Elementary algebra [by J. Martin].

The Royal School Series.

ELEMENTARY ALGEBRA.

SECOND BOOK.

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PREFACE.

An effort is made in the work, of which this is the Second Part, so to simplify the subject of Algebra as to bring it within the capacity of very young pupils. Its various processes are unfolded in a carefully graduated series of paragraphs; the chief object throughout being to lay a *broad* foundation, so that the learner may afterwards pursue the study with profit in works of a more advanced character.

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CONTENTS.

[The Paragraphs and the Exercises are numbered in continuation of those in the First Book.]

ELEMENTARY ALGEBRA.

SECOND BOOK.

28. The Multiplication of algebraical quantities has been referred to in paragraph 7. In addition to what was therein stated, two other things remain to be determined:

(1.) The **numeral** coefficient of the **product.**

(2.) The **sign** of the product.

The first of these is easily determined, the coefficient of the product being the product of the coefficients.

Thus, the numerical coefficient of $7a \times 4bc = 28$,

$4b \times c = 4$;

or, of $3a \times 4b \times 12c = 144$.

29. In order to determine the **signs** of the product, it must be remembered that Multiplication is a shortened form of Addition. Thus, in ordinary Arithmetic, in order to find 7 times 365, instead of adding 365 seven times together, or *vice versâ*, we merely multiply the numbers together. So in Algebra:

(1.) b, that is $+b$, multiplied by $+a$, means that b is to be added to itself a times. Thus the product is $+ab$.

(2.) $-b$ multiplied by $+a$, means that $-b$ is to be added to itself a times. Thus the product is $-ab$.

(3.) b, that is $+b$, multiplied by $-a$, means that b is to be subtracted as many times as there are units in a. Thus the product is $-ab$.

(4.) $-b$ multiplied by $-a$, means that $-b$ is to be subtracted as many times as there are units in a. But subtracting a negative quantity is the same as adding a positive quantity (paragraph 20). Hence the product is $+ab$.

We thus deduce the following rule: that in the multiplication of algebraical quantities **like** signs produce $+$, and **unlike** signs $-$.

EXERCISE XXIX.

(1.) Multiply $8ab$ by $2cd = 16abcd$.
(2.) Multiply $7bc$ by $-5xy = -35bcxy$.
(3.) Multiply $-2de$ by $7ab$.
(4.) Multiply $5xy$ by $-3bc$.
(5.) Multiply $3cd$ by $2ax$.
(6.) Multiply $-4df$ by $-ac$.
(7.) Multiply $7ab$ by $-4dx$.
(8.) Multiply $5bc$ by $-9ay$.
(9.) Multiply $-7cd$ by $-ax$.
(10.) Multiply $5acd$ by $7xy$.

30. It has been stated in paragraph 11 that $a^2 = a \times a$, $b^3 = b \times b \times b$, &c., &c.

Hence $b^3 \times b^2 = (b \times b \times b) \times (b \times b) = b^5$; *i.e.*, b^{3+2}.

Similarly, $b^m \times b^n = b \times b \ldots$ to m factors $\times\, b \times b \ldots$ to n factors

$= b \times b \ldots$ to $(m+n)$ factors $= b^{m+n}$.

Thus the **powers** of any quantity are multiplied together by **adding the indices.**

Note 1.—From the foregoing it may be seen that $b^m \times b^n = b^{m+n}$; *i.e.*, when m and n are **positive integers.**

Note 2.—The same rule is assumed to hold good with m and n when either or both are **negative.**

Hence (1.) $b^{-m} \times b^{-n} = b^{-m-n}$.

(2.) $b^{-m} \times b^{n} = b^{-m+n}$.

(3.) $b^{m} \times b^{-n} = b^{m-n}$.

EXERCISE XXX.

(1.) Multiply $-4bc^2$ by $3b^2c$.
(2.) Multiply $9ax^3$ by $-5x^2y$.
(3.) Multiply $-4ay^3$ by $-2a^4y$.
(4.) Multiply $7a^2y$ by $-a^3x$.
(5.) Multiply $-5bc^2x^2$ by $7b^2cx^3$.
(6.) Multiply $8a^2c$ by $-6a^3c^2$.
(7.) Multiply $-7ay$ by $-9a^2y^3$.
(8.) Multiply $3a^2bx^2$ by a^2bx^2.
(9.) Multiply $-4ac^2y^2$ by $-4a^3cy^4$.
(10.) Multiply $-13x^3yz^4$ by $8y^2z^3$.
(11.) Multiply $-5a^3y$ by $4a^2y^3$.
(12.) Multiply $6ab^2$ by $-5a^3b^4$.
(13.) Multiply $-7a^2x$ by $-8a^3x^3$.
(14.) Multiply $9b^3y$ by $-b^ny^m$.
(15.) Multiply $-10ax^2y$ by $-4a^5x^3y^2$.
(16.) Multiply $15b^3y$ by $-2by^4$.
(17.) Multiply a^mc^n by a^nc^{-m}.
(18.) Multiply $a^{-m}b^n$ by $-a^nb^m$.

(19.) Multiply $24x^m y^n$ by xy^m.

(20.) Multiply $-c^n d^m$ by $-4c^{-p}d^q$.

31. When the **multiplicand** (the quantity to be multiplied) is a **binomial** (paragraph 14), each term must be multiplied **singly** by the common multiplier; and the sum of the two products is the **complete** product. Thus:

$$2a+3b\times 4a \ \ = 8a^2+12ab.$$
$$3c^2-2cd\times 5c = 15c^3-10c^2d.$$

EXERCISE XXXI.

(1.) Multiply $5a^2b+2bc^2$ by $-4b^2c$.

(2.) Multiply $3ax^2-2xy^2$ by $3ax^3$.

(3.) Multiply $-5ac+3cx^2$ by $-8c^2x$.

(4.) Multiply $8a^2x-3xy^2$ by $6a^2y^2$.

(5.) Multiply $4bc^2+2x^2y$ by $-5b^2y^3$.

(6.) Multiply $-7ac^2-3c^2d$ by $2c^2d$.

(7.) Multiply $8bc^2+c^2d^2$ by $-7b^2c^3$.

(8.) Multiply $3ax+2x^2y$ by $9ax^3$.

(9.) Multiply $-5xy-3yz^3$ by $8y^3z^2$.

(10.) Multiply $8xy-x^2y^2z^2$ by $-x^2y^3$.

(11.) Multiply $4a^3x-5b^3y$ by $-a^2y^n$.

(12.) Multiply $8a^2x^2y+2x^my$ by $-ax^ny$.

(13.) Multiply $10a^3xy-x^3y^n$ by $-5x^ny^m$.

(14.) Multiply $-9a^2b+4a^mb$ by $3a^3b$.

(15.) Multiply $8x^3y^4+10x^m$ by $6x^2y^m$.

(16.) Multiply $9x^{-m}y-10x^{-n}y$ by $8xy^{-n}$.

(17.) Multiply $-6a^3b^5+9x^m$ by $-5a^mb^m$.

(18.) Multiply $-5a^mb^{-n}-6x^3y^{-n}$ by $-2a^my^{-n}$.

Note.—Precisely the same method is to be followed when the multiplicand is a **trinomial** or **multinomial**.

EXERCISE XXXII.

(1.) Multiply $7a+3b-2c^2$ by $4bc^2$.
(2.) Multiply $-3a-4ac-5ad$ by $-3a^3d$.
(3.) Multiply $4bc+3xy+3bc^2$ by $7c^2y$.
(4.) Multiply $3x^3y-2xy^2-9a^2y$ by $-8x^2y^2$.
(5.) Multiply $-2xy-4xy^2-4a^2y$ by $6xy^3$.
(6.) Multiply $7y+3abc+3b^2c$ by $-4bc^2$.
(7.) Multiply $4z-7xy-5x^2$ by $2x^2z$.
(8.) Multiply $-3ab-3ad+d^2x$ by $-d^3x^2$.
(9.) Multiply $ax+4ac-a^3c$ by $9a^2c^3$.
(10.) Multiply $bc-7xy+y^2z$ by $8xy^2z$.

32. When the **multiplier** is a **binomial**, the process of multiplication is usually performed thus:

(1.) Multiply $a+b$ by $a+b$.

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

In this example we first multiply $a+b$ by a, and obtain the partial product a^2+ab. We then multiply $a+b$ by b, and obtain the partial product $ab+b^2$. These partial products are then added together, and we obtain the **complete** product $a^2+2ab+b^2$.

(2.) Multiply $a-b$ by $a-b$.

$$\begin{array}{l} a-b \\ a-b \\ \hline a^2-ab \\ \quad -ab+b^2 \\ \hline a^2-2ab+b^2 \end{array}$$

In each of the above examples we have found the **square** of a quantity; that is to say, we have multiplied a quantity by itself (paragraphs 11 and 19). Thus:

$(a+b)(a+b)=(a+b)^2=a^2+2ab+b^2$ or a^2+b^2+2ab.
$(a-b)(a-b)=(a-b)^2=a^2-2ab+b^2$ or a^2+b^2-2ab.

On looking at the result of Example 1, it will be seen that the square of $\overline{a+b}$ = the square of a + the square of b + twice the product of a and b.

On looking at the result of Example 2, it will be seen that the square of $\overline{a-b}$ = the square of a + the square of b — twice the product of a and b.

From these results we may find the square of **any** binomial **without actual multiplication.** Thus:

$$(a+x)^2 = a^2+2ax+x^2,$$
$$(2b-c)^2 = 4b^2-4bc+c^2,$$
$$(3c-4d)^2 = 9c^2-24cd+16d^2.$$

EXERCISE XXXIII.

Write down the squares of

(1.) $a+x$.

(2.) $b-y$.

(3.) $2a-x$.

(4.) $a+2x$.

(5.) $4a-3b$.

(6.) $2a+3b^2$.

(7.) $6c-7d^2$.

(8.) $4ax+b$.

(9.) $8xy+7yz^2$.

(10.) $9xy^3-12y^2z$.

33. (1.) Multiply $a+b$ by $a-b$.

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad\;\, -ab-b^2 \\ \hline a^2 \qquad\;\; -b^2 \end{array}$$

(2.) Multiply $3x+2y$ by $3x-2y$.

$$\begin{array}{l} 3x+2y \\ 3x-2y \\ \hline 9x^2+6xy \\ \quad\;\;\, -6xy-4y^2 \\ \hline 9x^2 \qquad\quad -4y^2 \end{array}$$

The preceding examples are similar in **one** respect; *viz.*, that in each case the sum of two quantities is multiplied by their difference.

The product of any **similar** binomials may thus be easily found **without actual multiplication.** Thus:

$$(4a+b)(4a-b) = 16a^2-b^2.$$
$$(7c^2-d)(7c^2+d) = 49c^4-d^2.$$
$$(7ac+2d)(7ac-2d) = 49a^2c^2-4d^2.$$

EXERCISE XXXIV.

Find the product of

(1.) $(a+x)(a-x)$.

(2.) $(x-1)(x-1)$.

(3.) $(2a+b)(2a+b)$.

(4.) $(7a-4c)(7a+4c)$.

(5.) $(3a-2c^2)(3a-2c^2)$.

(6.) $(10a^2b-4b^3)(10a^2b+4b^3)$.

(7.) $(8a^2c-2c^2d)(8a^2c-2c^2d)$.

(8.) $(7ab+3x^2)(7ab+3x^2)$.

(9.) $(8a^2x-7b^2y)(8a^2x+7b^2y)$.

(10.) $(4a^2xy+3a)(4a^2xy-3a)$.

EXERCISE XXXV.

(1.) Mult. x^2+xy-y^2 by $x-y$.

(2.) Mult. x^2-4x+3 by $x-2$.

(3.) Mult. $2a-b$ by $c-3d$.

(4.) Mult. $4a^2-6a+9$ by $2a+3$.

(5.) Mult. $5a^2+5ax+5x^2$ by $2a^2-2ax$.

(6.) Mult. $6a^2+10ax-4x^2$ by $4a-2x$.

(7.) Mult. $a^4-2a^3b+4a^2b^2-8ab^3+16b^4$ by $a+2b$.

(8.) Mult. $a^4+a^3b+a^2b^2+ab^3+b^4$ by $a-b$.

(9.) Mult. $27x^3+9x^2y+3xy^2+y^3$ by $3x-y$.

(10.) Mult. $x^4-2ax^3+4a^2x^2-8a^3x+16a^4$ by $2a+x$.

34. The following results should be noticed:

(1.) $(x+a)(x+b) = x^2+(a+b)x+ab$.

(2.) $(x+a)(x-b) = x^2+(a-b)x-ab$.

(3.) $(x-a)(x+b) = x^2-(a-b)x-ab$.

(4.) $(x-a)(x-b) = x^2-(a+b)x+ab$.

Hence:

(1.) $(x+3)(x+2) = x^2+(3+2)x+6$
$= x^2+5x+6$.

(2.) $(x+3)(x-2) = x^2+(3-2)x-6$
$= x^2-x-6$.

(3.) $(x-3)(x+2) = x^2-(3-2)x-6$
$= x^2-x-6$.

(4.) $(x-3)(x-2) = x^2-(3+2)x+6$
$= x^2-5x+6$.

From a careful inspection of the above, the following sums may be worked **without actual multiplication.**

EXERCISE XXXVI.

Find the product of

(1.) $(x+a)(x+c)$.

(2.) $(x+7)(x-3)$.

(3.) $(x-7)(x+3)$.

(4.) $(x+5)(x-4)$.

(5.) $(x-8)(x-2)$.

(6.) $(x+3)(x-5)$.

(7.) $(x-5)(x-6)$.

(8.) $(x-8)(x+7)$.

(9.) $(x-3)(x-6)$.

(10.) $(x+8)(x-7)$.

35. When the multiplier is a **trinomial**, the process of multiplication is similar to that to which the pupil is now accustomed.

EXERCISE XXXVII.

(1.) Mult. $x+y+z$ by $x-y-z$.

(2.) Mult. $a^2+2ab+b^2$ by $a^2-2ab+b^2$.

(3.) Mult. $2a+bc-2b^2$ by $2a-bc+2b^2$.

(4.) Mult. $a^2+4b^2+9c^2+2ab+3ac-6bc$ by $a-2b-3c$.

(5.) Mult. $a^4-2a^3b+3a^2b^2-2ab^3+b^4$ by $a^2+2ab+b^2$.

(6.) Mult. $9a^2-3ab+b^2-6a-2b+4$ by $3a+b+2$.

36. Sometimes it is required to find the **continued** product of three or more factors; that is to say, after finding the product of **any** two, we must multiply that result by the third or remaining factors. It is often unnecessary to **actually** multiply all the factors together, because frequently some **two** of the factors fall under one of the **formulae.**

(1.) $(a+b)^2 = a^2+2ab+b^2$;

(2.) $(a-b)^2 = a^2-2ab+b^2$;

(3.) $(a+b)(a-b) = a^2-b^2$; or,

(4.) $(x+a)(x+b) = x^2+(a+b)x+ab$.

In working the following, the above results should be remembered.

EXERCISE XXXVIII.

Find the continued product of

(1.) $(a+b)(a-b)(a^2+b^2)$.

(2.) $(x-10)(x+1)(x+4)$.

(3.) $(a-b)(a+b)(-a+b)$.

(4.) $(x+a)(x-a)(x+2a)(x-2a)$.

(5.) $(x+1)(x+2)(x+3)(x+4)$.

(6.) $(a-1)(a-2)(a-3)$.

37 (A). It has been stated in paragraph 19, that by the aid of brackets or a vinculum two or more terms may be considered as forming but **one** quantity. Hence, the **actual** process of multiplication may often be considerably reduced. Thus:

$$a+b-c=(a+b)-c \text{ or } \overline{a+b}-c;$$

$$\text{and } a+b+c=(a+b)+c \text{ or } \overline{a+b}+c.$$

Then considering $\overline{a+b}$ as one quantity, we have

$$(\overline{a+b}-c)\,(\overline{a+b}+c)=(a+b)^2-c^2,$$
$$=a^2+2ab+b^2-c^2.$$

It will be often necessary for the pupil **first** to rearrange the terms, so as to give a familiar appearance to the expression. He must also remember the force of the **minus** sign before a bracket or a vinculum.

EXERCISE XXXIX.

(1.) Multiply $a-b+c$ by $a-b-c$.

(2.) Multiply $a+x-1$ by $a+x+1$.

(3.) Multiply $2a-x+2$ by $2a+2+x$.

(4.) Multiply a^2+x^2+ax by a^2+x^2-ax.

(5.) Multiply $2a+b-3c$ by $2a-b+3c$.
(6.) Multiply $2a-b-3c$ by $b-3c-2a$.
(7.) Multiply $2a+bc-2b^2$ by $2a-bc+2b^2$.
(8.) Multiply $2a^2-3ab+b^2$ by $2a^2+3ab+b^2$.

EXERCISE XL. (A.)

(1.) Multiply $x^2-axy+by^2$ by $x-cy$.
(2.) Multiply x^2-ax+b by $x-1$.
(3.) Multiply x^2-ax+b by $x-c$.
(4.) Multiply $a+bx$ by $a+cx$.
(5.) Multiply $x^2+(a+b)x+ab$ by $x-c$.
(6.) Multiply $x^2-(a-p)x+a^2-ap+q$ by $x+a$.
(7.) Multiply $2ax-3cy$ by $bx+dy$.
(8.) Multiply $a^{2n}+x^{2n}$ by $2a^{2n}-2x^{2n}$.
(9.) Multiply $a^{n-1}b-a^{n-2}b^2+ab^{n-1}$ by ab.
(10.) Multiply $(1+a)a^2y+y^2+ay^2$ by a^2-y.
(11.) Multiply $x^2+(a+b)x+ab$ by $x-c$.
(12.) Multiply $a^{2m}-2a^{3n}$ by $a^{2m}+2a^{3n}$.
(13.) Multiply $a^m+3b^n-2c^p$ by $a^m-3b^n+2c^p$.

37 (B). In all the preceding examples the coefficients have been integral (paragraph 23). We now proceed to the consideration of **fractional** coefficients; *e.g.*, $\frac{1}{2}a$, $\frac{3}{4}a$, $\frac{2}{3}b$, $\frac{4}{5}cd$, &c., or as these quantities are often expressed, $\frac{a}{2}$, $\frac{3a}{4}$, $\frac{2b}{3}$, $\frac{4cd}{5}$, &c., &c.

From the working of the following sum it will be seen that the treatment is the same as that of sums in vulgar fractions (arithmetic).

EXERCISE XL. (B).

(1.) Multiply $x^2-\frac{1}{3}x+\frac{1}{4}$ by $\frac{1}{2}x+\frac{1}{4}$.

$$\begin{array}{l} x^2-\frac{1}{3}x+\frac{1}{4} \\ \frac{1}{2}x+\frac{1}{4} \\ \hline \frac{1}{2}x^3-\frac{1}{6}x^2+\frac{1}{8}x \\ \qquad +\frac{1}{4}x^2-\frac{1}{12}x+\frac{1}{16} \\ \hline \frac{1}{2}x^3+\frac{1}{12}x^2+\frac{1}{24}x+\frac{1}{16}. \end{array}$$

$$\text{or, } \frac{x^3}{2}+\frac{x^2}{12}+\frac{x}{24}+\frac{1}{16}.$$

(2.) Multiply $\frac{3}{2}x^3-5x^2+\frac{1}{4}x+9$ by $\frac{1}{2}x^2-x+3$.

(3.) Multiply $1+\frac{x}{2}+\frac{x^2}{4}$ by $1-\frac{x}{2}+\frac{x^3}{8}-\frac{x^4}{16}$.

(4.) Multiply $x^2-\frac{1}{2}x+\frac{2}{3}$ by $\frac{1}{3}x+2$.

(5.) Multiply $1+\frac{1}{2}a+\frac{1}{3}b$ by $1-\frac{1}{2}a+\frac{1}{3}b$.

(6.) Multiply $a^3-\frac{3}{2}a^2x+2ax^2-\frac{x^3}{6}$ by $a^2-\frac{ax}{3}+x^2$.

(7.) Multiply $15x^2+18bx+14b^2$ by $x^2-\frac{bx}{2}-\frac{b^2}{4}$.

(8.) $mx^{n+2}-nx^n+bx^n+1$ by nx^n-1.

38. From a careful study of paragraph 34, it will be seen that sometimes trinomials may be resolved into their elementary factors.

(1.) As **like** signs produce $+$; when the last term of the trinomial is **positive**, the last terms of the two factors must be either $+$ or $-$.

(2.) The sign of the middle term of the trinomial will obviously show which they are.

(3.) As **unlike** signs produce $-$; when the last term of the trinomial is **negative**, one of the last terms of the two factors must be $+$, and the other $-$.

(4.) The last term of the trinomial is always the product of the last terms of the two factors.

(5.) The coefficient of the middle term of the trinomial equals the **sum** of the last terms of the two factors.

(6.) The **sign** of the middle term of the trinomial will indicate which of the last terms is positive, and which is negative.

(7.) The first term of the trinomial is the product of the first terms of the two factors. Thus:

(*Ex.* 1.) Resolve $x^2+7x+12$ into its elementary factors.

Here $+12$ the third term $= +4\times+3$, or -4×-3. But as the sign of the middle term is **positive**, the terms of the factors are clearly $+4$, and $+3$; and the **sum** of 4 and $3 = 7$, the coefficient of the middle term. Hence the factors are $(x+4)(x+3)$.

But $+12$ the third term $= +6\times+2$, or -6×-2. As in this case the sum of 6 and 2 do not make up the coefficient of the middle term; $(x+6)(x+2)$ cannot be the factors required.

(*Ex.* 2.) Resolve $x^2-3x-10$ into its elementary factors.

Here -10 the third term $= -5\times+2$, or $+5\times-2$. But -3 the coefficient of the middle term $= -5+2$. Hence the factors are $(x-5)(x+2)$.

EXERCISE XLI.

Resolve into elementary factors the following **trinomials**:

(1.) $x^2+8x+15$.

(2.) $x^2+2x-15$.

(3.) x^2+8x+7.

(4.) $4x^2-4x-3$.

(5.) x^2-8x-9.

(6.) $x^2-13x+40$.

(7.) $x^2-5x-104$.

(8.) $3x^2-2x-5$.

(9.) $12x^2-5x-2$.

(10.) $12a^4+a^2x^2-x^4$.

(11.) $a^3-a^2x-6ax^2$.

(12.) $6a^4x^2+a^3x-a^2$.

EXERCISE XLII.

Simplify the following expressions:

(1.) $4a^3-2a^2+a+1-(3a^3-a^2-a-7)-(a^3-4a^2+2a+8)$.

(2.) $a+2x-6a-\{3x-(6a-6x)\}$.

(3.) $(a+x)(x+c)-(c+d)(d+a)-(a+c)(x-d)$.

(4.) $(a+x+c)^2-a(x+c-a)-x(a+c-x)-c(a+x-c)$.

(5.) $(a^2+b^2+x^2)^2+(a+b+x)(a+b-x)(a+x-b)(b+x-a)$.

(6.) $(a+x)(a+c)+(x-c)(x-a)$.

(7.) $(b-y)^2+(b+y)^2+2\{(b-y)(b+y)+(b+y)(b-y)\}$.

(8.) $(2a+3x)(3+4c)-(4c-x)(6+2a)-(2a+4c)(3x-6)$.

(9.) $(a^2+b^2+c^2)^2-(a^2+b^2+c^2)$.

(10.) $(x+1)(x+2)(x+3)-(x+1)(x+2)-x^3$.

39. We now proceed to the **Division** of algebraical quantities. The pupil being familiar with the signs of Addition, Subtraction, and Multiplication, is now introduced to the sign of Division.

The sign $\div$ (read **divided by**) signifies that the quantity which precedes it is to be divided by the quantity which follows it. Thus, if a represent 12, and b represent 4, then $a \div b = 12 \div 4 = 3$, which is called the **quotient.**

But the sign of division is often dispensed with, the quantity to be divided (called the **dividend**) being placed over the divisor with a line between them. The **whole** then assumes a **fractional** form. Thus, $a \div b$ may be written $\frac{a}{b}$.

40. In the division of **simple** algebraical quantities **three** things have to be considered:

(1.) The **numerical** coefficient of the quotient.
(2.) The **literal** part of the quotient.
(3.) The **sign** of the quotient.

In order to determine the **first,** the process is precisely similar to that of ordinary division.

To determine the **second,** any letters which are **common** to both dividend and divisor are necessarily omitted in the quotient; for, taking the **literal** factors of Example 2, $axy = a \times x \times y$ and $xy = x \times y$. Hence, $a \times x \times y \div x \times y$, or $\frac{a.x.y}{x.y} = a$; that is to say, a is the quantity which when multiplied by $x.y$ gives axy.

To determine the **third,** the **same** rule must be observed as in Multiplication; *viz.*, that **like** signs produce $+$, and unlike signs $-$. This rule is clearly deduced from that of Multiplication; for,

as $+a \times +b = +ab$, then $+ab \div +b = +a$;
as $+a \times -b = -ab$, then $-ab \div -b = +a$;
as $-a \times +b = -ab$, then $-ab \div +b = -a$;
as $-a \times -b = +ab$, then $+ab \div -b = -a$.

EXERCISE XLIII.

(1.) Divide $8ax$ by $4x = 2a$.
(2.) Divide $-9axy$ by $3xy = -3a$.
(3.) Divide $15xy$ by $-5y$.
(4.) Divide $14ac$ by $7c$.
(5.) Divide $-12bcx$ by $-6cx$.
(6.) Divide $18abx$ by $-3ab$.
(7.) Divide $-30bcd$ by $-5d$.
(8.) Divide $4ax$ by $-2x$.
(9.) Divide $-14ay$ by $2y$.
(10.) Divide $6xyz$ by $-3xy$.

41. In paragraph 30 it was shown that $b^3 \times b^2 = b^5$. Hence, $b^5 \div b^2 = b^3$, or $b^5 \div b^3 = b^2$; *i.e.*, b^{5-3}.

Similarly, $b^m \div b^n$, or $\frac{b^m}{b^n} = \frac{b \times b \times b \ldots \text{to } m \text{ factors}}{b \times b \times b \ldots \text{to } n \text{ factors}}$
$= b^{m-n}$; if m be greater than n, or if $m > n$.

