A TRANSLATION

OF

M. POINSOT'S ELEMENTS OF STATICS,

Poinsot

BY T. SUTTON, Esq., B.A.

CAIUS COLLEGE, CAMBRIDGE.

IN FIVE PARTS.

PART I.

PRICE FOUR SHILLINGS.

ELEMENTS OF STATICS.

ELEMENTS OF STATICS.

ELEMENTS OF STATICS.

ELEMENTS OF STATICS.

ELEMENTS OF STATICS.

ELEMENTS OF STATICS.

THE ELEMENTS

OF

STATICS:

TRANSLATED FROM THE FRENCH OF M. POINSOT.

TO WHICH ARE ADDED,

EXPLANATORY NOTES, EXPLANATION OF A FEW FAMILIAR PHENOMENA, AND EXAMPLES ILLUSTRATIVE OF THE DIFFERENT THEOREMS AS THEY OCCUR.

BY THOMAS SUTTON, Esq., B.A.
CAIUS COLLEGE, CAMBRIDGE.

IN FIVE PARTS.

PART I.

CAMBRIDGE:
PRINTED AT THE UNIVERSITY PRESS:
FOR J. & J. J. DEIGHTON AND W. P. GRANT.
LONDON: JOHN W. PARKER.
M.DCCC.XLVII.

PREFACE.

THE great celebrity of M. Poinsot's Treatise on Statics, not merely on the Continent, but also in this University, has induced me to offer a Translation to the English Student. In the course of the work I have often had occasion to admire the excellence of the plan, and the elegance of the demonstrations; and I feel sure that these will equally gratify any person fond of simple and *natural* reasoning, and sincerely desirous of obtaining a thorough practical knowledge of the subject.

I have thought it better, for many reasons, to publish each Chapter (of which there are four) separately, than to wait until the whole was completed; particularly as each Chapter is in itself a complete treatise upon that branch of the subject to which it relates. Of these four Chapters, together with M. Poinsot's notes, will consist the five parts into which I have divided the subject.

The contents of Part I will be found in the table.

Part II will be devoted to the Discussion of the Conditions of Equilibrium, and will contain numerous examples and problems, illustrative of this part of the subject.

Part III will be a complete Treatise upon the Center of Gravity; and here much new matter will be added, in which the varied applications of the Integral Calculus to this problem will be exhibited.

Part IV will relate to Machines. In the appendix of this part will be described many ingenious contrivances in modern mechanism; and also some of the more prominent of the applications of Statics to Architecture and Engineering.

Part V will be little more than a translation of M. Poinsot's very valuable notes, upon the Conservation of Moments and Areas; the Theory of the Motion and Equilibrium of Systems; the Principle of Virtual Velocities; New Theory of the Rotation of Bodies, &c. All that will be added here, will be possibly a few notes, and problems, solved by means of the principle of Virtual Velocities.

The whole, when completed, will be about equal to three moderate-sized octavo volumes.

Any suggestions, or good original problems, will be thankfully received and acknowledged.

T. S.

CAMBRIDGE,
Dec. 1846.

CONTENTS OF PART I.

ARTS.

Introduction .... 1—11

CHAPTER I.

SECTION I.

COMPOSITION AND DECOMPOSITION OF FORCES.

Composition of Forces which act in parallel directions .... 19—32

Composition of Forces whose lines of action all pass through a point .... 33—46

SECTION II.

COMPOSITION AND DECOMPOSITION OF COUPLES. 47

Transformation of Couples; their measure .... 50—53

Composition of Couples, situated either in the same plane, or in parallel planes 54

Composition of Couples, situated in any planes whatever........ 55—59

A more simple method of expressing the Theorems which relate to the Composition of Couples 60

The Parallelogram of Couples 61

GENERAL CONCLUSION OF THIS CHAPTER.

Composition of Forces, directed in any manner in space 68

Corollary, which contains the Laws of the Equilibrium of any free rigid system 69

Corollary, which contains the necessary conditions in order that Forces situated in any manner in Space may have a single resultant 71

Remarks........ 73—76

APPENDIX.

PAGE

Notes 63

Explanation of a few Familiar Phenomena 66

Examples 73

Concluding observations 79

INTRODUCTION.

I.

1. The idea which we have of bodies is such, that we do not suppose them to have need of *motion* in order to exist. For, although possibly there may not be a single atom in the universe absolutely motionless, yet for a very short time, we do not conceive less clearly that a body may exist at rest. Note 1 Appendix.

But if the body be once at rest it will for ever remain so, provided that no foreign cause interfere to disturb it; for since motion can only take place in some one direction, there will be no reason why the body should move itself one way rather than another, and consequently it will not move at all.

Hence, if a body previously at rest begin to move, we may be certain that this is only in virtue of some extraneous cause acting upon it. This cause, whatever it may be, and which is known to us only by its effects, we call *Force.*

Force then is any cause whatever of motion. Note 2.

II.

2. Without understanding the precise nature of Force we still conceive very clearly that it acts in a certain direction and with a certain intensity.

We acquire, almost from our birth, the idea of the direction of Force and of its intensity. The sensation of weight, which always solicits us the same way; the sight of a body which falls, or remains suspended at the end of a line; the difference in weights which the hand experiences, and a multitude of other phenomena as simple, give us an idea of the direction and intensity of Force as incontestible as that of our own existence.

Note 3. So that we shall consider as self-evident, *That every force acts at the point at which it is applied in a certain direction and with a certain intensity.*

III.

3. Now, if we represent the directions of Forces by straight lines, and their intensities by proportional lengths taken upon these lines, or by numbers, it is manifest that the Forces can be made the subject of calculation the same as all other magnitudes; and hence this general problem, the solution of which is the object of mechanics.

Any body or system of bodies being solicited by certain given forces, to find what will be the body's motion in space.

And, reciprocally, *What relations should exist amongst the forces which act upon a system, in order that the system may receive a given motion in space;* which comes in the end to the same question as the preceding.

4. In order to solve this general problem, we begin by solving that particular case, in which it is demanded, what ought to be the relations amongst the forces in order that the system to which they are applied may receive a motion equal to zero, that is to say, may remain in equilibrium. This problem once solved, it is very easy thence to deduce the other. And this is why we generally commence the study of Mechanics with that of Statics, which is defined to be *The science of the equilibrium of Forces.*

The other part of Mechanics treats of all those questions which relate to the motion of bodies. It is called *Dynamics*, or the science of motion. But in this treatise we are only concerned with the science of equilibrium.

IV.

5. Remark first, that in Statics properly so called it is not necessary to know the actual effect of forces upon matter, that is to say, the different *motions* which they are capable of imparting to it, corresponding to their intensities and direc-

tions; but that it is sufficient to consider the forces as simple homogeneous magnitudes, and therefore admitting of comparison, and to assign the relations which should exist amongst them in order that they may counteract each other. When we pass from the theory of equilibrium to that of motion we must have new principles of the estimation of forces; for, calulating no more then than their effects, we must know there how to compare and deduce them; to ascertain, for instance, if a double force produces upon the same body a double velocity, or if the same force applied to a body of double the mass produces but half the velocity, &c. But here, whatever the action of forces upon bodies may be, be they proportional or not to their sensible effects, still the truths which we are about to expound will remain no less the same, because these truths result from the sole actual presence of forces which do not produce any effect, but which obviously counteract each other, so that the state of equilibrium of bodies is as a single instant of the state of motion in which the measure of the forces by their effects, and their effects themselves, have disappeared.

6. Strictly speaking, a body in equilibrium is in the same state as if it were at rest, for the effect of the forces being annihilated for ever, or vanishing every instant if the forces be perpetually renewable, every body in equilibrium is actually capable of being moved by a single force in precisely the same manner as if acted upon by the same force when at rest. Still we may distinguish equilibrium from rest: in that in the latter case the body is not solicited by any force whatever, whereas in the former it is solicited by forces which counteract each other.

This distinction, which is null in the exact state of things, becomes sensible in the equilibria which nature presents. Scarcely any body is accurately in equilibrium, and when it appears to be in this situation, there exists amongst the forces which solicit it a perpetual contest which gives it an indefinitely small oscillation, and continually brings it back to one particular position which it is perpetually abondoning. But in the mathematical solution of problems we must regard a body in equilibrium as if it were at rest; and reciprocally, if a body be at rest or solicited by any forces, we may suppose any such new forces as we choose, themselves in equilibrium, to be applied to it, and the state of the body will not be altered.

We shall soon see numberless applications of this remark.

V.

7. These preliminary notions being established, let us see how we can proceed in the discovery of the conditions of equilibrium, for any rigid system of bodies whatever solicited by any forces P, Q, R, &c. applied at given points a, b, c, &c. of the system.

We shall suppose first that all the bodies are without weight, that is to say, such as they would be if they existed alone in space, so that there will only be to consider the effects of the forces P, Q, R, &c. which should mutually counterbalance in the case of equilibrium.

Then it is easy to see that it will be sufficient to find the conditions of equilibrium for the simple system of the points of application a, b, c, &c. regarded as an assemblage of points rigidly connected together.

For, if we indicate by a', b', c', &c. the same points a, b, c, &c. of the system, but considered solely as points connected by straight rigid and inextensible lines, and if we suppose that the forces P, Q, R, &c. maintain *them* in equilibrium, it is evident that the same forces P, Q, R, &c. would also maintain *the system* in equilibrium, for we might imagine the system to be placed upon the points a', b', c', &c.; so that the points a, b, c, &c. might coincide with them. The system being left at rest in this situation, the equilibrium of the points a', b', c', &c. will not be disturbed. But it is clear that the equilibrium would still subsist, if, instead of supposing the points a and a', b and b', &c. coincident, we were to suppose them rigidly connected, so that a could not separate from a', b from b', &c. and similarly for the others. Whence it results that the conditions of equilibrium amongst the forces P, Q, R, &c. applied to any system of bodies, are the same conditions as would obtain amongst the same forces P, Q, R, &c. applied to a simple system of points of application a, b, c, &c. rigidly connected together.

So that, when we seek the relations amongst certain forces which are in equilibrium upon a rigid system we shall be able to abstract all the bodies of the system and to suppose that there remain only the points of application, which we shall imagine so connected together as to be unable to change their relative position.

By these considerations we disengage from the problem both the weights and the volume of the bodies, and the question becomes more simple.

Afterwards, we shall restore to the bodies their weight and then consider their respective weights as new forces which we must combine with the others in order to equilibrium. We shall be able in this way to apply the results of Statics to the equilibrium of natural bodies which all have weight.

VI.

8. Now, since we only have to consider three things in the equilibrium of forces, viz., their intensities, their directions, and their points of application, it is manifest that the conditions of equilibrium are merely the mutual relations which should exist amongst these three things, in order that there may be equilibrium in the system. And we may already perceive, and shall soon see, that these relations may be expressed by means of equations, in which the *intensities* of the forces enter directly,—their *directions* by means of the angles which they make with fixed straight lines in space,—and their *points of application* by means of the coordinates which determine their respective positions.

And thus it is that we are enabled to form an idea of the problem of Statics, and to comprehend the real state of the question.

But it may be observed that all which we have said is only applicable to the case of a body *free in space*, whilst we can easily conceive that a body may be subjected to certain conditions; for instance, to revolve about a fixed point, or a fixed axis, or to remain constantly upon an impenetrable surface, &c. But we shall see in the sequel, that the resistances which a body experiences in consequence of being subjected to extraneous conditions may always be replaced by suitable forces, and that after this substitution of forces for resistances the body may be regarded as free in space; so that it was needless to embarrass the commencement of the question.

VII.

9. In order actually to discover the course which may conduct us to the conditions of equilibrium, let us conceive a body or system kept in equilibrium by any forces *P*, *Q*, *R*, &c. directed as we choose in space.

Since all these forces are in equilibrium, we see that any one of them, *P* suppose, opposes itself singly to the action of all the rest, *Q*, *R*, &c. Whence it appears that the effect of these latter is to solicit the system in the same manner precisely as a single force equal and opposite to the force *P*.

This is in fact what takes place, and it may be proved most fully by means of the preceding remark (6), and by this axiom, That two equal and opposite forces are necessarily in equilibrium, (12).

For, suppose we apply to the system a force *P′* exactly equal and opposite to the force *P*. The forces *P* and *P′* being in equilibrium, their effect is zero, and we may consider the body as only submitted to the action of the forces *Q*, *R*, &c. But on the other hand, the force *P* being in equilibrium with the forces *Q*, *R*, &c., their effect is also zero, and we may consider the body as only submitted to the action of the single force *P′*. The state of the body is therefore identically the same, when we suppose it solicited by the forces *Q*, *R*, &c., as when we suppose it solicited by the single force *P′* equal and opposite to that which was in equilibrium with them.

Hence, since it appears that a single force is capable of producing upon a body the same effect as several forces, and that it may exactly supersede them: our first business must be to seek to reduce the applied forces to the smallest number possible, and to observe particularly *the law* of this reduction. Thus, the conditions of equilibrium amongst all the forces will reduce themselves to the conditions of equilibrium amongst these latter forces equivalent to the former, and they will become more easy to express.

10. This force which is capable of producing upon a body the same effect as several other combined forces, and which can singly replace them, is called *their Resultant*. Whence we see, by calling to mind what has been before said, *that if several forces are in exact equilibrium upon a body, any one of them is equal and opposite to the resultant of all the rest.*

The other forces, with respect to the resultant, are called *Components*. The law by which we find the resultant of several forces is called *The Composition of Forces*. The same law (but taken inversely) by which we substitute for a single force, several others capable of producing jointly the same effect, the former being their resultant, is called *The Decomposition of Forces*.

