THE JUNIOR STUDENT'S ALGEBRA

ALEXANDER WILSON, M.A.

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JUNIOR STUDENT'S ALGEBRA.

BEING

A MANUAL OF THE SCIENCE TO THE END OF SIMPLE EQUATIONS.

FOR THE USE OF JUNIOR STUDENTS PREPARING FOR THE OXFORD AND CAMBRIDGE LOCAL EXAMINATIONS.

BY

ALEXANDER WILSON, M.A.

WITH AN APPENDIX

CONTAINING

THE LOCAL EXAMINATION QUESTIONS,

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PREFACE.

THIS manual is designed for the use of young pupils beginning Algebra: and I have deemed it advisable not to go beyond Simple Equations, the first distinct stage in the study of the subject. No attempt has been made to present a logical exposition of the science; but such familiar explanations have been given as, it is hoped, will aid both teacher and pupil. Most of the works on the subject now in use have been written for students of more mature years; and teachers who use these in schools, find that the text is useless in the hands of beginners, while books written specially for beginners are too meagre, and omit the consideration of most important matter. The text of this book is such that, if the teacher will have it read carefully in class, I venture to think he will find that it will materially help him in explaining the various operations, and that his pupils will be able to refer to it intelligently, when left to themselves to work out the examples. The subjects are so treated as fully to meet the requirements of the Local Examinations, and others of the same scope; and the number of examples given will be found amply sufficient for practice.

To those who know by experience the difficult and irksome task of getting young pupils to understand and use Algebraic formulæ, I specially commend that part of the book which treats of factors.

I acknowledge with much pleasure my obligations to many professional friends for valuable suggestions, and especially to Alexander Grant, M.A., Head Master of Audley Grammar School, to H. Sharratt, B.A., late Scholar of Emmanuel College, Cambridge, and to G. C. Graves, B.A., Oxon, for material assistance in revising and verifying the Examples.

I shall be grateful to anyone using the book who will point out to me such inaccuracies as may be discovered.

ALEXANDER WILSON.

ST. LEONARDS ON SEA,
Dec., 1878.

ALGEBRA.

I.—DEFINITIONS AND SIGNS.

1.—ALGEBRA is that science which treats of numbers by means of symbols. The symbols used are the letters of the alphabet and certain signs, of which the following are those to be first considered :—

$=$, which is read EQUAL TO.

$+$, which is read PLUS, and indicates ADDITION.

$-$, which is read MINUS, and indicates SUBTRACTION.

$\times$, the sign of MULTIPLICATION.

$\div$, the sign of DIVISION.

Sometimes we write $\because$ for *because*, and $\therefore$ for *therefore.*

2.—The letters of the alphabet are put for the quantities reasoned about; and sometimes they have particular numerical values assigned to them, or stand for *any* numbers, according to circumstances.

If we write $a = x$, we mean that the value of a is equal to the value x.

$a + b$ means that the values of a and b are to be added together. If $a = 6$ and $b = 2$, then $a + b = 8$.

$a - b$ means that the value of b is to be taken from the value of a. Suppose $a = 10$, and $b = 8$, then $a - b = 2$.

3.—And here we meet with a use of the sign —, and results [illegible] operations of subtraction, which are not [illegible] subtractions. Suppose we have [illegible] and then [illegible] while [illegible]. It is [illegible] arithmetical idea, the operation [illegible] would be impossible. In Algebra [illegible] as the signs + and − are opposite [illegible] destroy each other until the [illegible] and then the result will be the difference [illegible] with the sign of the greater put before [illegible] will therefore result in a quantity [illegible] This is called a negative quantity, while [illegible] is called a positive quantity. [illegible] quantity is written with neither + nor [illegible] to be positive.

[illegible] is used to denote Multiplication [illegible] Thus [illegible] of writing a multiplied by [illegible] The most usual way of [illegible] the quantities together [illegible] Thus ab means that a is to be [illegible]

[illegible] we should have

[illegible]

[illegible] is called a [illegible] is called a coefficient or [illegible] however, we shall have [illegible] Thus, the [illegible] When no numerical [illegible] is never written.

[illegible] divided by b. This is [illegible]

[illegible]

7.—If $a=1$, $b=2$, $c=3$, $d=4$, and $e=5$, find the value of

$$ab+bc-bcd+abcde-ad$$
$$=1\times 2+2\times 3-2\times 3\times 4+1\times 2\times 3\times 4\times 5-1\times 4.$$
$$=2+6-24+120-4.$$

Adding all the positive quantities together, and all the negative quantities together, respectively, we have $128-28=100$.

Any combination of algebraic symbols is called an *expression.* ab, bc, bcd, $abcde$, and ad are called the *terms* of the above *expression.*

A quantity of more than one term is called a *compound expression.*

8.—If $a=6$, $b=2$, $x=4$, and $y=0$, find the value of

$$ax+by-2bx+ay+abxy+ab-abx.$$

☞ When any factor in a term represents 0, the term in which it occurs will disappear, since 0 multiplied by any number = 0. The above expression will therefore become

$$24+0-16+0+0+12-48.$$

Collecting terms (that is, adding together the positive and negative terms respectively), we have $36-64=-28$, the value required.

Examples I.

Find the value of the following expressions, when $a=2$, $b=3$, $x=4$, $y=5$, and $z=0$.

1. $a+b+x+y+z.$
2. $a+x-y+b-z.$
3. $3a+4b-2x-y.$
4. $6y-12+3z.$
5. $ab+ax-ay.$
6. $3a-y-2x+2z+1.$
7. $ax-ay+bx-3.$
8. $by+xy-az+yz.$
9. $abx-12y-1.$
10. $axy-abxy+xyz-4bxy.$
11. $-13xy-ab+12bxy.$
12. $-byz+5ax-ay-bx-6b.$

When $m=1$, $n=0$, $p=2$, and $x=y=3$, find the value of

13. $px-py+m.$
14. $m+my-ny-2x.$
15. $xy-mn.$
16. $2xy+px+pm+1.$

of the operation of subtraction, which are not introduced into arithmetical calculations. Suppose we have the expression $x - y$ to consider, and that $x = 6$, while $y = 8$. It is evident that, according to arithmetical ideas, the operation $x - y$, that is 8 subtracted from 6, would be impossible. In Algebra, however, we consider that, as the signs + and − are opposite in their nature, they will destroy each other until the smaller quantity is exhausted, and that the result will be the difference between the quantities, with the sign of the greater put before it. Our example $x - y$ will therefore result in a quantity which we write as -2. This is called a *negative* quantity, while any quantity before which + stands is called a *positive* quantity.

☞ If a quantity is written with neither + nor − before it, it is understood to be positive.

4.—The sign × is used to denote Multiplication, and is read *into* or *multiplied by*. Thus, if $a = 3$ and $b = 4$, $a \times b = 12$. Also, $a \, . \, b$ is another form of writing a multiplied by b, though not in such general use. The most usual way of expressing multiplication is to write the quantities together without any sign between them. Thus, ab means that a is to be multiplied by b, and is the same as $a \times b$ or $a \, . \, b$.

Suppose $a = 1$, $b = 2$, $c = 3$, we should have

$$a \times b \times c = 6, \quad a \cdot b \cdot c = 6, \text{ and } abc = 6.$$

5.—Each of the quantities a, b, c, is called a *factor* of abc. Each of them also is called a *co-factor* or CO-EFFICIENT of the others. At present, however, we shall have most to do with numerical co-efficients. In $3a$, $6ab$, $7ax$, the numerical co-efficients are 3, 6, and 7. When no numerical co-efficient is written, 1 is understood, but never written.

6. $a \div b$ means a divided by b. This is more usually written $\frac{a}{b}$. Thus if $x = 12$ and $y = 2$, then $x \div y = 6$, and $\frac{x}{y} = 6$.

7.—If $a=1$, $b=2$, $c=3$, $d=4$, and $e=5$, find the value of

$$ab+bc-bcd+abcde-ad$$
$$=1\times2+2\times3-2\times3\times4+1\times2\times3\times4\times5-1\times4.$$
$$=2+6-24+120-4.$$

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EXAMPLES I.

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12. $-byz+5ax-ay-bx-6b.$

When $m=1$, $n=0$, $p=2$, and $x=y=3$, find the value of

13. $px-py+m.$
14. $m+my-ny-2x.$
15. $xy-mn.$
16. $2xy+px+pm+1.$

17. $-mx - 2pnx - 4p$.

18. $mnpxy - mpxy + pm + 2$.

19. $m + px + py + mxy - 11p$.

20. $mpy - 30m + 5y + 10$.

21. $11mnp - 5mnx + 6npy - 9pxy + 7mxy - 15mpxy$.

22. $1 - 12m + 17p - 13xy + 8nxy - 3mxy + 7pxy$.

23. $-mnp - 50mpxy - 180mx + 27nx - 45mnxyp$.

24. $-13yx + 12ypm - 11mx + 5my + 63$.

When $a = 5,\ b = 2,\ c = 1$, and $d = 0$, find the value of

$$\frac{a+b}{c} - \frac{a-b}{b+c} + \frac{ab}{a+b+3c} - \frac{bcd}{a-b}$$

$$= \frac{5+2}{1} - \frac{5-2}{2+1} + \frac{10}{5+2+3} - 0$$

$$= \frac{7}{1} - \frac{3}{3} + \frac{10}{10} = 7 - 1 + 1 = 7.$$

Examples II.

With the values of a, b, c, and d given above, find the value of each of the following expressions :—

1. $\dfrac{a+c}{b}$

2. $\dfrac{a+b+c}{2b}$

3. $\dfrac{b+c}{a-b}$

4. $\dfrac{abc}{2b+c}$

5. $\dfrac{a-b}{b+c}$

6. $\dfrac{c+d}{c}$

7. $\dfrac{a+b+c+d+2}{b}$

8. $\dfrac{b}{c} - \dfrac{a}{2b+c} - \dfrac{a-b-c}{2c}$

9. $\dfrac{bc+ab}{2a+2c} - \dfrac{a}{c} + \dfrac{5ac-8b}{b+c}$

10. $\dfrac{3a+b-2c}{a+bd} + \dfrac{19bc}{8a-2} + \dfrac{3bcd}{2a+3b+4c-5d}$

When $a = 2,\ b = 3,\ x = 4,\ y = 5,\ z = 0$, find the values of

11. $\dfrac{xy}{2} + \dfrac{bx}{6} - \dfrac{24}{a} + \dfrac{6a}{bx} + \dfrac{a}{2}$.

12. $\dfrac{ay}{5} \times \dfrac{24}{b} \times \dfrac{a+b+x+y}{a+y+bz}$.

13. $\dfrac{a+b+y}{2x+2} \cdot \dfrac{ab}{a+x} \div \dfrac{b+x}{a+y}$.

14. $\dfrac{2a+2b}{x+y+1} - \dfrac{xy-2ab}{2a+x} + \dfrac{3yz}{ab+bx-yz}$.

15. $\dfrac{bx}{a} \cdot \dfrac{b-a}{x-b} - \dfrac{10xy}{4a+b+x+y+z}$.

16. $\dfrac{10x+b+y}{2y-a-x+z} \div \dfrac{y-z-1}{x-b-z}$.

17. Find the value of $\dfrac{16a+7y-b+z}{2abz+x+y} - yz$, when $x=0$, $y=1$, and $z=a=b=3$.

With the same values prove

18. $\dfrac{a+b}{b} + \dfrac{xy+ab}{az} = \dfrac{x+y+z}{2-y} - y$.

19. $xyz \cdot \dfrac{a+b+y}{y+z-a} = \dfrac{ab-z}{x+z} - \dfrac{a+y}{y+1}$.

20. $\dfrac{ab+bz}{a+b+z} \times \dfrac{a-y}{b-y} = \dfrac{xy+az+1}{a+2y} \div \dfrac{a-2y}{ax+y}$.

21. $yz \times \dfrac{6a+7b}{2a+b+y+z} + \dfrac{ax+by+yz}{ab+ay-2z} - \dfrac{4abz-2a-2}{a+b+y+z} = 0$.

22. $\dfrac{2ab}{a+b} \cdot \dfrac{3az}{a+b+z} \div \dfrac{6xy+2yz}{a+b+x} \times \dfrac{b-y}{ab-az+y} = 2ab$.

23. $\dfrac{az+bz+xz+yz+9}{ab-x+y} \div \dfrac{13az+14ab}{3abz} = 1$.

24. $\dfrac{3a}{b} - \dfrac{3b}{a} + \dfrac{4a}{z} + \dfrac{2ax+3by-z}{a+b} = 5y$.

9.—A POWER of a quantity is the quantity multiplied by itself a given nnmber of times.

$2 \times 2 = 4 =$ the second power or *square* of 2, written 2^2.

$3 \times 3 = 3^2 = 9$: $2 \times 2 \times 2 =$ the *cube* of 2, written 2^3.

$a \times a = a^2$, the square of a : $a \times a \times a = a^3$, the cube of a.

$a \times a \times a \times a = a^4 =$ the fourth power of a.

It will be observed that the small figure is used to denote *the number of a.'s that are multiplied into each other.* This figure is called the *index* (plural *indices).*

The *first* power of a quantity is the quantity itself. The index 1, however, is never written.

(Here the pupil should be exercised in finding mentally the squares and cubes of the numbers from 0 to 12.)

Ex. (i.) When $x = 2$, $y = 3$, and $a = 0$, find the value of

$$x^2y - 2x^2y^2 + xy^2 - 3ax^2y^2$$
$$= 4 \times 3 - 2 \times 4 \times 9 + 2 \times 9 - 0.$$
$$= 12 - 72 + 18 = 30 - 72 = -42.$$

Ex. (ii.) When $a = 3$, $b = 2$, $c = 1$, $d = 0$, find the value of

$$\frac{a^2 + b^2 + c^2}{3a - b + d} - \frac{3a^2 - ad + d^2}{3a^2 + ad - d^2}$$
$$= \frac{9 + 4 + 1}{9 - 2 + 0} - \frac{27 - 0 + 0}{27 + 0 - 0} = \frac{14}{7} - \frac{27}{27} = 2 - 1 = 1.$$

EXAMPLES III.

Find the value of the following expressions, when $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 0$.

1. $a^2 + b^2$.
2. $a^3 + b^3$.
3. $d^2 - b^3$.
4. $c^2 + d^2 - e^2$.
5. $ab^3 - 3d^2 + 2c^2$.
6. $d^3 - ab + bc$.
7. $4c^3 - d^2 + 3e^3$.
8. $a^2b^2 + a^2b^3 - c^2d^3$.
9. $ab^4 - c^4 + b^5$.

Find the values of the following expressions, when $a = 6$, $b = 5$, $m = 2$, $n = 3$, $x = 1$, $y = 0$.

10. $a^2 + 2ab + b^2$.

11. $a^3 - 3a^2b + 3ab^2 - b^3$.

12. $m^2 + n^2 - 10x^2 + xy - abmn$.

13. $x^2 . \dfrac{ab}{mn} - bx$.

14. $\dfrac{m^2n^2x^2}{a + b + x} - \dfrac{m^2n^2}{a^2}$.

15. $\dfrac{a^2 - b^2}{a + m + n} + \dfrac{a + b + m + n^3}{a + b - n + 6m}$.

16. $\dfrac{x^3 + y^3}{a - b} + \dfrac{m^2n^2 - ab}{mn - n} - \dfrac{m + n - x}{bm^2 - mn^2}$.

17. With the same values prove

$$\frac{ab - m^2n^2 + 2mn}{2n^3 - 13nx - 7m} = \frac{b^2 + m + n}{m + n}.$$

When $a = 2$, $b = 3$, $c = 4$, $d = 5$, and $e = 0$, find the value of

18. $\frac{a^2b^2}{9c} + \frac{a^2 + b^2 + c^2 - d^2}{c} + \frac{a^2b^2c^2}{b^2c - 2c^2}.$

19. $\frac{a + b - c}{c - b} + \frac{2a - b + c}{a + b} + \frac{2a^2b^2}{a^2}.$

20. $\frac{a^2b^2}{a + c^2} \div \frac{a + b + d}{5} \times \frac{a^2}{c}.$

21. $\frac{a^3 - b}{a + b} \times \frac{abd}{3} \times \frac{cde}{a^2 + b^2 + c^2}.$

22. $\frac{ac + bc + de}{a + b} \times \frac{abc}{ae + bc} \times \frac{ab}{a + c}.$

23. $\frac{b^2 - a^2}{d} \times \frac{d^2e^2 + ac^2}{b^2 + d^2 - a} \div \frac{abc}{a + b + c + 3d}.$

24. $\frac{a^4 - b^3 + c^2}{d} \times \frac{b^3 - d^2}{a} + \frac{3a^2b^3 - c^3d}{a^5 - b^4 + 2d^2} \div \frac{2b^2 - a^4}{c^2 - 3d}.$

25. $\frac{c^3d^3 - a^4b^3c^2}{a^3 + ab^3cd} \times \frac{c^2}{a^3} + \frac{d^3e}{b^3} - \frac{d^3 - a - b}{abcd}.$

26. $a^5 \cdot \frac{c^3 - 7b^2}{d^3 - 4b^3 - c^2} - \frac{a^3b^2c - 2d^3 - ab}{b^2 - a^3} - \frac{ad^3 - acd^2}{c^3 - a^4 + a}.$

27. $d^a - c^b - a^c.$ 28. $\frac{b^a - a^b}{c^a - bd}.$ 29. $\frac{b^a c^a}{d^a - a^b + 7}.$

Find the value of—

30. $\frac{a^2 - 4}{a - 2} - \frac{a^3 - 8}{a - 2} \times \frac{a^2 + a - 6}{a + 3}$, when $a = 3$, and when $a = 10$.

10.—Since, by multiplying quantities by themselves, we obtain powers of these quantities, we can perform the converse operation and obtain the *roots* from which these powers arise.

Thus 4 being the square of 2, 2 is called the SQUARE ROOT of 4. Also 3 is the square root of 9, 8 of 64, and so on. Again

2 is the CUBE ROOT of 8, and 5 of 125, because 2^3 and 5^3 are respectively 8 and 125.

11.—To express that a certain root of a number is to be obtained, we make use of the sign $\sqrt{}$, which is called the RADICAL SIGN. When the simple sign is written before a quantity, *it is understood that the square root is meant.* Other roots are indicated by small figures, called RADICAL INDICES placed within the radical sign; thus $\sqrt[3]{}$ and $\sqrt[4]{}$ indicate the third and fourth roots.

$\sqrt{a}$ signifies the square root of a.

$\sqrt[3]{a+b}$ signifies the cube root of $a+b$.

The radical sign is extended by means of a line to include several terms. If $a=9$, $x=4$, $y=5$,

$$\text{Ex. (i.)}\quad \sqrt{a+x+2y+2} = \sqrt{9+4+10+2} = \sqrt{25} = 5.$$

$$\text{Ex. (ii.)}\quad 6\sqrt[3]{y^2+x^2-3a-6} = 6\sqrt[3]{25+16-27-6} = 6\sqrt[3]{8} = 6\times 2 = 12.$$

(Before proceeding to the next Exercise the pupil should be able to recognise as *roots* the numbers whose powers were found in **9**).

EXAMPLES IV.

When $a=1$, $b=2$, $c=3$, find the values of

1. $\sqrt{6bc}$. 2. $\sqrt{2c^2+2c+a}$. 3. $\sqrt{5c+a}$.

4. $3\sqrt[3]{4ab}$. 5. $2\sqrt[4]{bc+5b}$.

6. $4\sqrt{c^2+12abc}$. 7. $\sqrt{b^3+c^3+1} - \sqrt{6bc}$.

8. $\sqrt[3]{b^2+10abc} + \sqrt[3]{c^3-3c+2b^2+a}$.

9. $\dfrac{a+b+c}{\sqrt{a}} - \dfrac{a^2+b^2-c+2}{\sqrt{a+c}}$.

10. $\dfrac{\sqrt{a^2+b^2+2b}}{a+b} - \dfrac{\sqrt{b+c-a}}{b}$.

When $a = 25$, $x = 16$, $y = z = 9$, find the values of

11. $\dfrac{3\sqrt{a} - 3\sqrt{x}}{\sqrt{y}}$. 12. $\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{a} - \sqrt{x}}$.

13. $\dfrac{\sqrt{a-x}}{\sqrt{x-y+2}} - \dfrac{\sqrt{xy}}{\sqrt{x}-1}$. 14. $\dfrac{a - \sqrt{2y+2z}+1}{y+z+\sqrt[4]{x}}$

When $x = 4$, $y = 2$, $z = 3$, prove that

15. $\dfrac{2\sqrt{x+y+z}}{x+y} - \dfrac{\sqrt{y+z-1}}{\sqrt{x}} = \dfrac{\sqrt{x}+\sqrt{6yz}}{\sqrt{x-z}} - xy.$

16. $\dfrac{x^2+y^2+z^2+y-x}{\sqrt[3]{xyz+z}} - \dfrac{\sqrt[3]{xyz+5xy}}{\sqrt[4]{xyz-2y^2}} - \sqrt{12x+1} = 0.$

17. $3\sqrt[3]{xy} + 6\sqrt[4]{x+y-2z} = \dfrac{yz\sqrt{x}}{\sqrt[3]{x^2-y^2-z^2+5}}$.

18. $\dfrac{\sqrt[5]{x-z}}{\sqrt[4]{z-y}} + \dfrac{z^3-yz^2}{x+y+z} = \dfrac{y^2z-\sqrt{x}}{\sqrt[3]{x^2z^2-xy^2-z}}$.

19. $z\sqrt[4]{y^2x} - y\sqrt[3]{xz^3-11x} = \dfrac{\sqrt[4]{x-y^2}}{\sqrt[5]{x^3-y^3+z^2}} - \sqrt[5]{4xy}.$

20. $\dfrac{10\sqrt[4]{y+z-x} + 11\sqrt[3]{z-y}}{x^2+y+z} = \dfrac{6\sqrt{x^2-5z}}{\sqrt{2x^2+y^2}}$.

BRACKETS.

12.—Brackets are used in pairs ( ), { }, and [ ], to include expressions of several terms, and to shew that they are to be subjected to some operation indicated outside the brackets.

Thus $(a + b)^2$ means that the value of $a + b$ is to be squared.

$a - (x + y)$ means that the value of $x + y$ is to be subtracted from the value of a.

$\sqrt[3]{(x + y + z)}$ means that the cube root of $x + y + z$ is to be found. A line called a *vinculum* is sometimes used instead of a pair of brackets. Thus

$$a - \overline{x + y} \text{ is the same as } a - (x + y).$$

13.—When expressions included in brackets are written together without any sign between the brackets, the multiplication of one expression by the other is signified. Thus $(a + b)(a - b)$ is the same as $(a + b) \times (a - b)$ and means that the value of $a + b$ is to be multiplied by the value of $a - b$.

When $a = 6$, $b = 4$, $c = 1$, find the values of

Ex. (i.) $a - b + c$ and $a - (b + c)$ respectively.

$$a - b + c = 6 - 4 + 1 = 3.$$
$$a - (b + c) = 6 - (4 + 1) = 6 - 5 = 1.$$

Ex. (ii.) $a^2 - (a - b)^2 + (a - b)(b - c)$.

$$= 36 - (6 - 4)^2 + (6 - 4)(4 - 1).$$
$$= 36 - 2^2 + 2 \times 3 = 36 - 4 + 6 = 38.$$

Ex. (iii.) $ab - \{a - (b - c)\} + [a + \{b + (a - c)\} - b]$.

Such an expression should be re-written, substituting numerical values first for the quantities contained in the inner brackets ( ), then for those within the next set { }, and lastly for those within the outer brackets [ ]. Then since $ab = 24$, $b - c = 3$, and $a - c = 5$, the expression becomes

$$24 - \{a - 3\} + [a + \{b + 5\} - b]$$
$$= 24 - 3 + [a + 9 - b]$$
$$= 24 - 3 + 11 = 32.$$

Examples V.

When $a = 1$, $b = 2$, $c = 3$, $d = 4$, and $x = 0$, find the values of

1. $(a - b) + c$. 2. $a - (b + c)$. 3. $(b + c)(c - b)$.
4. $\overline{a + b} \times \overline{c - a}$. 5. $\overline{d - x} \times \overline{c - a}$. 6. $(b - a)^2 + bc - (b + c)$.
7. $x(a + b) - c(d - a)$. 8. $\sqrt{(b + c - a)} + \sqrt{(b + c + d)}$.
9. $\sqrt{(b + c)(d - b) - a} - 3\sqrt{(c - b)(c - 2a)}$.
10. $a^2 + \{a - (b + c) + b^2\} - (a + b)$.
11. $a^2 - \{(b + c)^2 + 1\} - [a + \{b + (a + c)\}]$.
12. $\{(d - c)^2 + (b - a)^2\}^2 - (b + c)(a + x)$.

13. $\sqrt[3]{\{(d-x)(c-a)\}} - \sqrt[3]{\{(a+b)(b+c+d)\}}$.

14. $\dfrac{(a+c)(c+d)}{(a+bc)(c-a)} + \dfrac{(bc+a)(ad-c)}{a+b+d}$.

15. $12a - [\{b+(d-c)\}^2 - \{(x+a)(x+b)+1\}]$.

If $a=1$, $x=2$, $y=3$, $z=4=p$, prove

16. $(a+x+y+z)^2 = \dfrac{p^2(p+a)^2}{4}$.

17. $a^2x^2 + [(a+x)(z-p) + \{y-(x-\overline{x-a})\}] = 2y$.

18. $x^2y^3[\{a^2+(y^2-\overline{y+z})\} - (a+x)] = 0$.

19. $a^2 - \left(a + [y - \{z - (2y - \overline{a+z})\}]\right) = 0$.

20. $a^2[\{(p-\overline{a+y})^2 + a^3\}^2 - (y-x)^2] = y - \overline{a+x}$.

Find the value of the following, when $a=1$, $b=2$, and $x=0$.

21. $\dfrac{ax+b}{b} - \dfrac{a-bx}{a} + \dfrac{abx}{a-x+b}$.

22. $(\sqrt{2ab} + \sqrt[3]{b-a})^2 + 3\sqrt{ax+bx}$.

23. $4\sqrt{\left\{(\sqrt[3]{a} + \sqrt{ax})^3 + (a+b+x)^2 - a\right\}}$

24. $\left(\dfrac{b^2-a}{a-b}\right)^2 - \dfrac{\sqrt{2ab^3}}{b-a+x} + a - \{b+(a-x)\}$.

25. $\dfrac{b(2a-b)^2+ab}{\sqrt[3]{b(2a+b)}} - \dfrac{\sqrt{(x-a+b)^3}}{b^2-3a^2+x^2}$

26. $\sqrt{\{(a+b+x)(a+b-x)\}} - 3[a+2\{b-(a+\overline{b-a})\}]$.

27. $\left(\dfrac{a^3b^3}{10a-b} + \dfrac{a+b+x}{a+b-x}\right)^3 - \dfrac{a^4b^4-x^4}{(4a-b)^2}$.

28. $\dfrac{x^2-4}{x+6} - \dfrac{x-1}{2x-11} \times \dfrac{x^2-x-9}{x-1}$, when $x=10$.

29. $x-(x-[x-\{x-(x-\overline{x-1})\}])$, when $x=12$.

30. $(a-x)^a - (a+x)^x$, when $a=3$, $x=2$.

II.—ADDITION AND SUBTRACTION.

ADDITION.

14.—LIKE quantities are those which consist of the same letters and powers, and differ only in their numerical co-efficients.

Thus $2a$, $3a$, and a are like quantities.

So also are a^2b, $3a^2b$, $15a^2b$.

a^2 and a^3 are unlike, and so are ab^2 and $2a^2b$.

15.— Only *like* quantities can be really added together. We have seen that the operation of addition is indicated by the sign $+$: and when unlike quantities are to be added together, such as a and b, or a^2 and a^3, all that can be done is to write $a + b$ cr $a^2 + a^3$.

When, however, we have to deal with *like* quantities, we can combine their co-efficients.

Thus $2ax + 3ax + ax = 6ax$, and $3x - 4x + x = 0$.

16.—It is usual to write the quantities to be added in lines, putting like quantities under each other.

The difference between the sum of the positive and that of the negative co-efficients may then be easily found and the result set down with the sign of the larger.

(i.) Add together—

$a + b + c$; $3a - 2b + 4c$; $6a - b - c$; $-8a + 2b - 3c$; $2a - 10b - c$.

$$\begin{array}{r}
a + b + c \\
3a - 2b + 4c \\
6a - b - c \\
-8a + 2b - 3c \\
2a - 10b - c \\
\hline
4a - 10b
\end{array}$$

(ii.) Add together—

$$a^2-3ab+b^2;\ 2ab-b^3;\ a^2+3b^2;\ 2ab-b^3+c.$$

$$\begin{array}{l}
a^2-3ab+\ b^2 \\
\quad\ \ 2ab\quad\ \ -\ b^3 \\
a^2\qquad\ \ +3b^2 \\
\quad\ \ 2ab\quad\ \ -\ b^3+c \\
\hline
2a^2+\ ab+4b^2-2b^3+c
\end{array}$$

Examples VI.

Find the sum of

1. a^2+2a+1; $3a^2-a-2$; $4a^2+6a-3$; $-a^2-a-2$.
2. $a^2+b^2+3a+3b$; $2a^2+b^2-3a+2b$; $-a^2-b^2-6a-6b$; $3a^2+2b^2-a-b$; $5a^2-5b^2-2a+2b$.
3. $x-y+z$; $-3y-3z+y^2-z^2$; $-6x+5y-4z-6y^2-4z^2$; $z-y^2-z^2$; $7x+3y$; $y+z+y^2-2z^2$; $8x+5y-10z-10y^2+8z^2$.
4. $a^2+b^2-c^2+4d$; $3b^2-20c^2-4d+x$; $10a^2-4b^2+15c^2$; $-12a^2+6c^2-x+y$; a^2+x-y.
5. $7a-3b+8c-x+y$; $6a-b+c+6x-4y$; $-5a+2b-3c-5x+3y$; $-6a-4b+4c+4x-y$; $-a+6b-10c-4x+2y$.
6. $a+b+x$; $3a-2b-y$; $4a+b+z$; $b-6y$; $3a+2b+5y$.
7. $a^2+b^2+c^2-2ab$; b^2+c^2-3ac; $3a^2-2b^2-c^2+ab-2bc$; $-c^2-ab+cd$.
8. $2ac-3b^2+3x^2y+5z^2$; $4b^2-x^2y-6z^2$; $4ac+6b^2+x^2y-5z^2$; $ac-b^2+3z^2$; $4z^2-b^2-x^2y$.
9. $a^2-ab+ax+ay$; $2a^2-3ax$; $4ab-3ay$; $6ax-7az$; $-3a^2-4ax$; $2ay-5ab$.
10. $a^3-3a^2b-2ab^2+b^3$; $3a^3+3a^2b+ab^2$; a^2-b^3; $ab^2+b^3-3a^3$.
11. $2x^2-3xy+y^2-z$; $xy-y^2+2z$; $2y^2-3z+z^2$; $x^2+3xy-2y^2+2z$; z^2-xy.

12 $a^2b^2-2cd+f-g$; a^2+g; $-a^2b^2-f$; $2cd-a^2$.

13. $x^2y^2+y^2z^2-xyz+yz-z^3$; $2x^2y^2-2y^2z^2+3xyz-2yz+2z^3$; $-x^2y^2+y^2z^2-5xyz+4yz-3z^3$; $6x^2y^2+3xyz-5yz+4z^3$; $x^2y^2+6yz+6z^3$.

14. $x^2+4y^2+z^2-2$; y^2-3z^2+1 ; $-2x^2-2y^2+3$; $x^2-4y^2-z^2$; $y^2+z^2+a^2+2$.

15. Add together $x^2-3xy+y^2+x+y-1$; $2x^2+4xy-3y^2-2x-2y+3$; $3x^2-5xy-4y^2+3x+4y-2$; and $10xy+5y^2+x+y$; and find the numerical value of the sum, when $x=1$, $y=2$.

SUBTRACTION.

17.—Subtraction is indicated by the sign —. Thus, if a is to be subtracted from b we write $b-a$. This is all that can be done when the quantities are *unlike.* When, however, the quantities are *like*, the difference of their co-efficients is found. Thus, to subtract $4a$ from $7a$, we write $7a-4a$, and the result or difference, is expressed as $3a$.

18.—When the expressions under consideration are compound, the expression to be subtracted is generally written under the other, and the following is the rule for their treatment.

Change the signs of the lower line and proceed as in Addition.

Thus, if from $3a^2+2ab-3b^2$ we have to subtract $a^2-3ab-6b^2$ we write

$$\begin{array}{r} 3a^2+2ab-3b^2 \\ a^2-3ab-6b^2 \\ \hline 2a^2+5ab+3b^2 \end{array}$$

and $2a^2+5ab+3b^2$ is the result.

19.—The reason for changing + into — is obvious enough: but the reason for changing — into + may not seem so clear. The following may serve to illustrate it:—

If we have to subtract $-a$ from $2a$, the assertion is that the result will be $3a$.

The student will observe that $2a + a - a$ is still $2a$ (since $+a$ and $-a$ destroy each other). If from this expression we subtract or take away $-a$, the result will be $2a + a = 3a$.

20.—Quantities occurring in either line unlike any others are to be set down in the difference, their signs being treated by the general rule.

(i.) Find the difference between

$$2a^2 - 3ab + 2b^2, \text{ and } a^2 + ab - b^2.$$

$$\begin{array}{r} 2a^2 - 3ab + 2b^2 \\ a^2 + \ ab - \ b^2 \\ \hline a^2 - 4ab + 3b^2 \end{array}$$

(ii.) From $3ax - 2by + c$
Take $-2ax - 2by + d$

$$\begin{array}{rl} \text{From} & 3ax - 2by + c \\ \text{Take} & -2ax - 2by + d \\ \hline & 5ax \quad\ + c - d \end{array}$$

Examples VII.

Find the difference between

1. $2a + 3b - c + d$, and $a - b + 3c - d$.
2. $x^2 - 2xy + 3y^2 - 1$, and $-2x^2 - 2xy + y^2 + 2$.
3. $m^2 - n^2 + 3mn - 2$, and $m^2 + n^2 - 3mn - 2$.
4. $x^3 + 3x^2y + xy^2 - y^3$, and $2x^3 - 3x^2 + xy^2 - 3y^3$.
5. $1 - 2m^2 + 3mn - n^2$, and $m^2 - mn + n^2$.
6. $40x^3 - 50y^3 + 10x^2y - 10xy^2$, and $20x^3 + 30y^3 - 15xy$.
7. Subtract both $2a - 3b + 4c + d$, and $-a + 2b - c - d$, from $6a + 4b - 2c - d$.
8. Subtract $m^2 + mn - n^2 + 1$, and $-mn + 2n^2 + 3$, from $3m^2 + 5mn + n^2$.
9. Prove that the difference between $x^3 - y^3$ and $x^3 - xy + y^3$ will be 0, when $x = 2$, and $y = 1$.
10. From $a^2 - 3axy + 3x^2y^2 - 1$ subtract $a^2 - axy - 2x^2y^2 - 2$, and from the remainder subtract $-3axy + 4x^2y^2 - y^3 + 1$.

11. From $2a - 3b + c$ subtract $a + x - y + z$.

12. From $10a^4 - 7a^3 + 8a^2 - 6a + 2$ subtract $6a^4 - 15a^3 + 8a^2 - 6$.

THE USE OF BRACKETS IN ADDITION AND SUBTRACTION.

21.—When we write the sign $+$ before an expression within brackets, we mean that the expression is to be added to some other quantity. When we write the sign $-$ before any expression within brackets, we mean that the expression is to be subtracted from some other quantity. Thus, $a + (b - c)$ signifies that $b - c$ is to be added to a; and $a - (b - c)$ signifies that $b - c$ is to be subtracted from a.

22.—It will be seen that if we wish actually to perform the addition indicated by such an expression as $a + (b - c)$ we have simply to remove the brackets and collect terms where possible. $a + (b - c)$ therefore will become $a + b - c$.

23.—In performing the subtraction indicated in the expression $a - (b - c)$ we must change the signs of the quantity to be subtracted, that is of the quantity within the brackets, and the expression becomes $a - b + c$. In removing brackets from quantities, therefore, we must observe that when the sign $+$ is written before a bracket, the signs of the included expression remain unchanged; but *when the sign $-$ precedes a bracket, all the signs must be changed on removing the bracket.*

In simplifying expressions in brackets, the inner brackets should be first removed, then those next in order, rewriting the expression on each removal as directed in Ex. iii., page 10.

Ex. (i.) Simplify

$$\begin{aligned} &a^2 - \{a^2 - (2ab - a^2)\} + \{2ab + a^2 - (ab + a^2)\} \\ &= a^2 - \{a^2 - 2ab + a^2\} + \{2ab + a^2 - ab - a^2\} \\ &= a^2 - a^2 + 2ab - a^2 + 2ab + a^2 - ab - a^2 \\ &= 3ab - a^2. \end{aligned}$$

Ex. (ii.) Simplify

$$a - \{a + (x - y)\} - [a - \{x - (y - 3) - 4\} - 1] + (a + x + y).$$
$$= a - \{a + x - y\} - [a - \{x - y + 3 - 4\} - 1] + a + x + y$$
$$= a - a - x + y - [a - x + y - 3 + 4 - 1] + a + x + y$$
$$= a - a - x + y - a + x - y + 3 - 4 + 1 + a + x + y$$
$$= x + y.$$

After a little practice the student will be able to collect terms, in examples such as these, at the intermediate steps, thus saving the trouble of re-writing all the terms so often.

24.—When an expression such as $a - (-x)$ occurs, it means the subtraction of $-x$ from a, and the result is $a + x$.

$a + (-x)$, and $a - (+x)$ each $= a - x$.

EXAMPLES VIII.

Simplify—

1. $a^2 + (a^2 + b^2) - (a^2 - b^2)$.
2. $x^2 - (x^2 - y^2) - (y^2 - x^2) - \{x^2 - (x^2 - y^2)\}$.
3. $x + y - \{(x + y) - (x - 3y)\} - \{x - (x - y)\}$.
4. $a + \{b - x - (a - b - x)\} - \{a - x - (a + b)\}$.
5. $a - [a - \{a - (a + b)\}] - \{a - (a - b)\}$.
6. $3a - [2x + \{3y - (2 - x)\}] - \{a + 1 - (x - 2)\}$.
7. $3x - [x - \{(x + 1) - (x - 1) + 2\}] - [x - \{x - (x - 1)\}]$.
8. $3a^2b^2 - \{a^2b^2 + x - (a^2b^2 + x) + 2x - (a^2b^2 + x) - x\}$.
9. $a - (-a - m) + a + (-a + m)$.
10. $a - [a - \{a - (a - \overline{a - b})\}]$.
11. $x - \overline{x + y} - [x - \{x + y - (x - \overline{x - y})\}]$.
12. $10a - [12b - \{3a + (4b - \overline{a - b} - a) + 2b\} - 3a]$.
13. $3ax - (2by - (2ax + by)) + ax - (-by)$.
14. $1 - (1 - [1 - \{1 - (1 - \overline{1 - a})\}])$.
15. $3y - [\{a^2x - (ax^2 + y) - \overline{a^2x - y}\} - \{a^2x - (ax^2 - \overline{a^2x - y})\}]$.

16. $3a-(3a-[3a-\{3a-(3a-\overline{3a-1})\}])$.

17. $12-[\{6a-(7-\overline{a-5})\}-\{5a+(3-\overline{2-a})\}]$.

18. $x^2-[x^2-y^2-\{x^2-(y^2-z^2)\}]-x^2+(y^2-z^2)$.

19. $m^2-[n^2-\{n^2-(m^2-n^2)\}]-[-\{-(-2m^2)\}]$.

20. $1-a-(1-[a-\{1-(a-\overline{1-a}-1)-a\}-1]-a)$.

21. From $a^2-\{3ab-(b^2-a^2)\}$ subtract $b^2-\{a^2+ab-(a^2+b^2)\}$.

22. Shew that the sum of $x^2-\{x-(x^2-3x+1)\}$ and $x-(2x^2-3x-1)$ will be 2, whatever be the value of x.

23. Subtract $2a^2-[3ab+\{b^2-(a^2+b^2)\}]$ from the sum of a^2-ab+b^2, $2ab-b^2$, and $3a^2-ab-2b^2$.

24. Shew that the difference between m^3-m^2+3m and $2m+3m^2+3m^3$ is equal to the expression $m^3-[3m^2+2m-\{m^3-(m^2-m)\}]$.

III.—MULTIPLICATION AND DIVISION.

MULTIPLICATION.

25.—The quantity by which we multiply is called the MULTIPLIER: the quantity multiplied the MULTIPLICAND: and the result of the multiplication the PRODUCT.

26.—When two quantities are multiplied together, if their signs are the same (that is, either both + or both —), the sign of the product will be positive. If, however, the signs of the two quantities are not the same, but one is + and the other —, then the sign of the product will be negative.

Like signs produce +, unlike signs produce —.

27.—In multiplying, we first multiply the numerical coefficients, if any, and then the letters of the multiplier and multiplicand by writing them together.

Ex. (i.) $3ab \times 2cd = 6abcd.$

Ex. (ii.) $-7xy \times mz = -7mxyz.$

28.—To multiply powers of the same letter, we *add the indices.*

Thus $a^2 \times a^3 = a^{2+3} = a^5$.

For since $a^2 = a \times a$; and $a^3 = a \times a \times a$

Then $a^2 \times a^3 = a \times a \times a \times a \times a = a^5$.

Ex. (iii.) $3ax^2y^3 \times 2a^2x^2yz = 6a^3x^4y^4z.$

EXAMPLES IX.

Multiply—

1. ax by b.
2. $3ax$ by $4by$.
3. $9amn$ by $-bc$.
4. $12ayz$ by $3abc$.
5. x^2 by x^3.
6. $-2x^3$ by $3x^2y$.
7. $-4xyz$ by $-3x^2y^2z$.
8. $8abm^2n$ by $-2a^2bn$.

9. $-ax^2byz$ by $3x^3y^2z^2$.
10. $-adc$, $-a^2c$, and dc^2 together.
11. m^2n, $-m$, and $3n^2$ together.
12. $2axby$ by $-6a^2bxy^2$.
13. $-3a^5b^6c^4$ by $2a^4b^3c$.
14. $-15m^3n^3$ by $-16m^8n^2p$.
15. $11a^2b^2m^5$ by $20m^4$.
16. $19a^2b^2c^3$ by $20a^3bxy$.
17. $-14abd^3$ by $2a^2bcx^2$.
18. $100p^2q^2$ by $-2p^3q^4rs$.
19. $18a^2m^2n$ by $-9amn^2$.
20. $-5x^3yz$ by $6x^2y^5z^2$.

29.—When the multiplicand is a compound expression, the terms must be separately multiplied as above, and connected by their appropriate signs.

Ex. (i.) Multiply $a^2x - 2ax - x^2$ by $3ax$.

$$\begin{array}{l} a^2x - 2ax - x^2 \\ 3ax \\ \hline 3a^3x^2 - 6a^2x^2 - 3ax^3 \end{array}$$

Ex. (ii.) $-2n(3m^2 - 2mn + n^3) = -6m^2n + 4mn^2 - 2n^4$.

Examples X.

Multiply—

1. $x + y - z$ by a.
2. $m^2 - 2m + 3$ by m.
3. $x^3 + 2x^2 + x + 1$ by $-x$.
4. $-3m^2n + 2mn^2 + 2$ by mn.
5. $x^3 - 3x^2y + xy^2 - y^4$ by $2x^2y$.
6. $16m^4 - 3m^3x + 2m^2x^2 - x^3$ by $4mx^3$.
7. $3a^3 - 4a^2b + 2ab^2 - 3ab^3 + b^4$ by $-5a^2b^5$.
8. $1 - 2a^2 + a^3 - 4a^4 + a^5$ by $6a^3b$.
9. $8m^2n^3p - 7mnp^2 + 3mp^3 - p^4$ by $-n^3p^2$.
10. $1 - ab^3 + 3a^2b^3 - 7a^3b^2$ by $-7a^5b$.
11. $15a^3x - 17a^2x^2 + 19mx - 20m^3$ by -1.
12. $18abxyz - 3a^2bxz - 9a^3z^2 - 36z^3$ by $-3ayz$.
13. $a^4b^3 - a^3b^2c + a^2bc^2 - ac^3 + c^4$ by $3a^2bc^2$.
14. $-5a^5 + 4a^4xy - 3a^3x^2y^2 + 2a^2x^3y^3 - 1$ by $-2a^2y$.
15. $36m^3n^4 - 15m^2n^5 + 19mn^6 - 4m$ by $3m^2n$.
16. $11xyz - 9x^2 + 10y^2 - 13z^2 + xy^3$ by $12x^2y$.
17. $3a(a^4 - 2a^3b + 3a^2b^2 - 4ab^3 + b^4)$.
18. $-a^2x(a^3 - 6a^2x + 2a^2x^2 - a^3x^3)$.

19. $5m^2n^3(m^3 - 3m^2n + 4mn^2 - 11n^3 + 1)$.
20. $7xy^2z^3(1 - 2x + 3y - 4z + xyz)$.

30.—When the multiplier and multiplicand are both compound expressions, the product will be found by multiplying each term of the multiplicand by each term of the multiplier, and combining the terms of these partial products.

Ex. (i.) Multiply $a^2 - 3a + 2$ by $2a - 4$.

$$\begin{array}{llll}
a^2 - 3a + 2 & & & \\
2a - 4 & & & \\
\hline
2a^3 - 6a^2 + 4a & & = & \text{the product of } a^2 - 3a + 2 \text{ by } 2a. \\
\quad\; - 4a^2 + 12a - 8 & & = & \text{the product of } a^2 - 3a + 2 \text{ by } -4. \\
\hline
\end{array}$$

By add., $2a^3 - 10a^2 + 16a - 8$ = the product of given quantities.

It will be observed that the product of $a^2 - 3a + 2$ by the second term of the multiplier, -4, is set down under the first product one place to the right. This arrangement is adopted for convenience in adding.

Ex. (ii.) Find the product of $a^2 + x^2 - 2y$ and $a^2 - x^2 + 2y$.

$$\begin{array}{lllll}
a^2 & + x^2 & - 2y & & \\
a^2 & - x^2 & + 2y & & \\
\hline
a^4 & + a^2x^2 & - 2a^2y & & \\
 & - a^2x^2 & - x^4 & + 2x^2y & \\
 & & + 2a^2y & + 2x^2y & - 4y^2 \\
\hline
a^4 & & - x^4 & + 4x^2y & - 4y^2
\end{array}$$

Examples XI.

Multiply—

1. $x + 3$ by $x + 6$.
2. $2x - 9$ by $3x - 2$.
3. $5x + 1$ by $3x - 4$.
4. $2x^2 - 3$ by $3x^2 - 4$.
5. $a - 12$ by $a + 13$.
6. $4a^2 - 16$ by $a^2 + 3$.
7. $3x^3 - 2$ by $x^3 - 1$.
8. $a^2 - ab + b^2$ by $a + b$.
9. $3a^3 + 2a^2b^2 + b^3$ by $a - b$.
10. $a^3 - 3a^2 + 2$ by $a^3 + 3a^2 - 2$.

11. a^2-ab+b^2 by a^2+ab+b^2.
12. $x^2-3bx+2b^2$ by $x^2+3bx-2b^2$.
13. x^2+ax+a^2 by $x^2-3ax-a^2$.
14. $12x^3+13x^2y+6xy^2+y^3$ by $2x-3y$.
15. $x^4+2x^3+3x^2+x+2$ by $2x-4$.
16. $x^2+y^2+z^2-xy-xz-yz$ by $x+y+z$.
17. $x^2+y^2+1+xy-x+y$ by $x-y+1$.
18. $1+a+a^2+a^3$ by $1-a$.
19. $4a^2-6ab-8b^2$ by $3a^2-2ab+b^2$.
20. x^3+3x^2+2x+1 by x^3-3x^2+2x-1.
21. $x^2-2ax+2a^2$ by $x^3-2ax^2+a^2x-a^3$.
22. $a^2b^2-3abx+x^2$ by $ab+x$.
23. $ax+by+ab+xy$ by $ax+by-ab-xy$.
24. $x^2y^2-z^2$ by $x^4y^4+x^2y^2z^2+z^4$.
25. $a^3b^3-2a^2b^2x+3abx^2-b^3$ by a^2b^2+2abx.
26. $ab+2x+3y$ by $3ab+2x+y$.
27. $a^3+3a^2b-2ab^2+3b^3$ by $a^2-3ab+2b^2$.

Find the continued product of—

28. $x+y$, $x-y$, and x^2+y^2.
29. $a+2$, $a+3$, and $a+6$.
30. $a+1$, $a+2$, $a-3$, and $a-4$.

Multiply—

31. $\frac{1}{2}a^2-\frac{3}{4}a+1$ by $\frac{1}{3}a-\frac{1}{2}$.
32. $\frac{2}{3}x^3-\frac{4}{5}x^2y+\frac{1}{2}y^3$ by $\frac{3}{5}x-2y$.
33. $\frac{3}{4}m^2-mn+\frac{1}{2}n^2$ by $\frac{2}{3}m-\frac{3}{4}n$.
34. $\frac{3}{2}x^3-6x^2-4x+\frac{3}{4}$ by $\frac{2}{3}x^2+\frac{4}{3}x+\frac{1}{3}$.