Thus the power of a quantity in the dividend is divided by a power of the **same** quantity in the divisor, by subtracting the index of the latter from that of the former.

$$\text{Again, } b^2 \div b^5, \text{ or } \frac{b^2}{b^5} = b^{2-5} = b^{-3};$$

$$\text{or } \frac{b^2}{b^5} = \frac{1 \times b^2}{b^3 \times b^2} = \frac{1}{b^3}.$$

Similarly, $b^m \div b^n$, or $\frac{b^m}{b^n} = \frac{b \times b \times b \ldots \text{to } m \text{ factors}}{b \times b \times b \ldots \text{to } n \text{ factors}}$
$= \frac{1}{b^{n-m}} = b^{m-n}$; if m be less than n,
or if $m < n$.

EXERCISE XLIV. (A.)

(1.) Divide $12a^2x^2b$ by $3axb = 4ax$.
(2.) Divide $-14a^3x$ by $-7a^2 = 2ax$.
(3.) Divide $16a^4x^5$ by $-4a^3x^3$.
(4.) Divide $-18x^2y^2z^3$ by $-9x^2y$.
(5.) Divide $20x^4y^5$ by $-4x^2y^2$.
(6.) Divide $-30a^2b^3$ by $5ab^3$.
(7.) Divide $49a^5b^2$ by $7a^2b^2$.
(8.) Divide $-81a^3b^3$ by $-3ab^2$.
(9.) Divide $16a^2b^2$ by $-8a^2b$.
(10.) Divide $-110a^3b^4$ by $11a^3b^3$.

EXERCISE XLIV. (B.)

(1.) Divide $-16c^4y^4$ by $-8c^8y^2$.
(2.) Divide $6a^{3m}$ by $3a^{2m}$.
(3.) Divide $a^{2m}x^m$ by a^mx^{3m}.
(4.) Divide $a^{m+2}x^m$ by $a^{m-2}x^2$.
(5.) Divide $12a^{2m}x^{m-1}$ by $4a^mx$.
(6.) Divide $12cx^{m-2}$ by $2x^{m-3}$.
(7.) Divide $14a^mb^nc^{2p+q}d^{2r-1}$ by $-7a^nb^nc^{p+q}d$.
(8.) Divide $6a^{3m}x^{2m}$ by $3a^mx^m$.
(9.) Divide $-3x^{m-1}$ by $3x^{m-2}$.
(10.) Divide $-6a^{2m}x^{2m}$ by $3a^{3m}x^m$.
(11.) Divide $8a^{m-1}$ by $4a^{m-3}$.
(12.) Divide $4a^{3m}x$ by $8a^{m+2}x^3$.

42. Hitherto the coefficients have been whole numbers (or **integers**). Moreover, the **literal** portion of the divisor has been always contained in the **literal** portion of the dividend. We now proceed:

(1.) To cases in which the coefficient of the quotient is a fraction, seeing that the coefficient of the divisor is greater than that of the dividend.

(2.) To cases in which the **literal** portion of the divisor is **not** contained in that of the dividend.

When the coefficient of the divisor is not contained in that of the dividend, it must be written under it. When the **literal** portion of the divisor is not contained in that of the dividend, it must also be written under it. Thus:

$$6ax \div 7xy = \frac{6a}{7y}.$$

EXERCISE XLV.

(1.) Divide $14ax$ by $42ac = \frac{14x}{42c}$.

(2.) Divide $-21bx$ by $42dx = -\frac{21b}{42d}$.

(3.) Divide $13dy$ by $-39cdy$.

(4.) Divide $-7abx$ by $-14acx$.

(5.) Divide $14ay$ by $-7bc$.

(6.) Divide $-39dx$ by $-13ac$.

(7.) Divide $42abc$ by $21bd$.

(8.) Divide $-6abx$ by $-30ax$.

(9.) Divide $30bcd$ by $6d$.

(10.) Divide $21cdx$ by $-7dx$.

[In the following remarks an elementary knowledge of vulgar fractions is assumed.]

43. In any fraction, if the numerator and denominator be multiplied by the **same** number or quantity, the value of the fraction remains the same. Thus:

$$\frac{2}{3}=\frac{2\times 5}{3\times 5}=\frac{10}{15},$$

$$\frac{6}{7}=\frac{6\times 3}{7\times 3}=\frac{18}{21},$$

$$\frac{4c}{3b}=\frac{4c\times b}{3b\times b}=\frac{4bc}{3b^2}.$$

Hence, in any fraction, if the numerator and denominator be **divided** by the **same** number or quantity, the value of the fraction remains unaltered. Thus:

$$\frac{14}{21}=\frac{7\times 2}{7\times 3}=\frac{2}{3},$$

$$\frac{21}{30}=\frac{7\times 3}{10\times 3}=\frac{7}{10}.$$

Note.—This process is called reducing fractions to their lowest terms.

EXERCISE XLVI.

(1.) Divide $14ab$ by $21bx=\frac{2a}{3x}$.

(2.) Divide $-7bx$ by $21bxy-\frac{1}{3y}$.

(3.) Divide $13cx$ by $26c$.

(4.) Divide $-22dx$ by $88cd$.

(5.) Divide $5ax^2$ by $-30ax$.

(6.) Divide $-9a^2x^2$ by $27ax^2$.

(7.) Divide $7b^2x$ by $14bx$.

(8.) Divide $-10b^2x^2$ by $-15bx$.

(9.) Divide $13x^2y$ by $-26xy$.

(10.) Divide $8y^2z^3$ by $20yz$.

44. Hitherto both dividend and divisor have been simple quantities. We now proceed to the case in which the dividend is a **binomial** (paragraph 14).

Here we must resolve the binomial into its component terms, and deal with each singly. Thus:

(1.) $\frac{a^2b-3a^2b^3}{3ab}=\frac{a^2b}{3ab}-\frac{3a^2b^3}{3ab}=\frac{a}{3}-ab^2.$

(2.) $\frac{x^2y^2-2bxy^2}{-4bxy^2}=-\frac{x^2y^2}{4bxy^2}+\frac{2bxy^2}{4bxy^2}=-\frac{x}{4b}+\frac{1}{2}.$

EXERCISE XLVII.

(1.) Divide $4a^2x^3-5x^3y$ by $4axy$.

(2.) Divide $7ax^3+3a^2b^3$ by $-3abx$.

(3.) Divide $2bx^4-9b^2c^3$ by $2bcx$.

(4.) Divide $8a^2b^3-2a^3x$ by $-2a^2b^2$.

(5.) Divide $10x^2y^3+8a^2b^3$ by $2abx$.

(6.) Divide $9xyz^3-7x^3y^3$ by $-7xy$.

(7.) Divide $8xy+6x^2y^4$ by $3xy$.

(8.) Divide $7b^2c^2-14b^3c^3$ by $-7bc^2$.

(9.) Divide $6abc^3-7b^2c^3$ by $6bc^3$.

(10.) Divide $5b^2c^4+15b^2c^3$ by $5b^2c$.

45. It was stated in paragraph 39 that $a \div b$ may be written $\frac{a}{b}$. Moreover, the force of the negative sign (—) before quantities enclosed within brackets, or connected with a **vinculum,** has been referred to in paragraph 22. Now the line which separates the numerator from the denominator in a fractional expression is a kind of vinculum, and the effect of the negative sign before it must be carefully noticed. Thus: it is required to find the value of

$$-\frac{15ab-20bc+35b^2d}{5bd}.$$

This is equivalent to

$$\text{Divide } -(15ab-20bc+35b^2d) \text{ by } 5bd$$
$$=-15ab+20bc-35b^2d \div 5bd$$
$$=-\frac{15ab}{5bd}+\frac{20bc}{5bd}-\frac{35b^2d}{5bd}$$
$$=-\frac{3a}{d}+\frac{4c}{d}-7b.$$

EXERCISE XLVIII.

Find the value of

(1.) $-\dfrac{28a^2b+35b^2c-42c^3d+49a^2b^3}{7abc}$.

(2.) $-\dfrac{12a^2x-21ax^2+30x^2y^3+33a^3x^4}{3axy}$.

(3.) $-\dfrac{30ax^3+20ay^3-80x^3y^2-10xy^4}{5axy}$.

(4.) $-\dfrac{21a^2x-28x^3y^2+35x^2y^3-14x^3y^4}{7xy}$.

(5.) $\dfrac{12a^2b-15b^3x-18a^2b^3x^3-21b^3y}{3bx}$.

(6.) $\dfrac{18b^2x+27b^3x^2+81b^2x^3-63b^2y^3}{9bx}$.

(7.) $-\dfrac{33a^2bc-44ab^3c^2+55a^3bc^3-77ac^3}{11abc}$.

46. We now proceed to the division of algebraical quantities where the **divisor** (as well as the dividend) is **a compound** quantity.

Note.—The **order** of the terms in an algebraical expression may be changed without altering the value of the whole expression, **provided that each term take its accompanying sign with it.** Thus:

$$3a - 2b - 3c + 4d$$
$$= 3a - 3c - 2b + 4d$$
$$= 4d + 3a - 3c - 2b = \&c.$$

Hence, before commencing to work sums of this class, we must (if necessary) arrange the terms of the dividend so that the powers of some letter, which is common to two or more terms, may follow in a **descending** order.

Thus $a^4 - a^2b + a^3$ should be written $a^4 + a^3 - a^2b$.

When **another** letter is common to two or more terms, the powers of that letter should follow (where practicable) in an **ascending** order.

Thus $a^4 - ab^3 - a^3b + a^2b^2 + b^4$ should be written $a^4 - a^3b + a^2b^2 - ab^3 + b^4$.

The same plan must be followed both in the divisor **and in the course of working the sum.**

The letter thus selected is termed the **letter of reference.**

47(A). In working sums of this kind the process is similar to that of Arithmetic. Thus:

(1.) Divide $a^2 + b^2 + 2ab$ by $a + b$;
or $a^2 + 2ab + b^2$ by $a + b$.

$$\begin{array}{r|l} a+b \;)\; a^2 + 2ab + b^2 & a+b \\ a^2 + ab & \\ \hline + ab + b^2 & \\ + ab + b^2 & \\ \hline * \quad * & \end{array}$$

(2.) Divide $a^2 - 2ab + b^2$ by $a - b$.

$$\begin{array}{r|l} a-b \;)\; a^2 - 2ab + b^2 & a-b \\ a^2 - ab & \\ \hline - ab + b^2 & \\ - ab + b^2 & \\ \hline * \quad * & \end{array}$$

(3.) Divide a^2-b^2 by $a-b$.

$$\begin{array}{l} a-b\,\big)\,a^2-b^2\,\big(\,a+b \\ \qquad\quad a^2-ab \\ \qquad\quad \overline{\quad +ab-b^2} \\ \qquad\qquad +ab-b^2 \\ \qquad\qquad \overline{\quad * \quad *} \end{array}$$

In working Example (1.), the steps are as follows:

(1.) a is contained in a^2, a times.

(2.) Put a in the quotient.

(3.) Then a times $a+b=a^2+ab$.

(4.) Subtract a^2+ab from a^2+2ab: the difference is ab.

(5.) Take down the next term, b^2.

(6.) a is contained in $+ab$, $+b$ times.

(7.) Put $+b$ in the quotient.

(8.) Then b times $a+b=ab+b^2$.

(9.) Take $ab+b^2$ from $ab+b^2$, and nothing remains.

(10.) Hence the quotient $a+b$ is the answer.

(4.) Divide a^5-x^5 by $a-x$.

$$\begin{array}{l} a-x\,\big)\,a^5-x^5\,\big(\,a^4+a^3x+a^2x^2+ax^3+x^4 \\ \qquad\quad a^5-a^4x \\ \qquad\quad \overline{\quad +a^4x-x^5} \\ \qquad\qquad +a^4x-a^3x^2 \\ \qquad\qquad \overline{\qquad +a^3x^2-x^5} \\ \qquad\qquad\qquad +a^3x^2-a^2x^3 \\ \qquad\qquad\qquad \overline{\qquad +a^2x^3-x^5} \\ \qquad\qquad\qquad\qquad +a^2x^3-ax^4 \\ \qquad\qquad\qquad\qquad \overline{\qquad +ax^4-x^5} \\ \qquad\qquad\qquad\qquad\qquad +ax^4-x^5 \\ \qquad\qquad\qquad\qquad\qquad \overline{\quad * \quad *} \end{array}$$

EXERCISE XLIX. (A.)

(1.) Divide $a^3-3a^2x+3ax^2-x^3$ by $a-x$.

(2.) Divide $a^3-9a^2+27a-27$ by $a-3$.

(3.) Divide $a^3-3a^2b+3ab^2-b^3$ by $a-b$.

(4.) Divide $6y^4-96$ by $3y-6$.

(5.) Divide a^4-b^4 by $a-b$.

(6.) Divide $6a^4-96$ by $3a-6$.

(7.) Divide $6x^6-6y^6$ by $2x^2-2y^2$.

(8.) Divide $2x^2+x-6$ by $2x-3$.

(9.) Divide $64-x^6$ by $2-x$.

(10.) Divide x^4-81 by $x-3$.

47 (B). We now consider the division of algebraical quantities having **fractional** coefficients.

From the working of the following sum, it will be seen that the treatment is the same as that of sums in vulgar fractions (arithmetic).

EXERCISE XLIX. (B.)

(1.) Divide $\frac{1}{2}x^3+\frac{1}{12}x^2+\frac{1}{24}x+\frac{1}{16}$ by $\frac{1}{2}x+\frac{1}{4}$.

$$\frac{1}{2}x+\frac{1}{4}\Big)\,\frac{1}{2}x^3+\frac{1}{12}x^2+\frac{1}{24}x+\frac{1}{16}\,\Big(x^2-\frac{1}{3}x+\frac{1}{4}$$

$$\frac{1}{2}x^3+\frac{1}{4}x^2$$

$$\overline{-\frac{1}{6}x^2+\frac{1}{24}x}$$

$$-\frac{1}{6}x^2-\frac{1}{12}x$$

$$\overline{+\frac{1}{8}x+\frac{1}{16}}$$

$$+\frac{1}{8}x+\frac{1}{16}$$

(2.) Divide $a^2-\frac{1}{20}ab-\frac{1}{20}b^2$ by $a-\frac{1}{4}b$.

(3.) Divide $\frac{4}{3}a^3-\frac{4}{3}a^2b+\frac{9}{4}ab^2-\frac{1}{2}b^3$ by $2a-\frac{1}{2}b$.

(4.) Divide $a^3-5a^2b-\frac{4a^2}{3}+\frac{20ab}{3}+\frac{3a}{4}-\frac{15b}{4}$ by $a-5b$.

(5.) Divide $x^4-\frac{5}{4}x^3+\frac{11}{8}x^2-\frac{1}{2}x$ by $x^2-\frac{1}{2}x$.

(6.) Divide $\frac{1}{3}-6a^2+27a^4$ by $\frac{1}{3}+2a+3a^2$.

(7.) Divide $\frac{x^4}{3}-\frac{11x^3}{12}+\frac{41x^2}{8}-\frac{23x}{4}+6$ by $\frac{2x^2}{3}-\frac{5x}{6}+1$.

(8.) Divide $\frac{1}{3}x^3+\frac{11}{6}x^2+\frac{7}{9}x+\frac{4}{3}$ by $\frac{1}{3}x+2$.

(9.) Divide $1-\frac{1}{4}a^2+\frac{2}{3}b+\frac{1}{9}b^2$ by $1-\frac{1}{2}a+\frac{1}{3}b$.

(10.) Divide $1-\frac{x^6}{64}$ by $1+\frac{x}{2}+\frac{x^2}{4}$.

(11.) Divide $\frac{3}{8}a^2-\frac{17}{4}ab+\frac{71}{8}ac^2+10b^2-29bc^2-3c^4$ by $\frac{3}{2}a-5b-\frac{1}{2}c^2$.

(12.) Divide $x^2-(a+b)x+ab$ by $x-b$.

(13.) Divide $\frac{1}{3}x^3+\frac{17}{6}x^2-\frac{5}{4}x+\frac{9}{4}$ by $\frac{1}{3}x+3$.

(14.) Divide $\frac{3}{8}a^2-\frac{17}{4}ab+10b^2+\frac{107}{12}ac^2-\frac{88}{3}bc^2-2c^4$ by $\frac{3}{2}a-5b-\frac{1}{3}c^2$.

48. It sometimes happens that in the division of algebraical quantities there is a **remainder**; that is to say, the divisor is not contained an exact number of times in the dividend. Thus:

Divide $39a^2-13a^4+12a^5-34a^3$ by $4a^2-7a$.

Arranging the terms of the dividend, we have

$$
\begin{array}{l}
4a^2-7a)\,12a^5-13a^4-34a^3+39a^2\,(\,3a^3+2a^2-5a+\dfrac{4a^2}{4a^2-7a} \\
\qquad\quad 12a^5-21a^4 \\
\hline
\qquad\qquad +8a^4-34a^3 \\
\qquad\qquad +8a^4-14a^3 \\
\hline
\qquad\qquad\qquad -20a^3+39a^2 \\
\qquad\qquad\qquad -20a^3+35a^2 \\
\hline
\qquad\qquad\qquad\qquad +4a^2
\end{array}
$$

Note.—Every dividend is equal to the **sum** of all its parts. In the preceding example the following quantities are its parts:

$$
\begin{array}{llll}
(1.) & 12a^5-21a^4 & = 4a^2-7a\times+3a^3. \\
(2.) & \quad +8a^4-14a^3 & = 4a^2-7a\times+2a^2. \\
(3.) & \qquad -20a^3+35a^2 & = 4a^2-7a\times-5a. \\
(4.) & \qquad\qquad +4a^2 & \text{Remainder.} \\
\hline
 & 12a^5-13a^4-34a^3+39a^2 & = \text{Dividend.}
\end{array}
$$

EXERCISE L.

(1.) Divide $6a^2bc-9abc^2-4ab^2c+12ac^3+6b^2c^2+15bc^3$ by $3ac-2bc$.

(2.) Divide $6a^2+5ax+8ay-6x^2+13xy$ by $2a+3x$.

(3.) Divide $14a^2-13ab+2ac+3b^2-3bc$ by $2a-b$.

(4.) Divide $12x^3-17x^2y+3xy^2+3y^3$ by $3x-2y$.

(5.) Divide $x^6-x^4y^2+x^2y^4-2y^6$ by $x+y$.

(6.) Divide $6x^4+x^3y+2x^2y^2-13xy^3+6y^4$ by $3x^2-4xy+y^2$.

(7.) Divide $4x^3-22x^2y+42xy^2-30y^3$ by $2x-3y$.

(8.) Divide $a^2b+a^2x-2abx+b^2x-2ax^2-3x^3$ by $b+x$.

(9.) Divide $x^7-7x^6+21x^5-17x^4-25x^3+6x^2-2x-6$ by x^2-2x-2.

(10.) Divide $a^3+b^3+c^3-4abc$ by $a+b+c$.

49. It has been shown that

$$a^2-b^2 \div a-b=a+b,$$
$$a^4-b^4 \div a-b=a^3+a^2b+ab^2+b^3,$$
$$a^5-b^5 \div a-b=a^4+a^3b+a^2b^2+ab^3+b^4.$$

By actual division it will also be found that

$$a^3-b^3 \div a-b=a^2+ab+b^2,$$
$$a^6-b^6 \div a-b=a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5,$$
$$a^7-b^7 \div a-b=a^6+a^5b+a^4b^2+\&c.,$$
$$a^8-b^8 \div a-b=a^7+a^6b+a^5b^2+\&c.,$$
$$\&c. = \&c.$$

Arranging these results thus:

$$a^2-b^2 \div a-b=a+b,$$
$$a^3-b^3 \div a-b=a^2+ab+b^2,$$
$$a^4-b^4 \div a-b=a^3+a^2b+ab^2+b^3,$$
$$a^5-b^5 \div a-b=a^4+a^3b+a^2b^2+ab^3+b^4,$$
$$a^6-b^6 \div a-b=a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5,$$
$$\&c. = \&c.$$

We observe:

(1.) That **every** dividend is **exactly** divisible by $a-b$.

(2.) That in the terms of the quotient the powers of a decrease in a regular order, while those of b increase in the same order.

(3.) That the terms in the quotient are all $+$.

Now, regarding the preceding as a model, the quotient may often be written down without the actual process of division. Thus:

$$8a^3-b^3=(2a)^3-b^3;$$

hence $8a^3-b^3 \div 2a-b = 4a^2+2ab+b^2$

$[a^3-b^3 \div \; a-b = a^2+ab+b^2]$;

and $a^4-16b^4 \div a-2b = a^3+2a^2b+4ab^2+8b^3$
$[a^4- \quad b^4 \div a- \quad b = a^3+a^2b+ab^2+b^3]$.

EXERCISE LI.

(1.) Divide $16a^4-81b^4$ by $2a-3b$.
(2.) Divide $64a^3-c^3$ by $4a-c$.
(3.) Divide $81x^4-y^4$ by $3x-y$.
(4.) Divide $16y^4-z^4$ by $2y-z$.
(5.) Divide $27a^3-b^3$ by $3a-b$.
(6.) Divide $32a^5-c^5$ by $2a-c$.
(7.) Divide $625c^4-81d^4$ by $5c-3d$.
(8.) Divide $4a^2-49c^2$ by $2a-7c$.
(9.) Divide $64a^3-x^3$ by $4a-x$.
(10.) Divide $64x^6-y^6$ by $2x-y$.

50. By actual division it will be found that

$a^2+b^2 \div a+b$ is indivisible,
$a^3+b^3 \div a+b = a^2-ab+b^2$,
$a^4+b^4 \div a+b$ is indivisible,
$a^5+b^5 \div a+b = a^4-a^3b+a^2b^2-ab^3+b^4$,
$a^6+b^6 \div a+b$ is indivisible,
$a^7+b^7 \div a+b = a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6$,
$a^8+b^8 \div a+b$ is indivisible,
&c. = &c.

Hence we observe:

(1.) That **every** dividend is not divisible by $a+b$.
(2.) That where the dividend **is** divisible, the index of each term in it is **odd.**
(3.) That the terms in the quotient are alternately $+$ and $-$.

Now, regarding the preceding as a model, the quotient may often be written down without the actual process of division. Thus:

$$27a^3+8b^3=(3a)^3+(2b)^3;$$

hence $\quad 27a^3+8b^3\div 3a+2b=9a^2-6ab+4b^2$

$$[a^3+\ b^3\div\ a+b\ =\ a^2-\ ab+b^2].$$

EXERCISE LII.

(1.) Divide $8a^3+b^3$ by $2a+b$.

(2.) Divide $243x^5+y^5$ by $3x+y$.

(3.) Divide $64x^3+y^3$ by $4x+y$.

(4.) Divide $243a^5+b^5$ by $3a+b$.

(5.) Divide $a^5+1024b^5$ by $a+4b$.

(6.) Divide $8a^3+x^3$ by $2a+x$.

(7.) Divide $243a^5+32b^5$ by $3a+2b$.

(8.) Divide $64a^3+x^3$ by $4a+x$.

(9.) Divide $27a^3+8y^3$ by $3a+2y$.

(10.) Divide $1024a^5+243y^5$ by $4a+3y$.

51. In paragraph 33 it was shown that $(a+b)(a-b)=a^2-b^2$. It may also be seen by actual multiplication that

$$(a^2+b^2)(a^2-b^2)=a^4-b^4,$$
$$(a^3+b^3)(a^3-b^3)=a^6-b^6,$$
$$(a^4+b^4)(a^4-b^4)=a^8-b^8,$$
$$\&c. = \&c.$$

Hence:

$$a^2-b^2=(a+b)(a-b),$$
$$a^4-b^4=(a^2+b^2)(a+b)(a-b),$$
$$a^6-b^6=(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2),$$
$$a^8-b^8=(a^4+b^4)(a^2+b^2)(a+b)(a-b),$$
$$\&c. = \&c.$$

Thus we see that when the index is **even**, expressions similar to a^2-b^2, a^8-b^8, &c., are divisible by both $a+b$ and $a-b$, the quotient being the product of the remaining factors.

The quotients, it may be seen, either by the multiplication of the remaining factors, or by actual division, will be as follows:

$$a^4-b^4 \div a+b = a^3-a^2b+ab^2-b^3,$$
$$a^6-b^6 \div a+b = a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5,$$
$$a^8-b^8 \div a+b = a^7-a^6b+a^5b^2-a^4b^3+a^3b^4-a^2b^5 +ab^6-b^7.$$

Hence, when the divisor is $a+b$, the terms of the quotient are **alternately** $+$ and $-$.

Note.—When the divisor is $a-b$, the terms of the quotient are all $+$ (see paragraph 49).

EXERCISE LIII.

(1.) Divide $16a^4-81b^4$ by $2a+3b$.
(2.) Divide $27x^3-y^3$ by $3x-y$.
(3.) Divide $16b^4-c^4$ by $2b+c$.
(4.) Divide b^3-27c^3 by $b-3c$.
(5.) Divide $16b^2-c^2$ by $4b+c$.
(6.) Divide $32x^5-y^5$ by $2x-y$.
(7.) Divide $1024x^5-y^5$ by $4x-y$.
(8.) Divide $125x^3+y^3$ by $5x+y$.
(9.) Divide $243x^5-y^5$ by $3x-y$.
(10.) Divide $16x^4-z^4$ by $2x-z$.

EXERCISE LIV.

(1.) Divide $a^5-5a^4b-b^5+10a^3b^2-10a^2b^3+5ab^4$ by $a^3-3a^2b+3ab^2-b^3$.
(2.) Divide $a^4+8a^3b+24a^2b^2+16b^4+32ab^3$ by $a+2b$.

(3.) Divide $a^3+5ab^2+5a^2b+b^3$ by a^2+b^2+4ab.

(4.) Divide $4a^6-25a^2b^4+20ab^5-4b^6$ by $2a^3+2b^3-5ab^2$.

(5.) Divide $a^6-x^6-3a^4x^2+3a^2x^4$ by $a^3-3a^2x+3ax^2-x^3$.