We shall commence then by these two inquiries, which in reality only include a single one, that of the law which connects the resultant with its components.

11. Often, for the sake of brevity, we shall call forces whose directions are parallel, parallel forces; and forces whose directions meet, forces which meet.

We shall designate forces generally by the letters P, Q, R, &c., placed upon the lines which represent their directions; and if a letter, A suppose, indicates the point of application of a force, P suppose, we shall always consider that the action of the force takes place from A towards the letter P, or that the force *draws* from A to P.

If in order to represent the magnitude of this force we take upon its direction, and starting from A, a certain finite line AB, we shall suppose also that this line is drawn from that side towards which A tends to move. Thus, when we simply say of a force, that it is represented in magnitude and direction by a certain finite line starting from the point of application, we must also understand, that the force *draws* the point towards the extremity of the line which represents it.

We might adopt the contrary hypothesis, that is to say, suppose that the force represented by the line *AB* *pushes* the point of application *A* so as to increase its distance from the extremity *B* of the line which represents it: for we are here making a convention which is perfectly arbitrary, and we might choose either of these definitions indifferently. But when the definition is once made, we must be careful to conform to it in the figure, for all the forces which we are considering, in order to give to each of them the meaning which it ought to have, and to the enunciation of the theorem its necessary exactness.

CHAPTER I.

FIRST PRINCIPLES.

SECTION I.

COMPOSITION AND DECOMPOSITION OF FORCES.

AXIOMS, LEMMAS, &c.

12. It is evident *that two equal and opposite forces applied at the same point are in equilibrium.*

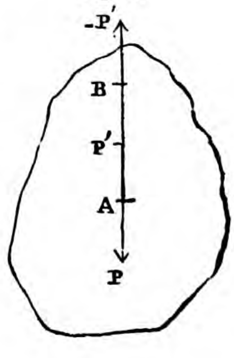
It is also evident *that two equal and opposite forces applied at the extremities of a straight line, considered as a rigid rod, and acting in the direction of this line, are in equilibrium,* for there is no reason why the motion should take place towards one side rather than another, as in the former axiom.

COROLLARY.

13. It is easy to deduce from this, that the effect of a force which solicits a body will not be altered whatever be the point in its line of action at which we suppose it to be applied, provided that this point be either a point in the body itself, or if without it be rigidly connected with it in some manner.

For, let any force P (Fig. 1) be applied at a point A of any body or system; if we take in the line of action of this force another point B rigidly connected with the system, so that the length AB remains always constant, and if we apply to the point B two forces P' and $-P'$, each equal to P and acting in the line AB, the point A will still be solicited as before, for the effect of the two forces P' and $-P'$ is nothing. But, considering the force P and

Fig. 1.



its equal and opposite, $-P'$ applied at B, it is manifest that their effect is also nothing. They may therefore be removed, and there only remains the force P', which is no other than the force P applied at the point B of its line of action, and the point A has not ceased to be solicited in the same manner as before.

Note 4. *We may therefore apply a force at any point whatever of its line of action, provided that this point be connected with the former point of application by a rigid and inextensible straight line.*

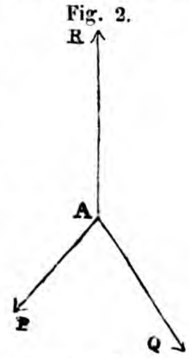
REMARK.

When we change in this way the points of application of forces, we shall not always repeat that we must suppose the new points rigidly connected with the former, but this must always be understood.

LEMMA.

14. When two forces, P and Q, are applied at the same point A under any angle, (Fig. 2) we easily conceive that a third force R suitably applied at A might be in equilibrium with the two forces P and Q, for in virtue of the combined efforts of P and Q the point A tends to quit the place in which it is, but it *can* only go one way, so that if we apply a proper opposite force the point will remain at rest.

Fig. 2.

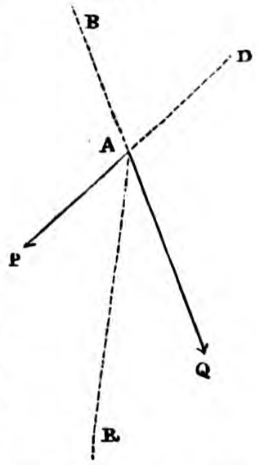


The forces P, Q, R being in equilibrium at the point A, the force R is equal and directly opposite to the resultant of the two others (10); thus two forces, P and Q, which meet have a single resultant.

In the next place, it is manifest that this resultant ought to act in the plane of their lines of action AP, AQ (Fig. 3); for there is no reason why it should act in a certain direction *above* the plane, rather than in the same perfectly symmetrical direction beneath it.

Moreover, it ought to lie *within* the angle PAQ, for it is clear that the point A cannot move in that part of the plane which is beyond the line AQ towards D, nor can it move beyond the line *AP* towards *B*, consequently it will only be able to move within the angle *PAQ*, and the resultant *R* ought to lie within this angle..

Fig. 3.



REMARK.

15. There is but one case in which we can see, *a priori*, what the direction of the resultant will be. It is that in which the two forces, *P* and *Q*, *are equal*. Then it is manifest that the line of action of the resultant *bisects* the angle *PAQ*, for there is no reason why the resultant should make with one of the components a smaller angle than with the other.

FUNDAMENTAL AXIOM.

16. *When two forces*, P *and* Q, *have the same sign and act in the same line, it is manifest, and we must admit it as an axiom, that these forces unite and produce a resultant equal to their sum* P + Q.

REMARK.

This axiom is the foundation of the whole science of equilibrium. We may regard it if we choose as a sort of definition or postulate which it is not necessary to attempt to prove, for it is included within the very idea of force considered as a magnitude, that is to say, as susceptible of increase or diminution. And, in fact, what idea *could* we form, for instance, of a force double or triple of another, if we did not regard this force as the actual union of two or of three equal forces which draw at the same time the same point the same way? It is this which has been naturally understood in what has preceded. Besides, this postulate is the only one which the science requires; after that, all the theorems of pure Statics are no more, at the bottom, than theorems of Geometry.

Corollary.

17. From the preceding axiom we conclude (by combining successively the forces two and two) that the resultant of as many forces as we choose, which act in the same line and have the same sign, is equal to their sum total and acts in the same direction.

Also, that when two unequal forces, P and Q, act in the same line but have not the same sign, their resultant is equal to their difference $P - Q$, and that it has the same sign as the greater force; for we can conceive within the greater, which we will suppose to be P, a force equal and opposite to Q, and which destroys it. These two forces may be removed, and the point is then solicited by the difference $P - Q$ of the two forces P and Q.

Whence we see, that in general, *the resultant of as many forces as we choose acting in the same line is equal to the excess of the sum of those which act one way over the sum of those which act the opposite way, and that it has the same sign as the greater sum.*

Remark.

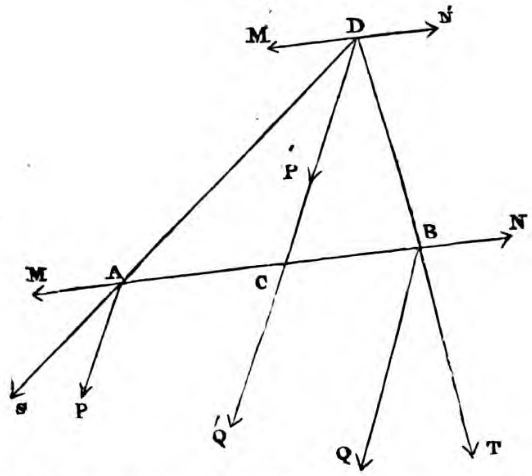
18. Such are some of the more elementary propositions whose truth we discover *a priori* and almost at first sight. The most simple case of the composition of forces, and at the same time that of which we know at once the resultant, is evidently the case of forces which act in the same line. We are going now to commence the composition of forces by means of propositions which are immediately deducible from what has preceded.

COMPOSITION OF FORCES WHICH ACT IN PARALLEL DIRECTIONS.

Theorem I.

19. *If any two forces,* P *and* Q, (Fig. 4), *parallel and of the same sign, be applied at the extremities* A *and* B *of a rigid straight line* AB, *we assert*

Fig. 4.



1*st. That these two forces have a resultant, and that this resultant ought to be applied to the line* AB *between the two points* A *and* B.

2*ndly. That this resultant is parallel to the components* P *and* Q, *and equal to their sum.*

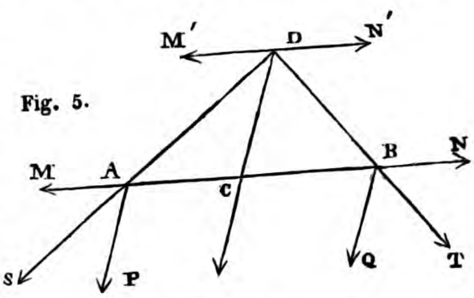
1st. Apply to the two points A and B any two equal and opposite forces M and N, which act in the line AB. These will produce no effect, and therefore the effect of the two forces P and Q will not be altered. But the two forces M and P applied at A have a resultant S applied at A and lying within the angle MAP (14). Similarly, the two forces N and Q have a resultant T applied at B and lying within the angle NBQ. Suppose that we have taken these two resultants, and have applied them both at a point D in which their lines of action must necessarily intersect; the resultant of the two forces S and T will be absolutely the same as that of the two forces P and Q; but being applied at D and obliged to lie within the angle ADB, it will intersect AB in some intermediate point C where we may suppose it to be applied.

2ndly. In order to shew that this resultant is parallel to the forces P and Q and equal to their sum, let us suppose that at the point D we decompose the force S into two components M' and P' respectively, equal and parallel to the former M and P, and similarly that we decompose the force T into two components N' and Q' respectively equal and parallel to the former N and Q. The two forces M', N' will be equal, and moreover directly opposite, since applied at the same point D they are parallel to the same straight line MN; they will therefore have no effect. There will then remain only the two forces P' and Q', respectively equal and parallel to the forces P and Q; but these two forces evidently acting in the same line will unite into a single force R equal to their sum $P' + Q'$ or $P + Q$. Q. E. D.

Corollary I.

20. If the two forces P and Q (Fig. 5) be equal, the point of application C of the resultant will bisect the line AB. Let us take the two arbitrary forces M and N equal to the two P and Q. The resultant S of the two equal forces M and P will bisect the angle MAP (15); and because DC is parallel to the line AP, the triangle ACD will be isosceles. By similar reasoning the triangle BCD will be isosceles, and we shall have $AC = CD$, and $CD = CB$; therefore $AC = CB$.

Fig. 5.



Corollary II.

21. It thence follows that the resultant of as many parallel forces as we choose, equal two and two, and symmetrically applied at equal distances from the middle of the same straight line, is equal to the sum of all the forces, is parallel to them, and passes through the middle point of the line of application. For in combining successively each pair of equal forces placed on opposite sides at equal distances from the centre of the straight line, their successive resultants will all pass through this same point, and will then unite, since they act the same way and in the same line.

22. And reciprocally, we can decompose a whole force P applied to a line into as many other parallel forces as we choose, applied at different points of this line, provided that

these forces be equal two and two, at equal distances fromthe point of application of the force *P*, and that their sum total be equal to this force.

THEOREM II.

23. *The point of application* C (Fig. 6) *of the resultant*

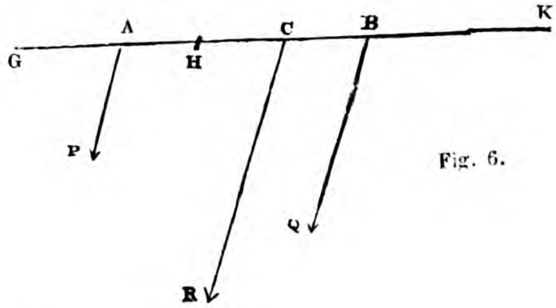


Fig. 6.

of two parallel forces P *and* Q *which act at the extremities* A *and* B *of an inflexible straight line* AB, *divides this straight line in the inverse ratio of* P : Q, *so that we have* P : Q = BC : AC.

Let us suppose first that the forces *P* and *Q* are commensurable, that is to say, are to one another as two whole numbers *m* and *n*.

Divide *AB* in *H* so that *AH* : *BH* = *P* : *Q*, and therefore = *m* : *n*. In *AB* produced take *AG* = *AH*, and *BK* = *BH*. The point *A* will be the middle point of *GH*, and *B* the middle point of *HK*.

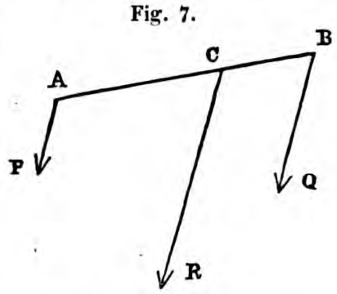
Now, since the forces *P* and *Q* are to one another as the lines *AH* and *BH*, they will also be to one another as the doubles of these lines, that is, as the lines *GH* and *HK*. And since there are by hypothesis, in the line *AH*, *m* such parts as that *BH* contains *n*, there will be 2*m* parts in *GH* and 2*n* equal parts in *HK*. But we can decompose the force *P* into 2*m* equal and parallel forces applied to the 2*m* points which are the centres of the several parts of the line *GH* (22), and the force *Q* into 2*n* parallel forces each equal to the former ones and applied to the 2*n* centres of the several parts of the line *HK*. Now all these equal forces being equidistant from one another will be placed in pairs at equal distances from the middle point *C* of the whole line *GK*, and therefore their general resultant which is that of the two forces P and Q will necessarily pass through C.