Additional exercises in Multiplication may be formed by multiplying the quotients given as answers to Ex. XIV. by the divisors. Also, each of the above examples may be varied by changing the places of the multiplier and multiplicand, the products being, of course, the same as those already found.

DIVISION.

31.—The quantity to be divided is called the DIVIDEND; the quantity by which we divide is called the DIVISOR; and the result of the division is called the QUOTIENT.

32.—In division we have the product of two factors (the dividend) and one of the factors (the divisor) presented to us that we may find the other factor.

Thus in $\frac{ab}{b}$, we have to consider by what factor we must multiply the divisor b, that the product may be ab. This factor is evidently a, and that is the quotient, $\therefore \frac{ab}{b} = a$.

When the dividend and divisor are powers of the same letter, the division is performed by subtracting the index of the divisor from the index of the dividend, the remainder being the index of the quotient.

Thus, $x^3 \div x^2 = x^{3-2} = x$.

For since $x^3 = x \,.\, x \,.\, x$, and $x^2 = x \,.\, x$, then

$$\frac{x^3}{x^2} = \frac{x \,.\, x \,.\, x}{x \,.\, x} = x.$$

33.—When the dividend and divisor are both simple quantities, the dividend may be written over the divisor with a line indicating division between them; and, on the factors common to both being removed, those left will form the quotient.

When the factors of the dividend and divisor cannot be so treated, the division cannot be actually performed, but can only be indicated. Thus the quotient of x divided by y can only be represented as $\frac{x}{y}$.

Numerical co-efficients are treated by the ordinary arithmetical process. Thus $\frac{12xy}{6y} = 2x$.

With regard to signs, when the signs of the dividend and divisor are the same, the sign of the quotient will be $+$; when the signs of the dividend and divisor are different, the sign of the quotient will be —.

Ex. (i.) $\frac{x^5}{x^3} = x^2$. Ex. (iv.) $\frac{16xyz^2}{-xz} = -16yz$.

Ex. (ii.) $\frac{3a^2x}{3ax} = a$. Ex. (v.) $\frac{-2ab^2}{-2ab^2} = 1$.

Ex. (iii.) $\frac{-10a^6b^2x^3}{2a^3x^3} = -5a^3b^2$. Ex. (vi.) $\frac{7a^2mn}{-7a.} = -amn$.

Examples XII.

Divide—

1. a^2 by a.
2. a^2b^2 by ab.
3. a^3b^2c by abc.
4. $6a^4b^2c^2$ by $3a^2c^2$.
5. $10a^3b^3x$ by $5ab$.
6. $121x^2yz$ by $11z$.
7. $98m^2n^2$ by $49m$.
8. $100a^2x^2$ by $-20ax$.
9. $-272a^{12}b^{10}c^6$ by $-17a^{10}b^{10}c^5$.
10. $196a^7x^8y^{10}$ by $49a^5x^6y^9$.
11. $16a^2bc$ by $-8abc$.
12. $-52x^4y^6z^2$ by $26x^2y^3z^2$.

34.—When the dividend is a compound quantity, each of its terms must be divided in succession by the process shown above.

Ex. (i.) $$\frac{a^2x^2y + x^2y - xy^2z}{xy} = a^2x + x - yz.$$

Ex. (ii.) $$\frac{12ax^2y - 4xy^2 + 8x^3y^3z}{-4xy} = -3ax + y - 2x^2y^2z.$$

Examples XIII.

Divide—

1. $a^2x^2 + 3axy - 4axy^2$ by ax.
2. $m^2n^2x^3 - m^2n^2x^2 + mnx$ by mnx.

3. $xy^2 + xy^3 + xy^4 + xy^5$ by xy.
4. $a^3b - a^2b^2 + ab^3$ by ab.
5. $20m^2n + 30mn^2 + 40mn$ by $5mn$.
6. $108a^5b^6 - 84a^4b^5 + 60a^3b^4$ by $-12a^3b^3$.
7. $156x^5 + 143x^4 + 130x^3 + 117x^2$ by $-13x^2$.
8. $19ax^2 - 38ax^3 + 57ax^4 - 76ax^5$ by $-19ax^2$.
9. $x^2y^2z - x^2y^2z^2 + x^2y^2z^3 - x^2y^2z^4$ by $-x^2y^2z$.
10. $85m^2n^2 - 68m^2n^2x + 51m^2n^2x^2$ by $17m^2n^2$.

35.—When both dividend and divisor are compound quantities, the following is the method of proceeding.

Ex. (i.) Divide $6a^2 + 5ab - 6b^2$ by $2a + 3b$.

$$\begin{array}{r|l|l}
2a + 3b & 6a^2 + 5ab - 6b^2 & 3a - 2b. \\
 & \underline{6a^2 + 9ab} & \\
 & \quad - 4ab - 6b^2 & \\
 & \quad \underline{- 4ab - 6b^2} &
\end{array}$$

Here the divisor is set down in the space to the left, the dividend in the middle, and the quotient, when it is obtained, to the right. We first consider the first term of the dividend and the first term of the divisor, $6a^2$ and $2a$, as if they were the only quantities under consideration. On dividing $6a^2$ by $2a$, we get the quotient $3a$, which we set down in the space reserved for it to the right. We now multiply the *whole of the divisor* by this quantity $3a$, and set down the product, $6a^2 + 9ab$, under the dividend and subtract. We thus get the remainder $-4ab - 6b^2$, and on dividing its first term $-4ab$ by $2a$, we obtain $-2b$, the next term in the quotient. Multiplying the divisor by this quantity $-2b$, we get the product $-4ab - 6b^2$, and this being set down under and subtracted from the last remainder, we find that the operation is finished. The process may be seen carried to greater extent in the next example.

Ex. (ii.) Divide $10a^3 - 19a^2x + 14ax^2 - 12x^3$ by $2a - 3x$.

$$
\begin{array}{r|l}
2a - 3x & 10a^3 - 19a^2x + 14ax^2 - 12x^3 \,\lfloor\, 5a^2 - 2ax + 4x^2. \\
 & \underline{10a^3 - 15a^2x} \\
 & \quad - 4a^2x + 14ax^2 - 12x^3 \\
 & \quad \underline{- 4a^2x + \;\; 6ax^2 \qquad\quad} \\
 & \qquad\qquad 8ax^2 - 12x^3 \\
 & \qquad\qquad \underline{8ax^2 - 12x^3}
\end{array}
$$

36.—It may here be noticed that algebraical expressions are usually arranged in a particular way, and that the terms do not follow each other by chance. If we look at Ex. (ii.) given above, we find that only two letters occur in the dividend, a and x. The highest power of a is put in the first term, the next highest in the second term, and so on to the last term, in which a does not appear at all. Similarly x does not occur in the first term, but in the second term its lowest power appears, and the next higher in the next term, and so on to the last in which its highest power occurs. The expression is therefore said to be arranged according to descending powers of a, or according to ascending powers of x. This arrangement does not affect the value of the expression, but it is of the utmost importance in the operation of division; as unless both dividend and divisor are arranged according to ascending or descending powers of the letters involved, the process cannot be carried out, and the student must particularly notice this before beginning to work out any example.

37.—A term of any expression is said to be of as many *dimensions* as the sum of the indices of its factors. Thus na, ax, and b^2, are each of two dimensions; and $6a^2xy$, and x^3y are each of four dimensions. When all the terms of a compound expression are of the same dimensions, the expression is said to be *homogeneous*. Thus the dividend in Ex. (ii.) given above is a homogeneous expression, each term being of three dimensions. The dividend in Ex. (iii.) given below is also homogeneous, being of four dimensions, and the divisor and quotient each of two.

Ex. (iii.) Divide $9a^2b^2 + a^4 - 4b^4 - 6a^3b$ by $a^2 + 2b^2 - 3ab$.

Arrange both expressions by *descending* powers of a; and observe the same arrangement in setting down each remainder.

$$
\begin{array}{l|l|l}
a^2 - 3ab + 2b^2 & a^4 - 6a^3b + 9a^2b^2 - 4b^4 & a^2 - 3ab - 2b^2 \\
 & \underline{a^4 - 3a^3b + 2a^2b^2} & \\
 & \quad - 3a^3b + 7a^2b^2 - 4b^4 & \\
 & \quad \underline{- 3a^3b + 9a^2b^2 - 6ab^3} & \\
 & \qquad\quad - 2a^2b^2 + 6ab^3 - 4b^4 & \\
 & \qquad\quad \underline{- 2a^2b^2 + 6ab^3 - 4b^4} &
\end{array}
$$

Ex. (iv.) Divide $1 - x^6$ by $x^2 + x + 1$.

Arrange by *ascending* powers of x.

$$
\begin{array}{l|l|l}
1 + x + x^2 & 1 - x^6 & 1 - x + x^3 - x^4 \\
 & \underline{1 + x + x^2} & \\
 & \quad - x - x^2 - x^6 & \\
 & \quad \underline{- x - x^2 - x^3} & \\
 & \qquad\quad x^3 - x^6 & \\
 & \qquad\quad \underline{x^3 + x^4 + x^5} & \\
 & \qquad\qquad - x^4 - x^5 - x^6 & \\
 & \qquad\qquad \underline{- x^4 - x^5 - x^6} &
\end{array}
$$

EXAMPLES XIV.

Divide—

1. $a^2 + 3a + 2$ by $a + 1$.
2. $a^2 + a - 2$ by $a - 1$.
3. $a^2 - 4a + 4$ by $a - 2$.
4. $a^2 + 5a + 6$ by $a + 3$.
5. $a^2 - 2ab + b^2$ by $a - b$.
6. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$.

7. x^4+4 by x^2-2x+2.
8. x^4-1 by x^3+x^2+x+1.
9. $3a^3-5a^2+3a-1$ by $3a^2-2a+1$.
10. $2x^3-13x^2+17x-10$ by $x-5$.
11. $2x^4-5x^3y+5x^2y^2-xy^3-4y^4$ by $2x^2-3xy+4y^2$.
12. $6a^3-5a^2b+3ab^2+6b^3$ by $3a+2b$.
13. $a^5-a^4b-2a^3b+a^2b^3+2a^2b^2-2b^4$ by a^2-2b.
14. $a^3b^3-2abc^2+21c^3$ by $ab+3c$.
15. $4x^2y+x^3+5y^3+8xy^2$ by $x+y$.
16. $3a^3+2b^3-5ab^2-4a^2b$ by $a-2b$.
17. $9a^3x^3+18a^2x^2y-16axy^2-7y^3$ by $3ax+y$.
18. $m^3n^3-mnx^2-6x^3$ by $mn-2x$.
19. $2a^4+a^3x+3a^2y-a^2x^2-3axy-2y^2$ by $2a^2-ax-y$.
20. $6x^3y^3-21x^2y^2z+21xyz^2-6z^3$ by $2xy-z$.
21. $5a^4b^2-5a^5b+2a^6+3a^2b^4-5a^3b^3$ by a^2+b^2.
22. $4a^6-9a^4+12a^3-16a^2+8a-4$ by $2a^3+3a^2-2a+2$.
23. $x-3x^2+x^5+1$ by $1+x^2-2x$.
24. $1-3x+6x^2-7x^3+5x^4-2x^5$ by x^2-x+1.
25. $2x^3y^3z^2+x^2y^2z^3+x^4y^4z+2xyz^4$ by $x^2y^2+z^2$.
26. $a^5-4a^4b+10a^3b^2-8a^2b^3+ab^4+12b^5$ by $a^2-2ab+3b^2$.
27. $6a^6-13a^5b+59a^2b^4-47a^3b^3+35a^4b^2-40ab^5$ by $2a^3+5ab^2-3a^2b$.
28. $15a^3-14a^2b+24ab^2-7b^3$ by $3a-b$.
29. $x^5+x^4-4x^3-10x^2-9x-3$ by x^2+2x+1.
30. $7x^4-10ax^3+3a^2x^2-4a^3x+4a^4$ by $x^2-2ax+a^2$.

Divide—

31. $\frac{1}{4}x^3+\frac{3}{16}xy^2+\frac{1}{8}y^3$ by $\frac{1}{2}x+\frac{1}{4}y$.
32. $m^3-\frac{33}{10}m^2+\frac{16}{5}m-3$ by $\frac{6}{5}m-3$.
33. $\frac{1}{3}x^4-\frac{41}{24}x^3+2x^2-\frac{5}{2}x+2$ by $\frac{1}{2}x-2$.

Additional examples in Division may be formed by dividing the products given as answers to Ex. XI. by either the multiplier or multiplicand. Also, each example in the Greatest Common Measure will furnish two examples in Division by dividing each of the given quantities by the G. C. M.

THE USE AND INTERPRETATION OF SYMBOLS.

38.—We must not lose sight of the fact that the operations which have been thus far explained are not mere manipulations of letters; but that they have a real signification as regards numbers; and, as we have now learned the language of Algebra to a certain extent, we may give some examples of how it may be employed.

Ex. (i.) Write expressions to signify the sum, difference, product, and quotient, of any two numbers.

a and b may be taken to represent any two numbers.

$a + b$ = their sum; $a - b$ = their difference; ab = their product; $\frac{a}{b}$ = their quotient.

Ex. (ii.) A father is three times as old as his son. Write expressions which may be put for the son's age, the father's age, and also for the sum of the two.

x may represent the son's age, as it is not definitely stated; then $3x$ will represent the father's age, and $x + 3x$, or $4x$ will represent the sum of the two.

Ex. (iii.) Express by symbols and prove the truth of the following:—

Take any number and add 2 to it: multiply the sum by the same number increased by 3; from this product take the square of the number, and also 5 times the number; and the remainder will always be 6.

Let x represent *any* number.

By adding 2 to it we have $x + 2$.

Multiplying this by the number increased by 3, we have $(x+2)(x+3)$. From this take x^2 and $5x$, then we have

$$(x + 2)(x + 3) - x^2 - 5x.$$

By actual multiplication of $x+2$ by $x+3$, the expression becomes $x^2+5x+6-x^2-5x$.

Collecting terms, the expression becomes 6 as asserted.

Examples XV.

1. Write an expression to signify the sum of any two numbers, and one to signify their product.

2. If 6 be taken from y, represent the remainder.

3. What do you understand by $\dfrac{a+b}{a}$?

4. Write an expression to signify the sum of the squares of two numbers, and one to signify the square of the sum of the same two numbers.

5. Write a number greater than x by 10, and one 10 times as great as x.

6. If $x=5$, how would you express 15 by means of x?

7. I walk x miles in an hour. How far do I walk in y hours?

8. Prove that if you divide the difference of the cubes of any two numbers by the difference of the numbers, the quotient will be the sum of the squares of the numbers together with the product of the numbers.

9. A boy earns x shillings per day and a man three times as much. How much do they earn together in a week of 6 days?

10. The square of any number, diminished by 1, and divided by the number increased by 1, will give for result 1 less than the number itself. Prove the truth of this.

11. Write in words the operation expressed by $3a^2-2a+1$.

12. If $x=y=b$, write b^2 in terms of x and y.

13. What is the next whole number greater, and the next less, than x?

14. Write the square of the sum of a, b, and c.

15. Write the square root of the sum of x and y divided by the product of x and y.

16. Add to x three times the difference between y and z, and divide the sum by the product of x, y, and z.

17. If $x = y$, what is the value of $(a + b)(x - y)$?

18. Write the square of n subtracted from the square of m, and 1 added to the remainder, which is then divided by the sum of m, n, and 2.

19. From x take y; from y take z; then multiply the remainders together.

20. Write the product of a and b taken 19 times.

21. The population of a certain parish at the beginning of a year was m. There were x births and y deaths during the year. One person removed from the parish, and none moved into it. State in Algebraic symbols the state of the population at the end of the year.

22. A cistern has x gallons drawn from it by one pipe and three times as much let into it by another in an hour. How much water will then be in the cistern when both have been running y hours?

23. Prove that if you subtract 18 from twice the square of any number, and divide the remainder by the number increased by 3, the result will be twice the original number less 6.

24. A boy who is x years old has a brother 3 years older and one 2 years younger. Express in symbols the age of each, and the sum of their ages.

25. What is the value of $(a + b) \div (a + x)$ when $x = b$?

26. Prove that the product of $a + b - x$, and $a + b + x$ is zero, when x is equal to the sum of a and b.

IV.—GENERAL RESULTS IN MULTIPLICATION AND DIVISION.

FACTORS AND CO-EFFICIENTS.

39.—When a compound expression contains some factor common to all its terms, it is sometimes convenient to express this by means of brackets. Thus in the expression $ab - ac + ay$, we observe that a is a factor of each term, and we may, therefore, write the expression $a(b - c + y)$.

This is sometimes called *collecting* the co-efficient of a, or *the resolution of the expression into its component factors.*

Ex. (i.) $ab + a = a(b + 1)$.

Ex. (ii.) $27ax^2 + 9x^3 = 9x^2(3a + x)$.

Examples XVI.

Resolve the following expressions into factors—

1. $ax + ay$.
2. $5ab^2 - 10b$.
3. $3ax^2 - 9axy + 27a^2x$.
4. $8a^3b - 16a^2b^2 + 24ab^3$.
5. $x^5 + x^4y + x^3z + x^2$.
6. $ax^4 - bx^3 + cx^2 + dx$.
7. $55a^5x^4y - 33a^4x^3z + 11a^3x^2$.
8. $14a^3b - 42a^4b^2 + 70a^5b^3$.
9. $6a^2bx^2 - 18a^2by^2 + 12a^2bz^2$.
10. $a^2b + a^3c + a^4d + a^5x$.
11. $a^2x^2 - 2ax^3 + x^4$.
12. $2m^2n^2 + 4mn^3 + 2n^4$.
13. $30a^3 + 45a^2b + 60ab^2$.
14. $-a^2x^2 - ax^3 + x^4$.
15. $-16x^3 + 32x^2 - 64x$.
16. $-x^4 - ax^3 - a^2x^2 - a^3x$.

Conversely, remove the brackets from the following expressions—

Ex. (iii.) $a^3 - a\{a - a(a - b)\} + a^2(b - 1)$.

$= a^3 - a\{a - a^2 + ab\} + a^2b - a^2$.

$= a^3 - a^2 + a^3 - a^2b + a^2b - a^2 = 2a^3 - 2a^2$.

17. $ab(a^2-3ab+b^2)$.
18. $a^2(a^2+ab+b^2)$.
19. $16x^2(1-2x+x^2)$.
20. $-5y^2(x^2-3xy+2y^2)$.
21. $3m^3-2m(m^2-mn+n^2)$.
22. $n^4-n^2(n^2-n-1)$.
23. $mn\{m^2-mn(m+n)\}$.
24. $a(a+b)-a^2+b(a-1)$.
25. $a\{a-b(a-1)\}+b(a+b)$.
26. $a-2\{a+2(b-a)\}-3\{(a-b)+(2a-b)\}$.
27. $a(b-1)-[a-\{2(b-a)-2(a+b)\}]$.
28. $a-[5x-\{a-3(x-y)-3y-(a-2x)\}]$.
29. $ax(a+1)-a[x-\{a(x-1)-ax\}]-a^2(x-1)$.
30. $m^2[1-\{1-(1-n)\}]-m\{m+n(n-1)\}+mn(m+n)$.

(The following, to the end of Exercise XVII., may be omitted on a First Reading.)

40.—This process of collecting co-efficients is often employed to shorten or condense the operations of Addition, Subtraction, Multiplication, and Division.

If we have to combine the terms $x+2x-3x+4x$, these will give as result $(1+2-3+4)x=4x$.

So also $ax-bx+cx-x$ may be written $(a-b+c-1)x$. Also $(a+b)x-(a-2b)x=(a+b-a+2b)x=3bx$, where the co-efficients $(a+b)$ and $-(a-2b)$ are combined in the ordinary way.

Ex. (i.) Add together ax^3+px^3+qx, bx^3-x^2-rx, cx^3+mx^2-sx, and $-(b+c)x^3-2px^2+2rx$.

$$\begin{array}{r}
ax^3+\ px^2+\ qx.\\
bx^3-\ \ x^2-\ rx.\\
cx^3+mx^2-\ sx.\\
-(b+c)x^3-2px^2+2rx.\\
\hline
ax^3+(m-p-1)x^2+(q+r-s)x.
\end{array}$$

Ex. (ii.) Subtract $(a-m)x^3+cx^2+nx$ from ax^3+bx^2-2x.

Co-efficient of $x^3=a-(a-m)=m$.

Co-efficient of $x^2=b-c$.

Co-efficient of $x = -2 - n$.

$\therefore$ the result will be $mx^3 + (b - c)x^2 - (n + 2)x$.

Ex. (iii.) Find the continued product of $x + a$, $x - b$, and $x + c$.

$$\begin{array}{l} x + a \\ \underline{x - b} \\ x^2 + ax \\ \underline{\quad - bx - ab} \\ x^2 + (a - b)x - ab \\ \underline{x + c \qquad\qquad\quad} \\ x^3 + (a - b)x^2 - abx \\ \underline{\qquad\quad + cx^2 + (ac - bc)x - abc} \\ x^3 + (a - b + c)x^2 - (ab - ac + bc)x - abc. \end{array}$$

Notice in the second line of the last multiplication how $(a - b)x$ is multiplied by c, producing $(ac - bc)x$.

Ex. (iv.) Divide $x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$ by $x + a$.

$$\begin{array}{l} x + a \,)\, x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc \,(\, x^2 + (b + c)x + bc. \\ \qquad\quad \underline{x^3 + ax^2 \qquad\qquad\qquad\qquad\qquad\qquad} \\ \qquad\qquad (b + c)x^2 + (ab + ac + bc)x \\ \qquad\qquad \underline{(b + c)x^2 + (ab + ac)x \qquad} \\ \qquad\qquad\qquad\qquad\qquad bcx + abc \\ \qquad\qquad\qquad\qquad\qquad \underline{bcx + abc} \end{array}$$

EXAMPLES XVII.

Add together—

1. $ax^2 + mx$, $bx^2 - nx$, $-cx^2 + 2mx$.
2. $ax^2 + by^2 + cz$, $bx^2 + cy^2 - az$, $cx^2 - ay^2 + bz$.

3. $x^3 - x^2 + x$, $(a-b)x^3 - (m+n)x^2 + px$, $(a+b)x^3 + (m-n)x^2 - qx$.
4. $(a+b)x^2 - (a-c)xy + (b-c)y^2$, $(a-b)x^2 + (2a-c)xy + (b+c)y^2$.
5. $(x+y)a^2 - (c+x)ab + (y-c)b^2$, $(z-x)a^2 + (x+y)ab + (z-y)b^2$, $-a^2 + 2acb - b^2$.

---

6. From $ax^3 - bx^2 + ax$, subtract $bx^3 + cx^2 - ax$.
7. From $(x-y)a^2 + 2(x-z)ab - (c-x)b^2$ subtract $2(x+y)a^2 + (x+z)ab - 2(c+x)b^2$.
8. From $2(m+n)x^3 - 3(a+b)x^2 - (a-b)x$ subtract $(m-n)x^3 - 2(b-a)x^2 + (a+c)x$.

Multiply—

9. $a+m$ by $a+n$.
10. $a+x$ by $a-y$.
11. $x^2+bx+bc$ by $x+a$.
12. $x^2-bx-bc$ by $x+a$.
13. $x^2-(a+b)x+ab$ by $x+c$.
14. $x^2+(a-b)x-ab$ by $x-c$.
15. m^2-mn+p by $m-q$.
16. x^2+ax+y by $x+z$.
17. Find the continued product of $x+a$, $x+b$, and $x+c$.

Divide—

18. $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$ by $x-c$.
19. $a^3+(x+y-z)a^2+(xy-xz-yz)a-xyz$ by $a-z$.
20. $x^3-3xyz+y^3+z^3$ by $x+y+z$.
21. $x^2(y+z)+y^2(x-z)+z^2(x-y)+xyz$ by $x+y+z$.
22. $m^3+m^2(n-p-q)-m(np+nq-pq)+npq$ by $m-p$.
23. $a^3-x^3-y^3-3axy$ by $a-b-c$.
24. $m^4-(a^2-b-c)m^2-(ab-ac)m+bc$ by m^2-am+c.

THE SQUARE OF A BINOMIAL.

41.—When we can express some general result of an operation in a convenient and easily remembered form, we call this result a FORMULA, and use it as a pattern for the expression of all quantities under similar circumstances. We shall now proceed to investigate a few of the simplest formulæ resulting from Multiplication and Division. The first of these relates to THE SQUARE OF A BINOMIAL.

42.—A BINOMIAL is a quantity which consists of two terms: as $x+y$, or $3a^2b-2ax$. A *trinomial* consists of three terms, and a *multinomial* of more than three.

We know by the ordinary process of multiplication that

$$(a+x)^2 = a^2+2ax+x^2,$$

$$\text{and } (a-x)^2 = a^2 - 2ax + x^2$$

Writing these results together, with the double sign $\pm$ (to be read *plus or minus*), we have the formula

$$(a \pm x)^2 = a^2 \pm 2ax + x^2;$$

i.e., the square of a binomial is equal to the square of the first term, plus or minus twice the product of the terms, plus the square of the second. Or, *The square of a binomial is equal to the sum of the squares of the terms with twice their product.*

Ex. (i.) $(a-y)^2 = a^2 - 2ay + y^2$.

(ii.) $(x+1)^2 = x^2 - 2x + 1$.

(iii.) $(1-a)^2 = 1 - 2a + a^2$.

(iv.) $(3a^2b+2x)^2 = 9a^4b^2 + 12a^2bx + 4x^2$.

EXAMPLES XVIII.

Write the following in their expanded form:—

1. $(x+a)^2$.
2. $(x-a)^2$.
3. $(a+1)^2$.
4. $(a-1)^2$.
5. $(1-x)^2$.
6. $(1-2x)^2$.
7. $(1+2x)^2$.
8. $(m-n)^2$.
9. $(y+x)^2$.
10. $(xy+a)^2$.
11. $(xy-a)^2$.
12. $(xy+1)^2$.

13. $(xy-1)^2$.
14. $(x^2-1)^2$.
15. $(2a-2x)^2$.
16. $(3x^2-1)^2$.
17. $(4x-5y^2)^2$.
18. $(3a^2-2b)^2$.
19. $(4m-n^2)^2$.
20. $(xy-ab)^2$.
21. $(1-5a^2)^2$.
22. $(1+4ab)^2$.
23. $(a+2b)^2$.
24. $(3mn-1)^2$.
25. $(1-2ax^2)^2$.
26. $(6x-a)^2$.
27. $(x^2-y^2)^2$.
28. $(3a^2-5)^2$.
29. $(3ax^2-1)^2$.
30. $(3a+2x)^2$.

Write the following in their contracted form—

31. $m^2+2mn+n^2$.
32. $p^2-2pq+q^2$.
33. $1-2a+a^2$.
34. $1+2r+r^2$.
35. $4-8m+4m^2$.
36. $9a^2+6a+1$.
37. $9a^2-12ab+4b^2$.
38. $a^2b^2-2abx+x^2$.
39. $1-6a^3+9a^6$.
40. $81a^4-18a^2b+b^2$.

As an additional exercise, the student may be required to state at sight the contracted forms of the answers to the first part of this exercise, 1 to 30.

Additional Examples for Oral Practice—

(a.)

1. $(m+n)^2$.
2. $(m+1)^2$.
3. $(1-a)^2$.
4. $(2a-3)^2$.
5. $(ab+x)^2$.
6. $(2x^2y+3)^2$.
7. $(a^3b^2x+ab^2x^3)^2$.
8. $(pm^2-4)^2$.
9. $(2m^2-3mn)^2$.
10. $(3m-2n)^2$.
11. $(10m-1)^2$.
12. $(1+2a)^2$.
13. $(3-4a)^2$.
14. $(3x^2-1)^2$.
15. $(10a-3)^2$.
16. $(m^6-1)^2$.
17. $(x-5yz)^2$.
18. $(3a^2b^2-1)^2$.
19. $(5a^2x-y^3)^2$.
20. $(7a^5+x^2y^3)^2$.
21. $(6a^3-7b^3)^2$.

(b.)

1. $(3a^3-5x^5)^2$.
2. $(4a^6+1)^2$.
3. $(x^3+y^3)^2$.
4. $(x^3-a^2)^2$.
5. $(x+2a)^2$.
6. $(a^3b^3+x^3y^3)^2$.
7. $(2a^7+1)^2$.
8. $(1-4a^3)^2$.
9. $(x^3-y^3)^2$.
10. $(x^2+a^2)^2$.
11. $(3x-2a)^2$.
12. $(a^2b^2-x^2y^2)^2$.
13. $(x^2-7)^2$.
14. $(mn^2-5)^2$.
15. $(x^2y^3+8)^2$.
16. $(m^3-7mn)^2$.
17. $(abx-2a)^2$.
18. $(5ax-6y)^2$.
19. $(d^2-3abc)^2$.
20. $(8m^3-7n^3)^2$.
21. $(3m^3-4n^4)^2$.

PRODUCT OF THE SUM AND DIFFERENCE OF TWO QUANTITIES.

43.—By actual multiplication we observe that the product of $a+x$ and $a-x$ (which are respectively the sum and difference of a and x) is a^2-x^2; we thus obtain the formula

$$(a+x)(a-x) = a^2-x^2.$$

i.e., The product of the sum and difference of any two quantities is equal to the difference of their squares.

Ex. (i.) $(x+y)(x-y) = x^2-y^2$.

(ii.) $(a+1)(a-1) = a^2-1$.

(iii.) $(m+n)(m-n)(m^2+n^2) = (m^2-n^2)(m^2+n^2) = m^4-n^4$.

Examples XIX.

Write the following products—

1. $(a+b)(a-b)$.
2. $(b+c)(b-c)$.
3. $(x+y)(x-y)$.
4. $(x+1)(x-1)$.
5. $(x+2)(x-2)$.
6. $(x-3)(x+3)$.
7. $(1-a)(1+a)$.
8. $(2-a)(2+a)$.
9. $(ab+c)(ab-c)$.
10. $(x^2+y)(x^2-y)$.
11. $(x-y^2)(x+y^2)$.
12. $(x^3+y)(x^3-y)$.
13. $(x^3+y^3)(x^3-y^3)$.
14. $(x^2+y^2)(x^2-y^2)$.
15. $(ab^2+1)(ab^2-1)$.
16. $(7x+1)(7x-1)$.
17. $(6m^2-2)(6m^2+2)$.
18. $(pm+q)(pm-q)$.
19. $(3m-2n)(3m+2n)$.
20. $(a+2xy)(a-2xy)$.
21. $(x+y)(x-y)(x^2+y^2)$.
22. $(m+n)(m-n)(m^2+n^2)$.
23. $(a+1)(a-1)(a^2+1)$.
24. $(a+2)(a-2)(a^2+4)$.
25. $(1+x)(1-x)(1+x^2)$.
26. $(x+3)(x-3)(x^2+9)$.
27. $(2-x)(2+x)(4+x^2)$.
28. $(1-3a)(1+3a)(9a^2+1)$.
29. $(4a^2+1)(2a+1)(2a-1)$.
30. $(x^3-1)(x^3+1)(x^6+1)$.

An additional exercise may be formed by simplifying out of brackets the expressions given as answers to Exercise XX.

44.—By means of this formula we can perform the converse operation, that is, resolve THE DIFFERENCE OF TWO SQUARES into factors.

Ex. (i.) $a^2 - x^2 = (a + x)(a - x)$.

(ii.) $a^2 - 1 = (a + 1)(a - 1)$.

(iii.) $4a^6 - 9 = (2a^3 - 3)(2a^3 + 3)$.

(iv.) $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y)$.

(v.) $x^3 - xy^2 = x(x^2 - y^2) = x(x + y)(x - y)$.

EXAMPLES XX.

Resolve the following expressions into factors—

1. $a^2 - b^2$.
2. $x^2 - y^2$.
3. $m^2 - 1$.
4. $p^2 - q^2$.
5. $x^2 - 4$.
6. $1 - a^2$.
7. $a^2 - 9$.
8. $16 - b^6$.
9. $x^6 - y^2$.
10. $m^2 - 100$.
11. $25 - 9x^2$.
12. $a^6 - x^6$.
13. $1 - x^6$.
14. $1 - 4a^2$.
15. $4 - 9x^2$.
16. $m^2n^2 - a^2$.
17. $a^2x^2 - y^2z^2$.
18. $x^4y^2 - 1$.
19. $m^2 - n^2$.
20. $p^2 - q^2$.
21. $a^2 - 9$.
22. $9 - a^2$.
23. $4 - x^2$.
24. $25 - a^2$.
25. $ax^2 - a$.
26. $a^2 - 4$.
27. $x - a^2x$.
28. $x^4 - y^4$.
29. $a^4 - 1$.
30. $1 - a^4$.
31. $x^8 - 1$.
32. $x^4 - 4$.
33. $4 - x^4$.
34. $x^6 - 9$.
35. $16 - x^4$.
36. $x^8 - a^8$.
37. $a^3 - ax^2$.
38. $a^2x - x^3$.
39. $ax^2 - a^3$.
40. $a^2x^2 - a^2$.
41. $x - x^3$.
42. $ax^4 - a^5$.

Simplify

43. $(a - b)^2 - (a + b)(a - b) + 2b(a - b)$.
44. $x^2 - y^2 - z^2$, where $x = 1 - a$, $y = 1 + a$, $z = a - 1$.
45. $4a - (a + 1)(a - 1) + (a - 2)^2$.
46. $(x - y)^2(x + y)^2 - x^2(x^2 - 2y^2)$.
47. $(x + y)^2 - 2y(x + y) - (x + y)(x - y)$.

THE MULTIPLICATION OF BINOMIALS.

45.—By multiplication we know that $(x+a)(x+y) = x^2+ax+xy+ay = x^2+x(a+y)+ay$. Now if, instead of a and y, we write any values of these letters, such as 2 and 3, the formula will become $(x+2)(x+3)=x^2+5x+6$.

Observe that the co-efficient of x in the product is the *sum* of the numbers, and the third term is their *product*. We can thus at once write such expressions as

Ex. (i.) $(a+1)(a+2) = a^2+3a+2$.

(ii.) $(x-1)(x-2) = x^2-3x+2$.

(iii.) $(a+3)(a-2) = a^2+a-6$.

Examples XXI.

Write the following products—

1. $(x+3)(x+4)$.
2. $(m+2)(m-1)$.
3. $(m+5)(m-4)$.
4. $(1+a)(2+a)$.
5. $(a-5)(a+4)$.
6. $(x^2+2)(x^2-3)$.
7. $(x^3+1)(x^3-10)$.
8. $(m+100)(m-10)$.
9. $(a-9)(a+4)$.
10. $(a+6)(a-3)$.
11. $(a^3-5)(a^3+7)$.
12. $(x-8)(x-2)$.
13. $(x-6)(x-5)$.
14. $(a-11)(a+7)$.
15. $(m^5-4)(m^5+3)$.
16. $(ab+3)(ab-4)$.
17. $(ax^2+2)(ax^2-1)$.
18. $(b-5)(b+4)$.
19. $(a^2+6)(a+2)(a-2)$.
20. $(a+4)(a-4)(a^2+10)$.
21. $(x^2+5)(x^2-2)$.
22. $(x^2-3)(x^2-4)$.
23. $(x^3+4)(x^3-6)$.
24. $(m^2-2)(m^2+10)$.
25. $(m^3+6)(m^3-5)$.
26. $(a^3+7)(a^3+6)$.
27. $(ax-11)(ax+12)$.
28. $(xy-9)(xy+8)$.
29. $(ab+3)(ab-2)$.
30. $(a^2x-2)(a^2x+3)$.

An additional exercise may be formed by multiplying out of brackets the expressions given as answers to the next Exercise.

THE RESOLUTION OF TRINOMIALS INTO BINOMIAL FACTORS.

46.—This is the converse of the operation performed by the last formula.

If we resolve $a^2 + 7a + 12$ into factors, it is evident that the first term of each factor will be a. The remaining terms of the two factors must be such that their *sum* will be 7 (the co-efficient of the middle term of the expression under consideration) and their *product* 12. The two numbers whose sum is 7, and product 12, are 3 and 4. The two factors, therefore, are $(a+3)$ and $(a+4)$, $\therefore a^2 + 7a + 12 = (a+3)(a+4)$.

Again in resolving $a^2 - a - 20$ into factors, we must find two numbers whose sum is -1, and whose product is -20. These are plainly 4 and -5, $\therefore a^2 - a - 20 = (a+4)(a-5)$.

In resolving such expressions, therefore, we must break up the last term into two factors whose sum will be the co-efficient of the middle term.

Ex. (i.) Resolve $x^2 + 6x + 8$ into factors.

Here we must find two factors of 8 whose sum will be 6. These are plainly 4 and 2.

$$\therefore x^2 + 6x + 8 = (x+4)(x+2).$$

Ex. (ii.) Resolve $x^2 - 5x + 6$ into factors.

We require the two factors of 6 whose sum is -5. These are -3 and -2.

$$\therefore x^2 - 5x + 6 = (x-3)(x-2).$$

Ex. (iii.) Resolve $x^2 - 2x - 8$ into factors.

Here 8 must be resolved into factors, which when combined, will give -2. These are -4 and $+2$.

$$\therefore x^2 - 2x - 8 = (x-4)(x+2).$$

☞ When, as in this case, the factors are of *different* signs, the sign of the middle term of the trinomial will indicate what the sign of the greater of the two factors should be.

EXAMPLES XXII.

Resolve into factors—

1. a^2+3a+2.
2. a^2-3a+2.
3. a^2+a-2.
4. a^2-a-2.
5. $x^2+3x-28$.
6. $x^2-2x-15$.
7. $a^2x^2+6ax+8$.
8. $x^2+9x+14$.
9. $x^2+5x-14$.
10. $x^2-9x+14$.
11. a^2x^2+ax-2.
12. a^2x^2-ax-2.
13. $a^2+15a+56$.
14. a^2+5a+6.
15. x^2-x-30.
16. $x^2-7x+12$.
17. $a^2-10a+16$.
18. $a^2+2a-24$.
19. x^4+x^2-2.
20. x^4+2x^2-24.
21. x^6-6x^3-16.
22. x^2+3a-4.
23. $x^2-3x-18$.
24. x^6-2x^3-15.
25. $a^2+10a+24$.
26. $a^2-2a-24$.
27. x^2+6x-7.
28. $p^2+5p-24$.
29. $x^2+x-9900$.
30. $x^2-2x-143$.

As additional exercises, the student may be required to resolve into factors at sight the answers to Ex. XXI.

EXPRESSIONS DIVISIBLE BY $x+y$ AND $x-y$.

47.—It is important to note that certain expressions are divisible by $x+y$, and $x-y$, as follows:—

(i.) Those divisible by $x+y$.

$\dfrac{x+y}{x+y}=1$. $x-y$ is not divisible.

x^2+y^2 is not divisible. $\dfrac{x^2-y^2}{x+y}=x-y$.

$\dfrac{x^3+y^3}{x+y}=x^2-xy+y^2$. x^3-y^3 is not divisible.

$x^4 + y^4$ is not divisible. $\quad \dfrac{x^4 - y^4}{x + y} = x^3 - x^2y + xy^2 - y^3.$

$\dfrac{x^5 + y^5}{x + y} = x^4 - x^3y + x^2y^2 - xy^3 + y^4.$ $\quad x^5 - y^5$ is not divisible.

48.—Notice that the examples to the left are the *sum* of the first, second, third, fourth, &c., powers of x and y; and those on the right the *difference* of the same powers.

From this we learn that *the sum of the odd powers, and the difference of the even powers of x and y are divisible by $x + y$; and the signs of the quotient are alternately positive and negative.*

49.— (ii.) Expressions divisible by $x - y$.

It will be found on trial that $x - y$ will not divide the *sum* of any powers of x and y. We, therefore, need consider only the difference of the powers.

$$\frac{x - y}{x - y} = 1.$$

$$\frac{x^2 - y^2}{x - y} = x + y.$$

$$\frac{x^3 - y^3}{x - y} = x^2 + xy + y^2.$$

$$\frac{x^4 - y^4}{x - y} = x^3 + x^2y + xy^2 + y^3.$$

And so on.

From this we learn that *the difference of all powers of x and y is divisible by $x - y$, and the signs of the quotient are all positive.*

The student should commit these formulæ to memory.

Ex. (i.) Resolve $a^3 + b^3$ into factors.

Since it is divisible by $a + b$,

$$a^3 + b^3 = (a^2 - ab + b^2)(a + b).$$

Ex. (ii.) Resolve $a^6 - b^6$ into factors.

$$a^6 - b^6 = (a^3 - b^3)(a^3 + b^3).$$

But $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$,

And $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$,

$$\therefore a^6 - b^6 = (a+b)(a^2 - ab + b^2)(a-b)(a^2 + ab + b^2).$$

Ex. (iii.) Resolve $a^3 - 27$ into factors.

$$a^3 - 27 = a^3 - (3)^3 = (a-3)(a^2 + 3a + 9).$$

These formulæ may also be employed conversely to obtain such products as the following :—

Ex. (iv.) $(x+y)(x^2 - xy + y^2) = x^3 + y^3$.

Ex. (v.) $(x+y)(x^3 - x^2y + xy^2 - y^3) = x^4 - y^4$.

Ex. (vi.) $(x-y)(x^2 + xy + y^2) = x^3 - y^3$.

Examples XXIII.

Write the following quotients without going through the process of division :—

Divide by $a + b$.	Divide by $a - b$.	Divide by $a + 1$.
1. $a^2 - b^2$.	7. $a^2 - b^2$.	13. $a^2 - 1$.
2. $a^3 + b^3$.	8. $a^3 - b^3$.	14. $a^3 + 1$.
3. $a^4 - b^4$.	9. $a^4 - b^4$.	15. $a^4 - 1$.
4. $a^5 + b^5$.	10. $a^5 - b^5$.	16. $a^5 + 1$.
5. $a^6 - b^6$.	11. $a^6 - b^6$.	17. $a^6 - 1$.
6. $a^7 + b^7$.	12. $a^7 - b^7$.	18. $a^7 + 1$.

Divide by $a - 1$.

19. $a^2 - 1$.	21. $a^4 - 1$.	23. $a^6 - 1$.
20. $a^3 - 1$.	22. $a^5 - 1$.	24. $a^7 - 1$.

Resolve into factors :—

25. $a^3 - b^3$.	32. $8 - x^3$.	39. $8x^3 + 27a^3$.
26. $a^3 + b^3$.	33. $1 + m^3$.	40. $1000a^3 + 1$.
27. $a^3 - 1$.	34. $m^5 + n^5$.	41. $x^6 - a^6$.
28. $1 - a^3$.	35. $m^5 - 1$.	42. $x^6 - 1$.
29. $x^3 + 1$.	36. $1 - m^5$.	43. $a^3 - 125$.
30. $x^3 + 8$.	37. $a^3 - 27$.	44. $1 - a^6$.
31. $x^3 - 8$.	38. $27 + x^3$.	45. $a^3 + 125$.

Write the following products :—

46. $(a+b)(a^2-ab+b^2)$.	55. $(a+2)(a^2-2a+4)$.
47. $(a+b)(a^3-a^2b+ab^2-b^3)$.	56. $(a+2)(a^3-2a^2+4a-8)$.
48. $(a-b)(a^2+ab+b^2)$.	57. $(a-2)(a^2+2a+4)$.
49. $(a-b)(a^3+a^2b+ab^2+b^3)$.	58. $(x-3)(x^2+3x+9)$.
50. $(a+1)(a^2-a+1)$.	59. $(2-x)(4+2x+x^2)$.
51. $(a+1)(a^3-a^2+a-1)$.	60. $(3-m)(9+3m+m^2)$.
52. $(1+x)(1-x+x^2)$.	61. $(x+5)(x^2-5x+25)$.
53. $(1-x)(1+x+x^2+x^3)$.	62. $(x-4)(x^2+4x+16)$.
54. $(1+m)(1-m+m^2-m^3)$.	63. $(ab+1)(a^2b^2-ab+1)$.

EXTENDED APPLICATION OF FORMULÆ.

(The following may be omitted on a first reading as far as the end of Ex. XXV.)

50.—We have hitherto applied the formulæ

$$(a \pm x)^2 = a^2 \pm 2ax + x^2,$$

$$\text{and } (a+x)(a-x) = a^2 - x^2,$$

only to binomial factors. We now proceed to apply them to expressions of more than two terms.

Ex. (i.) Let it be required to find the square of $a+b+c$. If we consider two of these three terms as one, we shall then have to deal with an ordinary binomial. Thus arrange the expression as $(a+b)+c$, and write its square by the formula as $(a+b)^2+2(a+b)c+c^2$.

We then simplify the expressions in brackets, and have

$$a^2+2ab+b^2+2ac+2bc+c^2.$$

51.—Should the student find any difficulty in following the steps of this process, let him adopt the following method. Taking the same example, let $a+b$ be represented by the single letter x. The expression then becomes $(x+c)^2 = x^2+2cx+c^2$. Now wherever x occurs in this expansion, write its value $a+b$. The expansion then becomes $(a+b)^2+2c(a+b)+c^2$

$$= a^2+2ab+b^2+2ac+2bc+c^2, \text{ as before.}$$

Ex. (ii.) Find the square of $a-b-x-y$.

Here arrange the terms thus $(a-b) - (x+y)$.

☞ Observe that $x-y$ becomes $x+y$ on account of the $-$ preceding the bracket.

Regarding each of the terms within brackets as a single letter, we have the square, $(a-b)^2-2(a-b)(x+y)+(x+y)^2$

$$=a^2-2ab+b^2-2ax+2bx-2ay+2by+x^2+2xy+y^2.$$

Or by the method of substitution, let $(a-b)=m$, and $(x+y)=n$, then $(a-b-x-y)^2 = (m-n)^2 = m^2-2mn+n^2$. Writing for m and n their values, we have

$$(a-b)^2-2(a-b)(x+y)+(x+y)^2$$

$$= a^2-2ab+b^2-2ax+2bx-2ay+2by+x^2+2xy+y^2, \text{ as before.}$$

Ex. (iii.) Expand $(a^3-a^2+a-1)^2$.

$$= \{(a^3-a^2)+(a-1)\}^2.$$
$$= (a^3-a^2)^2+2(a^3-a^2)(a-1)+(a-1)^2.$$
$$= a^6-2a^5+a^4+2a^4-4a^3+2a^2+a^2-2a+1.$$
$$= a^6-2a^5+3a^4-4a^3+3a^2-2a+1.$$

Examples XXIV.

Expand—

1. $(a+b-c)^2$.
2. $(x+y+z)^2$.
3. $(a^2-a-1)^2$.
4. $(a-x-y)^2$.
5. $(x-y-z)^2$.
6. $(x^2-x+2)^2$.
7. $(a^2-ab+b^2)^2$.
8. $(2a^2-a+1)^2$.
9. $(3a^2+2ab-2)^2$.
10. $(x^2+y^2-z^2)^2$.
11. $(m^2-5m+7)^2$.
12. $(m^2-2mn+3n^2)^2$.
13. $(a+1)^4$.
14. $(x-2)^4$.
15. $(p+q-r)^2$.
16. $(p-q+r-s)^2$.
17. $(m^2+m+n-1)^2$.
18. $\{(a+2)(a-3)\}^2$.
19. $\{(x-4)(x+5)\}^2$.
20. $\{(x+1)(x-1)(x^2+4)\}^2$.
21. $\{(a+2)(a-2)(a^2+3)\}^2$.
22. $\{(x+3)(x-3)(x^2+1)\}^2$.
23. $\{(x^2+2)(x-1)(x+1)\}^2$.
24. $(2x^3-3x^2+2x-1)^2$.
25. $(x+a-b+c)^2$.
26. $(p-q+x-y)^2$.
27. $(m^2-n^2-n-1)^2$.
28. $(a^2-x^2-y+z)^2$.

52.—We now come to the consideration of our other formula, $(x+y)(x-y) = x^2 - y^2$, as applied to factors of more than two terms.

Ex. (i.) Multiply $x+y+a$ by $x+y-a$.

If we regard $x+y$ as a single term, we find we have to deal with the sum of this term and a, and the difference of this term and a. For we can write the first expression as $(x+y)+a$ and the other as $(x+y)-a$. We know by our formula that the product of these two will be the difference of their squares,

$$\therefore \{(x+y)+a\}\{(x+y)-a\} = (x+y)^2 - a^2$$
$$= x^2 + 2xy + y^2 - a^2.$$

Or, by the method of substitution, let $x+y = m$, then we have

$$(m+a)(m-a) = m^2 - a^2 = (x+y)^2 - a^2$$
$$= x^2 + 2xy + y^2 - a^2, \text{ as before.}$$

Ex. (ii.) Multiply $a+b+c+d$ by $a+b-c-d$.

Here arrange the expressions thus,

$$(a+b)+(c+d) \text{ and } (a+b)-(c+d),$$

and we see that we have the sum of two quantities and also their difference. Representing the multiplication by means of brackets, we have by the formula

$$\{(a+b)+(c+d)\}\{(a+b)-(c+d)\} = (a+b)^2 - (c+d)^2$$
$$= a^2 + 2ab + b^2 - c^2 - 2cd - d^2.$$

Or, by the method of substitution, let $a+b=x$, and $c+d=y$. Then $(x+y)(x-y) = x^2 - y^2 = (a+b)^2 - (c+d)^2$

$$= a^2 + 2ab + b^2 - c^2 - 2cd - d^2, \text{ as before.}$$

Ex. (iii.) Resolve $x^2 - (a+y)^2$ into factors.