(6.) Divide $x^3+y^3+z^3-3xyz$ by $x+y+z$.

(7.) Divide $51a^2b^2+10a^4-48a^3b-15b^4+4ab^3$ by $4ab-5a^2+3b^2$.

(8.) Divide $a^4-6a^3+5a^2+24a-36$ by a^2-4.

(9.) Divide $a^3+28b^3-5ab^2+a^2b$ by $4b+a$.

(10.) Divide $6a^4+4x^4+13ax^3-a^3x+2a^2x^2$ by $2a^2-3ax+4x^2$.

(11.) Divide $a^4+a^3x-8a^2x^2+19ax^3-15x^4$ by a^2-5x^2+3ax.

(12.) Divide $2a^7x-5a^6x^2-11a^5x^3+5a^4x^4-26a^3x^5+7a^2x^6-12ax^7$ by $a^4-3ax^3+a^2x^2-4a^3x$.

MISCELLANEOUS EXAMPLES.

(A)

(1.) Multiply $-3mn$ by $4m^2n$, and $5m^2n^3p$ by $-mnp^2$.

(2.) Find the product of $3a^2x^2-6axy+3y^2$ and $5ax+5y$.

(3.) Find by inspection $(x^2+1)^2$ and $(3x-2y)^2$.

(4.) Divide a^3b^5 by ab^3, and a^4b^2 by $-ab^2$.

(5.) Divide $x^{12}-6x^{10}+15x^8-20x^6+15x^4-6x^2+1$ by x^2-1.

(6.) Find by inspection $x^{10}-y^8\div x^5+y^4$ and $x^9-y^6\div x^3-y^2$.

(7.) Simplify $(x+y)^3+(x+y)^2y+(x+y)y^2-\{3x^2y+5y^2x+2y^3\}$.

(8.) Simplify $(a+b)(b+c)-(c+d)(d+a)-(a+c)(b-d)$.

(B)

(1.) Multiply $4ab-2ac$ by $6ab+3ac$.

(2.) Find the product of a^2+ax+x^2 and a^2-ax+x^2.

(3.) Find the continued product of $x+1$, $x+2$, $x+3$, and $x+4$.

(4.) Divide a^5+b^5 by $a+b$.

(5.) Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.

(6.) Resolve $x^2-15x+50$ into factors.

(7.) Multiply x^2+xy+y^2 by x^2-xy-y^2.

(8.) Simplify $2\{(x-a)(x-b)+(a-x)(a-b)+(b-x)(b-a)\}-\{(a-b)^2+(x-a)^2+(x-b)^2\}$.

(C)

(1.) Find the continued product of $2ac-3$, $2ac+3$, and $4a^2c^2+9$.

(2.) Find the coefficient of x^4 in the product of $x^4-ax^3+a^2x^2-a^3x+a^4$ and x^2+ax+a^2.

(3.) Write down the squares of $3bx^2-3c^2y^2$ and $4ax^4-bx^2$.

(4.) Resolve into their elementary factors x^8-a^8 and x^4+5x^2+4.

(5.) Simplify $3a^3-5a^2b-(2a-b)^2(a-2b)+a^3-2b^3$.

(6.) Divide $a^2(b+c)+b^2(a-c)+c^2(a-b)+abc$ by $a+b+c$.

(7.) Divide $x^3+y^3+3xy-1$ by $x+y-1$.

(8.) Simplify $(3b+2a)(3b-2a)+(3a-b)(3a+b)-2(b-a)(b+a)$.

(D)

(1.) Multiply $8a^3+4a^2x+2ax^2+x^3$ by $2a-x$.

(2.) Find by inspection $(x^2-x+1)(x+1)$ and $(1+x+x^2)(1-x)$.

(3.) Divide $a^{10}-x^{15}$ by a^2-x^3.

(4.) By what quantities are x^3+1 and x^6-1 divisible?

(5.) Find by inspection $(3x-4y)^2$, $(ax-by)^2$.

(6.) Divide $a^8+a^6b^2-a^2b^6-b^8$ by $a^3-a^2b+ab^2-b^3$.

(7.) Simplify $(a+b+c)^2-\{a(b+c-a)+b(a+c-b)+c(a+b-c)\}$.

(8.) Simplify $(1-ax)(1-bx)(1-cx)-(1-cx)^2+(1-ax)(bx-1)$.

(E)

(1.) Multiply $x^2+(x+y)^2+y^2+x+y+1$ by $x+y-1$.

(2.) Divide $a^5b-a^4b^2+a^3b^3-a^2b^4+ab^5$ by $-ab$.

(3.) Divide $x^4+y^4-z^4+2x^2y^2-2z^2-1$ by $x^2+y^2-z^2-1$.

(4.) Resolve into factors $x^2-8xy-48y^2$, and x^3+343.

(5.) Simplify $(x+8)^2+3(x+8)(x-8)+(x-1)^2$.

(6.) If $x=a+b$, $y=a-b$, and $z=a^2-b^2$, find the value of $x^2-y^2+z^2$.

(7.) Decompose into two simple binomial factors $x^2-3x-40$, and $x^2+24x+135$.

(8.) Find the product of x^2-ax+b, and $x-1$.

(F)

(1.) Multiply $a^2+b^2+c^2-ab-bc-ca$ by $a+b+c$.

(2.) Multiply x^3+x^2+x-1 by $x-1$.

(3.) Divide a^6-b^6 by $a^3-2a^2b+2ab^2-b^3$.

(4.) Simplify $\{x(x+a)-a(x-a)\}\{x(x-a)-a(a-x)\}$.

(5.) Divide $(x-y)^3-2y(x-y)^2+y^2(x-y)$ by $(x-2y)^2$.

(6.) Resolve $3x^3-14x^2-24x$ into its simple factors.

(7.) Divide x^9-1 by x^3-1.

(8.) Simplify $(x^2-9x+20)(x^2-5x)-(x^2-13x+42)(x^2-6x)$.

(G)

(1.) Multiply $14ac-3ab+2$ by $ac-ab+1$.

(2.) Divide $\frac{2a^3}{x}-\frac{35ax}{6}+\frac{3x^3}{a}$ by $4a^2-9x^2$.

(3.) Resolve into elementary factors $-60a^9b^3+72a^6b^4-84a^3b^6$.

(4.) Divide $x^3-(a-1)x^2-(a-1)x+1$ by $x+1$.

(5.) Write down the result of $\left(\frac{1}{2}a-2b^2\right)\left(\frac{1}{2}a+2b^2\right)$.

(6.) Multiply together $x-a$, $x-b$, $x-c$.

(7.) Simplify $\{6a+2b-(3a+2b)\}-\{2a+4b-(4a-b)\}$.

(8.) Simplify $(a+b)^2-2b(a+b)+2b^2$.

---

52. It has been stated that a collection of terms is called an **expression.**

When two expressions are equal to each other, and are connected by the sign of equality (=), the **whole** is termed an **equation.** Thus:

(1.) $(a+b)(a+b) = a^2+2ab+b^2$.
(2.) $x+3 = 9$.

Note 1.—The former of the above examples is different from the latter, inasmuch as the equation holds good, **whatever** values are assigned to a and b.

Thus, suppose $a=3$, $b=2$, then we have

$$(3+2)(3+2) = 3^2+2\times3\times2+2^2;$$

or $$5\times5 = 9+12+4;$$

or $$25 = 25.$$

[Such equations are termed **identical.**]

Note 2.—The latter equation holds good **only** when a **particular** value is assigned to x, which in this case $=6$; *e.g.*, $6+3=9$.

An equation contains some **unknown** quantity or quantities, which when found out, or **solved,** will satisfy the equation.

Thus, $2x+3=x+5$ is an equation, in which x represents some **unknown** quantity, which satisfies the equation.

The quantity in question here is evidently 2, since $2\times 2+3=7$, and $2+5=7$.

Note 1.—When an equation contains the **unknown** quantity or quantities in its **simple** form—that is to say, when it contains no **power** of the unknown quantity higher than the first (paragraph 11)—it is termed a **simple equation.**

Note 2.—In the following pages the term "equation" means simple equation.

53. In order to **solve** an equation, the following **axioms** (or self-evident truths) must be strictly adhered to:

(A.) If equals be added to equals, the sums are equal.

(B.) If equals be subtracted from equals, the differences are equal.

(C.) If equals be multiplied by equals, the products are equal.

(D.) If equals be divided by equals, the quotients are equal.

Or, in other words, **in order to maintain the equality subsisting between the two sides, or members, of an equation, whatever is done to the one side must also be done to the other.**

Note.—In an equation, **unknown** quantities are usually denoted by the last letters of the alphabet (x, y, z, &c.).

(*Ex.* 1.) Given $4x=20$; to find x.

Here, dividing both sides of the equation by 4, we get $\frac{4x}{4} = \frac{20}{4}$; but $\frac{4x}{4} = x$, and $\frac{20}{4} = 5$,

$$\therefore x = 5.$$

[The symbol ($\therefore$) means **therefore.**]

(*Ex.* 2.) Given $6x + 2x = 32$; to find x.

Here, collecting the terms which contain x,

we have $6x + 2x = 8x$,

$$\therefore 8x = 32;$$

$\therefore$ dividing each member of the equation by 8,

we have $x = 4$.

EXERCISE LV.

(1.) Given $3x = 18$; to find the value of x.

(2.) Given $7x = 35$; to find the value of x.

(3.) Given $5x + 3x = 32$; to find the value of x.

(4.) Given $7x + 8x = 30$; to find the value of x.

(5.) Given $9x + 4x = 39$; to find the value of x.

(6.) Given $2x + 3x + 4x = 45$; to find the value of x.

(7.) Given $5x + 2x + 7x = 70$; to find the value of x.

(8.) Given $9x + 3x + 8x = 80$; to find the value of x.

(9.) Given $8x - 3x - 4x = 4$; to find the value of x.

(10.) Given $7x + 6x - 4x = 72$; to find the value of x.

(11.) Given $10x + 5x - 11x = 36$; to find the value of x.

(12.) Given $6x + 8x - 12x = 22$; to find the value of x.

54. In the preceding examples, the term or terms containing the **unknown** quantity have all been on one side of the equation, and the known quantities on the other.

We now proceed to examples in which the **known** quantities are in both members of the equation. Thus:

Given $3x + 5 = 26$; to find the value of x.

Here, taking 5 from each side of the equation, we have

$$3x+5-5 = 26-5;$$

$$\text{but } 5-5 = 0, \therefore 3x = 26-5 = 21,$$

$$\therefore x = 7.$$

Note.—From the above we see that a term may be transposed from one side of an equation to the other, by **changing its sign**; *e.g.*, $3x+5 = 26$ becomes

$$3x = 26-5.$$

EXERCISE LVI.

(1.) Given $5x+6 = 26$; to find the value of x.

(2.) Given $x+8 = 15$; to find the value of x.

(3.) Given $9x+13=31$; to find the value of x.

(4.) Given $8x+2x+3 = 43$; to find the value of x.

(5.) Given $9x+2x+1 = 56$; to find the value of x.

(6.) Given $7x+3x+3 = 33$; to find the value of x.

(7.) Given $8x+7x+9 = 39$; to find the value of x.

(8.) Given $4x+3x+x+3 = 19$; to find the value of x.

(9.) Given $2x+3x+5x+1=21$; to find the value of x.

(10.) Given $x+10x+2x+3=29$; to find the value of x.

55. Given $3x-7 = 17$; to find the value of x.

Here, adding 7 to each side of the equation, we have

$$3x-7+7 = 17+7;$$

$$\text{but } -7+7 = 0, \therefore 3x = 17+7 = 24,$$

$$\therefore x = 8.$$

In this example also we see that a term may be transposed from one side of an equation to the other, **by changing its sign**; *e.g.*, $3x-7 = 17$ becomes

$$3x = 17+7.$$

EXERCISE LVII.

(1.) Given $4x-5=3$; to find the value of x.
(2.) Given $9x-3=33$; to find the value of x.
(3.) Given $10x-7=43$; to find the value of x.
(4.) Given $6x+x-4=24$; to find the value of x.
(5.) Given $3x+7x-5=5$; to find the value of x.
(6.) Given $8x+10x-5=31$; to find the value of x.
(7.) Given $7x+5x-5=43$; to find the value of x.
(8.) Given $8x+x-11=16$; to find the value of x.
(9.) Given $5x+7x-4=56$; to find the value of x.
(10.) Given $3x+11x-1=55$; to find the value of x.

56. The effect of placing a quantity outside a **bracketed** expression has been explained in paragraph 23.

Note.—In the following examples, the value of x will often be a **fractional** quantity.

(*Ex.* 1.) Given $3(x+3)+2(x-2=18$.

Clearing this expression of brackets, we have

$$3x+9+2x-4=18,$$
or
$$5x+5=18;$$
$$\therefore 5x=18-5=13,$$
$$\therefore x=\tfrac{13}{5}=2\tfrac{3}{5}.$$

(*Ex.* 2.) Given $2x-4(3x-2)=16$.

Clearing this expression of brackets, we have

$$2x-12x+8=16,$$
or
$$-10x=16-8=8;$$
$$\therefore 10x=-8, \text{ and}$$
$$x=-\tfrac{8}{10}=-\tfrac{4}{5}.$$

Note.—The equation $-10x=8$ becomes $10x=-8$ by multiplying **both** sides of the equation by -1, an operation which should always be performed when the first side of the equation is **negative.**

EXERCISE LVIII.

(1.) Given $2x-4+3(x+5)=36$; to find the value of x.

(2.) Given $7x+4(2x-3)+12=63$; to find the value of x.

(3.) Given $5x+3(2x-2)+14=14$; to find the value of x.

(4.) Given $2(2x-3)+4(x-4)=27$; to find the value of x.

(5.) Given $3(4x-3)+2(x-3)=55$; to find the value of x.

(6.) Given $5(2x-3)-4(x+2)=21$; to find the value of x.

(7.) Given $6(x+4)+3(2x-3)=18$; to find the value of x.

(8.) Given $7(3x-4)+4(2x-3)=5$; to find the value of x.

(9.) Given $4(x-3)-2(x-5)=14$; to find the value of x.

(10.) Given $2x-3x+\{4-(2x-3)\}=-2$; to find the value of x.

57. Hitherto the unknown quantity has been found only on **one** side of the equation.

We now proceed to examples in which the unknown quantity is to be found in **both** members of the equation.

(1.) Given $2(x+3)-3x=4(x-2)$.

Here, multiplying the bracketed expression by the common factor, we have

$$2x+6-3x = 4x-8,$$
or $$2x-3x-4x = -8-6,$$
or $$-5x = -14;$$
multiplying both sides by -1, $\quad 5x = 14$;
$$\therefore x = \tfrac{14}{5} = 2\tfrac{4}{5}.$$

EXERCISE LIX.

(1.) Given $3(x-3)+2(x-4) = 4(x-5)$; to find the value of x.

(2.) Given $2(3-x)-3(4-x) = 5(x+3)$; to find the value of x.

(3.) Given $4(3-x)+3(2+x) = 4(3+x)$; to find the value of x.

(4.) Given $2(x-1)-3(x+4) = 2(x+3)$; to find the value of x.

(5.) Given $5(x-4)+3(x+2) = x-5$; to find the value of x.

(6.) Given $x+3-3(x-4) = 12(x+1)$; to find the value of x.

(7.) Given $4(x-3)+2(x+7) = 3(x-3)$; to find the value of x.

(8.) Given $5(x+3)+4(x-3) = 4(x+3)$; to find the value of x.

(9.) Given $2(x-3)+3(x+3) = 3(x-3)$; to find the value of x.

(10.) Given $4(x+3)+2(x-3) = 5(x+2)$; to find the value of x.

58. It was stated in paragraph 4 that **unknown** quantities are usually represented by the **last** letters of the alphabet, x, y, z, &c. Hitherto **known** quantities have been invariably represented by **numbers**; but they are sometimes represented by **letters**, whose numerical values are known, or supposed to be so. For this purpose, the **first** letters of the alphabet are used, as a, b, c, &c.

(1.) Given $x - a = b - 3x$; to find the value of x.

Here, collecting the unknown quantities on one side of the equation, and the known quantities on the other, we have

$$x + 3x = a + b,$$

or

$$4x = a + b;$$

$$\therefore x = \frac{a+b}{4}, \text{ or } \tfrac{1}{4}(a+b).$$

(2.) Given $3x - 2b = 3a + 5b - 2x$; to find the value of x.

Here $3x + 2x = 3a + 5b + 2b$,

or

$$5x = 3a + 7b;$$

$$\therefore x = \frac{3a+7b}{5}.$$

EXERCISE LX.

(1.) Given $2x + 3b = x - 2c$; to find the value of x.

(2.) Given $x - a + b = 3x + b$; to find the value of x.

(3.) Given $x + 3a - b = 2x + c$; to find the value of x.

(4.) Given $3x + x - c = 4a + b$; to find the value of x.

(5.) Given $2a - x + c = 4x - 3c$; to find the value of x.

(6.) Given $4x - 3b = 2a + 2c - x$; to find the value of x.

(7.) Given $x - 3a = 4x - b + 2a$; to find the value of x.

(8.) Given $4x - 3b = 2a - x - 3b$; to find the value of x.

(9.) Given $5x - 2c = 3a - 3x - 4b$; to find the value of x.

(10.) Given $4x - 3a = 3b + a - 2x$; to find the value of x.

59. We now proceed to the solution of problems producing equations. It must be distinctly understood that, in order to solve **problems**, no general rules can be laid down, as in the case of solving **equations** (paragraph 53); but, generally speaking, each problem requires particular treatment to bring it into an algebraical form. **When that is done,** the ordinary rules for solving an equation are applicable.

Note.—The pupil will be better able to understand the force of the preceding remarks when he has followed the working of a few problems.

(1.) Find a number, which being multiplied by 5, and having 13 added to the product, the sum shall be 43.

Let x = number required,
then $5x$ = required number multiplied by 5,
and $5x+13$ = product $+13$.
But by the question, the product $+13 = 43$;

$$\therefore 5x+13 = 43.$$

We have thus formed an equation, by solving which it will be found that x, the required number, $= 6$.

(2.) Divide the number 18 into two parts, so that one of them shall exceed the other by 5.

Let x = one part (the less);
then by the question $x+5$ = the other part (the greater).
But the two parts make the whole number which, by the question, $= 18$;

$$\therefore x+\overline{x+5} = 18;$$

whence $x = 6\frac{1}{2}$ the less part,
and $6\frac{1}{2}+5 = 11\frac{1}{2}$ the greater part.

(3.) Divide £600 amongst *A*, *B*, and *C*, so that *A* may receive twice as much as *B*, and *C* as much as *A* and *B* together.

Let x = *B*'s share in £;
then by the question,

$2x$ = *A*'s share in £,
and $x+2x$ = *C*'s share in £;

$$\therefore x+2x+\overline{x+2x} = \text{the shares of } B, A, \text{ and } C \text{ in £}.$$

But the shares of A, B, and C = the whole sum = £600;

$\therefore\ x+2x+x+2x = 600$ £.

Whence $x = 100$ £ = B's share;

$2x = 200$ £ = A's share;

$x+2x = 300$ £ = C's share.

(4.) A and B began to play with equal sums; A won 30 shillings, and then 7 times A's money was equal to 13 times B's money. What had each at first?

Let x = sum that A had at first in shillings; then by the question,

x also = sum that B had at first in shillings;

then $x+30$ = A's sum when he has won 30 shillings;

and $x-30$ = B's sum when he has lost 30 shillings;

and by the question,

$$7(x+30) = 13(x-30):$$

whence $x = 100$ shillings.

(5.) A and B have together 16 shillings; A and C have 20 shillings; B and C have 24 shillings. What have they each?

Let x = what A has;

then by the question,

$16-x$ = what B has,

and $20-x$ = what C has.

But by the question, B and C have 24 shillings;

$$\therefore\ \overline{16-x}+\overline{20-x} = 24:$$

whence $x = 6$ shillings, *i.e.* A's sum;

$16-x = 10$ shillings, *i.e.* B's sum;

$20-x = 14$ shillings, *i.e.* C's sum.

(6.) A fish was caught, the tail of which weighed 10 lb., its head weighed as much as its tail and half its body, and its body weighed as much as its head and tail. Find the weight of the fish.

Let $2x =$ weight of body in ℔.;
then $x =$ weight of half its body.
By the question, the tail weighs 10 ℔.;
$\therefore 10 + x =$ weight of its head in ℔.
Now the body weighs as much as the head and tail;

$$\therefore 2x = \overline{10 + x} + 10:$$

whence $x = 20$;

$\therefore 2x = 40$ ℔., the weight of the body;
$10 + x = 10 + 20 = 30$ ℔., the weight of the head;
$\therefore$ weight of the **whole** fish $= 40 + 30 + 10 = 80$ ℔.

60. In every problem, some quantities are known, and others are unknown. Those which are given are said to be known; the unknown are to be found.

In forming an equation, we denote the unknown quantity by some letter (say x). We then consider its relations to the quantities which are given, and express the same in algebraical language. We thus form an equation, from which the unknown quantity may be found.

61. In the following exercise, the pupil is required to show the connection between the problem and its corresponding equation, by inserting the necessary steps (paragraph 59), and then to solve the equation.

EXERCISE LXI.

(1.) What two numbers are those whose difference is 5, and if 3 times the greater be added to 4 times the less, the sum shall be 43?

$$3(x + 5) + 4x = 43.$$

(2.) A horse, a cow, and a sheep were bought for £36; the cow cost £6 more than the sheep, and the horse £15 more than the cow. Find the price of the sheep.

$$x + \overline{x + 6} + \overline{x + 6 + 15} = 36.$$

(3.) How much tea at 4s. 6d. must be mixed with 25 ℔. at 6s., that the mixture may be sold at 5s. 6d?

$$9x+12\times 25 = 11(x+25).$$

(4.) *A* and *B* lay out equal sums of money in trade; *A* gains £63, *B* loses £87: *A*'s money is now 4 times *B*'s. What did each lay out?

$$x+63 = 4(x-87).$$

(5.) The difference between two numbers is 3, and their product exceeds the square of the less by 12. Find the numbers.

$$x(x+3) = x^2+12.$$

(6.) An express set out to travel 480 miles in 4 days; but, owing to the badness of the roads, it was found that it must go 10 miles the second day, 18 miles the third, and 28 miles the fourth, less than the first. How many miles did it travel each day?

$$x+x-10+x-18+x-28 = 480.$$

EXERCISE LXII.

(1.) A boy at play wins 15 marbles, and then has 4 times as many as he began with. How many had he at first?

(2.) Divide £100 between *A* and *B*, so that *A* may have three times as much as *B*.

(3.) The sum of two numbers is 85, and their difference is 27: find the numbers.

(4.) Divide a line 12 feet long into three parts, such that the middle one shall be double the least, and the greatest triple the least.

(5.) Find a number, which being multiplied by 6, and having 10 added to the product, the sum shall be 64.

(6.) *A* and *B* have together £25, and three times *A*'s money and four times *B*'s would together make £85. How much has each?

(7.) Find two numbers whose difference shall be 6, and their sum 16.

(8.) The sum of £155 was raised by *A*, *B*, and *C* together: *B* contributed £15 more than *A*, and *C* £20 more than *B*. How much did each contribute?

(9.) Divide a line of 60 inches into three such parts that the second may be three times the first, and the third double the second.

(10.) What two numbers are those whose difference is 10, and if 15 be added to their sum, the whole will be 43?

(11.) *A* and *B* have together £60, and three times what *A* has would be £4 more than what *B* has. How much has each?

(12.) There are four brothers, each of whom is three years younger than his next eldest brother, and the eldest brother is four times as old as the youngest. What is the age of each?

(13.) The sum of £76 was raised by *A*, *B*, and *C* together; *B* contributed as much as *A* and £10 more, and *C* as much as *A* and *B* together. How much did each contribute?

(14.) The difference between two numbers is 2, and their product exceeds the square of the less by 8. What are the numbers?

(15.) The ages of two men differ by 10 years; and 15 years ago the elder was just twice as old as the younger. Find the ages of the men.

(16.) A person has £3, 7s. 6d., made up of half-crowns and half-sovereigns: he has 12 coins in all. How many has he of each?

(17.) A bill of £1, 6s. 6d. was paid in florins and half-crowns: the total number of coins was 12. Required the number of each sort.

(18.) A person distributes 20 shillings among 20 persons, giving sixpence each to some, and sixteenpence each to the rest. How many persons received sixpence each?

(19.) It is required to divide £470 among three persons, so that the second may have £10 more than the first, and the third £30 more than the second.

(20.) Divide 19 guineas among three persons, so that the first shall have twice as much as the second, and the third 5s. less than the second.

(21.) A grocer mixes tea at 6s. per ℔. with an equal weight of tea at 3s. per ℔. How many ℔. must there be of each kind, so that the whole may sell for 36s.?

(22.) *A* has £60 and *B* £20; *A* spends 5 times as much as *B*, and then they have equal sums. How much has each spent?

(23.) How much wine at 12s. a gallon must be mixed with 30 gallons at 15s., that the mixture may be worth 14s. a gallon?

(24.) Divide the number 75 into two parts, such that 3 times the greater may exceed 7 times the less by 15.

(25.) The sum of £154 was paid in half-sovereigns, half-crowns, and fourpenny-pieces, and an equal number of each of these coins was used. What was the number?

(26.) A man and his wife usually drank a cask of beer in 12 days; but when the man was from home, it lasted the woman 30 days. How long would the man be in drinking it?

(27.) A gentleman gave away 27s. among 9 persons; to a certain number he gave 2s. each, and to the rest 5s. each. How many were there in each class?

(28.) A farmer has 7 times as many sheep as cows, and 3 times as many cows as horses. They number 200 altogether. How many has he of each?

(29.) Divide £520 among A, B, and C, so that A may have twice as much as B, and C half as much again as A and B together.

(30.) A is twice as old as B, and seven years ago their united ages amounted to as many years as now represent the age of A. Find the ages of A and B.

(31.) The sum of two numbers is 10, and their product is equal to the excess of 60 above the square of the greater. Required the numbers.