Note 5. But because $GC = AB$, therefore taking away the common part AC, $BC = AG = AH$, and adding CH to each $AC = BH$, so that since $P : Q = AH : BH$, we get $P : Q = BC : AC$.

Let us suppose next, that the two forces P and Q are *not* commensurable.

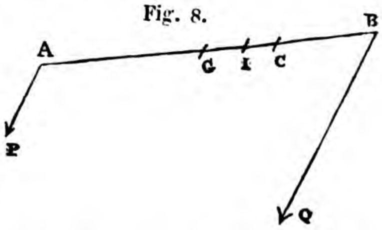
But first let us observe, that if the resultant of any two forces P and Q (Fig. 7) applied at the points A and B acts at C, then the resultant of a force P and a force $Q + I$ greater than Q will act between the point C and the point B, that is to say, that the point of application of the resultant will approach the point of application of that component which has been augmented. In fact, in order to find the resultant of two components P and $Q + I$, we may first find the resultant R of P and Q, which acts at C by hypothesis, and then that of R and of I, the point of application of which will be between C and B. (19)

Fig. 7.



Now if the resultant of two incommensurable forces P and Q (Fig. 8) do *not* act at the point C which is such that $P : Q = BC : AC$, it will act at some other point situated either between A and C, or between B and C. Let us suppose it to be at G between A and C. Divide the line AB into equal parts, each of them smaller than GC, there will be at least one point of division between C and G; let this point be I; then the two lines AI and BI will be commensurable, and the point I may be considered as the point of application of the resultant of two forces P and Q', such that $P : Q' = BI : AI$, which gives Q' less than Q (because we have by hypothesis $P : Q = BC : AC$). But the resultant of the two forces P and Q' acting at I, that of the two forces P and Q

Fig. 8.

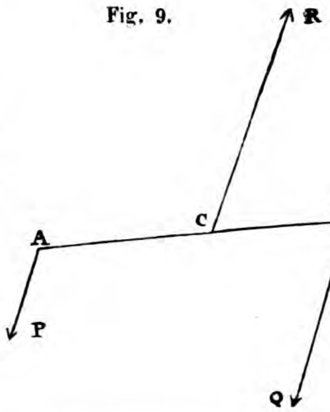


greater than Q' will act between I and B, and will *not* act at G, which is contrary to the hypothesis.

In the same manner we might shew that it could not act between C and B, and therefore it must necessarily act at C. Ex. 1.

Corollary I.

Fig. 9.



24. When three parallel forces P, Q, R (Fig. 9) are in equilibrium upon a line AB, one of them is equal and opposite to the resultant of the other two. Thus, the force Q, for instance, acting the opposite way is the resultant of the two forces P and R. Since the two forces P and Q act the same way, the force R is equal to $P + Q$, and therefore $Q = R - P$; whence it follows, that the resultant of two parallel forces which act opposite ways is equal to their difference, and acts the same way as the greater force.

25. The two forces P and R being given, also the distance AC between their points of application, if it be required to find the point of application of the resultant Q, we shall obtain this proportion $P : Q = BC : AC$, whence we get $P + Q : Q = BC + AC : AC$, that is to say, $R : Q = AB : AC$, a proportion which gives AB, and therefore the point B. Ex. 2.

Corollary II.

26. Suppose that the two forces P and R are equal, the resultant Q will be zero, and the distance AB of its point of application will be by the preceding proportion equal to $\frac{R \times AC}{0}$, that is to say, infinite.

If the two forces P and R, instead of being equal were to differ by a very small quantity, the resultant Q, which is equal to this difference, would be very small, and the distance $AB = \frac{R \times AC}{Q}$ would be very great, because the denominator Q is very small, thus the more nearly the two forces approach to equality, the smaller the resultant becomes, and the greater the distance of its point of application becomes. So that when the two forces become perfectly equal the resultant vanishes, and the distance of its point of application becomes infinite; which seems to indicate that there is no longer then a single resultant, or, to announce in the clearest terms, that we cannot actually find a single force which can maintain in equilibrium two equal parallel and opposite forces.

27. But to leave *no* uncertainty as to this last inference, let us imagine, if it be posssible, that a single force R *can* maintain in equilibrium two forces P and $-P$, perfectly equal, parallel and opposite.

First then, whatever may be the position of this single force with respect to the given ones, we shall find instantly, opposed to it, another position perfectly symmetrical with respect to the same forces, for there is equality every way. If, then, a force R can equilibrate the two forces P and $-P$, there is also *another* force $-R$ equal, opposite and parallel, which can equilibrate them. Add then this second force $-R$, and in order that nothing may be altered, destroy it instantly by a force R' equal and opposite; there will then be equilibrium amongst the five forces R, P, $-P$, $-R$ and R'. But there is equilibrium amongst the three forces P, $-P$, and $-R$; there will therefore be equilibrium between the two remaining forces R and R', which is impossible, since these two equal and parallel forces have the same sign. (19)

Thus the two forces P and $-P$ cannot be held in equilibrium by any single force, and, consequently, they cannot have a single resultant.

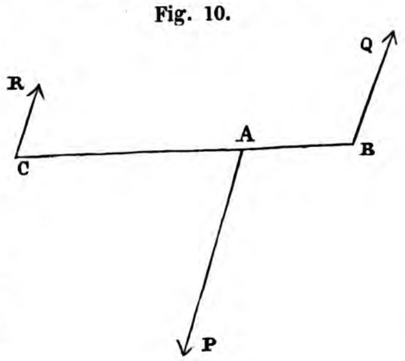
We shall return soon to these kinds of forces, the consideration of which, (which would not have occurred in this place except as a particular case), will form the second essential part of our elements.

COROLLARY III.

28. In the same manner that we compound into one, two parallel forces which act at given points in a line, we may also decompose any force whatever R applied at a point C of an inflexible line into two others P and Q, which may be parallel to it, and which act at given points A and B of this line. In order to this it is only required to divide the force R into two others which may be in the ratio of the distances BC and AC, and in order to find the force Q, for instance, we must make use of the proportion $R : Q = AB : AC$, in which there is only Q unknown. The force P will be equal to $R - Q$. Ex. 3.

If the point of application C of the force R, (Fig. 10) which we wish to decompose, do not fall between A and B, the given points of application of the components P and Q, which we are seeking, we should still have the proportion $R : Q = AB : AC$, which would give the force Q, but the force P would be equal to $R + Q$. Ex. 4.

Fig. 10.

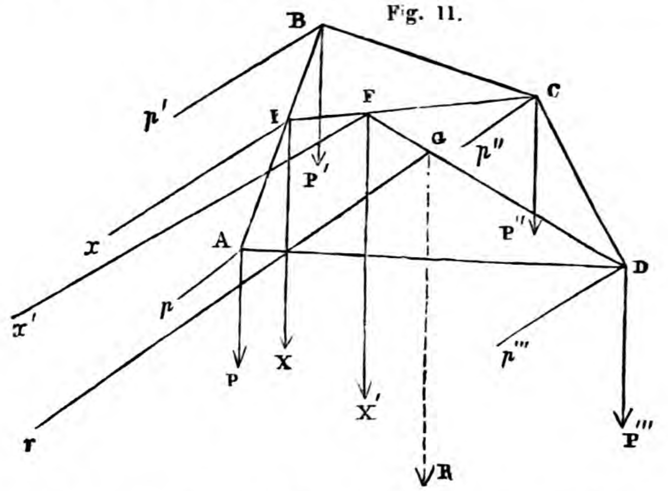


COROLLARY IV.

29. When we know how to determine the resultant of two parallel forces, we can easily find that of as many parallel forces as we choose applied at different points of any rigid system.

For instance, let there be four parallel forces P, P', P'', P''', applied (Fig. 11) respectively at the four points A, B, C, D, situated in any manner in space and rigidly connected in some way. By considering these forces two at a time they will be situated in the same plane. Thus, we first take the resultant X of the two forces P and P'; it will be equal to their sum $P + P'$, and will act at the point I of the line AB, which we shall find by dividing AB in the inverse ratio of P to P'. The resultant X being thus determined, we join the point I at which it acts, with the point C of the third force P''. The two forces X and P'' being parallel, we may obtain the resultant X', as we have just done. This resultant will be

Fig. 11.



equal to their sum $X + P''$, and the point F at which it should be applied will be found by dividing the line CI in the inverse ratio of X to P''. Finally, joining the point F with the point of application D of the fourth force P''', and dividing the line FD into two parts inversely proportional to the forces X' and P''', we obtain the point of application G of the general resultant R, which will be parallel to the two forces X' and P''', and therefore to all the components, and equal to their sum, and therefore to the sum of all the components.

This process may manifestly be extended to any number whatever of parallel forces.

If amongst the forces P, P', P'', P''', &c. some act one way, and some the opposite way, we should commence by finding the resultant of all those that acted one way, and then the resultant of those which acted the opposite way. And all the forces being then reduced to two parallel and opposite forces, we should find the resultant of them by what has preceded.

30. We can, then, in general determine the position and magnitude of the resultant of as many parallel forces as we please. *This resultant will be parallel to the forces, and equal to the excess of the sum of those which act one way, above the sum of those which act the opposite way.*

We say in general, because it *may* happen that the resultant of the forces which act one way may be exactly equal to the resultant of those which act the other way, without having the same line of action, and then there will be no single resultant, as we have already seen. Exs. 5, 6, 7.

COROLLARY V.

31. Let us suppose that the four forces P, P', P'', P''', without changing their magnitude, or ceasing to be parallel and acting at the same points respectively A, B, C, D, should take the positions p, p', p'', p''' in space.

If we seek their resultant, by the same method as before, we shall find first, that the resultant x of p and p' acts at the same point I as the resultant X of P and P', and that it is equal to it. It will act at the same point, because its point of application should divide the same straight line AB in the inverse ratio of p to p', which is the same as that of P to P'. It will be equal to it, because we have $P+P'=p+p'$. We shall find similarly, that the resultant x' of x and p'' acts at the same point F as the resultant X' of X and P'', and that it will be equal to it, and so on for the rest; so that the general resultant of the four forces p, p', p'', p''' will act at the same point as the resultant of the four forces P, P', P'', P''', and this is universally true whatever may be the number of the forces whence we deduce this remarkable theorem.

32. *If we take any system whatever of parallel forces, applied at an assemblage of points* A, B, C, D, *&c., and if we successively turn the whole system of forces in different directions, so that the same forces always act at the same points, preserving the same magnitudes and their parallelism, then, the general resultant which we successively obtain in each of these cases will act at the same point.*

This point of application of the successive resultants is called *the centre of parallel forces*. We shall have occasion to speak of it more at large when we come to the problem of the centre of gravity.

We may remark, in conclusion, that in the preceding demonstration, it is not necessary to suppose that the forces always preserve the same magnitudes: it is sufficient that, in the successive positions of the group they remain proportional.

COMPOSITION OF FORCES WHOSE LINES OF ACTION PASS THROUGH THE SAME POINT.

THEOREM III.

33. *The resultant of any two forces* P *and* Q (Fig. 12) *applied at the same point* A, *under any angle acts in the direction of the diagonal of the parallelogram* ABCD *constructed upon the two lines* AB, AC, *which represent the forces* P *and* Q *in magnitude and direction.*

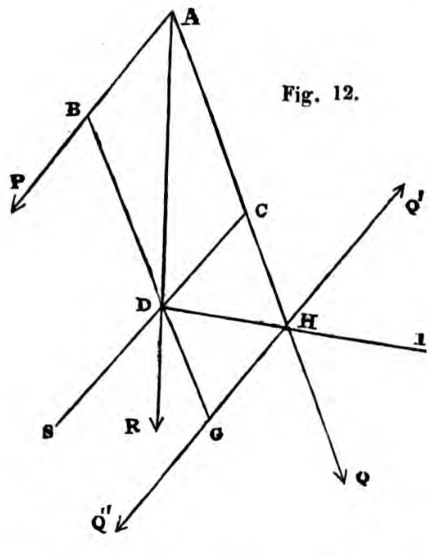


Fig. 12.

First, we have seen (14), that this resultant must be in the plane of the two forces P and Q; and secondly, that it must be applied at the point A, since this resultant by hypothesis is to solicit the point in exactly the same manner as the two forces P and Q.

We now assert that its line of action must pass through D, the extremity of the diagonal AD.

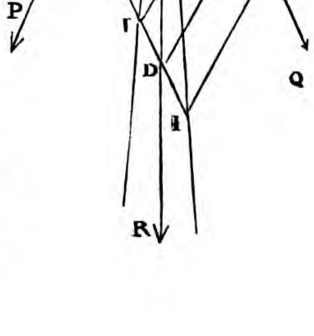
Take in the line BD produced, the part $DG = DC$, and complete the parallelogram $CDGH$. Apply at G and H and in the line GH, two opposite forces Q'' and Q' each equal to Q. It is easy to see that the resultant of the four forces P, Q, Q', Q'', must act at D; for first, since $Q'' = Q$ the two parallel forces P and Q'' are as the sides AB, AC, or as DC and DB, or, since $DC = DG$, as the lines DG and DB, and consequently (23) their resultant S acts at D; secondly, the two forces Q and Q' being equal, their resultant I produced bisects the angle CHG of the rhombus $CDGH$, and therefore passes through D, where we may suppose it to be applied. Hence, the general resultant, which is that of the two forces S and I, acts at the point D.

