

x 4. 6.

y 14. 5.

(SP,) and likewise  $SP \perp TQ$ , and  $TQ \perp R$ .  
 And because the angles  $DZC, DZN, DZS, DZP$   
z 3 def. 11.  $z$  are right, and the sides  $DC, DN, DS, DP,$  a 15 def. 11.  $a$  equally,  
b 47. 1. and  $DZ$  common, b thence  $ZC, ZN, ZS, ZP$  are e-  
 quall one to the other; and consequently about the  
 quadrilaterall  $NCPS$  c 15 def. 1. i a circle may be described, in  
d constr. which (because  $NS, NC, CP,$  are d equally, and  $NC$   
e 18. 3.  $\perp SP$ )  $NC$  e subtends more then the quadrant,  
f 31. 9.  $f$  therefore the ang.  $NZC$  at the center is obtuse.  
g 12. 2.  $g$  therefore  $NCq \perp z ZCq$  ( $ZCq + ZNq$ .) Let  
 $NI$  be drawn perpendicular to  $AC$ . therefore since  
 the angle  $ADN$  (h 32. 1.  $h$   $DNC + DCN$ ) k 9 ax. 1.  $k$  is obtuse, the  
l 5. 1. half of it  $DCN$  shall be greater then the half of a  
 right angle; and so that which remains of the right  
 ang.  $CNI$  shall be lesse then it. n 19. 1.  $n$  whence  $IN \perp IC$ .  
o 47. 1. therefore  $NCq$  ( $NIq + ICq$ ) p 47. 1.  $p$   $\perp z INq$ . there-  
q cor. 19. 11. fore  $IN \perp ZC$ . and consequently  $DZP \perp DI$ .  
 but the point  $I$  is r 47. 1.  $r$  without the sphere  $EFGH$ . & so  
 much more the point  $Z$ . wherefore the plane  $NC$ -  
 $PS$ , (whose r next point to the center is  $Z$ ) does  
 not touch the sphere  $EFGH$ . And if a perpendicu-  
 lar  $Dd$  be drawn to the plane  $SPQT$ , the point  $d$ , &  
 so also the plane  $SPQT$  is yet further removed  
 from the center. which is also true of the other  
 planes of the polyedron. Therefore the polyedron  
 $ORQPCN$ , &c. inscribed in the greater sphere, does  
 not touch the lesser. *W.W. to be Done.*

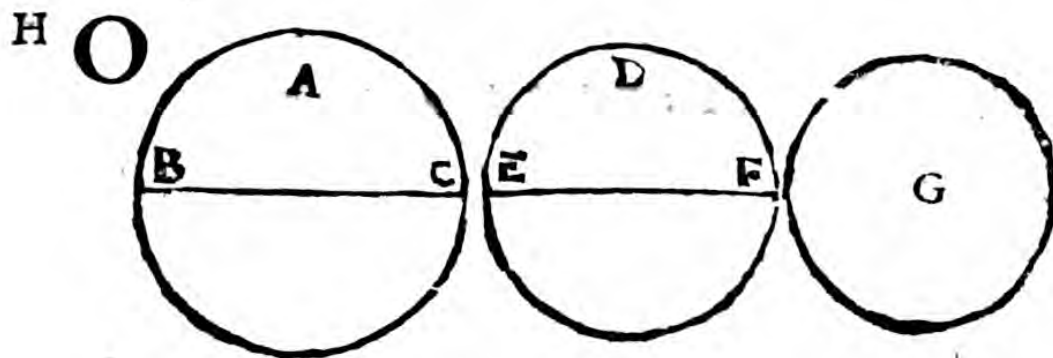
## Coroll.

Hence it follows, that if in any other sphere a solid polyedron, like to the abovesaid solid polyedron, be inscribed, the proportion of the polyedron in one sphere to the polyedron in the other is triple of that of the diameters of the spheres.

For if right lines be drawn from the centers of the spheres to all the angles of the bases of the said polyedrons, then the polyedrons will be divided into pyramids equal in number and like; whose ho-  
 mo-

homologous sides are semidiameters of the spheres; as appears, if the lesser of these spheres be conceived described within the greater about the same center. For the right lines drawn from the center of the sphere to the angles of the bases will agree one to the other by reason of the likeness of the bases; and so will like pyramids be made. Wherefore since every pyramide in one sphere to every pyramide like it in the other sphere <sup>a</sup> has proportion triple to that <sup>cor. 8.12.</sup> of the homologous sides, that is, of the semidiameters of the spheres; and <sup>b</sup> as one pyramide is to one <sup>b 12. 5.</sup> pyramide, so all the pyramids, that is, the solid polyedron composed of these, are to all the pyramids, that is, the solid polyedron composed of the others; therefore the polyedron of one sphere shall have to the polyedron of the other sphere, proportion triple of that of the semidiameters, <sup>c</sup> and so of the diameters <sup>c 15. 5.</sup> of the spheres.

P R O P. XVIII.



*Spheres BAC, EDF, are in triple proportion one to the other of that in which their diameters BC, EF, are.*

Let the sphere BAC be to the sphere G in triple proportion of that of the diameter BC to the diameter EF. I say  $G = EDF$ . For if it be possible, let G be  $\square EDF$ . and conceive the sphere G concentricall with EDF. In the sphere EDF <sup>a</sup> inscribe a <sup>a 17. 12.</sup> polyedron not touching the sphere G, and a like po-



b cor. 17. 12.  
c hyp.  
d 14. 5.

polyedron in the sphere BAC. These polyedrons *b* are in triple proportion of the diameters BC, EF, *c* that is, of the sphere BAC to G. *d* Consequently the sphere G is greater then the polyedron inscribed in the sphere EDF, the part then the whole.

e hyp. inverf.  
f 14. 5.

Again, if it be possible, let the sphere G be  $\square$  EDF. and as the sphere EDF is to another sphere H, so let G be to BAC, *e* that is, in triple proportion of the diameter EF to BC. therefore since BAC  $f \square$  H, we shall incur the absurdity of the first part. wherefore rather the sphere G  $=$  EDF. *W.W.*  
*to be Demonstrated.*

*Coroll.*

Hence, As one sphere is to another sphere, so is a polyedron described in that to a like polyedron described in this.

*The End of the twelfth Book.*

THE

# THE THIRTEENTH BOOK OF EUCLIDE'S ELEMENTS.

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## P R O P. I.

**L**et a right line  $z$  be divided according to extreme and mean proportion ( $z.a :: a.e.$ ) the square of the half of the whole line  $z$ , and of the greater segment  $a$ , as one line, is quintuple to that which is described of half of that whole line  $z$ .



I say  $Q. a + \frac{1}{2}z$

$z = 5 Q. \frac{1}{2}z$ , that is,  $aa + \frac{1}{4}zz +$

$za = zz + \frac{1}{4}zz$ . *b* or  $aa + za = zz$ . For  $ze + za = zz$ . and  $ze = aa$ . *e* therefore  $aa + za = zz$ . *W.W. to be Dem.*

*a 4. 2.  
b 3. ax. 1.  
c 2. 2.  
d hyp. and  
16. 6.  
e 2. ax. and  
1. ax.*

## P R O P. II.

*See the I. Scheme.*

If a right line  $\frac{1}{2}z + a$  be in power quintuple to a segment of it self  $\frac{1}{2}z$ . the line double of the said segment ( $z$ ) being divided according to extreme and mean proportion, the greater segment is ( $a$ ) the other part of the right line at first given  $\frac{1}{2}z + a$ .

I say  $z.a :: a.e$ . For because by the hyp.  $*aa + \frac{1}{4}zz + za = zz + \frac{1}{4}zz$ ; or  $aa + za = zz$   $a = ze + za$ , *b* thence shall  $aa = ze$ , *c* wherefore  $z.a :: a.e$ . *W.W. to be Dem.*

*\* 4. 2.  
a 2. 2.  
b 3. ax. 1.  
c 17. 6.*

P R O P.

P R O P. III.

If a right line  $z$  be divided according to extreme and mean proportion ( $z.a :: a.e.$ ) the line made of the lesser segment  $e$  and half of the greater segment  $a$ , is in power quintuple to the square, which is described of the half line of the greater segment  $a$ .

a 4. 2.  
b 3. ax.  
c 3. 2.  
d hyp. and  
17. 6.



I say  $Q:e + \frac{1}{2}a = 5Q:$

$\frac{1}{2}a$ .  $Q$  that is  $ee + \frac{1}{4}aa + ea = aa + \frac{1}{4}aa$ . b or  $ee +$

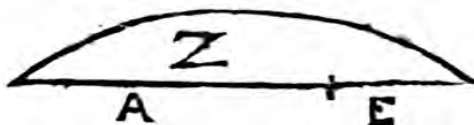
$ea = aa$ . For  $ee + ea = ze = aa$ . Which was to be Dem.

P R O P. IV.

If a right line  $z$  be cut according to extreme and mean proportion ( $z.a :: a.e.$ ) the square made of the whole line  $z$ , and that made of the lesser segment  $e$ , both together, are triple of the square made of the greater segment  $a$ .

a 4. 1.

b 3. 2.  
c 17. 6.  
d 2. ax.



I say  $zz + ee = 3$

$aa$ .  $a$  or  $aa + ee + 2$

$ae + ee = 3aa$ . For

$ae + ee = ze =$

$aa$ . d therefore  $aa + 2ae + 2ee = 3aa$ . W.W. to be Dem.

P R O P. V.

**D A C B** If a right line  $AB$  be cut according to extreme and mean proportion in  $C$ , and a line  $AD$ , equal to the greater segment  $BC$ , added to it, the whole right line  $DB$  is divided according to extreme and mean proportion; and the greater segment is the right line  $AB$  given at the beginning.

a hyp.

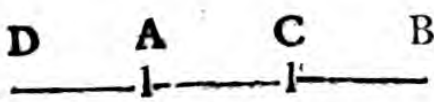
For because  $AB.AD :: AC.CB$ . and by inversion  $AD.AB :: CB.AC$ . therefore by composition  $DB.AB :: AB.AC$  (AD.) W.W. to be Dem.

Schol.