(32.) Two shepherds, owning a flock of sheep, agree to divide its value: A takes 72 sheep, whilst B takes 94 sheep, and pays A £35. Find the value of a sheep.

(33.) A butcher bought a sheep and a cow for 96 shillings: now he paid 7 times as much for the cow as he paid for the sheep. Find the cost of each.

(34.) Two trains start at the same time from the same place, and travel in opposite directions; the former travels at the rate of 30 miles per hour, and the latter at the rate of 40. How long must they travel before they are 350 miles apart?

62. Before we proceed to the solution of equations, the terms of which contain one or more **fractional** quantities, it will be necessary to consider the treatment of **algebraical fractions.**

It was stated in paragraph 43, that if both the numerator and denominator of a fraction be multiplied **or** divided by the **same** quantity, the value of the fraction is unchanged. Hence fractions may often assume a simpler form by dividing both numerator and denominator by the same quantity.

(*Ex.* 1.) Reduce $\frac{14b^3+7ab+21b^2}{35b^2}$ to its lowest terms.

On **inspection,** we see that $7b$ is contained in **every** term of the fraction. Hence, the expression might be written $\frac{7b(2b^2+a+3b)}{7b\times 5b}$, in which the quantity $7b$ is a **factor common** to both numerator and denominator. Then, dividing both numerator and denominator by this common factor ($7b$), we find that

$$\frac{14b^3+7ab+21b^2}{35b^2}=\frac{2b^2+a+3b}{5b}.$$

(*Ex.* 2.) Reduce $\frac{a^3-ab^2}{a^4-b^4}$ to its lowest terms.

By referring to paragraph 5, the pupil will readily see the following steps in simplifying this expression:

$$\frac{a^3-ab^2}{a^4-b^4}=\frac{a(a^2-b^2)}{(a^2-b^2)(a^2+b^2)}=\frac{a}{a^2+b^2},$$

by dividing both numerator and denominator by the common factor (a^2-b^2).

EXERCISE LXIII.

Reduce the following fractions to their lowest terms:

(1.) $\frac{14a^2b^2-21a^3b^2}{7a^3b}$.

(2.) $\frac{3b^2+3bx}{b^2-x^2}$.

(3.) $\frac{ab^2+b^2}{3bc-bd}$.

(4.) $\frac{x^4-y^4}{x^5-x^3y^2}$.

(5.) $\frac{11a^2+22ax}{33(a^2-4x^2)}$.

(6.) $\frac{a^3+2a^2b+ab^2}{a^3+b^3}$.

(7.) $\frac{a^4-b^4}{a^6-a^4b^2}$.

(8.) $\frac{x^3+2x^2}{x^2+4x+4}$.

(9.) $\frac{9a^2b^3-15ab^4}{12a^2b^2-21ab^3}$.

(10.) $\frac{2a^3+5a^2-3a}{a^3+2a^2-3a}$.

Note.—The preceding fractions may be easily reduced by **inspection**; that is to say, the **common factor** can be readily determined.

63. In paragraph 8 it was stated that a **product** is made up of **factors.** Thus:

$$28 = 7 \times 4;$$
$$88 = 11 \times 8;$$
$$a^6 = a^2 \times a^4;$$
$$a^4b^5 = a^2b^2 \times a^2b^3;$$
$$10a^4y^2 = 5a^2y \times 2a^2y.$$

Hence the quantities 7, 11, a^2, a^2b^2, $5a^2y$ must be contained in the quantities 28, 88, a^6, a^4b^5, $10a^4y^2$ an **exact** number of times; or, in other words, they are called meaures of 28, 88, a^6, &c. Again:

(1.) $\left.\begin{aligned} 18 &= 6 \times 3 \\ 24 &= 6 \times 4 \end{aligned}\right\}$ (2.) $\left.\begin{aligned} 9a^4x^2 &= 3a^4x \times 3x \\ 12a^5x &= 3a^4x \times 4a \end{aligned}\right\}$

Hence, in Example (1), 6 is a measure of both 18 and 24; in Example (2), $3a^4x$ is a measure of both $9a^4x^2$ and $12a^5x$. The quantities, 6 and $3a^4x$, are therefore **common measures** of 18, 24, and $9a^4x^2$, $12a^5x$ respectively. Further:

(1.) $\left.\begin{aligned} 18 &= 2 \times 9 \\ 24 &= 2 \times 12 \end{aligned}\right\}$ (2.) $\begin{aligned} 9a^4x^2 &= 3a^2 \times 3a^2x^2; \\ 12a^5x &= 3a^2 \times 4a^3x. \end{aligned}$

Hence, it will be seen that there may be more than one common measure of any two or more quantities, and that in this case the **greatest** common measures (G.C.M.) are 6 and $3a^4x$.

64. We now proceed to the method for determining the G.C.M. of two **compound** quantities, such as

$$x^2 + x - 2, \text{ and } x^2 - 3x + 2;$$
$$\text{or } 2x^3 + 10x^2 + 14x + 6, \text{ and } x^3 + x^2 + 7x + 39.$$

The following is the rule:

"Arrange the expressions according to the dimensions of some one letter contained in them; divide the one of higher degree by the lower, or, if they be of the same degree, either of them by the other; then the preceding divisor by the last remainder; and so on, till there be no remainder: the last divisor will be the greatest common measure. If the terms of one of the expressions contain a factor which is not contained in the terms of the other expression, such factor may be neglected at any stage of the operation, and also the terms of either expression may be multiplied by any factor not common to the terms of the other expression, without altering the greatest common measure. The same may also be done with respect to any remainder during the process, and the remainder of which it is the divisor or dividend."

(*Ex.* 1.) Find the G.C.M. of x^2+x-2 and x^2-3x+2.

$$x^2+x-2\,)\,x^2-3x+2\,(\,1$$
$$x^2+x-2$$
dividing by $-4\,)\,\overline{-4x+4}$
$$x-1\,)\,x^2+x-2\,(\,x+2$$
$$x^2-x$$
$$\overline{2x-2}$$
$$2x-2$$
$$\overline{\ast\quad\ast}$$

$\therefore$ G.C.M. $= x-1$.

(*Ex.* 2.) Find the G.C.M. of $2x^3+10x^2+14x+6$ and $x^3+x^2+7x+39$.

Here $2x^3+10x^2+14x+6 = 2(x^3+5x^2+7x+3)$.

[The factor 2 may therefore be omitted.]

Hence we have

$$x^3+5x^2+7x+3)\,x^3+x^2+7x+39\,(1$$
$$x^3+5x^2+7x+3$$
dividing by $-4)\,-4x^2 \quad +36$

$$x^2 \quad -9)\,x^3+5x^2+7x+3\,(x+5$$
$$x^3 \quad -9x$$
$$5x^2+16x+3$$
$$5x^2 \quad -45$$
dividing by $16)\,16x+48$
$$x+3$$

$$x+3)\,x^2-9\,(x-3$$
$$x^2+3x$$
$$-3x-9$$
$$-3x-9$$
$$*\quad *$$

$\therefore$ G.C.M. is $x+3$.

EXERCISE LXIV.

Find the G.C.M. of

(1.) x^3+2x^2+2x+1 and x^3-2x-1.

(2.) $3a^3-3a^2b+ab^2-b^3$ and $4a^2-5ab+b^2$.

(3.) $a^6+4a^5-3a^4-16a^3+11a^2+12a-9$ and $6a^5+20a^4-12a^3-48a^2+22a+12$.

(4.) $3a^3-3a^2b+ab^2-b^3$ and $4a^3-a^2b-3ab^2$.

(5.) $6x^2-x-2$ and $21x^3-26x^2+8x$.

(6.) $a^3+a^2b+ab^2+b^3$ and $a^4+a^3b+ab^3-b^4$.

(7.) $2a^3+6a^2+6a+2$ and $6a^3+6a^2-6a-6$.

(8.) $6a^5-4a^4-11a^3-3a^2-3a-1$ and $4a^4+2a^3-18a^2+3a-5$.

(9.) $3x^3-13x^2+23x-21$ and $6x^3+x^2-44x+21$.

(10.) $x^4-bx^3-b^2x^2-b^3x-2b^4$ and $3x^3-7bx^2+3b^2x-2b^3$.

65. All fractions cannot be reduced to their lowest terms by **inspection**; but in order to determine the factor common to both numerator and denominator, we must find their G.C.M. When that is found, both numerator and denominator may be divided by it, since it is a **measure** of both.

(*Ex.* 1.) Reduce $\frac{a^2+a-12}{a^3-5a^2+7a-3}$ to its lowest terms.

Here it may be readily found that the G.C.M. of a^2+a-12 and a^3-5a^2+7a-3 is $a-3$.

The above fraction thus equals $\frac{(a-3)(a+4)}{(a-3)(a^2-2a+1)}$

$$=\frac{a+4}{a^2-2a+1}.$$

EXERCISE LXV.

Reduce the following fractions to their lowest terms:

(1.) $\frac{2a^3+5a^2-3a}{a^3+2a^2-3a}$.

(2.) $\frac{10x-24x^2+14x^3}{15-24x+3x^2+6x^3}$.

(3.) $\frac{x^4+bx^3-9b^2x^2+11b^3x-4b^4}{x^4-bx^3-3b^2x^2+5b^3x-2b^4}$.

(4.) $\frac{7a^2-23ab+6b^2}{5a^3-18a^2b+11ab^2-6b^3}$.

(5.) $\frac{a^3+3a^2-4}{a^3-1}$.

(6.) $\frac{x^4+b^2x^2+b^4}{x^4+bx^3-b^3x-b^4}$.

(7.) $\frac{2a^3+9a^2+7a-3}{3a^3+5a^2-15a+4}$.

(8.) $\frac{a^3+2a^2+2a}{a^5+4a}$.

(9.) $\frac{6b^3-5b^2+4}{2b^3-b^2-b+2}$.

(10.) $\frac{ax+2}{2a+(a^2-4)x-2ax^2}$.

66. The expression $\frac{a^2+ab+c}{a}=\frac{a^2}{a}+\frac{ab}{a}+\frac{c}{a}=a+b+\frac{c}{a}$;

also $\frac{16a^2+3x-y}{4a}=\frac{16a^2}{4a}+\frac{3x}{4a}-\frac{y}{4a}=4a+\frac{3x-y}{4a}$.

Hence we see that an **improper** algebraical fraction (that is, one whose numerator is divisible by the denominator) may often be reduced to a mixed quantity.

In the above examples, $a+b$ and $4a$ are the **integral** parts of the quantity, and $\frac{c}{a}$, $\frac{3x-y}{4a}$ the **fractional.**

EXERCISE LXVI.

Reduce the following fractions to whole or mixed quantities:

(1.) $\frac{25x^3-3a+2b}{5x}$.

(2.) $\frac{b^2+3x^2}{b+x}$.

(3.) $\frac{x^4}{x^2-b^2}$.

(4.) $\frac{2a^2+5}{a-3}$.

(5.) $\frac{10a^2y+3a^3-2b^2}{a^2}$.

(6.) $\frac{a^2-2ay+y^2-y^2}{a-y}$.

(7.) $\frac{3a^2}{a^2-1}$.

(8.) $\frac{4a^3+2a^2-3}{2a^2}$.

67. In the last paragraph it was shown how improper fractions are reduced to mixed quantities. We now proceed to show how mixed quantities are reduced to fractions.

(*Ex.* 1.) $2a+\frac{5b}{4a}=\frac{2a}{1}+\frac{5b}{4a}=\frac{8a^2}{4a}+\frac{5b}{4a}=\frac{8a^2+5b}{4a}$.

(*Ex.* 2.) $7x-\frac{x-3}{5}=\frac{7x}{1}-\frac{x}{5}+\frac{3}{5}=\frac{35x}{5}-\frac{x}{5}+\frac{3}{5}$

$$=\frac{35x-x+3}{5}=\frac{34x+3}{5}.$$

Note.—The effect of the negative sign before a fraction has been explained in paragraph 45.

Hence we see that the integral part of the mixed quantity is multiplied by the denominator of the fractional part; to this product the numerator, with its proper sign, is annexed; and underneath the whole expression the former denominator is placed.

EXERCISE LXVII.

Reduce the following mixed quantities to improper fractions:

(1.) $a-x+\frac{a^2-ax}{x}$.

(2.) $3a^2-\frac{4a-9}{10}$.

(3.) $a^2-2ab+4b^2-\frac{6b^3}{a+2b}$.

(4.) $a+b+\frac{a^2+b^2}{a-b}$.

(5.) $ax+4-\frac{ax^2-y}{x+y}$.

(6.) $1-\frac{x+y+z}{x+z}$.

(7.) $1-\frac{a^2-2ab+b^2}{a^2+b^2}$.

(8.) $x+2+\frac{2}{x-1}$

68.

$$28 = 7 \times 4;$$
$$88 = 11 \times 8;$$
$$a^6 = a^2 \times a^4;$$
$$a^4b^5 = a^2b^2 \times a^2b^3;$$
$$10a^4y^2 = 5a^2y \times 2a^2y.$$

On looking at the above, it will be seen that 28, 88, &c., contain 7, 11, &c., an **exact** number of times; the numbers 28, 88, &c., are accordingly called **multiples** of 7, 11, &c.

Again, 18 is a multiple of 9 and 2, therefore it is a **common** multiple of 9 and 2, **or** 6 and 3.

So, $36ab^2$ is a **common** multiple of $6ab$, $9b^2$, $4ab$, $12a$, &c.

Further, it is very clear that $36ab^2$, if multiplied by any quantity, its product must also be a **common** multiple of $6ab$, $9b^2$, &c.; but $36ab^2$ is the **least** common multiple (L.C.M.).

The least common multiple of two quantities is found

(1.) By dividing their product by their G.C.M.; or,

(2.) By dividing one of the quantities by their G.C.M., and then multiplying the other by the quotient.

(*Ex.* 1.) Find the L.C.M. of $12a^4x^6$ and $18a^2x^2$.

Here G.C.M. $= 6a^2x^2$.

Then $\dfrac{12a^4x^6}{6a^2x^2} = 2a^2x^4$.

$\therefore$ L.C.M. $= 18a^2x^2 \times 2a^2x^4 = 36a^4x^6$.

(*Ex.* 2.) Find the L.C.M. of a^2-x^2 and a^3-x^3.

Here $a^2-x^2 = (a-x)(a+x)$,

and $a^3-x^3 = (a-x)(a^2+ax+x^2)$.

$\therefore$ G.C.M. $= a-x$.

Then $\dfrac{a^2-x^2}{a-x} = a+x$.

$\therefore$ L.C.M. $= (a^3-x^3)(a+x) = a^4+a^3x-ax^3-x^4$.

EXERCISE LXVIII.

Find the L.C.M. of

(1.) $7a^2y$ and $3ay^2$.

(2.) $12a^2b^2c$ and $20a^3bc^2$.

(3.) axy and $a(xy-y^2)$.

(4.) $2(a+x)$ and $3(a^2-x^2)$.

(5.) $4(x^2-x)$ and $6(x^2+x)$.

(6.) a^3-1 and a^2+a-2.

(7.) $a^3+2a^2b-ab^2-2b^3$ and $a^3-2a^2b-ab^2+2b^3$.

(8.) x^2+xy and $(x+y)^2$.

(9.) a^3-a^2b and a^2-b^2.

(10.) $(a+1)^2$ and a^3+1.

69. It is often necessary to convert fractions having

different denominators into their equivalent fractions, having a **common** denominator. Thus:

(1.) $\frac{3a}{2x}$ and $\frac{b}{2y}$ are fractions having **different** denominators.

Multiplying both terms of the first fraction by y, and both terms of the second fraction by x, they become $\frac{3ay}{2xy}$ and $\frac{bx}{2xy}$, thus having a **common** denominator ($2xy$).

(2.) $\frac{2}{ab}$, $\frac{3}{bc}$, and $\frac{4}{a^2}$, have **different** denominators.

Multiplying the first by ac, the second by a^2, and the third by bc, they become

$$\frac{2ac}{a^2bc},\ \frac{3a^2}{a^2bc},\ \text{and}\ \frac{4bc}{a^2bc}.$$

Note.—The quantities $2xy$ and a^2bc are the L.C.M. of the denominators $2x$, $2y$, and ab, bc, a^2 respectively.

70. Where the least common denominator cannot be readily found by **inspection,** we must adopt the following rule:

"Multiply the numerator of each fraction by all the denominators except its own for the new numerator of that fraction, and all the denominators together for a new denominator."

Thus, it is required to reduce $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{x}{y}$ to equivalent fractions having the least common denominator.

$$\text{Here } \frac{a}{b} = \frac{a \times d \times y}{b \times d \times y} = \frac{ady}{bdy}.$$

$$\frac{c}{d} = \frac{c \times b \times y}{d \times b \times y} = \frac{bcy}{bdy}.$$

$$\frac{x}{y} = \frac{x \times b \times d}{y \times b \times d} = \frac{bdx}{bdy}.$$

Hence the required fractions are $\frac{ady}{bdy}$, $\frac{bcy}{bdy}$, and $\frac{bdx}{bdy}$.

Note.—In each case, both terms of the fraction being multiplied by the **same** quantities, the value is unaltered.

EXERCISE LXIX.

Reduce to equivalent fractions having the least common denominator

(1.) $\frac{5xy}{8a}$ and $\frac{3y}{4a^2x^2}$.

(2.) $\frac{a}{3b}$, $\frac{3b}{4a}$, and $\frac{6(a-b)}{15(a+b)}$.

(3.) $\frac{a+b}{a-b}$, $\frac{a-b}{a+b}$, and $\frac{a^2+b^2}{a^2-b^2}$.

(4.) $\frac{c^2}{a^2+b^2}$ and $\frac{d^2}{a^2-b^2}$.

(5.) $\frac{4x^2}{3(a+c)}$ and $\frac{xy}{6(a^2-c^2)}$.

(6.) $\frac{1}{a-b}$ and $\frac{1}{a^2-b^2}$.

(7.) $\frac{x}{a-x}$, $\frac{2}{a^2x-x^3}$, and $\frac{a}{x}$.

(8.) $\frac{c}{a^2+c^2}$ and $\frac{c}{a^2-c^2}$.

71. Addition of fractions is performed thus:

(1.) $\frac{5}{6a}+\frac{2}{3b}=\frac{5b}{6ab}+\frac{4a}{6ab}=\frac{5b+4a}{6ab}$.

(2.) $\frac{5+x}{y}+\frac{3-ax}{ay}+\frac{b}{3a}=\frac{3a(5+x)}{3ay}+\frac{3(3-ax)}{3ay}+\frac{by}{3ay}$

$=\frac{15a+3ax}{3ay}+\frac{9-3ax}{3ay}+\frac{by}{3ay}$

$=\frac{15a+3ax+9-3ax+by}{3ay}=\frac{15a+by+9}{3ay}$.

(3.) $\frac{x}{a^2}+\frac{a-x}{a(a+x)}=\frac{x(a+x)}{a^2(a+x)}+\frac{a(a-x)}{a^2(a+x)}=\frac{ax+x^2}{a^2(a+x)}$

$+\frac{a^2-ax}{a^2(a+x)}=\frac{ax+x^2+a^2-ax}{a^2(a+x)}=\frac{a^2+x^2}{a^2(a+x)}$.

In the preceding examples, we first reduce the fractions to a **common** denominator. We then find the sum of the numerators, which, being placed over the common denominator, gives the required sum.

EXERCISE LXX.

Find the sum of

(1.) $\frac{a+x}{a-x}+\frac{a-x}{a+x}$.

(2.) $\frac{2a+1}{3}+\frac{4a+2}{5}+\frac{a}{7}$.

(3.) $\frac{a-b}{ab}+\frac{a+b}{b^2}+\frac{b}{a^2}+\frac{a^2-b^2}{4ab^2}$.

(4.) $\frac{1+a}{1+a+a^2}+\frac{1-a}{1-a+a^2}$.

(5.) $\frac{a}{2x}+\frac{a+x}{3(a-x)}$.

(6.) $\frac{a-x}{ax}+\frac{c-a}{ac}+\frac{x-c}{cx}$.

(7.) $\frac{a^2+x^2}{a^2-x^2}+\frac{a-x}{a+x}$.

(8.) $\frac{1}{a+b}+\frac{1}{a-b}+\frac{2a}{a^2+b^2}$.

(9.) $\frac{a-x}{2(a+x)}+\frac{a^2+x^2}{a^2-x^2}$.

(10.) $\frac{x}{2x-2y}+\frac{y}{2y-2x}$.

(11.) $\frac{2a+3b}{2a-3b}+\frac{2a-3b}{2a+3b}$.

(12.) $\frac{2a^2+4ab+2b^2}{2a^2+2ab}+\frac{a}{a-b}$.

72. Subtraction of fractions is performed thus:

(1.) $$\frac{5}{6a}-\frac{2}{3b}=\frac{5b}{6ab}-\frac{4a}{6ab}=\frac{5b-4a}{6ab}.$$

(2.) $$\frac{5+x}{y}-\frac{3-ax}{ay}=\frac{a(5+x)}{ay}-\frac{3-ax}{ay}$$
$$=\frac{5a+ax-3+ax}{ay}=\frac{5a+2ax-3}{ay}.$$

(3.) $$\frac{a^2+x^2}{a^2-x^2}-\frac{a-x}{a+x}=\frac{a^2+x^2}{a^2-x^2}-\frac{(a-x)(a-x)}{(a-x)(a+x)}=\frac{a^2+x^2}{a^2-x^2}$$
$$-\frac{a^2-2ax+x^2}{a^2-x^2}=\frac{a^2+x^2-a^2+2ax-x^2}{a^2-x^2}=\frac{2ax}{a^2-x^2}.$$

In the above examples, we first reduce the fractions to a **common** denominator. We then find the difference of the numerators, which, being placed over the common denominator, gives the required difference.

EXERCISE LXXI.

Find the value of

(1.) $\frac{4a}{7}-\frac{3a-7}{8}$.

(2.) $\frac{1}{a-x}-\frac{1}{a+x}$.

(3.) $\frac{2a^2-2ab+b^2}{a^2-ab}-\frac{a}{a-b}$.

(4.) $\frac{b}{a^2}-\frac{a+b}{a(a-b)}$.

(5.) $\frac{1+a}{1+a+a^2}-\frac{1-a}{1-a+a^2}$.

(6.) $\frac{3x^2-4ax+a^2}{a^2-x^2}-\frac{a-x}{a+x}$.

(7.) $\frac{5a-3}{a+1}-\frac{2a^2-13a+1}{a^2-1}$.

(8.) $\frac{x}{(x+y)^2}-\frac{y}{x^2-y^2}$.

(9.) $\frac{4x+2}{3}-\frac{2x-3}{3x}$.

(10.) $\frac{2}{a}-\frac{3}{2a-1}-\frac{2a-3}{4a^2-1}$.

(11.) $\frac{1}{x+y}+\frac{y}{x^2-y^2}-\frac{x}{x^2+y^2}$.

(12.) $\frac{3x-4y}{7}-\frac{2x-y-z}{3}+\frac{15x-4z}{12}-\frac{x-4y}{21}$.

73. Multiplication of fractions is performed thus:

(1.) $\frac{5}{6a}\times\frac{2}{3b}=\frac{5\times 2}{6a\times 3b}=\frac{10}{18ab}=\frac{5}{9ab}$.

(2.) $\frac{5+x}{y}\times\frac{3-ax}{ay}=\frac{(5+x)(3-ax)}{y\times ay}=\frac{15+3x-5ax-ax^2}{ay^2}$.

(3.) $\frac{a^2-x^2}{4}\times\frac{a+x}{(a-x)^2}=\frac{(a-x)(a+x)}{4}\times\frac{a+x}{(a-x)(a-x)}$

$$=\frac{(a-x)(a+x)(a+x)}{4(a-x)(a-x)}=\frac{(a+x)^2}{4(a-x)}.$$

In the above examples, we multiply the numerators together for a new numerator, and the denominators for a new denominator.

EXERCISE LXXII.

Find the product of

(1.) $\frac{2a}{a-x}\times\frac{a^2-x^2}{8}$. (7.) $\frac{a^2+ab}{a-b}\times\frac{(a-b)^2}{a^4-b^4}$.

(2.) $\frac{3a^2-a}{5}\times\frac{10}{2a^2-4a}$. (8.) $\frac{a^4-x^4}{a^2-2ax+x^2}\times\frac{a-x}{a^2+ax}$.

(3.) $\frac{a-x}{x^2}\times\frac{a^2}{a^2-x^2}$. (9.) $\frac{2x^2}{b^3+x^3}\times\frac{x+b}{x}$.

(4.) $\frac{3(a+b)}{2}\times\frac{4b}{a+b}$. (10.) $\frac{1-x^2}{1+b}\times\frac{1-b^2}{x+x^2}\times 1+\frac{x}{1-x}$.

(5.) $\frac{ax}{(a-x)^2}\times\frac{a^2-x^2}{ab}$. (11.) $\frac{3ax}{4by}\times\frac{a^2-x^2}{c^2-x^2}\times\frac{bc+bx}{a^2+ax}\times\frac{c-x}{a-x}$

(6.) $\frac{a^3-b^3}{a^3+b^3}\times\frac{(a+b)^2}{(a-b)^2}$. (12.) $\frac{y(a-y)}{a^2+2ay+y^2}\times\frac{a(a+y)}{a^2-2ay+y^2}$.

74. Division of fractions is performed thus:

(1.) $\frac{5}{6a}\div\frac{2}{3b}=\frac{5}{6a}\times\frac{3b}{2}=\frac{15b}{12a}=\frac{5b}{4a}$.

(2.) $\frac{5+x}{y}\div\frac{3-ax}{ay}=\frac{5+x}{y}\times\frac{ay}{3-ax}=\frac{5ay+axy}{3y-axy}$.

(3.) $\frac{a^2-x^2}{4}\div\frac{a+x}{(a-x)^2}=\frac{a^2-x^2}{4}\times\frac{(a-x)^2}{a+x}$

$$=\frac{(a+x)(a-x)}{4}\times\frac{(a-x)(a-x)}{a+x}$$

$$=\frac{(a+x)(a-x)(a-x)(a-x)}{4(a+x)}=\frac{(a-x)^3}{4}.$$

In the above examples, we first invert the divisor, then proceed as in multiplication.