But the two forces Q'' and Q' applied at the points G, H, being exactly equal and opposite have no effect, and the resultant of the four forces, P, Q, Q', Q'', is identically the same as that of the two forces P and Q. Therefore since this latter acts at D, that of the two forces P and Q also acts at D.

Corollary.

34. Hence we conclude, that if we only knew the lines of action of the two forces P and Q and that of their resultant R, we could determine the ratio of the force P to the force Q. For, taking any point D in the line of action of the resultant, and through this point drawing DC, DB, parallel to the lines of action of the components P and Q, and meeting them in the points C and B, we shall necessarily have $P : Q = AB : AC$. Otherwise we should have P to Q as AB to a line AO, (Fig. 13), less or greater than AC, and then the resultant of the two forces P and Q would act in the direction of the diagonal AI of a parallelogram $AOIB$ different from the parallelogram $ABDC$, which is contrary to the hypothesis.

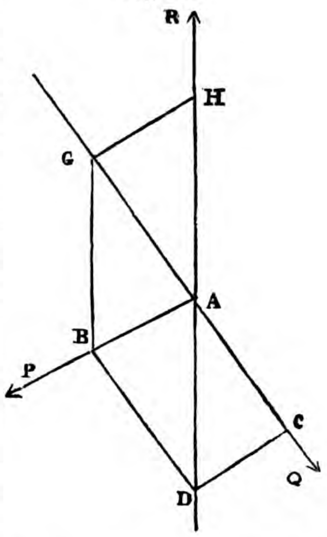
Fig. 13.



THEOREM IV.

35. *The resultant of any two forces* P *and* Q (Fig. 14), *applied at the same point* A, *is represented in magnitude and direction by the diagonal of the parallelogram* ABDC, *constructed upon the lines* AB, AC, *which represent these forces* Note 6. *in magnitude and direction.*

Fig. 14.



We have already seen that the diagonal represents the resultant *in direction;* it now remains to shew that it also represents it in magnitude.

Let R be this resultant, and suppose it applied at A in the direction of the diagonal DA produced, and opposite to its own action. The three forces P, Q, R, will be in equilibrium on the point A: one of them then, Q for instance, will be equal and exactly opposite to the resultant of the other two, P and R. Hence the line of action of the force Q produced, will be that of the resultant of the two forces P and R. Therefore, if through the point B, we draw BG parallel to AR, and meeting QA produced in G, and from the point G the line GH parallel to AP, and meeting AR in H, the two forces P and R will be to one another as the sides AB, AH, of the parallelogram $ARGH$ (34). But the line AB actually represents the force P, therefore AH represents the force R. Exs. 8, 9. But, by parallels, we have $AH = BG = AD$; therefore &c Q. E. D.

COROLLARY I.

36. Since the three forces P, Q, R are to one another as the lines AB, AC, AD, and that in the parallelogram $ABDC$, we have $AB = CD$, we may say that these three forces are to one another as the three sides CD, CA, and AD of the triangle ACD. But these three sides are to one another as the sines of the opposite angles CAD, CDA, ACD, and by parallels the angle CDA = the angle BAD, and the angle ACD is the

supplement of the angle BAC, and therefore has the same sine; hence we have

$$P : Q : R = \sin CAD : \sin BAD : \sin BAC.$$

Whence we conclude that the resultant of two forces P and Q being represented by the sine of the angle formed by their lines of action, the two forces are represented reciprocally by the sines of the two angles adjacent to the line of action of the resultant; or thus, if we choose, *each of the forces* P, Q, R, *is proportional to the sine of the angle formed by the lines of action of the other two.*

Remark.

37. We see from this, and better still by the direct consideration of the parallelogram of forces, that when two forces act at the same point under an angle which is not equal to two right angles, they can never have a resultant zero, unless they are themselves each zero.

For, if neither of the two forces were zero we might construct a parallelogram upon the two lines which represent them in magnitude and direction, and the diagonal of this parallelogram would be the resultant.

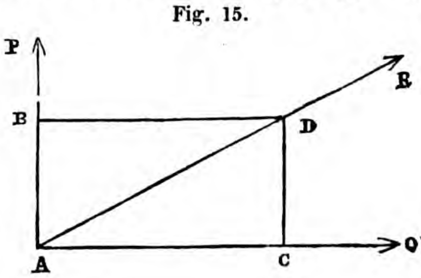
If one of them only were zero, the other would be the resultant; and, consequently, the resultant cannot vanish, unless *both* the components vanish at the same time.

When the two components act under an angle equal to two right angles, they are opposite forces, and the case in which both components separately vanish is not the *only* case in which the resultant vanishes. There is another case, viz. that in which the forces are equal.

Corollary II.

38. We can always decompose a given force R into two others, P and Q, having given lines of action AP, AQ, (Fig. 15) provided that these lines and that of R lie in the same plane and meet at A; for, taking in the line of action of the force R a part AD which represents its magnitude, and through D drawing the lines DC, DB parallel to the lines

Fig. 15.



AP, AQ, we shall form a parallelogram $ABDC$, whose sides AB, AC, will represent the required forces.

If we want to calculate directly their magnitudes, we may obtain the following proportions:

$$R : P = \sin BAC : \sin CAD,$$

$$R : Q = \sin BAC : \sin BAD,$$

in which P and Q only are unknown.

Remark.

39. If the angle BAC be a right angle the preceding proportions become

$$R : P = 1 : \cos BAD,$$

$$R : Q = 1 : \cos CAD;$$

therefore $P = R \cos BAD$, and $Q = R \cos CAD$.

From which it follows that when we decompose a force into two others whose lines of action are at right angles to one another, we find either component by multiplying the given force by the cosine of the angle which it makes with the line of action of that component.

Each component is represented by the projection of the resultant upon its line of action, and this is often called *the resolved part of the force in that direction*. Thus, $R \cos BAD$ or the component P is the force R resolved in the direction AP.

Corollary III.

40. When we know how to determine the resultant of two forces applied at a point, we can determine that of as many forces P, Q, R, &c. as we please, applied at the same point A and whose lines of action take any direction in space. For, considering first any two of these forces, P and Q for instance, these two forces will be in the same plane, and we may determine their resultant as we have just done. Let X be this resultant. We may find the resultant of this force X and of one of the others as R; next combining this resultant, which we will call Y, with a new force S we may get their

resultant Z which will be that of the four forces P, Q, R, S, and by continuing this process we may arrive at length at the general resultant.

If all the forces P, Q, R, S, &c. are in the same plane, the successive resultants X, Y, Z, &c. will be in the same plane, and consequently the general resultant will be also.

If all the forces are in equilibrium, the general resultant will be zero.

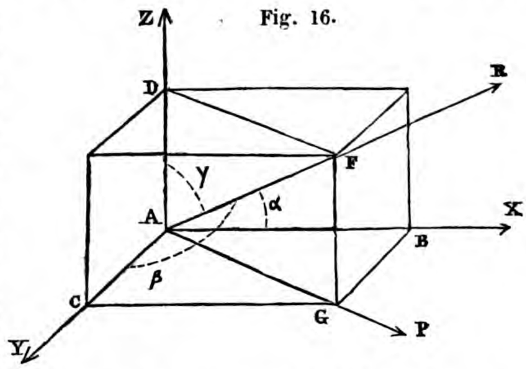
By this successive composition of many forces at the same point, we also see that, if we describe in space a polygonal outline whose sides are successively parallel and proportional to these forces, the straight line which encloses this outline and thus completes the polygon is parallel and proportional to the general resultant of all the forces, so that if the polygon Note 7. encloses itself, there is no resultant and all the forces are in equilibrium on the point which they solicit.

The following theorem is nothing more in reality than a particular case of this elegant proposition ; but as it is of frequent use in Mathematics, we shall enunciate it and demonstrate it expressly.

THEOREM V.

41. If three forces X, Y, Z, applied at the same point A (Fig. 16) in space be represented by three straight lines AB, AC, AD, and the parallelopiped AF be completed, the resultant R of these forces will be represented by the diagonal AF of this parallelopiped.

Fig. 16.



For the two forces X and Y which are represented by the two sides of the parallelogram $ABGC$ will have for their resultant a force P represented by the diagonal AG of this parallelogram.

Then, because AD is equal and parallel to GF the figure $AGFD$ will be a parallelogram, and consequently the two forces P and Z will have a resultant R represented by the diagonal AF, which is at the same time the diagonal of the proposed parallelopiped.

Remark.

42. Let us observe now, as in Art. 37, that so long as the three forces X, Y, Z, are not in the same plane, they will never give a resultant zero unless they are themselves each zero.

For if none of them be zero, we may construct the parallelopiped upon the lines which represent them in magnitude and direction, and the diagonal will be the resultant.

If one of them only be zero, the two others, which by hypothesis are not in the same line, will have a resultant.

Lastly, if two only be zero, the third will be the resultant, and, consequently, the components X, Y, Z, must each be separately zero, in order that the resultant may be zero.

Corollary I.

43. We see by the preceding theorem (which is called *the parallelopiped of forces*,) that any given force R is always decomposable into three others, X, Y, Z, respectively parallel to their given straight lines in space, provided that no two of these be parallel.

For, by taking the part AF to represent the magnitude of the force R and drawing through A the point of application, three lines parallel to the given lines each to each, we shall pass through the point A three indefinite planes XY, XZ, YZ, then through F we may take three other planes respectively parallel to the former, and these six planes will determine the parallelopiped whose three contiguous edges AB, AC, AD will represent the three components X, Y, Z.

Corollary II.

44. If the parallelopiped be rectangular we shall have in the rectangle $ADFG$, $AF^2 = AD^2 + AG^2$; but in the rectangle $AGBC$ we have $AG^2 = AC^2 + AB^2$, therefore by substituting this value of AG^2 we get

$$AF^2 = AD^2 + AC^2 + AB^2,$$

and, consequently,

$$R^2 = X^2 + Y^2 + Z^2:$$

which gives $R = \sqrt{X^2 + Y^2 + Z^2}$ as the value of the resultant in terms of the three components.

45. If we want the three components in terms of the resultant and the angles which they make with it, by first calling a the angle which the resultant R makes with the component X, we have manifestly $AF : AB = 1 : \cos a$, and, consequently,

$$R : X = 1 : \cos a,$$

whence we obtain $X = R \cos a$.

Similarly, calling β and γ the angles which the resultant makes respectively with the components Y and Z, we obtain $Y = R \cos \beta$, $Z = R \cos \gamma$; whence it follows that we shall obtain the values of the three respective components, by multiplying the resultant by the respective cosines of the three angles which its line of action makes with the lines of action of its components.

REMARK.

46. Since we have found that $R^2 = X^2 + Y^2 + Z^2$, by substituting for X, Y, Z, their respective values $R \cos a$, $R \cos \beta$, $R \cos \gamma$, we obtain

$$R^2 = R^2 \cos^2 a + R^2 \cos^2 \beta + R^2 \cos^2 \gamma,$$

$$\text{or } R^2 = R^2 (\cos^2 a + \cos^2 \beta + \cos^2 \gamma);$$

$$\text{therefore } \cos^2 a + \cos^2 \beta + \cos^2 \gamma = 1,$$

a relation which always exists between the angles which a straight line makes with three rectangular axes in space. Note 8.

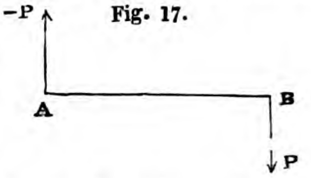
SECTION II.

COMPOSITION AND DECOMPOSITION OF COUPLES.

---

47. For the sake of brevity, a system of two forces, such as P and $-P$ (Fig. 17), equal, parallel and opposite, but not applied at the same point, is called *a couple*. The common perpendicular AB drawn between the lines of action of the two forces, is called *the arm* of the couple, and the product of either of the forces and the arm, that is, $P \times AB$, is called *the moment* of the couple.

Fig. 17.



Whatever the action of two such forces as P and $-P$ may be, upon the body to which they are applied, we have seen (27) that this action cannot be counterbalanced by that of any single force, applied any where to the same body, and that consequently the action of a couple cannot be compared in any manner with that of a single force. In order to distinguish this new cause of motion, which is of somewhat a peculiar nature, we should give it a particular name; but that of *couple* is sufficient, and denotes well enough for the time, the combination of the two opposite forces of which it is composed, and the kind of effort to which that couple gives birth.

Besides, as we shall see presently, that the effort of a couple is measured by its *moment*, we shall often be able to substitute this word for the former, or to use them sometimes for one another.

The composition of couples will form the second essential part of our principles of Statics, and will occur in the course of this treatise nearly as often as the composition of forces. We shall see soon how, by means of it, to deduce the laws of equilibrium, in a manner so natural and so simple, that we shall be excused for having appeared to linger here in the discussion of a particular case, whilst we are perhaps taking the most direct course possible towards the main object.

What we are going to say about couples, is quite independent of the effect which they actually produce upon bodies: but when we shall require to form a notion of the respective *signs of* different couples situated in the same plane, we shall picture to ourselves that the centres of their arms are fixed; thus the effect of each couple will be visibly to make the body rotate about the centre of its arm, and we shall easily distinguish the sign of couples by distinguishing between the couples which tend to make it turn one way, and those which make it turn the opposite way. But not forgetting that there will not really *be* any fixed point (at least that we should not have expressly hinted at it), and that the idea of rotation, which as yet is purely accessory, is only to help to furnish a picture for the occasion.