*Schol.*

But if  $BD \cdot BA :: BA \cdot AD$ . then shall be  $BA \cdot AD :: AD \cdot BA - AD$ . For by division is  $BD - BA (AD) \cdot BA :: BA - AD \cdot AD$ . therefore inverfely  $BA \cdot AD :: AD \cdot BA - AD$ .

P R O P. VI.



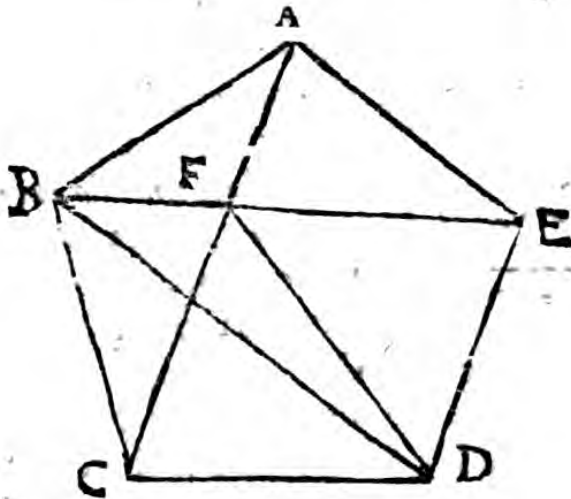
If a ratiounall right line AB be cut according to extreme and mean proportion

in C, either of the segments (AC, CB) is an irratiounall line of that kind which is called apotome or residuall.

To the greater segment AC a adde AD =  $\frac{1}{2}$  AB. b therefore DCq =  $\frac{5}{4}$  DAq. c therefore DCq =  $\frac{5}{4}$  DAq. consequently d since AB, e and so the half thereof DA are  $\rho$ , likewise DC is  $\rho$ . But because  $\frac{5}{4} \cdot 1 ::$  not Q. Q. f thence is DC =  $\frac{5}{4}$  DA. g therefore DC - AD, that is, AC, is a residuall line. Further, because ACq = AB x BC, and AB is  $\rho$ , k likewise BC is a residuall line. *W.W. to be Dem.*

a 3. 1.  
b 1. 13.  
c 6. 10.  
d hyp.  
e *sch.* 12. 10.  
f 9. 10.  
g 74. 10.  
h 17. 6.  
k 98. 10.

P R O P. VII.



If three angles of an equilaterall Pentagone ABCDE, whether they follow in order, (EAB, ABC, BCD,) or not, (EAB, BCD, CDE) be equall, the pentagone ABCDE shall be equiangular.

Let

Let the right lines  $BE, AC, BD$ , be subtended to the equall angles in order.

a hyp.  
b 4. 1.  
c 4 and 5. 1.  
d 6. 1.  
e 3. ax. 11

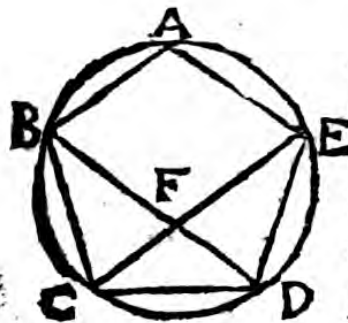
f 8. 1.  
g 2. ax. 1.

Being the sides  $EA, AB, BC, CD$ , and the included angles  $a$  are equall,  $b$  therefore shall the bases  $BE, AC, BD$ ,  $c$  & the angles  $AEB, ABE, BAC, BCA$ , be equall,  $d$  wherefore  $BF = FA$ ,  $e$  and consequently  $FC = FE$ ; therefore the triangles  $FCD, FED$ , are equilaterall one to the other:  $f$  whence the angle  $FCD = FED$ .  $g$  consequently the ang.  $AED = BCD$ . In like manner the ang.  $CDE$  is equall to the rest; wherefore the pentagone is equiangular. *W.W. to be Dem.*

h 4. 1.  
k 5. 1.  
l 2. ax.

But if the angles  $EAB, BCD, CDE$ , which are not in order, be supposed equall,  $i$  then shall the ang.  $AEB = BDC$ , and  $BE = BD$ .  $k$  and thence the ang.  $BED = BDE$ .  $l$  consequently the whole ang.  $AED = CDE$ . therefore because the angles  $A, B, D$ , in order, are equall, as before, the pentagone shall be equiangular. *W.W. to be Dem.*

P R O P. VIII.



If in an equilaterall and equiangular Pentagone  $ABCDE$ , two right lines  $BD, CE$ , subtend two angles  $BCD, CDE$ , following in order, those lines do cut one another according to extreme and mean proportion; and their greater segments  $BF$  or  $EF$  are equall to the

side of the Pentagone  $BC$ .

a 14. 4.  
b 28. 3.  
c 27. 3.  
d 32. 1.  
e 33. 6.

f 6. 1.

g 27. 3.  
h 4. 6.

$a$  Describe about the pentagone the circle  $ABD$ .  $b$  The arch  $ED$  is  $= BC$ ,  $c$  therefore the angle  $FCD = FDC$ .  $d$  therefore the ang.  $BFC = 2 FCD$  ( $FCD + FDC$ .) But the arch  $BAE$   $b$  is  $= 2 ED$ , and consequently the angle  $BCF$   $e = 2 FCD = BFC$ .  $f$  wherefore  $BF = BC$ . Which was to be Dem.

Moreover because the triangles  $BCD, FCD$ , are  $g$  equiangular,  $h$  therefore  $BD. DC. (BF.) :: CD. (BF)$

(BF.) FD. and likewise EC.EF :: EF. FC. *W.W.to be Dem.*

PROP. IX.



If the side of an Hexagone BE, and the side of a Decagone AB, both described in the same circle ABC, be added together the whole right line AE is cut according to extreme and mean proportion (AE. BE :: BE. AB,) and the greater segment thereof is the side

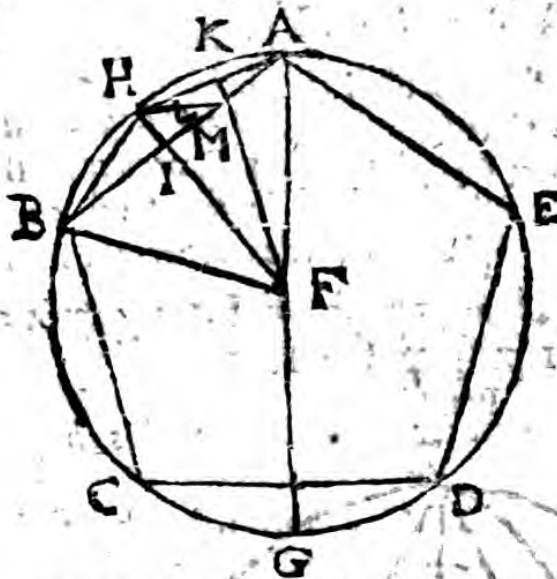
of the Hexagone BE.

Draw the diameter ADC, and join the right lines DB, DE. Because the ang. BDC  $a = 4$  BDA, a hyp. and 27 3. and the ang. BDC  $b = 2$  DBA (DAB + DBA) b 32. 1. thence shall DBA  $(b$  BDE + BED)  $c$  be  $= 2$  BDA c 7 ax. 14  $d = 2$  BDE. whence the ang. DBA or DAB  $e =$  ADE. Therefore the triangles ADE, ADB, are equiangular: d 5. 1. wherefore AE. AD  $(g$  BE)  $::$  AD. e 4. 6. (BE.) AB. *W.W.to be Dem.* g cor. 15. 4.

Coroll.

Hence, If the side of a hexagone in a circle be cut according to extreme and mean proportion; the greater segment thereof shall be the side of the Decagone in the same circle.

PROP. X.



If an equilateral Pentagone ABCDE be described in a circle ABCE, the side of the pentagone AB containeth in power both the side of a hexagone FB, and the side of a decagone AH described in the same circle.

Draw the diameter AG. and bisect equally the arch AH in K. and draw FK, FH, FB, BH, HM.

The semicircle AG — the arch AC<sup>a</sup> = AG — AD. that is, the arch CG = GD<sup>b</sup> = AH = HB. therefore the arch BCG = 2 BHK; <sup>c</sup> and so the ang. BFG = 2 BFK. <sup>d</sup> but the ang. BFG = 2 BAG. <sup>e</sup> therefore the ang. BFK = BAG. Wherefore the triangles BFM, FAB, <sup>f</sup> are equiangular. <sup>g</sup> whence AB. BF :: BF. BM. <sup>h</sup> therefore AB x BM = BFq. Moreover, the ang. AFK<sup>k</sup> = HFK, and FA = FH. <sup>m</sup> wherefore AL = LH, <sup>n</sup> and the angles FLA, FLH<sup>a</sup> are equall, and so right angles. therefore the ang. LHM<sup>m</sup> = LAM<sup>n</sup> = HBA. therefore the triangles AHB, AMH, <sup>o</sup> are equiangular. wherefore AB. AH :: AH. AM. <sup>q</sup> therefore AB x AM = AHq. So that seeing ABq<sup>r</sup> = AB x BM + AB x AM, <sup>s</sup> thence ABq = BFq + AHq. *W.W. to be Dem.*

a 28. 3. and  
3. ax.  
b hyp. and 7.  
ax.  
c 33. 6.  
d 20. 3.  
e 1. ax. 1.  
f 31. 1.  
g 4. 6.  
h 17. 6.  
k 27. 3.  
m 4. 1.

n 17. 3.  
o 32. 1.  
p 4. 6.  
q 17. 6.  
r 2. 2.  
s 2. ax.

Coroll.