EXERCISE LXXIII.

Find the value of

(1.) $\frac{4a+2}{3} \div \frac{2a+1}{5a}$.

(2.) $\frac{2x^2}{a^3+x^3} \div \frac{x}{a+x}$.

(3.) $\frac{a^2-b^2}{(a-b)^2} \div \frac{a^2+ab}{a-b}$.

(4.) $\frac{a^2+x^2}{a^2-x^2} \div \frac{a-x}{a+x}$.

(5.) $\frac{a^3+b^3}{a^2-b^2} \div \frac{a^2-ab+b^2}{a-b}$.

(6.) $\frac{x^2+xy}{x-y} \div \frac{x^4-y^4}{(x-y)^2}$.

(7.) $\frac{2x^2}{b^3+x^3} \div \frac{x}{b+x}$.

(8.) $\frac{x^4-y^4}{x^2-2xy+y^2} \div \frac{x^2+y^2}{x-y}$.

(9.) $\frac{a^3-b^3}{a^2b^2-b^4} \div \frac{a^2+ab+b^2}{ab^2+b^3}$.

(10.) $\left\{\frac{a+b}{a-b}+\frac{a-b}{a+b}\right\} \div \left\{\frac{a+b}{2a-2b}-\frac{a-b}{2a+2b}\right\}$.

(11.) $\left\{\frac{2a}{a^2+1}+\frac{2a}{a^2-1}\right\} \div \left\{\frac{a}{a^2+1}-\frac{a}{a^2-1}\right\}$.

(12.) $\left\{\frac{x+y}{x-y}+\frac{x^2+y^2}{x^2-y^2}\right\} \div \left\{\frac{x^3-y^3}{x^3+y^3}-\frac{x-y}{x+y}\right\}$.

EXERCISE LXXIV.

Simplify the following expressions:

(1.) $\frac{a^3-3a^2x+3ax^2-x^3}{a^2-x^2} \div \frac{2ax-2x^2}{3} \times \frac{a^2+ax}{a-x}$.

(2.) $\left\{\frac{a^2}{b^3}+\frac{1}{a}\right\} \div \left\{\frac{a}{b^2}-\frac{1}{b}+\frac{1}{a}\right\}$.

(3.) $\left\{\frac{x+2y}{x+y}+\frac{x}{y}\right\} \div \left\{\frac{x+2y}{y}-\frac{x}{x+y}\right\}$.

(4.) $\frac{1}{2}\cdot\frac{3m+2n}{3m-2n}-\frac{1}{2}\cdot\frac{3m-2n}{3m+2n}$.

(5.) $\frac{3}{4(1-x)^2}+\frac{3}{8(1-x)}+\frac{1}{8(1+x)}-\frac{1-x}{4(1+x^2)}$.

(6.) $\dfrac{\frac{a+b}{a-b}+\frac{a-b}{a+b}}{\frac{a+b}{a-b}-\frac{a-b}{a+b}}$ (7.) $\dfrac{\frac{a^2+x^2}{2a^2}-\frac{2x^2}{a^2+x^2}}{\frac{a^2+x^2}{2x^2}-\frac{2a^2}{a^2+x^2}}$.

(8.) $\left\{\frac{1}{a+b}+\frac{2}{a-b}-\frac{9}{3a-b}\right\}\div\frac{b^2-9a^2}{8b}$.

(9.) $\dfrac{\frac{a^2+b^2}{b}-a}{\frac{1}{b}-\frac{1}{a}}\times\frac{a^2-b^2}{a^3+b^3}$.

(10.) $\frac{3xyz}{yz+xz-xy}-\dfrac{\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}}{\frac{1}{x}+\frac{1}{y}-\frac{1}{z}}$.

75. We now proceed to the solution of equations, the members of which contain one or more **fractional** quantities.

(1.) Given $\frac{x}{5}=6$; to find x.

Multiplying both sides of the equation by 5 (in order to clear it of fractions), we have

$$\frac{x\times 5}{5}=6\times 5. \qquad \therefore x=30.$$

(2.) Given $\frac{x}{3}+\frac{x}{4}=7$; to find x.

Multiplying each side of the equation by 3, we have

$$x+\frac{3x}{4}=21.$$

Again, multiplying each side of the equation by 4, we have

$$4x+3x=84,$$
$$\therefore\ x=12;$$

or we might multiply both sides of the equation **at once** by 12, the L.C.M. of 3 and 4, when we should have as before

$$4x+3x=84.$$

(3.) Given $4x-\frac{x-1}{2}=x+\frac{2x-2}{5}+24$; to find x.

Multiplying every term by 10, the L.C.M. of 5 and 2, we have

$$40x-5(x-1)=10x+2(2x-2)+240,$$
$$\text{or, } 40x-5x+5=10x+4x-4+240;$$
$$\therefore\ 40x-5x-10x-4x=240-5-4;$$
$$\therefore\ 21x=231;$$
$$\therefore\ x=11.$$

Note.—The above equation might be worked as follows:

$$4x-\frac{x-1}{2}=x+\frac{2x-2}{5}+24.$$

Taking x from each side,

$$3x-\frac{x-1}{2}=\frac{2x-2}{5}+24.$$

Multiplying each side by 10,

$$30x-5x+5=4x-4+240;$$
$$\therefore\ 25x+5=4x+236;$$
$$\therefore\ 21x=231;$$
$$\therefore\ x=11.$$

(4.) Given $\frac{7x+9}{4}-\left(x-\frac{2x-1}{9}\right)=7$; to find x.

Removing the bracket, we have,

$$\frac{7x+9}{4}-x+\frac{2x-1}{9}=7;$$

multiplying both sides by 4, we have

$$7x+9-4x+\frac{8x-4}{9}=28,$$

$$\therefore\ 3x+9+\frac{8x-4}{9}=28,$$

$$\therefore\ 3x+\frac{8x-4}{9}=19,$$

$$\therefore\ 27x+8x-4=171,$$

$$\therefore\ 35x-4=171,$$

$$\therefore\ 35x=175,$$

$$\therefore\ x=5.$$

EXERCISE LXXV.

(1.) Given $\frac{1}{2}x+\frac{1}{3}x=\frac{1}{4}x-32+x$; to find x.

(2.) Given $\frac{7-x}{2}-\frac{9-2x}{3}=5\frac{1}{2}$; to find x.

(3.) Given $\frac{2x+7}{7}-\frac{9x-8}{11}=\frac{x-11}{2}$; to find x.

(4.) Given $\frac{1}{4}x+\frac{1}{3}x=x-5$; to find x.

(5.) Given $\frac{1}{6}x+\frac{1}{8}x-7=0$; to find x.

(6.) Given $\frac{8x}{9}-\frac{7x}{12}-\frac{x}{6}=3\frac{3}{4}$; to find x.

(7.) Given $\frac{x}{4}+\frac{5x}{6}-3=\frac{7x}{9}+\frac{5}{2}$; to find x.

(8.) Given $\frac{1}{4}(3x+5)+\frac{1}{6}(3+5x)=16$; to find x.

(9.) Given $\frac{1}{2(x-3)}-\frac{1}{3(x-2)}=\frac{x-5}{(x-2)(x-3)}$; to find x.

(10.) Given $\frac{1}{3}x-\frac{1}{4}x+\frac{1}{6}=\frac{1}{8}x+\frac{1}{12}$; to find x.

(11.) Given $\frac{3x}{4}+5=\frac{5x}{6}+2$; to find x.

(12.) Given $\frac{x}{3}-\frac{1}{3}-\frac{x}{4}+\frac{1}{4}=\frac{x}{5}-\frac{1}{5}-\frac{x}{6}+\frac{1}{6}$; to find x.

(13.) Given $\frac{1}{7}(3x-4)+\frac{1}{3}(5x+3)=43-5x$; to find x.

(14.) Given $\frac{1}{6}(8-x)+x-1\frac{2}{3}=\frac{1}{2}(x+6)-\frac{x}{3}$; to find x.

(15.) Given $\frac{5x-3}{7}-\frac{9-x}{3}=\frac{5x}{2}+\frac{19}{6}(x-4)$; to find x.

(16.) Given $\frac{1}{2}x-\frac{1}{4}x+15=\frac{1}{8}x-\frac{1}{6}x+22$; to find x.

(17.) Given $\frac{x+1}{2}+\frac{x+2}{3}=16-\frac{5x+1}{4}$; to find x.

(18.) Given $\frac{3}{4}-\frac{1}{3}(x-2)=\frac{5}{4}-\frac{1}{4}(x+3)$; to find x.

(19.) Given $\frac{x+6}{4}-\frac{16-3x}{12}=\frac{25}{6}$; to find x.

(20.) Given $\frac{2x+7}{7}-\frac{9x-8}{11}=\frac{x-11}{2}$; to find x.

(21.) Given $\frac{1}{23}(7x+5)+\frac{1}{10}(9x-1)-\frac{1}{5}(x-9)$
$+\frac{1}{15}(2x-3)=23\frac{1}{3}$; to find x.

(22.) Given $52-5(2x-1)=27$; to find x.

(23.) Given $\frac{4}{5}x - \frac{5}{4}x + 18 = \frac{1}{9}(4x+1)$; to find x.

(24.) Given $\frac{1}{12}x - \frac{1}{8}(8-x) - \frac{1}{4}(5+x) + \frac{11}{4} = 0$; to find x.

(25.) Given $4(5x-3) - 64(3-x) - 3(12x-4) = 96$; to find x.

(26.) Given $3x - 4 - \frac{4}{5}\cdot\frac{7x-9}{3} = \frac{4}{5}\left(6 + \frac{x-1}{3}\right)$; to find x.

(27.) Given $\frac{x}{a+3} + \frac{x}{a-3} = \frac{4a^2}{a^2-9}$; to find x.

(28.) Given $\frac{x-7}{x+7} - \frac{2x-15}{2x-6} = -\frac{1}{2(x+7)}$; to find x.

76. In solving the following equations, remember that quantities which are known, or supposed to be so, are sometimes represented by the letters a, b, c, &c. (paragraph 58).

(1.) Given $\frac{x}{a} = c$; to find the value of x.

Multiplying both sides of the equation by a, we have

$$\frac{x \times a}{a} = c \times a; \quad \therefore \ x = ac.$$

(2.) Given $\frac{x}{a} + b = \frac{x}{c} + e$; to find x.

Multiplying each term by ac, the L.C.M. of a and c,

$$cx + abc = ax + ace.$$

Transposing and changing signs,

$$ax - cx = abc - ace;$$

$$\text{or} \quad x(a-c) = ac(b-e);$$

$$\therefore \ x = \frac{ac(b-e)}{a-c}.$$

(3.) Given $\frac{a}{cx} + \frac{c}{ax} = a^2 + c^2$; to find x.

Multiplying each term by acx, the L.C.M. of cx an ax,

$$a^2+c^2=a^3cx+ac^3x;$$
$$\text{or}\quad a^3cx+ac^3x=a^2+c^2;$$
$$\text{or}\quad x(a^3c+ac^3)=a^2+c^2;$$
$$\therefore\ x=\frac{(a^2+c^2)}{ac(a^2+c^2)}=\frac{1}{ac}.$$

EXERCISE LXXVI.

(1.) Given $\frac{x}{c}-\frac{x-b}{a}=\frac{de}{ac}$; to find x.

(2.) Given $\frac{(a-b)x}{a+b}+e=\frac{(a+b)x}{a-b}-d$; to find x.

(3.) Given $\frac{3x}{a}-e+\frac{x}{b}=4x+\frac{2x}{d}$; to find x.

(4.) Given $a+\frac{d-x}{x}=b-1+\frac{c}{x}$; to find x.

(5.) Given $\frac{2x+c}{d}-\frac{x-d}{c}=\frac{3cx+(c-d)^2}{cd}$; to find x.

(6.) Given $\frac{x}{a}+\frac{x}{c-a}=\frac{a}{c+a}$; to find x.

(7.) Given $\frac{ax+c}{b}-\frac{a}{c}=\frac{bx+d}{e}$; to find x.

(8.) Given $\frac{x}{a+x}=\frac{a+x}{x}-\frac{2a-c}{2x}$; to find x.

(9.) Given $\frac{ax}{d(x+e)}+\frac{dx}{a(x+e)}=1$; to find x.

(10.) Given $ax+c=\frac{x}{a}+\frac{1}{c}$; to find x.

(11.) Given $(a+x)(d+x)-a(d+e)=\frac{a^2e}{d}+x^2$; to find x.

(12.) Given $\frac{ax+b}{c}+\frac{ax+b}{cx+b}=\frac{2ax+d}{2c}+\frac{b}{c}$; to find x.

77. We now proceed to the solution of problems producing equations which involve fractions.

(*Ex.* 1.) What number is that, to which 10 being added, $\frac{4}{5}$ of the sum shall be 72?

Let $\quad x =$ number required;
then $\quad x+10 =$ required number, with 10 added to it;
then $\frac{4}{5}$ of sum $=\frac{4(x+10)}{5}$; *i.e.*, $\frac{4}{5}\times\overline{x+10}$.

But by the question, $\quad \frac{4(x+10)}{5}=72$;

$$\therefore\ 4(x+10)=360;$$

whence $\quad x=80$.

(*Ex.* 2.) A post is $\frac{1}{3}$ in the earth, $\frac{3}{7}$ in the water, and 10 ft. out of the water. Find the length of the post.

Let $x =$ length of post in feet;

then $\frac{x}{3}=$ the part of it in the earth;

$\frac{3x}{7}=$ the part of it in the water;

$10=$ the part of it out of the water.

But part in the earth + part in the water + part out of the water = the whole post;

$$\therefore\ \frac{x}{3}+\frac{3x}{7}+10=x.$$

Multiplying by 21, the L.C.M. of 3, and 7,

$$7x+9x+210=21x;$$

whence $\quad x=42$ feet.

78. In the following exercise, the pupil is required to show the connection between the problem and its corresponding equation, by inserting the necessary steps, and then to solve the equation.

EXERCISE LXXVII.

(1.) After paying away one-fourth and one-seventh of my money, I had £119 left in my purse. How much money had I at first?

$$x-\frac{x}{4}-\frac{x}{7}=119.$$

(2.) After paying away one-fifth of my money, I had 6d. more than the half of my money left. How much money had I at first?

$$x-\frac{x}{5}=\frac{x}{2}+6.$$

(3.) John's age is now four-fifths of James's, and twenty-one years ago it was only half as much as James's. Find the age of each person.

$$\frac{4x}{5}-21=\frac{x-21}{2}.$$

(4.) Out of a cask of wine, of which a fourth part had leaked away, 12 gallons were drawn, and then it was two-thirds full. How much did it contain?

$$\frac{3x}{4}-12=\tfrac{2}{3}x.$$

(5.) A cistern was found to be three-fourths full of water; but after running off 880 quarts, it was found to be one-fifth full. How many gallons could it contain?

$$\frac{3x}{4}-220=\frac{x}{5}.$$

EXERCISE LXXVIII.

(1.) A person, after spending one-fifth of his income, *plus* £20, had then remaining one-half of it, *plus* £70. Required his income.

(2.) One-third of a cask of beer is first drawn off, and afterwards one-half of the remainder; and then 20 gallons remain in the cask. How much did the cask contain at first?

(3.) A person rows 60 miles down a river and back in 14 hours: he rows 2 miles against the stream in the same time that he rows 5 miles with it. Find the rates of going and returning.

(4.) There are two numbers whose sum is 37; and if 3 times the less be subtracted from 4 times the greater, and this difference be divided by 6, the quotient will be 6. Find the numbers.

(5.) A man and his wife lived in wedlock one-third of his age, and one-fourth of hers. Now the man was 6 years older than his wife at marriage, and she survived him 20 years. What were their ages at marriage?

(6.) A and B have the same income: A saves one-fifth of his income; but B, by spending £50 a year more than A, at the end of four years is £100 in debt. What is their income?

(7.) A post is one-fourth of its length in the mud, three-sevenths in the water, and 27 feet out of the water. Find the length of the post.

(8.) Divide £4400 among three persons, so that the first may have three-fifths of the second's share, and the second three-fourths of the third's share.

(9.) A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3 leaps; but two of the greyhound's equal 3 of the hare's. How many leaps must the greyhound take to catch the hare?

(10.) A boy spent one-half of his money in oranges, one-third in nuts, and found he had $4\frac{1}{2}$d. left. How much did he spend?

(11.) Two-thirds of a certain number of persons received eighteenpence each, and one-third received half-a-crown each. The whole sum spent was £2, 15s. How many persons were there?

(12.) A smuggler had a quantity of brandy which he expected would raise £9, 18s. After he had sold 10 gallons, a revenue officer seized one-third of the remainder, in consequence of which he makes only £8, 2s. Required the number of gallons he had, and the price per gallon.

(13.) From a sum of money one-fourth is taken away, one-fifth of the remainder is added, and the result is less than the original sum by £30. What was the original sum?

(14.) At what price must a farmer purchase a flock of 100 sheep, that expending £10 in feeding them and losing 9, he may sell the remainder at £2 each, and gain £20?

(15.) *A* alone can perform a piece of work in 9 days, and *B* alone can perform it in 12 days. In what time will they perform it if they work together?

(16.) *A* can do half as much work as *B*, *B* can do half as much as *C*, and together they can complete a piece of work in 24 days. In what time could each alone complete the work?

(17.) A basket weighs 3 ℔.; a fish in it weighs 7 ℔., and half its own weight. Find the weight of the whole.

(18.) *A* can build a wall in 10 days; *B* in 15 days; *A*, *B*, and *C* together in 4 days. In what time could *C* alone do it?

(19.) A cistern was found to be three-fourths full of water, but after running off 220 gallons it was found to be one-fifth full. How many gallons could it contain?

(20.) An uncle divided £375 among three nephews: to the first he gave four-fifths of the sum given to the second, and the third received a fifth part of the second's share more than the second. How much did each receive?

(21.) A person passed one-sixth of his age in childhood, one-twelfth in youth, one-seventh + 5 years in matrimony; he had then a son whom he survived 4 years, and who reached only one-half the age of his father. Find the son's age when he died.

(22.) If a train which travels at the rate of 35 miles an hour start one quarter of an hour after a luggage-train, and overtake it in 10 minutes, find the speed of the luggage-train.

(23.) The wages of a servant were £40 a year, and a suit of livery: being discharged at the end of 5 months, he received the livery and £6, 3s. 4d. in money. What was the value of the livery?

(24.) When a body of men are to be formed into a square, it is found that there are 50 men over; but on attempting to add one man to each side of the square, it is found that there are 213 men too few to complete the square. How many men were there?

(25.) Two persons, *A* and *B*, have the same income: *A* lays by one-fifth of his; but *B*, by spending £60 per annum more than *A*, at the end of three years finds himself £100 in debt. What is the income of each?

(26.) It is between 11 and 12 o'clock, and it is observed that the number of minute spaces between the hands is two-thirds of what it was ten minutes previously. Find the time.

(27.) Half a gallon more than half the number of gallons in a cask are sold, and then half a gallon more than half the remainder, and then it is empty. How many gallons were in it?

(28.) A person has a certain number of shillings in each hand; and if he take 8 from the left and put them in the right hand, he would then have 4 times as many in his right hand as in his left; but at first he had 5 more in his right hand than in his left. How many had he in each at first?

79. It was stated in paragraph 11, that

$a^2 = a \times a$, and is called the **second** power of a;
$a^3 = a \times a \times a$, and is called the **third** power of a;
$a^4 = a \times a \times a \times a$, and is called the **fourth** power of a;
&c. = &c.

When any quantity is thus multiplied by **itself**, it is said to be raised or **involved**, and the process is called **involution.**

When the sign of a quantity is $+$, **all** its powers are positive; but if the sign be $-$, then the signs of the **even** powers only will be $+$, while those of the odd powers will be $-$. Thus:

$$-a \times -a \times -a = -a^3.$$

80. A **binomial** expression may be raised in a similar manner. Thus:

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2;$$

$(a+b)^3 = (a+b)(a+b)(a+b) = a^3+3a^2b+3ab^2+b^3;$
$(a+b)^4 = (a+b)(a+b)(a+b)(a+b)$
$= a^4+4a^3b+6a^2b^2+4ab^3+b^4;$
&c. = &c.:

or

$(a-b)^2 = (a-b)(a-b) \qquad = a^2-2ab+b^2;$
$(a-b)^3 = (a-b)(a-b)(a-b) = a^3-3a^2b+3ab^2-b^3;$
$(a-b)^4 = (a-b)(a-b)(a-b)(a-b)$
$= a^4-4a^3b+6a^2b^2-4ab^3+b^4;$
&c. = &c.

Again: $(a^2)^2 = a^2 \times a^2 \qquad = a^4;$
$(a^2)^3 = a^2 \times a^2 \times a^2 \qquad = a^6;$
$(a^2)^4 = a^2 \times a^2 \times a^2 \times a^2 = a^8.$

Hence we see that, in order to obtain any **power of a power** of a quantity, we **multiply** together the indices of the two powers.

Note.— $a^3 \times a^2 = a^5$ (paragraph 30);
$(a^3)^2 = a^3 \times a^3 = a^6.$

Fractional quantities are involved by raising both numerator and denominator to the required power. Thus:

$$\left(\frac{a}{b}\right)^3 = \frac{a}{b}\times\frac{a}{b}\times\frac{a}{b} \qquad = \frac{a^3}{b^3};$$

$$\left(\frac{2a}{3b}\right)^3 = \frac{2a}{3b}\times\frac{2a}{3b}\times\frac{2a}{3b} = \frac{8a^3}{27b^3}.$$

81. A **trinomial** expression may be thus raised:

$a+b+c = a+(b+c)$, thus regarding $(b+c)$ as **one** term;
then $(a+b+c)^2 = \{a+(b+c)\}^2 = a^2+2a(b+c)+(b+c)^2$
$= a^2+2ab+2ac+b^2+2bc+c^2;$
$(a+b+c)^3 = \{a+(b+c)\}^3 = a^3+3a^2(b+c)+3a(b+c)^2$
$+(b+c)^3$
$= a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2+b^3+3b^2c+3bc^2+c^3$

Again: $a-b-c=a-(b+c)$;

then $(a-b-c)^2=\{a-(b+c)\}^2=a^2-2a(b+c)+(b+c)^2$
$= a^2-2ab-2ac+b^2+2bc+c^2$;

$(a-b-c)^3=\{a-(b+c)\}^3=a^3-3a^2(b+c)+3a(b+c)^2$
$-(b+c)^3$
$=a^3-3a^2b-3a^2c+3ab^2+6abc+3ac^2-b^3-3b^2c-3bc^2-c^3$

Regarding the preceding **formulae** as models, we may raise binomial and trinomial expressions to the second, third, or fourth power, without **actual multiplication.** Thus:

$$[(a-b)^3=a^3-3a^2b+3ab^2-b^3]$$
$$(2a-3b)^3=(2a)^3-3.(2a)^2.3b+3.(2a).(3b)^2-(3b)^3$$
$$=8a^3-9b.4a^2+6a.9b^2-27b^3$$
$$=8a^3-36a^2b+54ab^2-27b^3.$$

EXERCISE LXXIX.

Find the value of

(1.) $(a+2)^3$.
(2.) $(2x-1)^3$.
(3.) $(bx+x^2)^4$.
(4.) $(a^2-2ax+3x^2)^2$.
(5.) $\left(-\frac{a^2x^3}{3}\right)^4$.
(6.) $(7a^2-12a+5)^2$.
(7.) $(2a^2-3b^2)^3$.
(8.) $(a-b+2d)^3$.
(9.) $(a^2-4a+2)^3$.
(10.) $\left(z+1-\frac{1}{z}\right)^3$.
(11.) $(2-3x)^4$.
(12.) $(3a-b)^3$.
(13.) $(x-y-z)^3$.
(14.) $(x^2-x+1)^3$.
(15.) $(2x^2-3x-4)^2$.
(16.) $(x+2y+3z)^2$.
(17.) $\left(\frac{1}{4}x-12y\right)^2$.
(18.) $\left(\frac{3(a-x)}{2(a+x)}\right)^2$.
(19.) $(a-x)^4$.
(20.) $\left(1+\frac{1}{2}x+\frac{1}{3}x^2\right)^2$.

82. Evolution is the converse of involution; that is to say, since the cube or third **power** of $a^2 = a^6$, therefore the cube or third **root** of $a^6 = a^2$.

Thus, by the process of evolution, we find a quantity such that a certain power of it is equal to the given quantity.

The term **root** is given to the quantity evolved.

As there are second, third, fourth, &c., **powers,** so there are second, third, fourth, &c., **roots.**

83. $(a^2)^2 = a^4$; $\qquad \therefore \sqrt{a^4} = a^2$.
$(a^2)^3 = a^6$; $\qquad \therefore \sqrt[3]{a^6} = a^2$.
$(a^2)^4 = a^8$; $\qquad \therefore \sqrt[4]{a^8} = a^2$.

Hence we see, that in order to obtain any **root of a power** of a quantity we **divide** the index of the power by that of the root.

Note.— $\quad a^6 \div a^2 = a^4$ (paragraph 41);
$\sqrt[2]{a^6} = a^3$.

The root of a **fractional** quantity is found by extracting the roots of both numerator and denominator. Thus:

$$\sqrt[3]{\frac{8a^3}{27b^3}} = \frac{2a}{3b}.$$

84. We now proceed to the evolution of **monomials.**

(1.) $125x^3 = 5x \times 5x \times 5x$;
$\therefore$ the cube root of $125x^3 = 5x$, or $\sqrt[3]{125x^3} = 5x$.

(2.) $\dfrac{16a^4b^4}{81x^4} = \dfrac{2ab}{3x} \times \dfrac{2ab}{3x} \times \dfrac{2ab}{3x} \times \dfrac{2ab}{3x}$;

$\therefore$ the fourth root of $\dfrac{16a^4b^4}{81x^4} = \dfrac{2ab}{3x}$, or $\sqrt[4]{\dfrac{16a^4b^4}{81x^4}} = \dfrac{2ab}{3x}$.

EXERCISE LXXX.