TRANSLATION OF COUPLES.

48. We have already seen that a force may be transferred to any point whatever of its line of action, provided that this point be rigidly connected in some way with the former. Here is an analogous proposition for couples, which is no less remarkable than the former, and of which we shall make great use in the sequel.

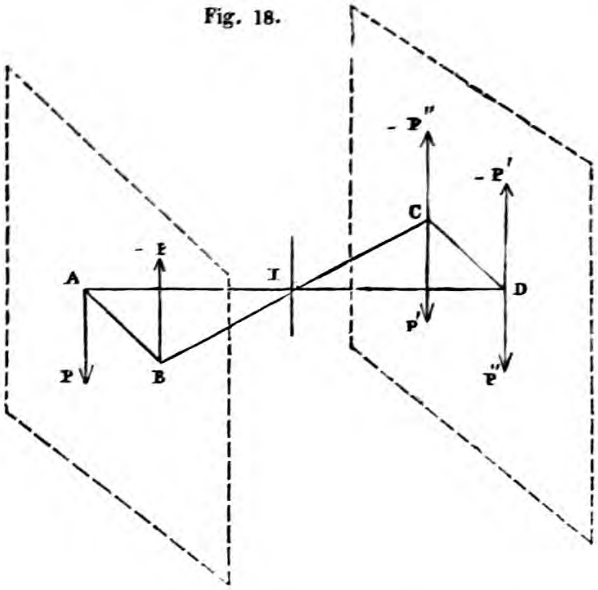
LEMMA.

49. *Any couple whatever may be transferred wherever we choose, in its own plane, or in any other parallel plane, and turned about as we choose in that plane, without altering its effect upon the body on which it is applied, provided that we suppose its new arm to be rigidly connected with the former.*

In order to demonstrate this proposition more easily, we shall divide it into two others.

First let the couple $(P, -P)$ (Fig. 18) be applied perpendicularly to AB; take any where in the plane of this couple, or in any other parallel plane, the straight line CD equal and parallel to AB, join AD and BC, which will be in the same plane, and will visibly bisect each other in the point I, and

Fig. 18.



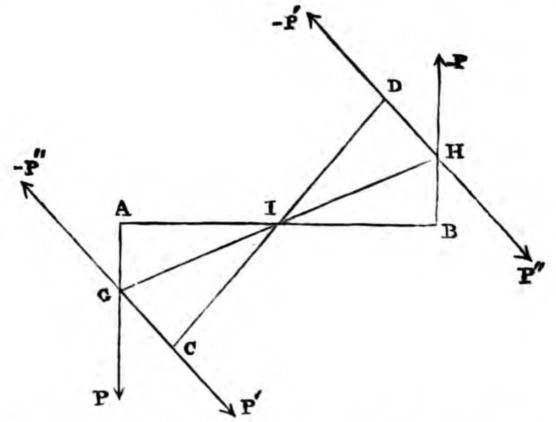
finally suppose the straight lines AB and CD to be rigidly connected together.

If we apply to the line CD, parallel to the forces P and $-P$, two opposite couples $(P', -P')$, $(P'', -P'')$ each equal to the given couple $(P, -P)$, it is evident that these two couples will destroy one another, and, consequently, that the effect of the couple $(P, -P)$ will not be altered. But on the other hand it is easy to see that the two couples $(P, -P)$ and $(P'', -P'')$ also destroy one another; for the point I being at once the centre of the two lines AD and BC, the two equal and parallel forces P and P'' applied to AD give a resultant exactly equal, and opposite to the resultant of the two forces $-P$ and $-P''$ applied to BC. We may therefore remove the two couples $(P, -P)$, $(P'', -P'')$ and there will only remain the couple $(P', -P')$ applied to CD, which is nothing else than the original couple which we have, so to speak, transferred parallel to itself, so that its arm AB has come into the parallel position CD.

Secondly, let the couple $(P, -P)$ (Fig. 19) be applied perpendicularly to AB. Draw, in the plane of this couple, and making any angle whatever with AB, the straight line

$CD = AB$, and let these two straight lines bisect each other in the point I, and be rigidly connected together.

Fig. 19.



If we apply at right angles to CD two opposite couples $(P', -P')$, $(P'', -P'')$ each equal to the given couple $(P, -P)$ these two couples will destroy each other, and consequently the effect of the couple $(P, -P)$ will not be altered. But, on the other hand, the two couples $(P, -P)$, $(P'', -P'')$ also destroy each other; for, with a little attention, we see that the two equal forces P and $-P''$, which meet in G, give a resultant equal and opposite to the resultant of the two forces $-P$ and P'' which meet in H. We may therefore remove the two couples $(P, -P)$, $(P'', -P'')$ and there will only remain the couple $(P', -P')$ applied at CD, which is nothing more, so to speak, than the original couple which we have turned in its own plane so that its arm AB has come into the oblique position CD.

By these two propositions jointly, we may conclude that any couple whatever, may, without having its effect altered, be transferred in its own plane, or in any other parallel plane, into any position we choose; for we can first transfer it parallel to itself in the given plane, so that the centre of its arm may coincide with any given point, and we can then turn it about this point, so as to place it in the required position; or, reciprocally, we can first turn it about in its own plane until its forces become parallel to the new directions which we wish to give them, and then we may directly transfer it into the required position.

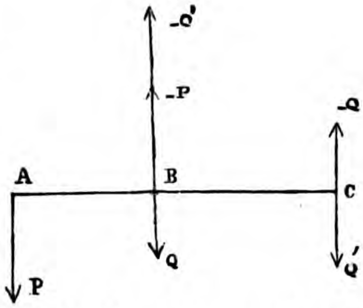
TRANSFORMATION OF COUPLES: THEIR MEASURE.

Lemma.

50. *Any couple whatever* (Fig. 20) (P, – P) *applied at an arm* AB *may be changed into another* (Q, – Q) *of the same sign, applied at an arm* BC *different from the former, provided we have* P : Q = BC : AB, *or*

$$P \times AB = Q \times BC,$$

Fig. 20.



that is to say, provided the moments of the two couples be equal.

Take, in *AB* produced, any part whatever *BC*, and apply to *BC*, parallel to the forces *P* and – *P*, two couples (*Q*, – *Q*), (*Q'*, – *Q'*) equal and opposite: they will have no effect, and consequently that of the couple (*P*, – *P*) will not be altered. But on the other hand, if we suppose that the forces *P* and *Q*, and consequently *P* and *Q'* are inversely as the lines *AB* and *BC*, their resultant, which is equal to *P* + *Q'*, acts at *B*, and manifestly destroys the opposite forces – *P*, – *Q'* which act there. We may therefore remove the four forces *P*, *Q'*, – *P'*, – *Q'*, and there will only remain the couple (*Q*, – *Q*) applied to *BC*, which replaces the given couple (*P*, – *P*) applied to *AB*.

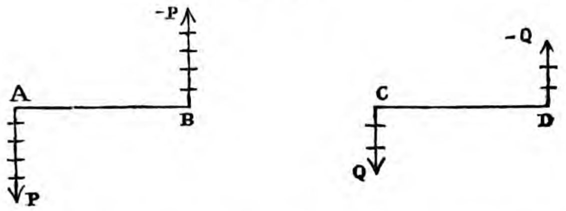
Corollary.

51. It is not difficult to conclude from this that the efforts of couples are proportional to their moments.

In fact, we may first see that two couples (*P*, – *P*), (*Q*, – *Q*) (Fig. 21) which act upon the equal arms *AB*, *CD* are to one another as the forces *P* and *Q* of these couples; for if we suppose the forces *P* and *Q* to be in the ratio of two whole numbers, 5 and 3 for instance, by dividing each force *P* and – *P* into 5 equal forces, and each force *Q* and

$-Q$ into three equal forces, each equal to the former, we shall be able to consider the couple $(P, -P)$ as the sum

Fig. 21.



of 5 equal couples of the same sign, exactly applied one upon another; and the couple $(Q, -Q)$ as the sum of 3 couples, each equal to the former, and also applied one upon another. The intensities of the couples $(P, -P)$, $(Q, -Q)$ will then be in the ratio of 5 to 3, or of P to Q. If the forces be immensurable, we shall employ the usual mode of Note 9.
reasoning, &c.

Now let there be any two couples whatever $(P, -P)$, $(Q, -Q)$; let p be the arm of the former, q the arm of the latter: the couple $(Q, -Q)$ acting in the line q is equivalent to the couple $\left(\frac{q}{p} Q, -\frac{q}{p} Q\right)$, acting on the line p, for their moments are equal, the former being Qq, and the latter being $\frac{q}{p} Qp = Qq$. Thus, instead of the two given couples, we have these two, $(P, -P)$, $\left(\frac{q}{p} Q, -\frac{q}{p} Q\right)$, which have the same arm p. But the intensities M and N of these two couples are as their forces, and consequently we have $M : N = P : \frac{q}{p} Q$, or $M : N = Pp : Qq$.

52. Since two couples are to one another in the ratio of their moments, it follows that the moment of a couple is the measure of its effect or intensity; for if we take as our unit of couple, that couple which is composed of two forces, each equal to the unit of force applied at an arm equal to the unit of length, the couple $(P, -P)$ at an arm p will contain as many units of couple as the moment $P \times p$ contains the moment 1×1, that is to say, unity.

Remark.

53. In order to compare the magnitudes or intensities of couples with one another, we might also take, instead of the products Pp, Qq, of the forces and their rectangular arms, the products of these same forces and their oblique arms. But it would be necessary that all the arms of the couples should make the same angle with the lines of action of the forces. It is clear that then the oblique arms would all be proportional to the rectangular arms, and that, consequently, the new moments would be proportional to the former.

We shall sometimes employ these new moments in the relative measure of different couples; but we shall always consider the others as the absolute measure of their intensities.

COMPOSITION OF COUPLES SITUATED IN THE SAME PLANE, OR IN DIFFERENT PLANES.

Theorem I.

54. *Two couples situated any where in the same plane or in parallel planes, can always be compounded into one which is equal to their sum, if they have the same sign; to their difference, if they have opposite signs.*

First, we may transfer the couples to the same plane; next, we may place them with their forces parallel; and, finally, we may change them into others having the same arm, and apply them one upon the other.

Let P and Q be the forces of the couples, p and q their respective arms, and let D be the length of the common arm of the two transformed couples. Instead of the couple $(P, -P)$ with moment Pp, we shall substitute the equivalent couple $(P', -P')$, whose moment $P'D$ will equal Pp; similarly instead of the couple $(Q, -Q)$, with moment Qq, we shall substitute the couple $(Q', -Q')$ with moment $Q'D = Qq$; and these two transformed couples being applied one upon the other to the same arm D, we shall have a single resultant couple $\{(P' + Q'), -(P' + Q')\}$, whose moment will be $(P' + Q')D$, or $P'D + Q'D = Pp + Qq$.

Thus, the resultant moment will be equal to the sum of the component moments, or to their difference, according as the forces P' and Q', which act at the same extremity of the arm D, have the same or the opposite sign.

Corollary.

We see then, that, by combining couples in this way, two and two, as many couples as we choose, situated in any manner whatever in the same, or in parallel planes, will always be reducible to a single one, equal to the sum of those of the same sign, minus the sum of those of the opposite sign.

And, reciprocally, we shall be able to decompose any single given couple into as many others as we choose; either situated in the same plane, or in parallel planes. We shall be able even to take at pleasure all these couples in different planes; for it will be sufficient that the sum of those of one sign minus the sum of those of the opposite sign, be equal to the given couple.

COMPOSITION OF COUPLES SITUATED IN ANY PLANES WHATEVER.

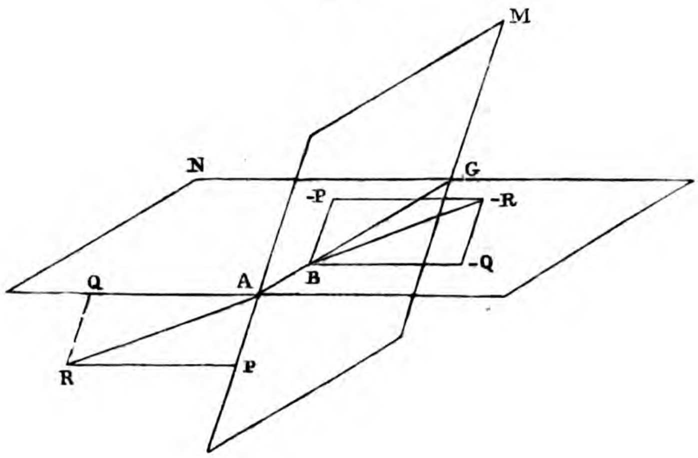
Theorem II.

55. *Two couples situated anywhere in two planes which intersect one another at any angle, can always be compounded into one.*

And if we represent the moments of these couples by the respective lengths of two straight lines drawn at an angle equal to that between the two planes, and complete the parallelogram, the moment of the resultant couple will be represented by the diagonal of this parallelogram, and the plane of this couple will divide the angle between the planes of the component couples in the same manner that the diagonal of the parallelogram divides the angle between the two adjacent sides.

Let the two given couples be situated in the two planes AGM, AGN, (Fig. 22) which intersect in AG; and suppose that we have previously changed these two couples into other equivalent couples, whose *arms* are equal.