1. Hence, a right line (FK) which being drawn from the center (F) divides an arch (HA) into two equall segments, do's also divide the right line (HA) sub-

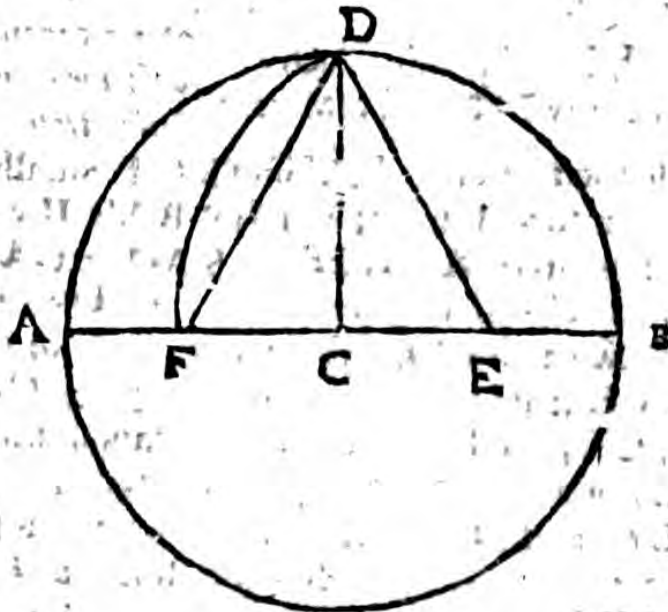
subtending that arch, perpendicularly into two equal segments.

2. The diameter of a circle (*AG*) drawn from any angle (*A*) of a pentagone, do's divide equally in two both the arch (*CD*) which the side of the pentagone opposite to that angle subtends, and also the opposite side it self (*CD*) and that perpendicularly.

*Schol.*

Here, according to our promise, we shall lay down a ready praxis of the 11. prop. of the 4. Book.

*Probl.*



To find out the side of a pentagone to be inscribed in a circle *ADB*.

Draw the diameter *AB*, to which erect a perpendicular *CD* at the center *C*, divide *CB* equally in *E*. and make *EF = ED*. then *DF* shall be the side of the pentagone.

For  $BF \times FC + ECq^a = EFq^b = EDqc = DCq + ECq^d$  therefore  $BF \times FC = DCq$  or  $BCq$ .  
 $e$  wherefore  $BF \cdot BC :: BC \cdot FC$ . therefore since *BC* is the side of a hexagone, *f FC* shall be the side of a decagone. Consequently  $DF^b = \sqrt{DCq + FCq}$   
*g* is the side of a pentagone. *W.W. to be Done.*

a 6. 2.  
 b confr.  
 c 47. 1.  
 d 3. ax.  
 e 17. 6.  
 f 9. 13.  
 g 10. 13.  
 h 47. 1.



PROP. XI.



If in a circle ABCD, whose diameter is rational AG, an equilateral pentagon be inscribed ABCDE; the side of the pentagon AB is an irrational line, of that kind which is called a minor line.

Draw the diameter BFH, and the right lines AC, AH; and \* make FL =  $\frac{1}{4}$  of the ray FH; and CM =  $\frac{1}{4}$  CA.

\* 10. 6.

a cor. 10. 13.  
b 32. 1.  
c 4. 6.

d 14. 5.  
e 18. 5.  
f 22. 6.  
g 1. 13.

h 9. 10.  
k 74. 10.  
l 9. 10.

m cor 8 6.  
and 17. 6.  
n 95. 10.

Because the angles AKF, AIC, are a right angles, & CAI commune, the triangles AKF, AIC, are b equiangular: c therefore CI. FK c :: CA. FA (FB) d :: CM. FL. therefore by permutation FK. FL :: CI. CM d :: CD. CK (2 CM) and so by e composition CD + CK. CK :: KL. FL. f consequently Q: CD + CK (g 5 CKq.) CKq :: KLq. FLq. therefore KLq = 5 FLq. wherefore if BH (p) be taken 8, FH shall be 4, FL 1, and FLq 1, BL 5, and BLq 25, KLq 5. by which it appears that BL and KL are p b  $\sqrt{\quad}$ , and so BK is a residuall, and KL its congruent or adjoining line. but being BLq - KLq = 20, l thence BL  $\sqrt{\quad}$   $\sqrt{BLq - KLq}$ . m whence BK shall be a fourth residuall line. Therefore because ABq m is = HB x BK, n shall AB be a minor line. which was to be Dem.

PROP. XII.



If in a circle ABEC an equilateral triangle ABC be inscribed, the side of that triangle AB is in power triple to the line AD drawn from D the centre of the circle to the circumference.

The diameter being extended to E, draw BE. Be-

cause the arch BE = EC, the arch BE is the sixt part of the circumference. therefore BE = DE. hence AEq = 4 DEq (4 BEq) = ABq + BEq (+ ADq) consequently ABq = 3 ADq. *W. W. to be Dem.*

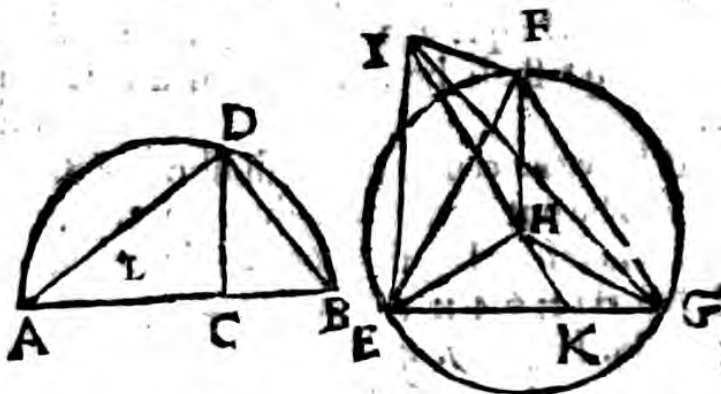
a cor. 10, 13.  
b cor. 15. 4.  
c 4. 2.  
d 47. 1.  
e 3. ex. 1.

Coroll.

- 1. AEq. ABq :: 4. 3.
- 2. ABq. AFq :: 4. 3. f For ABq. AFq :: AEq. ABq.
- 3. DF = FE. For the triangle EBD g is equilateral, and BF perpendicular to ED. therefore EF = FD.
- 4. Hence, AF = DE + DF = 3 DE.

f cor. 8. 6.  
and 22. 6.  
g cor. 15. 4.  
h cor. 3. 3.

PROP. XIII.



To describe a pyramide EGFI, and comprehend it in a sphere given: and to demonstrate that the diameter of the sphere AB is in power sesquialtera of the side EF of the pyramide EGFI.

X 3

About

a 10. 6.

About AB describe the semicircle ADB ; a and let AC be = 2 CB. from the point C erect the perpendicular line CD; and join AD, DB. then at the intervall of the ray HE = CD describe the circle HEFG. b wherein inscribe the equilaterall triangle EFG. from H erect IH = CA perpendicular to the plane EFG. produce IH to K, d so that IK = AB; and join the right lines IE, IF, IG. Then EFGI shall be the pyramide required.

b cor. 15. 4.  
e 12. 11.  
d 3 14

e const.  
f 41. 1.  
g 20. 6.

h 2. ax.  
k 12. 13.  
l 1. ax. 1.

m 8. ax.

n 15. def. 1.  
o 31. def. 11.

p cor. 8. 6.  
q const.

For because the angles ACD, IHE, IHF, IHG, e are right angles; & CD, HE, HF, HG e equall, e & IH = AC; f therefore AD, IE, IF, IG shall be equall among themselves. But being AC (2 CB.) CB g :: ACq. CDq. thence shall ACq be = 2 CDq, therefore ADq f = ACq + CDq = 3 CDq = 3 HEq k = EFq. l therefore AD, EF, IE, IF, IG are equall, and so the pyramide EFGI is equilateral. But if the point C be placed upon H, and AC upon HI, the right lines AB, IH, m shall agree, as being equall. Wherefore the semicircle ADB being drawn about the axis AB or IK n shall passe by the points E, F, G, o and so the pyramide EFGI shall be inscribed in a sphere. W. W. to be Done.

Also it is manifest that BAq. ADq o :: BA.AC, q :: 3. 2. W. W. to be Dem.

Coroll.

r 11. 13.

s const.

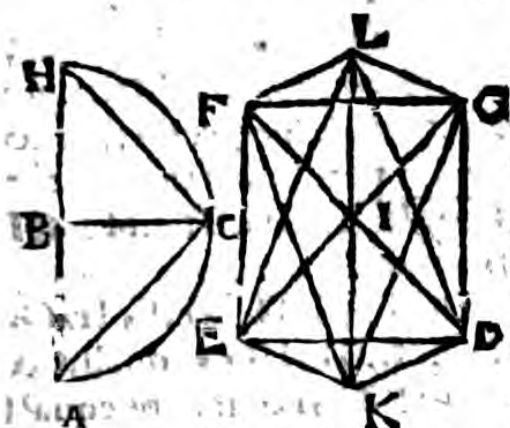
1. ABq. HEq :: 9. 2. For if ABq be put 9, then ACq (EFq) shall be 9, consequently HEq shall be 2.

2. If L be the center, then shall AB. LC :: 6. 1. For if AB be put 6, then AL shall be 3. r and thence AC 4. wherefore LC shall be 1. Hence

3. AB.HI :: 6. 4 :: 3. 2. whence

4. ABq. HIq :: 9. 4.

PROP. XIV.



To describe an Octaedron KEFGDL, and comprehend it in the given sphere, wherein a pyramide is; and to demonstrate that AH the diameter of the sphere is in power double of AC the side of that Octaedron.

About AH describe the semicircle ACH, and from the center B erect the perpendicular BC. draw AC, HC. then upon ED = AC<sup>a</sup> make the square EFGD, whose diameters DF, EG, cut in the center I, from I draw IL = AB<sup>b</sup> perpendicular to the plane EFGD. produce IL, <sup>c</sup> till IK = IL. and join KE, KF, KG, KD, LE, LF, LG, LD; then shall KEFGDL be the Octaedron required.

For AB, BH, FI, IE, &c. being semidiameters of equall squares are equall one to the other. <sup>d</sup> whence the bases LF, LE, FE, &c. of the rightangled triangles LIE, LIF, FIE, &c. are equall, and consequently the eight triangles LFE, LFG, LGD, LDE, KEF, KFG, KGD, KDE, are equilaterall, <sup>e</sup> and make an Octaedron, which may be inscribed in a sphere, whose center is I, and IL or AB the radius. (because AB, IL, IF, IK, &c. fare equall.) *W. W. to be Done.* Moreover, it is evident that AHq (LKq)g = 2 ACq (2 LDq.) *W. W. to be Dem.*

- Coroll.

1. Hence it is manifest, that in the octaedron the three diameters EG, FD, LK doe cut one the other perpendicularly in the center of the sphere.