Here we have the difference of two squares, which by the formula will give the product of the sum and difference of the square roots. The square root of the first term is x, and that of the second term $a+y$. The sum of these is $x+a+y$, and their difference $x-a-y$.

$$\therefore x^2 - (a+y)^2 = (x+a+y)(x-a-y).$$

Ex. (iv.) Resolve $a^2 - 2ab + b^2 - c^2$ into factors.

We remark that $a^2 - 2ab + b^2 = (a-b)^2$.

$$\therefore (a-b)^2 - c^2 = (a-b+c)(a-b-c).$$

Ex. (v.) Resolve $(a-b)^2 - (c-d)^2$ into factors.

The squares here are $(a-b)^2$ and $(c-d)^2$, whose square roots are respectively $a-b$ and $c-d$. The sum of these is $a-b+c-d$, and their difference $a-b-c+d$,

$$\therefore (a-b)^2 - (c-d)^2 = (a-b+c-d)(a-b-c+d).$$

Ex. (vi.) Resolve $m^2 + n^2 - p^2 - q^2 + 2mn + 2pq$.

The terms of this expression may be arranged as follows

$$m^2 + 2mn + n^2 - p^2 + 2pq - q^2;$$

when the expression $m^2 + 2mn + n^2$ is plainly seen to be the square of $m+n$. Putting the remaining terms in the form $-(p^2 - 2pq + q^2)$ we see that this also is the square of $p-q$. Our original expression therefore becomes $(m+n)^2 - (p-q)^2$, the difference of the squares of $m+n$, and $p-q$,

$$\therefore (m+n)^2 - (p-q)^2 = (m+n+p-q)(m+n-p+q).$$

Examples XXV.

Find the following products—

1. $(a+x+y)(a+x-y)$.
2. $(x+y+1)(x+y-1)$.
3. $(a+x-b)(a+x+b)$.
4. $(a+b+c)(a-b-c)$.
5. $(x^2+x+1)(x^2-x-1)$.
6. $(x+y+z)(x-y+z)$.
7. $(x^2+x+1)(x^2-x+1)$.
8. $(x^2+2x+3)(x^2-2x+3)$.
9. $(x^2+xy+y^2)(x^2+xy-y^2)$.
10. $(x+y)^2(x-y)^2$.
11. $(a+x+y)(x+y-a)$.
12. $(a^2+a+1)(1+a-a^2)$.
13. $(x^2+3a-2)(x^2+3a+2)$.
14. $(1-x+x^2)(x^2+x+1)$.
15. $(2+x+x^2)(x^2+x-2)$.
16. $(p+q+r+s)(p+q-r-s)$.
17. $(a^3-a^2+a-1)(a^3-a^2-a+1)$.
18. $(a^3-2a^2+2a+1)(a^3-2a^2-2a-1)$.
19. $(x^3+3x^2+2x+1)(x^3-3x^2+2x-1)$.
20. $(x^3-2x^2+4x-2)(x^3+2x^2+4x+2)$.

Resolve into factors—

21. $(a+b)^2-c^2$.
22. $m^2-(m+n)^2$.
23. $c^2-(x-d)^2$.
24. $(a-b)^2-1$.
25. $(a+b)^2-4$.
26. $9-(a+b)^2$.
27. $1-(x-y)^2$.
28. $a^2+2ab+b^2-x^2$.
29. $m^2-2mn+n^2-p^2$.
30. $x^2-(y^2-2yz+z^2)$.
31. $a^2-b^2+2bc-c^2$.
32. a^2-x^2+2x-1.
33. $1-a^2-x^2+2ax$.
34. $1+2mn-m^2-n^2$.
35. $a^2+2ab+b^2-x^2-2xy-y^2$.
36. $p^2+2pq+q^2-x^2+2xy-y^2$.
37. $a^2+b^2-c^2-d^2-2ab+2cd$.
38. $(a+b-1)^2-(x-q-1)^2$.
39. $2xy-x^2-y^2+p^2+q^2-2pq$.
40. $(3x+2)^2-(5x-4)^2$.

Divide—

41. $(a+b)^2-x^2$ by $a+b+x$.
42. $(x-y)^2-z^2$ by $x-y+z$.
43. $(m+n)^2-1$ by $m+n+1$.
44. $(x-y)^2-(a-b)^2$ by $x-y-(a-b)$.
45. $x^2-(a-b)^2$ by $x-a+b$.
46. $(a+b)^2-(x-y)^2$ by $a+b-x+y$
47. $(a-m)^2-(x-n)^2$ by $a-m+x-n$.
48. $4(a+x)^2-1$ by $2a+2x+1$.
49. $(a+b)^3+x^3$ by $a+b+x$.
50. $(a+b)^3-c^3$ by $a+b-c$.
51. $(x+y)^4-a^4$ by $x+y+a$.
52. $8(a+b)^3-c^3$ by $2(a+b)-c$.
53. $a^3-(x+y)^3$ by $a-x-y$.
54. $(x+y)^3-1$ by $(x+y)^2+x+y+1$.

V.—INVOLUTION AND EVOLUTION.

INVOLUTION.

53.—Involution is the operation of raising a quantity to some given power; and is only a particular case of Multiplication. A quantity of one term is raised to any given power by raising each factor separately: and this is effected by multiplying the index of each factor by the index of the power to which it is to be raised.

Thus, $(a^2)^3 = a^2 \, . \, a^2 \, . \, a^2 = a^{2 \times 3} = a^6.$

$(4a^3b^2c)^2 = 16a^6b^4c^2.$

54.—Any power of a positive quantity is positive; but, since a^2 may arise from the multiplication of either a or $-a$ by itself, it follows that when the original quantity is negative, the *even* powers will be positive, and the *odd* powers negative.

Thus $(-a)^2 = a^2$, $(-a)^3 = -a^3$.

$(-2a^3)^2 = 4a^6$, $(-2a^2)^3 = -8a^6$.

55.—We already have a formula for the square of a binomial; and for the cube by multiplication we have

$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3.$$

$$(a-x)^3 = a^3 - 3a^2x + 3ax^2 - x^3.$$

Writing these together with the double sign we have

$$(a \pm x)^3 = a^3 \pm 3a^2x + 3ax^2 \pm x^3,$$

which should be committed to memory. The fourth power may be obtained by squaring the square; the fifth power by multiplying the square and cube together; the sixth power by squaring the cube, and so on.

The Square of a Multinomial.

56.—If we investigate the form of the square of a multinomial, we shall obtain a formula by which we can at once write such a square. The student is recommended to find by actual multiplication that $(a+b+x+y)^2 = a^2 + 2ab + b^2 + 2ax + 2ay + 2bx + 2by + x^2 + 2xy + y^2$. Arranging these in another order we have $(a+b+x+y)^2$

$$=a^2+b^2+x^2+y^2+2ab+2ax+2ay+2bx+2by+2xy.$$

Or the expression might be written thus,

$$a^2+b^2+x^2+y^2+2a(b+x+y)+2b(x+y)+2xy.$$

That is, *The square of a multinomial is equal to the square of each term singly, together with twice the product of each term by those that follow it.*

The application of this formula leads to the same results as those given in No. **50.**

Ex. (i.) $(a^2+ab-1)^2 = a^4+a^2b^2+1+2a^3b-2a^2-2ab$
$$= a^4+2a^3b+a^2b^2-2a^2-2ab+1.$$

Ex. (ii.) $(1-a)^6 = \{(1-a)^3\}^2$
$$=(1-3a+3a^2-a^3)^2$$
$$=1+9a^2+9a^4+a^6-6a+6a^2-2a^3-18a^3+6a^4-6a^5.$$
$$=1-6a+15a^2-20a^3+15a^4-6a^5+a^6$$

EXAMPLES XXVI.

1. Raise to the third power a, $-a$, $2a^2$, $-2a^3$, $3a^4$, $-2a^2$.
2. Simplify $(-a)^2$, $(-2a)^4$, $(3a^2bx^2)^3$, $(-a)^{12}$, $(-x^n)^2$.
3. $(-a^2)^5$, $(-ab^2c^3)^3$, $(a^2bc^2)^3$, $(-3x^3y^2)^3$, $(x^n)^m$.

Expand the following by the formula given above—

4. $(a+x+y+z)^2$.
5. $(a-b-c-m)^2$.
6. $(c-d+x-y)^2$.
7. $(3a-2b+1)^2$.
8. $(a^2-4b^2+c^2)^2$.
9. $(3ab-2ac+ad)^2$.
10. $(1+2a+3a^2)^2$.
11. $(1-a+2a^2)^2$.
12. $(a^2-3a-1)^2$.
13. $(a-1)^4$.

14. Write the cubes of $a+y$, $x-y$, $m+n$, $m-1$, $1+x$, $1-x$, and $3a-2b$.

15. Obtain the sixth power of $1+x$; $x-y$; and $x-2$.

16. Find the fifth power of $x-y$.

Additional exercises in the application of the formula for squaring a multinomial may be formed by expanding the examples in XXIV. by means of it.

EVOLUTION.

57.—Evolution, or the extraction of roots, is the converse of Involution, and is the process by which we find a quantity such that when multiplied by itself a certain number of times, it will produce a given quantity.

58.—To obtain a root of a quantity of one term, we divide the indices by the index of the root required. Thus—

Ex. (i.) $\sqrt{a^2} = \pm a : \sqrt{16a^6b^2} = \pm 4a^3b$.

(ii.) $\sqrt[3]{a^3} = a : \sqrt[3]{(-a^3)} = -a : \sqrt[3]{(27a^3b^6)} = 3ab^2$.

(iii.) $\sqrt[4]{a^4} = \pm a : \sqrt[4]{16a^4b^8} = \pm 2ab^2$.

(iv.) $\sqrt[5]{-a^5b^{10}} = -ab^2 : \sqrt[5]{-32b^{10}} = -2b^2$.

59.—Because a^2 may arise from the multiplication of either a or $-a$ by itself, we must prefix to the square root and to all *even* roots the double sign $\pm$.

Odd roots must have the same sign as the given quantity.

There can be no even root of a negative quantity, because no negative quantity raised to an even power can give a negative result.

Examples XXVII.

Find the square root of—

1. $4a^2b^2$.	3. $25a^2b^4c^6$.	5. $169m^6n^4$.
2. $9a^6b^6$.	4. $100a^2b^4c^8$.	6. $400a^4b^6c^8$.

Extract the cube root of—

7. $a^3b^3c^6$.	9. $-64a^9b^3c^3$.	11. $1000a^3x^6y^9$.
8. $-a^6x^9y^{12}$.	10. $-27m^{12}n^6x^3$.	12. $216m^6n^3$.

Simplify—

13. $\sqrt[4]{a^4m^4y^8}$. 14. $\sqrt[5]{-a^{10}b^5}$. 15. $\sqrt{\dfrac{a^2b^2c^4}{x^2y^2}}$.

16. $\sqrt[3]{\dfrac{-a^3b^6}{c^6d^9}}$. 17. $\sqrt[n]{x^{2n}}$. 18. $\sqrt[3]{x^{3n}}$.

THE SQUARE ROOT OF COMPOUND EXPRESSIONS.

Ex. (i.) Find the square root of $a^2 - 2ax + x^2$.

$$
\begin{array}{r|l}
a & a^2 - 2ax + x^2 \;\lfloor\, a - x \\
 & a^2 \\
\hline
2a - x & \quad - 2ax + x^2 \\
 & \quad - 2ax + x^2 \\
\hline
\end{array}
$$

60.—Here we extract the square root of the first term, a^2, which root we find to be a. We set this quantity a in the spaces both to the right and left; and considering that on the left as a *divisor*, and that on the right as a *quotient*, we multiply them together and put their product, a^2, under the first term of the quantity which is under consideration.

We then subtract, and bring down the next *two* terms. Then we double the last quantity, a, in the divisor, and thus obtain a new divisor, $2a$, which will divide into $-2ax$, and give a quotient $-x$. This is put into both divisor and quotient; and then the new divisor $2a - x$ is multiplied by $-x$, and the product, $-2ax + x^2$, set down under $-2ax + x^2$. Subtracting, we find that the operation is complete; and the quantity in the place of the quotient, $a - x$, is the square root required. This process will be seen carried to greater length in the next example.

Ex. (ii.) Find the square root of

$$6a^3b^2 - 4ab^3 + 9a^6 - 12a^4b + 4a^2b^2 + b^4.$$

Arrange the terms according to descending powers of a.

$$
\begin{array}{r|l}
3a^3 & 9a^6 - 12a^4b + 6a^3b^2 + 4a^2b^2 - 4ab^3 + b^4 \;\lfloor\; 3a^3 - 2ab + b^2. \\
 & 9a^6 \\
\hline
6a^3 - 2ab & -12a^4b + 6a^3b^2 \\
 & -12a^4b + 4a^2b^2 \\
\hline
6a^3 - 4ab + b^2 & 6a^3b^2 - 4ab^3 + b^4 \\
 & 6a^3b^2 - 4ab^3 + b^4 \\
\hline
\end{array}
$$

Examples XXVIII.

Extract the square roots of the following—

1. $a^2 + 2ab + b^2$.
2. $9a^2 + 36ab + 36b^2$.
3. $4a^4x^2 - 12a^3x^3 + 9a^2x^4$.
4. $a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + b^4$.
5. $1 - 2a + a^2 + 2b - 2ab + b^2$.
6. $x^4 + 2x^2y^2 + y^4 - 6x^2 - 6y^2 + 9$.
7. $9a^4 - 30a^3 + 37a^2 - 20a + 4$.
8. $1 - 2a^2 + 2a^3 + a^4 - 2a^5 + a^6$.
9. $9a^6 + 12a^5 + 4a^4 - 30a^3 - 20a^2 + 25$.
10. $9a^6 - 12a^5x + 28a^4x^2 - 28a^3x^3 + 24a^2x^4 - 16ax^5 + 4x^6$.
11. $4 - 8a + 8a^3 - 3a^4 - 2a^5 + a^6$.
12. $a^6 - 4a^5b + 8a^4b^2 - 10a^3b^3 + 8a^2b^4 - 4ab^5 + b^6$.
13. $4x^4 - 12x^3y + 37x^2y^2 - 42xy^3 + 49y^4$.
14. $16a^6 - 16a^5b + 20a^4b^2 - 40a^3b^3 + 20a^2b^4 - 16ab^5 + 16b^6$.
15. $27x^2 - 14x + 1 + 121x^4 + 154x^3$.
16. $1 - 8x^3 + 18x^6 - 8x^9 + x^{12}$.

61.—When $a = 1$, $b = 2$, $x = -1$, and $y = 0$, find the value of

Ex. (i.) $(a - b)^2 + (x - y)^2 - (b - x)^3$.

$= (-1)^2 + (-1)^2 - (2 + 1)^3$.

$= 1 + 1 - 27 = -25$.

Ex. (ii.) $\dfrac{\sqrt{(4x^2+5ab+3b^2+x)}}{(a+b+x-3)^3}+\dfrac{ax+by+2x^3}{\sqrt{(a^2x^2+2abx-6bx)}}$

$$=\frac{\sqrt{(4+10+12-1)}}{(1+2-1-3)^3}+\frac{-1+0-2}{\sqrt{(1-4+12)}}$$

$$=\frac{\sqrt{25}}{(-1)^3}+\frac{-3}{\sqrt{9}}=\frac{5}{-1}+\frac{-3}{3}=-5-1=-6.$$

Examples XXIX.

When $a=4$, $b=2$, $c=-1$, and $d=-2$, find the value of the following expressions—

1. $(a-b)^2+(c-d)^2-d^3$.
2. $(a+d)^2-(c+d)^2$.
3. $(b+c+d)^3-(a-b+c+d)^3$.
4. $\dfrac{b^3-c^3}{c^3+d^3}+\dfrac{a^2-b^2}{4(b^2-c^2)}$.
5. $\dfrac{\sqrt{2(a-b)^3}}{\sqrt[3]{(a+c+2d)}}+\dfrac{\sqrt[3]{(a^2b-2cd-1)}}{\sqrt[3]{(b+c+d)}}$.
6. $\dfrac{(a+d)(b-c)}{b(c+d)}-\dfrac{ab+bc+cd}{c-(a+b)-1}$.
7. $\dfrac{c(a-b)+c^2(b-a)}{a(c+d)+2bd^2}$.
8. $\dfrac{(b+c+d)^2}{(a+c+d)^2}-\dfrac{b^2-d^3}{(b+c)^2}$.
9. Find the value of $x^2-\dfrac{2x-7}{x+4}+\sqrt{x+1}$, when $x=8$, and when $x=-1$.
10. Find the value of $x^4-2x^2(x-2)+3x$, when $x=1$, when $x=-1$, and when $x=3$.
11. Find the value of $a^3-3a^2b+3ab^2-b^3$, when $a=2$, $b=1$; when $a=1$, $b=2$, and when $a=b$.
12. Find the value of $ab\{(a^2-ab+b^2)^2+(a^3-b^3)\}$, when $a=-1$, $b=-2$, and when $a=b$.

FRACTIONAL AND NEGATIVE INDICES.

62.—The full consideration of fractional and negative indices cannot be entered upon here, but enough may be said respecting them to enable the student to understand their meaning, and to apply to them those processes which have so far been explained. What follows may either be taken at this point or deferred till the student has become acquainted with algebraic fractions.

63.—Fractional indices arise as follows. If we wish to express such a quantity as *the cube root of the square of a*, we use the index 2, and the radical sign and index for the cube root, thus, $\sqrt[3]{a^2}$. Instead of the sign $\sqrt[3]{}$, however, we may take another way of expressing the same meaning, by writing the index of the root to be extracted as the denominator of a fraction. Thus $a^{\frac{2}{3}}$ signifies that a is to be squared, and then the cube root is to be taken, and is precisely the same as $\sqrt[3]{a^2}$. The form $a^{\frac{2}{3}}$ is much the more convenient for purposes of Multiplication, Division, &c.

Ex. (i.) Find the value of $a^{\frac{1}{3}}b^{\frac{3}{2}}$, when $a = 8$, $b = 4$.

$$a^{\frac{1}{3}}b^{\frac{3}{2}} = \sqrt[3]{8} \times \sqrt{4^3} = 2 \times 8 = 16.$$

Ex. (ii.) $a^{\frac{1}{2}}b^{\frac{2}{3}} \times a^{\frac{1}{4}}b^{\frac{1}{3}} = a^{\frac{1}{2}+\frac{1}{4}} \times b^{\frac{2}{3}+\frac{1}{3}} = a^{\frac{3}{4}}b.$

Ex. (iii.) Multiply $2a^{\frac{2}{3}} - 3a^{\frac{1}{3}}b^{\frac{1}{2}} + 3b$ by $a^{\frac{1}{3}} + b^{\frac{1}{2}}$.

$$\begin{array}{lllll}
2a^{\frac{2}{3}} & - 3a^{\frac{1}{3}}b^{\frac{1}{2}} & + 3b & & \\
a^{\frac{1}{3}} & + b^{\frac{1}{2}} & & & \\
\hline
2a & - 3a^{\frac{2}{3}}b^{\frac{1}{2}} & + 3a^{\frac{1}{3}}b & & \\
 & + 2a^{\frac{2}{3}}b^{\frac{1}{2}} & - 3a^{\frac{1}{3}}b & + 3b^{\frac{3}{2}} & \\
\hline
2a & - a^{\frac{2}{3}}b^{\frac{1}{2}} & & + 3b^{\frac{3}{2}} & \\
\hline
\end{array}$$

Ex. (iv.) Divide $a - 5a^{\frac{1}{2}} + 2a^{\frac{1}{4}}$ by $a^{\frac{1}{2}} - 2a^{\frac{1}{4}}$.

$$
\begin{array}{r|l}
a^{\frac{1}{2}} - 2a^{\frac{1}{4}} & a - 5a^{\frac{1}{2}} + 2a^{\frac{1}{4}} \quad \big\lfloor\; a^{\frac{1}{2}} + 2a^{\frac{1}{4}} - 1. \\
 & \underline{a - 2a^{\frac{3}{4}}\qquad\qquad} \\
 & \quad 2a^{\frac{3}{4}} - 5a^{\frac{1}{2}} + 2a^{\frac{1}{4}} \\
 & \quad \underline{2a^{\frac{3}{4}} - 4a^{\frac{1}{2}}\qquad\quad} \\
 & \qquad\quad - a^{\frac{1}{2}} + 2a^{\frac{1}{4}} \\
 & \qquad\quad \underline{- a^{\frac{1}{2}} + 2a^{\frac{1}{4}}}
\end{array}
$$

Examples XXX.

Multiply—

1. $a^{\frac{1}{2}} - a^{\frac{1}{4}} + 1$ by $a^{\frac{1}{2}} + a^{\frac{1}{4}} - 1$.
2. $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.
3. $\frac{1}{8}x^{\frac{3}{2}} - \frac{1}{12}xy^{\frac{1}{2}} + \frac{1}{18}x^{\frac{1}{2}}y - \frac{1}{27}y^{\frac{3}{2}}$ by $\frac{1}{2}x^{\frac{1}{2}} - \frac{1}{3}y^{\frac{1}{2}}$.
4. $a^{\frac{2}{3}} + 5a^{\frac{1}{2}} + 3a^{\frac{1}{3}} - a^{\frac{1}{6}} + 3$ by $a^{\frac{1}{3}} - 5a^{\frac{1}{6}} - 3$.
5. $(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{1}{2}} - y^{\frac{1}{2}})(x + y)$.

Divide—

6. $2x - 3x^{\frac{3}{4}} + 3x^{\frac{1}{2}} - 2x^{\frac{1}{4}}$ by $x^{\frac{1}{4}} - 1$.
7. $9x^{\frac{3}{4}} + 5x^{\frac{1}{4}}y^{\frac{1}{2}} + 6y^{\frac{3}{4}}$ by $3x^{\frac{1}{4}} + 2y^{\frac{1}{4}}$.
8. $m^{\frac{7}{6}} + m^{\frac{1}{2}}n^{\frac{1}{3}} - 4m^{\frac{1}{6}}n^{\frac{2}{3}} + 2n^{\frac{2}{3}}$ by $m^{\frac{1}{2}} + 2n^{\frac{1}{3}}$.
9. $x^{\frac{4}{3}} - 5x + 9x^{\frac{2}{3}} - 8x^{\frac{1}{3}} + 2$ by $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 2$.
10. $a^{\frac{2}{3}} - b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.

Find the square root of—

11. $a - 4a^{\frac{3}{4}} + 6a^{\frac{1}{2}} - 4a^{\frac{1}{4}} + 1$.
12. $a^{\frac{1}{2}} - 4a^{\frac{7}{12}}b^{\frac{1}{3}} + 4a^{\frac{2}{3}}b^{\frac{2}{3}} + 2a^{\frac{1}{4}}b^{\frac{1}{4}} - 4a^{\frac{1}{3}}b^{\frac{7}{12}} + b^{\frac{1}{2}}$.

Find the value of—

13. $a^{\frac{2}{3}}b^{\frac{1}{2}}$, when $a = 8$, $b = 4$.

14. $(x^{\frac{1}{4}}y^{\frac{1}{3}})^{\frac{1}{2}} + (x-y)^{\frac{1}{3}}$, when $x = 16$, $y = 8$.

15. $(x+y)^{\frac{1}{3}} + (x-y)^{\frac{1}{2}} - (\overline{x-y+3})^{\frac{4}{5}}$, when $x=6$, $y=2$.

64.—To learn the meaning of such an expression as a^{-2}, let us first multiply some other power of a, for instance a^5, by it. By the rule for multiplication by the addition of indices in **28**,

$$a^5 \times a^{-2} = a^{5-2} = a^3.$$

$$\text{But } a^5 \div a^2 = a^{5-2} = a^3.$$

It is plain, therefore, that $a^5 \times a^{-2} = a^5 \div a^2$; and that the expression a^{-2} is only another way of representing a^2 as a divisor. Therefore, any factor which has a negative index may be represented as a divisor with a positive index. Thus—

$$xy^{-1} = \frac{x}{y} : xy^{-2} = \frac{x}{y^2} : x^{-1} = \frac{1}{x}.$$

65.—☞ Notice the following—

$$x^a \times x^o = x^{a+o} = x^a$$

$$\therefore x^o = 1,$$

which should be remembered as an important fact. Again—

$$x^a \times x^{-a} = x^{a-a}$$

$$= x^o = 1.$$

$$\therefore x^{-a} = \frac{1}{x^a}.$$

This shews that any factor may be removed from the numerator of a fraction to the denominator, or conversely from the denominator to the numerator, by changing the sign of its index.

Ex. (i.) Find the value of $ab^{-2} + 16a^{-2}b$, where $a=4$, $b=2$.

$$ab^{-2} + 16a^{-2}b = \frac{a}{b^2} + \frac{16b}{a^2}$$

$$= \frac{4}{4} + \frac{32}{16} = 1 + 2 = 3.$$

Ex. (ii.) Multiply $a+a^{\frac{1}{3}}+a^{-\frac{1}{3}}$ by $a^{\frac{1}{3}}+a^{-\frac{1}{3}}$.

$$\begin{array}{l}
a+a^{\frac{1}{3}}+a^{-\frac{1}{3}} \\
a^{\frac{1}{3}}+a^{-\frac{1}{3}} \\
\hline
a^{\frac{4}{3}}+a^{\frac{2}{3}}+1 \\
\quad +a^{\frac{2}{3}}+1+a^{-\frac{2}{3}} \\
\hline
a^{\frac{4}{3}}+2a^{\frac{2}{3}}+2+a^{-\frac{2}{3}} \\
\hline
\end{array}$$

Ex. (iii.) Divide $a^{-1}-1$ by $a^{-\frac{2}{3}}+a^{-\frac{1}{3}}+1$.

$$\begin{array}{l}
a^{-\frac{2}{3}}+a^{-\frac{1}{3}}+1 \,)\, a^{-1}-1 \,(\, a^{-\frac{1}{3}}-1 \\
\qquad a^{-1}+a^{-\frac{2}{3}}+a^{-\frac{1}{3}} \\
\hline
\qquad -a^{-\frac{2}{3}}-a^{-\frac{1}{3}}-1 \\
\qquad -a^{-\frac{2}{3}}-a^{-\frac{1}{3}}-1 \\
\hline
\end{array}$$

Examples XXXI.

1. Find the value of $x^{-1}y-2xy^{-2}$, when $x=2$, $y=1$.

Multiply—

2. $a^{-2}+a^{-1}+1$ by $a^{-1}-1$.
3. $x^2+2x^{-1}-x^{-2}$ by $x-x^{-2}$.
4. $a^{-\frac{2}{3}}+a^{-\frac{1}{3}}+1$ by $a^{-\frac{1}{3}}-1$.
5. $(x^{-3}+a^{-2})(x^{-3}-a^{-2})$.

Divide—

6. a^2-a^{-2} by $a-a^{-1}$.
7. m^2-m^{-2} by $m+m^{-1}$.
8. a^4+1+a^{-4} by a^2-1+a^{-2}.
9. a^2+1+a^{-2} by $a-1+a^{-1}$.
10. Simplify $a^{x+y+z}\,.\,a^{x+y-2z}\,.\,a^{z-2x-2y}$.

VI.—GREATEST COMMON MEASURE AND LEAST COMMON MULTIPLE.

THE GREATEST COMMON MEASURE.

66.—A MEASURE of a quantity is some other quantity which will divide it without a remainder. Thus 1, 2, 3, 4, 6, and 12 are all measures of 12, because they will each divide 12 without a remainder. a, a^2, ab, a^2b, and a^2b^2 are in the same way measures of a^2b^2.

When the same quantity is a measure of more quantities than one, it is said to be a measure *common* to those quantities. Thus, since 6 is a measure of 6, 12, and 18, 6 is said to be a measure common to these numbers; and it will be found on trial that no number greater than 6 will measure 6, 12, and 18. Therefore 6 is called the Greatest Common Measure of 6, 12, and 18. As a definition, we may say that *the Greatest Common Measure of a series of quantities is the greatest quantity that will divide each of the series without a remainder.*

67.—In Algebra, however, we must bear in mind that the word *greatest* does not mean numerically greatest, but that which has the highest powers of the letters involved in it.

68.—To find the G.C.M. of simple expressions, it is evident that if we observe those letters which are common to all the expressions and take the *lowest* powers of these letters that occur, we shall obtain the highest factor required: because any factor of higher power than the lowest contained in any one of the given expressions would make our G.C.M. too great.

Numerical co-efficients must be treated by the ordinary Arithmetical process.

Ex. (i.) Find the G.C.M. of x^2y^3 and x^2yz.

Here x and y are the only letters that occur in each of the given expressions, and the lowest given powers are x^2 and y,

$$\therefore \text{ G.C.M.} = x^2y.$$

Ex. (ii.) Find the G.C.M. of $12a^2b^2c$, $60ab^2c$, and $24a^2b^2c^2$.

G.C.M. of 12, 60, and 24 = 12. a, b, and c occur,

$$\therefore \text{ G.C.M.} = 12ab^2c.$$

69.—In finding the G.C.M. of compound expressions, when these can be resolved into factors at sight, the factors should be set down in some convenient form so that those which occur in each of the given quantities may at once be written as factors of the G.C.M.

Ex. (iii.) Find the G.C.M. of $(a+b)^2$ and a^2-b^2.

$$(a+b)^2 = (a+b)(a+b)$$
$$a^2-b^2 = (a+b)(a-b)$$

Here $a+b$ is the only factor that occurs in each of the expressions,

$$\therefore \text{ G.C.M.} = a+b.$$

Ex. (iv.) Find the G.C.M. of $12a^3(x-y)^2$ and $4a(x^4-y^4)$.

Observing that the G.C.M. of those factors outside the brackets is $4a$, we write

$$(x-y)^2 = (x-y)(x-y)$$
$$x^4-y^4 = (x+y)(x-y)(x^2+y^2),$$
$$\therefore \text{ the required G.C.M.} = 4a(x-y).$$

Examples XXXII.

Find the G.C.M. of—

1. a^2b and ab^2.
2. abc and abd.
3. $m^4n^3x^2$ and $m^2n^2x^4$.
4. xyz and $4x^2yz$.
5. $3x^2y$ and $18x^2y^2z^2$.
6. $60m^2p$ and $195p^2q^2$.
7. $100a^2b^2c^3d^4$ and $25abd^2$.
8. $19x^2y^5z^6$ and $57xy^6z^5$.
9. $34m^2n^2pq$ and $51np^2q^2$.
10. $14abxy$ and $105x^3$.

11. a^2b^2c, ab^3c^2, and a^2b^2.
12. $13m^2n^2pq$, $52mnp$, $65mp$.
13. $a^2b^2c^4d$, ab^2c^2x, b^2c^2xy.
14. $46a^6x$, $23a^2x^3$, and a^3x^2y.
15. $11a^3$, $22a^5b^3$, and $33b^3c^2$.
16. a^2, and $a(x-y)$.
17. ax^2, and $ax(y-z)$.
18. $3(x-y)$ and $12a(x+y)$.
19. $12a^2b^2$, and $4ab(x+y)$.
20. x^3, and ax^2+bx^2.
21. a^2-b^2 and $a-b$.
22. x^2-y^2 and $(x-y)^2$.
23. $a^2-2ax+x^2$ and $a-x$.
24. $a^2+2ax+a^2$ and $a+x$.
25. x^4-1 and $(x-1)^2$.
26. x^2-y^2 and x^4-y^4.
27. x^3-y^3 and x^2-y^2.
28. x^3+y^3 and $x^2+2xy+y^2$.
29. $3(x^2-y^2)$ and $x+y$.
30. $2(x-y)$ and x^2-y^2.
31. x^2+3x+2 and x^2-1.
32. x^2-x-12, $x^2+7x+12$.
33. $a^2-2ab+b^2$ and $a^3-3a^2b+3ab^2-b^3$.
34. a^2-3a+2 and a^2+3a-4.
35. $m^2-5mn+6$ and $m^2+9mn-36$.
36. a^2-1, a^2+4a+3, and a^3+1.
37. $2n+2$, $2n^2+4n+2$, and n^2+5n+4.
38. $1-n^2$, $(1+n)^2$, and $1+n^3$.
39. m^2-6m+5, m^2-25, m^2+m-30.
40. x^2+5x+6, x^2+4x+3, x^2-x-12.

70.—The method of finding, otherwise than by inspection, the G.C.M. of compound Algebraical quantities is to divide one by the other, using each remainder as it is obtained for a new divisor, and the preceding divisor for a new dividend. When the process comes to an end for want of a remainder, it will be found that the last divisor is the G.C.M. of the quantities under consideration.

Ex. (i.) Find the G.C.M. of x^2+6x+5 and x^2+7x+6.

$$\begin{array}{rrl}
x^2+6x+5 \,) & x^2+7x+6 & (\,1 \\
 & x^2+6x+5 & \\
\hline
 & x+1 \,)\; x^2+6x+5 & (\,x+5 \\
 & x^2+x & \\
\hline
 & 5x+5 & \\
 & 5x+5 & \\
\hline
\end{array}$$

G.C.M. $= x+1$.

When one of the expressions contains a factor which does not occur in the other, this factor may at once be removed, as it is plain that it cannot be a factor of the G.C.M., which must divide both.

Ex. (ii.) Thus, in a^2-5a+6 and $3a^4-15a^3-24a-24$, we observe that 3 is a factor of the second expression $3(a^4-5a^3-8a-8)$, but not of the first. We therefore remove this factor, 3, and write the quantities as follows :—

$$
\begin{array}{r|l}
a^2-5a+6 & a^4-5a^3-8a-8 \;\lfloor\; a^2-6 \\
 & \underline{a^4-5a^3+6a^2} \\
 & \qquad -6a^2-8a-8 \\
 & \qquad \underline{-6a^2-30a+36} \\
22 & \qquad 22a-44 \\
\end{array}
$$

Remove the factor 22 from the remainder $22a-44$.

$$
\begin{array}{r|l}
a-2 & a^2-5a+6 \;\lfloor\; a-3 \\
 & \underline{a^2-2a} \\
 & \quad -3a+6 \\
 & \quad \underline{-3a+6} \\
\end{array}
$$

G.C.M. $= a-2$.

71.—If, however, we strike out the *same factor from both* quantities in their original forms, it is evident that the factor so struck out will be a factor of the G.C.M. Were the two quantities of the last example presented to us in the form of

$$4a^2-20a+24 \text{ and } 4a^4-20a^3-32a-32,$$

we should notice that 4 is a factor of *both* quantities. We therefore remove it at present, remembering that 4 will be a factor of the G.C.M.

Finding the G.C.M. of the two expressions as before to be $a-2$, we now write the G.C.M. of the two quantities presented to us as $4(a-2)$, thus replacing the factor, 4, which was found to be a factor of each at the beginning.

72.—It will sometimes happen that the numerical co-efficient of the first term of one of the quantities will not divide that of the other exactly, as for instance, in finding the G.C.M. of

$$2a^4 - 3a^3 + a^2 + a - 1 \text{ and } 3a^3 - 4a^2 + 3a - 2.$$

Putting the latter quantity for a divisor, it is found that 2 cannot be divided by 3. We therefore multiply the dividend by the smallest number that will make the numerical co-efficient of the first term divisible by 3. In this case the number is 3.

$$\begin{array}{r|l|l}
 & 2a^4 - 3a^3 + a^2 + a - 1 & \\
 & 3 & \\
\hline
3a^3 - 4a^2 + 3a - 2 & 6a^4 - 9a^3 + 3a^2 + 3a - 3 & 2a \\
 & 6a^4 - 8a^3 + 6a^2 - 4a & \\
\hline
☞ & \quad - a^3 - 3a^2 + 7a - 3 &
\end{array}$$

When, as in this case, the remainder begins with a negative sign, it is usual to change all its signs on proceeding to divide by it.

$$\begin{array}{r|l|l}
a^3 + 3a^2 - 7a + 3 & 3a^3 - 4a^2 + 3a - 2 & 3 \\
 & 3a^3 + 9a^2 - 21a + 9 & \\
\hline
 & \quad - 13a^2 + 24a - 11 &
\end{array}$$

$$\begin{array}{r|l|l}
13a^2 - 24a + 11 & a^3 + 3a^2 - 7a + 3 & a + 4 \\
 & 13 & \\
\hline
 & 13a^3 + 39a^2 - 91a + 39 & \\
 & 13a^3 - 24a^2 + 11a & \\
\hline
 & \quad 63a^2 - 102a + 39 & \\
 & \quad 52a^2 - 96a + 44 & \\
\hline
 & \quad 11a^2 - 6a - 5 &
\end{array}$$

$$
\begin{array}{rl}
11a^2-6a-5 \,\big)\, & 13a^2-\ \ 24a+\ 11 \,\big(\, 13 \\
& 11 \\
\hline
& 143a^2-264a+121 \\
& 143a^2-\ \ 78a-\ \ 65 \\
\hline
-186 \,\big|\, & -186a+186 \\
\hline
\end{array}
$$

$$
\begin{array}{rl}
a-1 \,\big)\, & 11a^2-\ \ 6a-5 \,\big(\, 11a+5 \\
& 11a^2-11a \\
\hline
& \qquad\quad 5a-5 \\
& \qquad\quad 5a-5 \\
\hline
\end{array}
$$

G.C.M. $= a-1$.

Examples XXXIII.

Find the Greatest Common Measure of—

1. a^3+1 and a^3+2a^2-1.
2. $2a^4+a^2-1$ and a^4-2a^2-3.
3. $2x^3-17x^2+19x+3$ and $3x^3-20x^2-10x-1$.
4. $a^3b^3-4a^2b^2+5ab-2$ and $a^3b^3-5a^2b^2+7ab-3$.
5. $x^3-2x^2y-xy^2+2y^3$ and $x^3-4x^2y+5xy^2-2y^3$.
6. $6x^3-7x^2y+4xy^2-y^3$ and $9x^3-12x^2y+7xy^2-2y^3$.
7. $2a^3-9a^2b+19ab^2-15b^3$ and $2a^3-3a^2b+ab^2+15b^3$.
8. $a^4+a^2b^2+b^4$ and $a^4+2a^3b+3a^2b^2+2ab^3+b^4$.
9. $2a^3-4a^2b-4ab^2+6b^3$ and $2a^3-8ab^2-6b^3$.
10. $8a^4x-6a^3x+3ax-2x$ and $4a^3x-11a^2x+8ax-4x$.
11. m^4-2m^3+4m-4 and $m^3n-2mn+4n$.
12. $9x^3-x+20$ and $9x^4-12x^3-3x^2+24x-30$.
13. $6a^3-7a^2y+4ay^2-y^3$ and $2a^3-3a^2y+5ay^2-2y^3$.
14. $2a^3-a^2-2a+1$ and $3a^3-4a^2+3a-2$.
15. $3x^3+5x^2+x-1$ and $6x^3+5x^2+4x+5$.
16. $2x^3+5x^2y+2xy^2-y^3$ and $3x^3+2x^2y+xy^2+2y^3$.
17. $5a^3+9a^2b+8ab^2-2b^3$ and $10a^3-12a^2b-3ab^2+b^3$.
18. $3x^3-4x^2-x-14$ and $6x^3-11x^2-10x+7$.

19. $8m^4 + m - 1$ and $4m^3 - 3m + 1$.
20. $2a^3 + a^2 - 2a - 6$ and $6a^3 - a^2 - 14a + 3$.
21. $8a^3 - 8a^2 - 4a - 3$ and $2a^4 + 3a^3 - 3a^2 - 7a - 3$.
22. $2x^5 + x^2 - 2x + 1$ and $x^5 - x^2 - x - 1$.
23. $6a^3 - 5a - 1$ and $6a^4 - 7a^3 + a^2 + 2a - 2$.
24. $2x^3 + x^2 + 2x - 12$ and $2x^3 - 7x^2 + 14x - 12$.
25. $a^3 - a^2b - ab^2 + b^3$ and $a^4 - 2a^3b + 2ab^3 - b^4$.
26. $a^4 + 67a^2 + 66$ and $a^4 + 2a^3 + 2a^2 + 2a + 1$.
27. $2x^4 + 4x^3 + 3x^2 - 2x - 2$ and $3x^4 + 6x^3 + 7x^2 + 2x + 2$.
28. $2x^3 - 3ax^2 - 7a^2x + 4a^3$ and $6x^3 - 7ax^2 - 4a^2x + 3a^3$.
29. $28a^2 + 37a - 21$ and $35a^2 + 62a - 33$.
30. $7m^3 - 13m^2 + 34m - 72$ and $7m^3 - 6m^2 + 35m - 36$.

73.—When the G.C.M. of more than two quantities is required, we find the G.C.M. of two quantities first, and then find the G.C.M. of this and the third quantity, and so on for as many quantities as are given.

Find the G.C.M. of

$a^3 - 4a^2 + 5a - 2$, $3a^3 - 8a^2 + 7a - 2$, and $2a^3 - 4a^2 + 3a - 1$

G.C.M. of $a^3 - 4a^2 + 5a - 2$ and $3a^3 - 8a^2 + 7a - 2$ may be found in the usual way to be $a^2 - 2a + 1$. We now find the G.C.M. of $a^2 - 2a + 1$ and $3a^3 - 4a^2 + 3a - 1$, by the same process, to be

$$a - 1,$$

$\therefore$ G.C.M. of the three quantities is $a - 1$.

Find the G.C.M. of—

31. $a^2 - 3a + 2$, $a^2 - 5a + 6$, and $a^2 + 15a - 34$.
32. $a^3 - 2a^2 + 2a - 1$, $a^3 - a^2 - 3a + 3$, and $a^2 - 7a + 6$.
33. $2x^3 + 5x^2 - 2x - 5$, $x^3 + 3x^2 - x - 3$, and $3x^3 + x^2 - x + 1$.
34. $2a^3 - 6a - 4$, $a^3 - 3a - 2$, and $3a^3 + 5a^2 + a - 1$.
35. $a^2 - 4ax + 3x^2$, $a^3 - 2a^2x + 2ax^2 - x^3$, and $a^3 - a^2x - ax + x^2$.

THE LEAST COMMON MULTIPLE.

74.—A *multiple* of a quantity is some other quantity which contains the given quantity an exact number of times.

Thus, 24 is a multiple of 24, 12, 8, 6, and 3; and a^3 is a multiple of a; $6a^2b^2c$ of $2ab$; and $a^2 - x^2$ of $a + x$.

75.—The LEAST COMMON MULTIPLE of a series of quantities is the smallest quantity that will contain each of the series an exact number of times.

Thus 30 is the L.C.M. of 5, 6, 3, 10, and 15; that is, 30 is the smallest number that is divisible by each of these numbers. Also, $6a^2bc$ is the L.C.M. of abc, $3a^2$, and $2ab$, that is, it is the quantity involving the *lowest powers* of the letters and the smallest numerical co-efficient, which will contain each of the quantities abc, $3a^2$, and $2ab$, an exact number of times.

Ex. (i.) Find the L.C.M. of

$$3a^2,\ 4a^2b^2,\ 6a^3b^4c,\ 2abc, \text{ and } 4b^2c.$$

We first note that the L.C.M. of the numerical co-efficients is 12.

We then examine the quantities presented to us, and find that they consist of powers of the letters a, b, and c. The highest power of a that occurs is a^3; of b, b^4; and of c, c. Inasmuch as our L.C.M. must contain each of these, it is evident that the L.C.M. must consist of those factors and the numerical co-efficient already noted, $\therefore$ L.C.M. $= 12a^3b^4c$.

76.—That is, the L.C.M. of a series of simple expressions may be found by taking, as factors, the L.C.M. of the numerical co-efficients and the highest power of each letter occurring in the series.

Ex. (ii.) Find the L.C.M. of $3a^2x$, $4ax^2$, $5x^3y$, and $10xy^3$.

L.C.M. of numerical co-efficients = 60. Highest powers of letters are a^2, x^3, y^3.

$$\therefore \text{ L.C.M.} = 60a^2x^3y^3.$$

EXAMPLES XXXIV.

Find the L.C.M. of—

1. $a,\ ab,\ a^2,\ b^2.$
2. $7xy^2,\ 3x^2y^3,\ 2xyz.$
3. $2a,\ 3b,\ a^2bc,\ 3a^2b^2,\ 2b^2c^2.$
4. $5a^2x,\ 3ay,\ 6a^2z^2,\ 7a^2xy.$
5. $6m^2,\ 7n^2,\ 2mn,\ 3m^3,\ 21.$
6. $18,\ 2a^2b,\ 3a^3b^2,\ 6abx,\ x^3.$
7. $11a^2b,\ 12b^2c,\ 66c^2d,\ 2ec.$
8. $9a^2,\ 6b^2,\ 18c^2,\ 2b^3,\ 3c^3.$
9. $3abcxy,\ 5a^3,\ 6x^4,\ 2y^3.$
10. $4x^2y^2z,\ 5abcx,\ 10b^2cxz.$

When one compound expression occurs in the series, it will appear as one of the factors of the L.C.M.

Ex. (iii.) Find the L.C.M. of $a^2b^2,\ ab^3,\ a^2(a-b).$

We have here the letters a and b, and the compound factor $a-b$. L.C.M. $= a^2b^3(a-b).$

11. $x^2y,\ xy^2,\ x+y.$
12. $x-y,\ x^2,\ axy.$
13. $x+y,\ x^2,\ y^2,\ xyz.$
14. $x^2(x+y),\ 4y^2,\ 5x^2y^2.$
15. $a^2+ab,\ b^2c,\ bc^3,\ abc.$
16. $a^2-a,\ b^2c^2,\ bc^3,\ b^3c^2.$
17. $x^2(1-x),\ yz,\ 6xy,\ 7z.$
18. $3a-3b,\ 4a^2b,\ 12ab^3.$
19. $a(x+y),\ b(x+y),\ c(x+y).$
20. $2ac(x-y),\ abc(x-y).$

77.—We now proceed to give a few examples of the method of finding the L.C.M. of compound quantities.

When the expressions are *prime* to each other (that is when they have no common factor), the L.C.M. will be their product.

Ex. (i.) Find the L.C.M. of $a+x$, $a-x$, and a^2+x^2.

Since these three expressions contain no common factor, the L.C.M. $= (a+x)(a-x)(a^2+x^2) = a^4-x^4.$

Ex. (ii.) Find the L.C.M. of

$$(a-x)^2,\ a^2-x^2,\ a^2+2ax+x^2, \text{ and } (a^4-x^4).$$

Here we must resolve each expression into its component factors that we may select only such as are necessary.

$$(a-x)^2 = (a-x)(a-x)$$
$$a^2-x^2 = (a+x)(a-x)$$
$$a^2+2ax+x^2 = (a+x)(a+x)$$
$$(a^4-x^4) = (a+x)(a-x)(a^2+x^2)$$

We now remove one of these factors *once* from each expression in which it occurs, writing it at the same time as a factor of the L.C.M. Remove the factor $a-x$ *once* from each expression, and write it as a factor of the L.C.M.

The series then stands thus—

$$a-x,\ a+x,\ (a+x)(a+x),\ (a+x)(a^2+x^2).$$

Now remove the factor $a+x$ *once* from each expression in which it occurs, writing it also as a factor of the L.C.M. The series then is $a-x$, 1, $a+x$, a^2+x^2, and these are now prime to each other. The L.C.M. will consist of those factors already written and those now left.

$$\begin{aligned}\text{L.C.M.} &= (a-x)(a+x)(a-x)(a+x)(a^2+x^2).\\ &= (a^2-x^2)(a^2-x^2)(a^2+x^2)\\ &= (a^2-x^2)(a^4-x^4)\end{aligned}$$

In practice, of course, the expressions need not be re-written. It will be enough to set down any factor in the L.C.M., and draw a line through it to signify its removal from any of the expressions in which it occurs.

Ex. (iii.) Find L.C.M. of x^2-xy, x^2y-y^3, x^3-y^3, and x^2+xy+y^2.

$$x^2-xy = x(x-y)$$
$$x^2y-y^3 = y(x^2-y^2) = y(x+y)(x-y)$$
$$x^3-y^3 = (x-y)(x^2+xy+y^2)$$
$$x^2+xy+y^2 = x^2+xy+y^2.$$

$$\text{L.C.M.} = xy(x-y)(x+y)(x^2+xy+y^2).$$

EXAMPLES XXXV.