Fig. 22.



Wherever the couple (P, $-P$) may be situated in the plane AGM, we shall be able to place it at right angles to the line AG, and so that its arm AB may be upon that line (49). Similarly, wherever the couple (Q, $-Q$) may be situated in the plane AGN, we shall be able to place it also at right angles, upon the same line of intersection, and so that its arm, equal to the former, may coincide with it in AB.

Then the two forces P and Q applied at A, will compound into a single one R applied at the same point A, and represented by the diagonal AR of the parallelogram constructed upon the two lines AP, AQ, which represent the forces P and Q. Also the two forces $-P$, $-Q$, applied at B will compound into a single force $-R$ applied at B exactly equal, parallel, and opposite to the former. And we shall have, instead of the two couples (P, $-P$), (Q, $-Q$), the single couple (R, $-R$) applied at the same arm AB.

Since then these three couples have the same arm, their respective moments are proportional to the three forces P, Q, R. Hence, if we represent the moments of the two component couples by the two lines AP, AQ, which are proportional to them, the moment of the resultant couple will be represented by the diagonal AR of the parallelogram $APRQ$, constructed upon these lines. Moreover, it is manifest that the angles formed by the three lines AP, AQ, AR, are the angles between the three planes. Therefore the plane of the resultant couple divides the angles between the other two planes, as the diagonal AR divides the angle PAQ between the two adjacent sides AP, AQ. Therefore, &c. Q. E. D.

Corollary.

56. We shall always be able therefore to reduce to a single one, as many couples as we choose, applied to a body in any manner whatever in space. For by successively compounding them, two and two, as we have just done, we shall at length necessarily arrive at a single couple, whose plane and magnitude we shall know, and which will be equivalent to all the others.

Reciprocally, we can always decompose a couple into two others situated in two given planes, provided that these planes and that of the proposed couple intersect in the same straight line, (or in parallel straight lines), for, by transforming the plane of one of these couples parallel to itself, which we are allowed to do (49), we shall reduce their parallel intersections to one.

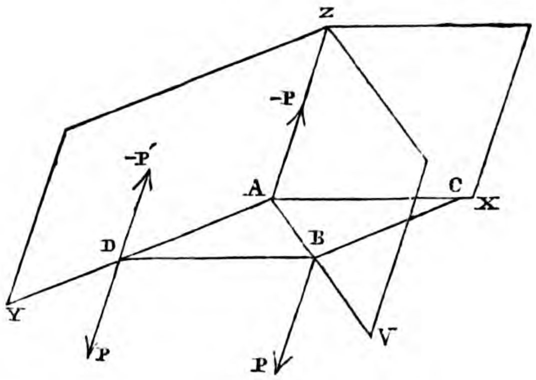
Remark I.

57. In order to effect this decomposition, we shall only have to follow, in an inverse order, the process which we have just given, for the composition of two couples; or we might employ the following method which is very simple, and of which we shall sometimes avail ourselves.

Let AZ (Fig. 23), be the common intersection of three planes; take arbitrarily a plane YAX cutting them in the three lines AY, AV, AX; and let ZAV be the plane of the proposed couple.

Wherever this couple $(P, -P)$ may be situated in the plane ZAV, we may place it so that its forces may be parallel to the line AZ, and the line of action of one of them, as that of $-P$, may coincide with this same line. Then the line of action of the other force P will intersect somewhere in B the straight line AV, and we shall have the couple $(P, -P)$, applied to AB at some angle as shewn in the figure. Now let us construct, with the directions AX, AY, and with AB

Fig. 23.



as diagonal the parallelogram $BCAD$, and at one of the angles C or D, D suppose, let us apply two opposite forces P' and $-P'$, equal and parallel to the forces P and $-P$ of the proposed couple. The effect of this couple will not be altered. But actually, instead of the couple $(P, -P)$ applied to the diagonal AB, we may consider the two others; viz. $(P', -P)$ applied to AD in one of the given planes ZAY, and $(P, -P')$ applied to BD parallel to the other plane ZAX. But this couple may be transferred parallel to itself into the plane ZAX, to the side $AC = BD$, and we shall then have instead of the couple $(P, -P)$ applied to the diagonal AB, two couples $(P', -P)$, $(P, -P')$, composed of forces equal and parallel to the former, applied in the two given planes to the sides AD, AC.

Remark II.

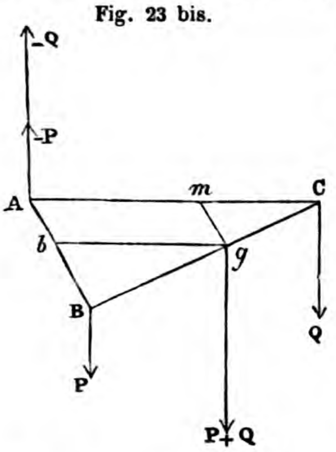
58. If we had supposed the plane YAX drawn perpendicularly to the common intersection AZ of the planes of the three couples, the forces of these couples would have been perpendicular to the lines AY, AV, AX, and as these forces are equal, the moments of the couples would have been proportional to their arms AD, AB, AC, and according to what we have just said, we should have fallen again upon the preceding theorem; which furnishes, as we see, a new demonstration of that theorem.

REMARK III.

59. This double demonstration proceeds from the two-fold manner in which we may transform the two couples before compounding them. In the former, we commence by giving them the same *arm* with different forces; in the latter by giving them the same *forces* with different arms.

There is a third demonstration which is conducted without changing anything in the two given couples. For, let $(P, -P)$ and $(Q, -Q)$ (Fig. 23 bis.) be two couples applied perpendicularly to the plane of the triangle ABC, on the respective arms AB, AC, and let us suppose that the two forces P and Q, which act at B and C, have the same sign. It is clear that these two forces compound into a single one $P + Q$ of the same sign, and applied at a point g which divides the base BC in the inverse ratio of P to Q. The two forces $-P$ and $-Q$ which act at A, also compound into a single one $-(P+Q)$, applied at the point A; and if for the sake of brevity we put $P + Q = R$, we have the resultant couple $(R, -R)$ applied to Ag in a plane perpendicular to the triangle ABC.

Fig. 23 bis.



Now, through the point g draw two lines parallel to the sides AB and AC, and thus complete the parallelogram $Algm$. It is required to prove that the moments of our three couples, viz.,

$$P \times AB, \quad Q \times AC, \quad R \times Ag,$$

are to one another as the sides Al, Am, and the diagonal Ag of this parallelogram. But this is easy, for putting, instead of the forces P, Q, R, the three lines Cg, Bg, BC which are proportional to them, we see that these moments are to one another as the three products

$$Cg \times AB, \quad Bg \times AC, \quad BC \times Ag.$$

But the similar triangles give $Cg \times AB = BC \times Al$, and

$Bg \times AC = BC \times Am$. Therefore, substituting these two new products for the former ones, and striking out the common factor BC, we have the three moments in the proportion of the simple lines Al, Am, Ag, which was what we had to prove.

We may still further vary these demonstrations; but there is a much more simple method of effecting the composition of couples, as will be seen in the next article.

MORE SIMPLE EXPRESSION OF THE THEOREMS WHICH RELATE TO THE COMPOSITION OF COUPLES.

60. Instead of determining the position of a couple by that of its plane, we may determine it by the direction of any straight line whatever, perpendicular to its plane, and which we shall call the *axis* of the couple. Since a couple may be supposed to be applied wherever we choose in its own plane, or in any other parallel plane (49), it is evident that we shall know the position of a couple in space when we know the direction of its axis; for, drawing wherever we choose a plane perpendicular to this axis, we shall be able to take this plane as that of the proposed couple.

Thus the position of different parallel couples can be given by that of a single straight line perpendicular to all of them, and which will be, so to speak, their common axis.

If the couples are situated in any planes whatever, we shall suppose first, for the sake of clearness, that they be transferred into planes respectively parallel to them, which all pass through one and the same point A taken arbitrarily in space, and which will become the common centre of all the couples; then if we take this point for the origin of the perpendiculars which we erect upon these respective planes, the position of the different couples will be determined by that of so many straight lines drawn through this single point, and making with one another the same angles that the planes of the given couples do.

Moreover, if starting from this point A we take upon these straight lines the lengths AL, AM, AN, &c. proportional to the respective moments of these couples, which we will designate by the simple letters *L*, *M*, *N*, &c.; each of these terminated straight lines, *AL* for instance, will suffice to determine at the same time the axis and the magnitude of the couple *L* which corresponds to it.

Finally, if we wish that the same straight line *AL* may also indicate the sign of the couple, which is necessary in order to the complete determination of the couple, we shall only have to make a convention very similar to that which relates to simple forces. But for a simple force *P* applied at *A*, and which we represent by a certain line *AP*, this convention consists, as we have said, in that the action of this force always takes place from *A* towards *P*, or that the force *draws* from *A* towards *P*. Here, for a couple *L* applied about a centre *A*, and whose axis and magnitude we represent by the terminated straight line *AL*, we shall always suppose that the sign of the couple, or of the rotation which it tends to produce, is such, that if we were to place ourselves at the point *L*, considered as the north, in order to look before us at the point *A*, considered as the south, we should see the rotation going on from east to west, or from the left to the right, in the same way as the motion of the Sun. This is, besides, the ordinary way in which the hand Note 10. turns round most instruments of rotation, and it is in this arbitrarily assumed way that the couple will act which we represent by the line *AL*.

We may adopt, if we choose, the opposite convention, provided that we keep to it with the same care for all couples in the same figure, or the enunciation of the same proposition.

Beside, we see that one of the two conventions, the former, for instance, will suffice; for if it were required to indicate in the figure a couple *L'* opposite to *L*, we should represent it of course by a line *AL'*, taken on the opposite side of the point *A* in the former line produced. It is obvious that this second couple, which, viewed from the point *L'* would give rotation in the manner agreed upon, that is to say, from left to right, would, when viewed from *L*, give rotation from right to left, and would be actually opposite to the former one.

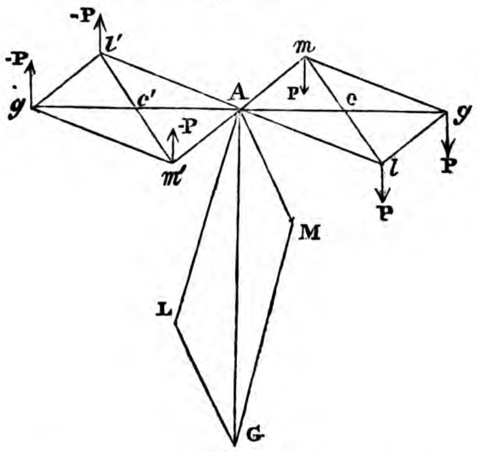
By this method, then, of determining couples, and of indicating their simultaneous signs, we see that the geometrical representation of as many couples as we choose applied to a body in any planes whatever, becomes exactly the same as that of so many simple forces applied to a point. And we are going to prove immediately that their composition may be expressed by laws exactly similar. It all reduces itself, in fact, to the demonstration of the following theorem, which replaces Theorem II., and which may very properly be called the "Parallelogram of Couples."

Theorem.

61. *If two couples,* L *and* M, *be represented in axis and in magnitude by the two sides* AL *and* AM *of a parallelogram* ALGM, *these two couples compound into a single one* G, *represented in axis and in magnitude by the diagonal* AG *of this parallelogram.*

Through the point A and in the plane of the parallelogram $ALMG$ (Fig. 24), draw the two lines ll', mm' respectively

Fig. 24.



perpendicular and proportional to the two sides AL and AM, and intersecting one another in their middle point A. Complete the parallelograms $Algm$, $Al'g'm'$, it is manifest that these parallelograms will be equal to one another, and similar to the former $ALGM$, and that, consequently, the line gg' will be also perpendicular and proportional to the diagonal AG, and bisected in the point A.

Now, upon the lines ll', mm', as arms, and in planes perpendicular to the figure, apply two couples, whose *forces* are equal, the former $(P, -P)$ to the line ll', the latter $(P, -P)$ to the line mm'; and let us suppose, in order to conform to the preceding convention (60), that these couples both tend to give rotation from left to right when we view them successively from the points L and M; it is evident that these two couples may be taken for those which the two sides AL and AM represent; for, firstly, they are situated in planes perpendicular to these sides; secondly, they have moments proportional to these same sides; and, thirdly, their simultaneous signs are conformable to the established convention. But it is easy to see that these two couples compound into a single one, represented in the same manner by the diagonal AG. For the two forces P and P applied at l and m compound into a single one $2P$, parallel and of the same sign, applied at the point c, which is the centre of lm, and consequently of Ag. And in the same way, the two forces $-P$ and $-P$ at l' and m' compound into a single one $-2P$ applied at c' the centre of Ag'; and we have the single resultant couple $(2P, -2P)$ applied to the line cc', or simply the couple $(P, -P)$ applied to twice this line gg'. But this couple is evidently perpendicular and proportional to the diagonal AG, and gives rotation from left to right when viewed from the point G. Therefore, &c. Q.E.D.

We might have deduced this theorem from one of the preceding ones, but we have preferred to give here an independent demonstration of it, with a new figure, in which we have seen very clearly how the signs of the respective couples which we have been considering, would affect one another.

Remark I.

62. We see here, by reasoning very similar to that of (57), that if two couples act in planes which intersect or which are not parallel, they cannot have a resultant couple zero, unless they be at the same time both zero.

Remark II.