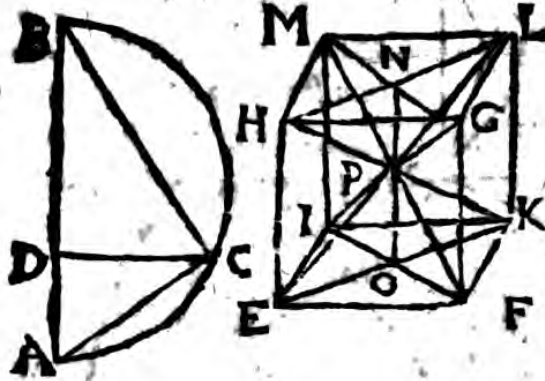
2. Also that the three planes EFGD, LEKG, LFKD are squares, cutting one another perpendicularly.

3. The Octaedron is divided into two like and equal pyramids EFGDL, and EFGDK, whose common base is the square EFGD.

15. 11.

4. Lastly, it follows that the opposite bases of the octaedron are parallel one to the other.

PROP. XV.



To describe a cube EFGHIKLM, and comprehend it in the same sphere wherein the former figures were; and to demonstrate that AB the diameter of the sphere is in power triple to

EF the side of that cube.

a 10. 6.

b 46. 11.

Upon AB describe a semicircle ACB, and make  $AB = 3 DA$  from D raise the perpendicular DC, & join BC and AC. Then upon  $EF = AC$  make the square EFGH, upon whose plane let the right lines EI, FK, HM, GL, stand perpendicular, being equal to EF, & connect them with the right lines IK, KL, LM, IM. The solid EFGHIKLM is a cube, as is sufficiently apparent from the construction.

e cor. 39 11.  
d 15. def. 11.  
and 14 def.  
11.

e 47. 1  
f confr  
g cor. 8. 6.  
h 14 5.

In the opposite squares EFKI, HGLM, draw the diameters EK, FI, HL, MG, by which let the planes EKLH, FIMG be drawn, cutting one another in the line NO. which shall divide equally in two parts the diameters of the cube EL, FM, GI, HK, in P the center of the cube. & therefore P shall be the center of a sphere passing by the angular points of the cube. Moreover,  $ELq = EKq + KLq = 3 KLq$ , for  $3 ACq$ . but  $ABq.ACq :: BA.DA :: 3.1.$  therefore  $AB = EL$  wherefore we have made a cube, &c. *W.W. to be Done.*

Coroll.

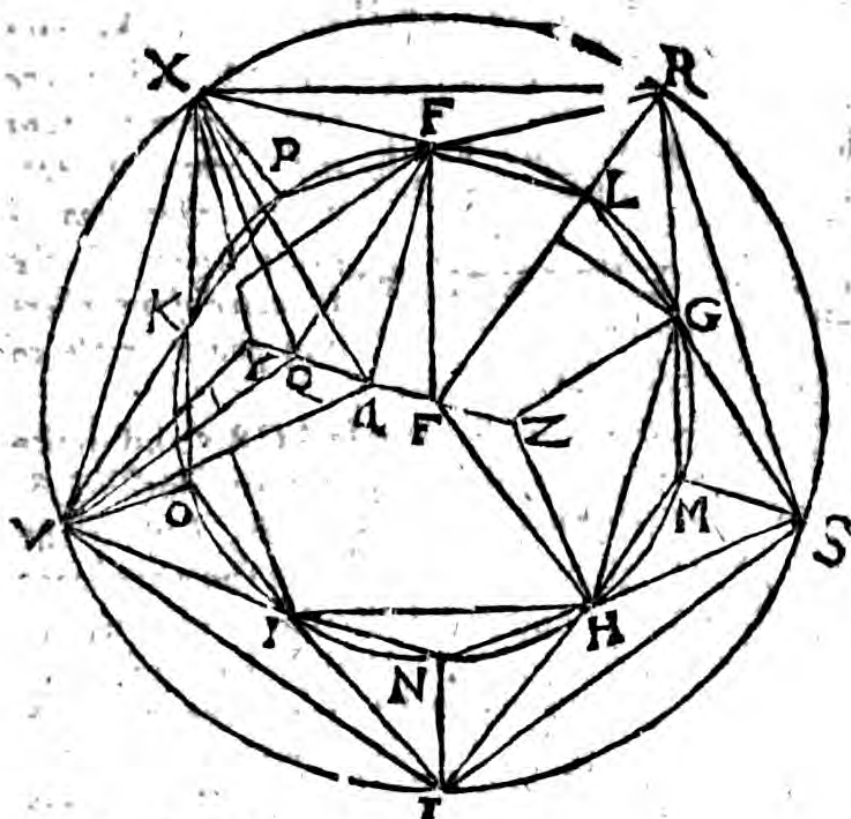
1. Hence it is manifest, that all the diameters of the cube are equal one to another, and do equally bisect one another in the center of the sphere. And by the same means the right lines which conjoin the cen-

centers of the opposite squares are equally bisected in the same center.

2. The diameter of a sphere containeth in power the side of a tetraedron and of a cube, viz.  $ABq = BCq + ACq$ .

k47. r.  
l 13, 15.  
m 19, 23.

PROP. XVI.



To describe an Icosaedron  $ZGHKIFYVXRST$ , and encompassse it in the sphere, wherein were contained the foresaid solids; and to demonstrate that  $FG$  the side of the Ifocaedron is that irrational line, which is called a minor line.

Upon  $AB$  the diameter of a sphere describe the semicircle  $ADB$ ; & make  $AB = 5 BC$ . then from  $C$  erect  $CD$  perpendicular, and draw  $AD$  and  $BD$ . At the distance  $EF = BD$  describe the circle  $EFG$ .



a 10. 6.

b 11. 4 NG; *b* wherein inscribe the equilaterall pentagone  
FKIHG. Divide equally in two parts the arches FG,  
GH, &c. and join the right lines FL, LG, &c. being  
the sides of a decagone. Then *c* erect EQ, LR, MS,  
c 12. 11. NT, OV, PX equall to EF, and perpendicular to the  
plane FKNG; and connect RS, ST, TV, VX, XR; as  
also FX, FR, GR, GS, HS, ST, HT, IT, IV, KV, KX.  
Lastly, produce EQ, and take QY = FL, and EZ  
= FL. and conceive the right lines ZG, ZH, ZI,  
ZK, ZF to be drawn; as also YV, YX, YR, YS,  
YT. Then I say the Icosaedron required is  
made.

d const.  
e 6 11.  
f 33. 1.

g 15. 11.

h 1. def. 3.

k 47. 1.  
l const.  
m 10. 13.  
n feb 48. 1.  
and 1 ax.

o cor. 14. 11.  
p 47. 1.  
q 10. 13.

r 15. def. 1.

s 1.

For because EQ, LR, MS, NT, OV, PX, are de-  
quall and *e* parallel, also those lines that join them  
EL, QR, EM, QS, EN, QT, EO, QV, EP, QX, *f* are  
equall & parallel. And thence likewise LM (or FG)  
RS, MN, ST, &c. are equall one to the other, *g* there-  
fore the plane drawn by EL, EM, &c. is distant e-  
qually from the plane passing by QR, QS, &c. *b* and  
the circle QXRSTV drawn from the center Q is e-  
quall to the circle EPLMNO; and RSTVX is an  
equilaterall pentagone. But EF, EG, EH, &c. and  
QX, QR, QS, &c. being conceived to be drawn; then  
because FRq<sup>k</sup> = FLq + LRq, <sup>l</sup> or EFq<sup>m</sup> = FGq,  
<sup>n</sup> therefore FR, FG, and so all RS, FG, FR, RG, GS,  
GH, &c. shall be equall one to the other. and conse-  
quently the ten triangles RFX, RFG, RGS, &c. are  
equilaterall and equall. Moreover, because XQY is a  
<sup>o</sup> right angle, therefore XYq<sup>p</sup> = QXq + QYq <sup>q</sup> =  
VXq or FGq. wherefore XY, VX, and likewise YV,  
YT, YS, YR, ZG, ZH, &c. are equall. Therefore o-  
ther ten triangles are made, equilaterall and equall  
both to one another, and to the ten former; and so  
an Icosaedron is made.

Moreover, divide equally EQ in *a*, draw the right  
lines *a*F, *a*X, *a*V; and because QX<sup>r</sup> = QV, and *a*Q  
the common side, and FQX, EQV are right angles,  
*s* therefore shall *a*X be = *a*V; and by the same rea-  
son all the lines *a*X, *a*R, *a*S, *a*T, *a*V, *a*F, *a*G, *a*H, *a*I, *a*K  
are

are equal. But because  $ZQ.QE :: QE.ZE$ . therefore  $Zaq^u = 5 Eaq^x = EQq (EEq) + Eaq^y = aFq$ . therefore  $Za = aF$ . & in like manner  $aF = Ya$ . therefore the sphere, whose center is  $a$ , and  $aF$  the ray, shall passe by the 12 angular points of the Icofaedron.

Lastly, because  $Za.aE :: ZY.QE$ ;  $a$  and so  $Zaq$ ,  $aEq :: ZYq.QEq$ .  $b$  therefore  $ZYq = 5 QEq$ , or  $5 BDq = :$  but  $ABq.BDq :: AB.BC :: 5.1$ .  $d$  therefore  $ZY = AB$ . *W.W. to be Done.*

Therefore if  $AB$  be put  $\dot{p}$ ,  $e$  then  $EF = \sqrt{ABq}$  shall be also  $\dot{p}$ . and consequently  $FG$  the side of the pentagone, and likewise of the Icofaedron,  $f$  is a minor line. *Which was to be Demonstrated.*

*Coroll.*

1. From hence is inferred, that the diameter of the sphere is in power quintuple of the semidiameter of the circle encompassing the five sides of the Icofaedron.

2. Also it is manifest that the diameter of the sphere is composed of the side of a hexagone, that is, of the semidiameter, and two sides of the decagone of a circle encompassing the five sides of the Icofaedron.

3. It appears likewise that the opposite sides of an Icofaedron, such as  $RX$ ,  $HI$ , are parallels. For  $RX$   $a$  is parall. to  $LP$ .  $b$  parall. to  $HI$ .