Find the value of

(1.) $\sqrt[4]{16x^4}$.	(5.) $\sqrt[3]{27x^6y^3}$.
(2.) $\sqrt[3]{1000a^3b^3}$.	(6.) $\sqrt{4a^4b^2}$.
(3.) $\sqrt[2]{16a^4c^2}$.	(7.) $\sqrt[3]{729x^3y^6}$.
(4.) $\sqrt[3]{\dfrac{27a^6b^3}{8x^3y^3}}$.	(8.) $\sqrt[4]{\dfrac{81a^8b^4x^4}{256c^8d^{12}}}$.

85. We proceed now to the evolution of **trinomials,** &c.

We know that $(a+b)^2 = a^2+2ab+b^2$;

$$\therefore \sqrt{a^2+2ab+b^2} = a+b.$$

(1.)

$$\begin{array}{r|l} & a^2+2ab+b^2 \,\big(\, a+b \\ & a^2 \\ 2a+b & +2ab+b^2 \\ & +2ab+b^2 \end{array}$$

(2.)

$$\begin{array}{r|l} & a^4-4a^3x+6a^2x^2-4ax^3+x^4 \,\big(\, a^2-2ax+x^2 \\ & a^4 \\ 2a^2-2ax & -4a^3x+6a^2x^2-4ax^3+x^4 \\ & -4a^3x+4a^2x^2 \\ 2a^2-4ax+x^2 & 2a^2x^2-4ax^3+x^4 \\ & 2a^2x^2-4ax^3+x^4 \end{array}$$

The following is the process employed in Example (2):

(1.) The terms are arranged according to the power of some letter or letters.

(2.) The square root of the first term (a^4) is a^2, which is thus the first term of the required root.

(3.) This square (a^4) is subtracted from the whole expression, the remainder being

$$-4a^3x+6a^2x^2-4ax^3+x^4.$$

(4.) The first term $(-4a^3x)$ is then divided by $2a^2$, that is, $a^2 \times 2$; the quotient $-2ax$ then becomes the second term of the root.

(5.) The sum of twice the first term and the second term, that is, $2a^2-2ax$, is then multiplied by the second term $(-2ax)$, and the product is subtracted, the second remainder being

$$2a^2x^2-4ax^3+x^4.$$

Note.—The remaining part of the process is similar to the foregoing steps.

EXERCISE LXXXI.

Extract the square root of the following expressions:

(1.) $9x^6+24x^3y^4+16y^8$.

(2.) $9x^4-12x^3y+34x^2y^2-20xy^3+25y^4$.

(3.) $a^4-8a^3b+24a^2b^2-32ab^3+16b^4$.

(4.) $4-12x+5x^2+14x^3-11x^4-4x^5+4x^6$.

(5.) $4x^4-12bx^3+25b^2x^2-24b^3x+16b^4$.

(6.) $x^6-6bx^5+15b^2x^4-20b^3x^3+15b^4x^2-6b^5x+b^6$.

(7.) $25a^4b^2-30a^3b^3+29a^2b^4-12ab^5+4b^6$.

(8.) $4x^4-12x^3+25x^2-24x+16$.

(9.) $16a^4-24a^3b+25a^2b^2-12ab^3+4b^4$.

(10.) $x^6+4x^5+10x^4+20x^3+25x^2+24x+16$.

86. It was stated in paragraph 82, that by the square root of a quantity we mean the finding of a quantity which, when multiplied by **itself,** produces the required quantity. Thus:

$\sqrt{4} \times \sqrt{4}=4$; $\therefore$ the square of $\sqrt{4}=4$.

$\sqrt{a} \times \sqrt{a}=a$; $\therefore$ the square of $\sqrt{a}=a$.

$\sqrt{a+b} \times \sqrt{a+b}=a+b$; $\therefore$ the square of $\sqrt{a+b}=a+b$.

87. In paragraph 53, four axioms (A. B. C. D.) were enunciated, in accordance with which the solution of equations is conducted. In addition to those four, the following must also be strictly adhered to:

(E.) If equals be raised to the **same** power, the powers are equal.

(F.) If the **same** roots of equals be extracted, the roots are equal.

88. We proceed now to the solution of equations in which the unknown quantity is contained under a radical sign.

(1.) Given $\sqrt{x}+2=5$; to find the value of x.

Here $\sqrt{x}=5-2=3$.

Squaring both sides of the equation, $x=3^2=9$.

(2.) Given $\sqrt{2x+2}+5=9$.

Here $\sqrt{2x+2}=9-5=4$.

Squaring both sides of the equation, $2x+2=16$;

$\therefore\ x=7$.

(3.) Given $x+a+\sqrt{2ax+x^2}=c$.

Here $\sqrt{2ax+x^2}=c-a-x=c-(a+x)$;

$$\therefore\ 2ax+x^2=\{c-(a+x)\}^2=c^2-2c(a+x)+(a+x)^2$$
$$=c^2-2ac-2cx+a^2+2ax+x^2;$$
$$\therefore\ 0=c^2-2ac-2cx+a^2:$$

whence $2cx=c^2-2ac+a^2=(c-a)^2$;

$$\therefore\ x=\frac{(c-a)^2}{2c}.$$

EXERCISE LXXXII.

(1.) Given $\sqrt{x+4}=5$; to find the value of x.

(2.) Given $\sqrt{x-1}=4$; to find the value of x.

(3.) Given $\sqrt{x^2+8x-5}=x+3$; to find the value of x.

(4.) Given $13+\sqrt{4x+1}=16$; to find the value of x.

(5.) Given $\sqrt{x^2+a^2}+x=c$; to find the value of x.

(6.) Given $\sqrt{ax+x^2}=1+x$; to find the value of x.

(7.) Given $b+x+\sqrt{b^2+x^2}=a$; to find the value of x.

(8.) Given $a+x+\sqrt{2ax+x^2}=b$; to find the value of x.

(9.) Given $a+x=\sqrt{x^2+5x+a}$; to find the value of x.

(10.) Given $\frac{x-bx}{\sqrt{x}}=\frac{\sqrt{x}}{x}$; to find the value of x.

89. In all the equations which have hitherto been given, **one** unknown quantity only, (x), has been required to be determined. We shall now consider the solution of equations, in which **two** unknown quantities are required to be determined.

In all the preceding examples, in order to determine the **one** unknown quantity, **one** equation was sufficient; but when **two** unknown quantities are to be found, we require **two** equations, each independent of the other, and **each** containing the two unknown quantities. The validity of this remark will be evident from the following considerations:

Supposing the **two** unknown quantities to be represented by x and y, let $x=18-2y$; then

if we suppose $y=1$, $x=16$;
if we suppose $y=2$, $x=14$;
if we suppose $y=3$, $x=12$; &c.:

that is to say, **whatever** value we assign to y, we shall always obtain a **corresponding** value for x.

Thus, with **one** equation only, the **number** of pairs of values may be **unlimited**, each pair satisfying the equation.

But if we have **two** independent equations, each containing the two unknown quantities, we shall obtain only **one** pair of values, which satisfies the equation. Thus let

$$2x = 4 + 3y \quad [1];$$
$$3x = 32 - 2y \quad [2].$$

From the foregoing it is clear that, taking these equations **separately**, whatever value we assign to y, we shall always obtain a corresponding value for x; but that taking them **conjointly**, there is only **one** pair of values common to both.

Note.—Equations which are satisfied by the **same values** of the unknown quantities are termed **simultaneous equations.**

90. There are **three** modes of solving equations of this class:

A. Given $3x - 5y = 13$ [1] to find x and y,
$2x + 7y = 81$ [2].

Multiplying eq. [1] by 2, we have
$6x - 10y = 26$ [3];

Multiplying eq. [2] by 3, we have
$6x + 21y = 243$ [4];

Subtracting eq. [3] from eq. [4], we have
$31y = 217$ [5]

[an equation which contains only **one** unknown quantity];

$$\therefore\ y = 7.$$

From eq. [1] $3x = 13 + 5y$; but $y = 7$,

$$\therefore\ 3x = 13 + 35 = 48;$$

whence $x = 16$.

Hence the required roots are $x = 16$, $y = 7$.

Note.—This plan is distinguished as the mode of **equalizing co-efficients.**

B. Given $3x - 5y = 13$ [1] to find x and y,
$2x + 7y = 81$ [2].

From eq. [1] $3x = 13 + 5y$, $\therefore\ x = \frac{13+5y}{3}$ [3].

From eq. [2] $2x = 81 - 7y$, $\therefore\ x = \frac{81-7y}{2}$ [4].

$\frac{13+5y}{3}$ and $\frac{81-7y}{2}$ being both equal to x,

are equal to each other. Thus:

$$\frac{13+5y}{3} = \frac{81-7y}{2} \quad [5];$$

$$\therefore\ 2(13+5y) = 3(81-7y);$$

$$\therefore\ y = 7.$$

Then from eq. [4] $x = \frac{81-49}{2} = \frac{32}{2} = 16.$

Hence the required roots are $x = 16 \quad y = 7.$

Note.—This plan is distinguished as the mode of **equating.**

c. Given $3x - 5y = 13$ [1]; to find x and y.
$2x + 7y = 81$ [2].

From eq. [1] $3x = 13 + 5y$, $\therefore\ x = \frac{13+5y}{3}$ [3].

Substituting this value of x in eq. [2], we have

$$\frac{2(13+5y)}{3} + 7y = 81;$$

$$\therefore\ 2(13+5y) + 21y = 243,$$

$$\text{or, } 26 + 10y + 21y = 243;$$

$$\text{whence } y = 7.$$

Then from eq. [3] $x = \frac{13+35}{3} = 16.$

Hence the required roots are $x = 16 \quad y = 7.$

Note.—This plan is distinguished as the mode of **substitution.**

91. It must have been noticed that the object aimed at in **all** the preceding modes for solving equations of this class is the same; viz., to form a **new** equation by some combination of the given equations, in which **one** unknown quantity only is found, the other being **eliminated,** or expelled.

Note.—The following equations may be solved by any or all of the methods just explained.

EXERCISE LXXXIII.

Find the values of x and y in the following equations:

(1.) $\begin{cases} 3x+5y = 8 \\ 5x+y = 6 \end{cases}$

(2.) $\begin{cases} 3x-2y = 11 \\ 2x+3y = 16 \end{cases}$

(3.) $\begin{cases} 5x+4y = 19 \\ 3x+7y = 16 \end{cases}$

(4.) $\begin{cases} 5x-2y = 17 \\ 2x-y = 6 \end{cases}$

(5.) $\begin{cases} 3x+2y = 19 \\ 2x-3y = 4 \end{cases}$

(6.) $\begin{cases} 9x-4y = 8 \\ 13x+7y = 101 \end{cases}$

(7.) $\begin{cases} 7y+6 = 4x \\ 2x+3y = 16 \end{cases}$

(8.) $\begin{cases} 4y-4x-4 = 0 \\ 25-3x = 4y \end{cases}$

(9.) $\begin{cases} 5x-7y = 33 \\ 11x+12y = 100 \end{cases}$

(10.) $\begin{cases} 21y+20x = 165 \\ 77y = 295+30x \end{cases}$

(11.) $\begin{cases} 9x-4y = 8 \\ 13x+7y = 101 \end{cases}$

(12.) $\begin{cases} 8y = 350-45x \\ 21y-13x = 132 \end{cases}$

92. When one or both of the simultaneous equations contains **fractional** quantities, these must be reduced as in the case of solving equations of **one** unknown quantity (paragraph 75).

Given $\frac{x+y}{5}-\frac{x-y}{2} = 3$ [1]; to find x.

$\frac{x-y}{2}+\frac{x+y}{10} = 0$ [2].

Multiplying eq. [1] by 10, we have

$2(x+y)-5(x-y) = 30;$

$$\therefore\ 2x+2y-5x+5y=30;$$

$$\therefore\ 7y-3x=30 \quad [3].$$

Multiplying eq. [2] by 10, we have

$$5(x-y)+x+y=0;$$

$$\therefore\ 5x-5y+x+y=0;$$

$$\therefore\ 6x-4y=0 \quad [4].$$

But $-6x+14y=60$ [5], by multiplying eq. [3] by 2;

$\therefore$ adding [4] and [5], we have

$$10y=60; \qquad \therefore\ y=6.$$

From [4] $6x=4y$; $\qquad \therefore\ 6x=24$; or, $x=4$.

Hence the required roots are $x=4$, $y=6$.

EXERCISE LXXXIV.

Find the values of x and y in the following equations:

(1.) $\begin{cases} \frac{x}{2}+\frac{2y}{7}=28 \\ 2x-y=7 \end{cases}$ (2.) $\begin{cases} 2x-\frac{3y}{2}=1 \\ 5x-2y=9\frac{1}{2} \end{cases}$

(3.) $\begin{cases} \frac{x+y}{4}-\frac{x-y}{6}=4 \\ \frac{x+y}{3}+\frac{x-y}{4}=11 \end{cases}$ (4.) $\begin{cases} \frac{x+11}{10}+\frac{y-4}{6}=x-7 \\ \frac{x+5}{7}-\frac{y-7}{3}=3y-x \end{cases}$

(5.) $\begin{cases} \frac{x+2}{14}+\frac{y-x}{8}=x-4 \\ \frac{2y-3x}{3}+2y=3x+4 \end{cases}$

(6.) $\begin{cases} \frac{7+x}{5}-\frac{2x-y}{4}=3y-5 \\ \frac{5y-7}{2}+\frac{4x-3}{6}=18-5x \end{cases}$

(7.) $\begin{cases} \frac{3x}{5}-\frac{2y}{15}-\frac{8}{9}=\frac{x}{6}-\frac{y}{9} \\ 4x-5\frac{1}{3}=\frac{x}{6}-\frac{2y}{15}+2\frac{1}{5} \end{cases}$

(8.) $\left\{\begin{array}{l} \frac{x}{3}-2+\frac{y}{4}+\frac{x}{2}=4-\frac{3y}{8}+\frac{1}{24} \\ \frac{y}{12}-\frac{x}{4}+1=\frac{1}{12}-x+3 \end{array}\right\}$

93. When **both** of the simultaneous equations contain **literal** quantities, they must be solved as in paragraph 58.

(*Ex.* 1.) Given $ax+by=d$ [1]; to find x and y.
$mx-ny=c$ [2].

Here adopting mode A., paragraph 90, we have

$$amx+bmy=dm \ [3];$$

and $$amx-any=ac \ [4].$$

Subtracting eq. [4] from eq. [3]

$$(an+bm)y=md-ac \ [5];$$

$$\therefore y=\frac{md-ac}{an+bm}.$$

From eq. [1] $ax=d-by$;

$$\therefore ax=d-\frac{b(md-ac)}{an+bm}=\frac{adn+bdm-bdm+abc}{an+bm};$$

$$\therefore ax=\frac{a(dn+bc)}{an+bm}; \qquad \therefore x=\frac{nd+bc}{mb+na}.$$

Hence the required values are,

$$x=\frac{nd+bc}{mb+na};\ y=\frac{md-ac}{an+bm}.$$

(*Ex.* 2.) Given $ax+by=c$ [1]; to find x and y.
$\frac{x}{b}-\frac{y}{a}=1.$ [2].

Here multiplying eq. [2] by ab, we have

$$ax-by=ab \ [3];$$

adding eq. [1] and eq. [3], $x=\frac{ab+c}{2a}$;

subtracting eq. [3] from eq. [1], $y=\frac{c-ab}{2b}$.

EXERCISE LXXXV.

Find the values of x and y in the following equations:

(1.) $\left\{\begin{array}{r} ax = cy \\ x+y = b \end{array}\right\}$

(2.) $\left\{\begin{array}{l} \frac{c}{x}+\frac{d}{y} = 1 \\ \frac{d}{x}+\frac{c}{y} = 1 \end{array}\right\}$

(3.) $\left\{\begin{array}{l} \frac{x}{a}-\frac{y}{b} = p \\ \frac{x}{c}+\frac{y}{d} = q \end{array}\right\}$

(4.) $\left\{\begin{array}{l} \frac{x}{m}+\frac{y}{n} = 1 \\ \frac{ax}{n}-\frac{my}{a} = 0 \end{array}\right\}$

(5.) $\left\{\begin{array}{l} x+y = c \\ ax+by = d \end{array}\right\}$

(6.) $\left\{\begin{array}{l} \frac{x}{c}+\frac{y}{d} = 1 \\ \frac{x}{d}-\frac{y}{c} = 1 \end{array}\right\}$

(7.) $\left\{\begin{array}{l} \frac{m}{x}+\frac{n}{y} = a \\ \frac{n}{x}+\frac{m}{y} = b \end{array}\right\}$

(8.) $\left\{\begin{array}{l} \frac{x}{a}+\frac{y}{b} = 1-\frac{x}{m} \\ \frac{y}{a}+\frac{x}{b} = 1+\frac{y}{m} \end{array}\right\}$

94. We now proceed to problems in equations of **two** unknown quantities.

(*Ex.* 1.) 7 ducks and 3 pigeons cost 27 shillings, and 5 ducks and 2 pigeons cost 19 shillings; how much did each cost?

Let x = cost of each duck in shillings,
and y = cost of each pigeon in shillings.
∴ Cost of the first purchase in shillings $= 7x+3y$,
and of the second purchase in shillings $= 5x+2y$.
But, by the question, the first purchase = 27 shillings,
and, by the question, the second purchase = 19 shillings.
Hence we have two equations,

$$7x+3y = 27 \quad [1];$$
$$5x+2y = 19 \quad [2].$$

Solving these equations by mode A, B, or C, we have

$$x = 3 \text{ shillings},$$
$$y = 2 \text{ shillings}.$$

(*Ex.* 2.) There are two numbers, such that 3 times the greater added to one-third the less equals 72; and if twice the greater be subtracted from 6 times the less, and the remainder divided by 8, the quotient will be 8. Find the numbers.

Let $x =$ the **greater** number,
$y =$ the **less** number.

Then three times the greater $= 3x$,

one-third the less $= \frac{y}{3}$.

Hence by the question $3x + \frac{y}{3} = 72$ [1].

Again, twice the greater $= 2x$,
six times the less $= 6y$.

Hence by the question $\frac{6y - 2x}{8} = 8$ [2].

These equations [1] and [2] being cleared of fractions, may be solved by any of the methods, when it will be found that $x = 22$, $y = 18$.

EXERCISE LXXXVI.

(1.) A room of rectangular form is 4 feet longer than it is broad; if the length and breadth were both 5 feet less, its breadth would be two-thirds of its length. Required its length and breadth.

(2.) A merchant has two kinds of tea, one worth 9s. 6d. per ℔., the other 13s. 6d. How many pounds of each must he take to form a chest of 208 ℔. which shall be worth £112?

(3.) A person has two kinds of coins in his pocket, half-sovereigns and fourpenny-pieces; by taking 170 of them he can discharge a bill of £18, 6s. 0d. How many of each must he take?

(4.) A person has two horses, and a saddle worth £10: if the saddle be put on the first horse, his value becomes double that of the second; but if the saddle be put on the second horse, his value will not amount to that of the first horse by £13. Find the value of each horse.

(5.) What numbers are those whose difference is 30, and the quotient of the greater by the less is 4?

(6.) A draper bought two pieces of cloth for £25, 6s. 0d., one being 8s. and the other 9s. per yard. He sold them each at an advanced price of 2s. per yard, and gained by the whole £6. What was the length of the pieces?

(7.) A grocer bought tea at 10s. per lb., and coffee at 2s. 6d. per lb., to the amount altogether of £15, 12s. 6d.; he sold the tea at 8s. per lb., and the coffee at 4s. 6d. per lb., and gained £2, 10s. 0d. by the bargain. How many pounds of each did he buy?

(8.) Five years hence John's age will be one-fourth of mine, and 15 years hence his age will be two-fifths of mine. What are our ages?

(9.) Find two numbers, such that if the first be added to four times the second, the sum is 29; and if the second be added to six times the first, the sum is 36.

(10.) Find two numbers, such that twice the first *plus* the second is equal to 20, and twice the second *plus* the first is equal to 22.

(11.) A cask, which held 100 gallons, was filled with a mixture of brandy, wine, and cider, in such proportions that the cider was 10 gallons more than the brandy, and the wine was as much as the cider and one-fifth of the brandy. How much was there of each?

(12.) A farmer wishing to purchase a number of sheep found that, if they cost him 20s. a head, he would be £2 short of money; but were they to be only 16s. a head, he would then have £1 over. How many sheep were there, and how much money had he?

(13.) *A* can buy a horse worth £33, if *B* give him one-third of his money; *B* can buy it, if *A* give him one-fourth of his. How much has each?

(14.) A person buys 8 ℔. of tea and 3 ℔. of sugar for £1, 2s. 0d., and at another time he buys 5 ℔. of tea and 4 ℔. of sugar for 15s. 2d. Find the price of tea and sugar per ℔.

(15.) A farmer sold to one person 30 bushels of wheat and 40 bushels of barley for £13, 10s. 0d.; to another person he sold 50 bushels of wheat and 30 bushels of barley for £17. Find the price of the wheat and barley per bushel.

(16.) Find two numbers such that half the first with a third of the second shall be equal to 9, and a fourth part of the first with a fifth part of the second shall be equal to 5.

(17.) A person wished to distribute 3d. apiece to some poor persons, but found he had not money enough in his pocket by 8d.; he therefore gave them each 2d., and found he had 3d. remaining. Required the number of poor people, and the money he had in his pocket.

(18.) A rectangular room having been measured, it was observed that if it were 5 feet broader and 4 feet longer, it would contain 116 square feet more; but if it were 4 feet broader and 5 feet longer, it would contain 113 square feet more. Find its length and breadth.

(19.) If *A*'s money be increased by 36s., he will have

3 times as much as B; but if B's money be diminished by 5s., he will have half as much as A. Find the sum possessed by each.

(20.) Two clocks are together at 12: one loses two seconds, while the other gains three; in twelve hours one is four minutes before the other. What is the time indicated by each clock?

95. Having shown how simultaneous equations of **two** unknown quantities are solved, we now proceed to the solution of equations containing **three** unknown quantities.

The process is as follows:

(1.) Given $2x+3y+4z=16$ [1]; to find $x, y,$ and z.

$$3x+2y+5z=18 \quad [2];$$
$$4x+3y+2z=20 \quad [3].$$

Eq. [1] $\times 3 = 6x+9y+12z=48$ [4];

Eq. [2] $\times 2 = 6x+4y+10z=36$ [5].

$\therefore$ Subtracting [5] from [4], we have

$$5y+2z=12 \quad (\text{A}).$$

Eq. [1] $\times 2 = 4x+6y+8z=32$ [6].

$\therefore$ Subtracting [3] from [6], we have

$$3y+6z=12 \quad (\text{B}).$$

We have thus formed **two** equations (A) and (B), in each of which **one** of the unknown quantities (x) is **eliminated.**

Then eq. (A) $\times 3 = 15y+6z=36$ (C);

$\therefore$ Subtracting (B) from (C), we have

$$12y=24.$$

We have thus formed **one** equation containing **one** unknown quantity only; $\therefore y=2$.

From (B) $6z=12-3y=12-6=6$; $\therefore z=1$.

From [1] $2x=16-3y-4z=16-6-4=6$; $\therefore x=3$.

Hence the required roots are $x=3$, $y=2$, $z=1$.

(2.) Given $3x+4z=57$ [1]; to find x, y, and z.

$5x+3y=65$ [2];

$2y-z=11$ [3].

From eq. [2] $3y=65-5x$; $\therefore y=\dfrac{65-5x}{3}$ [4].

From eq. [1] $4z=57-3x$; $\therefore z=\dfrac{57-3x}{4}$ [5].

Substituting these values for y and z in equation [3], we have

$$2\times\frac{65-5x}{3}-\frac{57-3x}{4}=11;$$

whence $x=7$.

From [4] $y=\dfrac{65-35}{3}=10$;

From [5] $z=\dfrac{57-21}{4}=9$.

Hence the required roots are $x=7$, $y=10$, $z=9$.

EXERCISE LXXXVII.

Find the values of x, y, and z, in the following equations:

(1.) $\left\{\begin{array}{l} 4x+8y+12z=68 \\ 2x+3y+z=12 \\ 3x+y+2z=13 \end{array}\right\}$

(2.) $\left\{\begin{array}{l} 6x+4y+2z=46 \\ 5x+2y+4z=46 \\ 10x+5y+4z=75 \end{array}\right\}$

(3.) $\left\{\begin{array}{l} 2x-y+z=9 \\ 4x-8y+12z=56 \\ 3x+4y-2z=7 \end{array}\right\}$

(4.) $\left\{\begin{array}{l} 4x-8y+12z=24 \\ 2x+3y-4z=20 \\ 3x-2y+5z=26 \end{array}\right\}$

(5.) $\left\{\begin{array}{l} \dfrac{2}{x}+\dfrac{1}{y}=\dfrac{3}{z} \\ \dfrac{3}{z}-\dfrac{2}{y}=2 \\ \dfrac{2}{x}+\dfrac{2}{z}=\dfrac{8}{3} \end{array}\right\}$

(6.) $\left\{\begin{array}{l} \dfrac{x}{a}+\dfrac{y}{b}=1 \\ \dfrac{x}{a}+\dfrac{z}{m}=1 \\ \dfrac{y}{b}+\dfrac{z}{m}=1 \end{array}\right\}$

96. We now proceed to problems in equations of **three** unknown quantities. Thus:

It is required to find three numbers, such that the first, with half the other two, shall be 41; the second, with half the other two, shall be 48; and the third, with half the other two, shall be 51.

Let $x =$ the first number;
$y =$ the second number;
$z =$ the third number.

Then by the question,

$$x + \frac{y+z}{2} = 41,$$

or $2x + y + z = 82$ [1];

$$y + \frac{x+z}{2} = 48,$$

or $2y + x + z = 96$ [2];

$$z + \frac{x+y}{2} = 51,$$

or $2z + x + y = 102$ [3].

Multiplying eq. [2] by 2, $4y + 2x + 2z = 192$ [4];
subtracting eq. [1] from [4], $3y + z = 110$ (A);
subtracting eq. [2] from [3], $-y + z = 6$ (B).
Subtracting (B) from (A), $4y = 104$; $\therefore\ y = 26$;
from B, $z = 6 + y = 6 + 26 = 32$;
from [2], $x = 96 - 2y - z = 96 - 52 - 32 = 12$.
Hence the required roots are $x = 12$, $y = 26$, $z = 32$.