63. When the planes of the component couples are at right angles to one another, the two axes AL and AM are also at right angles; and in the rectangle $ALGM$ we have $AG^2 = AL^2 + AM^2$. Further, if we call α and β the angles which the diagonal AG makes with the two adjacent sides AL, AM, we have

$$AL = AG \cos \alpha, \quad AM = AG \cos \beta.$$

Hence, denoting simply the three respective moments by the letters L, M, G, we have for the moment G, $G^2 = L^2 + M^2$, whence

$$G = \sqrt{L^2 + M^2}$$

And for the angles α and β, which this axis makes with the two others, $L = G \cos \alpha$, $M = G \cos \beta$; therefore

$$\cos \alpha = \frac{L}{G}, \qquad \cos \beta = \frac{M}{G}.$$

Remark III.

In general, if we call ϕ the angle between the two component couples, or their axes AL and AM, we shall have in the parallelogram $ALGM$

$$AG^2 = AL^2 + AM^2 + 2AL \times AM \cos \phi,$$

and therefore,

$$G^2 = L^2 + M^2 + 2LM \cos \phi,$$

which gives the resultant couple G in terms of the component couples L, M, and their mutual inclination ϕ.

If the angle ϕ be zero, we have $\cos \phi = 1$; and then

$$G = L + M,$$

which agrees with what we have already seen; for the two couples are then in the same plane and of the same sign, and thus compound into a single one equal to their sum.

If the angle ϕ be equal to two right angles, we have $\cos \phi = -1$, and therefore $G = L - M$, as it ought to do, for then the two couples have opposite signs, and they compound into a single one equal to their difference.

When ϕ is a right angle, $\cos \phi = 0$, and then we have $G = \sqrt{L^2 + M^2}$ as before.

Remark IV.

From the composition of two couples, it is very easy to proceed to the composition of as many couples as we choose; and it is evident that we shall have theorems very like those which relate to simple forces at a point; still we ought to enunciate and demonstrate the following theorem, because of the great use we may make of it in mechanics.

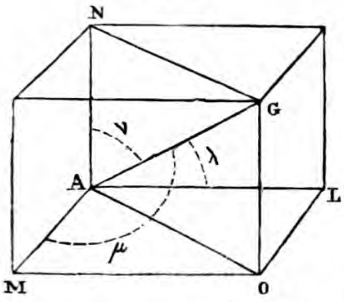
Theorem.

64. *Three couples represented in axis and in magnitude by the three edges of a parallelopiped, may always be compounded into one represented in axis and in magnitude by the diagonal of this parallelopiped.*

Let AG (Fig. 25) be a parallelopiped, and AL, AM, AN, the sides which represent at the same time the axes and the moments of three couples.

The two couples represented by the two sides AL, AM, of the parallelogram $ALOM$ will compound into one represented in axis and in magnitude by the diagonal AO of this parallelogram. Now this couple, and the third represented by AN, will compound into one represented by the diagonal AG of the parallelogram $ANGO$. But this diagonal is at the same time that of the parallelopiped. Therefore, &c. Q. E. D.

Fig. 25.



65. We also see here, by the same reasoning as that of (42) that if three couples act in three planes which form a solid angle, or which intersect in a single point, they can never have a resultant couple zero, unless they are themselves all zero at the same time.

Remark.

66. When the parallelopiped is rectangular, calling L, M, N, the component moments, and G the resultant moment, we have, manifestly,

$$G^2 = L^2 + M^2 + N^2;$$

denoting by λ, μ, ν, the three angles which the diagonal, or rather which the axis of the resultant couple makes with the three axes of the component couples, we have

$$L = G\cos\lambda, \quad M = G\cos\mu, \quad N = G\cos\nu.$$

Therefore,

$$\cos\lambda = \frac{L}{G}, \quad \cos\mu = \frac{M}{G}, \quad \cos\nu = \frac{N}{G}.$$

Therefore, if it is required to calculate the resultant moment G of the three moments L, M, N, whose axes are rectangular, we shall have for its value, $G = \sqrt{L^2 + M^2 + N^2}$, and for the angles λ, μ, ν, which its axis makes with the three axes of component moments,

$$\cos\lambda = \frac{L}{\sqrt{L^2 + M^2 + N^2}},$$

$$\cos\mu = \frac{M}{\sqrt{L^2 + M^2 + N^2}},$$

$$\cos\nu = \frac{N}{\sqrt{L^2 + M^2 + N^2}}.$$

If, on the contrary, it be required to decompose a couple G into three others, situated in three planes at right angles to one another, or whose three axes are at right angles, we shall have for the three values respectively of the component moments,

$$L = G\cos\lambda, \quad M = G\cos\mu, \quad N = G\cos\nu,$$

λ, μ, ν, being the angles which the axis of the given couple makes with those of the required components.

67. As for the rest, we shall not stop about these details, but shall simply remark, that between the seven quantities, L, M, N, G, $\cos\lambda$, $\cos\mu$, $\cos\nu$, we have four equations, which are $G^2 = L^2 + M^2 + N^{2\prime}$, $L = G\cos\lambda$, Note 11.
$M = G\cos\mu$, $N = G\cos\nu$, by means of which, knowing any three of these quantities, we shall be able to determine the other four.

We must always except the case in which we only know the three angles λ, μ, ν; for then we shall only be able to obtain the *ratios* of the moments L, M, N.

GENERAL CONCLUSION OF THIS CHAPTER.

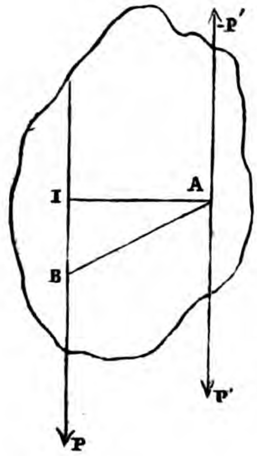
COMPOSITION OF FORCES DIRECTED IN ANY MANNER IN SPACE.

---

68. Let there be any number of forces, P, P', P'', &c. applied in any manner whatever in space, to a body or free system.

First consider any one of them, P, for instance (Fig. 26), which is applied at the point B. Then at the point A arbitrarily taken in the body, or without the body (provided we suppose it to be rigidly connected therewith) apply two opposite forces, P', $-P'$, equal and parallel to the force P. It is clear that nothing will be altered in the state of the system. But we may now consider, instead of the force P applied at B, the force P' applied at A, and the couple $(P, -P')$ acting on the straight line AB. If, for the sake of greater clearness, we transfer this couple elsewhere, into any plane whatever, parallel to its own, there will only remain at the

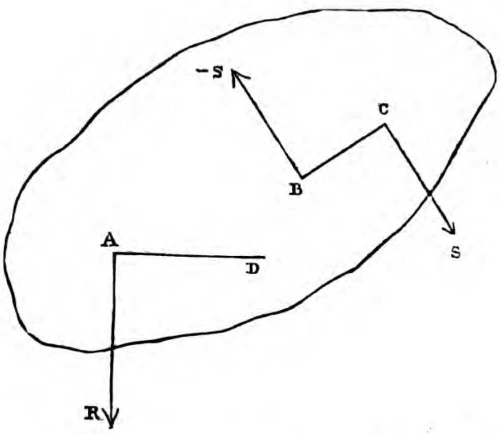
Fig. 26.



point A the force P', equal and parallel to the force P, which is nothing more than this force P which we have transferred parallel to itself from B to A.

If we make the same transformations for all the forces of the system, with respect to the same point A, it is manifest that all those forces will be assembled there parallel to themselves, but that there will also be in the system as many couples applied, in consequence of each transformation. But all the forces applied at A will compound into a single one

Fig. 27.



R, and all the couples into a single couple $(S, -S)$ (Fig. 27) applied to a certain straight line BC.

Hence we learn that, *any number of forces applied in any manner to a body, can always be reduced to a single force which passes through any proposed point, and to a single couple, whose plane will be, in general, inclined to the direction of the force.*

Observe, now, that the magnitude, the direction, and the sign of the resultant R will be always *the same*, in whatever position we take the point A. By varying the position of this point, the resultant R will only transfer itself parallel to itself to different points in space; but the plane and the magnitude of the resultant couple $(S, -S)$ will necessarily be changed.

But amidst this infinity of reductions, relative to all the points A in space, there is one distinguished from all the rest in that the plane of the resultant couple is perpendicular to the line of action of the resultant. We may demonstrate this at once in a very ready manner: for all being already reduced to the single force R, and the single couple $(S, -S)$, with respect to some known point A, suppose that we decompose this couple $(S, -S)$ into two others, the former $(T, -T)$, which falls in a plane perpendicular to the line of action of the resultant, and the other $(V, -V)$ in a plane which passes through this line of action AR. In this plane we find at the same time the couple and the force R; we may therefore transfer this force parallel to itself from A to a point O, so situated, and at such a distance AO as that the couple $(R, -R)$ generated by this translation, may be equal and opposite to the couple $(V, -V)$ and may destroy it: then will there only remain the single force R applied at the new point O, and the couple $(T, -T)$, which is in a plane perpendicular to the direction of this force. Thus,

Any number of forces are always reducible to a single force, and a single couple whose plane is perpendicular to the direction of the force. So that there is always in space a certain determinate straight line OR, which may serve at the same time to represent the line of action of the resultant, and the axis of the resultant couple. Note 12.

This reduction is unique; that is to say, there exists no other position in space at which we could find the resultant couple perpendicular to the resultant force. For transfer the force R wherever we choose out of the actual position OR, it will produce a couple $(R, -R)$ perpendicular to the couple $(T, -T)$, and these two couples, on being compounded into one, will give the new resultant couple necessarily inclined to the couple $(T, -T)$, and therefore always greater, because the two components are at right angles to one another. Whence we see not only that the couple $(T, -T)$ is the *only* one which can be perpendicular to the direction of the resultant, but that it is also *the least* of all the resultant couples, which can be formed relative to all points in space. We see, at the same time, that for points taken about OR at equal distances from this straight line, the resultant couples have equal values and are in different planes, but equally inclined to this axis OR, which we may therefore call the *central axis* of the couples of the system; the further we get from this axis the greater the couples become, and they increase without limit, but they have all this common property, that each of them, *resolved* upon the plane perpendicular to the constant direction of the force R, gives the same couple $(T, -T)$, whence we see that the value of this *minimum* couple is always obtained by taking any resultant couple whatever $(S, -S)$ and multiplying it by the cosine of its inclination to the plane of the other.

We only just touch upon this *central axis*, it gives us so luminous a reduction of all the forces of the system as to throw a light at the same time upon all the other equivalent reductions, and to group them, so to speak, into a single picture, in which we see at once order and mutual dependence.

This theory will be more fully developed in the *note upon moments and areas.* But we must here confine ourselves to the general corollaries which have more immediate connexion with the elements of statics.

Corollary I.

Which contains the laws of the equilibrium of any free system.

69. Since a couple can never be kept in equilibrium by any simple force, directed in any manner in space, it follows, from what we have just said, that there will never be equilibrium in the system, unless both the resultant R of the forces be itself equal to zero, and the resultant couple $(S, -S)$ be at the same time zero.

So that, *all the forces applied to the system being transferred parallel to themselves to any point in the system, or in space, must be in equilibrium with one another; and all the couples which they produce by their transference to this point, must be also in equilibrium with one another.*

Remark.

70. Such are, for any free rigid system whatever, the two necessary and sufficient conditions of equilibrium; that is to say, *without* which equilibrium cannot subsist, but that it *will* take place if they are complied with.

In order to develope these two conditions, we must get to the value of the resultant R, and to that of the resultant couple $(S, -S)$, by observing the laws which connect the resultant with its components, and the resultant couple with its component couples; and then, putting the resultant R and the couple $(S, -S)$ separately zero, observe what relations this establishes amongst the primitive forces applied to the system. In this way we shall obtain the conditions of equilibrium, expressed by means of single forces given immediately by the state of the question, which is the solution of the problem which we were contemplating. Exs. 10, 11, 12.

But all these investigations, which, according to the principles already laid down, become no more than an affair of geometry and of calculation, will form the subject of the following chapter.

Corollary II.

Which contains the necessary conditions in order that all the forces applied to the system may have a single resultant when they are not in equilibrium.

71. All the forces applied to the system being reduced, as we have just seen, to a single force and a couple, let us suppose that this force R and the couple $(S, -S)$ are reducible to a single force; or, if we choose, that a single force R' may be in equilibrium with the couple $(S, -S)$ and the force R.

Since, then, there is equilibrium between the two forces R, R', and the couple $(S, -S)$, it is clear that the two forces R, R', ought to form a couple equal and opposite to the couple $(S, -S)$, and situated in the same plane, or, which is the same thing, in a parallel plane.

For there can only be three cases; firstly, that in which the forces R and R' are capable of being reduced to a single one, and then this force will not be able to maintain in equilibrium the couple $(S, -S)$; secondly, that in which they reduce themselves to a single one, with a couple, and where this couple and the proposed one $(S, -S)$ combine into a single one, which cannot be in equilibrium with the force; thirdly, that in which they reduce themselves to a single couple, and this is all that can happen.