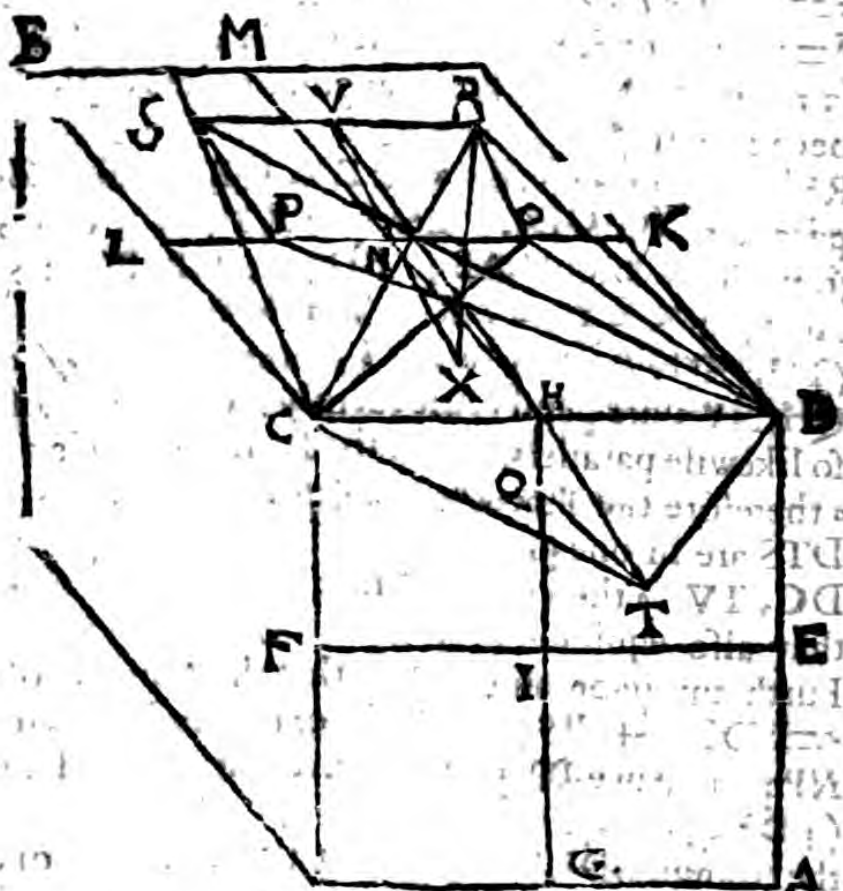
t 9. 13.  
u 3. 13.  
x 4. 2.  
y 47. 1.  
z 15. 5.

a 12. 6. 1  
b 14. 5.  
c cor. 8. 6.  
d 1. ax. 1.  
e feb. 12. 10.  
f 11. 13.

a 33. 5.  
b feb. 26. 3.



## PROP. XVII.



To describe a Dodecaedron, and comprehend it in the Sphere wherein the former figures were comprehended: and to demonstrate that the side RS of the Dodecaedron is an irrationall line of that sort which is called an apotome or residuall line.

a 30. 6.

Let AB be a cube inscribed in the given sphere, and let all the sides thereof be divided equally in the points E, H, F, G, K, L, &c. and join the right lines KL, MH, HG, EF. make  $HI.IQ :: IQ.QH$ ; and take  $NO, NP, = IQ$ . then erect OR, PS, perpendicular to the plane DB, and QT to the plane AC; and let OR, PS, QT, be equall to IQ, NO, NP. whence DR, RS, SC, CT, DT, being connected, DRSC shall be a pentagone of the dodecaedron required. For draw NV parallel to OR, and having drawn NV out as farre as the center of the cube X, join

join the right lines DS, DO, DP, CR, CP, HV, HE, RX. Because  $DOq = DKq$  (b KNq) +  $KOq$  a 47. 1.  $c = 3 ONq$  (3 ORq) & thence  $DRq = 4 ORq$  b 7. 1.  $e = OPq$ , or  $RSq$ . therefore  $DR = RS$ . By the same reason  $DR, RS, SC, CT, TP$  are equal. But because  $ORf$  is = and  $g$  parallel to  $PS$ , therefore  $RS, OP$ , and  $b$  consequently  $RS, DC$  shall be also parallels.  $b$  therefore these with them that conjoin them  $DK, CS, VH$ , are in one and the same plane. Moreover, because  $HI, IQk :: IQ$  (TQ.)  $QHk :: HN, NV$ , and both  $TQ, HN$ , and  $QH, NV$  are perpendicular to the same plane, and so likewise parallels,  $m$   $THV$  shall be a right line.  $n$  therefore the Trapezium  $DRSC$ , and the triangle  $DTS$  are in one plane extended by the right lines  $DC, TV$ . therefore  $DCPSR$  is a pentagone, and that also equilaterall, by what is shewn already. Furthermore, because  $PK, KN :: KN, NP$ ; and  $DSq = DPq + PSq$  (PNq) =  $DKq + PKq + NPq$ ,  $q$  thence  $DSq = DKq + 3 KNq = 4 DKq$  (4 DHq) =  $DCq$ . therefore  $DS = DC$ . whence the triangles  $DRS, DCT$ , are equilaterall one to another.  $f$  therefore the angle  $DRS = DTC$ , and likewise the ang.  $CSR = DCT$ . therefore the pentagone  $DTCSR$  is also equiangular. Moreover, because  $AX, DX, CX, &c.$  are semidiameters of the cube,  $r$  thence is  $XN = IH$  or  $KN$ ,  $u$  and so  $XV = KP$ ; wherefore because  $RVX$ , is a  $x$  right angle,  $z$  thence  $RXq = XVq + RVq$  (NPq) =  $KPq + NPq$   $a = 3 KNq$   $b = AXq$  or  $DXq$ , &c. therefore  $RX, AX, DX$ , and by the same reason  $XS, XT, AX$ , are equal one to another. And if by the same method, whereby the pentagone  $DTCSR$  was made, twelve like pentagones, touching the twelve sides of the cube, be made, they shall compose a Dodecaedron, and a sphere passing by their angular points, whose ray is  $AX$  or  $BX$ , shall comprehend that Dodecaedron. *W. W. to be Done.*

Lastly, because  $KN, NOc :: NO, OK$ .  $d$  thence  $c$  *constr.*  $KL$ .  $d$  15. 5.





Schol.

*It is very apparent that besides the five aforesaid figures, there cannot be described any other regular solid figure (viz such as may be contained under ordinate and equall plane figures.)*

a & b. 11.  
b See schol.  
32. 1.

For three plane angles at least are required to the constituting of a solid angle; *a* all which must be lesse then four right angles. *b* But 6 angles of an equilaterall triangle, 4 of a square, and six of a hexagon, do severally equall 4 right angles; & 4 of a pentagon, 3 of a heptagon, 3 of an octagons, &c. do exceed 4 right angles: Therefore only of 3, 4, or 5 equilaterall triangles, of 3 squares, or 3 pentagones, it is possible to make a solid angle. Wherefore besides the five above mentioned, there cannot be any other regular bodies.

### Out of P. Herigon.

*The Proportions of the sphere and the five regular figures inscribed in the same.*

Let the diameter of the sphere be 2. then shall

The Peripherie or circumference of the greater circle, be 6 28318.

The superficies of the greater circle, 3 14159.

The superficies of the sphere, 12 756637.

The solidity of the sphere, 4 1879.

The side of the tetraedron, 4 62299.

The

**EUCLIDE'S Elements.**

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The superficies of the tetraedron, 4 6188.

The solidity of the tetraedron, 0 15132.

The side of the Hexaedron, 1 1547.

The superficies of the hexaedron, 8.

The solidity of the hexaedron, 1 5396.

The side of the Octaedron, 1 41421.

The superficies of the octaedron, 6 9282.

The solidity of the octaedron, 1 33333.

The side of the Dodecaedron, 0 71364.

The superficies of the dodecaedron, 10 51462.

The solidity of the dodecaedron, 2 78516.

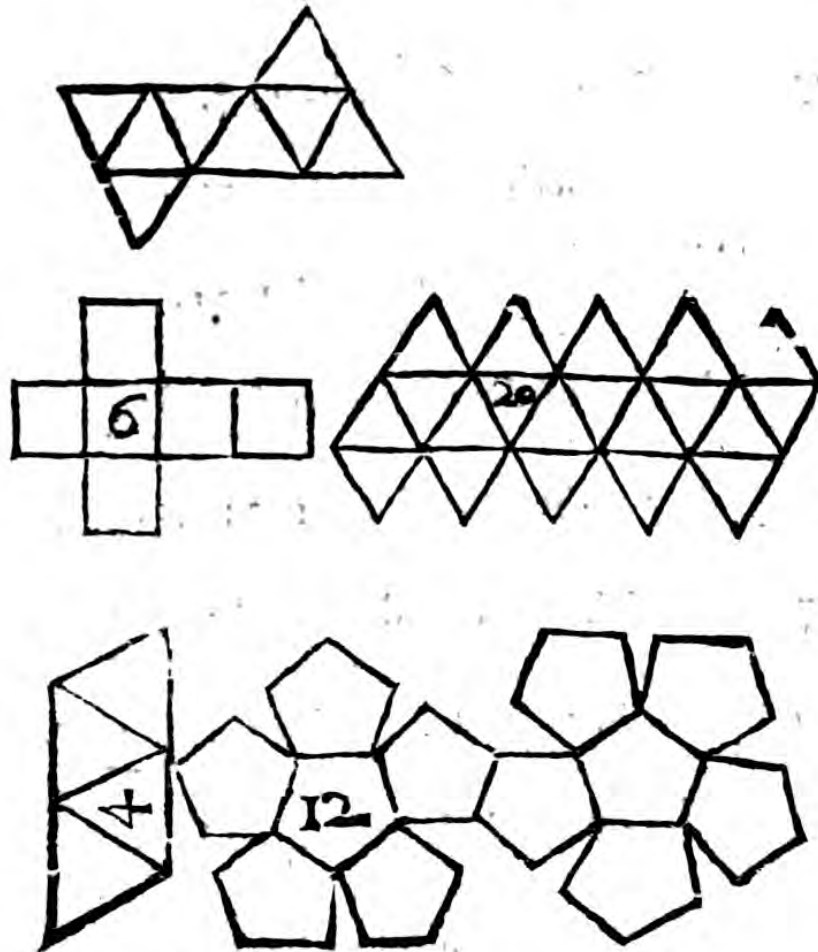
The side of the Icosaedron, 1 05146.