Find the L.C.M. of—

1. $pq^2 - pr$, $mq^2 - mr$, and $q^2 - r$.
2. $am^2 - an^2$, $3(m + n)$, $2(m - n)$.
3. $2c^2 - 8d^2$, $c + 2d$, $3c - 6d$.
4. $a^2 - 1$, $4a^2 - 4$, $3a + 3$, $3a - 3$.
5. $y^2 - z^2$, $y + z$, $y^4 - z^4$, $y - z$, $y^2 + z^2$.
6. $a^5 - a^3b^2$, $(a^3 - a^2b)3a$, $a^2 + 2ab + b^2$.
7. $3ab(a^2 - ab)$, $ab^3(a^2x^2 - b^2x^2)$, $2b^2a + 2b^2b$.
8. $a^2 - b^2$, $a^2 - 2ab + b^2$, $(a + b)^2$, $a^4 - b^4$.
9. $a^2 - 4$, $(a + 2)^2$, $(a - 2)^2$.
10. $a^2 - 7a + 12$, $4a - 16$, $3a - 9$.
11. $x^2 + 6x + 8$, $x^2 + 2x - 8$, $x^2 - 4$.
12. $(x - 1)^3$, $x^2 - 2x + 1$, $x^2 - 1$, $x + 1$.
13. $a^3 - x^3$, $a^2 - x^2$, $x + a$, $a - x$.
14. $a^2 + a - 20$, $a^2 + 4a - 5$, $a^2 + 3a - 28$.
15. $a^4 - 5a^2 + 4$, $a^2 + a - 2$, $a^2 - a - 2$.
16. $x^2 - 1$, $(x + 1)(x + 2)$, $(x - 1)(x - 2)$, $x^2 + 4$.
17. $(x + 1)(x + 2)$, $(x + 2)(x + 3)$, $x(x + 3)$, $y(x + 4)$.
18. $x^2(x^2 - y^2)$, $x(x + y)$, $x^2(x - y)$, $x^3 - y^3$.
19. $1 - a$, $1 - a^2$, $1 - a^3$, $1 - a^4$, $a + a^2$.
20. $3a^2 - 3x^2$, $6(a - x)^2$, $(a + x)^3$, $a - x$.
21. $a^2 - 1$, $a^3 - 1$, $a^3 + 1$.
22. $2(x^2 - y^2)$, $x^3 - 2x^2y + xy^2$, $(2x + 2y)^2$.
23. $(2m + 2)^2$, $(m + 1)^3$, $5 + 5m$, $m^2 - 1$.
24. $a^2 + 3a + 2$, $a^2 + 5a + 6$, $a^2 + 4a + 3$.
25. $a^2 - 1$, $a^2 + a - 2$, $a^2 + 3a + 2$, $a^2 - 2a + 1$.
26. $3m^2 - 12$, $6m^2 + 18$, $12m + 24$, $18m - 36$.
27. $(x - 1)^3$, $(2x - 2)^2$, $x^4 - 2x^2 + 1$.
28. $15x - 45$, $x^2 - 9$, $27 + 9x$, $5x - 15$.
29. $(6a - 6x)^2$, $12(a^2 - x^2)$, $54a + 54x$, $27a$.
30. $17a^2 - 68$, $51a - 102$, $a^2 + 3a + 2$, $a^2 - a - 2$.

78.—When it is required to find the L.C.M. of two expressions which cannot be readily resolved into factors, we find their G.C.M.; and dividing one of the expressions by the G.C.M., we multiply the quotient by the other expression, and the product is the L.C.M. required. Thus, to find the L.C.M. of $x^2+xy-2y^2$ and $5x^2-8xy+3y^2$ we first find their G.C.M. to be $x-y$; then

$$\frac{x^2+xy-2y^2}{x-y} = x+2y\text{ ; or } x^2+xy-2y^2 = (x-y)(x+2y).$$

But the other expression also is made up of $x-y$ and some other factor, since it is divisible by $x-y$,

$$\therefore\ (x+2y)(5x^2-8xy+3y^2)\text{ is the L.C.M.,}$$

since it contains all the necessary factors. When it is required to find the L.C.M. of three or more such expressions, we find the L.C.M. of two of them, then the L.C.M. of this and the third, and so on.

Find the L.C.M. of—

31. $2x^2-x-3$ and $3x^2+4x+1$.
32. $4x^2-x-3$ and $2x^2-7x+5$.
33. $1-9x+14x^2$ and $2-x-6x^2$.
34. $6x^2+11xy+4y^2$ and $4x^2-8xy-5y^2$.
35. $7x^2+6xy-y^2$ and $6x^2+xy-5y^2$.
36. $3x^2+5xy-2y^2$ and $x^2+5xy+6y^2$.
37. x^3+x^2-5x+3 and x^3-3x^2+3x-1.
38. $x^3+xy^2+2y^3$ and $x^3+x^2y+4y^3$.
39. x^2-2x-2, x^3-4x^2+3, and x^3-3x^2+2.
40. x^3+2x^2+9, x^3-8x+3, and x^2-3x+1.

VII.—FRACTIONS.

79.—A *fraction* is a part or parts of some unit.

We shall have a much better idea of an Algebraic fraction, and one which will facilitate the understanding of many of the operations about to be explained, if we bear in mind that it consists of two quantities, *one of which is to be* DIVIDED *by the other.*

The quantity to be divided is called the NUMERATOR.

The quantity which divides is called the DENOMINATOR.

Thus, in the fraction $\frac{a+x}{a-x}$, the numerator $a+x$ is supposed to be divided by the denominator, $a-x$.

When the numerator is less than the denominator, the fraction is called a *proper fraction.* When the numerator is greater than the denominator, and the division can be actually performed, either with or without a remainder, the fraction is called an *improper* fraction; since it is really not a fraction, but either a *complete quotient,* generally called a WHOLE NUMBER; or a whole number and a fraction, generally called a MIXED NUMBER, or MIXED FRACTION. Thus, $\frac{a^2-x^2}{a+x}$ is an improper fraction; because the division may be actually performed, giving the quotient $a-x$.

A *complex fraction* is one in which a fraction occurs in either the numerator or denominator, or in both. Thus,

$\frac{x}{x+\frac{a}{x}}$, $\frac{\frac{a}{a-b}}{a^2+b^2}$, and $\frac{\frac{a}{b}}{\frac{a}{a-b}}$ are complex fractions.

The RECIPROCAL of a fraction is unity divided by the fraction.

We shall presently learn that this is equivalent to inverting the fraction. Thus, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$; and the reciprocal of $\frac{a}{x+y}$ is $\frac{x+y}{a}$.

REDUCTION OF FRACTIONS TO THEIR LOWEST TERMS.

80.—When the same quantity is a factor of both the numerator and denominator of a fraction, it may be removed from both, since in the one its effect is to multiply and in the other to divide. When the greatest factor common to both has been thus removed, the fraction is said to be reduced to its lowest terms. The method of discovering this greatest common factor we have discussed under the head of the G.C.M.

Reduce to their lowest terms—

$$\text{Ex. (i.)} \quad \frac{a^2bx^2}{a^2x^3y}.$$

Here the G.C.M. is a^2x^2; and on removing this factor from numerator and denominator, the fraction becomes $\frac{b}{xy}$.

$$\text{Ex. (ii.)} \quad \frac{a^2-x^2}{a^2+2ax+x^2} = \frac{(a+x)(a-x)}{(a+x)(a+x)}.$$

Removing the common factor $a+x$, the fraction becomes $\frac{a-x}{a+x}$.

$$\text{Ex. (iii.)} \quad \frac{a^3-3a^2b+ab^2+b^3}{2a^3-3a^2b+2ab^2-b^3}.$$

Here the G.C.M. is found, by the process of division, to be $a-b$. The numerator being divided by $a-b$ gives $a^2-2ab-b^2$ as quotient, and the denominator being divided by $a-b$ gives

$2a^2 - ab + b^2$. The fraction then becomes, in its lowest terms, $\dfrac{a^2 - 2ab - b^2}{2a^2 - ab + b^2}$.

EXAMPLES XXXVI.

Reduce the following fractions to their lowest terms—

1. $\dfrac{x^2y}{a^2y}$.
2. $\dfrac{a^2b}{a^2c}$.
3. $\dfrac{lm^2n^2}{lmn}$.
4. $\dfrac{pq}{2p^2q^2}$.
5. $\dfrac{6c^2d^2}{12c}$.
6. $\dfrac{12ab(x-y)}{18ab(x-y)}$.
7. $\dfrac{3(a+b)}{6(a-b)}$.
8. $\dfrac{3x^2y^2z}{axyz - 2x^2yz^2}$.
9. $\dfrac{x^2 - xy}{ax - ay}$.
10. $\dfrac{lm - ln}{pm - pn}$.
11. $\dfrac{3xy - 6xy^2}{6axy - 12x^2y}$.
12. $\dfrac{24x^2y - 24xy^2}{24(x^2y^2 - xy)}$.
13. $\dfrac{ab^2 - a}{b^2c - c}$.
14. $\dfrac{a^2 + a^2y - ab}{ab + aby - b^2}$.
15. $\dfrac{x^2 - y^2}{(x-y)^2}$.
16. $\dfrac{x+y}{x^2 - y^2}$.
17. $\dfrac{a^2 + 2ax + x^2}{a^2 - x^2}$.
18. $\dfrac{(a+b)^3}{(a+b)^4}$.
19. $\dfrac{a^2 - b^2}{a+b}$.
20. $\dfrac{a-x}{a^3 - x^3}$.
21. $\dfrac{a^2b - b^3}{a^2c - b^2c}$.
22. $\dfrac{ab + ac}{ab^2 - ac^2}$.
23. $\dfrac{a+x}{a^2 + 2ax + x^2}$.
24. $\dfrac{a-x}{(a-x)^3}$.
25. $\dfrac{b^2 - c^2}{b^2 - 2bc + c^2}$.
26. $\dfrac{6ab - 8}{36a^2b^2 - 64}$.
27. $\dfrac{a^2 + 5a + 6}{a^2 + a - 6}$.
28. $\dfrac{a^2 - 6a - 7}{a^2 - 9a + 14}$.

29. $\dfrac{3x^2 - 3x - 18}{3(x+2)^2}$.

30. $\dfrac{a^2b^2 - 4}{a^2b^2 - 4ab + 4}$.

31. $\dfrac{x^2 + x(y+z) + yz}{(x+y)(y-1)}$.

32. $\dfrac{a^2 + 4a + 3}{a^2 + 6a + 5}$.

33. $\dfrac{9a^2 + 12a + 4}{15a^2 + 13a + 2}$.

34. $\dfrac{6a^2 + a - 1}{9a^2 - 12a + 3}$.

35. $\dfrac{5x^2 - 16x + 12}{10x^2 - 19x - 2}$.

36. $\dfrac{a^2 - 18ab + 17b^2}{17a^2 - 14ab - 3b^2}$.

37. $\dfrac{2x^2 - 5xy + 3y^2}{5x^2 - 8xy + 3y^2}$.

38. $\dfrac{7x^2 - 79x - 156}{77x^2 + 216x + 144}$.

39. $\dfrac{15a^2 - 13ax + 2x^2}{27a^2 - 3ax - 10x^2}$.

40. $\dfrac{a^2x^2 - 3abx + 2b^2}{3a^2x^2 - 2abx - b^2}$.

41. $\dfrac{x^3 - x^2y + y^3}{x^4 - x^2y^2 + xy^3 + y^4}$.

42. $\dfrac{2x^3 + x^2 - 2x - 1}{6x^3 + 13x^2 + 9x + 2}$.

Additional exercises may be formed by writing the examples of G.C.M. in the form of fractions to be reduced to their lowest terms.

REDUCTION OF FRACTIONS TO THEIR LOWEST COMMON DENOMINATOR.

81.—The reduction of a series of fractions to others of equal value, having a denominator common to all, is in some respects the converse of the operation of reducing fractions to their lowest terms. In this operation we multiply the numerator and denominator of each fraction by some quantity which will cause all the denominators to be the same. This common denominator ought to be the least possible quantity, and yet it must contain each of the given denominators. But the least quantity which will contain a series of numbers has been already treated of under the head of the L.C.M. Therefore the L.C.M. of the given denominators will be the denominator common to each of the new series of fractions.

Ex. (i.) Reduce $\frac{a}{b}, \frac{b}{c}, \frac{c^2}{b^2}, \frac{a^2}{b^3}, \frac{b^2}{c^2}$, to others of equal value, having a common denominator.

Here the L.C.M. of the denominators is b^3c^2. The first denominator, b, is contained b^2c^2 times in b^3c^2. Multiplying the numerator by b^2c^2, the first fraction becomes $\frac{ab^2c^2}{b^3c^2}$. Repeating this process for each of the fractions, the series becomes

$$\frac{ab^2c^2}{b^3c^2}, \frac{b^4c}{b^3c^2}, \frac{bc^4}{b^3c^2}, \frac{a^2c^2}{b^3c^2}, \frac{b^5}{b^3c^2}.$$

These are generally written

$$\frac{ab^2c^2,\ b^4c,\ bc^4,\ a^2c^2,\ b^5}{b^3c^2}.$$

Ex. (ii.) $\frac{a}{a+x}, \frac{x}{a-x}, \frac{1}{2(a^2-x^2)}$.

L.C.M. of denominators $= 2(a^2-x^2)$.

Dividing $2(a^2-x^2)$ by each denominator in succession, we have the quotients $2(a-x)$, $2(a+x)$, and 1. Multiplying the numerators by these, the fractions become

$$\frac{2a(a-x)}{2(a^2-x^2)}, \frac{2x(a+x)}{2(a^2-x^2)}, \frac{1}{2(a^2-x^2)}.$$

82.—Observe that when the L.C.M. consists simply of the denominators multiplied together, *i.e.*, when no two denominators have a common factor, the operation becomes in effect to *multiply each numerator by all the denominators except its own.*

Ex. (iii.) $\frac{a}{a-b}, \frac{b}{b-c}, \frac{c}{a-c}$.

These may at once be written

$$\frac{a(b-c)(a-c)}{(a-b)(b-c)(a-c)}, \frac{b(a-b)(a-c)}{(a-b)(b-c)(a-c)}, \frac{c(a-b)(b-c)}{(a-b)(b-c)(a-c)}$$

Examples XXXVII.

Reduce to fractions in their lowest common denominator—

1. $\frac{a}{b}, \frac{a}{bc}, \frac{b}{ac}$.

2. $\frac{x}{y}, \frac{x^2}{y^2}, \frac{y}{x^2}, \frac{y^2}{x}$.

3. $\frac{a}{b}, \frac{b}{c}, \frac{c}{d}, \frac{d}{a}$.

4. $\frac{a+b}{b}, \frac{c+d}{d}, \frac{a+c}{c}$.

5. $\frac{x-y}{x}, \frac{x+y}{y}, \frac{x+y}{xy^2}$.

6. $\frac{a+x}{ab}, \frac{a+y}{bc}, \frac{y-x}{ac}, \frac{1}{abc}$.

7. $\frac{a^2}{b^2}, \frac{b^2}{c^2}, \frac{a+b}{b}, \frac{a-b}{c}$.

8. $\frac{x-y}{3a}, \frac{x+y}{6b}, \frac{a+b}{4a^2}, \frac{a-b}{3b^2}$.

9. $\frac{x^2}{a^2}, \frac{y^2}{b^2}, \frac{x-y}{c^2}, \frac{x^2+y^2}{abc}$.

10. $\frac{m+n}{5a}, \frac{m-n}{4}, \frac{m}{10a^2}$.

11. $\frac{x}{a^2}, \frac{y}{b^2}, \frac{x-y}{a-b}$.

12. $\frac{a+x}{a-x}, \frac{a-x}{a+x}, \frac{1}{a^2+x^2}$.

13. $\frac{x}{x-y}, \frac{(x-y)^2}{x^2}, x^3$.

14. $\frac{x}{a+x}, \frac{y}{a-x}, \frac{z}{a^2-x^2}, \frac{1}{x^2}$.

15. $\frac{1-x}{1+x}, \frac{1}{1-x^2}, \frac{1+x}{(1-x)^2}$.

16. $\frac{1}{(x+y)^2}, \frac{1}{x^2-y^2}, \frac{1}{x-y}$.

17. $\frac{a}{b+c}, \frac{b}{c+d}, \frac{c}{d+e}$.

18. $\frac{1}{ax-a^2}, \frac{1}{ax+a^2}, \frac{x-1}{x^2-a^2}$.

19. $\frac{2a}{a^4-9}, \frac{x}{a^2+3}, \frac{1}{a^5-9a}$.

20. $\frac{x}{a^2-ab}, \frac{x^2}{a^2-b^2}, \frac{1}{a^2x-b^2x}$.

ADDITION AND SUBTRACTION OF FRACTIONS.

83.—When fractions are to be added or subtracted, we must first reduce them to their common denominator, and then combine their numerators according to the signs. The result will give the numerator of the new fraction, and the common denominator will be its denominator.

We proceed to give examples of the most usual forms of fractions that occur in practice; and on carefully studying those that are worked out, the student will experience little difficulty in dealing with the others.

Ex. (i.) Simplify $\frac{a}{b} + \frac{b}{c} - \frac{c}{a}$.

Here the L.C.M. is abc, and the fractions may at once be written—

$$\frac{a^2c + ab^2 - bc^2}{abc}.$$

Ex. (ii.) Simplify $\frac{a^2 + b^2}{a^2} + \frac{a - b}{a}$.

$$\text{L.C.M.} = a^2 \therefore \frac{a^2 + b^2}{a^2} + \frac{a - b}{a} = \frac{a^2 + b^2 + a(a - b)}{a^2}$$

$$= \frac{a^2 + b^2 + a^2 - ab}{a^2} = \frac{2a^2 - ab + b^2}{a^2}$$

Ex. (iii.) $\frac{3a - 2x}{4} + \frac{3x - 2a}{3} - \frac{x - a}{12}$.

$$= \frac{3(3a - 2x) + 4(3x - 2a) - (x - a)}{12}$$

$$= \frac{9a - 6x + 12x - 8a - x + a}{12} = \frac{2a + 5x}{12}.$$

84.—With a little practice, the student will be able to pass over the first step of the operation as shewn above, and to multiply the numerators without using brackets. Care must be taken, however, *when a negative sign occurs before a fraction, to change the signs of the numerator of that fraction.*

Ex. (iv.) $\frac{a + 1}{a - 1} + \frac{a - 1}{a + 1} - \frac{1 - a}{a^2 - 1}$.

L.C.M. $= a^2 - 1$.

The fractions may therefore be written—

$$\frac{(a + 1)^2 + (a - 1)^2 - 1 + a}{a^2 - 1}$$

$$= \frac{a^2 + 2a + 1 + a^2 - 2a + 1 - 1 + a}{a^2 - 1} = \frac{2a^2 + a + 1}{a^2 - 1}.$$

Ex. (v.) $\dfrac{x-y}{xy} - \dfrac{x-z}{xz} + \dfrac{y-z}{yz} = \dfrac{(x-y)z-(x-z)y+(y-z)x}{xyz}$

$$= \frac{xz-yz-xy+yz+xy-xz}{xyz} = \frac{0}{xyz} = 0.$$

Examples XXXVIII.

Simplify—

1. $\dfrac{a}{x} + \dfrac{b}{y} - \dfrac{c}{z}$.

2. $\dfrac{a-b}{a} + \dfrac{c-a}{c} + \dfrac{b-a}{b}$.

3. $\dfrac{a+1}{2} - \dfrac{a-2}{3} + \dfrac{a-3}{4}$.

4. $\dfrac{3a-b}{5} + \dfrac{b-a}{10}$.

5. $\dfrac{2x^2+3y^2}{2x^2} + \dfrac{2x-y}{3x} - \dfrac{5x^2+3y^2}{3x^2}$.

6. $\dfrac{a+b-1}{16} - \dfrac{b-a+2}{8} + \dfrac{1-a-b}{32}$.

7. $\dfrac{3(a-b)+2}{5} + \dfrac{4(b-a)-5}{6} + 1$.

8. $\dfrac{1-x+y}{3} - \dfrac{1+x-y}{5} - \dfrac{x+y-1}{6}$.

9. $\dfrac{x}{x+y} + \dfrac{x}{x-y}$.

10. $\dfrac{x+y}{x-y} - \dfrac{x-y}{x+y}$.

11. $\dfrac{1}{x-1} + \dfrac{1}{x+1} - \dfrac{1}{x^2-1}$.

12. $\dfrac{x}{x(x+1)} - \dfrac{1}{x-1} + \dfrac{x}{x^2-1}$.

13. $\dfrac{a+1}{a-b} + \dfrac{a-1}{a+b} + \dfrac{2a^2}{a^2-b^2}$.

14. $\dfrac{x}{x+y} - \dfrac{y}{x-y} - 1$.

15. $x - \dfrac{x(x-y)}{(x+y)}$.

16. $\dfrac{a}{a-x} + \dfrac{ax}{a^2-2ax+x^2}$.

17. $\dfrac{a+x}{(a+x)^2} - \dfrac{a^2}{(a+x)^3} + \dfrac{1}{a+x}$.

18. $\dfrac{x+y}{(x-y)^2} + \dfrac{x-y}{(x+y)^2}$.

19. $\dfrac{a-b}{ab} + \dfrac{c-a}{ac} + \dfrac{b-c}{bc}$.

20. $\dfrac{a}{2a-b} - \dfrac{a^3-b}{2a^3} - \dfrac{b}{2(2a-b)}$.

85.—Observe the following—

$$\frac{2}{a-1}-\frac{2}{a+2}-\frac{3}{(a+1)(a+2)}$$

In such examples as this, labour and confusion may often be avoided by simplifying only *two* of the fractions at first, and combining the result with the third. Thus, in the above, taking only the first two together, we have

$$\frac{2}{a-1}-\frac{2}{a+2}=\frac{2a+4-2a+2}{(a-1)(a+2)}=\frac{6}{(a-1)(a+2)}$$

Combining this with the third fraction, the original expression may now be written

$$\frac{6}{(a-1)(a+2)}-\frac{3}{(a+1)(a+2)}$$

$$=\frac{6(a+1)-3(a-1)}{(a-1)(a+1)(a+2)}=\frac{6a+6-3a+3}{(a-1)(a+1)(a+2)}$$

$$=\frac{3a+9}{(a-1)(a+1)(a+2)}.$$

21. $\frac{2}{1+x}+\frac{1}{1-x}+\frac{x}{1-x^2}$.

22. $\frac{x}{x+y}-\frac{2x+y}{2(x-y)}+\frac{x^2+3xy+2y^2}{2x^2-2y^2}$.

23. $\frac{4a-3b}{3(1-b)}-\frac{a+3b}{3-3b}+\frac{2b}{1-b}$.

24. $\frac{x}{x-y}+\frac{3x}{x+y}-\frac{2xy}{x^2-y^2}$.

25. $\frac{x}{y(x-y)}-\frac{y}{x(x-y)}$.

26. $\frac{a}{a+b}+\frac{ab}{a^2-b^2}-\frac{a^2}{a^2+b^2}-\frac{a^2b^2}{a^4-b^4}$.

27. $\frac{1}{(x-1)(x-2)}+\frac{1}{(x-2)(x-3)}$.

28. $\frac{1}{(a+1)(a+2)}-\frac{1}{(a+1)(a+2)(a+3)}$.

29. $\dfrac{x}{(x+3)(x-2)}+\dfrac{x}{(x-2)(x-3)}-\dfrac{x}{x^2-9}$.

30. $\dfrac{1}{(a+b)(a+c)}-\dfrac{1}{(a+c)(a+d)}$.

31. $\dfrac{1}{(a-x)(x-y)}+\dfrac{1}{(y-a)(a-x)}+\dfrac{1}{(x-y)(y-a)}$.

32. $\dfrac{x+2}{x+3}+\dfrac{x-3}{x-4}+\dfrac{x+4}{x-5}$.

33. $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}+\dfrac{x-3}{x-4}$.

34. $\dfrac{1}{3(x+1)}+\dfrac{1}{2(x-1)}-\dfrac{1}{x^2}$.

35. $\dfrac{x+y}{(x-y)^2}+\dfrac{x-y}{(x+y)^2}-\dfrac{2x}{x^2-y^2}$.

36. $\dfrac{x}{a+1}+\dfrac{x}{a-1}-\dfrac{ax}{a^2+2}$.

37. $a+b-\dfrac{2a}{a+b}+\dfrac{a^2-3ab}{a^2-b^2}$.

38. $1-\dfrac{b}{a+b}-\dfrac{b+a}{a-b}-\dfrac{3a}{b+a}$.

39. $\dfrac{a+b}{a^2+ab+b^2}-\dfrac{2a^2+ab}{a^3-b^3}+\dfrac{1}{a-b}$.

40. $\dfrac{1}{a^2+3a+2}-\dfrac{1}{a^2-a-2}+\dfrac{1}{a^2-1}$.

41. $\dfrac{x-1}{x^2-5x+6}+\dfrac{x-2}{x^2-4x+3}-\dfrac{2(x-3)}{x^2-3x+2}$.

42. $\dfrac{1}{a^2-2a-8}-\dfrac{1}{a^2-a-6}-\dfrac{2}{a^2-7a+12}$.

43. $\dfrac{1}{a-x}-\dfrac{1}{a+x}-\dfrac{2x-a}{a^2-x^2}+\dfrac{a}{a^2+x^2}$.

44. $1-\dfrac{1}{x-1}+\dfrac{x}{x+1}-\dfrac{2}{x^2-1}$.

45. $\dfrac{1}{a+b} + \dfrac{b}{a^2-b^2} - \dfrac{ab^4}{a^6-b^6}$.

46. $1 + \dfrac{3a}{2b} - \dfrac{6a}{2b+3a} - \dfrac{8b(b-3a)+18a^2}{2(4b^2-9a^2)}$.

47. $\dfrac{a^2+b^2}{a^2-b^2} - \dfrac{b}{a-b} - \left(\dfrac{a}{a+b} - \dfrac{a^2}{a^2-b^2}\right)$.

48. $\dfrac{2x^2-2x+1}{x^2-x} - \dfrac{x}{x-1} + \dfrac{1}{x}$.

49. $\dfrac{1}{2(3a-2b)} - \dfrac{1}{2(3a+2b)} + \dfrac{3a}{9a^2-4b^2}$.

50. $\dfrac{1}{2(a-1)} - \dfrac{1}{a-2} + \dfrac{1}{2(a-3)}$.

86.—The following is very important, and should have strict attention.

$$\frac{x}{(a-x)(x-b)} + \frac{b}{(b-x)(a-b)}.$$

Here there are apparently four factors in the denominators; but it will be observed that two of then, $x-b$ and $b-x$, differ only in their signs. If we can put $b-x$ into the form $x-b$ then we shall have only three factors to deal with.

87.—First let us note that we may change all the signs in the numerator and denominator of any fraction without altering the value of the fraction. This is obvious, inasmuch as it is equivalent to multiplying both numerator and denominator by -1.

Also let us note that if we change the signs of any one factor of a product, we change all the signs of the product. This may be verified by actual multiplication, thus—

$$(a+b)(a-c) = a^2+ab-ac-bc,$$

and, changing the signs of the second factor, $(a+b)(c-a) = ac-a^2+bc-ab$. Now in the expression

$$\frac{x}{(a-x)(x-b)}+\frac{b}{(b-x)(a-b)}$$

if we write $x-b$ in the denominator of the second fraction, instead of $b-x$, we change all the signs of that denominator, and must also change the signs of the numerator that the value of the fraction may remain unchanged. The expression may therefore be written

$$\frac{x}{(a-x)(x-b)}+\frac{-b}{(x-b)(a-b)}.$$

This is more usually written

$$\frac{x}{(a-x)(x-b)}-\frac{b}{(x-b)(a-b)}.$$

The L.C.M. $=(a-x)(x-b)(a-b)$. The expression therefore becomes

$$\frac{x(a-b)-b(a-x)}{(a-x)(x-b)(a-b)}=\frac{ax-bx-ab+bx}{(a-x)(x-b)(a-b)}$$

$$=\frac{ax-ab}{(a-x)(x-b)(a-b)}=\frac{a(x-b)}{(a-x)(x-b)(a-b)}=\frac{a}{(a-x)(a-b)}.$$

From this example, then, we learn that we may change the signs of any factor in the denominator of a fraction, if we change the sign preceding the fraction.

51. $\dfrac{1}{(x-a)(a-b)}+\dfrac{1}{(x-b)(b-a)}.$

52. $\dfrac{a}{(a-x)(a-y)}-\dfrac{b}{(b-x)(y-a)}.$

53. $\dfrac{1}{x+1}-\dfrac{1}{1-x}+\dfrac{x}{x^2+1}.$

54. $\dfrac{1}{(a+b)(a-b)}-\dfrac{1}{(a+b)(b-a)}.$

55. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(c-a)(b-c)}.$

56. $\dfrac{1}{a(a-b)(a-c)} + \dfrac{1}{b(a-b)(c-a)}$.

57. $\dfrac{1}{(x-y)(x-z)} + \dfrac{1}{(y-z)(y-x)} + \dfrac{1}{(z-x)(z-y)}$.

58. $\dfrac{a}{(a-b)(a-c)} + \dfrac{a}{(a-b)(c-a)} + \dfrac{a}{(b-c)(c-a)}$.

59. $\dfrac{x}{(a-b)(b-c)} + \dfrac{x}{(b-a)(a-c)} + \dfrac{x}{(c-a)(b-c)}$.

60. $\dfrac{1}{(x-1)(x-2)} + \dfrac{1}{(x-3)(1-x)} + \dfrac{1}{x(2-x)(3-x)}$.

MULTIPLICATION AND DIVISION OF FRACTIONS.

88.—To multiply fractions, multiply the numerators together for a new numerator, and the denominators together for a new denominator.

Thus : $\dfrac{a}{b} \times \dfrac{c}{d} = \dfrac{ac}{bd}$.

This may be proved as follows :—

Let $\dfrac{a}{b} = m$, and $\dfrac{c}{d} = n$; then $a = bm$, and $c = dn$. Multiplying these together, we get $ac = bdmn$.

Dividing by bd, $\dfrac{ac}{bd} = mn$; but $mn = \dfrac{a}{b} \times \dfrac{c}{d}$, since $\dfrac{a}{b} = m$, and $\dfrac{c}{d} = n$, $\quad \therefore \dfrac{a}{b} \times \dfrac{c}{d} = \dfrac{ac}{bd}$.

89.—To divide one fraction by another, invert the divisor and proceed as in Multiplication.

Thus : $\dfrac{a}{x} \div \dfrac{y}{z} = \dfrac{a}{x} \times \dfrac{z}{y} = \dfrac{az}{xy}$.

This may be proved as follows :—

Let $\dfrac{a}{x} = p$, and $\dfrac{y}{z} = q$; then $a = px$, and $y = qz$.

Multiplying these expressions by z and x respectively, we have $az = pxz$, and $xy = qxz$.

By division, $\frac{az}{xy} = \frac{pxz}{qxz} = \frac{p}{q}$. But since $\frac{p}{q} = p \div q = \frac{a}{x} \div \frac{y}{z}$

$$\therefore \frac{a}{x} \div \frac{y}{z} = \frac{az}{xy}.$$

90.—The process of Division may thus at once be transformed into that of Multiplication by inverting the divisor.

Before multiplying the factors to form the new numerator and denominator respectively, any factor occurring in each may be struck out, so that the new fraction may be presented in its lowest terms.

☞ Care must be taken not to treat *terms* as factors.

Ex. (i.) Multiply $2a$ by $\frac{x}{a^2}$.

$$2a \times \frac{x}{a^2} = \frac{2a}{1} \times \frac{x}{a^2} = \frac{2x}{a},$$

where the factor a has been struck out of the numerator and denominator.

Ex. (ii.) $\frac{x}{a^2} \div 2x = \frac{x}{a^2} \div \frac{2x}{1} = \frac{x}{a^2} \times \frac{1}{2x} = \frac{1}{2a^2}$.

Ex. (iii.) Multiply $\frac{6m^2}{n}$ by $-\frac{n}{12m}$.

$$= -\frac{6m^2}{n} \times \frac{n}{12m} = -\frac{m}{2}.$$

Ex. (iv.) $\frac{2ax}{3y} \div \frac{2x^2+2xy}{6ay+3y^2} = \frac{2ax}{3y} \times \frac{3y(2a+y)}{2x(x+y)} = \frac{a(2a+y)}{x+y}$

Ex. (v.) $\frac{a^2-x^2}{ax} \times \frac{x}{a+x} = \frac{(a+x)(a-x)x}{(a+x)ax} = \frac{a-x}{a}$.

Ex. (vi.) $\frac{(a+x)^2}{a-x} \div \left(1 + \frac{a}{x}\right) = \frac{(a+x)^2}{a-x} \div \frac{x+a}{x}$

$$= \frac{(a+x)^2}{a-x} \times \frac{x}{a+x} = \frac{(a+x)x}{a-x}.$$

Ex. (vii.) $\left(\frac{a+b}{a-b} + \frac{a-b}{a+b}\right) \div \left(\frac{a+b}{a-b} - \frac{a-b}{a+b}\right)$.

$$= \frac{(a+b)^2+(a-b)^2}{a^2-b^2} \div \frac{(a+b)^2-(a-b)^2}{a^2-b^2}$$

$$= \frac{2a^2+2b^2}{a^2-b^2} \times \frac{a^2-b^2}{4ab} = \frac{2(a^2+b^2)}{4ab} = \frac{a^2+b^2}{2ab}.$$

Examples XXXIX.

Simplify the following :—

1. $\frac{2a}{3x} \times \frac{b}{c}$.

2. $\frac{x}{y} \times \frac{y}{z} \times \frac{z}{x}$.

3. $\frac{6ax}{5y} \times \frac{y^2}{3a} \times \frac{10a}{y^3}$.

4. $\frac{m^2}{np} \times \frac{n^2}{mp} \times \frac{p^2}{mn}$.

5. $\frac{12ab}{c^2} \times \frac{a^2}{6b^2} \times \frac{b^2c^2}{2a}$.

6. $\frac{am^2}{bn^2} \div \frac{m}{n}$.

7. $\frac{ax}{by} \times \frac{bx}{ay} \div \frac{x^2}{y^2}$.

8. $\frac{a+x}{bc} \times \frac{b^2}{ac} \times \frac{c^2}{a+x}$.

9. $\frac{a+1}{b} \times \frac{a}{b+1} \times \frac{b+1}{a+1}$.

10. $\frac{(a+b)^2}{a^2(a-b)} \times \frac{b^2}{a+b} \times \frac{a-b}{a+b}$.

11. $\left(1 - \frac{b}{a}\right)\left(\frac{a}{a-b}\right)$.

12. $\frac{a^2-b^2}{a^2} \times \frac{b^2}{a+b} \div \frac{a-b}{a^2}$.

13. $\left(\frac{x}{y} + \frac{y}{x} + 1\right) \div \left(\frac{x}{y} + \frac{y}{x} - 1\right)$.

14. $\frac{a(a+b)}{a^2-2ab+b^2} \times \frac{a(a-b)}{a^2+2ab+b^2}$.

15. $\frac{a^2+ab}{a^2-ab+b^2} \times \frac{a^3+b^3}{a}$.

16. $\frac{x^4-y^4}{xy} \times \frac{x}{x^2+y^2} \div \frac{x-y}{y}$.

17. $\frac{a+b}{a-b} \times \frac{(a-b)^2}{a^3+a^2b} \div \frac{ab-b^2}{a}$.

18. $\dfrac{x^2-3x+2}{x^2+x-12} \times \dfrac{x-3}{x-2}$.

19. $\dfrac{a^3-3a^2b+3ab^2-b^3}{a^2-b^2} \times \dfrac{a^2+ab}{(a-b)^2}$.

20. $\dfrac{a^2-3b}{(a^2-b^2)^2} \times \dfrac{(a+b)^2}{a^2+3b} \div \dfrac{1}{a^2-2ab+b^2}$.

21. $\dfrac{a+3}{a^2+4} \div \dfrac{a^2-4}{a^2-3a} \times \dfrac{a}{a^2-9}$.

22. $\dfrac{x^2-x-2}{x^2+xy} \times \dfrac{x}{xy-y^2} \div \dfrac{x-2}{y}$.

23. $\dfrac{2x^2(1-ax)}{x^2-a} \times \dfrac{1}{2-2ax} \div \dfrac{x^2+a}{x^4+a^2}$.

24. $\left(\dfrac{1}{1+a} + \dfrac{a}{1-a}\right) \div \left(\dfrac{1}{1-a} - \dfrac{a}{1+a}\right)$.

25. $\left(\dfrac{1}{a} + 1\right) \times \left(1 - \dfrac{1}{a}\right) \div \left(a - \dfrac{1}{a}\right)$.

26. $\dfrac{x^2-xy+y^2}{x^2-y^2} \div \dfrac{x-y}{x+y} \times \dfrac{(x-y)^2}{x^3+y^3}$.

27. $\dfrac{x^2+5x+6}{x^2-3x-4} \div \dfrac{2x+4}{x+1} \times \dfrac{x-4}{x+3}$.

28. $\dfrac{(x-3)^2}{x-1} \div \dfrac{x^2-9}{x^2-2x+1} \times \dfrac{x+3}{x^2-4x+3}$.

29. $\left(\dfrac{x^2+xy+y^2}{x^4-y^4} \times \dfrac{x^3-xy^2}{x^3-y^3}\right) \div \left(\dfrac{(x-y)^2}{x^2-xy+y^2} \times \dfrac{x^3+y^3}{x}\right)$.

30. $\left(\dfrac{a}{b} + \dfrac{b}{a} - 2\right) \times \left(\dfrac{a}{b} + \dfrac{b}{a} + 2\right) \times \dfrac{a^2b^2}{a^2-b^2}$.

COMPLEX FRACTIONS.

91.—Referring to the definition of a complex fraction in **79**, we shall now be able to simplify such expressions by the processes already explained.

Ex. (i.) $$\frac{a-\frac{x^2}{a}}{1+\frac{x}{a}}=\frac{\frac{a^2-x^2}{a}}{\frac{a+x}{a}}=\frac{a^2-x^2}{a}\div\frac{a+x}{a}=\frac{a^2-x^2}{a}\times\frac{a}{a+x}$$
$$=(a-x).$$

92.—In any expression of the form of $\frac{\frac{a}{b}}{\frac{c}{d}}$, b and c are called the *means*; and a and d the *extremes*. The product of the *extremes* becomes the numerator of the expression, and the product of the *means* the denominator. If, therefore, any factor be common to both extremes and means, it is evident that it may be removed.

Ex. (ii.) Simplify $\dfrac{\frac{a+b}{a-b}+\frac{a-b}{a+b}}{\frac{a+b}{a-b}-\frac{a-b}{a+b}}$.

Simplifying the numerator first, we have
$$\frac{a+b}{a-b}+\frac{a-b}{a+b}=\frac{(a+b)^2+(a-b)^2}{a^2-b^2}$$
$$=\frac{a^2+2ab+b^2+a^2-2ab+b^2}{a^2-b^2}=\frac{2a^2+2b^2}{a^2-b^2}.$$

Again, for the denominator,
$$\frac{a+b}{a-b}-\frac{a-b}{a+b}=\frac{(a+b)^2-(a-b)^2}{a^2-b^2}$$
$$=\frac{a^2+2ab+b^2-a^2+2ab-b^2}{a^2-b^2}=\frac{4ab}{a^2-b^2},$$

which is the denominator in its simplified form. The whole expression therefore becomes
$$\frac{\frac{2a^2+2b^2}{a^2-b^2}}{\frac{4ab}{a^2-b^2}}=\frac{2a^2+2b^2}{a^2-b^2}\times\frac{a^2-b^2}{4ab}=\frac{a^2+b^2}{2ab}.$$

Ex. (iii.) Simplify $\dfrac{1}{x-1+\dfrac{1}{1+\dfrac{x}{4-x}}}$

Beginning with the part, $1+\dfrac{x}{4-x}$, we write that in the form $\dfrac{4-x+x}{4-x}=\dfrac{4}{4-x}$.

The original expression is then written $\dfrac{1}{x-1+\dfrac{1}{\dfrac{4}{4-x}}}$

And since $\dfrac{1}{\dfrac{4}{4-x}}=\dfrac{4-x}{4}$, the expression becomes $\dfrac{1}{x-1+\dfrac{4-x}{4}}$

Since $x-1+\dfrac{4-x}{4}=\dfrac{4x-4+4-x}{4}=\dfrac{3x}{4}$, the expression becomes $\dfrac{1}{\dfrac{3x}{4}}=\dfrac{4}{3x}$.

Examples XL.

Simplify—

1. $\dfrac{1+\dfrac{b}{a}}{a^2-b^2}$.

2. $\dfrac{x-\dfrac{2x}{3}}{x+\dfrac{1}{3}}$.

3. $\dfrac{\dfrac{x}{3}+\dfrac{1}{2}}{\dfrac{x}{6}}$.

4. $\dfrac{\frac{1}{3}(x-1)+2}{1\frac{1}{3}}$.

5. $\dfrac{\frac{1}{2}(2x-1)}{\dfrac{x}{4}+3\frac{1}{4}}$.

6. $\dfrac{\dfrac{a}{b}-\dfrac{b}{a}}{a+b}$.

7. $\dfrac{\dfrac{1}{\frac{1}{2}(x^2-y^2)}}{\dfrac{x-y}{x+y}}$.

8. $\dfrac{\dfrac{x^2-y^2}{x}}{x+y}$.

9. $\dfrac{a-\dfrac{1}{a^2}}{1-\dfrac{1}{a}}$.

10. $\dfrac{1-\dfrac{b^2}{a^2}}{\dfrac{b}{a}+1}$.

11. $\dfrac{\dfrac{b}{a}-\dfrac{a}{b}}{\dfrac{1}{a}+\dfrac{1}{b}}$.

12. $\dfrac{\dfrac{2a}{3}-\dfrac{b^2}{a+\dfrac{a}{2}}}{(a-b)^2}$.

13. $\dfrac{\dfrac{2(x-y)}{x+y}}{x-y}$.

14. $\dfrac{\dfrac{a+b}{b}-\dfrac{a+b}{a}}{ab-\dfrac{b^3}{a}}$.

15. $\dfrac{\dfrac{a}{b}+\dfrac{b}{c}}{\dfrac{a}{b}-\dfrac{b}{c}}$.

16. $\dfrac{\dfrac{3}{4}x-\dfrac{1}{2}(x-1)}{x-\dfrac{4}{x}}$.

17. $\dfrac{1-\dfrac{1}{2}x}{\dfrac{1}{3}x^2-\dfrac{1}{2}(x^2-1)+\dfrac{1}{6}}$.

18. $\dfrac{\dfrac{m}{m-1}+\dfrac{m}{m+1}}{\dfrac{m}{m+1}-\dfrac{m}{m-1}}$.

19. $\dfrac{\dfrac{1}{1-x}-\dfrac{1}{1+x}}{\dfrac{1}{1+x}-\dfrac{1}{1-x}}$.

20. $\dfrac{\dfrac{1}{3}x+\dfrac{x}{12}(x-5)}{x-\left(2-\dfrac{1}{x}\right)}$.

21. $\dfrac{\dfrac{1}{a-b}-\dfrac{1}{a-c}}{\dfrac{b^2}{a^2-a(b+c)+bc}}$.

22. $\dfrac{\dfrac{x-1}{x+1}+\dfrac{x+1}{x-1}}{\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}}$.

23. $\dfrac{1}{1-\dfrac{1}{x}}$.

24. $\dfrac{1}{1+\dfrac{1}{x-1}}$.

25. $\dfrac{x}{x-1+\dfrac{1}{x}}$.

26. $\dfrac{x-1}{x-2+\dfrac{1}{x}}$.

27. $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}$.

28. $\dfrac{1}{1-\dfrac{1}{1+\dfrac{1}{x}}}-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}$.

29. $\dfrac{1-\dfrac{1-\dfrac{1}{x}}{x}}{\dfrac{1}{x^2}}$.

30. $x^2-\dfrac{x^2-1}{1+\dfrac{x}{1+\dfrac{x}{1-x}}}$.

31. $\dfrac{2}{2-\dfrac{2}{2+\dfrac{2}{2-x}}}$.

32. $\dfrac{\dfrac{a-b}{a+b}+\dfrac{a+b}{a-b}}{\dfrac{a-b}{a+b}-\dfrac{a+b}{a-b}}$.

33. $\dfrac{\dfrac{a+b}{b}+\dfrac{2(b-a)}{a-b}}{\dfrac{1}{a}-\dfrac{1}{b}}$.

34. $\dfrac{\dfrac{m}{n}+\dfrac{n}{m}-2}{m-n}$.

35. $\dfrac{2+\dfrac{x}{y}+\dfrac{y}{x}}{x+y}$.

36. $\dfrac{\left(\dfrac{x}{y}+1\right)^2-\left(\dfrac{x}{y}-1\right)^2}{\left(\dfrac{x}{y}-1\right)^2+\left(\dfrac{x}{y}+1\right)^2}$.

37. $\dfrac{1+\dfrac{a^2}{b^2}+\dfrac{b^2}{a^2}}{\dfrac{a}{b}+\dfrac{b}{a}-1}$.

38. $\dfrac{\left(\dfrac{x}{y}+\dfrac{y}{x}\right)^2+\left(\dfrac{x}{y}-\dfrac{y}{x}\right)^2}{2\left(\dfrac{x}{y}\right)^2+2\left(\dfrac{y}{x}\right)^2}$.

39. $$\frac{\dfrac{x}{1+\dfrac{1}{x}}+\dfrac{x}{x+1}}{1-\dfrac{1}{x+1}}.$$

40. $$\frac{x^2-\dfrac{1}{x^2}}{x+\dfrac{1}{x}+\dfrac{x^2+1}{x^2}}.$$

MISCELLANEOUS FRACTIONS.

93.—The processes which have been explained may also be applied to the working of such examples as the following, which involve no new principle.

Ex. (i.) Multiply $\frac{a^2}{2}-\frac{a}{3}-\frac{1}{2}$ by $\frac{a}{3}-\frac{1}{2}$.

$$\begin{array}{llll}
\frac{a^2}{2} & -\frac{a}{3} & -\frac{1}{2} & \\
\frac{a}{3} & -\frac{1}{2} & & \\
\hline
\frac{a^3}{6} & -\frac{a^2}{9} & -\frac{a}{6} & \\
 & -\frac{a^2}{4} & +\frac{a}{6} & +\frac{1}{4} \\
\hline
\frac{a^3}{6} & -\frac{13a^2}{36} & & +\frac{1}{4}
\end{array}$$

Ex. (ii.) Divide $\frac{x^3}{6}-\frac{17x^2}{72}+\frac{x}{12}+1-\frac{2}{3x}$ by $\frac{x}{2}-\frac{1}{3}$.

94.—Here we note that both dividend and divisor are arranged by descending powers of x; and that, when x^3, x^2, and x have appeared, there occurs a term in which x does not appear at all, and after that it appears in the denominators of the fractions. That is, we have an expression in which we carry the powers of x lower than x itself, and in which it ceases to be a multiplying factor, and becomes a dividing one. The same order of descending powers might be represented by means of negative indices, as in **64** and **65**; and the student is recommended to compare the results there obtained with those given here, and to work some of the examples by both methods.

$$
\begin{array}{r|l}
\frac{x}{2} - \frac{1}{3} & \frac{x^3}{6} - \frac{17x^2}{72} + \frac{x}{12} + 1 - \frac{2}{3x} \quad \Big(\frac{x^2}{3} - \frac{x}{4} + \frac{2}{x} \\
& \frac{x^3}{6} - \frac{x^2}{9} \\
\hline
& \quad - \frac{x^2}{8} + \frac{x}{12} + 1 - \frac{2}{3x} \\
& \quad - \frac{x^2}{8} + \frac{x}{12} \\
\hline
& \qquad\qquad\quad 1 - \frac{2}{3x} \\
& \qquad\qquad\quad 1 - \frac{2}{3x} \\
\hline
\end{array}
$$

The same results might be arrived at by combining the expressions into single fractions, and treating them by the ordinary processes of multiplication and division of fractions; but the methods given here, following those for whole numbers, are preferable.

For instance, Example (i.), given above, might be written in the form

$$\frac{3a^2 - 2a - 3}{6} \times \frac{2a - 3}{6}$$

$$= \frac{6a^3 - 13a^2 + 9}{36} = \frac{a^3}{6} - \frac{13a^2}{36} + \frac{1}{4}.$$

Ex. (iii.) Square root—

$$
\begin{array}{l|l}
x & x^2 - 2 + \frac{4}{x} + \frac{1}{x^2} - \frac{4}{x^3} + \frac{4}{x^4} \; \Big(x - \frac{1}{x} + \frac{2}{x^2} \\
 & x^2 \\
\hline
2x - \frac{1}{x} & -2 + \frac{4}{x} + \frac{1}{x^2} \\
 & -2 \qquad + \frac{1}{x^2} \\
\hline
2x - \frac{2}{x} + \frac{2}{x^2} & \frac{4}{x} - \frac{4}{x^3} + \frac{4}{x^4} \\
 & \frac{4}{x} - \frac{4}{x^3} + \frac{4}{x^4} \\
\hline
\end{array}
$$

Examples XLI.

Multiply—

1. $x + \frac{1}{x}$ by $x - \frac{1}{x}$.
2. $\frac{2x}{3} - \frac{3}{2x}$ by $\frac{2x}{3} + \frac{3}{2x}$.
3. $x^2 + \frac{1}{x^2}$ by $x^2 - \frac{1}{x^2}$.
4. $\frac{x^2}{2} + \frac{x}{3} - \frac{1}{2}$ by $\frac{x}{3} + \frac{1}{4}$.
5. $\frac{x}{2} - \frac{1}{x} + \frac{1}{x^2}$ by $\frac{x}{3} - \frac{1}{x}$.
6. $\frac{a}{b} - 1 + \frac{b}{a}$ by $\frac{a}{b} + 1$.
7. $x^2 + x + \frac{1}{x} + \frac{1}{x^2}$ by $x - \frac{1}{x}$.
8. $\frac{x}{3} - \frac{2}{3x} + \frac{1}{x^2}$ by $\frac{x}{3} - \frac{2}{3x} - \frac{1}{x^2}$.
9. $\frac{x^2}{4} - \frac{2x}{3} + 1 - \frac{3}{2x}$ by $\frac{2x}{3} - 1 + \frac{2}{x}$.
10. $2 - \frac{2}{a} + \frac{3}{a^2} - \frac{3}{a^3}$ by $1 - \frac{1}{a} + \frac{2}{a^2}$.