EXERCISE LXXXVIII.

(1.) Find three numbers, such that the sum of the first and second shall be 9, the sum of the first and third 10, and the sum of the second and third equal to 11.

(2.) Find three numbers, A, B, and C, such that A with half of B, B with a third of C, and C with a fourth of A, may each be 500.

(3.) Find three numbers, such that the first, with half the sum of the second and third shall be 120; the second, with two-fifths of the difference of the third and first shall be 75; and half the sum of the three numbers shall be 95.

(4.) A person purchased three jewels: the price of the first, with half the price of the other two, was £25; the price of the second, with a third of that of the first and third, was £26; and the price of the third, with half the price of the other two, was £29. Find the price of each.

(5.) A and B can perform a piece of work in 8 days, A and C together in 9 days, and B and C together in 10 days. In what time could each alone perform it?

(6.) A certain number of sovereigns, shillings, and sixpences amount to £8, 6s. 6d.; the amount of the shillings is a guinea less than that of the sovereigns, and a guinea and a half more than that of the sixpences. Find the number of each coin.

(7.) A certain sum of money was divided between A, B, and C, so that A's share exceeded four-sevenths of the shares of B and C by £30; also B's share exceeded three-eighths of the shares of A and C by £30; and C's share exceeded two-ninths of the shares of A and B by £30. Find the share of each person.

(8.) A certain number is composed of three digits: the sum of the digits is 11; the digit in the place of units is double that in the place of hundreds; and if 297 be added to the number, its digits are inverted. Find the number.

EXAMINATION PAPERS.

(A)

(1.) Divide $2x^m - 3x^{m-1} + 5x^{m-2}$ by $3x^{m-2}$.

(2.) Divide $4a^6 - 25a^2x^4 + 20ax^5 - 4x^6$ by $2a^3 - 5ax^2 + 2x^3$.

(3.) Find the G.C.M. of $6a^4 - 5a^2x^2 - 6x^4$ and $4a^5 - 6a^3x^2 - 2a^2x^3 + 3x^5$.

(4.) Reduce $\frac{4a^4 - 4a^2x^2 + 4ax^3 - x^4}{6a^4 + 4a^3x - 9a^2x^2 - 3ax^3 + 2x^4}$ to its simplest form.

(5.) From $3a - 2x - \frac{ax - x^2}{x^2 - 1}$ take $2a - x - \frac{a - x}{x + 1}$.

(6.) Divide $\frac{x^2 - y^2}{(x - y)^2}$ by $\frac{x^2 + xy}{x - y}$.

(7.) $\sqrt{\{6 + \sqrt{(x - 1)}\}} = 3$; find x.

(8.) $\left\{\begin{array}{l} \frac{x}{5} + \frac{y}{2} = 14 \\ \frac{x}{9} - \frac{y}{5} = 3 \end{array}\right\}$ find x and y.

(B)

(1.) If $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$, find the numerical value of the following expressions:

$$\sqrt{(e^2 + d^2 + c^2 - a^2)} \text{ and } \sqrt[4]{(2b^2 + c^2 - a)}.$$

(2.) From $7x^3 - 2x^2 + 2x + 2$ take $4x^3 - 2x^2 - 2x - 14$, and from the remainder subtract $2x^3 - 8x^2 + 4x + 16$.

(3.) Multiply the following expressions together: $x - 2a$, $x - a$, $x + a$, $x + 2a$.

(4.) Find the value of $x - \frac{x^2}{x - 1} - \frac{x}{x + 1}$.

(5.) Divide $\left(1+\frac{x}{y}\right)\left(1-\frac{x}{y}\right)$ by $\frac{y}{x^2+y^2}$.

(6.) Find the value of the following expression:

$$\frac{a^2x+b^2y}{x+y} \text{ when } a=\frac{2}{3} \text{ and } b=\frac{2}{3}.$$

(7.) $\frac{1}{2}x-\frac{1}{3}(x-2)=\frac{1}{4}(x+3)-\frac{2}{3}$; find x.

(8.) Divide the number 60 into two parts, such that a seventh of one part may be equal to an eighth of the other part.

(C)

(1.) Find the value of $\frac{a+\sqrt{a^2+b^2}}{a^3-2b(a^2-b^2)}$ when $a=-4$ and $b=-3$.

(2.) From $a+b$ take $\frac{a}{2}-\frac{b}{2}$.

(3.) Find the product of $(x-10)(x+1)(x+4)$.

(4.) Divide $mpx^3+(mq-np)x^2-(mr+nq)x+nr$ by $mx-n$.

(5.) Find the L.C.M. of $4(1-x)^2$, $8(1-x)$, $8(1+x)$, and $4(1+x^2)$.

(6.) Reduce to its lowest terms $\frac{4x^2-12ax+9a^2}{8x^3-27a^3}$.

(7.) $\frac{a}{bx}+\frac{b}{ax}=a^2+b^2$; find x.

(8). A boy is exactly one-third the age of his father, and has a brother one-sixth his own age; the ages of all three amount to 50 years. Find the age of each.

(D)

(1.) Show that the following expression is numerically equal, when $a=10$, $b=4$, $c=1$:

$$(a^2+b^2+c^2-ab-ac-bc)(a+b+c)=a^3+b^3+c^3-3abc.$$

(2.) Multiply $x^3-5x^2+2x-10$ by x^3+2x^2+3x-1.

(3.) Resolve into factors $x^2+5ax-14a^2$.

(4.) Simplify the following expression:

$$\left\{\frac{x^4-y^4}{x^2-y^2}-(x-y)^2\right\}\times\frac{x^3-y^3}{x-y}.$$

(5.) Find the quotient of $\left(a+\frac{b^2}{a+2b}\right)\div\left(b+\frac{a^2}{2a+b}\right)$.

(6.) $\frac{2(3-4x)}{3-x}+\frac{3}{1-x}=8$; find x

(7.) $\left\{\begin{aligned}\frac{9}{x}-\frac{4}{y}&=1\\ \frac{18}{x}+\frac{20}{y}&=16\end{aligned}\right\}$; find x and y.

(8.) The price of wheat per quarter is 15s. more than that of barley, and the value of 5 quarters of barley exceeds that of 3 quarters of wheat by 15s. Find the price of each per quarter.

(E)

(1.) Simplify $\dfrac{\frac{x}{a}+\frac{a}{x}-2}{x-a}+\dfrac{\frac{x}{a}+\frac{a}{x}+2.}{x+a}$.

(2.) Simplify $\frac{2a^2}{b^2-4a^2}-\frac{b}{b+2a}+\frac{a}{2a-b}$.

(3.) Find the G.C.M. of x^4+67x^2+66 and $x^4+2x^3+2x^2+2x+1$.

(4.) Find the L.C.M. of x^3-a^3, x^3+a^3, $x^4+a^2x^2+a^4$, $x^3-ax^2-a^2x+a^3$, and $x^3+ax^2-a^2x-a^3$.

(5.) Resolve $3x^3-14x^2-24x$ into its simple factors.

(6.) $\left\{\begin{aligned}2(x-y)&=3(x-4y)\\ 14(x+y)&=11(x+8)\end{aligned}\right\}$; find x and y.

(7.) $\frac{1}{x}+\frac{1}{2x}-\frac{1}{3x}=\frac{7}{3}$; find x.

(8.) A man has a sum of money amounting to £23, 15s. 0d., consisting only of half-crowns and florins; in all he has 200 pieces of money. How many has he of each sort?

(F)

(1.) Write down the squares of the following expressions: x^2+x+1 and $ax+by+c$.

(2.) Simplify the following expression, and find its numerical value when $x=y=\frac{1}{2}$: $\{(x+y)^2+(x-y)^2\}$ $\{(x+y)^2-(x-y)^2\}$.

(3.) Divide $(x^2-4)(x^2-4x)$ by x^2+2x.

(4.) Find the value of $(a^2-ab+b^2)(a+b)-(a-b)^3$ $+(3a^2-2b^2)b$.

(5.) Find the square root of $x^6-2x^5+3x^4-4x^3$ $+3x^2-2x+1$.

(6.) Simplify $\frac{3a-4b}{7}-\frac{2a-b-c}{3}+\frac{15a-4c}{12}-\frac{a+64b}{84}$.

(7.) $\left\{\begin{array}{l} x+y+z=90 \\ 2x+40=3y+20 \\ 2x+40=4z+10 \end{array}\right\}$; find x, y, and z.

(8.) A person bought a certain number of sheep for £94; having lost 7 of them, he sold one fourth of the remainder, at prime cost, for £20. How many sheep had he at first?

ANSWERS.

Exercise XXIX.

(1.) $16abcd$. (2.) $-35bcxy$. (3.) $-14abde$. (4.) $-15bcxy$.
(5.) $6acdx$. (6.) $4acdf$. (7.) $-28abdx$. (8.) $-45abcy$.
(9.) $7acdx$. (10.) $35acdxy$.

Exercise XXX.

(1.) $-12b^3c^3$. (2.) $-45ax^5y$. (3.) $8a^5y^4$.
(4.) $-7a^5xy$. (5.) $-35b^3c^3x^5$. (6.) $-48a^5c^3$.
(7.) $63a^3y^4$. (8.) $3a^4b^2x^4$. (9.) $16a^4c^3y^6$.
(10.) $-104x^3y^3z^7$. (11.) $-20a^5y^4$. (12.) $-30a^4b^6$.
(13.) $56a^5x^4$. (14.) $-9b^{n+3}y^{m+1}$. (15.) $40a^6x^5y^3$.
(16.) $-30b^4y^5$. (17.) $a^{m+n}c^{-m+n}$. (18.) $-a^{-m+n}b^{m+n}$.
(19.) $24x^{m+1}y^{m+n}$ (20.) $4c^{n-p}d^{m+q}$.

Exercise XXXI.

(1.) $-20a^2b^3c-8b^3c^3$. (2.) $9a^2x^5-6ax^4y^2$.
(3.) $40ac^3x-24c^3x^3$. (4.) $48a^4xy^2-18a^2xy^4$.
(5.) $-20b^3c^2y^3-10b^2x^2y^4$. (6.) $-14ac^4d-6c^4d^2$.
(7.) $-56b^3c^5-7b^2c^5d^2$. (8.) $27a^2x^4+18ax^5y$.
(9.) $-40xy^4z^2-24y^4z^5$. (10.) $-8x^3y^4+x^4y^5z^2$.
(11.) $-4a^5xy^n+5a^2b^3y^{n+1}$. (12.) $-8a^3x^{n+2}y^2-2ax^{m+n}y^2$.
(13.) $-50a^3x^{n+1}y^{m+1}+5x^{n+3}y^{m+n}$. (14.) $-27a^5b^2+12a^{m+3}b^2$.
(15.) $48x^5y^{m+4}+60x^{m+2}y^m$. (16.) $72x^{-m+1}y^{-n+1}-80x^{-n+1}y^{-n+1}$.
(17.) $30a^{m+3}b^{m+5}-45a^mb^mx^m$. (18.) $10a^{2m}b^{-n}y^{-n}+12a^mx^3y^{-2n}$.

Exercise XXXII.

(1.) $28abc^2+12b^2c^2-8bc^4$. (2.) $9a^4d+12a^4cd+15a^4d^2$.
(3.) $28bc^3y+21c^2xy^2+21bc^4y$. (4.) $-24x^5y^3+16x^3y^4+72ax^2y^3$.
(5.) $-12x^2y^4-24x^2y^5-24a^2xy^4$. (6.) $-28bc^2y-12ab^2c^3-12b^3c^3$.
(7.) $8x^2z^2-14x^3yz-10x^4z$. (8.) $3abd^3x^2+3ad^4x^2-d^5x^3$.
(9.) $9a^3c^3x+36a^3c^4-9a^5c^4$. (10.) $8bcxy^2z-56x^2y^3z+8xy^4z^2$.

Exercise XXXIII.

(1.) $a^2+2ax+x^2$. (2.) $b^2-2by+y^2$.
(3.) $4a^2-4ax+x^2$. (4.) $a^2+4ax+4x^2$.
(5.) $16a^2-24ab+9b^2$. (6.) $4a^2+12ab^2+9b^4$.
(7.) $36c^2-84cd^2+49d^4$. (8.) $16a^2x^2+8abx+b^2$.
(9.) $64x^2y^2+112xy^2z^2+49y^2z^4$. (10.) $81x^2y^6-216xy^5z+144y^4z^2$.

Exercise XXXIV.

(1.) a^2-x^2. (2.) x^2-2x+1. (3.) $4a^2+4ab+b^2$.
(4.) $49a^2-16c^2$. (5.) $9a^2-12ac^2+4c^4$. (6.) $100a^4b^2-16b^6$.
(7.) $64a^4c^2-32a^2c^3d+4c^4d^2$. (8.) $49a^2b^2+42abx^2+9x^4$.
(9.) $64a^4x^2-49b^4y^2$. (10.) $16a^4x^2y^2-9a^2$.

Exercise XXXV.

(1.) $x^3-2xy^2+y^3$. (2.) $x^3-6x^2+11x-6$.
(3.) $2ac-bc-6ad+3bd$. (4.) $8a^3+27$.
(5.) $10a^4-10ax^3$. (6.) $24a^3+28a^2x-36ax^2+8x^3$.
(7.) a^5+32b^5. (8.) a^5-b^5. (9.) $81x^4-y^4$. (10.) $32a^5+x^5$.

Exercise XXXVI.

(1.) $x^2+(a+c)x+ac$. (2.) $x^2+4x-21$. (3.) $x^2-4x-21$.
(4.) x^2+x-20. (5.) $x^2-10x+16$. (6.) $x^2-2x-15$.
(7.) $x^2-11x+30$. (8.) x^2-x-56.
(9.) $x^2-9x+18$. (10.) x^2+x-56.

Exercise XXXVII.

(1.) $x^2-y^2-z^2-2yz$. (2.) $a^4-2a^2b^2+b^4$.
(3.) $4a^2-b^2c^2+4b^3c-4b^4$. (4.) $a^3-8b^3-27c^3-18abc$.
(5.) $a^6+2a^3b^3+b^6$. (6.) $27a^3+b^3+8-18ab$.

Exercise XXXVIII.

(1) a^4-b^4. (2.) $x^3-5x^2-46x-40$.
(3.) $-a^3+ab^2+a^2b-b^3$. (4.) $x^4-5a^2x^2+4a^4$.
(5.) $x^4+10x^3+35x^2+50x+24$. (6.) $a^3-6a^2+11a-6$.

Exercise XXXIX.

(1.) $a^2-2ab+b^2-c^2$. (2.) $a^2+2ax+x^2-1$.
(3.) $4a^2+8a-x^2+4$. (4.) $a^4+a^2x^2+x^4$.
(5.) $4a^2-b^2+6bc-9c^2$. (6.) $-4a^2+4ab-b^2+9c^2$.
(7.) $4a^2-b^2c^2+4b^3c-4b^4$. (8.) $4a^4-5a^2b^2+b^4$.

Exercise XL. (A.)

(1.) $x^3-(a+c)x^2y+(b+ac)xy^2-bcy^3$.

(2.) $x^3-(a+1)x^2+(a+b)x-b$.

(3.) $x^3-(a+c)x^2+(ac+b)x-bc$. (4.) $a^2+(ab+ac)x+bcx^2$.

(5.) $x^3+(a+b-c)x^2+(ab-ac-bc)x-abc$.

(6.) $x^3+px^2+qx+a^3-a^2p+aq$.

(7.) $2abx^2-(3bc-2ad)xy-3cdy^2$.

(8.) $2a^{4n}-2x^{4n}$. (9.) $a^nb^2-a^{n-1}b^3+a^2b^n$. (10.) $(1+a)y(a^4-y^2)$.

(11.) $x^3+(a+b-c)x^2+(ab-ac-bc)x-abc$.

(12.) $a^{4m}-4a^{6n}$. (13.) $a^{2m}-9b^{2n}+12b^nc^p-4c^{2p}$.

Exercise XL. (B.)

(2.) $\frac{3}{4}x^5-4x^4+\frac{77}{8}x^3-\frac{43}{4}x^2-\frac{33}{4}x+27$. (3.) $1-\frac{x^6}{64}$.

(4.) $\frac{1}{3}x^3+\frac{11}{6}x^2+\frac{7}{9}x+\frac{4}{3}$. (5.) $1-\frac{1}{4}a^2+\frac{2}{3}b+\frac{1}{9}b^2$.

(6.) $a^5-\frac{11}{6}a^4x+\frac{7}{2}a^3x^2-\frac{7}{3}a^2x^3+\frac{37}{18}ax^4-\frac{x^5}{6}$.

(7.) $15x^4+\frac{21}{2}bx^3-\frac{107}{4}b^2x^2+\frac{5}{2}b^3x+\frac{7}{2}b^4$.

(8.) $mnx^{2(n+1)}+2nx^n-nx^{2n}(n-b)-x^n(mx^2+b)-1$.

Exercise XLI.

(1.) $(x+5)(x+3)$. (2.) $(x+5)(x-3)$. (3.) $(x+7)(x+1)$.

(4.) $(2x-3)(2x+1)$. (5.) $(x-9)(x+1)$. (6.) $(x-8)(x-5)$.

(7.) $(x-13)(x+8)$. (8.) $(3x-5)(x+1)$. (9.) $(4x+1)(3x-2)$.

(10.) $(4a^2-x^2)(3a^2+x^2)$. (11.) $(a-3x)(a+2x)a$.

(12.) $a^2(3ax-1)(2ax+1)$.

Exercise XLII.

(1.) $3a^2$. (2.) $a-7x$. (3.) x^2-d^2.

(4.) $2(a^2+x^2+c^2)$. (5.) $4(b^2x^2+a^2x^2+a^2b^2)$. (6.) $a^2+2ac+x^2$.

(7.) $2(3b^2-y^2)$. (8.) $18a+15x-4ax$.

(9.) $a^4+2a^2b^2+b^4+2a^2c^2+2b^2c^2+c^4-a^2-b^2-c^2$. (10.) $5x^2+8x+4$.

Exercise XLIII.

(1.) $2a$. (2.) $-3a$. (3.) $-3x$. (4.) $2a$. (5.) $2b$.

(6.) $-6x$. (7.) $6bc$. (8.) $-2a$. (9.) $-7a$. (10.) $-2z$.

Exercise XLIV. (A.)

(3.) $-4ax^2$. (4.) $2yz^3$. (5.) $-5x^2y^3$. (6.) $-6a$.
(7.) $7a^3$. (8.) $27a^2b$. (9.) $-2b$. (10.) $-10b$.

Exercise XLIV. (B.)

(1.) $\frac{2y^3}{c^4}$. (2.) $2a^m$. (3.) $\frac{a^m}{x^{2m}}$. (4.) a^4x^{m-2}.
(5.) $3a^mx^{m-2}$. (6.) $6cx$. (7.) $-2a^{m-n}c^pd^{2r-2}$.
(8.) $2a^{2m}x^m$. (9.) $-x$. (10.) $-\frac{2x^m}{a^m}$. (11.) $2a^2$. (12.) $\frac{a^{2(m-1)}}{2x^2}$.

Exercise XLV.

(3.) $-\frac{13}{39c}$. (4.) $\frac{7b}{14c}$. (5.) $-\frac{2ay}{bc}$. (6.) $\frac{3dx}{ac}$.
(7.) $\frac{2ac}{d}$. (8.) $\frac{6b}{30}$. (9.) $5bc$. (10.) $-3c$.

Exercise XLVI.

(3.) $\frac{x}{2}$. (4.) $-\frac{x}{4c}$. (5.) $-\frac{x}{6}$. (6.) $-\frac{a}{3}$.
(7.) $\frac{b}{2}$. (8.) $\frac{2bx}{3}$. (9.) $-\frac{x}{2}$. (10.) $\frac{2yz^2}{5}$.

Exercise XLVII.

(1.) $\frac{ax^2}{y}-\frac{5x^2}{4a}$. (2.) $-\frac{7x^2}{3b}-\frac{ab^2}{x}$. (3.) $\frac{x^3}{c}-\frac{9bc^2}{2x}$.
(4.) $-4b+\frac{ax}{b^2}$. (5.) $\frac{5xy^3}{ab}+\frac{4ab^2}{x}$. (6.) $-\frac{9z^3}{7}+x^2y^2$.
(7.) $\frac{8}{3}+2xy^3$. (8.) $-b+2b^2c$. (9.) $a-\frac{7b}{c}$. (10.) c^3+3c^2.

Exercise XLVIII.

(1.) $-\frac{4a}{c}-\frac{5b}{a}+\frac{6c^2d}{ab}-\frac{7ab^2}{c}$. (2.) $-\frac{4a}{y}+\frac{7x}{y}-\frac{10xy^2}{a}-\frac{11a^2x^3}{y}$.
(3.) $-\frac{6x^2}{y}-\frac{4y^2}{x}+\frac{16x^2y}{a}+\frac{2y^3}{a}$. (4.) $-\frac{3a^2}{y}+4x^2y-5xy^2+2x^2y^3$.
(5.) $\frac{4a^2}{x}-5b^2-6a^2b^2x^2-\frac{7b^2y}{x}$. (6.) $2b+3b^2x+9bx^2-\frac{7by^3}{x}$.
(7.) $-3a+4b^2c-5a^2c^2+\frac{7c^2}{b}$.

Exercise XLIX. (A.)

(1.) $a^2-2ax+x^2$. (2.) a^2-6a+9. (3.) $a^2-2ab+b^2$.
(4.) $2y^3+4y^2+8y+16$. (5.) $a^3+a^2b+ab^2+b^3$.
(6.) $2a^3+4a^2+8a+16$. (7.) $3x^4+3x^2y^2+3y^4$. (8.) $x+2$.
(9.) $32+16x+8x^2+4x^3+2x^4+x^5$. (10.) $x^3+3x^2+9x+27$.

Exercise XLIX. (B.)

(2.) $a+\frac{1}{5}b$. (3.) $\frac{2a^2}{3}-\frac{ab}{2}+b^2$. (4.) $a^2-\frac{4a}{3}+\frac{3}{4}$.

(5.) $x^2-\frac{3}{4}x+1$. (6.) $1-6a+9a^2$. (7.) $\frac{x^2}{2}-\frac{3x}{4}+6$.

(8.) $x^2-\frac{1}{2}x+\frac{2}{3}$. (9.) $1+\frac{1}{2}a+\frac{1}{3}b$. (10.) $1-\frac{x}{2}+\frac{x^3}{8}-\frac{x^4}{16}$.

(11.) $\frac{1}{4}a-2b+6c^2$. (12.) $x-a$. (13.) $x^2-\frac{1}{2}x+\frac{3}{4}$.

(14.) $\frac{1}{4}a-2b+6c^2$.

Exercise L.

(1.) $2ab-3bc+4c^2+\frac{23bc^3}{3ac-2bc}$. (2.) $3a-2x+4y+\frac{xy}{2a+3x}$.

(3.) $7a-3b+c-\frac{2bc}{2a-b}$. (4.) $4x^2-3xy-y^2+\frac{y^3}{3x-2y}$.

(5.) $x^5-x^4y+xy^4-y^5-\frac{y^6}{x+y}$.

(6.) $2x^2+3xy+4y^2+\frac{2y^4}{3x^2-4xy+y^2}$.

(7.) $2x^2-8xy+9y^2-\frac{3y^3}{2x-3y}$. (8.) $a^2-2ax+bx-x^2-\frac{2x^3}{b+x}$.

(9.) $x^5-5x^4+13x^3-x^2-x+2-\frac{2}{x^2-2x-2}$.

(10.) $a^2+b^2+c^2-bc-ca-ab-\frac{abc}{a+b+c}$.

Exercise LI.

(1.) $8a^3+12a^2b+18ab^2+27b^3$. (2.) $16a^2+4ac+c^2$.
(3.) $27x^3+9x^2y+3xy^2+y^3$. (4.) $8y^3+4y^2z+2yz^2+z^3$.
(5.) $9a^2+3ab+b^2$. (6.) $16a^4+8a^3c+4a^2c^2+2ac^3+c^4$.
(7.) $125c^3+75c^2d+45cd^2+27d^3$. (8.) $2a+7c$.
(9.) $16a^2+4ax+x^2$. (10.) $32x^5+16x^4y+8x^3y^2+4x^2y^3+2xy^4+y^5$.

Exercise LII.

(1.) $4a^2-2ab+b^2$. (2.) $81x^4-27x^3y+9x^2y^2-3xy^3+y^4$.
(3.) $16x^2-4xy+y^2$. (4.) $81a^4-27a^3b+9a^2b^2-3ab^3+b^4$.
(5.) $a^4-4a^3b+16a^2b^2-64ab^3+256b^4$. (6.) $4a^2-2ax+x^2$.
(7.) $81a^4-54a^3b+36a^2b^2-24ab^3+16b^4$. (8.) $16a^2-4ax+x^2$.
(9.) $9a^2-6ay+4y^2$. (10.) $256a^4-192a^3y+144a^2y^2-108ay^3+81y^4$.

Exercise LIII.

(1.) $8a^3-12a^2b+18ab^2-27b^3$. (2.) $9x^2+3xy+y^2$.
(3.) $8b^3-4b^2c+2bc^2-c^3$. (4.) $b^2+3bc+9c^2$.
(5.) $4b-c$. (6.) $16x^4+8x^3y+4x^2y^2+2xy^3+y^4$.
(7.) $256x^4+64x^3y+16x^2y^2+4xy^3+y^4$. (8.) $25x^2-5xy+y^2$.
(9.) $81x^4+27x^3y+9x^2y^2+3xy^3+y^4$. (10.) $8x^3+4x^2z+2xz^2+z^3$.

Exercise LIV.

(1.) $a^2-2ab+b^2$. (2.) $a^3+6a^2b+12ab^2+8b^3$.
(3.) $a+b$. (4.) $2a^3+5ab^2-2b^3$.
(5.) $a^3+3a^2x+3ax^2+x^3$. (6.) $x^2+y^2+z^2-xy-xz-yz$.
(7.) $-2a^2+8ab-5b^2$. (8.) a^2-6a+9.
(9.) $7b^2-3ab+a^2$. (10.) $3a^2+4ax+x^2$.
(11.) $a^2-2ax+3x^2$. (12.) $x(2a^3+3a^2x-ax^2+4x^3)$.