The superficies of the Icosaedron, 9 57454.

The solidity of the Icosaedron, 2 53615.

*The thirteenth Book of, &c.*

*If five equilaterall and equiangular figures, like these in the schemes beneath, be made of paper, and rightly folded, they will represent the five regular bodies.*



*The End of the thirteenth Book.*

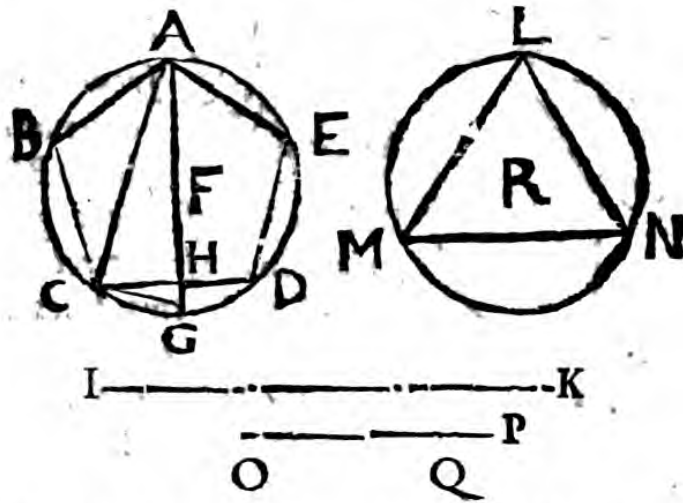




a 17. 6.  
b 8. 2.  
c 2. ax. 1.  
d 22. 5. and  
22. 6.  
e 22. 5.  
f 17. 5.

Take  $BC = BG$ ; and  $EF = EH$ . Then  $AB \times BG$  is  $a = 5$   $AGq$ . wherefore  $ACq \times b = 4$   $ABG + AGq$   $c = 5$   $AGq$ . In like manner shall  $DFq \times b = 5$   $DHq$ .  $d$  therefore  $AC \cdot AG :: DF \cdot DH$ . whence by addition  $AC + AG \cdot AG :: DF + DH \cdot DH$ . that is,  $2$   $AB \cdot AG$   $:: 2$   $DE \cdot DH$ .  $e$  consequently  $AB \cdot AG :: DE \cdot DH$ ;  $f$  whence by division  $AG \cdot GB :: DH \cdot HE$ . *W.W. to be Dem.*

PROP. III.



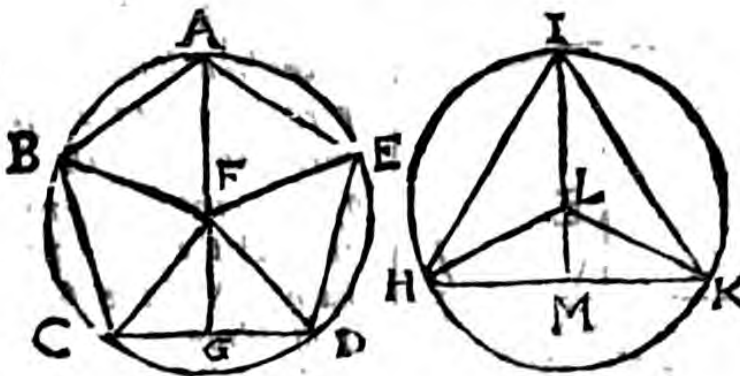
The same circle ABD comprehends both ABCDE the pentagone of a Dodecaedron, and LMN the triangle of an Icofaedron inscribed in the same sphere.

a feb. 47. 1.  
b 30. 6.  
c 47. 1.  
d 4. 2.  
e 10. 13.  
f 1. & 3. ax.  
g 8. 13.  
h 2. 13. & 16.  
i 5.  
k 22. 6. and  
4. 5.  
l 15. 13.  
m const.  
n cor. 16. 13.  
o 12. 13.  
p 10. 13.  
q 16. 5.  
r before.  
s 1. ax. 1. and  
feb. 48. 1.  
t 1. def. 3.

Draw the diameter  $AG$ , and the right lines  $AC$ ,  $CG$ . and let  $IK$  be the diameter of the sphere,  $a$  and  $IKq = 5$   $OPq$ .  $b$  & make  $OP \cdot OQ :: OQ \cdot QP$ . Because  $ACq + CGq = AGq = 4$   $FGq$ ; &  $ABq = FGq = CGq$ .  $f$  thence  $ACq + ABq = 5$   $FGq$ . moreover, because  $CA \cdot AB \cdot g :: AB \cdot CA - AB$ ; and  $OP \cdot OQ :: OQ \cdot QP$ .  $b$  and so  $CA \cdot CP :: AB \cdot OQ$ .  $k$  therefore  $3$   $ACq$ . ( $l$   $IKq$ .)  $5$   $OPq$  ( $m$   $IKq$ )  $:: 3$   $ABq$ .  $5$   $OQq$ . therefore  $3$   $ABq = 5$   $OQq$ . But because  $ML$  is the side of a pentagon inscribed in the circle, whose ray is  $OP$ , thence  $15$   $RMq = 5$   $MLq$   $p = 5$   $OPq + 5$   $OQq = 3$   $ACq + 3$   $ABq$   $q = 15$   $FGq$ .  $r$  therefore  $RM = FG$ .  $f$  and consequently the circle  $ABD$  is  $=$  to the circle  $LMN$ . *W.W. to be Dem.*

PROP.

PROP. IV.



If from F the center of a circle encompassing the pentagone of a dodecaedron ABCDE, a perpendicular line FG be drawn to one side of the pentagone CD; the rectangle contained under the said side CD and the perpendicular FG, being thirty times taken, is equal to the superficies of the Dodecaedron. Also,

If from the center L of a circle inclosing the triangle of an Icosaedron HIK, a perpendicular line LM be drawn to one side of the triangle HK, the rectangle contained under the said side HK, & the perpendicular LM, being thirty times taken, shall be equal to the superficies of the Icosaedron.

Draw FA, FB, FC, FD, FE. <sup>a</sup> then shall the triangles CFD, DFE, EFA, AFB, BFC be equal. but CD <sup>a</sup> 8. 1.  
 $\times FG$  <sup>b</sup> = 2 triangles CFD. therefore  $30 CD \times GF$  <sup>b</sup> 4. 1.  
<sup>c</sup> = 60 CFD <sup>d</sup> = 12 pentagones ABCDE <sup>e</sup> = to <sup>c</sup> 15 5.  
the superficies of the dodecaedron. *W.W. to be Dem.* <sup>d</sup> 6 ax.

Draw LI, LM, LK, then HK  $\times LM$  <sup>f</sup> is = 2 triang. <sup>e</sup> 17. 3.  
LHK. therefore  $30 HK \times LM$  <sup>g</sup> = 60 HLK = 20 <sup>f</sup> 41. 1.  
HIK <sup>b</sup> = to the superficies of the Icosaedron. *W.W.* <sup>g</sup> 15. 5.  
to be Dem. <sup>h</sup> 16. 13.

Coroll.

$CD \times FG. HK \times LM$  <sup>k</sup> :: the superficies of the dodecaedron to the superf. of the Icosaedron. <sup>k</sup> 15. 5.



P R O P. VI.



If a right line AB be cut in extreme and mean proportion, then as the right line BF, containing in power that which is made of the whole line AB, and that which is made of the greater segment AC, is to the right line E containing in power

that which is made of the whole line AB, and that which is made of the lesser segment BC; so is the side of the cube BG to the side of an Icosaedron BK inscribed in the same sphere with the cube.

In the circle, whose semidiameter is AB, inscribe BFGHI the pentagone of a dodecaedron, and BKL the triangle of an Icosaedron. <sup>a</sup> wherefore BG shall be the side of a cube inscribed in the same sphere. therefore BK <sup>b</sup> = 3 AB <sup>c</sup>; and Eq <sup>d</sup> = 3 AC <sup>e</sup>. wherefore by inversion BG <sup>f</sup> . BK :: BF . E. *W.W. to be Dem.*

<sup>a</sup> cor. 17. 13.  
<sup>b</sup> 12. 13.  
<sup>c</sup> 4. 13.  
<sup>d</sup> 15. 5.  
<sup>e</sup> 2. 14.  
<sup>f</sup> 22. 6.

P R O P. VII.

A Dodecaedron is to an Icosaedron, as the side of a Cube is to the side of an Icosaedron, inscribed in one and the same sphere.

Because <sup>a</sup> the same circle comprehends both the pentagone of a dodecaedron, and the triangle of an Icosaedron, <sup>b</sup> the perpendiculars drawn from the center of the sphere to the planes of the pentagone and triangle, shall be equall one to another. Therefore if the dodecaedron and Icosaedron be conceived divided into pyramids, right lines being drawn from the center of the sphere to all the angles, the altitudes of all the pyramids shall be equall one

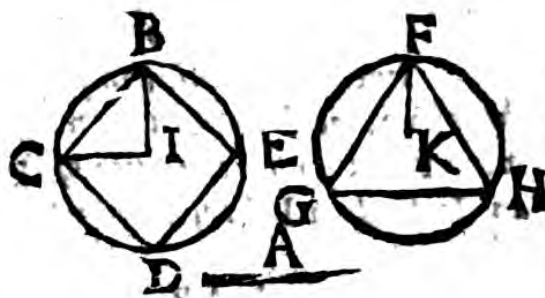
<sup>a</sup> 3. 14.  
<sup>b</sup> 47. 1.

c 5, and 6.  
12.

to the other. Wherefore since the pyramids *c* of equal height are one to another as their bases, & the superficies of the dodecaedron is equal to twelve pentagones, and the superficies of the Icosaedron to twenty triangles, the dodecaedron shall be to the Icosaedron, as the superficies of the dodecaedron is to the superficies of the Icosaedron, *d* that is, as the side of the cube is to the side of the Icosaedron.

d 5: 14

P R O P. VIII.