Divide—

11. $x^3 - \frac{1}{x^3}$ by $x - \frac{1}{x}$.
12. $x^2 - \frac{1}{x^2}$ by $x + \frac{1}{x}$.

13. $4 - \dfrac{1}{x^2}$ by $2 + \dfrac{1}{x}\cdot$

14. $x^3 + \dfrac{8}{x^3}$ by $x + \dfrac{2}{x}\cdot$

15. $\dfrac{1}{x^2} - \dfrac{1}{y^2}$ by $\dfrac{1}{x} + \dfrac{1}{y}\cdot$

16. $\left(\dfrac{1}{a} + \dfrac{1}{b}\right)^2 - 1$ by $\dfrac{1}{a} + \dfrac{1}{b} - 1.$

17. $8a^3 - \dfrac{27}{a^3}$ by $2a - \dfrac{3}{a}\cdot$

18. $\dfrac{1}{16}a^4 - \dfrac{1}{a^2}$ by $\dfrac{a^2}{4} + \dfrac{1}{a}\cdot$

19. $a^3 - \dfrac{a^2}{3} + a - \dfrac{4}{3} - \dfrac{1}{a^2}$ by $a + \dfrac{1}{a}\cdot$

20. $2a^3 - 3a^2 - 2a + 5 - \dfrac{4}{a} + \dfrac{2}{a^2}$ by $a - \dfrac{2}{a}\cdot$

21. $\dfrac{x^2}{3} - \dfrac{2x}{3} + \dfrac{2}{3} + \dfrac{3}{x} - \dfrac{2}{x^2}$ by $x + 1 - \dfrac{1}{x}\cdot$

22. $x^6 - 4 + \dfrac{4}{x^3} - \dfrac{1}{x^6}$ by $x^3 - 2 + \dfrac{1}{x^3}\cdot$

23. Expand $\left(x - \dfrac{1}{x}\right)^2 : \left(\dfrac{2a}{3} + \dfrac{3}{4a}\right)^2 : \left(x - 2 + \dfrac{1}{x}\right)^2\cdot$

Find the square root of—

24. $\dfrac{a^2}{9} - 2 + \dfrac{9}{a^2}\cdot$

25. $\dfrac{x^2}{4} + 2 + \dfrac{4}{x^2}\cdot$

26. $9x^2 - 18 + \dfrac{9}{x^2}\cdot$

27. $x^6 - 4 + \dfrac{4}{x^6}\cdot$

28. $x^2 - 2x - 1 + \dfrac{2}{x} + \dfrac{1}{x^2}\cdot$

29. $\dfrac{x^2}{4} - 3x + 11 - \dfrac{12}{x} + \dfrac{4}{x^2}\cdot$

30. $a^4 + 2a^2 + 3 + \dfrac{2}{a^2} + \dfrac{1}{a^4}\cdot$

31. $\dfrac{a^2}{b^2} - \dfrac{2a}{b} + 3 - \dfrac{2b}{a} + \dfrac{b^2}{a^2}\cdot$

EXAMPLES XLII.

Find the value of—

1. $\dfrac{a^2 + b^2 - c^2}{ab - ac - bc}$, when $a = 1$, $b = \frac{1}{2}$, $c = -1$.

2. $\dfrac{x^2 - xy}{x^2 + y^2} - \dfrac{\sqrt{a^3 + b^3}}{2(a^2 - b)}$, when $a = x = 1$, $b = y = -1$.

3. $\left(\dfrac{a+b}{2a-b}+\dfrac{a-b}{2a+b}\right)\div\dfrac{ab}{a-b}$, when $a=\frac{1}{2}$, $b=-\frac{1}{2}$.

4. $\dfrac{x^3+y^3+9x^2}{x^3+y+4z}$, when $x=\frac{1}{3}$, $y=-1$, $z=\frac{1}{4}$.

5. $\dfrac{x-\dfrac{1}{x}}{x+\dfrac{1}{x}}$, when $x=-\dfrac{2}{3}$.

6. $\left(\dfrac{1}{1+x}+\dfrac{x}{1-x}\right)\div\left(\dfrac{1}{1-x}-\dfrac{x}{1+x}\right)$, when $x=1$, or 2, or *any* number.

7. $\dfrac{a+x}{ab-x}$, when $x=\dfrac{a}{b}$.

8. $\dfrac{\dfrac{a}{b}+\dfrac{2b}{a}}{a-b}$, when $a=2$, $b=\dfrac{1}{2}$.

9. $\dfrac{\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{a}{c}}{\dfrac{(a-c)^2}{c^2-a^2}}$, when $a=1$, $b=2$, $c=3$.

10. $\dfrac{x}{x+a}+\dfrac{x}{2x-a}$, when $x=\dfrac{2}{3}a$.

11. $\dfrac{x^2+1-\dfrac{3}{y^2}}{2x-y}$, when $x=\dfrac{1}{y}$.

12. Shew that the difference between any number and its reciprocal, divided by the sum of the reciprocal and 1, will give as a result one less than the original number.

13. Prove that the sum of any two numbers divided by their product is equal to the sum of their reciprocals.

14. Prove that if to any fraction its reciprocal and 2 be added, the sum will be equal to the square of the sum of the terms of the fraction, divided by their product.

15. If p be the difference between any fraction and its reciprocal, and q the sum of the same, prove that the fraction itself is equal to $\frac{1}{2}(p+q)$.

16. Prove that the sum of any two fractions, divided by the sum of their reciprocals, will be equal to the product of the fractions.

Simplify—

17. $\left(\frac{a}{b} - \frac{b}{c}\right)^2 - \left(\frac{a}{b} + \frac{b}{c}\right)^2.$

18. $\dfrac{1 - \dfrac{1}{x}}{1 - \dfrac{2}{x + \dfrac{1}{x+1}}}.$

19. $\dfrac{x^2 - y^2}{\dfrac{1}{x^2} - \dfrac{2}{xy} + \dfrac{1}{y^2}} \times \dfrac{y-x}{x^2y^2}.$

20. $\dfrac{x^3 - 4x + 15}{x^4 + x^2 + 25}.$

21. $\dfrac{1}{1 - \dfrac{1}{x}} - \dfrac{1}{x(x-1)} - 1.$

22. $\left(\dfrac{1 - \dfrac{b}{a}}{1 + \dfrac{b}{a}}\right)^2 - \left(\dfrac{1 - \dfrac{a}{b}}{1 + \dfrac{a}{b}}\right)^2.$

23. $1 - \dfrac{2ax}{(a+x)^2} + \dfrac{a-x}{a+x}.$

24. $\left(1 + \frac{a}{b}\right)\left(1 - \frac{a}{b}\right)\left(\frac{a^2+b^2}{b}\right).$

25. $\dfrac{x}{(x-a)(x-b)}+\dfrac{a}{(a-x)(x-b)}.$

26. $\dfrac{\dfrac{a}{b}+\dfrac{b}{a}-2}{a-b}+\dfrac{\dfrac{a}{b}+\dfrac{b}{a}+2}{a+b}.$

27. $\dfrac{a}{2(x-1)}-\dfrac{a}{x-2}+\dfrac{a}{2(x-3)}.$

28. $\dfrac{1}{x+2}+\dfrac{1}{x-2}+\dfrac{1+2x-x^2}{4-x^2}.$

29. $\dfrac{6m^3+m^2-1}{2m^3+3m^2+4m-3}.$

30. $a-\dfrac{1}{b+\dfrac{1}{a+\dfrac{ab}{a-b}}}.$

31. $\dfrac{1}{a+b}+\dfrac{b}{a^2-b^2}-\dfrac{ab^4}{a^6-b^6}.$

32. $\left(\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{b}{a-b}\right)-\left(\dfrac{a}{a+b}-\dfrac{a^2}{a^2-b^2}\right).$

33. $\dfrac{3a(3a-2b)+b^2}{2a-b}\div\dfrac{3a-b}{1-\dfrac{a}{2a-b}}.$

34. $\dfrac{1}{x(x-y)(x-z)}+\dfrac{1}{y(y-z)(y-x)}+\dfrac{1}{z(z-x)(z-y)}.$

35. $\dfrac{b}{2(a-b)}-\dfrac{b}{2(a+b)}-\dfrac{b^2}{a^2}-\dfrac{b^4}{a^2(a^2-b^2)}.$

36. $\dfrac{1}{a^2+4a+3}-\dfrac{1}{a^2-a-2}-\dfrac{1}{a^2+a-6}.$

37. $\dfrac{\{(x+y)^2-4xy\}\{(x-y)^2+4xy\}}{(x-y)^2}.$

38. $\dfrac{1-\dfrac{a^2}{(a+x)^2}}{x^2-a^2}\div\dfrac{x^2+2ax}{(x-a)(x+a)^3}.$

VIII.—SIMPLE EQUATIONS.

95.—An EQUATION is the Algebraical expression of an equality existing between two quantities. An expression of Equality which is universally true, whatever may be the value of the quantities composing it, is called an IDENTICAL EQUATION, or IDENTITY. The following are identities :—

$$a^2 - x^2 = (a + x)(a - x).$$

$$\frac{a^2 + 2ax + x^2}{a + x} = a + x.$$

$$(x + 1)(x - 3) = x^2 - 2x - 3.$$

AN EQUATION OF CONDITION is one in which the equality holds good only when the letters have a particular value assigned to them. Thus the equation,

$$x + 2 = 5,$$

is true only when $x = 3$, and under no other circumstances. It is with such equations that we have here to deal.

96.—That value of x which makes the expressed equality true is said to *satisfy the equation*, and it is called the ROOT of the equation. In finding this root from the given equation, we are said to *solve* the equation.

97.—When the equation contains only the first power of the unknown quantity x, it is called a SIMPLE EQUATION, an equation of *One Dimension*, or an equation of the *First Degree*. When the square of the unknown quantity is involved in the equation, it is called a *Quadratic Equation*, an equation of *two* dimensions, or an equation of the *Second Degree*. An equation is said to be of as many dimensions as the highest power of the unknown quantity involved in it.

98.—The following axioms should be observed. The equality expressed by an equation is not affected :—

I.—If the same or equal quantities be added to each side.

II.—If the same or equal quantities be taken from each side.

III.—If each side be multiplied by the same or by equal quantities.

IV.—If each side be divided by the same or by equal quantities.

On these axioms depend the various methods for the solution of equations which we shall proceed to notice.

99.—*Any term of an equation may be transposed from one side to the other, if its sign be changed.*

Let $x - a = b$.

By axiom I., add a to each side, then

$$x - a + a = b + a,$$

$$\therefore\ x = b + a,$$

which is equivalent to transposing a from one side to the other with its sign changed.

Again in the original equation, subtract b from each side by axiom II., then, $\quad x - a - b = 0,$
which is equivalent to transposing b with its sign changed.

100.—*All the signs on each side of an equation may be changed without interfering with the equality.*

Let $x - a = b - c$.

Then, by transposing all the terms, we have

$$c - b = a - x,$$

which, of course, is the same as

$$a - x = c - b,$$

the original equation with all its signs changed.

101.—Solve the following equations :—

Ex. (i.) $3x - 2 = 2x + 2.$

Transpose the known quantities to the right hand side of the equation, and the unknown quantities to the left hand side.

We then have $3x - 2x = 2 + 2.$

Combining the terms, $x = 4,$

and thus the value of x in the original equation is found.

Ex. (ii.) $6x - 2 = 7x - 7.$

Transposing, $6x - 7x = -7 + 2.$

Combining terms, $-x = -5.$

Changing signs, $x = 5.$

Ex. (iii.) $x + 4 = 2x + 7.$

$$x - 2x = 7 - 4.$$

$$-x = 3.$$

Changing signs, $x = -3.$

Ex. (iv.) $15x - 3 = 12x - 6.$

$$15x - 12x = -6 + 3.$$

$$3x = -3.$$

Dividing by 3, $x = -1.$

Ex. (v.) $5x - (2x + 20) - 10 = 0.$

Removing brackets, $5x - 2x - 20 - 10 = 0.$

Transposing, $5x - 2x = 20 + 10$

$$3x = 30$$

$$\therefore x = 10.$$

Ex. (vi.) $(x + 2)(x + 3) - 2 = x^2 + 6x.$

$$x^2 + 5x + 6 - 2 = x^2 + 6x$$

$$x^2 - x^2 + 5x - 6x = 2 - 6$$

$$-x = -4$$

$$x = 4.$$

☞ Observe that when x^2 or any higher power of x occurs in a simple equation, it will disappear in the process of solution.

EXAMPLES XLIII.

Solve the following equations :—

1. $3x - 5 = 2x - 4.$
2. $6 - 7x = 1 - 2x.$
3. $7x - 40 = x + 2.$
4. $12 - 13x = 18 - 10x.$
5. $17x - 123 = 20x - 150.$
6. $5x - (2x + 20) - 10 = 30.$
7. $3x + 2 = 2(x + 6) - 1.$
8. $x + 3(x - 16) - (x + 15) = 2(x - 20) - 2.$
9. $x - 2\{x - (2x - 1)\} = 2(x - 1).$
10. $(x + 1)(x - 2) = x^2 - 5(x - 3) - 1.$
11. $(x + 2)(x - 7) = x(x + 3) + x - 5.$
12. $(x + 6)(x + 7) = x(x - 1) - 14x - 42.$
13. $x + 2 - (x - 50) - (x - 49) = 0.$
14. $3x + 2 - (1 - x) = 2(x + 1) + 5.$
15. $x - 3\{x + 2(x - 1) - 3\} = 2x - 3(2 + x).$
16. $3(1 - 3x) - 4(x - 1) = 5(2 - x) + 5.$
17. $6(2 - 3x) - 4(10 - 2x) + 9x + 2(x - 3) = 0.$
18. $x(2 - x) - 4 + x(2 + x) - (x - 1) = 0.$
19. $6(x - 3) - 2(1 - x) = 3(x - 1) - 2.$
20. $x - (3 - 6x) - (21 + 5x) = 0.$
21. $x + 1 - (x - 12)(x - 13) = x(17 - x) - (x - 5).$
22. $x^2 - 3x + (11 - x) = x(x - 2) + 1.$
23. $(x + 1)(x - 3) - (x + 2)(x - 5) = 7.$
24. $x(x + 2) = x(x + 3) + 13(x + 8) - x.$

EQUATIONS CONTAINING FRACTIONS.

102.—Equations which contain fractions may be cleared of those fractions by multiplying every term of the equation by the L.C.M. of the denominators of the fractions, which we do in accordance with axiom III. given above.

Ex. (i.) $\frac{x}{2} + \frac{x}{3} = x - 7.$

Multiply by 6, the L.C.M. of 2 and 3, then

$$3x + 2x = 6x - 42.$$
$$x = 42.$$

Ex. (ii.) $\frac{x-1}{2} - \frac{x+1}{5} + \frac{2x-5}{6} = 2\frac{7}{15}.$

Here we write $\frac{37}{15}$ for $2\frac{7}{15}$, and multiply by L.C.M., 30.

$$15(x-1) - 6(x+1) + 5(2x-5) = 74$$
$$15x - 15 - 6x - 6 + 10x - 25 = 74$$
$$19x = 120$$
$$x = 6\tfrac{6}{19}.$$

103.—After a little practice the student may omit the first step given here, in which the multiplication is expressed by brackets; but, in doing so, he must be particularly careful to *change the signs of a fraction preceded by a negative sign.*

Ex. (iii.) $\frac{8}{5}(x-3) - \frac{3}{4}(3x-19) = \frac{3}{10}(x-13) + 1.$

Here brackets are used instead of the line usually employed in fractional expressions. The expression $\frac{8}{5}(x-3)$, for instance, is only another way of writing $\frac{8(x-3)}{5}$.

Multiplying by 20, we have

$$32(x-3) - 15(3x-19) = 6(x-13) + 20.$$
$$32x - 96 - 45x + 285 = 6x - 78 + 20$$
$$-19x = -247$$
$$x = 13.$$

104.—Equations are sometimes proposed in which the value of x is to be found in letters instead of numbers. These are called *literal* equations.

Ex. (iv.) $ax + bc = bx + ac.$

Putting all terms containing x on the left side, and all others on the right side, we have

$$ax - bx = ac - bc.$$

Collecting the co-efficient of x, we have

$$x(a-b) = c(a-b).$$

Dividing by $a-b$, $x = c$.

Observe particularly the process of collecting and dividing by the co-efficient $a-b$.

EXAMPLES XLIV.

Equations involving easy fractions :—

1. $x - \dfrac{2}{3} + \dfrac{x}{2} = 3x + \dfrac{5}{6}.$

2. $\dfrac{3x}{2} + \dfrac{2x}{3} = \dfrac{5x}{6} + 2\dfrac{2}{3}.$

3. $\dfrac{23x}{24} - \dfrac{3x}{8} = \dfrac{16x}{3} - 4\dfrac{3}{4}.$

4. $\dfrac{x-1}{2} + \dfrac{x-2}{3} = \dfrac{x-4}{6} + 1\dfrac{1}{3}.$

5. $\dfrac{x+2}{5} - \dfrac{x-3}{15} = \dfrac{3x-4}{10} + 7.$

6. $\dfrac{2x-1}{9} + \dfrac{3x+1}{4} - \dfrac{11x-5}{12} = 0.$

7. $\dfrac{2x-1}{7} - \dfrac{16x-3}{5} = \dfrac{2(2x-1)}{35}.$

8. $\dfrac{7x-3}{5} - \dfrac{4x-2}{9} = \dfrac{x}{5} + \dfrac{1-2x}{45}.$

9. $\dfrac{8-x}{9} + \dfrac{4-7x}{2} - \dfrac{17}{18} + 5x = 0.$

10. $\frac{1}{2}x + \frac{2}{3}(x-1) - \frac{x-4}{7} = x.$

11. $\frac{1}{5}(19x-3) - \frac{4}{15}(14x+1) = \frac{x-2}{10} - \frac{1}{6}(x-4).$

12. $\frac{3}{2}(1-3x) - 2(6x+5) = \frac{17}{2}(x-1).$

13. $\frac{2}{3}(x+2) + \frac{3}{4}(x-4)(x+3) = \frac{x(18x-1)}{24}.$

14. $\frac{1}{8}(3x-2) - \frac{3}{4}(x+2) + \frac{5}{16}(x+10) = \frac{1}{4}(x-2).$

15. $\frac{x}{9} + \frac{3}{4}(x-1) = \frac{1}{12}(x-9) + x - 2.$

16. $\frac{1}{7}x + \frac{x+2}{5} - x = \frac{1-2x}{3}.$

17. $\frac{1}{2}x + \frac{3x-1}{7} - 2(2x-1) = \frac{2}{3}x - 9\frac{1}{6}.$

18. $\frac{x-40}{5} = \frac{1}{10}x + \frac{1}{2}x - \frac{17}{20}x + 1.$

19. $\frac{2x}{3} + \frac{7(2x-1)}{2} + \frac{14x-3}{9} - \frac{22x-5}{4} = 3.$

20. $\frac{15(x-1)}{4} - \frac{3(x-2)}{5} + \frac{x+3}{20} = \frac{1}{10}(x+7).$

21. $ax - b = bx + a.$

22. $ax - 2b = c - bx.$

23. $x - a = bx - c.$

24. $\frac{x}{a} - \frac{a}{b} = \frac{x}{b} - \frac{a}{c}.$

25. $\frac{a}{x} - 1 = \frac{b}{x} + 1.$

26. $x(x+a) - x(x-a) = 1.$

27. $(a+x)^2 - 1 = (a-x)^2 + 1.$

28. $(x+a)(x+b) = x(x-a+b)$.

29. $\dfrac{x-1}{4} - \left(\dfrac{x-2}{6} - \dfrac{x-3}{8}\right) = \dfrac{1}{8}$.

30. $\dfrac{x+1}{2} + \dfrac{3x-1}{4} - (x+1) = \dfrac{2x-3}{3} - 1$.

31. $\dfrac{2x-3}{4} - \left(\dfrac{6x-5}{5} - \dfrac{3x-2}{10} + \dfrac{7}{20}\right) = 0$.

32. $\dfrac{x}{6} - \left\{\dfrac{2}{3} - \left(\dfrac{2x-1}{3} + \dfrac{x-4}{2}\right)\right\} = 1$.

33. $\dfrac{x+a}{b} - \dfrac{x+b}{a} = \dfrac{2(a^2-b^2)}{ab}$.

34. $\dfrac{3(x-1)}{2} - \left(\dfrac{2(x+2)}{3} - \dfrac{x-3}{4}\right) = 4$.

35. $\dfrac{1}{6}(x+3) = \dfrac{1}{3}(x-1) - \left\{\dfrac{1}{4}(x-2) - \dfrac{1}{5}(x-3)\right\}$.

36. $x - \left(7 - \dfrac{x-12}{3}\right) = \dfrac{x-4}{2} + \dfrac{x-8}{4}$.

37. $\dfrac{2x-a}{b} - \dfrac{3x-b}{a} = \dfrac{3a}{b} - \dfrac{8b}{a}$.

38. $\dfrac{1}{c}(ax-b) + \dfrac{1}{a}(bx-c) + \dfrac{1}{b}(cx-a) = 0$.

39. $\dfrac{x}{a} + \dfrac{x}{b} + \dfrac{x}{c} - \left(\dfrac{1}{ac} + \dfrac{1}{ab} + \dfrac{1}{bc}\right) = 0$.

40. $(x-a)(x-b) + a(a+b) = (x+b)(x-b)$.

105.—Sometimes it happens that the L.C.M. of *all* the denominators is too large to be conveniently used. Thus in the equation,

Ex. (i.) $\dfrac{x-2}{4} - \dfrac{3x-2}{16} = \dfrac{3x+1}{19} - \dfrac{x+2}{8}$

the L.C.M. of all the denominators is 304, which would be a very inconvenient number to multiply by. If, however, we omit the 19 for the present, and multiply by 16, which is the L.C.M. of the other denominators, we shall be enabled to collect some of the terms, and thus avoid using large numbers.

Therefore, multiplying by 16, we have

$$4x - 8 - 3x + 2 = \frac{16(3x+1)}{19} - 2x - 4.$$

Collecting terms, we have

$$3x - \frac{16(3x+1)}{19} = 2.$$

Multiplying by 19, the only remaining denominotor—

$$57x - 48x - 16 = 38.$$

$$9x = 54.$$

$$x = 6.$$

The student is recommended to solve this equation by using the L.C.M. 304, and to compare his work with that given above, that he may fully appreciate the advantage of employing this, the first artifice that has been presented to him to facilitate the solution of an equation.

EXAMPLES XLV.

1. $\frac{7x+4}{10} + \frac{3x+1}{5} - \frac{15x-4}{17} = \frac{x+2}{20}$.

2. $\frac{x+19}{5} + \frac{2(x-1)}{5} - \frac{10x}{11} = \frac{x-11}{10}$.

3. $\frac{6x+5}{13} - \frac{4x+7}{5} = \frac{x+3}{10}$.

4. $\frac{3x}{10} - \frac{x-3}{17} = \frac{x-20}{2} + \frac{x+5}{5}$.

5. $\frac{1}{5}(2x-3) + \frac{19}{30}x = \frac{1}{15}(7x+8) + \frac{1}{17}(x-2)$.

6. $\dfrac{10x-7}{11} + \dfrac{x-4}{22} + \dfrac{x}{4} = x + \dfrac{x-4}{23}$.

7. $\dfrac{3x-1}{20} + \dfrac{x-7}{13} = \dfrac{x+1}{2} - \dfrac{2x+1}{5}$.

106.—This method may be advantageously used when one of the denominators is a compound quantity.

Ex. (ii.) $\dfrac{6x-7}{7} + \dfrac{7-2x}{4} = \dfrac{x-2}{x+3} + \dfrac{5x}{14}$.

Multiplying by 28,

$$24x - 28 + 49 - 14x = \frac{28(x-2)}{x+3} + 10x.$$

$$21 = \frac{28(x-2)}{x+3}$$

$$21x + 63 = 28x - 56$$

$$7x = 119.$$

$$x = 17.$$

8. $5\frac{1}{2} - \dfrac{3-5x}{4-x} + \dfrac{6x+1}{12} = \dfrac{4x+7}{8}$.

9. $\dfrac{12x-5}{9} - \dfrac{7x-3}{5} = \dfrac{5}{6-x} - \dfrac{x+14}{15}$.

10. $\dfrac{1-3x}{32} + \dfrac{3x-2}{16} - \dfrac{3x+1}{4(x+2)} = \dfrac{3(2x-3)}{64}$.

11. $\dfrac{x+2}{8} - \dfrac{x-6}{12} = \dfrac{x-5}{x-10} + \dfrac{x}{24}$.

12. $\dfrac{3x-1}{5} - \dfrac{2x-3}{3} + \dfrac{x}{15} = \dfrac{x+2}{3x+2}$.

13. $\dfrac{2x-1}{7} + \dfrac{3x+2}{14} - \dfrac{x}{2} = \dfrac{x-4}{x-5}$.

14. $\dfrac{x-1}{2} - \dfrac{x+2}{x-5} + \dfrac{x}{6} = \dfrac{2(x+3)}{3}$.

15. $\dfrac{2x-1}{6} - \dfrac{3x+2}{9} + \dfrac{7}{18} = \dfrac{2}{x-6} - 2.$

16. $\dfrac{2(x+1)-3(x-2)}{7} - \dfrac{8(x+3)-5(x+4)}{9} + \dfrac{1}{63} = 0.$

17. $\dfrac{10x+17}{18} - \dfrac{13x-2}{17x-32} = \dfrac{7x}{12} - \dfrac{x+16}{36}.$

18. $\dfrac{2x-a}{b} - \dfrac{3x-b}{a} = \dfrac{3a^2-8b^2}{ab}.$

19. $\dfrac{6x+7}{13} + \dfrac{8x+1}{9} = 3 - \dfrac{2x+5}{7}.$

20. $x - \dfrac{2x+1}{11} - \left(\dfrac{2(x-4)}{3} + \dfrac{x-5}{3(x-1)}\right) = \dfrac{5(x+17)}{33}.$

21. $\dfrac{2(x-2)}{5} - \left(\dfrac{2(x-3)}{6} + \dfrac{x-3}{15}\right) = \dfrac{2x-(x+12)}{3(x+9)}.$

22. $x - \left(\dfrac{x-1}{5} - \dfrac{x-6}{2}\right) - \left(x - \dfrac{2x-1}{11}\right) = 0.$

23. $\dfrac{x}{2} - \left\{\dfrac{x}{3} - \left(\dfrac{x-6}{4} - \dfrac{x-3}{5}\right)\right\} = \dfrac{5}{6}.$

24. $\dfrac{x}{13} - \left\{\dfrac{x-2}{12} - \dfrac{x-1}{18}\right\} = 1 + \dfrac{7x}{36}.$

25. $\dfrac{x-2}{14} + \left(\dfrac{x-1}{3} - \dfrac{x-2}{7}\right) = \dfrac{7(6x-5)-5}{28}.$

26. $\dfrac{3}{x-1} - \left\{x - \left(\dfrac{x-3}{2} + \dfrac{x-2}{3}\right)\right\} + \dfrac{x}{6} = 0.$

27. $\dfrac{a-x}{2} + \dfrac{2(x-a)}{3} + \dfrac{17a}{12} = \dfrac{3(x-2a)}{4}.$

28. $\dfrac{x-(a-2x)}{5} - \left(\dfrac{4(x+3a)}{15} - \dfrac{3x-a}{17} + \dfrac{x-3a}{3}\right) = a.$

29. $\dfrac{1}{2}(x-a) + \dfrac{1}{3}(x-b) - \dfrac{5}{12}\{x-(a-b)\} + \dfrac{3b}{4} = \dfrac{a}{3}.$

30. $\dfrac{x-(a+b)}{5} - \left(\dfrac{a-(b+x)}{6} + \dfrac{x+a+b}{30}\right) = \dfrac{1}{3}b.$

107.—We shall now give a few examples containing fractions of a more complex nature. These may be easily solved by the methods adopted in the simplification of fractions.

Ex. (i.) $\frac{2}{3}\left(\frac{x}{4}+\frac{x}{3}-1\right)=\frac{5}{6}\left(\frac{x}{5}+2\right)+\frac{x}{3}-2.$

Removing the brackets, we have

$$\frac{x}{6}+\frac{2x}{9}-\frac{2}{3}=\frac{x}{6}+\frac{5}{3}+\frac{x}{3}-2.$$

Omitting $\frac{x}{6}$ which appears on both sides, and multiplying by 9,

$$2x-6=15+3x-18$$
$$x=-3.$$

Ex. (ii.) $5x+\frac{7x+9}{4x+3}=9+\frac{10x^2-18}{2x+3}.$

Here reducing the improper fraction $\frac{10x^2-18}{2x+3}$ to the form $5x-\frac{15x+18}{2x+3}$, the equation may be written

$$5x+\frac{7x+9}{4x+3}=9+5x-\frac{15x+18}{2x+3}.$$

Omitting $5x$, we have

$$\frac{7x+9}{4x+3}=9-\frac{15x+18}{2x+3}$$
$$=\frac{18x+27-15x-18}{2x+3}$$

that is, $\frac{7x+9}{4x+3}=\frac{3x+9}{2x+3}.$

Observe that this is in general the most convenient form to which equations with compound denominators can be reduced.

We now multiply the numerator of one fraction by the denominator of the other, thus:

$$\begin{aligned}(7x+9)(2x+3) &= (3x+9)(4x+3).\\ 14x^2+39x+27 &= 12x^2+45x+27\\ 2x^2 &= 6x\end{aligned}$$

Dividing by $2x$, $\qquad x = 3.$

Ex. (iii.) $\dfrac{x-2}{x-3} - \dfrac{x-3}{x-4} = \dfrac{x-5}{x-6} - \dfrac{x-6}{x-7}.$

Simplifying the left side only, it becomes by multiplication,

$$\frac{(x-2)(x-4)-(x-3)^2}{(x-3)(x-4)} \text{ that is, } \frac{x^2-6x+8-x^2+6x-9}{(x-3)(x-4)}$$

Collecting terms in numerator, we have

$$\frac{-1}{(x-3)(x-4)}$$

The right side in a similar way becomes

$$\begin{aligned}\frac{(x-5)(x-7)-(x-6)^2}{(x-6)(x-7)} &= \frac{x^2-12x+35-x^2+12x-36}{(x-6)(x-7)}\\ &= \frac{-1}{(x-6)(x-7)}\end{aligned}$$

$$\therefore \frac{-1}{(x-3)(x-4)} = \frac{-1}{(x-6)(x-7)}.$$

Changing signs and multiplying,

$$\begin{aligned}(x-6)(x-7) &= (x-3)(x-4)\\ x^2-13x+42 &= x^2-7x+12\\ 6x &= 30\\ x &= 5.\end{aligned}$$

EXAMPLES XLVI.

1. $\frac{1}{2}\left(\frac{x}{2}+\frac{x}{3}\right) = \frac{1}{3}\left(\frac{x}{2}-6\right)+\frac{3}{4}x.$
2. $\frac{1}{3}\left(\frac{x}{4}+1\right)+\frac{1}{2}\left(\frac{x}{2}-\frac{x}{4}\right) = x-\left(\frac{x}{2}+2\right).$

3. $\frac{1}{2}\left(\frac{x-1}{2}\right)+\frac{1}{3}\left(\frac{x+1}{2}\right)=\frac{1}{4}(13-x).$

4. $\frac{1}{3}\left(x-\frac{2}{3}\right)+\frac{1}{2}\left(x+\frac{1}{2}\right)=\frac{1}{6}\left(\frac{2x}{3}-1\right)-\frac{19}{36}.$

5. $\frac{2}{5}\left(\frac{x}{3}+\frac{x}{12}\right)-\frac{\frac{2}{3}x-\frac{x}{12}}{5}=21.$

6. $\frac{x-\frac{1}{2}}{\frac{1}{3}}+\frac{2x-\frac{3}{4}}{\frac{1}{4}}=10\frac{1}{2}x.$

7. $\frac{2}{3}\left(\frac{x}{2}-\frac{x}{3}\right)-\frac{3}{4}\left(\frac{x-1}{2}-\frac{x-2}{3}\right)=\frac{5\frac{1}{2}}{36}.$

8. $\frac{x-\frac{1}{2}}{2\frac{1}{2}}-\frac{x-\frac{1}{2}}{3\frac{1}{2}}-\frac{6}{7}=\frac{x-8}{2}.$

9. $\frac{x-a}{10}+\frac{x-6a}{6}=x-\frac{11a}{30}.$

10. $2x-(a+b)=a-(b-x)-\{a-(b-a)\}.$

11. $\frac{2}{3}\left(\frac{3x}{2}-\frac{3}{4}\right)+\frac{x}{2}=5\frac{1}{2}.$

12. $\frac{5x}{6}-\frac{x}{4}=\frac{5}{12}\left(x+\frac{4}{5}\right)-\left(\frac{x}{2}-\frac{3x}{4}\right).$

13. $\frac{x-2}{1\frac{1}{5}}-\frac{\frac{x}{4}+3}{4}=x-\frac{x-4}{1\frac{1}{2}}.$

14. $\frac{x}{2}\left(\frac{x}{3}-\frac{4}{x}\right)-\frac{5}{6}\left(\frac{3x}{2}-1\frac{1}{2}\right)=\frac{x^2}{6}-\left(\frac{3x}{4}+\frac{5}{6}\right).$

15. $\frac{5x-1}{15}+\left(\frac{x-2}{4}-3\right)=\frac{3}{4}\left(\frac{4x-20}{15}\right)-\frac{4}{15}.$

16. $\frac{1}{6}\left(x-\frac{1}{3}\right)+6\left(\frac{x}{3}-\frac{1}{6}\right)-\left(x+\frac{x}{9}\right)=0.$

17. $\frac{1}{2}\left(\frac{x}{4}+\frac{x}{2}\right)-\left(\frac{x}{8}+1\right)=\frac{2}{3}x-\left(\frac{x}{2}+\frac{4}{3}\right).$

18. $\frac{1}{4}\left(\frac{x+1}{3}\right)+\frac{2}{3}\left(\frac{x+7}{2}\right)-\frac{x+1}{4}=4.$

19. $\frac{x}{4} - \left(\frac{x}{7} - \frac{x}{4}\right) - \frac{1}{4}\left(\frac{3x}{2} - \frac{2x}{14}\right) + \frac{1}{28} = 0.$

20. $\frac{1}{5}\left(\frac{x-2}{6}\right) - \frac{1}{4}\left(\frac{x-12}{5}\right) + \left(\frac{x}{16} - \frac{x-22}{5}\right) = 0.$

21. $\frac{1}{4}\left(\frac{x+1}{2} - \frac{x-3}{6}\right) - \left(\frac{x-1}{8} + \frac{x-9}{12}\right) = 0.$

22. $\frac{1}{a}\left(\frac{2x}{b} - 1\right) - \frac{1}{b}\left(\frac{x}{a} + 1\right) = 1.$

23. $\frac{a}{b}\left(\frac{x+a}{a} + \frac{x-b}{b}\right) = \frac{a+b}{b^2}.$

24. $(x+\frac{1}{2})(x-\frac{7}{5}) = (x-1)(x-\frac{1}{2}).$

25. $\frac{a}{b}\left(\frac{b^2+x^2}{x}\right) = \frac{a}{b}(bc+x).$

26. $\frac{x+a}{a-b} - \frac{x+a}{a+b} = \frac{2b}{a+b}.$

27. $\frac{ab}{c}\left(\frac{x}{a} + \frac{x}{b} + \frac{x}{ab}\right) = \frac{a+b+1}{c}.$

28. $x(x+2a) - x(x+2b) = 2(a-b).$

29. $\frac{ab+x}{b} - \frac{ab-x}{a} - (a-b) = 1.$

30. $\frac{a}{b} - \frac{1}{ab}(a^2+bx) + \frac{1}{a}(a-x) + 1 = 4.$

31. $\frac{2x^2-a}{ax} - \frac{x-b}{a} + \frac{a-b}{b} = \frac{x}{a}.$

32. $\frac{1}{x+a} - \frac{1}{x+b} = \frac{b-a}{x^2-ab}.$

33. $a\left(\frac{x}{2} + \frac{3}{4}\right) + x = 2 + \frac{a(2a+3)}{4}.$

34. $\frac{x+1}{a+1} + \frac{x+2}{a+2} = 2.$

35. $\frac{1}{2}\left(x - \frac{a}{5}\right) - \frac{1}{3}\left(x - \frac{a}{4}\right) = \frac{1}{7}(x+a).$

36. $\frac{1}{2x}(2a-b) = -\frac{1}{x}(a+x).$

37. $(x-b)(x-c)+b^2+c^2 = x^2-bc.$

38. $\frac{5x-\frac{7}{3}}{4} - \frac{5-\frac{2}{3}x}{3} = 11.$

39. $\frac{x-\frac{5}{2}}{3} - \frac{3(x+1\frac{1}{3})}{5} + 3\frac{1}{2} = 0.$

40. $\frac{1}{6}(x+5) - \frac{2}{3}(3+2x) + \frac{1}{9}\left(\frac{x}{2} + \frac{2}{5}\right) = \frac{43x-110}{30}.$

41. $\frac{8x+12}{7} - \left(\frac{3x+1}{2} - \frac{1}{4}\right) = \frac{3-x}{2} - \frac{1}{3}\left(\frac{3-2x}{4}\right).$

42. $m\left(\frac{x}{2} - 1\right) + x = 3 + \frac{m}{2}.$

43. $\frac{x+1}{6} - \frac{2x-6}{7} = 6 - \frac{4x+13\frac{2}{3}}{14}.$

44. $\frac{\frac{2}{3}x-1}{3} - \frac{\frac{5}{6}(3x-14)}{9} = \frac{x-\frac{1}{2}}{2} + \frac{2\frac{3}{4}}{27}.$

45. $\frac{1}{x} + \frac{1}{2x} = \frac{1}{3x} + 3\frac{1}{2}.$

46. $\frac{1}{x-1} - \frac{1}{x+7} = \frac{1}{7(x-1)}.$

47. $\frac{x-2}{x-3} - \frac{x+2}{x+3} = \frac{3x+7}{2(x^2-9)}.$

48. $\frac{7}{x+1} - \frac{3}{x-1} = \frac{14}{x^2-1}.$

49. $\frac{1}{2(x+1)} - \frac{1}{3(x-1)} = \frac{x}{x^2-1}.$

50. $\frac{3}{x+1} - \frac{2}{x-1} = \frac{x}{(x+1)^2}.$

51. $\frac{2x^2-x+1}{x^2-1} = \frac{x}{x+1} + \frac{x}{x-1}.$

PROBLEMS PRODUCING SIMPLE EQUATIONS.

108.—There is a large class of problems, the solution of which by ordinary Arithmetical means would be very difficult, if not impossible, because some one particular always appears to be wanting to render the data complete. By the substitution of x for this missing quantity, and by then treating it as a real and given quantity, we can form an equation. This equation, on being solved, will give the value of x, which, if it is not the actual answer to the proposed problem, will enable us to find the answer with very little trouble. No rule can be given for the solution of such problems. A few examples will enable the student to understand the method of dealing with them; and if care be taken to follow the question closely after the substitution of x, it will be found that the statements in the question, on being set down in Algebraic form, will give an equation.

EXAMPLES XLVII.

Easy Problems not requiring the use of Fractions, and which may be attempted as soon as the Student has solved the Equations in Ex. XLIII.

Ex. (i.) To a certain number add 13, then twice the sum will be equal to 36. Find the number.

Let x = the number.

Then $x + 13$ is the *sum* referred to in the question. $2(x + 13)$ is an expression representing twice this sum. But this is said to be equal to 36. We thus form the equation

$$2(x + 13) = 36$$

$$i.e., \ 2x + 26 = 36$$

$$\therefore x = 5, \text{ the number required.}$$

1. What number is that to which, if 16 be added, twice the sum will be equal to 62?

2. If 5 be subtracted from a certain number, three times the remainder will be equal to 63. Find the number.

3. If 5 be added to a certain number, four times the sum will be equal to five times the original number. Find it.

4. A merchant loses £50 and then succeeds in doubling the money he has left. His whole money now is £150 more than he had at first. What was his original capital?

5. From the double of a certain number I subtract 1; and then twice the remainder is equal to 38. Find the original number.

6. If 7 be added to a certain number, three times the sum and 1 more will amount to 76. Find the number.

Ex. (ii.) Find two numbers whose sum is 20 and their difference 6.

Taking x to represent one of the numbers, we can obtain an expression to represent the other; for, since their *sum* is 20, by subtracting one of the numbers from this sum we leave the other.

$$\text{Then if } x = \text{one of the numbers,}$$
$$20 - x = \text{the other.}$$

But we are told that the difference between these is 6; and $x - (20 - x)$ represents this difference.

$$\therefore x - (20 - x) = 6;$$
$$\text{Whence } x = 13, \text{ the one number,}$$
$$\text{and } 20 - 13 = 7, \text{ the other.}$$

In the above notice particularly how the expression for the second number is obtained.

Find two numbers

7. Whose sum is 31, and difference 3.
8. Whose sum is 43, and difference 9.
9. Whose difference is 9, and sum 201.

10. One of which is greater than the other by 5, and which together amount to 33.

11. One of which is less than the other by 6, and whose sum is 56.

12. A and B enter into partnership with a joint capital of £625, A's capital being £105 more than B's. What capital has each?

13. There is a difference of 3 years in the ages of two brothers, whose united ages amount to 31. What is the age of each?

Ex. (iii.) A is five years younger than B, and C's age is equal to A's and B's together. Their united ages amount to 90 years. What is the age of each?

Let x = A's age.

$x + 5$ = B's age, since he is 5 years older than A.

But A's age and B's age added together will give C's age.

That is, $2x + 5$ = C's age.

Now all these ages added together amount to 90.

$$\therefore x + (x + 5) + (2x + 5) = 90.$$

$$\begin{aligned} \text{Whence } x &= 20, \text{ A's age,} \\ 20 + 5 &= 25, \text{ B's age.} \\ 20 + 25 &= 45, \text{ C's age.} \end{aligned}$$

14. There are three brothers, of whom the eldest is 4 years older than the second, and the second 2 years older than the third. Their united ages amount to half a century. What is the age of each?

15. Three traders start with a capital of £2400. The first has £50 more than the second, and the second £50 more than the third. How much has each?

16. A man buys a suit of clothes for two guineas. For the trousers he gives 6 shillings more than for the vest, and for the jacket 6 shillings more than for the trousers. What did each cost?

Ex. (iv.) Divide 63 into three parts, so that the first part may be three times the second, and the second twice the third.

$$\text{Let } x = \text{the third part.}$$
$$\text{Then } 2x = \text{the second.}$$
$$\text{Also } 2x \times 3 = 6x = \text{the first part.}$$

All the parts together make the number 63.

$$\therefore\ x + 2x + 6x = 63\text{; whence } x = 7.$$

17. Divide 110 into three parts, so that the greatest may be twice the second, and that three times the third.

18. Divide £24 between A, B, and C, giving A twice as much as B, and C the same amount as A and B together.

19. Divide 121 into 2 parts, so that one part shall be three times the other and 1 more.

20. Divide 1375 into 4 parts, so that the third may be double of the fourth, the second equal to the third and fourth together, and the first equal to the second and third together.

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21. Find two numbers whose sum is 24, one being twice the other.

22. There are two numbers, one of which is greater than the other by 5; and their sum is 75. What are the numbers?

23. There are two numbers, one of which is three times as great as the other, and their difference is 14. Find them.

24. Divide 60 into two parts, so that one may be double of the other.

25. Divide 65 into three parts, so that the first may be twice the second, and the second 4 times the third.

26. Divide 23 into two parts, so that one may be 1 less than 3 times the other.

27. Divide a sovereign among three persons, giving the second twice as much as the first, and the third as much as the first and second together and two shillings more.

28. At the Oxford Local Examinations in a certain year there were 1221 Candidates, and the number of Juniors was three times that of the Seniors and 17 more. How many were there of each?

29. What number is that to which if 21 be added twice the sum is equal to nine times the original number?

30. Three times a certain number, less 1, are equal to twice the sum of the number and 8. Find the number.

31. There are two numbers whose difference is 2, and 4 times the smaller number are equal to 3 times the larger. Find them.

32. Find a number whose double added to 10 will require just the original number still to be added to make 100.

33. The double of a certain number is as much under 100 as the number itself is over 20. Find it.

34. There are two numbers whose sum is 35 and difference 1. Find them.

35. Find two numbers whose sum is 53 and difference 21.

36. Find two numbers whose sum is 160 and difference 80.

37. A and B have equal sums of money. They dine together, and A pays 10 shillings for the dinner. B gives a shilling to the waiter, and then he has four times as much money as A. How much had each at first?

38. A man has three sons, each 3 years older than the one next to him. Their united ages amount to 27 years. How old is each?

39. A father's age is 4 times that of his two sons together. One of the brothers is four years older than the other: and the united ages of all three amount to 50 years. What is the age of each?

40. Four partners raise a capital of £2200. B contributes twice as much as A and £50 more, while C contributes as much as A and B together, and D contributes £200 more than the other three together. What is the share of each?

Ex. (v.) B is 3 times as old as A. Ten years hence he will be only twice as old as A. Find their present ages.

$$\begin{aligned} \text{Let } x &= \text{A's present age.} \\ \text{Then } 3x &= \text{B's present age.} \\ x+10 &= \text{A's age 10 years hence.} \\ 3x+10 &= \text{B's age 10 years hence.} \end{aligned}$$

But we are told that B's age 10 years hence will be twice A's.

$$\therefore\ 3x+10 = 2(x+10)\,;\ \text{whence } x = 10.$$

Notice how the expressions for the ages at a future time are found. Had the problem related to a *past* instead of a future time, these would have been found by subtracting the number of years. Such a problem as the above may be varied in many ways. For instance, the problem, using the same numbers, might be stated as follows :—

A had three times as much money as B. When each had received 10 shillings, A had twice what B then had. How much had each at first?

The method of solution would be precisely that given above.

Ex. (vi.) I have a sum of £2 12s. 0d., which is composed of *an equal number* of Sovereigns, Crowns, and Shillings. How many are there of each?

$$\begin{aligned} \text{Let } x &= \text{the number of each.} \\ \text{Then } 20x &= \text{the value of the sovereigns expressed } \textit{in shillings}. \\ 5x &= \text{the value of the crowns expressed in shillings.} \\ x &= \text{the value of the shillings.} \end{aligned}$$

But all these together amount in value to 52 shillings.

$$\therefore\ 20x+5x+x = 52\,;\ \text{whence } x = 2.$$

☞ Notice particularly here that, in dealing with sums of money, they must be expressed in the same denomination. In this example, we express the sums of money as if they had all been given in shillings.

41. The elder of two brothers is twice as old as the younger: five years ago he was three times as old. Find the present ages of the brothers.

42. Divide £40 among three persons, so that the first may have five guineas less than the second, and the third ten shillings more than the first and second together.

43. Having £23, I gave a sovereign, a crown, a half-crown, a shilling, and a three-penny piece to each of a certain number of persons, and have nothing left. How many persons were there?

44. A and B play for a stake of ten shillings, A having three times as much money as B before they begin. B wins, and then they have equal sums. How much had each at first?

45. I bought for £10 15s. 0d. two sets of books, making in all 50 volumes. For the first set I gave 5s. 9d., and for the second set 3s. 4d. per vol. How many were there in each set?

46. I bought 20 volumes of books, part at 2s. per volume, and the rest at 1s. 6d. Had I bought one more volume at the former price I should have paid the same sum for each of the lots. How many were there in each lot?

47. A gives B £3, and then has six times as much money as B. Between them they have £28. How much had each at first?

48. A merchant doubled his capital with £50 more in one year. Next year he doubled the capital he then had. If he had gained £100 more, he would have had just five times what he began with. What was his original capital?

49. A hundredweight of tea, worth £19 12s. 0d., is composed of two sorts, part worth 4s. per lb., and the rest worth 2s. per lb. How much is there of each sort?

50. A barrel, holding 120 gallons, is to be filled with beer worth 1s. 2d. per gallon. The beer is drawn from two sorts, one worth 1s. 6d. and the other worth 1s. per gallon. How much of each sort will be required?

EXAMPLES XLVIII.

Problems introducing Easy Fractions.

Ex. (i.) Find a number greater than its third part by 10.

$$\text{Let } x = \text{the number.}$$

$$\frac{x}{3} = \text{its third part.}$$

$$\frac{x}{3} + 10 = x; \text{ whence } x = 15.$$

1. Find a number which is greater than its fourth part by 24.

2. Find a number which, increased by its sixth part, is equal to 35.

3. Find a number such that its third and fourth parts are together equal to 21.

4. On multiplying a certain number by 5, and dividing the result by 3, I get 20 as a quotient. What is the number?

5. The sum of a certain number and 5, being divided by 4 times the number less 15, gives a fraction equal to $\frac{1}{3}$. Find the number.

6. A certain number, less its third and sixth parts, is equal to $16\frac{1}{2}$. Find it.

7. Two-thirds of the difference between a certain number and 15 are equal to 8. Find the number. Shew that there are two solutions of this problem.

8. What number is that which is less by 7 than the half, third, and three-fourth parts of itself added together.

9. Find two numbers, whose difference is 4, such that the fourth part of the greater may be equal to the third part of the less.