MISCELLANEOUS EXAMPLES.

(A)

(1.) $-12m^3n^2$; $-5m^3n^4p^3$. (2.) $15a^3x^3-15a^2x^2y-15axy^2+15y^3$.
(3.) x^4+2x^2+1; $9x^2-12xy+4y^2$. (4.) a^2b^2; $-a^3$.
(5.) $x^{10}-5x^8+10x^6-10x^4+5x^2-1$. (6.) x^5-y^4; $x^6+x^3y^2+y^4$.
(7.) $x^3+x^2y+xy^2+y^3$. (8.) b^2-d^2.

(B)

(1.) $6a^2(4b^2-c^2)$. (2.) $a^4+a^2x^2+x^4$.
(3.) $x^4+10x^3+35x^2+50x+24$. (4.) $a^4-a^3b+a^2b^2-ab^3+b^4$.
(5.) $a^2+b^2+c^2-ab-ac-bc$. (6.) $(x-5)(x-10)$.
(7.) $x^4-x^2y^2-2xy^3-y^4$. (8.) 0.

(C)

(1.) $16a^4c^4-81$. (2.) a^2.
(3.) $9b^2x^4-18bc^2x^2y^2+9c^4y^4$; $16a^2x^8-8abx^6+b^2x^4$.
(4.) $(x-a)(x+a)(x^2+a^2)(x^4+a^4)$; $(x^2+1)(x^2+4)$.
(5.) $ab(7a-9b)$. (6.) $ab+ac-bc$.
(7.) $x^2-xy+x+y^2+y+1$. (8.) $7a^2+6b^3$.

(D)

(1.) $16a^4-x^4$. (2.) x^3+1; $1-x^3$.
(3.) $a^8+a^6x^3+a^4x^6+a^2x^9+x^{12}$. (4.) $x+1$; $x-1$.
(5.) $9x^2-24xy+16y^2$; $a^2x^2-2abxy+b^2y^2$.
(6.) $a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5$.
(7.) $2(a^2+b^2+c^2)$. (8.) $-abcx^3+(ac+bc-c^2)x^2+cx-1$.

(E)

(1.) $2x^3+2y^3+4x^2y+4xy^2-x^2-y^2-1$.
(2.) $-a^4+a^3b-a^2b^2+ab^3-b^4$. (3.) $x^2+y^2+z^2+1$.
(4.) $(x-12y)(x+4y)$; $(x+7)(x^2-7x+49)$.
(5) $5x^2+14x-137$. (6.) $a^4-2a^2b^2+4ab+b^4$.
(7.) $(x-8)(x+5)$; $(x+9)(x+15)$. (8.) $x^3-(a+1)x^2+(a+b)x-b$.

(F)

(1.) $a^3-3abc+b^3+c^3$. (2.) x^4-2x+1.
(3.) $a^3+2a^2b+2ab^2+b^3$. (4.) x^4-a^4.
(5.) $x-y$. (6.) $x(3x+4)(x-6)$.
(7.) x^6+x^3+1. (8.) $5x^3-55x^2+152x$.

(G)

(1.) $14a^2c^2-17a^2bc+16ac+3a^2b^2-5ab+2$. (2.) $\frac{a}{2x}-\frac{x}{3a}$.
(3.) $-12a^3b^3(5a^6-6a^3b+7b^3)$. (4.) x^2-ax+1.
(5.) $\frac{1}{4}a^2-4b^4$. (6.) $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$.
(7.) $5(a-b)$. (8.) a^2+b^2.

Exercise LV.

(1.) 6. (2.) 5. (3.) 4. (4.) 2. (5.) 3. (6.) 5.
(7.) 5. (8.) 4. (9.) 4. (10.) 8. (11.) 9. (12.) 11.

Exercise LVI.

(1.) 4. (2.) 7. (3.) 2. (4.) 4. (5.) 5.
(6.) 3. (7.) 2. (8.) 2. (9.) 2. (10.) 2.

Exercise LVII.

(1.) 2. (2.) 4. (3.) 5. (4.) 4. (5.) 1.
(6.) 2. (7.) 4. (8.) 3. (9.) 5. (10.) 4.

Exercise LVIII.

(1.) 5. (2.) $4\frac{1}{5}$. (3.) $\frac{6}{11}$. (4.) $6\frac{1}{3}$. (5.) 5.
(6.) $7\frac{1}{3}$. (7.) $\frac{1}{4}$. (8.) $1\frac{16}{29}$. (9.) 8. (10.) 3.

Exercise LIX.

(1.) —3. (2.) $-5\frac{1}{4}$. (3.) $1\frac{1}{5}$. (4.) $-6\frac{2}{3}$. (5.) $1\frac{2}{7}$.
(6.) $\frac{3}{14}$. (7.) $-3\frac{2}{3}$. (8.) $1\frac{4}{5}$. (9.) —6. (10.) 4.

Exercise LX.

(1.) $-(3b+2c)$. (2.) $-\frac{a}{2}$. (3.) $3a-b-c$. (4.) $\frac{1}{4}(4a+b+c)$.

(5.) $\frac{2}{5}(a+2c)$. (6.) $\frac{2a+3b+2c}{5}$. (7.) $-\frac{1}{3}(5a-b)$.

(8.) $\frac{2a}{5}$. (9.) $\frac{1}{8}(3a-4b+2c)$. (10.) $\frac{4a+3b}{6}$.

Exercise LXI.

(1.) 4, 9. (2.) £3. (3.) $12\frac{1}{2}$ ℔.
(4.) £137 (5.) 4, 7. (6.) 134, 124, 116, 106.

Exercise LXII

(1.) 5. (2.) *A*, £75; *B*, £25. (3.) 29 and 56.
(4.) 2 ft., 4 ft., 6 ft. (5.) 9 (6.) *A*, £15; *B*, £10.
(7.) 5 and 11. (8.) *A*, £35; *B*, 50; *C*, £70.
(9.) 6 in., 18 in., 36 in. (10.) 9 and 19. (11.) *A*, £16; *B*, £44.
(12.) 3, 6, 9, and 12 yr. (13.) *A*, £14; *B*, £24; *C*, £38.
(14.) 4 and 6. (15.) 35 yr., 25 yr. (16.) 7 half-cr., 5 half-sov.
(17.) 7 fl., 5 half-cr. (18.) 8. (19.) £140, £150, £180.
(20.) £10, 2s.; £5, 1s.; £4, 16s. (21.) 4 ℔. (22.) *A*, £50; *B*, £10.
(23.) 15 gal. (24.) 54 and 21. (25.) 240. (26.) 20 days.
(27.) 6 and 3. (28.) 8 horses, 24 cows, 168 sheep.
(29.) £$69\frac{1}{3}$, £$138\frac{2}{3}$, £312. (30.) *A*, 28 yr.; *B*, 14 yr. (31.) 4 and 6.
(32.) £$3\frac{2}{11}$. (33.) Sheep, 12s.; cow, 84s. (34.) 5 hr.

Exercise LXIII.

(1.) $\frac{2b-3ab}{a}$. (2.) $\frac{3b}{b-x}$. (3.) $\frac{ab+b}{3c-d}$. (4.) $\frac{x^2+y^2}{x^3}$.

(5.) $\frac{a}{3(a-2x)}$. (6.) $\frac{a(a+b)}{a^2-ab+b^2}$. (7.) $\frac{a^2+b^2}{a^4}$. (8.) $\frac{x^2}{x+2}$.

(9.) $\frac{3ab-5b^2}{4a-7b}$. (10.) $\frac{2a-1}{a-1}$.

Exercise LXIV.

(1.) $x+1$. (2.) $a-b$. (3.) a^3+a^2-5a+3. (4.) $a-b$.

(5.) $3x-2$. (6.) a^2+b^2. (7.) $2(a^2+2a+1)$.
(8.) $2a^3-4a^2+a-1$. (9.) $3x-7$. (10.) $x-2b$.

Exercise LXV.

(1.) $\frac{2a-1}{a-1}$. (2.) $\frac{14x^2-10x}{6x^2+9x-15}$. (3.) $\frac{x+4b}{x+2b}$.

(4.) $\frac{7a-2b}{5a^2-3ab+2b^2}$. (5.) $\frac{a^2+4a+4}{a^2+a+1}$. (6.) $\frac{x^2-bx+b^2}{x^2-b^2}$.

(7.) $\frac{2a+3}{3a-4}$. (8.) $\frac{1}{a^2-2a+2}$. (9.) $\frac{3b+2}{b+1}$. (10.) $\frac{1}{a-2x}$.

Exercise LXVI.

(1.) $5x^2-\frac{3a-2b}{5x}$. (2.) $b-x+\frac{4x^2}{b+x}$. (3.) $x^2+b^2+\frac{b^4}{x^2-b^2}$.

(4.) $2a+6+\frac{23}{a-3}$. (5.) $10y+3a-\frac{2b^2}{a^2}$. (6.) $a-y-\frac{y^2}{a-y}$.

(7.) $3+\frac{3}{a^2-1}$. (8.) $2a+1-\frac{3}{2a^2}$.

Exercise LXVII.

(1.) $\frac{a^2-x^2}{x}$. (2.) $\frac{30a^2-4a+9}{10}$. (3.) $\frac{a^3+2b^3}{a+2b}$. (4.) $\frac{2a^2}{a-b}$.

(5.) $\frac{axy+4x+5y}{x+y}$. (6.) $-\frac{y}{x+z}$. (7.) $\frac{2ab}{a^2+b^2}$. (8.) $\frac{x^2+x}{x-1}$.

Exercise LXVIII.

(1.) $21a^2y^2$. (2.) $60a^3b^2c^2$. (3.) ax^2y-axy^2. (4.) $6(a^2-x^2)$.
(5.) $12x(x^2-1)$. (6.) $(a^3-1)(a+2)$. (7.) $(a^2-b^2)(a^2-4b^2)$.
(8.) $x^3+2x^2y+xy^2$. (9.) $a^4-a^2b^2$. (10.) a^4+a^3+a+1.

Exercise LXIX.

(1.) $\frac{5ax^3y}{8a^2x^2}, \frac{6y}{8a^2x^2}$. (2.) $\frac{20a^2(a+b)}{60ab(a+b)}, \frac{45b^2(a+b)}{60ab(a+b)}, \frac{24ab(a-b)}{60ab(a+b)}$.

(3.) $\frac{(a+b)^2}{a^2-b^2}, \frac{(a-b)^2}{a^2-b^2}, \frac{a^2+b^2}{a^2-b^2}$. (4.) $\frac{a^2c^2-b^2c^2}{a^4-b^4}, \frac{a^2d^2+b^2d^2}{a^4-b^4}$.

(5.) $\frac{8ax^2-8cx^2}{6(a^2-c^2)}, \frac{xy}{6(a^2-c^2)}$. (6.) $\frac{a+b}{a^2-b^2}, \frac{1}{a^2-b^2}$.

(7.) $\frac{ax^2+x^3}{x(a^2-x^2)}, \frac{2}{x(a^2-x^2)}, \frac{a^3-ax^2}{x(a^2-x^2)}$. (8.) $\frac{a^2c-c^3}{a^4-c^4}, \frac{a^2c+c^3}{a^4-c^4}$.

Exercise LXX.

(1.) $\frac{2a^2+2x^2}{a^2-x^2}$. (2.) $\frac{169a+77}{105}$. (3.) $\frac{8a^2b-5ab^2+5a^3+4b^3}{4a^2b^2}$.

(4.) $\frac{2}{1+a^2+a^4}$. (5.) $\frac{3a^2-ax+2x^2}{6x(a-x)}$. (6.) 0.

(7.) $\frac{2a^2-2ax+2x^2}{a^2-x^2}$. (8.) $\frac{4a^3}{a^4-b^4}$. (9.) $\frac{3a^2-2ax+3x^2}{2(a^2-x^2)}$.

(10.) $\frac{1}{2}$. (11.) $\frac{8a^2+18b^2}{4a^2-9b^2}$. (12.) $\frac{2a^2-b^2}{a^2-ab}$.

Exercise LXXI.

(1.) $\frac{11a+49}{56}$. (2.) $\frac{2x}{a^2-x^2}$. (3.) $\frac{a-b}{a}$.

(4.) $\frac{a^2+b^2}{a^2(b-a)}$. (5.) $\frac{2a^3}{1+a^2+a^4}$. (6.) $-\frac{2x}{a+x}$.

(7.) $\frac{3a^2+5a+2}{a^2-1}$. (8.) $\frac{x^2-2xy-y^2}{(x+y)^2(x-y)}$. (9.) $\frac{4x^2+3}{3x}$.

(10.) $-\frac{2}{a(4a^2-1)}$. (11.) $\frac{2xy^2}{x^4-y^4}$. (12.) $\frac{81x-4y}{84}$.

Exercise LXXII.

(1.) $\frac{a^2+ax}{4}$. (2.) $\frac{3a-1}{a-2}$. (3.) $\frac{a^2}{x^2(a+x)}$. (4.) $6b$. (5.) $\frac{ax+x^2}{b(a-x)}$.

(6.) $\frac{a^3+2a^2b+2ab^2+b^3}{(a-b)(a^2-ab+b^2)}$. (7.) $\frac{a}{a^2+b^2}$. (8.) $\frac{a^2+x^2}{a}$.

(9.) $\frac{2x}{b^2-bx+x^2}$. (10.) $\frac{1-b}{x}$. (11.) $\frac{3x}{4y}$. (12.) $\frac{ay}{a^2-y^2}$.

Exercise LXXIII.

(1.) $\frac{10a}{3}$. (2.) $\frac{2x}{a^2-ax+x^2}$. (3.) $\frac{1}{a}$. (4.) $\frac{a^2+x^2}{(a-x)^2}$.

(5.) 1. (6.) $\frac{x}{x^2+y^2}$. (7.) $\frac{2x}{b^2-bx+x^2}$. (8.) $x+y$.

(9.) 1. (10.) $\frac{a^2+b^2}{ab}$. (11.) $-2a^2$. (12.) $\frac{x^4+x^2y^2+y^4}{xy(x-y)^2}$.

Exercise LXXIV.

(1.) $\frac{3a}{2x}$. (2.) $\frac{a+b}{b}$. (3.) 1. (4.) $\frac{6mn}{9m^2-4n^2}$.

(5.) $\frac{1+x+x^2}{1-x-x^4+x^5}$. (6.) $\frac{a^2+b^2}{2ab}$. (7.) $\frac{x^2}{a^2}$.

(8.) $\frac{64b^3}{(3a-b)(a^2-b^2)(b^2-9a^2)}$. (9.) a. (10.) $\frac{yz+xz+xy}{yz+xz-xy}$.

Exercise LXXV.

(1.) $76\frac{4}{5}$. (2.) 30. (3.) 7. (4.) 12. (5.) 24. (6.) 27.
(7.) 18. (8.) 9. (9.) 6. (10.) 2. (11.) 36. (12.) 1.
(13.) 6. (14.) 5. (15.) 2. (16.) 24. (17.) 7. (18.) 11.
(19.) 8. (20.) 7. (21.) 19. (22.) 3. (23.) 20. (24.) 12.
(25.) 6. (26.) $7\frac{1}{13}$. (27.) $2a$. (28.) 8.

Exercise LXXVI.

(1.) $\frac{bc-de}{c-a}$. (2.) $\frac{(c+d)(a^2-b^2)}{4ab}$. (3.) $\frac{abde}{3bd+ad-4abd-2ab}$.

(4.) $\frac{c-d}{a-b}$. (5.) $\frac{2cd}{c+d}$. (6.) $\frac{a^2(c-a)}{c(c+a)}$.

(7.) $\frac{abe+bcd-c^2e}{c(ae-b^2)}$. (8.) $-\frac{ac}{2a+c}$. (9.) $\frac{ade}{a^2-ad+d^2}$.

(10.) $\frac{a(1-c^2)}{c(a^2-1)}$. (11.) $\frac{ae}{d}$. (12.) $\frac{bd-2bc}{2ac-cd}$.

Exercise LXXVII.

(1.) £196. (2.) 20 pence. (3.) James, 35; John, 28.
(4.) 144 gallons. (5.) 400 gallons.

Exercise LXXVIII.

(1.) £300. (2.) 60 gallons.
(3.) 15 miles in going; 6 miles in returning. (4.) 21, 16.
(5.) 28, 22. (6.) £125. (7.) 84 ft. (8.) £900, £1500, £2000.
(9.) 300. (10.) 1s. $10\frac{1}{2}$d. (11.) 30. (12.) 22 gal.; 9s. per gal.
(13.) £300. (14.) £152. (15.) $5\frac{1}{7}$ days. (16.) 168, 84, and 42 days.
(17.) 17 lb. (18.) 12 days. (19.) 400 gal. (20.) £100, £125, £150.
(21.) 42 years. (22.) 14 miles per hour. (23.) £18. (24.) 17211.
(25.) £$133\frac{1}{3}$. (26.) 40 min. past 11. (27.) 3 gal. (28.) 15 and 20.

Exercise LXXIX.

(1.) $a^3+6a^2+12a+8$. (2.) $8x^3-12x^2+6x-1$.
(3.) $b^4x^4+4b^3x^5+6b^2x^6+4bx^7+x^8$.

(4.) $a^4-4a^3x+10a^2x^2-12ax^3+9x^4$. (5.) $\frac{a^8x^{12}}{81}$.
(6.) $49a^4-168a^3+214a^2-120a+25$.
(7.) $8a^6-36a^4b^2+54a^2b^4-27b^6$.
(8.) $a^3-3a^2b+3ab^2-b^3+8d^3+6a^2d-12abd+6b^2d+12ad^2-12bd^2$.
(9.) $a^6-12a^5+54a^4-112a^3+108a^2-48a+8$.
(10.) $z^3+3z^2-5+\frac{3}{z^2}-\frac{1}{z^3}$. (11.) $16-96x+216x^2-216x^3+81x^4$.
(12.) $27a^3-27a^2b+9ab^2-b^3$.
(13.) $x^3-3x^2y+3xy^2-y^3-3x^2z+6xyz-3y^2z+3xz^2-3yz^2-z^3$.
(14.) $x^6-3x^5+6x^4-7x^3+6x^2-3x+1$.
(15.) $4x^4-12x^3-7x^2+24x+16$.
(16.) $x^2+4y^2+9z^2+4xy+6xz+12yz$.
(17.) $\frac{1}{16}x^2-6xy+144y^2$. (18.) $\frac{9a^2-18ax+9x^2}{4a^2+8ax+4x^2}$.
(19.) $a^4-4a^3x+6a^2x^2-4ax^3+x^4$. (20.) $1+x+\frac{11}{12}x^2+\frac{1}{3}x^3+\frac{1}{9}x^4$.

Exercise LXXX.

(1.) $2x$. (2.) $10ab$. (3.) $4a^2c$. (4.) $\frac{3a^2b}{2xy}$.
(5.) $3x^2y$. (6.) $2a^2b$. (7.) $9xy^2$. (8.) $\frac{3a^2bx}{4c^2d^3}$.

Exercise LXXXI.

(1.) $3x^3+4y^4$. (2.) $3x^2-2xy+5y^2$. (3.) $a^2-4ab+4b^2$.
(4.) $2-3x-x^2+2x^3$. (5.) $2x^2-3bx+4b^2$. (6.) $(x-b)^3$.
(7.) $5a^2b-3ab^2+2b^3$. (8.) $2x^2-3x+4$. (9.) $4a^2-3ab+2b^2$.
(10.) x^3+2x^2+3x+4.

Exercise LXXXII.

(1.) 21. (2.) 17. (3.) 7. (4.) 2.
(5.) $\frac{c^2-a^2}{2c}$. (6.) $\frac{1}{a-2}$. (7.) $\frac{a}{2}\left\{1-\frac{b}{a-b}\right\}$.
(8.) $\frac{(a-b)^2}{2b}$. (9.) $\frac{a^2+a}{5-2a}$. (10.) $\frac{1}{1-b}$.

Exercise LXXXIII.

(1.) $x=1,\ y=1$. (2.) $x=5,\ y=2$. (3.) $x=3,\ y=1$.
(4.) $x=5,\ y=4$. (5.) $x=5,\ y=2$. (6.) $x=4,\ y=7$.
(7.) $x=5,\ y=2$. (8.) $x=3,\ y=4$. (9.) $x=8,\ y=1$.
(10.) $x=3,\ y=5$. (11.) $x=4,\ y=7$. (12.) $x=6,\ y=10$.

Exercise LXXXIV.

(1.) $x=28,\ y=49.$ (2.) $x=3\frac{1}{2},\ y=4.$ (3.) $x=18,\ y=6.$
(4.) $x=9,\ y=4.$ (5.) $x=5,\ y=9.$ (6.) $x=3,\ y=2.$
(7.) $x=2,\ y=-1.$ (8.) $x=2,\ y=7.$

Exercise LXXXV.

(1.) $x=\frac{bc}{a+c},\quad y=\frac{ab}{a+c}.$ (2.) $x=c+d,\quad y=c+d.$

(3.) $x=\frac{ac(bp+dq)}{ad+bc},\ y=\frac{bd(cq-ap)}{ad+bc}.$ (4.) $x=\frac{mn^2}{a^2+n^2},\ y=\frac{a^2n}{a^2+n^2}.$

(5.) $x=\frac{d-bc}{a-b},\ y=\frac{ac-d}{a-b}.$ (6.) $x=\frac{cd(c+d)}{c^2+d^2},\ y=\frac{cd(c-d)}{c^2+d^2}.$

(7.) $x=\frac{m^2-n^2}{ma-nb},\quad y=\frac{m^2-n^2}{mb-na}.$

(8.) $x=\frac{(ab+am-bm)abm}{a^2b^2+a^2m^2-b^2m^2},\quad y=\frac{(am-ab-bm)abm}{a^2b^2+a^2m^2-b^2m^2}.$

Exercise LXXXVI.

(1.) 17 ft.; 13 ft. (2.) 66 at 13s. 6d.; 142 at 9s. 6d.
(3.) 32 half-sovereigns; 138 fourpences. (4.) £56, £33.
(5.) 40, 10. (6.) 34 yd.; 26 yd. (7.) 20 ℔.; 45 ℔.
(8.) 5, 35. (9.) 5, 6. (10.) 6, 8.
(11.) Brandy, 25 gal.; cider, 35 gal.; wine, 40 gal.
(12.) 15 sheep; money, £13. (13.) *A*, £24; *B*, £27.
(14.) 30d.; 8d. (15.) 5s.; 3s. (16.) 8 and 15.
(17.) 11 poor persons; 25 pence. (18.) 12 ft.; 9 ft.
(19.) *A* had 42s.; *B* had 26s.
(20.) 11 hr. 58 min. 24 sec.; 12 hr. 2 min. 24 sec.

Exercise LXXXVII.

(1.) $x=1,\ y=2,\ z=4.$ (2.) $x=4,\ y=3,\ z=5.$
(3.) $x=3,\ y=2,\ z=5.$ (4.) $x=8,\ y=4,\ z=2.$
(5.) $x=\frac{7}{6},\ y=-\frac{7}{2},\ z=\frac{21}{10}.$ (6.) $x=\frac{a}{2},\ y=\frac{b}{2},\ z=\frac{m}{2}.$

Exercise LXXXVIII.

(1.) 4, 5, 6. (2.) $A=320$; $B=360$; $C=420$. (3.) 50, 65, 75.
(4.) £8, £18, and £16. (5.) $14\frac{34}{49}$, $17\frac{23}{41}$, and $23\frac{7}{31}$ days.
(6.) 4, 59, 55. (7.) £150, £120, £90. (8.) 326.

EXAMINATION PAPERS.

(A)

(1.) $\frac{2x^2}{3}-x+\frac{5}{3}$. (2.) $2a^3+5ax^2-2x^3$. (3.) $2a^2-3x^2$.

(4.) $\frac{2a^2-2ax+x^2}{3a^2-ax-2x^2}$. (5.) $a-x+\frac{x-a}{x^2-1}$. (6.) $\frac{1}{x}$.

(7.) 10. (8.) $x=45,\ y=10$.

(B)

(1.) 7, 2. (2.) x^3+8x^2. (3.) $x^4-5a^2x^2+4a^4$. (4.) $\frac{2x^2}{1-x^2}$.

(5.) $\frac{y^4-x^4}{y^3}$. (6) $\frac{4}{9}$. (7.) 7. (8.) 28 and 32.

(C)

(1.) $-\frac{1}{22}$. (2.) $\frac{a}{2}+\frac{3b}{2}$. (3.) $x^3-5x^2-46x-40$.

(4.) px^2+qx-r. (5.) $8(1-x)(1-x^4)$. (6.) $\frac{2x-3a}{4x^2+6ax+9a^2}$.

(7.) $\frac{1}{ab}$. (8.) 36, 12, 2 years.

(D)

(2.) $x^6-3x^5-5x^4-22x^3-9x^2-32x+10$. (3.) $(x+7a)(x-2a)$.

(4.) $2xy(x^2+xy+y^2)$. (5.) $\frac{2a+b}{a+2b}$. (6.) $\frac{3}{5}$.

(7.) $x=3,\ y=2$. (8.) 45s. and 30s.

(E)

(1.) $\frac{2}{a}$. (2.) $\frac{ab-b^2}{b^2-4a^2}$. (3.) x^2+1. (4.) $(x^2-a^2)(x^6-a^6)$.

(5.) $x(3x+4)(x-6)$. (6.) $x=20,\ y=2$. (7.) $\frac{1}{2}$. (8.) 150, 50.

(F)

(1.) $x^4+2x^3+3x^2+2x+1$ and $a^2x^2+b^2y^2+c^2+2abxy+2acx+2bcy$.

(2.) $8xy(x^2+y^2)$ and 1. (3.) x^2-6x+8. (4.) $6a^2b-3ab^2$.

(5.) x^3-x^2+x-1. (6.) $a-b$. (7.) $x=35,\ y=30,\ z=25$. (8.) 47.