The same circle BCDE comprehends both the square of the cube BCDE; and the triangle of the octaedron FGH inscri-

bed in one and the same sphere.

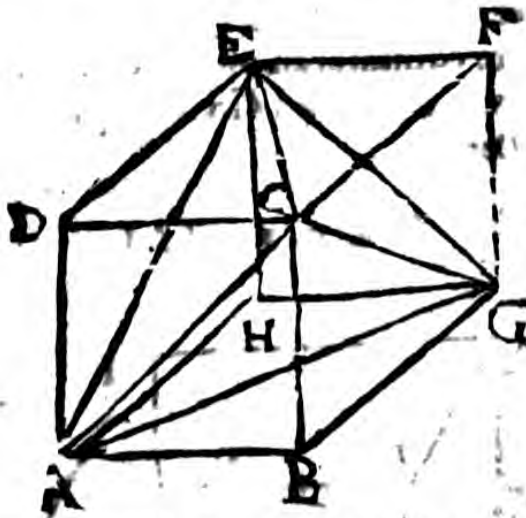
a 15. 13.  
b 47. 1  
c 14. 13.  
d 22. 13.  
e 2. def. 3.

Let A be the diameter of the sphere. Because  $Aq = 3 BCq$   $b = 6 BIq$ ; and also  $Aqc = 2 GFq$   $d = 6 KFq$ ; thence shall  $BI = KF$ . *e* therefore the circle  $CBED = GFH$ . *W.W.* to be Demonstrated.

The End of the fourteenth Book.

THE FIFTEENTH BOOK  
OF  
EUCLIDE'S ELEMENTS.

PROPOSITION I.



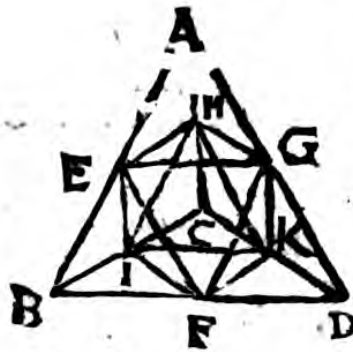
**I**N a cube given  $ABGHDCFE$  to describe  
a pyramide  $AGEC$ .

From the angle  $C$  draw the diameters  $CA, CG, CE$ ; and connect them with the diameters  $AG, GE, EA$ . All which are  $a$  equal among themselves, as being the diameters of equal squares: therefore the triangles  $CAG, CGE, CEA, EAG$  are equilaterall and equal; and consequently  $AGEC$  is a pyramide, which insits upon the angles of the cube, and therefore  $b$  is inscribed in the same. *W.M. to be Done.*

PROP.

PROP. II.

In a pyramide given  $ABDC$  to describe an octaedron  $EGK-IFH$ .



a 10. 1.

b 4. 11

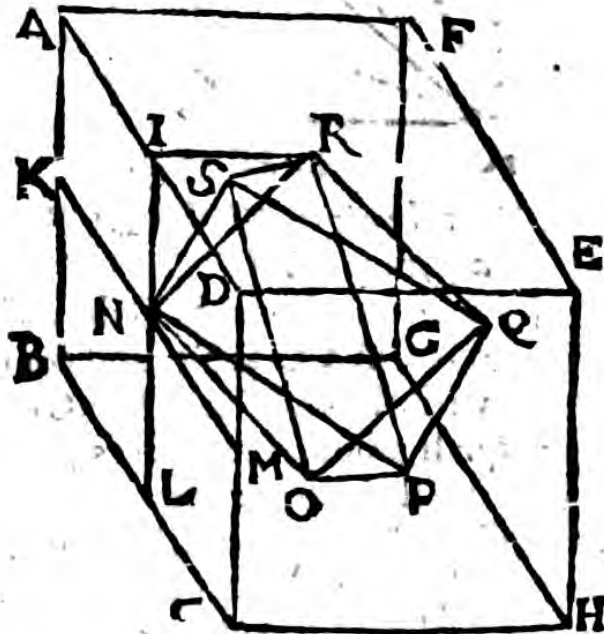
a Bisect the sides of the pyramide in the points,  $E, I, F, K, G, H$ , which join with the right lines  $EF, FG, GE, \&c.$  All these are  $b$  equal one

to the other ; consequently the 8 triangles  $EHI, IHK, \&c.$  are equilaterall and equall , and so make  $c$  an octaedron described  $d$  in the given pyramide.

c 27. def. 19.  
d 31. def. 11.

W.W. to be Done.

PROP. III.



In a cube given  $CHGBDEFA$  to describe an octaedron  $NPQSOR$ .

Connect \* the centers of the squares  $N, P, Q, S, O, R$  with the twelve right lines  $NP, PQ, QS, \&c.$  which are  $a$  equal among themselves ; and so make 8 equilaterall and equall triangles : wherefore  $b$  the Octaedron  $NPQSOR$   $b$  is inscribed in the cube.

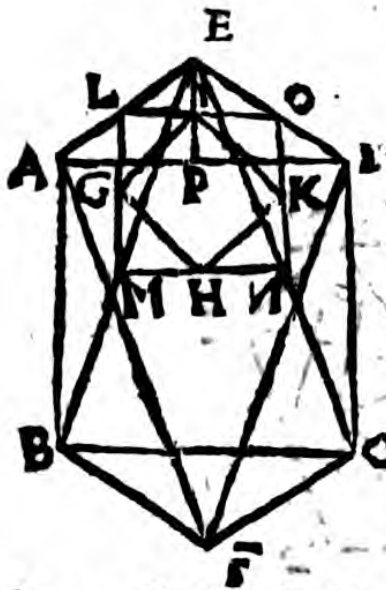
a 8. 4.

a 4. 1.  
b 31. and 27.  
def. 11.

W.W. to be Done.

PROP.

P R O P. I V.



*In an Octaedron given ABCDEF, to inscribe a cube.*

Let the sides of the pyramide EABCD, whose base is the square ABCD, be equally bisected by the right lines, LM, MN, NO, OL, which are *a* equall and *b* parallel to the sides of the square ABCD. *c* then the quadrilaterall LMNO is a square. In like manner, if the sides of the square LMNO

*a* 4.1.  
*b* 2.6.  
*c* 29. def. 1.

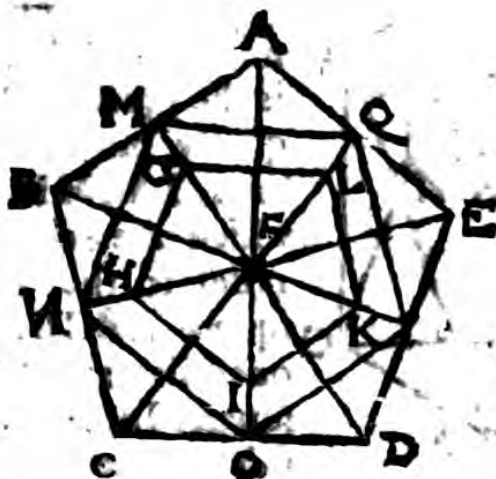
be equally bisected in the points G, H, K, I, & GH, HK, KI, IG connected, GHKI shall be a square. And if in the other 5 pyramides of the octaedron, the centers of the triangles be in the same sort conjoined with right lines, then other squares will be described like and equall to the square GHKI, wherefore six such squares shall make a cube, which shall be described within an octaedron, *d* being its eight angles touch the eight bases of the octaedron in their centers. *W.W. to be Done.*

*d* 31. def. 11

P R O P.



PROP. V.



In an Icosaedron given to inscribe a Dodecaedron.

Let ABCDEF be a pyramide of the Icosaedron, whose base is the pentagone ABCDE; and the centers of the triangles G, H, I, K, L; which connect with the right lines GH, HI, IK, KL, LG. Then GHIKL shall be a pentagone of the dodecaedron to be inscribed.

\* 5. 4.

a cor. 3. 3.

b 4. 1.

c 4. 1.

d 8. 1.

e 4. 1.

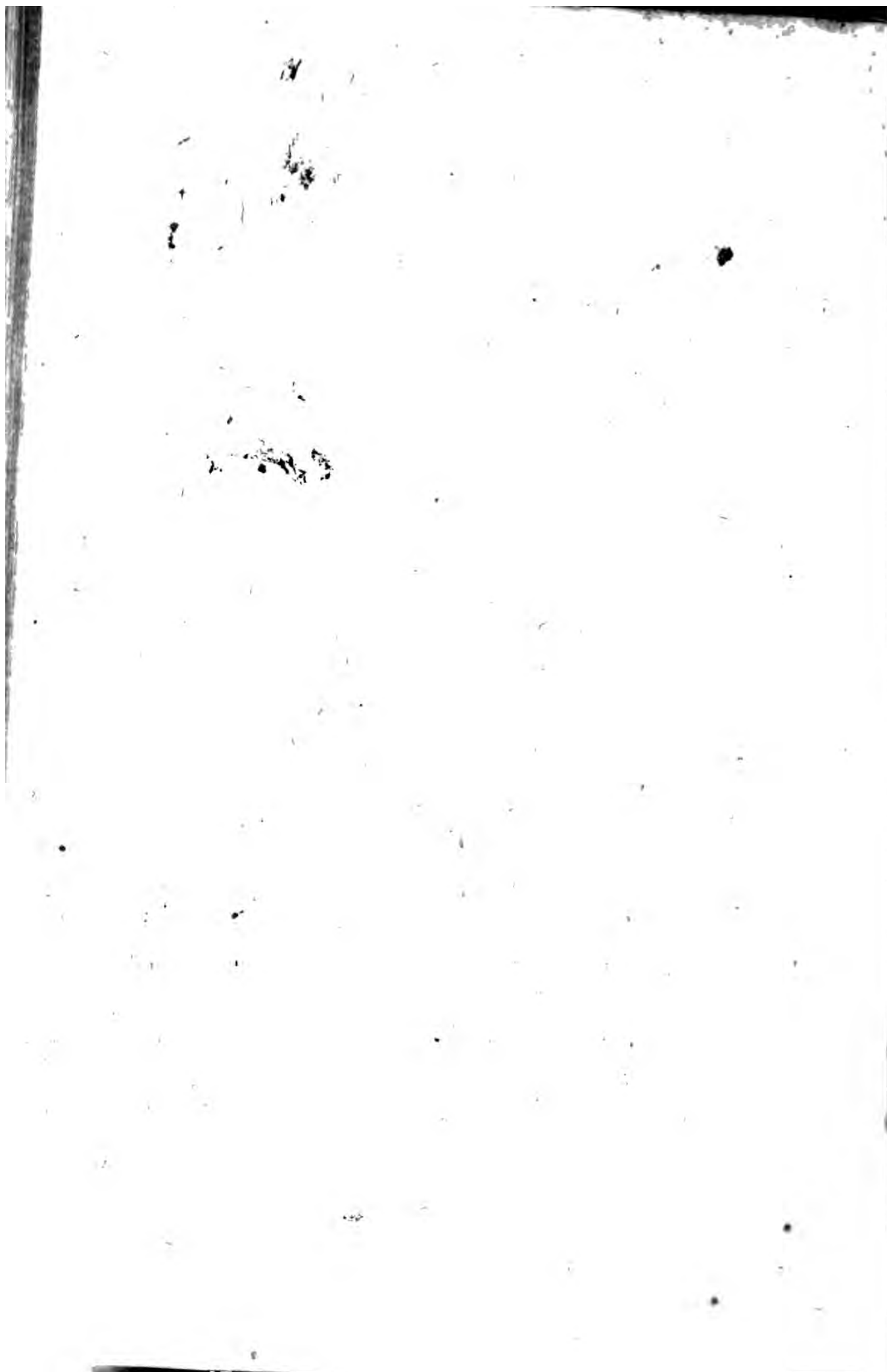
f 12. 13.