10. A certain number, multiplied by 3, and divided by itself less 10, gives the quotient 9. Find the number.

11. A certain number less 5, and divided by 4, is equal to the number less 1, divided by 5.

12. Out of a certain sum of money I pay a bill of £5, and with two-thirds of what is left I pay another bill of £14. What had I at first?

13. There is a *proper* fraction, the difference betweeen whose numerator and denominator is 7; and if the numerator be increased by 1, the fraction is equal to $\frac{1}{2}$. Find the fraction.

14. Find two numbers, whose difference is 96, such that if the larger be divided by the smaller the quotient will be 3.

15. Find two numbers, whose difference is 6, such that if the smaller be divided by the larger the result will be $\frac{2}{3}$.

Ex. (ii.) Divide 48 into three parts, so that the second part may be half the first, and the third part one-fifth of the second.

$$\text{Let } x = \text{the first part.}$$

$$\text{Then } \frac{x}{2} = \text{the second part.}$$

$$\text{And } \frac{x}{2} \times \frac{1}{5} = \frac{x}{10} = \text{the third part.}$$

But all these parts together = 48.

$$\therefore x + \frac{x}{2} + \frac{x}{10} = 48; \text{ whence } x = 30.$$

16. Divide 135 into three parts, so that the second may be one-third of the first, and the third half the second.

17. Divide 153 into three parts, so that the first part may be half the second, and the third double of the first and second together.

18. A sum of £70 is divided among A, B, and C. B's share is three times C's, and only half A's. How much does each get?

19. Divide 511 into three parts, so that the second may be one-third of the first, and the third three-fourths of the first and second together.

20. A father dies, leaving £6300 to be divided among his family of two sons and a daughter. The daughter's share is less by one-fifth than that of her younger brother; and the elder brother gets a third as much again as the younger brother and daughter together. What does each receive?

Ex. (iii.) A basket contained three apples for every two oranges. A dozen of each is taken out, and then there are twice as many apples as oranges. How many of each were there at first?

$$\begin{aligned} \text{Let } x &= \text{the number of apples.} \\ \text{Then } \frac{2}{3}x &= \text{the number of oranges.} \\ x - 12 &= \text{the diminished number of apples.} \\ \frac{2}{3}x - 12 &= \text{the diminished number of oranges.} \\ \text{Then } x - 12 &= 2\left(\frac{2}{3}x - 12\right) \\ \text{Whence } x &= 36, \text{ the original number of apples,} \\ \text{And } \frac{2}{3} \text{ of } 36 &= 24, \text{ the original number of oranges.} \end{aligned}$$

21. An angler, at the conclusion of a day's sport, found that he had 45 fish. There were five perch to every three roach, and the perch and roach together were 8 times as many as the chub. How many were there of each?

22. Find two numbers, in the proportion of 7 and 9, whose difference is 12.

23. In a railway train there were 240 passengers. For every three third-class passengers there were two second-class, and there were half as many first as second. How many were there of each class?

Ex. (iv.) A can do a piece of work in 8 days which B can do in 6, and C in 12. In what time will they finish it working together.

Let x = the number of days in which they can perform the work, working together.

Since A can do $\frac{1}{8}$ of the whole in 1 day, he can do $\frac{x}{8}$ of the whole in x days.

Similarly B can do $\frac{x}{6}$ in x days; and C can do $\frac{x}{12}$ in x days.

All together in x days can do $\frac{x}{8} + \frac{x}{6} + \frac{x}{12}$; and these parts are together equal to the whole work.

$$\therefore \frac{x}{8} + \frac{x}{6} + \frac{x}{12} = 1.$$

Whence $x = 2\frac{2}{3}$ days, the time required.

24. A can do a piece of work in 10 days which B can do in 12 days, and C in 6. In how many days can they do it working together?

25. A and B working together can do a piece of work in 3 days. B alone can do it in 5 days. In what time can A do it?

26. A cistern can be filled by one pipe in 4 hours, and by a second in 6 hours; and it can be emptied by a third in 5 hours. In what time can the cistern be filled when all three pipes are running.

27. A starts running at the rate of $8\frac{1}{2}$ miles an hour; and half a minute later B starts to overtake him at the rate of 9 miles an hour. How long and how far will A run before he is overtaken?

28. A and B trade with equal sums of money. On comparing results at the end of the first year, it is found that A has gained £100, and that B has lost one-third of what he had at first, and has now only half what A has. How much had each at first?

29. Divide £230 among a man, his wife, and two children, so that the woman's share may be two-thirds of the man's, the man's and woman's shares together ten times the elder child's, whose share is to be twice the younger's.

30. A man has two sons, whose united ages are five years less than their father's. The father's age also is twice that of the elder son, and the younger son's is five-sixths of his brother's. Find the age of each.

31. A post is one-fourth of its length in mud, a sixth of its length in water, and 7 feet above water. What is its length?

32. In a national school three-twentieths of the pupils are in the infant department, five-twelfths in the girls' department, and 130 in the boys' department. What is the total number of the pupils?

33. In a regiment of soldiers five per cent. are in hospital, ten per cent. on leave of absence, five-sixths are on active service, and twelve are undergoing punishment. Find the total number in the regiment.

34. From a cask of wine a twelfth part was lost by leakage. Fifty gallons were then drawn, and the cask was found to be half full. How much did it hold?

35. A bath can be filled by the cold water pipe in 10 minutes, and by the hot water pipe in 15 minutes. It can be emptied by the waste pipe in 8 minutes. A servant having left all these pipes running, returns after a time and finds the bath half full. How long was the servant absent?

36. Suppose that in the preceding problem it had been stated that the servant returned after an absence of 16 minutes. In what condition would the bath be found?

37. A and B play for a stake of two shillings, A's money before the game being two-thirds of B's. A wins, and then he has seven-eighths of what B then has. How much had each at first?

38. Divide 156 into two parts, so that the remainder, on subtracting one-sixth of the greater part from one-third of the smaller, may be 7.

39. A grocer has some tea worth 2s. 9d. and some worth 2s. per lb. How much of each sort should be used to make a mixture of 60 lbs., worth 2s. 6d. per lb.

40. A can do a piece of work in 20 hours. B can do only two-thirds as much as A in the same time. How long will they take to do the work together?

Examples XLIX.

Miscellaneous Problems.

1. A trader by his first venture gains ten per cent. on his capital. By his second venture he loses a third of what he then has, and finds that he has remaining £800 less than he had at first. What was his original capital?

2. Two hundred and fifty tons of goods are to be conveyed in two sets of railway waggons. For this purpose 25 waggons of each set, or 40 of the first set and 15 of the second, will suffice. How many tons will a waggon of each set convey?

3. The first (*i.e.*, the left-hand) digit of a number consisting of two digits is three times the second. If the order of the digits be inverted the number will be decreased by 36. Find the number.

(Note. The left-hand digit must be multiplied by 10, when we wish to represent a number of two digits in symbols. Thus, if x be the first and 5 the second digit of a number, the number will be written $10x+5$.)

4. A person bought a certain number of eggs for 6s. Having broken a dozen and a half, he sold half the remainder, at the same rate at which he bought them, for 2s. 3d. How many had he at first?

5. In a regiment of soldiers, five-sixths of the whole are on duty, one-tenth are sick, and of the rest one-half are absent. What is the full strength of the regiment, including 30 officers?

6. A and B have the same annual income. A is extravagant, and contracts an annual debt amounting to one-seventh of his income, while B saves one-fifth of his. At the end of 10 years B lends A money enough to pay his debts, and has £160 to spare. What is the income of each?

7. At a railway station the number of tickets issued in a week gave a certain daily average; but Sunday, with its 50 tickets, being omitted, this average is increased by $16\frac{2}{3}$. How many tickets were issued in the week?

Ex. (i.) Find the time between 3 and 4 o'clock when the minute hand of a clock will be exactly over the hour hand.

Let x = the distance, as marked by minutes, travelled by the minute hand past three o'clock.

Then $\frac{x}{12}$ = the distance travelled by the hour hand, since it moves only one-twelfth as fast as the other.

But the minute hand starts 15 minutes behind the hour hand, and consequently has this distance to make up.

Or, in other words, when the hands are together, one has travelled 15 minutes more than the other.

But we have found that one travels x minutes and the other $\frac{x}{12}$ minutes, $\qquad \therefore x = \frac{x}{12} + 15.$

$$\text{whence } x = 16\frac{4}{11} \text{ minutes, the time past three o'clock.}$$

This problem may be varied in many ways, but if it be borne in mind that the minute hand moves 12 times as fast as the hour hand, reasoning similar to the above will make the solution easy enough.

8. At eleven o'clock the minute hand of a watch is 5 minutes in advance of the hour hand. When will it be the same distance behind?

Also, when will the hands be (i.) at right angles to each other, (ii.) exactly opposite to each other?

9. At what time after 6 o'clock will the hands of a watch be (i.) over each other, (ii.) at right angles; and at what o'clock will they next be exactly opposite to each other?

10. A sum of £350 is to be divided among 10 men, 30 women, and 50 children. If the 10 men receive as much as the 50 children, and each woman gets half a man's share, what will each individual receive?

11. A merchant sells a certain quantity of goods at £2 10s. 0d. per cwt., and 30 cwt. at £5 10s. 0d. per cwt., and he receives in all as much as if he had sold the whole at £4 6s. 0d. per cwt. How much did he sell at the first price?

12. A starts walking at the rate of $3\frac{1}{4}$ miles an hour. Twenty minutes afterwards B starts to overtake him at the rate of $3\frac{3}{4}$ miles per hour. How far will A walk before he is overtaken?

13. A garrison of 500 men is provisioned for 60 days. At the end of ten days a reinforcement arrives, and then the provisions will last only 40 days. What was the number of the reinforcement?

14. A workman was to be paid five shillings per day for his labour, and was to be fined one shilling for every day on which he should be absent. At the end of 10 days he had £1 12s. 0d. to receive. How many days did he work, and on how many was he fined?

15. The product of two numbers is 735. If the greater number be diminished by 5, and the smaller by 1, the product will be diminished by the greater of the numbers and 100 more. Find the numbers.

16. The difference between two digits of a number is 2; and if their order be inverted the number is increased by three times the greater digit. Find the number.

17. A railway waggon carries 10 tons of goods from London to Liverpool, at a total charge of £13 12s. 0d. per ton. The goods are of four kinds; and of the first kind there are 4 tons, of the second 2, of the third 3, and of the fourth 1. Goods of the first kind are charged only half, and those of the third kind four-fifths the rate of the fourth kind; and those of the second kind are charged 8s. per ton more than the first. Find the rate at which each kind is carried.

18. A can do a piece of work in three days and a half, which B can do in two days and three-quarters. In what time can they do it working together?

19. A and B can do a piece of work in $1\frac{1}{6}$ days, which B and C can do in $1\frac{2}{5}$ days, and A and C in $1\frac{3}{4}$ days. In what time can each do the work separately?

20. Divide the number 96 into two parts, so that the difference between the greater and 80 may be equal to five times the difference between the less and 40.

21. A person pays income tax at the rate of 6d. in the pound, and for rent and other charges 20 per cent. of the remainder. He then has £468 left. What was his original income?

22. A person buys oranges at the rate of three for twopence. He sells them, half at the rate of a penny each, and half at the rate of two for a penny, and gains a penny by the transaction. How many did he buy?

23. A person bought some eggs, half at 2 a penny, and half at 3 a penny. He sold them at the rate of 5 for twopence, and lost a penny by the transaction. How many did he buy?

24. A person bought some eggs at the rate of 5 for twopence, and sold half of them at 2 a penny, and half of them at 3 a penny, and thus gained twopence. How many eggs were there?

25. An army in a defeat loses one-tenth of its number. It is then reinforced by a number equal to one-sixth of the remainder. It is again defeated, and this time loses one-ninth of its number, when it is found that there remain 56,000 men. What was its original strength?

26. A grocer mixed tea at 2s. 3d. per lb. with 100 lbs. at 3s. per lb., so that the mixture was worth 2s. 6d. per lb. What quantity of the former sort was used?

27. A profit of £80 upon a capital of £4000 is divided between two partners, in proportion to their shares of the capital. One partner gets £16 more than the other. Find how much of the capital belonged to each.

28. A cistern leaks, so that it empties itself in 16 hours. Two taps are then turned on, one of which would fill it in 5 hours and 20 minutes, and the other in 3 hours and 12 minutes. What time will it take to fill the cistern, which continues to leak?

29. A gardener, planting cabbages in the form of a square, finds that he has eleven plants left after completing the square of the size originally intended; and that, if he had twenty more, he could increase the side of his square by one. How many plants had he at first, and how many did he plant in a row?

30. A dealer bought some sheep at £2 each. He sold a fourth of them at 45s. each, three-eighths at 50s. each, and the remainder at 55s. each, clearing £17 by the transaction. How many sheep did he buy?

SIMULTANEOUS EQUATIONS:

OR,

EQUATIONS CONTAINING MORE THAN ONE UNKNOWN QUANTITY.

109.—When it is required to find the value of more than one unknown quantity, we have presented to us for solution two or more equations, according to the number of unknown quantities.

110.—If a single equation contain more than one unknown quantity, it is evident that by assigning *any* value to one of the unknowns, we can obtain a value of the other. And because we may assign an infinite number of values to the one quantity, we shall have an infinite number of values of the other, and thus the solution of the equation would be impossible.

111.—We shall see, however, that if we have given to us a pair of equations or more, expressing different relations between the unknown quantities, we can so combine these that we shall have only *one* Equation containing only *one* unknown quantity, the solution of which by the means already known will enable us to find values of the other unknown quantities. Equations of this kind, in which the values of the unknown quantities depend on each other, are called SIMULTANEOUS EQUATIONS.

First Method of Solution.

112.—Let it be required to find the value of x and y from the equations

$$\begin{cases} 3x + 2y = 7 \\ 2x + 3y = 8 \end{cases}$$

Multiply the first equation by 2, and the second by 3, so that the co-efficient of x in each may be the same.

$$\text{(i.)}\quad 6x + 4y = 14.$$
$$\text{(ii.)}\quad 6x + 9y = 24.$$

Subtracting (i.) from (ii.), we *eliminate* x, and form one equation of one unknown quantity, $5y = 10$, whence $y = 2$.

Substituting this value of y in the first equation, we have

$$3x + 4 = 7.$$
$$\therefore x = 1, \text{ and } y = 2.$$

Another example of the same method :—

$$\begin{cases} \text{(i.)}\quad 3x - 2y = 1 \\ \text{(ii.)}\quad 4x + y = 16 \end{cases}$$

Multiply (ii.) by 2,
$$3x - 2y = 1$$
$$8x + 2y = 32$$

Add these equations, then

$$11x = 33; \text{ whence } x = 3.$$

Substituting this value in (i.), we have

$$9 - 2y = 1, \therefore y = 4.$$

113.—In this method we make the co-efficient of one of the unknown quantities the same in each of the equations, by multiplication; and then by addition or subtraction, according as the signs are the same or different, we *eliminate* one of the unknown quantities, and form a new equation, containing only the other. There are two other methods, which will be mentioned presently.

EXAMPLES L.

1. $3x + y = 9$
   $2x + 3y = 13.$

2. $x + y = 7$
   $x - y = 1.$

3. $2x + y = 0$
   $3x - 2y = -14.$

4. $4x - 11y = 9$
   $2x + 3y = 13.$

5. $2x - 3y = 1$
$2y - x = 6.$

6. $2x = 11 + 9y$
$3x - 12y = 15.$

7. $9x - 4y - 8 = 0$
$13x + 4y - 80 = 0.$

8. $x - 2y = 2$
$2x - y = 16.$

9. $5x + 3y - 75 = 0$
$6x + y - 51 = 0.$

10. $x = y$
$12x - 9 = y + 2.$

11. $6x + 2 = 4y$
$x + y = 8.$

12. $3x + 2y = 42$
$2x + 3y = 48.$

13. $2x - y = 13$
$3x - 2y = 7.$

14. $3x + 2y = 17$
$5x - 7y = 49.$

15. $7x + 3y = 2$
$2x - 4y = -14.$

16. $3x - 2y = 96$
$y - 5x = -20.$

17. $17x - 14y = 11$
$15y - 13x = -17.$

18. $19x + 3y = 17$
$14x + 4y = 0.$

19. $5x + 11y = 51$
$4x + 5y = 18.$

20. $7x + 4y = 15$
$13x - 13y = 130.$

114.—Equations of the following form must be cleared of fractions in the usual way, and then they may be treated by the method already given.

$$\text{Ex. (i.)} \begin{cases} \text{(i.)} & \dfrac{x+2}{4} = \dfrac{y-1}{2} \\[2ex] \text{(ii.)} & \dfrac{x+y}{5} = \dfrac{2y-3}{3} \end{cases}$$

Multiplying (i.) by 4, $2x + 4 = 4y - 4$,
and (ii.) by 15, $3x + 3y = 10y - 15$.

That is,
$$2x - 4y = -8$$
$$3x - 7y = -15$$
whence $x = 2$, and $y = 3$.

115.—We may now introduce the other methods of solving Simultaneous Equations.

Second Method.

Ex. (ii.) $\begin{cases} \text{(i.)} & ax = by \\ \text{(ii.)} & x + y = c. \end{cases}$

From (i.) we have $x = \dfrac{by}{a}$

And from (ii.) $x = c - y$.

Now, these two values of x may be written equal to each other, and thus we form the new equation,

$$\frac{by}{a} = c - y,$$

which contains only one unknown quantity. Solving the equation, we have

$$\begin{aligned} by &= ac - ay \\ ay + by &= ac \\ y(a + b) &= ac \\ y &= \frac{ac}{a + b} \end{aligned}$$

which is the required value of y.

Returning to the original pair of equations, we have

$$\begin{aligned} \text{from (i.) } y &= \frac{ax}{b} \\ \text{from (ii.) } y &= c - x. \\ \therefore \frac{ax}{b} &= c - x \\ ax &= bc - bx \\ ax + bx &= bc \\ x(a + b) &= bc \\ x &= \frac{bc}{a + b} \end{aligned}$$

which is the value of x required.

Thus in this method we find the value of one of the unknown quantities in each equation; and by equating these values, we obtain a new equation, whence we obtain the ultimate value of the other.

116.— *Third Method.*

Ex. (iii.) $\begin{cases} \text{(i.)} & \dfrac{x}{3} + \dfrac{y}{5} = 8 \\ \text{(ii.)} & \dfrac{x}{9} - \dfrac{y}{10} = 1. \end{cases}$

From (i.) we have

$$x + \frac{3y}{5} = 24$$

$$x = 24 - \frac{3y}{5}.$$

Substituting this value of x in (ii.), we have

$$\frac{24 - \frac{3y}{5}}{9} - \frac{y}{10} = 1.$$

whence $y = 10$.

Substituting this value in (i.), we have

$$\frac{x}{3} + \frac{10}{5} = 8$$

$$\therefore x = 18.$$

Instead of thus substituting the value of y, we might have proceeded to find x independently, as in the Second Method.

In the Third Method, then, we find a value of one unknown quantity in terms of the other from one equation, and substitute this value in the other equation.

Any one of the three methods may be employed in the solution of any Simultaneous Equation.

Examples LI.

1. $\dfrac{x}{3} + y = 6$

   $\dfrac{y}{2} + x = 8.$

2. $\dfrac{x}{4} + \dfrac{y}{5} = 4$

   $\dfrac{x}{2} + y = 14.$

3. $\frac{x}{7} + y = 18$

$\frac{1}{2}x = y - 9.$

4. $\frac{x}{3} - \frac{y}{2} = 3\frac{1}{2}$

$\frac{x}{2} + \frac{1}{2} = y + 14.$

5. $\frac{x}{5} - 2 = \frac{y}{10}$

$\frac{y}{5} - 6 = 2x - 3y.$

6. $\frac{x+y}{3} = 1$

$\frac{x+2y}{2} = 2.$

7. $\frac{2x+y}{2} = 2(y-x)$

$x + y = 1\frac{1}{2}.$

8. $\frac{x-y}{4} - \frac{2x+y}{12} = 0$

$x + y = 20.$

9. $\frac{x-y}{9} = \frac{x-2}{8}$

$\frac{y-1}{6} = x - 10y.$

10. $\frac{x}{2} + \frac{1}{4} = y + \frac{1}{3}$

$x + y = 1\frac{1}{6}.$

11. $\frac{x+y}{2} - \frac{x-y}{3} = 8$

$\frac{x+y}{3} + \frac{x-y}{4} = 11.$

12. $\frac{3x-1}{4} = \frac{68-7y}{3}.$

$x = 3y.$

13. $\frac{x+y}{2} - \frac{x-y}{3} = 13$

$\frac{x+y}{3} - \frac{x-y}{4} = 7\frac{5}{6}.$

14. $\frac{x+y}{8} - \frac{y-x}{2} = 1$

$\frac{x+1}{4} = \frac{y+1}{5}.$

15. $\frac{y-x}{3} = \frac{x+y}{8}$

$\frac{y-6}{5} = y - 2x.$

16. $\frac{2x+6y}{5} = \frac{x+3y}{4}$

$2x + 3y = 4.$

17. $\frac{x}{6} + \frac{y}{3} = x + 2$

$\frac{x}{3} - \frac{y}{9} = 0.$

18. $\frac{x}{5} - \frac{y}{3} = x - (y+4).$

$x - 2 = y + 2.$

19. $\dfrac{3x-y}{4}=\dfrac{1}{20}$

$\dfrac{y-x}{12}=\dfrac{3}{20}.$

20. $\dfrac{x+3}{10}+2=\dfrac{y+3}{6}$

$y-x=\dfrac{x+7}{6}.$

21. $\dfrac{x-y}{2}-\dfrac{x+y}{3}=\dfrac{y}{2}-2$

$\dfrac{2}{3}(x+y)=x.$

22. $\dfrac{x+y}{9}+\dfrac{x+y}{3}=x-6$

$\dfrac{x+2y}{8}+y=0.$

23. $2x-\dfrac{y+31}{4}=\dfrac{3y-2x}{5}$

$5y-\dfrac{8-x}{3}=24.$

24. $\dfrac{19y-5x}{14}=y-x-1.$

$\left(\dfrac{2}{3}x+\dfrac{3}{4}y\right)=2(x+y).$

25. $\dfrac{x}{2}-\left(\dfrac{y}{4}+1\right)=2$

$\dfrac{x}{4}+\dfrac{y}{12}=\dfrac{y-x}{3}+5.$

26. $\dfrac{2x+y}{2}-\dfrac{x+y}{3}=\dfrac{1}{2}+y$

$\dfrac{2x-y+3}{3}=2y-x+1$

27. $2(y-x)=\dfrac{2(x+y)}{3}+4$

$4(3x-y)+12=\dfrac{4}{11}(2y+x).$

28. $\dfrac{3(x-y)}{2}+x+y=7\dfrac{1}{2}$

$\dfrac{2}{3}(x-y)+x+y=8\dfrac{1}{3}.$

29. $\dfrac{x+y}{5}-\dfrac{x-2y}{3}=0$

$\dfrac{x+1}{2}-\dfrac{x-y}{3}=3\dfrac{1}{3}.$

30. $x+\dfrac{2}{3}y=\dfrac{2}{3}$

$\dfrac{x+y}{4}+\dfrac{1}{16}=x-y.$

31. $ax+by=c$

$ax-by=d.$

32. $ax+by=m$

$cx+dy=n.$

33. $ax+y=b$

$ay+x=c.$

34. $ax+by=1$

$cx+dy=2.$

35. $x+y=m$

$x-y=n.$

36. $ay+bx=ab$

$ax+by=\dfrac{a^2+b^2}{2}.$

37. $bx + ay = ab = ax + by.$

38. $bx + ay = 2ab$
$bx = ay.$

39. $x + y = m + n$
$my + nx = 2mn.$

40. $ax - by = 0$
$cx + by = bc.$

41. $\frac{x}{a} + \frac{y}{b} = c$
$\frac{x}{b} - \frac{y}{a} = d.$

42. $\frac{x}{a} = \frac{y}{b}$
$\frac{x}{b} - 1 = \frac{y}{a} + 1.$

43. $\frac{x}{a} + \frac{y}{b} = 1$
$\frac{x}{a} - \frac{y}{b} = 0.$

44. $a(x - c) = b(y - c)$
$ay = bx.$

118.—Observe the following equations, in which the unknown quantities appear in the denominators of fractions.

Ex. (iv.) $\begin{cases} \text{(i.)} & \frac{a}{x} + \frac{b}{y} = c \\ \text{(ii.)} & \frac{b}{x} + \frac{a}{y} = d. \end{cases}$

Multiply (i.) by b, and (ii.) by a, then

$$\text{(i.)} \quad \frac{ab}{x} + \frac{b^2}{y} = bc$$

$$\text{(ii.)} \quad \frac{ab}{x} + \frac{a^2}{y} = ad.$$

Subtracting (ii.) from (i.), the term $\frac{ab}{x}$ disappears, and we have

$$\frac{b^2}{y} - \frac{a^2}{y} = bc - ad.$$

That is, $y(bc - ad) = b^2 - a^2$,

and $y = \dfrac{b^2 - a^2}{bc - ad}$.

This process may be repeated, multiplying (i.) by a, and (ii.) by b, to enable us to eliminate y, and so to find the value of x.

Ex. (v.) $\begin{cases} \text{(i.)} & \dfrac{21}{x} + \dfrac{10}{y} = 5 \\ \text{(ii.)} & \dfrac{7}{x} + \dfrac{15}{y} = 4 \end{cases}$

Multiply (ii.) by 3, (i.) $\dfrac{21}{x} + \dfrac{10}{y} = 5$

(ii.) $\dfrac{21}{x} + \dfrac{45}{y} = 12$

Subtracting (i.) from (ii.), $\dfrac{35}{y} = 7$

That is, $7y = 35$

$y = 5.$

Substituting this value of y in (i.), we have $x = 7$.

45. $\dfrac{12}{x} + \dfrac{10}{y} = 9$

$\dfrac{6}{x} + \dfrac{8}{y} = 7.$

46. $\dfrac{a}{x} - \dfrac{b}{y} = c$

$\dfrac{c}{x} - \dfrac{a}{y} = b.$

47. $\dfrac{3}{x} + \dfrac{1}{y} = \dfrac{3}{5}$

$\dfrac{12}{x} - \dfrac{20}{y} = 0.$

48. $\dfrac{m-n}{x} + \dfrac{m+n}{y} = 1$

$\dfrac{m+n}{x} - \dfrac{m-n}{y} = 0.$

49. $\dfrac{64}{x} - \dfrac{7}{y} = \dfrac{1}{2}$

$\dfrac{32}{x} + \dfrac{14}{y} = 4.$

50. $\dfrac{17}{x} - \dfrac{9}{y} = -1$

$\dfrac{51}{x} - \dfrac{25}{y} = 1.$

51. $\dfrac{8}{x} - \dfrac{7}{y} = 9$

$\dfrac{1}{x} + \dfrac{2}{y} = \dfrac{2}{7}.$

52. $\dfrac{12}{x} + \dfrac{14}{y} = 5$

$\dfrac{18}{x} - \dfrac{21}{y} = \dfrac{3}{2}.$

53. $\dfrac{a}{x} + \dfrac{b}{y} = c$

$\dfrac{a}{x} - \dfrac{b}{y} = d.$

54. $\dfrac{4}{ax} + \dfrac{2}{by} = 6$

$\dfrac{3}{ax} - \dfrac{2}{by} = 1.$

EQUATIONS OF THEEE UNKNOWN QUANTITIES.

119.—If we have three equations expressing different relations between three unknown quantities, we proceed by reducing any two of the equations to one equation containing two only of the unknown quantities; and then, by reducing the third equation and one of the former to one equation containing only the same two unknowns, we have a pair of equations which may be solved by methods already known.

Ex. (i.)
$$\begin{cases} \text{(i.)} & x + y + z = 0 \\ \text{(ii.)} & 5x + 4y + 3z = 7 \\ \text{(iii.)} & 3x + 2y + 2z = 2. \end{cases}$$

$$\begin{aligned} &\text{(i.)} & 3x + 3y + 3z &= 0 \\ &\text{(iii.)} & 3x + 2y + 2z &= 2. \end{aligned}$$

By subtraction, $y + z = -2,$
which is a new equation, containing only two unknown quantities, y and z.

Again,
$$\begin{aligned} &\text{(i.)} & 5x + 5y + 5z &= 0 \\ &\text{(ii.)} & 5x + 4y + 3z &= 7 \end{aligned}$$

By subtraction, $y + 2z = -7.$
which is another new equation, containing the same two unknown quantities, y and z.

The two equations now are

$$\begin{aligned} y + z &= -2 \\ y + 2z &= -7. \end{aligned}$$

By subtraction, $z = -5.$

Writing this value instead of z in either of the new equations, we have $y = 3$.

Substituting these values of y and z in (i.), we have $x = 2$.

Then $x = 2$, $y = 3$, $z = -5$.

Examples LII.

1. $x+y+z = 6$
   $x-y+z = 0$
   $x+y-z = 4.$

2. $3x+2y+5z = 5$
   $7x-5y-2z = 2$
   $3x-4y+7z = -1.$

3. $x+y = 1$
   $y+z = -3$
   $x+z = 0.$

4. $2x+3y-2z = 12$
   $3x-2y+3z = 11$
   $x-2y+z = 1.$

5. $x+\frac{1}{2}y+\frac{1}{3}z = 10$
   $\frac{1}{2}x+y+\frac{1}{2}z = 9$
   $\frac{1}{4}x-2y-\frac{1}{4}z = -7.$

6. $6x-5y-7z = -4$
   $5x-2y+4z = 1$
   $10x+20y-17z = 50.$

7. $2x-3y+5z = -8$
   $3x+2y-20z = 71$
   $5x-2y-14z = 47.$

8. $2x+3y+4z = 3$
   $4x+6y-2z = 3\frac{1}{2}$
   $6x-9y+12z = 2.$

9. $x+y = z+8$
   $z-x = 2(y-x)$
   $x+z = y+10.$

10. $x+2y+z = 1$
    $3x+6y+z = 1$
    $x+y-9z = 1.$

11. $\frac{x+y}{3} = \frac{z+12}{4}$
    $\frac{y+z}{11} = 4-\frac{1}{4}x$
    $x+z = 25-\frac{1}{2}y.$

12. $\frac{1}{4}(2x+y) = 3-\frac{1}{6}z$
    $\frac{1}{4}\left(x-\frac{z}{3}\right) = 4-y$
    $\frac{1}{5}\left(x+\frac{z}{2}\right) = 5-y.$

PROBLEMS LEADING TO SIMPLE EQUATIONS OF MORE THAN ONE UNKNOWN QUANTITY.

EXAMPLES LIII.

Ex. (i.) Find two numbers whose sum is 25, and difference 11.

$$\begin{aligned} \text{Let } x &= \text{one of the numbers,} \\ \text{and } y &= \text{the other.} \\ \text{Then } x+y &= 25 \\ x-y &= 11, \\ \text{whence } x &= 18, \text{ and } y = 7. \end{aligned}$$

1. Find two numbers whose sum is 75, and difference 15.

2. Find two numbers whose sum is 30, and difference 4.

3. Find two numbers such that three times their sum may be 48, and four times their difference 32.

4. Find two numbers such that the first together with half the second may be equal to 14; and the second with one-fourth of the first may also be equal to 14.

5. There are two numbers whose sum is equal to eighteen times their difference; and twice the first is equal to the sum of the second and 21. Find the numbers.

6. Two-thirds of the difference between two numbers is 8; and three-fouths of their sum is $46\frac{1}{2}$. Find the numbers.

Ex. (ii.) What fraction is that which becomes 1, if 1 be added to its numerator; and $\frac{1}{2}$, if 1 be added to its denominator?

$$\begin{aligned} \text{Let } x &= \text{the numerator of the fraction,} \\ \text{and } y &= \text{its denominator,} \\ \text{then } \frac{x}{y} &= \text{the fraction.} \end{aligned}$$

By the question, $\dfrac{x+1}{y} = 1,$

$$\text{and } \frac{x}{y+1} = \tfrac{1}{2},$$

whence $x = 2$, and $y = 3$, and the fraction is $\frac{2}{3}$.

7. What fraction is that which becomes 1 if 1 be added to its numerator; and $\frac{1}{2}$ if 2 be added to its denominator?

8. What fraction is that which is equal to $\frac{7}{9}$ if 3 be taken from its numerator, and $\frac{8}{9}$ if 2 be added to its numerator?

9. The sum of two numbers, divided by their difference, gives the quotient 5; and twice the less is equal to the greater and 4 more. Find the numbers.

10. Find a proper fraction, the sum of whose numerator and denominator is equal to twice their difference and 9 more, and when the numerator is increased by 1, the fraction becomes $\frac{1}{2}$.

Ex. (iii.) It is found that 9 carts and 12 waggons will carry 22 tons; and that 12 carts and 9 waggons will carry 20 tons. What will each cart and each waggon carry?

$$\begin{aligned} \text{Let } x &= \text{what one cart will carry,} \\ \text{and } y &= \text{what one waggon will carry,} \\ \text{then } 9x + 12y &= 22, \\ \text{and } 12x + 9y &= 20, \\ \text{whence } x &= \tfrac{2}{3}, \text{ and } y = 1\tfrac{1}{3} \text{ tons.} \end{aligned}$$

11. Twenty-two tons of goods are to be carried in carts and waggons. Sixteen carts and seven waggons, or twenty carts and six waggons will suffice. What does each cart and each waggon carry?

12. Hampers of two sizes are used to pack cabbages for market. Ten large hampers and twelve small ones will hold 38 dozen; or four small and sixteen large will hold the same number. How many cabbages will a hamper of each size hold?

13. A certain number of two digits being divided by 23 gives as quotient half the smaller digit; and on the digits being inverted, the number is increased by 27. Find the number.

14 A and B play for a stake of five shillings. If A wins, he will then have three-fifths of what B then has; but if he loses, he will have only one-third of what B then has. How much had each at first?

15. Boxes of two sizes are used to pack cocoa nuts and oranges. Each large box will hold 10 cocoa nuts or 25 oranges; and each small box will hold 8 cocoa nuts or 20 oranges. When the large boxes are filled with cocoa nuts, and the small ones with oranges, there are 200 nuts and oranges together; but when the large boxes are filled with oranges, and the small ones with nuts, there are 206. How many boxes of each sort are there?

16. A certain number consists of three digits; and when the order of the digits is reversed, the number is diminished by 297. The sum of the digits is 8; and the left hand one, less the sum of the others, is 0. Find the number.

17. 10 sheep and 15 lambs cost £55 10s. 0d.; 8 sheep and 18 lambs cost £51 12s. 0d. What does one of each cost?

18. A person rows 6 miles with the stream in an hour; but, on returning, his time is an hour and a half. Find the rate of rowing, and the rate of the stream.

19. A person rows from Cambridge to Ely, a distance of 20 miles, and back, in 10 hours, and finds that he can row 2 miles against the stream in the same time that he rows 3 miles with it. Find the rate of the stream, and the time of going and returning.

20. A company at a tavern found when they paid their bill that if there had been 3 more persons each would have paid a shilling less, but if there had been 2 less, each would have paid a shilling more. Find the number of persons, and the amount each paid.

APPENDIX

CONTAINING THE

OXFORD AND CAMBRIDGE LOCAL EXAMINATION PAPERS.

EXAMINATION PAPERS.

OXFORD, 1868.

1. If $a = 1$, $b = -1$, $c = 2$, $d = 0$, find the value of

$$\frac{a^2 - b^2}{a^2 + b^2} + \frac{b^2 - cd}{2b^2 + cd} + \frac{c^3 - b^3}{3abc}.$$

2. Divide $a^4 + a^2b^2 + b^4$ by $a^2 + ab + b^2$.

3. Simplify

(1) $$\frac{2}{x^2 - 6x + 8} + \frac{3}{x^2 - 11x + 28} - \frac{5}{x^2 - 9x + 14}.$$

(2) $$\frac{x - 2}{x + \dfrac{1}{x - 1}} + \frac{2x^2 + x - 1}{x^3 + 1}.$$

4. Extract the square root of

$$36x^4 + 36x^3 - 15x^2 - 12x + 4.$$

5. Solve

(1) $9(x - 1) - 7(x - 6) - 5(x + 6) = 0.$

(2) $$\frac{x - 3}{5} - \frac{2 - x}{3} - \frac{1 - 2x}{15} = 0.$$

(3) $$\begin{cases} 3x - 4y = 25. \\ 5x + 2y = 7. \end{cases}$$

6. A is twice as old as B; twenty years ago he was three times as old. Find their ages.

OXFORD, 1869.

1. If $a = 1$, $b = -1$, $c = 2$, $d = 0$, find the value of

$$\frac{a+b}{a-b} + \frac{c+d}{c-d} + \frac{ad-bc}{bd+ac} - \frac{c^2-d^2}{a^2+b^2}.$$

2. Divide $a^4 + 4b^4$ by $a^2 - 2ab + 2b^2$.

3. Simplify

$$(1)\quad \frac{2a^2-2a+1}{a^2-a} - \frac{a}{a-1} + \frac{1}{a};$$

$$(2)\quad \frac{x^2-9x-36}{x^3+3x^2-9x-27}.$$

4. Extract the square root of

$$x^4 + 8x^3 + 24x^2 + 32x + 16.$$

5. Solve

$$(1)\quad 5(x-4) - 3(x-5) - \tfrac{1}{4}(x-6) = 0;$$

$$(2)\quad \frac{x^2-5x+4}{x-4} + \frac{x^2-5x+6}{x-3} = 3;$$

$$(3)\quad (x-a)(x-b) + a^2 + b^2 = x^2 - ab;$$

$$(4)\quad \begin{cases} \dfrac{x}{2} + \dfrac{y}{3} = 8 \\[2mm] \dfrac{x}{4} - \dfrac{y}{12} = 1. \end{cases}$$

6. Divide 24 into two parts, so that one part be seven times the other part.

OXFORD, 1870.

1. Divide

$$a^4 - a^2b^2 - a^2c^2 + b^2c^2 \text{ by } a^2 - ab + ac - bc.$$

2. Simplify

$$(1)\quad \frac{2a-1}{2a} - \frac{2a}{2a-1} + \frac{\frac{1}{2}}{2a^2-a};$$

$$(2)\quad \frac{4x^3+4x^2-7x+2}{4x^3+5x^2-7x-2}.$$

3. Find the square root of

$$a^{12} - 8a^9 + 18a^6 - 8a^3 + 1.$$

4. Solve the equations :

$$(1)\quad x - 6 - \frac{x-12}{3} = \frac{x-4}{2} + \frac{x-8}{4};$$

$$(2)\quad \begin{cases} x+y-z = 0, \\ x-y+z = 4, \\ 5x+y+z = 20. \end{cases}$$

5. Find two numbers, such that their sum is 51, and their difference is 15.

OXFORD, 1871.

1. Divide

$$6x^4 + 5x^3y + 6x^2y^2 + 5xy^3 + 6y^4 \text{ by } 2x^2 + 3xy + 2y^2.$$

2. Simplify

$$(1)\quad \frac{2x^2-x-15}{5x^2-13x-6};$$

$$(2)\quad \frac{1}{2(3x-2y)} - \frac{1}{2(3x+2y)} + \frac{3x}{9x^2-4y^2}.$$

3. Find the square root of

$$9x^4 - 30x^3y + 31x^2y^2 - 10xy^3 + y^4.$$

4. Solve the equations :

$$(1)\quad 12(x-3) - 3(2x-1) + 5x = 22;$$

(2) $\frac{1}{3}(x-\frac{5}{2})-\frac{3}{5}(x+\frac{4}{3})+\frac{7}{2}=0$;

(3) $\begin{cases} 2x-3=y, \\ 3y+4=5x. \end{cases}$

5. Find two numbers in the proportion of 5 to 8, and such that the greater of them exceeds the less by 9.

OXFORD, 1872.

1. Find the value of

$$\frac{(a-b)^2}{a+b}+\frac{(b-c)^2}{b+c}+\frac{(a-c)^2}{a+c}$$

when $a=5$, $b=3$, $c=1$.

2. What is the least common multiple of

$$12x(x-1)^2,\ 15(x^5-x^3),\ \text{and}\ 30x^2(x+1)\ ?$$

3. Simplify

(1) $\dfrac{3x^3-6x^2+x-2}{x^3-7x+6}$;

(2) $1\frac{1}{2}(3x-1)+2\frac{2}{3}(2x+3)-2\frac{1}{3}(x-\frac{3}{7})$.

4. Solve the equations:

(1) $\dfrac{2x-a}{b}-\dfrac{3x-b}{a}-\dfrac{3a^2-8b^2}{ab}=0$;

(2) $\sqrt{x+25}-\sqrt{x}=1$;

(3) $2x-3y+4=2x+6y-5=11$.

5. Two sums of money are together equal to £54 12s., and there are as many pounds in the one as shillings in the other. What are the sums.

OXFORD, 1873.

1. Multiply

$$9x^2 + 4y^2 + z^2 - 2yz - 3xz - 6xy \text{ by } 3x + 2y + z.$$

2. Find the greatest common measure of

$$3x^4 - 3x^3 - 6x^2 \text{ and } 6x^4 - 12x^3 + 6x - 12.$$

3. Extract the square root of

$$\frac{4x^2}{y^2} + \frac{4y^2}{x^2} - 8.$$

4. Simplify

$$(1)\ \frac{3x+2}{(x-1)^2} - \frac{6}{x^2-1} - \frac{3x-2}{(x+1)^2};$$

$$(2)\ \frac{1}{1+\frac{1}{a}} \times \frac{1}{1-\frac{1}{a}} \div \frac{1}{a-\frac{1}{a}}.$$

5. Solve the equations:

$$(1)\ \tfrac{1}{2}(x-1) + \tfrac{2}{3}(x+2) = \tfrac{9}{4}(x-3);$$

$$(2)\ 3x + \tfrac{1}{2}y = 21,\ 2y + \tfrac{1}{3}x = 14.$$

6. Find three numbers, whose sum is 21, and of which the greatest exceeds the least by 4, and the remaining one is half the sum of the other two.

OXFORD, 1874.

1. Multiply

$$a^2 + b^2 + c^2 + d^2 - a(b+c+d) - b(c+d) - cd \text{ by } a+b+c+d.$$

2. Extract the square root of

$$x^4 + \frac{1}{x^4} + 2\left(x^2 + \frac{1}{x^2}\right) + 3.$$

3. Simplify $\dfrac{x^2}{a + \dfrac{x^2}{a + \dfrac{x^2}{a}}}$.

4. Find the L.C.M. of

$$a^2 - x^2\,;\ a^2 + 3ax + 2x^2\,;\ a^2 - 4x^2\,;$$

and the G.C.M. of

$$2x^3 - 16x + 6 \text{ and } x^4 + 3x^3 + x + 3.$$

5. Solve the equations:

(1) $\dfrac{x+4}{8} + \dfrac{x+5}{15} = \dfrac{x+1}{6} + \dfrac{x+6}{9}$;

(2) $\dfrac{x}{8} + \dfrac{y}{12} = 1 = \dfrac{x}{2} - \dfrac{y}{6}$.

6. Find the time between 2 and 3 o'clock when the hands of a watch are together.

OXFORD, 1875.

1. Find the value of $3a - \frac{1}{2}\{3b - 7(c-d)\}$ when $a=7$, $b=4$, $c = -3$, $d = -5$.

2. Multiply $a^4 - 3a^2 + 2 + \dfrac{2}{a^2} + \dfrac{3}{a^4}$ by $a - \dfrac{1}{a}$; and divide

$$x^4 + 2x^3y + xy^3 + 2y^4 - 3x^3z - 3y^3z \text{ by } x + 2y - 3z.$$

3. Find the L.C.M. of $x^2 - 1$, $x^2 - 2x - 3$, $x^2 - 3x + 2$; and the G.C.M. of $x^4 - 1$ and $x^4 + 2x^3 + 4x^2 + 2x + 3$.

4. Simplify $\dfrac{a^3 - x^3}{ax}\left(\dfrac{a+x}{a^2 - x^2} - \dfrac{a-x}{a^2 + ax + x^2}\right)$.

5. Find the square root of

$$\frac{4x^4}{9} - 8x^3 + \frac{112}{3}x^2 - 12x + 1.$$

6. Solve the equations :

$$2x - \tfrac{1}{3}(x + 27) = 16\,;$$

$$x = \frac{2}{3} + \frac{x^2 - \frac{8}{3}}{x - 1}.$$

7. Divide a yard into two parts such that half of one part with 22 inches may be double the other part.

OXFORD, 1876.

1. If $a = 4$, $b = 1$, $c = -1$, $d = 0$, find the value of

$$(3a - 2b + 2c) - \{4b - (6c - 6d)\}.$$

2. Multiply $x + 1 + \frac{1}{x}$ by $x - 1 + \frac{1}{x}$; and divide

$$x^3 + y^3 + z^3 - 3xyz \text{ by } x + y + z.$$

3. Simplify

$$(1)\ \frac{x}{1 - \dfrac{1}{1+x}}\,; \qquad (2)\ \frac{a^2 - x^2}{\dfrac{1}{a^2} - \dfrac{2}{ax} + \dfrac{1}{x^2}} \times \frac{\dfrac{1}{a^2x^2}}{(a+x)}.$$

4. Find the square root of

$$x^4 + 2x^2 + 3 + \frac{2}{x^2} + \frac{1}{x^4}.$$

5. Find the L.C.M. of

$$6a^2(a^2 - b^2),\ 18ab(a^3 - b^3),\ 9b^2(a^3 + b^3)\,;$$

and the G.C.M. of

$$x^3 - 8 \text{ and } x^4 - x^3 - x^2 - x - 2.$$

6. Solve the equations:

$$(1)\quad 3x-2 = \tfrac{1}{2}(x+4)-2\tfrac{1}{3};$$

$$(2)\quad x+\frac{1}{x} = 2+\frac{1}{9x}.$$

OXFORD, 1877.

1. If $a=1$, $b=-2$, $c=0$, $d=4$, find the value of

$$\{b^2-(c+a)^2\}-\{d-(a-\overline{b-c})\}\{d+(b-\overline{c-a})\}.$$

2. Multiply $a+b-\frac{1}{c}$ by $a-b-\frac{1}{c}$; and divide

$$y^2z-yz^2+z^2x-zx^2 \text{ by } x-y.$$

3. Solve the equations:

$$(1)\quad \frac{x}{2}+\frac{3x}{4}-\frac{5x}{6} = 3\tfrac{1}{12}-\frac{(x+15)}{6};$$

$$(2)\quad x^2-3x = 3(9+x).$$

4. Find the square root of $\dfrac{y^2}{x^2}+2\dfrac{y}{x}+3+2\dfrac{x}{y}+\dfrac{x^2}{y^2}$.

5. Find the L.C.M. of

$$32x^3y^3(x-y)^2,\ 18x^2y(x^3-y^3),\ 9xy^2(x^2+xy+y^2);$$

and the G.C.M. of

$$3x^3+6x^2+2x-11 \text{ and } 4x^3+6x^2+2x-12.$$

6. Simplify:

$$(1)\quad \frac{\{(x+y)^2-xy\}\{(x-y)^2+xy\}}{x^4+x^2y^2+y^4};$$

$$(2)\quad \frac{1}{x^2-4x+3}-\frac{1}{x^2-3x+2}-\frac{1}{x^2-5x+6}.$$

OXFORD, 1878.

1. Find the value of

$$\frac{abc + bcd}{3c(a+b)} + \frac{ab}{d} + \frac{(b+c)d}{b}$$

when $a = 0$, $b = -1$, $c = 1$, and $d = 3$.

2. Multiply $x^2 + y^2 + z^2 - yz - zx - xy$ by $x + y + z$; and divide

$$x^3 + 8x^{-3} \text{ by } x + 2x^{-1}.$$

3. Simplify:

$$(1)\ \frac{12x^2 + x - 1}{1 - 8x + 16x^2} \div \frac{1 + 6x + 9x^2}{16x^2 - 1};$$

$$(2)\ \frac{1}{1 - \dfrac{x}{x-1}} - \frac{1}{\dfrac{x}{x+1} - 1}.$$

4. Find the G.C.M. of

$$3x^3 - x^2 + 3x - 1 \text{ and } 3x^3 + 2x^2 - 4x + 1;$$

and the L.C.M. of

$$2(a^2b - ab^2),\ 6(a^2b - b^3),\ 9(a^3 - a^2b + ab^2),$$
$$\text{and } 15(a^3 + a^2b + ab^2).$$

5. Extract the square root of

$$4x^4 + 4x^3 - \frac{x}{2} + \frac{1}{16}.$$

6. Solve the equations:

$$(1)\ \frac{x}{2} + \frac{11}{2} = \frac{3x+5}{2} + \frac{3x-8}{4} + \frac{1}{3};$$

$$(2)\ \begin{cases} (x+4)(y+5) = (x-4)(y-5) + 44. \\ 4x + 5y = 23 \end{cases}$$

CAMBRIDGE, 1868.