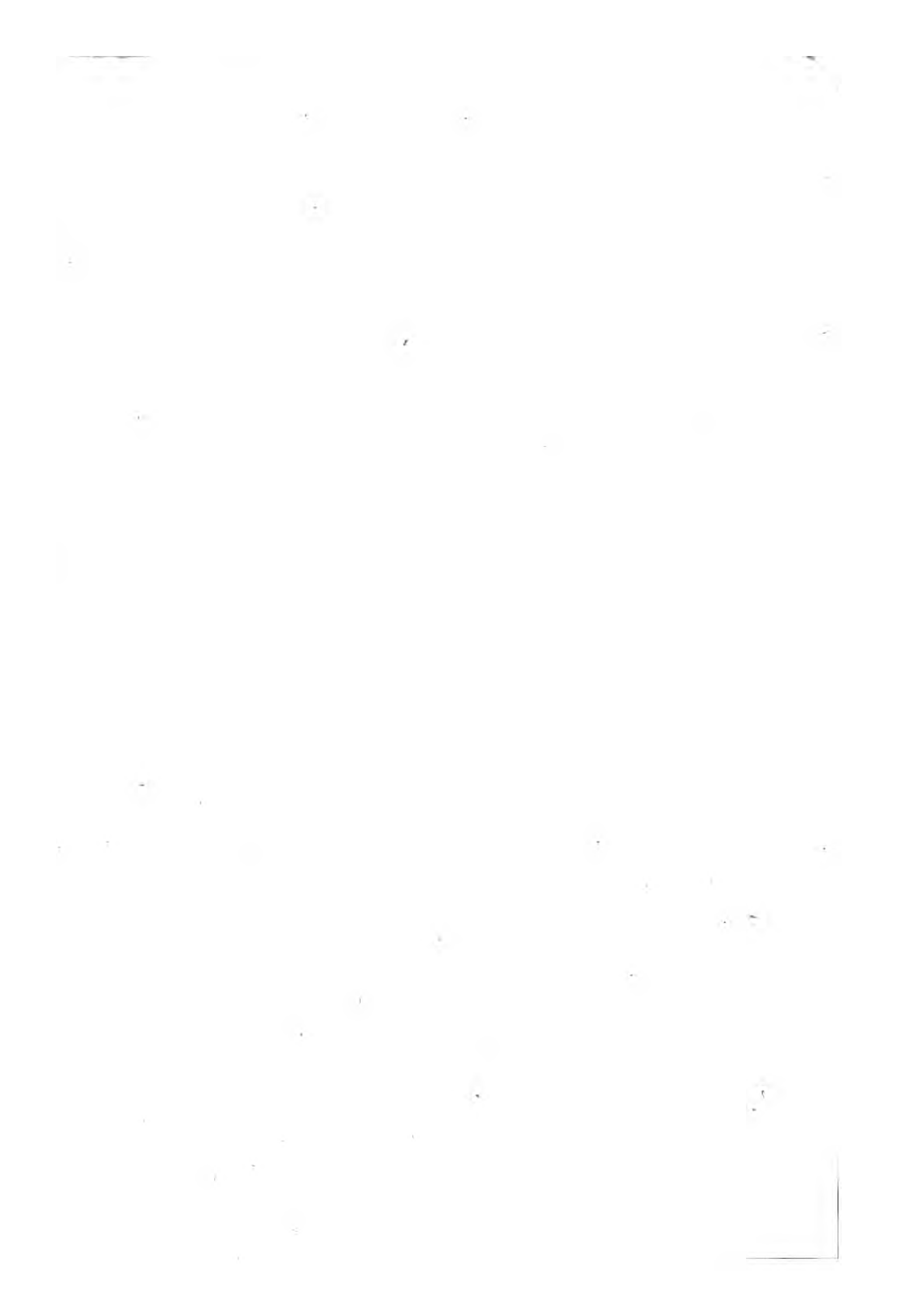
For the right lines FM, FN, FO, FP, FQ, passing by the centers of the triangles, *a* do equally divide their bases into two parts. *b* therefore the right lines MN, NO, OP, PQ, QM *c* are equal one to the other; *d* whence also the angles MFN, NFO, OFP, PFQ, QFM are equal. therefore the pentagone GHIKL is equiangular. *e* and consequently equilaterall, being FG, FH, FI, FK, FL *f* are equal. And if in the other eleven pyramids of the Icosaedron, the centers of the triangles be in like sort conjoined with right lines, then will pentagones equal and like to the pentagone GHIKL be described. Wherefore 12 of such pentagones shall constitute a dodecaedron; which also shall be described  
in

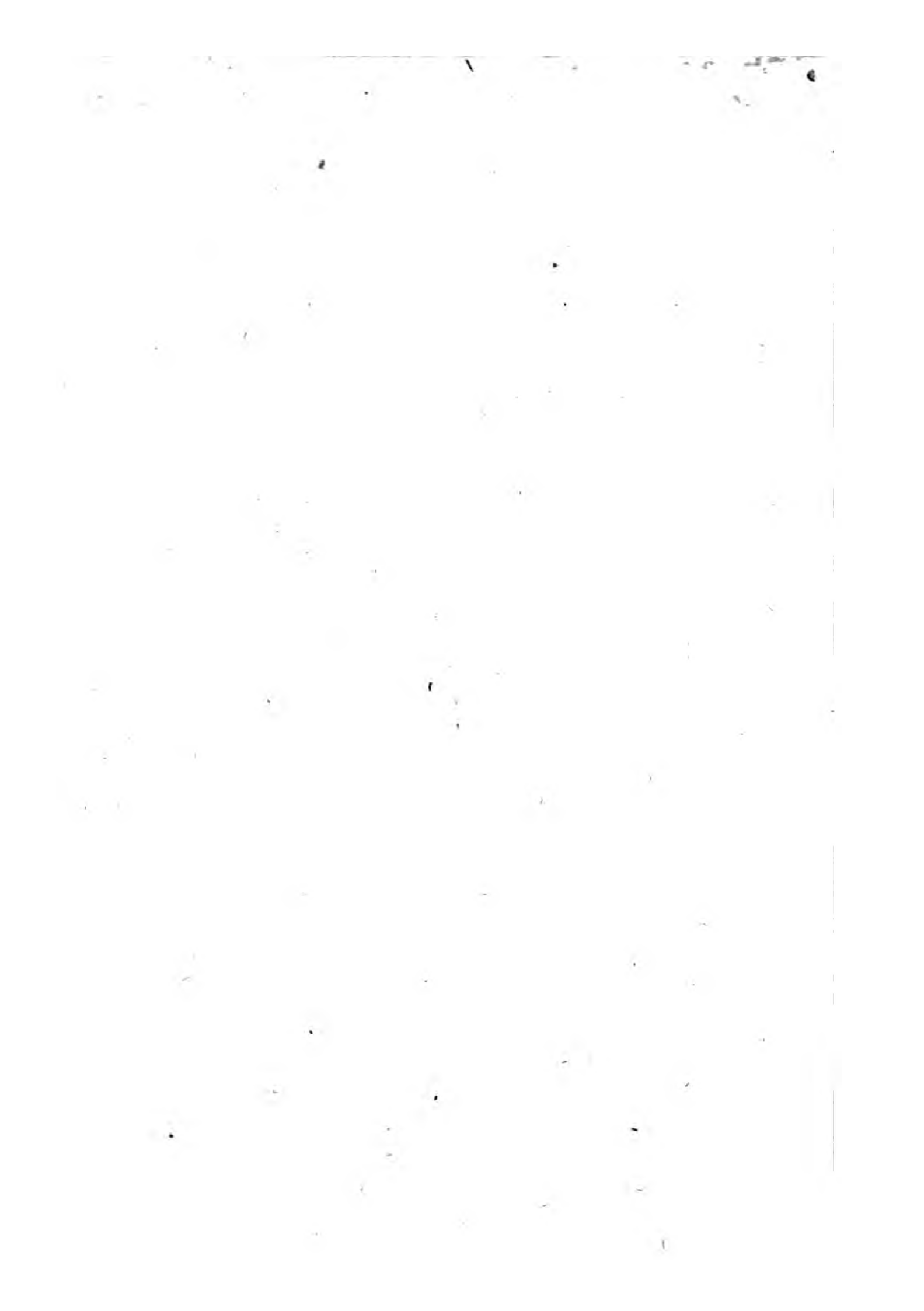
in the Icosaedron, seeing the twenty angles of the dodecaedron consist upon the centers of the twenty bases of the Icosaedron. Whereby it appears that we have described a dodecaedron in an Icosaedron given. *Which was to be Done.*

F I N I S.











\_\_\_\_\_

100

100

100

100

100