1. Add together $a-b+c$, $2a+2b-3c$, $-3a+4b+4c$; and subtract the sum from $4a+5b-5c$.

2. Reduce the following expressions to their simplest forms:

(1) $a-b-2\{a-3b-4(a-5b)\}$,

(2) $(x+y)^2-(x+y)(x-y)+(x-y)^2$,

(3) $$\frac{1}{x+\cfrac{1}{x+\cfrac{2}{x}}}.$$

3. Multiply $x^2-ax+2a^2$ by $x^2+3ax+4a^2$, and divide
$8x^4-2ax^3+3a^2x^2-2a^3x+a^4$ by $2x^2+ax+a^2$.

4. Prove that the quotient of the difference of the cubes of any two numbers by the difference of the numbers is always equal to the sum of the squares and product of the numbers.

5. Find the greatest common measure of
$6x^3+x^2-11x-6$ and $6x^4-x^3-14x^2-x+6$.

6. Add together $\frac{1}{a+b}$, $\frac{2b}{a^2-b^2}$, $\frac{1}{a-b}$.

7. Solve the following equations:

(1) $$\frac{x-6}{3}-\frac{x-5}{2}=1,$$

(2) $$\frac{ax-b}{c}+\frac{bx-c}{a}+\frac{cx-a}{b}=0,$$

(3) $$\begin{cases} \dfrac{x-y}{2}+\dfrac{x+y}{3}=2\frac{1}{2}, \\[2ex] \dfrac{x+y}{2}+\dfrac{x-y}{3}=4\frac{1}{6}. \end{cases}$$

8. A person possesses £5000 stock, some at 3 per cent., four times as much at $3\frac{1}{2}$ per cent., and the rest at 4 per cent.; find the amount of each kind of stock when his income is £176.

9. A railway train, after travelling an hour, is detained 15 minutes, after which it proceeds at three-fourths of its former rate, and arrives 24 minutes late. Had the detention taken place 5 miles further on, the train would have arrived 3 minutes sooner than it did. Find the original speed of the train, and the distance travelled.

CAMBRIDGE, 1869.

1. Find the value of each of the following expressions when $a = 10$, $b = 3$, $c = 7$:

$$\text{(i.)}\quad \frac{b+c}{2c-3b},$$

$$\text{(ii.)}\quad \frac{3c}{4a+2}+\frac{4bc}{10a-16},$$

$$\text{(iii.)}\quad \sqrt{\frac{3b+c}{c-b}}-\frac{\sqrt[3]{9b+7c-12}}{c-b}.$$

2. State the rules for removing brackets from algebraical expressions.

Simplify $2[4x-\{2y+(2x-y)-(x+y)\}]$.

3. Multiply $2x^3-3x^2+4x-5$ by $7x+9$, and

$$\frac{2}{3}a^3-\frac{4}{5}a^2x+\frac{1}{2}x^3 \text{ by } \frac{3}{5}a-2x.$$

4. Divide $x^6-2a^3x^3+a^6$ by $x^2-2ax+a^2$.

Find the values of a and b which will make x^3+c^3 exactly divisible by x^2+ax+b^2.

5. Prove that the sum of the squares of any two different numbers is always greater than twice their product.

6. Reduce the following fractions to lowest terms:

(i.) $\dfrac{20abx}{15a^2}$, (ii.) $\dfrac{a^2 - x^2}{a + x}$,

(iii.) $\dfrac{x^3 + a^3}{x^2 + 2ax + a^2}$, (iv.) $\dfrac{4 + 12x + 9x^2}{2 + 13x + 15x^2}$.

7. Solve the following equations:

(i.) $x + 1 + 2(x + 3) = 4(x + 5)$,

(ii.) $\dfrac{x + 1}{a + 1} + \dfrac{x + 2}{a + 2} = 2$,

(iii.) $\left.\begin{aligned} 4x + 9y &= 12, \\ 6x - 3y &= 7. \end{aligned}\right\}$

8. A certain fraction becomes $\frac{1}{3}$ if 1 be added to its numerator, and if 1 be added to its denominator it becomes $\frac{1}{4}$; what is the fraction?

9. A sum of £119 is to be divided among 10 men, 32 women, and 48 children; if each man's share is to be equal to the shares of two women, and if the 32 women are to have twice as much as the 48 children, how much will the several individuals receive?

CAMBRIDGE, 1870.

1. Find the value of each of the following expressions, when $a = 5$, $b = 4$, $c = 3$:

(i.) $\dfrac{7b + c}{3a + 4b}$. (ii.) $\dfrac{7a}{11b - 3c} + \dfrac{11b}{8b - 7c} - \dfrac{10c}{7a - 5b}$.

(iii.) $\dfrac{a^3 + b^3 + c^3 - 3abc}{a^2 + b^2 + c^2 - bc - ca - ab}$,

2. State and explain the rule for the subtraction of two algebraical expressions.

Subtract $3a^3 + 4x^2y - 7xy^2 + 10y^3$ from $4x^3 - 2x^2y + 4xy^2 + 4y^3$, and simplify $7a - 2[3a - 2b + \{(a+b) - (a-b)\}]$.

3. Multiply $x^3 + 3x^2 + 4x + 2$ by $x^2 - 2x + 2$; and determine a and b such that in the product of $x^2 + x + 1$ and $x^3 + ax^2 + bx + c$ the co-efficients of x^4 and x^3 may vanish.

4. Divide $x^6 - 20a^3x^3 + 343a^6$ by $x^2 + ax + 7a^2$; and divide the quotient by $x^2 + 4ax + 7a^2$.

5. If $\dfrac{a+b}{1-ab} + \dfrac{c+d}{1-cd} = 0$, prove that

$$a + b + c + d = abcd\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right).$$

6. Simplify the following expressions:

(i.) $\dfrac{168a^3b^2c}{48a^2bc^3}$; (ii.) $\dfrac{b^2-c^2}{b+c} + \dfrac{c^2-a^2}{c+a} + \dfrac{a^2-b^2}{a+b}$;

(iii.) $\dfrac{x^4+x^2+1}{x^3-1}$; (iv.) $\dfrac{20x^2+27x+9}{15x^2+19x+6} + \dfrac{20x^2+27x+9}{12x^2+17x+6}$.

7. What is meant by a simple equation? Prove that a simple equation cannot be satisfied by two different values of the unknown quantity.

Solve the equations:

(i.) $3(x-2) + 4(x+3) = \frac{5}{3}(4x+5)$.

(ii.) $a\dfrac{x-b}{a-b} + b\dfrac{x-a}{b-a} = 1$.

(iii.) $\dfrac{9}{x-4} + \dfrac{3}{x-8} = \dfrac{4}{x-9} + \dfrac{8}{x-3}$.

8. Two persons A and B play a match of four games on the following terms: when A wins he receives from B half the money B has and one shilling over; when B wins he receives from A one shilling less than half the money A has; A having won the first and third games and lost the second and fourth, A has lost ten shillings, and B's money exceeds A's by two shillings more than half of what A had at first. Find what money each had at first.

CAMBRIDGE, 1871.

1. Add together

$$2a^2 + ab - 3b^2,\ a^2 - 2ab + 4b^2,\ - 4a^2 + 5ab - 6b^2.$$

2. Subtract $2a - 4b + 5c - 6d$ from $3a + 6b - 8c - d$; and simplify

(i.) $2x^2 + 4xy - 3y^2 - (x - 2y)^2$.

(ii.) $3\{x - 2(y - z)\} - [4x + \{2y - (z - x)\}]$.

3. If $a = 1$, $b = 2$, $c = - 3$, find the value of

(i.) $4a + 3b + 2c$.

(ii.) $a^3 + 8b^3 + c^3 - 6abc$.

Also divide the last expression by $a + 2b + c$.

4. Multiply together $x + a - b$ and $x - a + b$; and shew that the product is zero if x is equal to the difference of the quantities a and b.

5. In the expression $x^3 - 2x^2 + 3x - 4$ substitute $a - 2$ for x, and arrange the result according to descending powers of a.

6. Prove the truth of this statement: Take any two proper fractions such that their sum is unity, subtract the square of the smaller from the square of the greater, and add unity to the remainder: and the result will always be equal to twice the greater.

7. Reduce to their simplest forms the following expressions:

(i.) $\dfrac{7a-4}{3} - \dfrac{11a-7}{5}$.

(ii.) $\dfrac{a^3 - 3a^2b + 3ab^2 - 2b^3}{a^4 + a^2b^2 + b^4}$.

(iii.) $\dfrac{1}{x+3y} + \dfrac{6y}{x^2 - 9y^2} - \dfrac{1}{3y - x}$.

8. State and explain the rule for transferring a quantity from one side of an equation to other side.

Solve the equations:

$$\text{(i.)} \quad x - 1 - 2(2x - 9) = \frac{2x}{5}.$$

$$\text{(ii.)} \quad \frac{x - \frac{1}{a}}{c} + \frac{x - \frac{1}{b}}{a} + \frac{x - \frac{1}{c}}{b} = 0.$$

$$\text{(iii.)} \quad \begin{cases} \dfrac{x - 2y}{6} - \dfrac{x + 3y}{4} = 1\frac{1}{8}. \\[2ex] \dfrac{2x - y}{6} - \dfrac{3x + y}{4} = \dfrac{5y}{4}. \end{cases}$$

9. A quantity of land, partly pasture and partly arable, is sold at the rate of £60 per acre for the pasture and £40 per acre for the arable, and the whole sum obtained is £10,000. If the average price per acre were £50, the sum obtained would be 10 per cent. higher. Find how much of the land is pasture and how much arable.

CAMBRIDGE, 1872.

1. Multiply together

$$a^2 - 3ab - 4b^2 \text{ and } 2a^2 - ab + 5b^2.$$

2. Simplify

$$2\{a - (b - c)\} - \{a + c - 3(b + d)\}.$$

3. Find the value, when $a = 1$, $b = -2$, $c = 3$, $d = 0$, of

$$2ab + 3b^2 + cd.$$

Also find the value, when $x = -\frac{1}{3}$, of

$$\frac{\sqrt{3 - 2x^2} - x}{x(1 + 3x) - x^3}.$$

4. What is meant by a co-efficient? Find the co-efficient of x in the quotient obtained by dividing

$$8x^4 + xy^3 - y^4 \text{ by } x - \frac{1}{2}y.$$

5. State and prove the rule for finding the L.C.M. of two algebraical expressions.

Add together the fractions

$$\frac{1}{x^3 - ax^2 - 4a^2x + 4a^3} \text{ and } \frac{1}{x^3 + 6ax^2 - 16a^3}.$$

6. Reduce to their simplest forms :—

$$\text{(i.)} \quad \frac{x-2y}{4} - \frac{x+2y}{5} + \frac{5y}{6}.$$

$$\text{(ii.)} \quad \frac{\frac{x}{y} + \frac{y}{x} + 2}{x+y} + \frac{\frac{x}{y} + \frac{y}{x} - 2}{x-y}.$$

$$\text{(iii.)} \quad \frac{2a}{(x-2a)^2} - \frac{x-a}{x^2 - 5ax + 6a^2} + \frac{2}{x-3a}.$$

7. Solve the equations :—

$$\text{(i.)} \quad \frac{x-3}{4} + \frac{x-1}{3} = x - 4.$$

$$\text{(ii.)} \quad \frac{a}{bx} + \frac{b}{ax} = \frac{1}{a} + \frac{1}{b}.$$

$$\text{(iii.)} \quad \begin{cases} \dfrac{x-y}{2} + \dfrac{x+y}{3} = \dfrac{25}{6}. \\ x + y - 5 = \dfrac{2}{3}(y - x). \end{cases}$$

8. A purse of sovereigns is divided amongst three persons, the first receiving half of them and one more, the second half of the remainder and one more, and the third six. Find the number of sovereigns the purse contained.

CAMBRIDGE, 1873.

1. Express in symbols the result of subtracting from unity the quotient obtained by dividing the sum of a and b by their product.

2. Subtract $2a - 3\{a - (b - a)\}$ from $2b - 3\{b - (a - b)\}$.

3. Multiply $3x^2 - xy - 2y^2$ by $x^2 + 2xy + 4y^2$.

4. Find the Greatest Common Measure of

$$x^4 - 21x + 8 \text{ and } 8x^4 - 21x^3 + 1.$$

5. Simplify

$$(1)\quad \frac{1}{3}(x - 3y) + \frac{2x - y}{2} - \frac{1}{12}(7x - 9y),$$

$$(2)\quad \frac{\dfrac{1}{x} - \dfrac{x + a}{x^2 + a^2}}{\dfrac{1}{a} - \dfrac{a + x}{a^2 + x^2}} + \frac{\dfrac{1}{x} - \dfrac{x - a}{x^2 + a^2}}{\dfrac{1}{a} - \dfrac{a - x}{a^2 + x^2}}.$$

and shew that the difference between $\dfrac{x}{x - a} + \dfrac{x}{x - b} + \dfrac{x}{x - c}$ and $\dfrac{a}{x - a} + \dfrac{b}{x - b} + \dfrac{c}{x - c}$ is the same whatever values be given to x.

6. What do you mean by *solving an equation?* Shew that 3 is a root of the equation $\sqrt{x^2 - 3x + 4} = \dfrac{3 + \sqrt[3]{x - 2}}{x^2 - 7}$.

7. Solve the equations:

$$(1)\quad \frac{2x}{3} - \frac{x - 3}{6} = \frac{1}{7}\{5x - 4\},$$

$$(2)\quad \left.\begin{aligned} 8x - 7y &= 12 \\ \frac{x - 2y}{4} + \frac{2x - y}{3} &= 1 \end{aligned}\right\},$$

$$(3)\quad x + \frac{ay}{a - b} = b = \frac{ax}{a + b} + y.$$

8. Large marbles are a penny a score dearer than small ones. A boy who has bought equal numbers of each kind finds that on the whole he has got eight for a penny. How much are small marbles per score?

CAMBRIDGE, 1874.

1. The difference between a number and half of ten more than twice the number is five. Express this algebraically.

2. Multiply $2x - 3y - 4(x - 2y) + 5\{3x - 2(x - y)\}$ by

$$4x - (y - x) - 3\{2y - 3(x + y)\}.$$

3. Resolve into factors $(4x^2 - 9y^2)$ and $(a^2 - ab - 6b^2)$.

4. Find the Least Common Multiple of a^4b^2, b^3c, a^2c^3; also of $3x^2 - 10xy + 3y^2$, $3x^2 - 4xy + y^2$, $x^2 - 4xy + 3y^2$.

5. Simplify

(i.) $\frac{1}{4}(2x - 3y) - \frac{1}{3}(3x + 2y) + \frac{7x - 5y}{12}$.

(ii.) $\frac{(x + y + z)(x^2 + y^2 + z^2)}{xyz} - \left(\frac{y + z}{x} + \frac{z + x}{y} + \frac{x + y}{z}\right)$.

6. Distinguish between an *equation* and an *identity*. Make an example of each. What value of c makes

$$(x - 2)^2 - (x - 1)(x - 3) = c$$

an identity? Can any value of c make it an equation?

7. Solve the equations :—

(i.) $\frac{3x - 4}{2} - \frac{4x - 3}{3} = \frac{2}{7}(x - 6)$.

(ii.) $\frac{2x - b}{a} = \frac{2y + a}{b} = \frac{3x + y}{a + 2b}$.

8. After walking at the rate of $3\frac{1}{2}$ miles per hour until the number of miles left of my journey was the same as the number of hours I had been walking, I quickened my pace to 4 miles per hour and accomplished the whole distance in 2 hours 55 minutes. What was the length of my journey?

CAMBRIDGE, 1875.

1. Express algebraically :—The fourth power of the sum of two numbers together with twice the product of their squares is equal to the sum of their fourth powers together with four times the product of their product and the square of their sum.

2. Verify the above, taking 2 and 3 as the numbers.

3. What is the meaning of a *term*, a *co-efficient*, an *index*, a *negative* quantity? Give an example of each.

4. Resolve into factors $6x^2 - 5xy - 6y^2$, shewing your method.

5. Find the L.C.M. of $2x^2 - 8$, $3x^2 - 9x + 6$, $6x^2 + 18x + 12$, and state the reason for the method you adopt.

6. Simplify :—

(i.) $$\frac{x+2}{x-2} - \frac{x-2}{2(x+2)} - \frac{1}{4}.$$

(ii.) $$\left(\frac{a^2+b^2}{2ab} + 1\right)\left(\frac{ab^2}{a^3+b^3}\right) \div \frac{4a(a+b)}{a^2-ab+b^2}.$$

7. Solve the equations :—

(i.) $$3x - 4(9 - \overline{2x+7} + 3x) = 13.$$

(ii.) $$ax + by = \frac{1}{2}(bx + ay) = a^2 - b^2.$$

8. I gave away a dole to a pauper at 3s. 6d. per week till the number of shillings I had left was the same as the number of weeks I had been giving it. I then raised the rate to 4s. per week, and the whole money was given away in 35 weeks. How many shillings had I to give away?

CAMBRIDGE, 1876.

1. If $a=2$, $b=3$, $x=5$, $y=1$, find the value of each of the following expressions :—

(i.) $(x+y)^3 - (x^3+y^3)$,

(ii.) $\dfrac{a^2+y^2}{(a+y)(x-y)} - \dfrac{a+b}{(b-y)(x+y)}$.

2. What is the use of brackets? What is the rule for removing brackets which are preceded by a *minus* sign?

Find the value of $1-[1-\{1-(1-\overline{1-1})\}]$.

3. Multiply

$$x^3+2x^2y+4xy^2+8y^3 \text{ by } x-2y,$$

and express without brackets

$$(x+y)(x+2y)(x-2y)(x-y).$$

4. Divide

$$a^3-2a^2b+\frac{11}{9}ab^2-\frac{2}{9}b^3 \text{ by } a^2-\frac{5}{3}ab+\frac{2}{3}b^2,$$

and

$$x^2-y^2 \text{ by } x^{\frac{1}{2}}-y^{\frac{1}{2}}.$$

5. What is the *reciprocal* of a number?

If any positive number be added to its reciprocal, prove that their sum cannot be less than 2.

6. Simplify the following expressions :—

(i.) $\dfrac{45a^3x}{36a^2x^3y}$, (ii.) $\dfrac{a^3-x^3}{a^2-x^2}$,

(iii.) $\dfrac{x^2-\frac{5}{6}x-1}{x^2+\frac{13}{6}x+1}$,

(iv.) $\dfrac{1-\dfrac{a^2}{(x+a)^2}}{(x+a)(x-a)} \div \dfrac{x(x+2a)}{(x^2-a^2)(x+a)^2}$.

7. Solve the equations :—

(i.) $\frac{2}{3}(x-2) - \frac{1}{7}(x+2) = 1$,

(ii.) $\dfrac{x-3}{x-1} + \dfrac{x-2}{x-3} = 2$,

(iii.) $\left.\begin{aligned} 3x-11y &= 4, \\ 5x-12y &= 13. \end{aligned}\right\}$

8. A sum of £3 is divided among 50 men and women, the men each receiving 1s. 6d. and the women 1s. Find the number of each sex.

9. A boy receives a fixed sum as pocket-money at the beginning of every week, and in each week he spends half of all that he had at its beginning. He had no money before the first pocket-money was given to him, and at the end of the third week he has 1s. 2d. What is his weekly allowance?

CAMBRIDGE, 1877.

1. If $x=4$ and $y=3$, find the values of the expressions :—

(i.) $\dfrac{x^3-y^3}{\sqrt{x^2+y^2}} - \dfrac{x^2-y^2}{\sqrt{x-y}}$,

(ii.) $\left(\dfrac{x^2}{y} - \dfrac{y^2}{x}\right)\left(\dfrac{x}{y^2} + \dfrac{y}{x^2}\right) \div \left(\dfrac{x^2}{y} + \dfrac{y^2}{x}\right)\left(\dfrac{x}{y^2} - \dfrac{y}{x^2}\right)$.

2. Multiply

$$4a^2+12ab+9b^2 \text{ by } 4a^2-12ab+9b^2, \text{ and}$$

$$\frac{1}{8}a^{\frac{3}{2}} - \frac{1}{12}ab^{\frac{1}{2}} + \frac{1}{18}a^{\frac{1}{2}}b - \frac{1}{27}b^{\frac{3}{2}} \text{ by } \frac{1}{2}a^{\frac{1}{2}} - \frac{1}{3}b^{\frac{1}{2}}.$$

3. Divide

$$x^2 - \left(a + \frac{1}{a}\right)x + 1 \text{ by } x - a.$$

The product of two factors is $(3x+2y)^3-(2x+3y)^3$, and one of the factors is $x-y$; find the other factor.

4. Prove that

$$(a-b)^3+(b-c)^3+(c-a)^3 = 3(a-b)(b-c)(c-a).$$

5. State the rule for finding the L.C.M. of two algebraical quantities.

Prove that the product of any two quantities is the same as the product of their L.C.M. and G.C.M.

Find the L.C.M. of

$$x^3+2ax^2+a^2x+2a^3 \text{ and } x^3-2ax^2+a^2x-2a^3.$$

6. Reduce to their simplest forms :—

(i.) $42\left\{\dfrac{4x-3y}{6} - \dfrac{3x-4y}{7}\right\} - 56\left\{\dfrac{3x-2y}{7} - \dfrac{2x-3y}{8}\right\}$.

(ii.) $\dfrac{209ab^3x^2}{133a^2bx^4}$, (iii.) $\left(\dfrac{x^{2p+q}x^{2q+r}x^{2r+p}}{x^p x^q x^r}\right)^{\frac{1}{2}}$, (iv.) $\dfrac{a^3}{(a-b)(a-c)} + \dfrac{b^3}{(b-a)(b-c)} + \dfrac{c^3}{(c-a)(c-b)}$.

7. Solve the equations :—

(i.) $\dfrac{2(2x-1)}{9} - \dfrac{3x-2}{13} = 1,$

(ii.) $(3x-8)(3x+2)-(4x-11)(2x+1) = (x-3)(x+7).$

(iii.) $\dfrac{bx+ay}{3ab} = \dfrac{x-y}{2(a-b)} = a+b.$

8. A sum of money is divided among three persons. The first receives £10 more than a third of the whole sum; the second receives £15 more than a half of what remains; and the third receives what is over, which is £70. Find the original sum.

9. A certain number consisting of two digits when divided by the sum of its digits has a quotient *seven* : if the digit in the units' place be decreased by two and that in the tens' place by one, and if the number thus formed be divided by the sum of its digits, the quotient is *ten*. Find the number.

ANSWERS TO THE EXAMPLES.

ANSWERS.

1. 14.	7. 7.	13. 1.	19. 0.
2. 4.	8. 35.	14. – 2.	20. 1.
3. 5.	9. – 37.	15. 9.	21. – 369.
4. 18.	10. – 320.	16. 27.	22. 5.
5. 4.	11. 454.	17. – 11.	23. – 1440.
6. – 6.	12. 0.	18. – 14.	24. 0.

II.

1. 3.	6. 1.	11. 2.	16. 3.
2. 2.	7. 5.	12. 32.	17. – 2.
3. 1.	8. 0.	13. 1.	18. 3.
4. 2.	9. – 1.	14. 0.	19. 0.
5. 1.	10. 4.	15. – 4.	20. 2.

III.

1. 5.	9. – 33.	16. 1.	23. 1.
2. 9.	10. 121.	17. 6.	24. 3.
3. 8.	11. 1.	18. 146.	25. 1.
4. 25.	12. – 177.	19. 20.	26. – 1.
5. – 22.	13. 0.	20. 1.	27. – 55.
6. 68.	14. 2.	21. 0.	28. 1.
7. 92.	15. 3.	22. 8.	29. 6.
8. – 564.			30. – 14.

IV.

1. 6.	6. 36.	11. 1.	16.
2. 5.	7. 0.	12. 1.	17. 6.
3. 4.	8. 7.	13. -3.	18. 2.
4. 6.	9. 4.	14. 1.	19. -2.
5. 4.	10. 0.	15. 0.	20. 1.

V.

1. 2.	9. 0.	16. 100.	23. 12.
2. -4.	10. -2.	17. 6.	24. 3.
3. 5.	11. -32.	18.	25. 0.
4. 6.	12. -1.	19.	26. 0.
5. 8.	13. -1.	20.	27. 4.
6. 2.	14. 3.	21. 0.	28. -3.
7. -9.	15. 6.	22. 9.	29. 1.
8. 5.			30. -24.

VI.

1. $7a^2 + 6a - 6$.	9. $-2ab - 7az$.
2. $10a^2 - 2b^2 - 9a$.	10. $a^3 + b^3 + a^2$.
3. $10x + 10y - 14z - 15y^2$.	11. $3x^2 + 2z^2$.
4. x.	12. 0.
5. $a + y$.	13. $9x^2y^2 + 4yz + 8z^3$.
6. $11a + 3b + x - 2y + z$.	14. $a^2 - 2z^2 + 4$.
7. $4a^2 - 2ab - 3ac - 2bc + cd$.	15. (i.) $6x^2 + 6xy - y^2 + 3x + 4y$
8. $7ac + 5b^2 + 2x^2y + z^2$.	(ii.) 25.

VII.

1. $a + 4b - 4c + 2d$.	7. $5a + 5b - 5c - d$.
2. $3x^2 + 2y^2 - 3$.	8. $2m^2 + 5mn - 4$.
3. $-2n^2 + 3mn$.	9.
4. $-x^3 + 3x^2 + 3x^2y + 2y^3$.	10. $axy + x^2y^2 + y^3$.
5. $1 - 3m^2 + 4mn - 2n^2$.	11. $a - 3b + c - x + y - z$.
6. $20x^3 - 80y^3 + 10x^2y - 10xy^2 + 15xy$.	12. $4a^4 + 8a^3 - 6a + 8$.

VIII.

1. a^2+2b^2.
2. x^2-y^2.
3. $x-4y$.
4. $3b+x$.
5. $-2b$.
6. $2a-2x-3y-1$.
7. $x+5$.
8. $4a^2b^2$.
9. $2a+2m$.
10. $a-b$.
11. $-y$.
12. $14a-5b$.
13. $6ax$.
14. a.
15. $2a^2x+2y$.
16. 1.
17. $25-a$.
18. y^2.
19. $2m^2+n^2$.
20. $4a-2$.
21. $-2ab-b^2$.
22.
23. $a^2+3ab-2b^2$.

IX.

1. abx.
2. $12abxy$.
3. $-9abcmn$.
4. $36a^2bcyz$.
5. x^5.
6. $-6x^5y$.
7. $12x^3y^3z^2$.
8. $-16a^3b^2m^2n^2$.
9. $3ax^5by^3z^3$.
10. $a^3d^2c^4$.
11. $-3m^3n^3$.
12. $-12a^3b^2x^2y^3$.
13. $-6a^9b^9c^5$.
14. $240m^{11}n^5p$.
15. $220a^2b^2m^9$.
16. $380a^5b^3c^3xy$.
17. $-28a^3b^2cd^3x^2$.
18. $-200p^5q^6rs$.
19. $-162a^3m^3n^3$.
20. $-30x^5y^6z^3$.

X.

1. $ax+ay-az$.
2. m^3-2m^2+3m.
3. $-x^4-2m^3-x^2-x$.
4. $-3m^3n^2+2m^2n^3+2mn$.
5. $2x^5y-6x^4y^2+2x^3y^3-2x^2y^5$.
6. $64m^5x^3-12m^4x^4+8m^3x^5-4mx^6$.
7. $-15a^5b^5+20a^4b^6-10a^3b^7+15a^3b^8-5a^2b^9$.
8. $6a^3b-12a^5b+6a^6b-24a^7b+6a^8b$.
9. $-8m^2n^6p^3+7mn^4p^4-3mn^3p^5+n^3p^6$.
10. $-7a^5b+7a^6b^3-21a^7b^4+49a^8b^3$.
11. $-15a^3x+17a^2x^2-19mx+20m^3$.
12. $-54a^2bxy^2z^2+9a^3bxyz^2+27a^4yz^3+108ayz^4$.
13. $3a^6b^4c^2-3a^5b^3c^3+3a^4b^2c^4-3a^3bc^5+3a^2bc^6$.
14. $10a^7y-8a^6xy^2+6a^5x^2y^3-4a^4x^3y^4+2a^2y$.

15. $108m^5n^5 - 45m^4n^6 + 57m^3n^7 - 12m^3n$.
16. $132x^3y^2z - 108x^4y + 120x^2y^3 - 156x^2yz^2 + 12x^3y^4$.
17. $3a^5 - 6a^4b + 9a^3b^2 - 12a^2b^3 + 3ab^4$.
18. $-a^5x + 6a^4x^2 - 2a^4x^3 + a^5x^4$.
19. $5m^5n^3 - 15m^4n^4 + 20m^3n^5 - 55m^2n^6 + 5m^2n^3$.
20. $7xy^2z^3 - 14x^2y^2z^3 + 21xy^3z^3 - 28xy^2z^4 + 7x^2y^3z^4$.

XI.

1. $x^2 + 9x + 18$.
2. $6x^2 - 31x + 18$.
3. $15x^2 - 17x - 4$.
4. $6x^4 - 17x^2 + 12$.
5. $a^2 + a - 156$.
6. $4a^4 - 4a^2 - 48$.
7. $3x^6 - 5x^3 + 2$.
8. $a^3 + b^3$.
9. $3a^4 - 3a^3b + 2a^3b^2 - 2a^2b^3 + ab^3 - b^4$.
10. $a^6 - 9a^4 + 12a^2 - 4$.
11. $a^4 + a^2b^2 + b^4$.
12. $x^4 - 9b^2x^2 + 12b^3x - 4b^4$.
13. $x^4 - 2a^3x - 3a^2x^2 - 4a^3x - a^4$.
14. $24x^4 - 10x^3y - 27x^2y^2 - 16xy^3 - 3y^4$.
15. $2x^5 - 2x^3 - 10x^2 - 8$.
16. $a^3 + y^3 + z^3 - 3xyz$.
17. $x^3 - y^3 + 1 + 3xy$.
18. $1 - a^4$.
19. $12a^4 - 26a^3b - 8a^2b^2 + 10ab^3 - 8b^4$.
20. $x^6 - 5x^4 - 2x^2 - 1$.
21. $x^5 - 4ax^4 + 7a^2x^3 - 7a^3x^2 + 4a^4x - 2a^5$.
22. $a^3b^3 - 2a^2b^2x - 2abx^2 + x^3$.
23. $a^2x^2 + b^2y^2 - a^2b^2 - x^2y^2$.
24. $x^6y^6 - z^6$.
25. $a^5b^5 - a^3b^3x^2 + 6a^2b^2x^3 - a^2b^5 - 2ab^4x$.
26. $3a^2b^2 + 8abx + 4x^2 + 10aby + 8xy + 3y^2$.
27. $a^5 - 9a^3b^2 + 15a^2b^3 - 13ab^4 + 6b^5$.
28. $x^4 - y^4$.
29. $a^3 + 11a^2 + 36a + 36$.
30. $a^4 - 4a^3 - 7a^2 + 22a + 24$.
31. $\frac{1}{6}a^3 - \frac{1}{2}a^2 + \frac{17}{24}a - \frac{1}{2}$.
32. $\frac{2}{5}x^4 - \frac{136}{75}x^3y + \frac{8}{5}x^2y^2 + \frac{3}{10}xy^3 - y^4$.
33. $\frac{1}{2}m^3 - \frac{59}{48}m^2n + \frac{13}{12}mn^2 - \frac{3}{8}n^3$.
34. $x^5 - 2x^4 - \frac{61}{6}x^3 - \frac{35}{6}x^2 - \frac{1}{3}x + \frac{1}{4}$.

XII.

1. a.
2. ab.
3. a^2b.
4. $2a^2b^2$.
5. $2a^2b^2x$.
6. $11x^2y$.
7. $2mn^2$.
8. $-5ax$.
9. $16a^2c$.
10. $4a^2x^2y$.
11. $-2a$.
12. $-2x^2y^3$.

XIII.

1. $ax+3y-4y^2$.
2. $mnx^2-mnx+1$.
3. $y+y^2+y^3$.
4. a^2-ab+b^2.
5. $4m+6n+8$.
6. $-9a^2b^3+7ab^2-5b$.
7. $-12x^3-11x^2-10x-9$.
8. $-1+2x-3x^2+4x^3$.
9. $-1+z-z^2+z^3$.
10. $5-4x+3x^2$.

XIV.

1. $a+2$.
2. $a+2$.
3. $a-2$.
4. $a+2$.
5. $a-b$.
6. $a^2+2ab+b^2$.
7. x^2+2x+2.
8. $x-1$.
9. $a-1$.
10. $2x^2-3x+2$.
11. x^2-xy-y^2.
12. $2a^2-3ab+3b^2$.
13. $a^3-a^2b+b^3$.
14. $a^2b^2-3abc+7c^2$.
15. $x^2+3xy+5y^2$.
16. $3a^2+2ab-b^2$.
17. $3a^2x^2+5axy-7y^2$.
18. $m^2n^2+2mnx+3x^2$.
19. $a^2+ax+2y$.
20. $3x^2y^2-9xyz+6z^2$.
21. $2a^4-5a^3b+3a^2b^2$.
22. $2a^3-3a^2+2a-2$.
23. $1+3x+2x^2+x^3$.
24. $1-2x+3x^2-2x^3$.
25. $x^2y^2z+2xyz^2$.
26. $a^3-2a^2b+3ab^2+4b^3$.
27. $3a^3-2a^2b+7ab^2-8b^3$.
28. $5a^2-3ab+7b^2$.
29. x^3-x^2-3x-3.
30. $7x^2+4ax+4a^2$.
31. $\frac{1}{2}x^2-\frac{1}{4}xy+\frac{1}{2}y^2$.
32. $\frac{5}{6}m^2-\frac{2}{3}m+1$.
33. $\frac{2}{3}x^3-\frac{3}{4}x^2+x-1$.

XV.

XVI.

1. $a(x+y)$.
2. $5b(ab-2)$.
3. $3ax(x-3y+9a)$.
4. $8ab(a^2-2ab+3b^2)$.
5. $x^2(x^3+x^2y+xz+1)$.
6. $x^3(ax-b)+x(cx+d)$.
7. $11a^3x^2(5a^2x^2y-3axz+1)$.
8. $14a^3b(1-3ab+5a^2b^2)$.
9. $6a^2b(x^2-3y^2+2z^2)$.
10. $a^2(b+ac)+a^4(d+ax)$.
11. $x^2(a^2-2ax+x^2)$.
12. $2n^2(m^2+2mn+n^2)$.
13. $15a(2a^2+3ab+4b^2)$.
14. $-x^2(a^2+ax-x^2)$.
15. $-16x(x^2-2x+4)$.
16. $-x(x^3+ax^2+a^2x+a^2)$.
17. $a^3b-3a^2b^2+ab^3$.
18. $a^4+a^3b+a^2b^2$.
19. $16x^2-32x^3+16x^4$.
20. $-5x^2y^2+15xy^3-10y^4$.
21. $m^3+2m^2n-2mn^2$.
22. n^3+n^2.
23. $m^3n-m^3n^3-m^2n^3$.
24. $2ab-b$.
25. $a^2-a^2b+2ab+b^2$.
26. $2b-6a$.
27. $ab-6a$.
28. $a-6x$.
29. a^2-a.
30. mn.

XVII.

1. $(a+b-c)x^2+(3m-n)x$.
2. $(a+b+c)x^2+(b+c-a)y^2+(c-a+b)z$.
3. $(2a+1)x^3-(2n+1)x^2+(p-q+1)x$.
4. $2ax^2+axy+2by^2$.
5. $(y+z-1)a^2+(c+y)ab+(z-c-1)b^2$.
6. $(a-b)x^3-(b+c)x^2+2ax$.
7. $-(x-3y)a^2+(x-3z)ab+(c+3x)b^2$.
8. $(m+3n)x^3-(5a+b)x^2-(2a-b+c)x$.
9. $a^2+(m+n)a+mn$.
10. $a^2+(x-y)a+xy$.
11. $x^3+x^2(a+b)+x(ab+bc)+abc$.
12. $x^3+x^2(a-b)-x(ab+bc)-abc$.
13. $x^3-x^2(a+b-c)+x(ab-ac-bc)+abc$.
14. $x^3+x^2(a-b-c)-x(ab+ac-bc)+abc$.
15. $m^3-m^2(n+q)+m(p+nq)-pq$.

16. $x^3 + x^2(a+z) + x(y+az) + zy$.
17. $x^3 + x^2(a+b+c) + x(ab+ac+bc) + abc$.
18. $x^2 - (a+b)x + ab$.
19. $a^2 + a(x+y) + xy$.
20. $x^2 - x(y+z) + y^2 - yz + z^2$.
21. $x(y+z) - yz$.
22. $m^2 + m(n-q) - nq$.
23. $a^2 + x(a+x) + y(a-x) + y^2$.
24. $m^2 + am + b$.

XVIII.

1. $x^2 + 2xa + a^2$.
2. $x^2 - 2xa + a^2$.
3. $a^2 + 2a + 1$.
4. $a^2 - 2a + 1$.
5. $1 - 2x + x^2$.
6. $1 - 4x + 4x^2$.
7. $1 + 4x + 4x^2$.
8. $m^2 - 2mn + n^2$.
9. $y^2 + 2yx + x^2$.
10. $x^2y^2 + 2axy + a^2$.
11. $x^2y^2 - 2axy + a^2$.
12. $x^2y^2 + 2xy + 1$.
13. $x^2y^2 - 2xy + 1$.
14. $x^4 - 2x^2 + 1$.
15. $4a^2 - 8ax + 4x^2$.
16. $9x^4 - 6x^2 + 1$.
17. $16x^2 - 40xy^2 + 25y^4$.
18. $9a^4 - 12a^2b + 4b^2$.
19. $16m^2 - 8mn^2 + n^4$.
20. $x^2y^2 - 2abxy + a^2b^2$.
21. $1 - 10a^2 + 25a^4$.
22. $1 + 8ab + 16a^2b^2$.
23. $a^2 + 4ab + 4b^2$.
24. $9m^2n^2 - 6mn + 1$.
25. $1 - 4ax^2 + 4a^2x^4$.
26. $36x^2 - 12ax + a^2$.
27. $x^4 - 2x^2y^2 + y^4$.
28. $9a^4 - 30a^2 + 25$.
29. $9a^2x^4 - 6ax^2 + 1$.
30. $9a^2 + 12ax + 4x^2$.
31. $(m+n)^2$.
32. $(p-q)^2$.
33. $(1-a)^2$.
34. $(1+r)^2$.
35. $(2-2m)^2$.
36. $(3a+1)^2$.
37. $(3a-2b)^2$.
38. $(ab-x)^2$.

XIX.

1. $a^2 - b^2$.
2. $b^2 - c^2$.
3. $x^2 - y^2$.
4. $x^2 - 1$.
5. $x^2 - 4$.
6. $x^2 - 9$.
7. $1 - a^2$.
8. $4 - a^2$.
9. $a^2b^2 - c^2$.
10. $x^4 - y^2$.
11. $x^2 - y^4$.
12. $x^6 - y^2$.
13. $x^6 - y^6$.
14. $x^4 - y^4$.
15. $a^2b^4 - 1$.
16. $49x^2 - 1$.

17. $36m^4-4$. 21. x^4-y^4. 24. a^4-16. 27. $16-x^4$.
18. $p^2m^2-q^2$. 22. m^4-n^4. 25. $1-x^4$. 28. $1-81a^4$.
19. $9m^2-4n^2$. 23. a^4-1. 26. x^4-81. 29. $16a^4-1$.
20. $a^2-4x^2y^2$. 30. $x^{12}-1$.

XX.

1. $(a+b)(a-b)$.
2. $(x+y)(x-y)$.
3. $(m+1)(m-1)$.
4. $(p+q)(p-q)$.
5. $(x+2)(x-2)$.
6. $(1+a)(1-a)$.
7. $(a+3)(a-3)$.
8. $(4+b^3)(4-b^3)$.
9. $(x^3+y)(x^3-y)$.
10. $(m+10)(m-10)$.
11. $(5+3x)(5-3x)$.
12. $(a^3+x^3)(a^3-x^3)$.
13. $(1+x^3)(1-x^3)$.
14. $(1+2a)(1-2a)$.
15. $(2+3x)(2-3x)$.
16. $(mn+a)(mn-a)$.
17. $(ax+yz)(ax-yz)$.
18. $(x^2y+1)(x^2y-1)$.
19. $(m+n)(m-n)$.
20. $(p+q)(p-q)$.
21. $(a+3)(a-3)$.
22. $(3+a)(3-a)$.
23. $(2+x)(2-x)$.
24. $(5+a)(5-a)$.
25. $a(x+1)(x-1)$.
26. $(a+2)(a-2)$.
27. $x(1+a)(1-a)$.
28. $(x+y)(x-y)(x^2+y^2)$.
29. $(a+1)(a-1)(a^2+1)$.
30. $(1+a)(1-a)(1+a^2)$.
31. $(x+1)(x-1)(x^2+1)(x^4+1)$.
32. $(x^2+2)(x^2-2)$.
33. $(2+x^2)(2-x^2)$.
34. $(x^3+3)(x^3-3)$.
35. $(2+x)(2-x)$.
36. $(x+a)(x-a)(x^2+a^2)(x^4+a^4)$
37. $a(a+x)(a-x)$.
38. $x(a+x)(a-x)$.
39. $a(x+a)(x-a)$.
40. $a^2(x+1)(x-1)$.
41. $x(1+x)(1-x)$.
42. $a(x^2+a^2)(x^2-a^2)$.
43. 0.
44. $-a(a+1)^2$.
45. 5.
46. y^4.
47. 0.

XXI.

1. $x^2+7x+12$. 4. $2+3a+a^2$. 7. x^6-9x^3-10.
2. m^2+m-2. 5. a^2-a-20. 8. $m^2+90m-1000$.
3. m^2+m-20. 6. x^4-x^2-6. 9. $a^2-5a-36$.

10. $a^2+3a-18$.
11. a^6+2a^3-35.
12. $x^2-10x+16$.
13. $x^2-11x+30$.
14. $a^2-4a-77$.
15. $m^{10}-m^5-12$.
16. $a^2b^2-ab-12$.
17. $a^2x^4+ax^2-2$.
18. b^2-b-20.
19. a^4+2a^2-24.
20. a^4-6a^2-160.
21. x^4+3x^2-10.
22. x^4-7x^2+12.
23. x^6-2x^3-24.
24. m^4+8m^2-20.
25. m^6+m^3-30.
26. a^6+13a^3+42.
27. $a^2x^2+ax-132$.
28. $x^2y^2-xy-72$.
29. a^2b^2+ab-6.
30. $a^4x^2+a^2x-6$.

XXII.

1. $(a+1)(a+2)$.
2. $(a-1)(a-2)$.
3. $(a-1)(a+2)$.
4. $(a+1)(a-2)$.
5. $(x-4)(x+7)$.
6. $(x+3)(x-5)$.
7. $(ax+2)(ax+4)$.
8. $(x+2)(x+7)$.
9. $(x-2)(x+7)$.
10. $(x-2)(x-7)$.
11. $(ax-1)(ax+2)$.
12. $(ax+1)(ax-2)$.
13. $(a+7)(a+8)$.
14. $(a+2)(a+3)$.
15. $(x+5)(x-6)$.
16. $(x-3)(x-4)$.
17. $(a-2)(a-8)$.
18. $(a-4)(a+6)$.
19. $(x+1)(x-1)(x^2+2)$.
20. $(x+2)(x-2)(x^2+6)$.
21. $(x^3+2)(x^3-8)$.
22. $(x-1)(x-4)$.
23. $(x+3)(x-6)$.
24. $(x^3+3)(x^3-5)$.
25. $(a+4)(a+6)$.
26. $(a+4)(a-6)$.
27. $(x-1)(x+7)$.
28. $(p-3)(p+8)$.
29. $(x-99)(x+100)$.
30. $(x+11)(x-13)$.

XXIII.

1. $a-b$.
2. a^2-ab+b^2.
3. $a^3-a^2b+ab^2-b^3$.
4. $a^4-a^3b+a^2b^2-ab^3+b^4$.
5. $a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5$.
6. $a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6$.
7. $a+b$.
8. a^2+ab+b^2.
9. $a^3+a^2b+ab^2+b^3$.
10. $a^4+a^3b+a^2b^2+ab^3+b^4$.
11. $a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5$.
12. $a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6$.

13. $a-1$.
14. a^2-a+1.
15. a^3-a^2+a-1.
16. $a^4-a^3+a^2-a+1$.
17. $a^5-a^4+a^3-a^2+a-1$.
18. $a^6-a^5+a^4-a^3+a^2-a+1$.
19. $a+1$.
20. a^2+a+1.
21. a^3+a^2+a+1.
22. $a^4+a^3+a^2+a+1$.
23. $a^5+a^4+a^3+a^2+a+1$.
24. $a^6+a^5+a^4+a^3+a^2+a+1$.
25. $(a-b)(a^2+ab+b^2)$.
26. $(a+b)(a^2-ab+b^2)$.
27. $(a-1)(a^2+a+1)$.
28. $(1-a)(1+a+a^2)$.
29. $(x^2+1)(x^2-x+1)$.
30. $(x+2)(x^2-2x+4)$.
31. $(x-2)(x^2+2x+4)$.
32. $(2-x)(4+2x+x^2)$.
33. $(1+m)(1-m+m^2)$.
34. $(m+n)(m^4-m^3n+m^2n^2-mn^3+n^4)$.
35. $(m-1)(m^4+m^3+m^2+m+1)$.
36. $(1-m)(1+m+m^2+m^3+m^4)$.
37. $(a-3)(a^2+3a+9)$.
38. $(3+x)(9-3x+x^2)$.
39. $(2x+3a)(4x^2-6ax+9a^2)$.
40. $(10a+1)(100a^2-10a+1)$.
41. $(x+a)(x-a)(x^2+ax+a^2)(x^2-ax+a^2)$.
42. $(x+1)(x-1)(x^2+x+1)(x^2-x+1)$.
43. $(a-5)(a^2+5a+25)$.
44. $(1+a)(1-a)(1-a+a^2)(1+a+a^2)$.
45. $(a+5)(a^2-5a+25)$.
46. a^3+b^3.
47. a^4-b^4.
48. a^3-b^3.
49. a^4-b^4.
50. a^3+1.
51. a^4-1.
52. $1+x^3$.
53. $1-x^4$.
54. $1-m^4$.
55. a^3+8.
56. a^4-16.
57. a^3-8.
58. x^3-27.
59. $8-x^3$.
60. $27-m^3$.
61. x^3+125.
62. x^3-64.
63. a^3b^3+1.

XXIV.

1. $a^2+b^2+c^2+2ab-2ac-2bc$.
2. $x^2+y^2+z^2+2xy+2xz+2yz$.
3. $a^4-2a^3-a^2+2a+1$.
4. $a^2+x^2+y^2-2ax-2ay+2xy$.

5. $x^2+y^2+z^2-2xy-2xz+2yz$.
6. $x^4-2x^3-3x^2+4x+4$.
7. $a^4-2a^3b+3a^2b^2-2ab^3+b^4$.
8. $4a^4-4a^3+5a^2-2a+1$.
9. $9a^4+12a^3b+4a^2b^2-12a^2-8ab+4$.
10. $x^4+y^4+z^4+2x^2y^2-2x^2z^2-2y^2z^2$.
11. $m^4-10m^3+39m^2-70m+49$.
12. $m^4-4m^3n+10m^2n^2-12mn^3+9n^4$.
13. $a^4+4a^3+6a^2+4a+1$.
14. $x^4-8x^3+24x^2-32x+16$.
15. $p^2+q^2+r^2+2pq-2pr-2qr$.
16. $p^2+q^2+r^2+s^2-2pq+2pr-2ps-2qr+2qs-2rs$.
17. $m^4+2m^3+2m^2n-m^2+n^2+2mn-2m-2n+1$.
18. $a^4-2a^3-11a^2+12a+36$.
19. $x^4+2x^3-39x^2-40x+400$.
20. $x^8+6x^6+x^4-24x^2+16$.
21. $a^8-2a^6-23a^4+24a^2+144$.
22. $x^8-16x^6+46x^4+144x^2+81$.
23. $x^8+2x^6-3x^4-4x^2+4$.
24. $4x^6-12x^5+17x^4-16x^3+10x^2-4x+1$.
25. $x^2+a^2+b^2+c^2+2ax-2bx+2cx-2ab+2ac-2bc$.
26. $p^2+q^2+x^2+y^2-2pq+2px-2py-2qx+2qy-2xy$.
27. $m^4-2m^2n^2+n^4-2m^2n-2m^2+2n^3+3n^2+2n+1$.
28. $a^4-2a^2x^2+x^4-2a^2y+2a^2z+2x^2y-2x^2z+y^2-2yz+z^2$.

XXV.

1. $a^2+2ax+x^2-y^2$.
2. $x^2+2xy+y^2-1$.
3. $a^2+2ax+x^2-b^2$.
4. $a^2-b^2-2bc-c^2$.
5. x^4-x^2-2x-1.
6. $x^2-y^2+2xz+z^2$.
7. x^4+x^2+1.
8. x^4+2x^2+9.
9. $x^4+2x^3y+x^2y^2-y^4$.
10. $x^4-2x^2y^2+y^4$.
11. $x^2+2xy+y^2-a^2$.
12. $1+2a+a^2-a^4$.
13. $x^4+6ax^2+9a^2-4$.
14. $1+x^2+x^4$.
15. $x^4+2x^3+x^2-4$.
16. $p^2+2pq+q^2-r^2-2rs-s^2$.

17. $a^6-2a^5+a^4-a^2+2a-1$.
18. $a^6-4a^5+4a^4-4a^2-4a-1$.
19. $x^6-5x^4-2x^2-1$.
20. $x^6+4x^4+8x^2-4$.

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21. $(a+b+c)(a+b-c)$.
22. $(m^2+m+n)(m^2-m-n)$.
23. $(c-d+x)(c+d-x)$.
24. $(a-b+1)(a-b-1)$.
25. $(a+b+2)(a+b-2)$.
26. $(3+a+b)(3-a-b)$.
27. $(1+x-y)(1-x+y)$.
28. $(a+b+x)(a+b-x)$.
29. $(m-n+p)(m-n-p)$.
30. $(x+y-z)(x-y+z)$.
31. $(a-b+c)(a+b-c)$.
32. $(a-x+1)(a+x-1)$.
33. $(1+a-x)(1-a+x)$.
34. $(1+m-n)(1-m+n)$.
35. $(a+b+x+y)(a+b-x-y)$.
36. $(p+q+x-y)(p+q-x+y)$.
37. $(a-b+c-d)(a-b-c+d)$.
38. $(a+b+x-q-2)(a+b-x+q)$.
39. $(p-q+x-y)(p-q-x+y)$.
40. $4(4x-1)(3-x)$.

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41. $a+b-x$.
42. $x-y-z$.
43. $m+n-1$.
44. $x-y+a-b$.
45. $x+a-b$.
46. $a+b+x-y$.
47. $a-m-x+n$.
48. $2a+2x-1$.
49. $a^2+2ab-b^2-ax-bx+x^2$.
50. $a^2+2ab+b^2+ac+bc+c^2$.
51. $(x+y)^3-a(x+y)^2+a^2(x+y-a)$.
52. $4a^2+8ab+4b^2+2ac+2bc+c^2$.
53. $a^2+ax+ay+(x+y)^2$.
54. $x+y-1$.

XXVI.

1. a^3, $-a^3$, $8a^6$, $-8a^3$, $27a^{12}$, $-8a^6$.
2. a^2, $16a^4$, $27a^6b^3x^6$, a^{12}, x^{2n}.
3. $-a^{10}$, $-a^3b^6c^9$, $a^6b^3c^6$, $-27x^9y^6$, x^{am}.
4. $a^2+x^2+y^2+z^2+2ax+2ay+2az+2xy+2xz+2yz$.
5. $a^2+b^2+c^2+m^2-2ab-2ac-2am+2bc+2bm+2cm$.
6. $c^2+d^2+x^2+y^2-2cd+2cx-2cy-2dx+2dy-2xy$.
7. $9a^2-12ab+6a+4b^2-4b+1$.

8. $a^4 - 8a^2b^2 + 16b^4 + 2a^2c^2 - 8b^2c^2 + c^4$.

9. $9a^2b^2 + 4a^2c^2 - 12a^2bc + 6a^2bd - 4a^2cd + a^2d^2$.

10. $1 + 4a + 10a^2 + 12a^3 + 9a^4$.

11. $1 - 2a + 5a^2 - 4a^3 + 4a^4$.

12. $a^4 - 6a^3 + 7a^2 + 6a + 1$.

13. $a^4 - 4a^3 + 6a^2 - 4a + 1$.

XXVII.

1. $2ab$.
2. $3a^3b^3$.
3. $5ab^2c^3$.
4. $10ab^2c^4$.
5. $13m^3n^2$.
6. $20a^2b^3c^4$.
7. abc^2.
8. $-a^2x^3y^4$.
9. $-4a^3bc$.
10. $-3m^4n^2x$.
11. $10ax^2y^3$.
12. $6m^2n$.
13. amy^2.
14. $-a^2b$.
15. $\dfrac{abc^2}{xy}$.
16. $\dfrac{-ab^2}{c^2d^3}$.

XXVIII.

1. $a + b$.
2. $3a + 6b$.
3. $2a^2x - 3ax^2$.
4. $a^2 - ab + b^2$.
5. $1 - a + b$.
6. $x^2 + y^2 - 3$.
7. $3a^2 - 5a + 2$.
8. $1 - a^2 + a^3$.
9. $3a^3 + 2a^2 - 5$.
10. $3a^3 - 2a^2x + 4ax^2 - 2x^3$.
11. $2 - 2a - a^2 + a^3$.
12. $a^3 - 2a^2b + 2ab^2 - b^3$.
13. $2x^2 - 3xy + 7y^2$.
14. $4a^3 - 2a^2b + 2ab^2 - 4b^3$.
15. $11x^2 + 7x - 1$.
16. $1 - 4x^3 + x^6$.

XXIX.

1. 13.
2. -5.
3. 0.
4. 0.
5. -7.
6. 0.
7. -1.
8. -11.
9. 66, -2.
10. 6, 4, 71.
11. 1, -1, 0.
12. 32; a^6 or b^6.

XXX.

1. $a - a^{\frac{1}{2}} + 2a^{\frac{1}{4}} - 1$.

2. $a + b$.

3. $\dfrac{1}{16}x^2 - \dfrac{1}{12}x^{\frac{3}{2}}y^{\frac{1}{2}} + \dfrac{1}{18}xy - \dfrac{1}{27}x^{\frac{1}{2}}y^{\frac{3}{2}} + \dfrac{1}{81}y^2$.

4. $a - 25a^{\frac{2}{3}} - a^{\frac{1}{2}} - a^{\frac{1}{3}} - 12a^{\frac{1}{6}} - 9$.
5. $x^2 - y^2$.
6. $2x^{\frac{3}{4}} - x^{\frac{1}{2}} + 2x^{\frac{1}{4}}$.
7. $3x^{\frac{1}{2}} - 2x^{\frac{1}{4}}y^{\frac{1}{4}} + 3y^{\frac{1}{2}}$.
8. $m^{\frac{2}{3}} - 2m^{\frac{1}{6}}n^{\frac{1}{3}} + n^{\frac{1}{3}}$.
9. $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} + 1$.
10. $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.
11. $a^{\frac{1}{2}} - 2a^{\frac{1}{4}} + 1$.
12. $a^{\frac{1}{4}} - 2a^{\frac{1}{8}}b^{\frac{1}{8}} + b^{\frac{1}{4}}$.
13. 8.
14. 4.
15. 3.

XXXI.

1. $-3\frac{1}{2}$.
2. $a^{-3} - 1$.
3. $x^3 + 1 - x^{-1} - 2x^{-3} + x^{-4}$.
4. $a^{-1} - 1$.
5. $x^{-6} - a^{-4}$.
6. $a + a^{-1}$.
7. $m - m^{-1}$.
8. $a^2 + 1 + a^{-2}$.
9. $a + 1 + a^{-1}$.
10. 1.

XXXII.

1. ab.	11. ab^2.	21. $a - b$.	31. $x + 1$.
2. ab.	12. $13mp$.	22. $x - y$.	32. $x + 3$.
3. $m^2n^2x^2$.	13. b^2c^2.	23. $a - x$.	33. $(a - b)^2$.
4. xyz.	14. a^2x.	24. $a + x$.	34. $a - 1$.
5. $3x^2y$.	15. 11.	25. $x - 1$.	35. $m - 3$.
6. $15p$.	16. a.	26. $x^2 - y^2$.	36. $a + 1$.
7. $25abd^2$.	17. ax.	27. $x - y$.	37. $n + 1$.
8. $19xy^5z^5$.	18. 3.	28. $x + y$.	38. $1 + n$.
9. $17npq$.	19. $4ab$.	29. $x + y$.	39. $m - 5$.
10. $7x$.	20. x^2.	30. $x - y$.	40. $x + 3$.

XXXIII.

1. $a + 1$.	6. $3x^2 - 2xy + y^2$.	11. $m^2 + 2m + 2$.
2. $a^2 + 1$.	7. $a^2 - 3ab + 5b^2$.	12. $3x^2 - 4x + 5$.
3. $x^2 - 7x - 1$.	8. $a^2 + ab + b^2$.	13. $2a - y$.
4. $a^2b^2 - 2ab + 1$.	9. $2(a^2 - ab + 3b^2)$.	14. $a - 1$.
5. $x^2 - 3xy + 2y^2$.	10. $x(4a^2 - 3a + 2)$.	15. $x + 1$.

16. $x+y$.
17. $5a-b$.
18. $3x-7$.
19. $2m-1$.
20. $2a-3$.
21. $2a-3$.
22. x^2+1.
23. $a-1$.
24. $2x-3$.
25. $a^3-a^2b-ab^2+b^3$.
26. a^2+1.
27. x^2+2x+2.
28. $2x-a$.
29. $7a-3$.
30. $7m^2+m+36$.
31. $a-2$.
32. $a-1$.
33. $x+1$.
34. $(a+1)^2$.
35. $a-x$.

XXXIV.

1. a^2b^2.
2. $42x^2y^3z$.
3. $6a^2b^2c^2$.
4. $210a^2xyz^2$.
5. $42m^3n^2$.
6. $18a^3b^2x^3$.
7. $132a^2b^2c^2de$.
8. $18a^2b^3c^3$.
9. $30a^3bcx^4y^3$.
10. $20ab^2cx^2y^2z$.
11. $x^2y^2(x+y)$.
12. $ax^2y(x-y)$.
13. $x^2y^2z(x+y)$.
14. $20x^2y^2(x+y)$.
15. $ab^2c^3(a+b)$.
16. $ab^3c^3(a-1)$.
17. $42x^2yz(1-x)$.
18. $12a^2b^3(a-b)$.
19. $abc(x+y)$.
20. $2abc(x-y)$.

XXXV.

1. $mp(q^2-r)$.
2. $6a(m^2-n^2)$.
3. $6(c^2-4d^2)$.
4. $12(a^2-1)$.
5. y^4-z^4.
6. $3a^3(a-b)(a+b)^2$.
7. $6a^2b^3l^2x^2(a^2-b^2)$.
8. $(a^2+b^2)(a^2-b^2)^2$.
9. $(a+2)^2(a-2)^2$.
10. $12(a-3)(a-4)$.
11. $(x^2-4)(x+4)$.
12. $(x+1)(x-1)^3$.
13. $(a^2-x^2)(a^2+ax+x^2)$.
14. $(a-1)(a-4)(a+5)(a+7)$.
15. $(a^2-1)(a^2-4)$.
16. $(x^2-1)(x^4-16)$.
17. $xy(x+1)(x+2)(x+3)(x+4)$.
18. $x^2(x^2-y^2)(x^2+xy+y^2)$.
19. $a(1-a^4)(1+a+a^2)$.
20. $6(a-x)^2(a+x)^3$.
21. a^6-1.
22. $2x(x^2-y^2)^2$.
23. $20(m+1)^3(m-1)$.
24. $(a+1)(a+2)(a+3)$.
25. $(a^2-1)(a+2)(a-1)$.
26. $36(m^2-4)(m^2+3)$.
27. $4(x-1)^3(x+1)^2$.

28. $45(x^2-9)$.
29. $108a(a-x)^2(a+x)$.
30. $51(a^2-4)(a+1)$.
31. $(2x-3)(3x^2+4x+1)$.
32. $(2x-5)(4x^2-x-3)$.
33. $(1-7x)(2-x-6x^2)$.
34. $(2x-5y)(6x^2+11xy+4y^2)$.
35. $(7x-y)(6x^2+xy-5y^2)$.
36. $(x+3y)(3x^2+5xy-2y^2)$.
37. $(x+3)(x^3-3x^2+3x-1)$.
38. $(x+y)(x^3+x^2y+4y^3)$.
39. $(x^3-4x^2+3)(x^2-2x-2)$.
40. $(x^3-8x+3)(x^2-x+3)$.

XXXVI.

1. $\frac{x^2}{a^2}$.
2. $\frac{b}{c}$.
3. mn.
4. $\frac{1}{2pq}$.
5. $\frac{cd^2}{2}$.
6. $\frac{2}{3}$.
7. $\frac{a+b}{2(a-b)}$.
8. $\frac{3xy}{a-2xz}$.
9. $\frac{x}{a}$.
10. $\frac{l}{p}$.
11. $\frac{1-2y}{2(a-2x)}$.
12. $\frac{x-y}{xy-1}$.
13. $\frac{a}{c}$.
14. $\frac{a}{b}$.
15. $\frac{x+y}{x-y}$.
16. $\frac{1}{x-y}$.
17. $\frac{a+x}{a-x}$.
18. $\frac{1}{a+b}$.
19. $a-b$.
20. $\frac{1}{a^2+ax+x^2}$.
21. $\frac{b}{c}$.
22. $\frac{1}{b-c}$.
23. $\frac{1}{a+x}$.
24. $\frac{1}{(a-x)^2}$.
25. $\frac{b+c}{b-c}$.
26. $\frac{1}{6ab+8}$.
27. $\frac{a+2}{a-2}$.
28. $\frac{a+1}{a-2}$.
29. $\frac{x-3}{x+2}$.
30. $\frac{ab+2}{ab-2}$.
31. $\frac{x+z}{y-1}$.
32. $\frac{a+3}{a+5}$.
33. $\frac{3a+2}{5a+1}$.

34. $\dfrac{2a+1}{3a-3}$.

35. $\dfrac{5x-6}{10x+1}$.

36. $\dfrac{a-17b}{17a+3b}$.

37. $\dfrac{2x-3y}{5x-3y}$.

38. $\dfrac{x-13}{11x+12}$.

39. $\dfrac{5a-x}{9a+5x}$.

40. $\dfrac{ax-2b}{3ax+b}$.

41. $\dfrac{1}{x+y}$.

42. $\dfrac{x-1}{3x+2}$.

XXXVII.

1. $\dfrac{a^2c, \&c.}{abc}$.

2. $\dfrac{x^3y, \&c.}{x^2y^2}$.

3. $\dfrac{a^2cd, \&c.}{abcd}$.

4. $\dfrac{cd(a+b), \&c.}{bcd}$.

5. $\dfrac{y^2(x-y), \&c.}{xy^3}$.

6. $\dfrac{c(a+x), \&c.}{abc}$.

7. $\dfrac{a^2c^2, \&c.}{b^2c^2}$.

8. $\dfrac{4ab^2(x-y), \&c.}{12a^2b^2}$.

9. $\dfrac{b^2c^2x^2, \&c.}{a^2b^2c^2}$.

10. $\dfrac{4a(m+n), \&c.}{20a^3}$.

11. $\dfrac{b^2x(a-b), \&c.}{a^2b^2(a-b)}$.

12. $\dfrac{(a+x)^2, \&c.}{a^4-x^4}$.

13. $\dfrac{x^3, \&c.}{x^2(x-y)}$.

14. $\dfrac{x^3(a-x), \&c.}{x^2(a^2-x^2)}$.

15. $\dfrac{(1-x)^3, \&c.}{(1+x)(1-x)^2}$.

16. $\dfrac{x-y, \&c.}{(x+y)^2(x-y)}$.

17. $\dfrac{a(c+d)(d+e), \&c.}{(b+c)(c+d)(d+e)}$.

18. $\dfrac{x+a, \&c.}{a(x^2-a^2)}$.

19. $\dfrac{2a^2, \&c.}{a^5-9a}$.

20. $\dfrac{x^2(a+b), \&c.}{ax(a^2-b^2)}$.

XXXVIII.

1. $\dfrac{ayz+bxz-cxy}{xyz}$.

2. $\dfrac{3abc-a^2b-a^2c-b^2c}{abc}$.

3. $\dfrac{5}{12}(a+1)$.

4. $\dfrac{5a-b}{10}$.

5. $\dfrac{3y^2-2xy}{6x^2}$.

6. $\dfrac{5a-3b-9}{32}$.

7. $-\dfrac{2a-2b-17}{30}$.

8. $\dfrac{9-21x+11y}{30}$.

9. $\dfrac{2x^2}{x^2-y^2}$.

10. $\dfrac{4xy}{x^2-y^2}$.

11. $\dfrac{2x-1}{x^2-1}$.

12. $\dfrac{x-2}{x^2-1}$.

13. $\dfrac{4a^2+2b}{a^2-b^2}$.

14. $-\dfrac{2xy}{x^2-y^2}$.

15. $\dfrac{2xy}{x+y}$.

16. $\dfrac{a^2}{(a-x)^2}$.

17. $\dfrac{a^2+4ax+2x^2}{(a+x)^3}$.

18. $\dfrac{2x^3+6xy^2}{(x^2-y^2)^2}$.

19. 0.

20. $\dfrac{b}{2a^3}$.

21. $\dfrac{3}{1-x^2}$.

22. $\dfrac{x-y}{2(x+y)}$.

23. $\dfrac{a}{1-b}$.

24. $\dfrac{4x}{x+y}$.

25. $\dfrac{x+y}{xy}$.

26. $\dfrac{a^2b^2}{a^4-b^4}$.

27. $\dfrac{2}{(x-1)(x-3)}$.

28. $\dfrac{1}{(a+1)(a+3)}$.

29. $\dfrac{x(x+2)}{(x-2)(x^2-9)}$.

30. $\dfrac{d-b}{(a+b)(a+c)(a+d)}$.

31. 0.

32. $\dfrac{3x^3-9x^2-23x+37}{x^3-6x^2-17x+60}$.

33. $\dfrac{x^3-8x^2+20x-14}{x^3-9x^2+26x-24}$.

34. $\dfrac{5x^3-5x^2+6}{6x^2(x^2-1)}$.

35. $\dfrac{8xy^2}{(x^2-y^2)^2}$.

36. $\dfrac{a^3x+5ax}{a^4+a^2-2}$.

37. $\dfrac{a^2-b^2-a}{a-b}$.

38. $-\dfrac{3a^2+b^2}{a^2-b^2}$.

39. 0.

40. $\dfrac{a(a-4)}{(a^2-1)(a^2-4)}$.

41. $\dfrac{6x-13}{(x-1)(x-2)(x-3)}$.

42. $-\dfrac{3}{(a+2)(a-3)(a-4)}$.

43. $\dfrac{2a^3}{a^4-x^4}$.

44. $\dfrac{2(x-2)}{x-1}$.

45. $\dfrac{a^3(a^2+b^2)}{a^6-b^6}$.

46. $\dfrac{6ab}{4b^2-9a^2}$.

47. $\dfrac{a^2}{a^2-b^2}$.

48. 1.

49. $\dfrac{1}{3a-2b}$.

50. $\dfrac{1}{(a-1)(a-2)(a-3)}$.

51. 0.

52. $\dfrac{2ab-ax-bx}{(a-x)(b-x)(a-y)}$.

53. $-\dfrac{x(1+3x^2)}{1-x^4}$.

54. $\dfrac{2}{a^2-b^2}$.

55. $\dfrac{b^2-ac}{(a-b)(a-c)(b-c)}$.

56. $\dfrac{b-a}{ab(a-b)(a-c)}$.

57. 0.

58. $-\dfrac{a}{(a-c)(b-c)}$.

59. 0.

60. $-\dfrac{1}{x(x-1)(x-2)(x-3)}$.

XXXIX.

1. $\dfrac{2ab}{3cx}$.

2. 1.

3. $\dfrac{4ax}{y}$.

4. 1.

5. a^2b.

6. $\dfrac{am}{bn}$.

7. 1.

8. $\dfrac{b}{a}$.

9. $\dfrac{a}{b}$.

10. $\dfrac{b^2}{a^2}$.

11. 1.

12. b^2.

13. $\dfrac{x^2+xy+y^2}{x^2-xy+y^2}$.

14. $\dfrac{a^2}{a^2-b^2}$.

15. $(a+b)^2$.

16. $x+y$.

17. $\dfrac{1}{ab}$.

18. $\dfrac{x-1}{x+4}$.

19. a.

20. $\dfrac{a^2-3b}{a^2+3b}$.

21. $\dfrac{a^2}{a^2-16}$.

22. $\dfrac{x-1}{x^2-y^2}$.

23. $\dfrac{x^2(x^4+a^2)}{x^4-a^2}$.

24. 1.

25. $\dfrac{1}{a}$.

26. $\dfrac{1}{x+y}$.

27. $\dfrac{1}{2}$.

28. 1.

29. $\dfrac{x^3}{(x^4-y^4)(x-y)^2}$.

30. a^2-b^2.

XL.

1. $\dfrac{1}{a(a-b)}$.

2. $\dfrac{x}{3x+1}$.

3. $\dfrac{2x+3}{x}$.

4. $\dfrac{x+5}{4}$.

5. $\dfrac{4x-2}{x+13}$.

6. $\dfrac{a-b}{ab}$.

7. $\dfrac{2}{(x-y)^2}$.

8. $\dfrac{x-y}{x}$.

9. $\dfrac{a^2+a+1}{a}$.

10. $\dfrac{a-b}{a}$.

11. $b-a$.

12. $\dfrac{2(a+b)}{3a(a-b)}$.

13. $\dfrac{2}{x+y}$.

14. $\dfrac{1}{b^2}$.

15. $\dfrac{ac+b^2}{ac-b^2}$.

16. $\dfrac{x}{4(x-2)}$.

17. $\dfrac{3}{2+x}$.

18. $-m$.

19. -1.

20. $\dfrac{x^3}{12(x-1)}$.

21. $\dfrac{b-c}{b^2}$.

22. $\dfrac{x^2+1}{2x}$.

23. $\dfrac{x}{x-1}$.

24. $\dfrac{x-1}{x}$.

25. $\dfrac{x^3}{x^2-x+1}$.

26. $\dfrac{x}{x-1}$.

27. $\dfrac{x+1}{2x+1}$.

28. $2x$.

29. x^2-x+1.

30. $\dfrac{1+x^3-x^4}{1+x-x^2}$.

31. $\dfrac{6-2x}{4-x}$.

32. $-\dfrac{a^2+b^2}{2ab}$.

33. $-a$.

34. $\dfrac{m-n}{mn}$.

35. $\dfrac{x+y}{xy}$.

36. $\dfrac{4y}{x+3y}$.

37. $\dfrac{a^2+ab+b^2}{ab}$.

38. 1.

39. $x+1$.

40. $x-1$.

XLI.

1. $x^2 - \dfrac{1}{x^2}$.

2. $\dfrac{4x^2}{9} - \dfrac{9}{4x^2}$.

3. $x^4 - \dfrac{1}{x^4}$.

4. $\dfrac{x^3}{6} + \dfrac{17}{72}x^2 - \dfrac{x}{12} - \dfrac{1}{8}$.

5. $\dfrac{x^2}{6} - \dfrac{5}{6} + \dfrac{1}{3x} + \dfrac{1}{x^2} - \dfrac{1}{x^3}$.

6. $\dfrac{a^2}{b^2} + \dfrac{b}{a}$.

7. $x^3 + x^2 - x + \dfrac{1}{x} - \dfrac{1}{x^2} - \dfrac{1}{x^3}$.

8. $\dfrac{x^2}{9} - \dfrac{4}{9} + \dfrac{4}{9x^2} - \dfrac{1}{x^4}$.

9. $\dfrac{x^3}{6} - \dfrac{25x^2}{36} + \dfrac{11x}{3} - \dfrac{10}{3} + \dfrac{7}{2x} - \dfrac{3}{x^2}$.

10. $2 - \dfrac{4}{a} + \dfrac{9}{a^2} - \dfrac{10}{a^3} + \dfrac{9}{a^4} - \dfrac{6}{a^5}$.

11. $x^2 + 1 + \dfrac{1}{x^2}$.

12. $x - \dfrac{1}{x}$.

13. $2 - \dfrac{1}{x}$.

14. $x^2 - 2 + \dfrac{4}{x^2}$.

15. $\dfrac{1}{x} - \dfrac{1}{y}$.

16. $\dfrac{1}{a} + \dfrac{1}{b} + 1$.

17. $4a^2 + 6 + \dfrac{9}{a^2}$.

18. $\dfrac{a^2}{4} - \dfrac{1}{a}$.

19. $a^2 - \dfrac{a}{3} - \dfrac{1}{a}$.

20. $2a^2 - 3a + 2 - \dfrac{1}{a}$.

21. $\dfrac{x}{3} - 1 + \dfrac{2}{x}$.

22. $x^3 + 2 - \dfrac{1}{x^3}$.

23. $x^2 - 2 + \dfrac{1}{x^2}$.

$\dfrac{4a^2}{9} + 1 + \dfrac{9}{16a^2}$:

$x^2 - 4x + 6 - \dfrac{4}{x} + \dfrac{1}{x^2}$.

24. $\dfrac{a}{3} - \dfrac{3}{a}$.

25. $\dfrac{x}{2} + \dfrac{2}{x}$.

26. $3x - \dfrac{3}{x}$.

27. $x^3 - \dfrac{2}{x^3}$.

28. $x - 1 - \dfrac{1}{x}$.

29. $\frac{x}{2} - 3 + \frac{2}{x}$. 30. $a^2 + 1 + \frac{1}{a^2}$. 31. $\frac{a}{b} - 1 + \frac{b}{a}$.

XLII.

1. 1.
2. 1.
3. -8.
4. 1.
5. $-\frac{5}{13}$.
6. 1.
7. $\frac{1}{b-1}$.
8. 3.
9. 3.
10. $2\frac{2}{5}$.
11. $-\frac{1}{y}$.

17. $-4\frac{a}{c}$.
18. $\frac{x^3-1}{x(x^2-x-1)}$.
19. $-(x+y)$.
20. $\frac{x+3}{x^2+3x+5}$.
21. $\frac{1}{x}$.
22. 0.
23. $\frac{2a^2}{(a+x)^2}$.
24. $\frac{b^4-a^4}{b^3}$.
25. $\frac{1}{x-b}$.
26. $\frac{2}{b}$.
27. $\frac{a}{(x-1)(x-2)(x-3)}$.
28. $\frac{x^2-1}{x^2-4}$.
29. $\frac{3m^2+2m+1}{m^2+2m+3}$.
30. $\frac{ab(a^2-1)}{a^2b+a-b}$.
31. $\frac{a^3}{a^4-a^2b^2+b^4}$.
32. $\frac{a^2}{a^2-b^2}$.
33. $\frac{(3a-b)(a-b)}{(2a-b)^2}$.
34. $\frac{1}{xyz}$.
35. 0.
36. $-\frac{a+6}{(a+1)(a-2)(a+3)}$.
37. $(x+y)^2$.
38. 1.

XLIII.

1. 1.
2. 1.
3. 7.
4. -2.
5. 9.
6. 20.
7. 9.
8. 21.
9. 0.
10. 4.
11. -1.
12. -3.
13. 101.
14. 3.
15. 3.
16. -1.
17. 34.
18. 1.
19. 3.
20. 12.
21. 16.
22. 5.
23. 0.
24. -8.

XLIV.

1. -1.	6. -10.	11. 10.	16. -7.
2. 2.	7. $\frac{9}{53}$.	12. 0.	17. $2\frac{149}{157}$.
3. 1.	8. $\frac{1}{2}$.	13. -184.	18. 20.
4. $2\frac{3}{4}$.	9. $-1\frac{2}{5}$.	14. 6.	19. $1\frac{1}{2}$.
5. -36.	10. 4.	15. 9.	20. 1.
21. $\frac{a+b}{a-b}$.	24. $\frac{a^2(c-b)}{c(b-a)}$.	27. $\frac{1}{2a}$.	
22. $\frac{2b+c}{a+b}$.	25. $\frac{1}{2}(a-b)$.	28. $\frac{b}{2}$.	
23. $\frac{a-c}{1-b}$.	26. $\frac{1}{2a}$.	29. 2.	
30. 3.	32. 3.	34. 7.	36. 12.
31. $-\frac{3}{4}$.	33. $a+b$.	35. 8.	37. $2a+3b$.
38. $\frac{ab^2+bc^2+a^2c}{a^2b+b^2c+ac^2}$.	39. $\frac{a+b+c}{ab+bc+ac}$.	40. $a+b$.	

XLV.

1. -2.	9. $\frac{39}{44}$.	16. $\frac{3}{2}$.	23. 8.
2. 11.	10. $-\frac{2}{9}$.	17. 4.	24. $-6\frac{2}{17}$.
3. -3.	11. -10.	18. $2a+3b$	25. 1.
4. 20.	12. $\frac{2}{7}$.	19. 1.	26. 3.
5. 2.	13. 4.	20. 5.	27. $\frac{33a}{7}$.
6. 4.	14. 3.	21. -24.	28. $6a$.
7. 7.	15. 7.	22. 6.	29. a.
8. $-54\frac{2}{7}$.			30. $\frac{6}{5}(a+b)$.

XLVI.

1. 4.	5. 420.	9. $-a$.	13. $6\frac{6}{7}$.
2. 8.	6. 9.	10. b.	14. $\frac{1}{6}$.
3. 5.	7. -20.	11. 4.	15. 6.
4. -1.	8. 8.	12. -4.	16. 1.

17. -4. 19. -2. 21. 9. 23. 1.

18. 11. 20. 32. 22. $a+b+ab$. 24. 2.

25. $\frac{b}{c}$. 29. $\frac{ab}{a+b}$. 33. $\frac{a^2+4}{a+2}$.

26. $-b$. 30. $-a$. 34. a.

27. 1. 31. $\frac{ab}{a^2+b^2-ab}$. 35. $\frac{67a}{10}$.

28. 1. 32. $-\frac{2ab}{a+b}$. 36. $\frac{b-4a}{2}$.

37. $b+c$. 41. 9. 45. $\frac{1}{3}$. 49. 1.

38. 9. 42. 3. 46. 49. 50. $-1\frac{2}{3}$.

39. 7. 43. 24. 47. 7. 51. 1.

40. 1. 44. 2. 48. 6.

XLVII.

1. 15.
2. 26.
3. 20.
4. £250.
5. 10.
6. 18.
7. 17 and 14.
8. 26 and 17.
9. 105, 96.
10. 19, 14.
11. 31, 25.
12. A., £365; B., £260.
13. 17, 14.
14. 20, 16, 14.
15. (1) £850; (2) £800; (3) £750.
16. Jacket, 20s. Trousers, 14s. Vest, 8s.
17. 66, 33, 11.
18. A., £8; B., £4; C., £12.
19. 91, 30.
20. 625; 375; 250; 125.
21. 16, 8.
22. 40, 35.
23. 21, 7.
24. 40, 20.
25. 40, 20, 5.
26. 17, 6.
27. 3s., 6s., 11s.
28. 920 Juniors, 301 Seniors.
29. 6.
30. 17.
31. 8, 6.
32. 30.
33. 40.
34. 18, 17.
35. 37, 16.
36. 120, 40.
37. 13s.
38. 12, 9, 6.
39. 40, 7, 3.
40. A., £150; B., £350; C., £500; D., £1200.
41. 10, 20.

42. (1) £7 5s. 0d.	44. 30s., 10s.	48. £200.
(2) £12 10s. 0d.	45. 20, 30.	49. 84, 28.
(3) £20 5s. 0d.	46. 8, 12.	50. 40 at 1s. 6d.
43. 16.	47. £27, £1.	80 at 1s.

XLVIII.

1. 32.	8. 12.	15. 18, 12.
2. 30.	9. 16, 12.	16. 90, 30, 15.
3. 26.	10. 15.	17. 17, 34, 102.
4. 12.	11. 21.	18. A., 42.
5. 75.	12. £26.	B., 21.
6. 33.	13. $\frac{5}{12}$.	C., 7.
7. 3 or 27.	14. 144, 48.	19. 219, 73, 219.

20. Elder Brother, £3600; Younger, £1500; Daughter, £1200.

21. 25 perch; 15 roach; 5 chub.

22. 42, 54.	29. Man, £120.	34. 120 gallons.
23. 40 1st class;	Woman, 80.	35. 12 min.
80 2nd class;	Elder child, 20.	36. $\frac{2}{3}$ full.
120 3rd class.	Younger, 10.	37. A., 12s.;
24. $2\frac{6}{7}$ days.	30. Father, 60.	B., 18s.
25. $7\frac{1}{2}$ days.	Elder Son, 30.	38. 90, 66.
26. $4\frac{8}{13}$ hours.	Younger, 25.	39. 40 at 2s. 9d.;
27. $8\frac{1}{2}$ minutes;	31. 12 feet.	20 at 2s.
$1\frac{49}{240}$ miles.	32. 300.	40. 12 hours.
28. £300.	33. 720.	

XLIX.

1. £3000.	3. 62.	5. 900.
2. 4 tons, 6 tons.	4. 72 eggs.	6. £280.

7. 1050 tickets.

8. 5′ behind at 11 h. $54\frac{6}{11}'$; Right angles, 11 h. $10\frac{10}{11}'$; Opposite, 11 h. $27\frac{3}{11}'$.

9. Together, 6 h. $32\frac{8}{11}'$; Right angles, 6 h. $16\frac{4}{11}'$; And 6 h. $49\frac{1}{11}'$; Opposite, 7 h. $5\frac{5}{11}'$.

10. Man's share, £10 ;
Woman's share, £5 ;
Child's share, £2.
11. 20 cwt.
12. 8 miles, 1 fur.
13. 125 men.
14. 7 working, 3 idle days.
15. 35, 21.
16. 46.
17. 1st, 20s. per ton.
2nd, 28s. „
3rd, 32s. „
4th, 40s. „
18. $1\frac{27}{50}$.
19. A$\frac{5}{14}$: B$\frac{1}{2}$: C$\frac{3}{14}$ days.
20. 50, 46.
21. £600.
22. 12.
23. 60.
24. 120.
25. 60,000.
26. 200 lbs.
27. £1600, £2400.
28. $2\frac{2}{7}$ hours.
29. 236.
30. 32.

L.

1. $2 : 3$.
2. $4 : 3$.
3. $-2 : 2$.
4. $5 : 1$.
5. $20 : 13$.
6. $1 : -1$.
7. $4 : 7$.
8. $10 : 4$.
9. $6 : 15$.
10. $1 : 1$.
11. $3 : 5$.
12. $6 : 12$.
13. $19 : 25$.
14. $7 : -2$.
15. $-1 : 3$.
16. $-8 : -60$.
17. $-1 : -2$.
18. $2 : -7$.
19. $-3 : 6$.
20. $5 : -5$.

LI.

1. $6 : 4$.
2. $8 : 10$.
3. $14 : 16$.
4. $-39 : -33$.
5. $\frac{175}{11} : \frac{130}{11}$.
6. $2 : 1$.
7. $\frac{1}{2} : 1$.
8. $16 : 4$.
9. $10 : 1$.
10. $\frac{5}{6} : \frac{1}{3}$.
11. $18 : 6$.
12. $15 : 5$.
13. $38 : 8$.
14. $7 : 9$.
15. $5 : 11$.
16. $4 : -1\frac{1}{3}$.
17. $12 : 36$.
18. $10 : 6$.
19. $1 : 2\frac{4}{5}$.
20. $17 : 21$.
21. $4 : 2$.
22. $10 : -1$.
23. $5 : 5$.
24. $-3\frac{9}{11} : 4\frac{4}{55}$.
25. $24 : 36$.
26. $7 : 5$.
27. $1 : 5$.
28. $4 : 5$.
29. $13 : 2$.
30. $\frac{1}{2} : \frac{1}{4}$.

31. $\frac{c+d}{2a} : \frac{c-d}{2b}$.

32. $\frac{md-nb}{ad-bc} : \frac{mc-an}{bc-ad}$.

33. $\frac{ab-c}{a^2-1} : \frac{ac-b}{a^2-1}$.

34. $\frac{d-2b}{ad-bc} : \frac{c-2a}{bc-ad}$.

35. $\frac{m+n}{2} : \frac{m-n}{2}$.

36. $\frac{1}{2}a : \frac{1}{2}b$.

37. $\frac{ab}{a+b} : \frac{ab}{a+b}$.

38. $a : b$.

39. $m : n$.

40. $\frac{bc}{a+c} : \frac{ac}{a+c}$.

41. $\frac{ab(bc+ad)}{a^2+b^2} : \frac{ab(ac-bd)}{a^2+b^2}$.

42. $\frac{2a^2b}{a^2-b^2} : \frac{2ab^2}{a^2-b^2}$.

43. $\frac{1}{2}a : \frac{1}{2}b$.

44. $\frac{ac}{a+b} : \frac{ab}{a+b}$.

45. $18 : \frac{6}{5}$.

46. $\frac{a^2-bc}{ac-b^2} : \frac{a^2-bc}{c^2-ab}$.

47. $6 : 10$.

48. $\frac{m^2+n^2}{m} : \frac{m^2+n^2}{n}$.

49. $32 : 4\frac{2}{3}$.

50. $1 : \frac{1}{2}$.

51. $\frac{23}{20} : -3\frac{20}{47}$.

52. $4 : 7$.

53. $\frac{2a}{c+d} : \frac{2b}{c-d}$.

54. $\frac{1}{a} : \frac{1}{b}$.

LII.

1. $2 : 3 : 1$.
2. $1 : 1 : 0$.
3. $2 : -1 : -2$.
3. $4 : 2 : 1$.
5. $7 : 4 : 3$.
6. $1 : 2 : 0$.
7. $19 : 17 : 1$.
8. $\frac{5}{12} : \frac{7}{18} : \frac{1}{4}$.
9. $9 : 10 : 11$.
10. $20 : -10 : 1$.
11. $8 : 10 : 12$.
12. $2 : 4 : 6$.

LIII.

1. 45, 30.
2. 17, 13.
3. 12, 4.
4. 8, 12.
5. 19, 17.
6. 37, 25.
7. $\frac{3}{4}$.
8. $\frac{38}{45}$.

9. 12, 8.
10. $\frac{11}{24}$.
11. Cart, $\frac{1}{2}$ ton ;
Waggon, 2 tons.
12. Large, 2 doz. ;
Small, $1\frac{1}{2}$ doz.
13. 69.
14. A., 25s. ; B., 55s.
15. 6 large, 7 small.
16. 431.
17. £3 15s. 0d., £1 4s. 0d.
18. Rate of rowing, 5 miles ;
Of stream, 1 mile an hour.
19. $\frac{5}{9}$, 6, and 4 miles an hour.
20. 12, 5s.

ANSWERS TO EXAMINATION PAPERS.

OXFORD, 1868.

1. -1.
2. $a^2 - ab + b^2$.
3. (1) $-\dfrac{4}{(x-2)(x-7)}$.
   (2) 1.
4. $6x^2 + 3x - 4$.
5. (1) $x = 1$.
   (2) $x = 2$.
   (3) $x = 3$;
   $y = -4$.
6. 80 : 40.

OXFORD, 1869.

1. 0.
2. $a^2 + 2ab + b^2$.
3. (1) 1.
   (2) $\dfrac{x-12}{x^2-9}$.
4. $x^2 + 4x + 4$.
5. (1) $x = 2$;
   (2) $x = 3$;
   (3) $x = a + b$;
   (4) $x = 8$, $y = 12$.
6. 21, 3.

OXFORD, 1870.

1. $a^2 + ab - ac - bc$.
2. (1) $-\dfrac{1}{a}$;
   $\dfrac{4x^2 - 4x + 1}{4x^2 - 3x - 1}$.
3. $a^6 - 4a^3 + 1$.
4. (1) $x = 24$;
   (2) $x = 2$; $y = 4$; $z = 6$.
5. 33, 18.

OXFORD, 1871.

1. $3x^2 - 2xy + 3y^2$.
2. (1) $\dfrac{2x+5}{5x+2}$.
   (2) $\dfrac{1}{3x-2y}$.
3. $3x^2 - 5xy + y^2$.
4. (1) $x = 5$;
   (2) $x = 7$;
   (3) $x = 5$, $y = 7$.
5. 15, 24.

OXFORD, 1872.

1. $4\frac{1}{6}$.
2. $60x^3(x+1)(x-1)^2$.
3. (1) $\dfrac{3x^2+1}{x^2+2x-3}$.
   (2) $45(x+1)$.
4. (1) $x = 2a + 3b$;
   (2) $x = 144$;
   (3) $x = 5$, $y = 1$.
5. £52; £2 12s.

OXFORD, 1873.

1. $27x^3 + 8y^3 + z^3 - 18xyz + 2y^2z$.
2. $3(x^2 - x + 1)$.
3. $\dfrac{2x}{y} - \dfrac{2y}{x}$.
4. (1) $\dfrac{10(x^2+1)}{(x^2-1)^2}$.
   (2) a.
5. (1) $x = 7$;
   (2) $x = 6$, $y = 6$.
6. 9, 7, 5.

OXFORD, 1874.

1. $a^3 + b^3 + c^3 + d^3 - 3a(bc + bd + ad) - 3bcd$.
2. $x^2 + 1 + \dfrac{1}{x^2}$.
3. $\dfrac{x^2(a^2+x^2)}{a(a^2+2x^2)}$.
4. $(a^2 - x^2)(a^2 - 4x^2)$.
5. (1) $x = 0$;
   (2) $x = 12$; $y = 30$.
6. $10\frac{10}{11}$ min. past 2 o'clock.

OXFORD, 1875.

1. 22.
2. $a^5 - 4a^3 + 5a + \frac{1}{a^3} - \frac{3}{a^5}$.
   $x^3 + y^3$.
3. L.C.M. $= (x^2 - 1)(x - 2)(x - 3)$.
   G.C.M. $= x^2 + 1$.
4. $a^3 - x^3$.
5. $\frac{2x^2}{3} - 6x + 1$.
6. $x = 15 : x = 2$.
7. 20 inches, 16 inches.

OXFORD, 1876.

1. -2.
2. $x^2 + \frac{1}{x^2} + 1$.
   $x^2 - xy - xz + y^2 - yz + z^2$.
3. (1) $1 + x$;
   (2) $-\frac{1}{x-a}$.
4. $x^2 + 1 + \frac{1}{x^2}$.
5. $18a^2b^2(a^6 - b^6)$.
6. (1) $x = \frac{2}{3}$:
   (2) $x = 1\frac{1}{3}$, or $\frac{2}{3}$.

OXFORD, 1877.

1. 0.
2. $a^2 - b^2 - \frac{2a}{c} + \frac{1}{c^2}$:
   $z^2 - z(x + y)$.
3. (1) $x = 1$:
   (2) $x = 9$, or -3.
4. $\frac{y}{x} + 1 + \frac{x}{y}$.
5. $288x^3y^3(x - y)(x^3 - y^3)$:
   $3x + 4$.
6. (1) 1;
   (2) $-\frac{1}{(x-1)(x-3)}$.

OXFORD, 1878.

1. 1.
2. $x^3 + y^3 + z^3 - 3xyz$: $x^2 - x + 4x^{-2}$.

3. (1) $\dfrac{16x^2-1}{(3x+1)(4x-1)}$: (2) 2.
4. $3x-1$: $90ab(a^2-b^2)(a^4+a^2b^2+b^4)$.
5. $2x^2+x-\frac{1}{4}$.
6. (1) $x = 2\frac{2}{3}$: (2) $x = 2$, $y = 3$.

CAMBRIDGE, 1868.

1. $5b+2c$.
2. (1) $7(a-5b)$: (2) x^2+3y^2 : (3) $\dfrac{x^2+2}{x^3+3x}$.
3. $x^4+2ax^3+3a^2x^2+2a^3x+8a^4$:
$4x^2-3ax+a^2$.
4. $\dfrac{a^3-x^3}{a-x} = a^2+ax+x^2$.
5. $2x^2+x-3$.
6. $\dfrac{2}{a-b}$.
7. (1) $x = -3$: (2) $x = \dfrac{ab^2+bc^2+a^2c}{a^2b+b^2c+ac^2}$: (3) $x=4$, $y=5$.
8. £800 at 3 p. c. : £3200 at $3\frac{1}{2}$ p. c. : £1000 at 4 p. c.
9. Original speed = $33\frac{1}{3}$ miles per hour; distance = $48\frac{1}{3}$ miles.

CAMBRIDGE, 1869.

1. (i.) 2. (ii.) $1\frac{1}{2}$. (iii.) 1.
2. $2x$.
3. $14x^4-3x^3+x^2+x-45$, and
$\frac{2}{5}a^4 - \frac{136}{75}a^3x + \frac{8}{5}a^2x^2 + \frac{3}{10}ax^3 - x^4$.

4. $x^4 + 2ax^3 + 3a^2x^2 + 2a^3x + a^4$;
$a = -c$, $b = \pm c$.
5.
6. (i.) $\dfrac{4bx}{3a}$. (ii.) $a - x$. (iii.) $\dfrac{x^2 - ax + a^2}{x + a}$. (iv.) $\dfrac{2 + 3x}{1 + 5x}$
7. (i.) -13. (ii.) a. (iii.) $x = \frac{3}{2}$, $y = \frac{2}{3}$.
8. $\frac{4}{15}$.
9.

CAMBRIDGE, 1870.

1. (i.) 1. (ii.) 3. (iii.) 12.
2. (i.) $x^3 - 6x^2y + 11xy^2 - 6y^3$. (ii.) a.
3. (i. $x^5 + x^4 + 4x + 4$. (ii.) 0.
4. $x^2 - 5ax + 7a^2$.
5.
6. (i.) $\dfrac{27ab}{2c^2}$. (ii.) 0. (iii.) $x + \dfrac{1}{x - 1}$. (iv.) 3.
7. (i.) $x = 7$. (ii.) $x = 1$.
8. A, 40s.; B, 42s.

CAMBRIDGE, 1871.

1. $-a^2 + 4ab - 5b^2$.
2. $a + 10b - 13c + 5d$;
(i.) $x^2 + 8xy - 7y^2$. (ii.) $-2x - 8y + 7z$.
3. (i.) 4. (ii.) 128. $a^2 - 2ab - ac + 4b^2 + c^2$.
4. $x^2 - a^2 + 2ab - b^2$.
5. $a^3 - 8a^2 + 23a - 26$.
6.

7. (i.) $\frac{2a+1}{15}$. (ii.) $\frac{a-2b}{a^2+ab+b^2}$. (iii.) $\frac{2}{x-3y}$.

8. (i.) 5. (ii.) 1. (iii.) $x=6$, $y=-\frac{3}{2}$.

9. 60 acres pasture, 160 acres arable.

CAMBRIDGE, 1872.

1. $2a^4 - 7a^2b - 11ab^3 - 20b^4$.

2. $a + b + c + 3d$.

3. 8, 54.

4. $2y^2$.

5.

6. (i.) $\frac{x}{20} - \frac{y}{15}$. (ii.) $\frac{2}{y}$. (iii.) $\frac{x}{(x-2a)^2}$.

7. (i.) 7. (ii.) $\frac{a^2+b^2}{a+b}$. (iii.) $x = 4$, $y = -5$.

8. 30 sovereigns.

CAMBRIDGE, 1873.

1. $\frac{ab-a-b}{ab}$.

2. $7a - 7b$.

3. $3x^4 + 5x^3y + 8x^2y^2 - 8xy^3 - 8y^4$.

4. $x^2 - 3x + 1$.

5. (1) $\frac{3}{4}(x-y)$. (2) 0.

6.

7. (1) 5. (2) $x = 12$, $x = 12$. (3) $\begin{cases} x = b+a \\ y = b-a. \end{cases}$

8. 2d. per score.

CAMBRIDGE, 1874.

1. $x - \frac{1}{2}(2x + 10) = 5$.
2. $42x^2 - 204xy - 30y^2$.
3. $(2x + 3y)(2x - 3y)$ and $(a + 2b)(a - 3b)$.
4. $a^4b^3c^3$; $(x - y)(x - 3y)(3x - y)$.
5. (i.) $x - 22y$. (ii.) $\dfrac{x^3 + y^3 + z^3}{xyz}$.
6. $c = 1$.
7. (i.) 6. (ii.)
8. $10\frac{1}{2}$ miles.

CAMBRIDGE, 1875.

1. $(a + b)^4 + 2a^2b^2 = a^4 + b^4 + 4ab(a + b)^2$.
2. Each side $= 697$.
3.
4. $(3x + 2y)(2x - 3y)$.
5. $6(x^2 - 1)(x^2 - 4)$.
6. (i.) $\dfrac{x^2 + 24x + 12}{4(x^2 - 4)}$. (ii.) $\dfrac{b}{8a}$.
7. (i.) -21. (ii.) $x = a - 2b$, $y = 2a - b$
8. 126 shillings.

CAMBRIDGE, 1876.

1. (i.) 90. (ii.) 0.
2. 0.
3. $x^4 - 16y^4$. $x^4 - 5x^2y^2 + 4y^4$.
4. $a - \frac{1}{3}b$. $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}$.
5.
6. (i.) $\dfrac{5a}{4x^2y}$. (ii.) $\dfrac{a^2 + ax + x^2}{a + x}$. (iii.) $\dfrac{2x - 3}{2x + 3}$. (iv.) 1.

7. (i.) 5. (ii.) 5. (iii.) $x = 5$, $y = 1$.
8. 20 men, 30 women.
9. Sixteen pence.

CAMBRIDGE, 1877.

1. (i.) $\frac{2}{5}$. (ii.) 1.
2. $16a^4 - 72a^2b^2 + 81b^4$;
$$\frac{1}{16}a^2 - \frac{1}{12}a^{\frac{3}{2}}b^{\frac{1}{2}} + \frac{1}{18}ab - \frac{1}{27}a^{\frac{1}{2}}b^{\frac{3}{2}} + \frac{1}{81}b^2.$$
3. $x - \frac{1}{a}$. $19(x+y)^2$.
4.
5. $x^4 - 3a^2x^2 - 4a^4$.
6. (i.) $-2y$. (ii.) $\frac{11b^2}{7ax^2}$. (iii.) $x^{\frac{3}{2}}$. (iv.) 1.
7. (i.) 5. (ii.) 4. (iii.) $x = a(2a+b)$, $y = b(a+2b)$.
8. £270.
9. 42.