It is necessary, then, at least, that the two forces R and R' should together form a couple. But in order that this couple may be in equilibrium with the couple $(S, -S)$, it is indispensable that it be situated either in the same plane or in a parallel plane, for otherwise these two couples would compound into a single one, which could never be zero (62), and there could not be equilibrium. Therefore the direction of the resultant R must be parallel to the plane of the resultant couple $(S, -S)$ and consequently, *the forces of the system will never be reducible to a single one, unless the resultant of these forces transformed parallel to themselves to the same point, have a direction parallel to the plane of the resultant couple, and that, whatever be the position of the point in space to which we may have transferred all the forces.*

This condition is necessary, and it is clear that it will be in general sufficient; for, so long as the resultant be not zero, we shall always have it in our power to apply to the system a force R', equal, parallel, and opposite to the force R, and forming with it a couple $(R, -R)$ of an opposite sign to that of the couple $(S, -S)$ and of equivalent moment. This force taken with an opposite sign will be the general resultant.

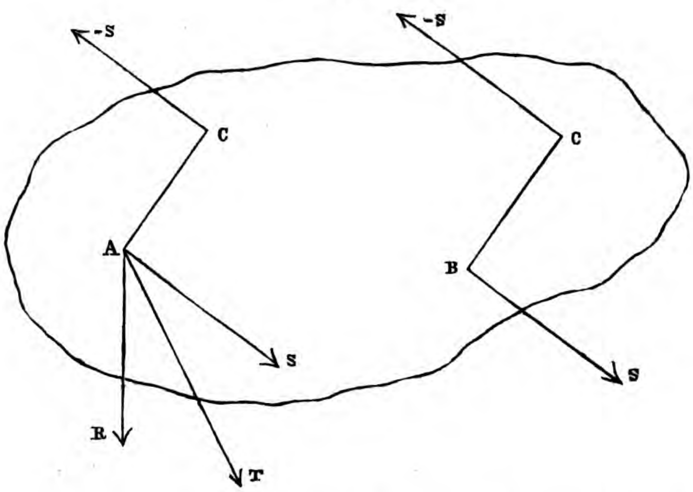
We shall always be able to find this resultant immediately; for, if the force R applied at A be parallel to the plane of the couple $(S, -S)$, we shall be able to transpose this couple into the same plane as the force R, and then the three forces R, S, and $-S$ being in the same plane will always compound into a single one equal and parallel to R, and which will be the single resultant of all the forces.

72. In the case in which the force R is equal to zero, there is no single resultant. For all the forces of the system are reduced to the single couple (S, $-S$), which can never reduce itself to a single force. Thus, to the preceding condition, which requires that the force R be parallel to the plane of the couple (S, $-S$), we must further add this as a particular condition, *that the force R be not equal to zero.* (Unless there be equilibrium, in which case the resultant force and the resultant couple being both zero, we may say that there is a single resultant which is zero, and which has besides any direction and position which we choose in space; but we have excluded the case of equilibrium.)

REMARK I.

73. When the resultant couple (S, $-S$) (Fig. 28) and the force R are not in parallel planes, there is never a sin-

Fig. 28.



gle resultant. Only, by transferring the couple (S, $-S$) parallel to its own plane, we may place the extremity B or C of its arm upon the point A, and then the two forces R and S, applied at A, compound into a single one T, and all the forces of the system are reduced to two others, T and $-S$, not situated in the same plane.

From which we learn, first, that *any number of forces* *directed arbitrarily in space, may always be reduced to two at most, not situated in the same plane.*

But it is clear that this reduction may be made in an infinite number of ways, even without altering the position of the point A, where all the forces are assembled; for the couple $(S, -S)$ may always be changed into an infinite number of other equivalent couples, and moreover turned upon its axis into any position, and we obtain in this way an infinity of different systems of two resultants not situated in the same plane.

In fact, we might select amidst these systems, that one in which one of the forces would be perpendicular to the plane of the couple, and the other directed in this same plane; for, imagine the resultant R decomposed into two forces; the one V perpendicular, the other U parallel to the plane of the couple $(S, -S)$. The force U and the parallel couple will always be reducible to one U', equal and parallel to U; and all the applied forces will be reduced to two others V and U' in directions perpendicular to one another in space. Thus, *any forces whatever may be reduced to two, whose directions are perpendicular to one another, and one of which passes through any given point A.* But this reduction itself, which suffers besides one exception, has scarcely more use than the preceding, and we shall dwell upon it no longer.

Remark II.

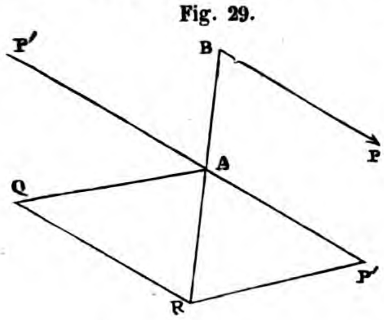
74. The only result which it may be as well to remark, is this other reciprocal proposition; *that two forces, not situated in the same plane, can never have a single resultant.*

And, in fact, we may always suppose that these two forces arise from some other force and a couple which is not parallel to it.

But if we would see this directly, let AB (Fig. 29) be a common perpendicular upon the lines of action of two forces, P and Q, not situated in the same plane, and neither of which is supposed to be zero. At A apply two forces P' and

$-P'$, equal and parallel to P; then P will be replaced by an equal and parallel force P' at A; and a couple $(P, -P')$. But at the point A, the forces P' and Q, which by hypothesis make with one another a certain angle QAP', compound into a single force R, directed within the interior of this angle. But this force R cannot be parallel to the plane of the couple $(P, -P')$, because it makes with this plane an angle RAP', which can never be zero, unless Q be zero, which is contrary to the supposition. Therefore (71) the two forces P and Q, not situated in the same plane, can never have a single resultant; a proposition which is commonly regarded as self evident, but which, nevertheless, needs demonstration.

Fig. 29.



Remark III.

75. This is, in short, the only general case in which we can be certain that forces are not reducible to a single one; for when we consider but three forces, it follows from our theory that they *may* have a single resultant, when, at the same time, their lines of action do not intersect in space.

Let there be three forces, P, Q, R, which are not situated in the same plane, or rather such that, if there were two in the same plane, the remaining one would not be in the same plane with either of them.

Suppose P and Q to be not situated in the same plane; transfer them to the same point A in the line of action of the third force R. These two forces, P and Q, will then compound into a third force V, and will give two couples which compound into a single one $(S, -S)$; the plane of which couple will not pass through the line of action AV of the force V (74).

This being done, if the resultant of the two forces V and R applied at A happen to be in the plane of the couple (S, $-S$), which passes through the same point, the three forces will be reducible to a single one (71). But, without changing the line of action of the force R, we may so dispose the sign and the magnitude of this force as that the resultant of V and R may turn about the point A into the plane of those two forces, and may have for its line of action the intersection of this plane with that of the couple (S, $-S$), and may thus fall in the same plane as that of this couple. Therefore, by suitably taking the magnitude and the sign of one of the three forces P, Q, R, without at all changing their respective lines of action, we may in general render these three forces reducible to a single one.

We say in general, because there is a particular case in which the thing could not happen, in consequence of there having been a relation given between P and Q, and the restriction to alter nothing but the magnitude of the third force R; for if, in consequence of this relation between P and Q, it should happen that the line of intersection of the plane VAR with the plane of the couple should be the same as the line of action AR of this third force, we should not be able to take for the line of action of the resultant of V and R the line AR, without making the component R infinite, which is impossible.

But in that particular case, in which we begin by changing the relation of the two forces P and Q, or simply the sign of one of them, the couple (S, $-S$) which results from their translation to the point A will no longer pass through the line of action of the third force R; for if the plane of this couple were still to pass through the same straight line AR it would follow that AR would be the common section of the two planes in which are situated the two component couples of (S, $-S$), and that thus the force R would be at the same time in the same plane as P and in the same plane as Q, which is contrary to the hypothesis.

Thus, *when of three forces* P, Q, R, *we can only find two at most that are situated at the same plane, it is still always possible to reduce these three forces to a single one, without in any manner altering their lines of action in space.*

The only case in which we can say of three forces, that their position alone renders them irreducible, is that in which, regarding these forces two and two, we only find one combination which presents two forces not situated in the same plane. In such a position, whatever relations we may give to the magnitudes of these forces, we shall never render them capable of being reduced to a single one.

REMARK IV.

76. Since it has been proved that a couple cannot be in equilibrium about a fixed point, the centre of its arm for instance, we remark this difference between the equilibrium of several *forces* applied to a body constrained to turn about a fixed point, and the equilibrium of several *couples* which might be applied to the same body.

In the former case, it is not necessary that the forces should have a resultant zero; it is sufficient that they have a resultant which passes through the fixed point, where it will be destroyed.

But in the latter case, it is absolutely necessary that the couples applied to a body should have a resultant couple zero, the same as if there had been no fixed point in the body; for, if this couple be not zero, place it, for the sake of clearness, so that the centre of its arm may be in the fixed point, and then it will be evident that its two forces will not be in equilibrium about this point.

Moreover, it is also manifest, that they would not be in equilibrium, even if there were in the body a fixed axis, provided that the plane of the couple did not pass through this axis, or were not parallel to its direction, which is the same thing (49).

So that, *when different couples, situated any where in space, solicit a body or system constrained to turn about a fixed point, the conditions of equilibrium will be exactly the same as if the body were perfectly free.*

And the same thing will happen in the case of a fixed axis if the applied couples be so disposed as that they never give a resultant couple parallel to this axis, which we can only be certain of, in a general way, when all the couples lie in parallel planes which intersect the fixed axis.

APPENDIX.

NOTES.

NOTE I.

MOTION is change of relative position, and is accomplished in *time*. By diminishing the time we diminish the motion, and by sufficiently diminishing the time we may render the motion inappreciable. In this state the body may be considered as at rest.

NOTE II.

Force is very often defined to be "any cause which changes or tries to change the state of a body's rest or motion."

NOTE III.

What is here called the direction of a force, is very frequently called the "line of action" of the force, whilst any line whatever in space parallel to the line of action, is called the "line of direction" of the force. These distinctions should be always rigorously attended to.

The "magnitude" or intensity of a force is estimated by the number of pounds weight which the force would support if it were made to act directly opposite to the force of gravity. Thus, when we call a statical force P, we mean that the force would sustain P pounds.

Forces are also affected with the signs plus and minus in order to indicate which way they act. It is quite arbitrary which direction we consider positive; but having made the selection, the opposite direction will be negative.

NOTE IV.

This principle is called "the principle of the transmission of force."

NOTE V.

GC and AB are each of them equal to half the line GK. They are therefore equal to one another.

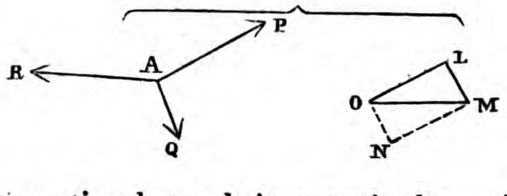
NOTE VI.

This important theorem is called the "Parallelogram of Forces."

Note VII.

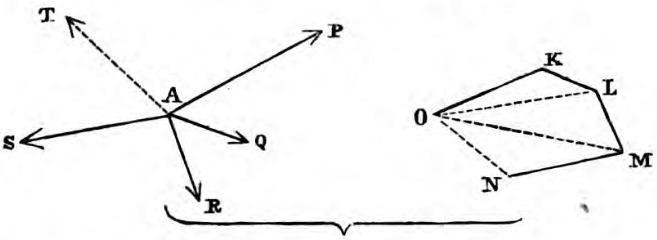
Let us first consider the case of three forces whose resultant is zero.

Let P, Q, R be in equilibrium on the point A. Take any point O and draw OL, LM, respectively parallel to the lines of action



of P, Q, and proportional to their magnitudes. Now, if we complete the parallelogram LN and join OM it will be manifest that OM is the line of direction, and represents the magnitude of the resultant of P and Q. It is therefore parallel to AR and represents R in magnitude.

Next, let us consider the case of several forces which act upon a point and whose lines of action are not in a plane.



Let P, Q, R, S be four forces acting upon a point A. Take any point O. Draw OK, KL, LM, MN, respectively parallel to the lines of action and proportional to the magnitudes of the forces P, Q, R, S, and join OL, OM, ON.

Since OK, KL, LO lie in a plane, it is manifest, by what has preceded, that OL is parallel and proportional to the resultant of P and Q; similarly, it appears that OM is parallel and proportional to the resultant of this last resultant, and of the force R, and therefore to the resultant of the forces P, Q, R; and finally, that OM which completes the polygon is parallel and proportional to the resultant of all the forces P, Q, R, S.

Hence the force T proportional to OM, and whose line of action AT is parallel to OM, will maintain in equilibrium the other four forces.

This reasoning may be extended to any number of forces.

NOTE VIII.

The reader must not imagine that there is any need of *Statics* in order to prove this. We have

$$AB^2 + AC^2 + AD^2 = AF^2.$$

Therefore dividing by AF^2, $\left(\frac{AB}{AF}\right)^2 + \left(\frac{AC}{AF}\right)^2 + \left(\frac{AD}{AF}\right)^2 = 1.$

Therefore $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Q.E.D.

NOTE IX.

The usual mode of reasoning here alluded to is this.

Let P' and Q' be two incommensurable forces; and let $mp = P'$, m being a whole number. Let n be another whole number. Now, by taking p sufficiently small and n correspondingly large, we may make np differ from Q' by a quantity less than any assignable quantity; so that np may be considered as equal to Q'. Hence any theorems which are true of two commensurable forces P, Q, will also be true for two incommensurable ones P', Q'.

NOTE X.

This will be the same way as that in which the hands of a watch rotate.

NOTE XI.

We have also the equation

$$\cos^2\lambda + \cos^2\mu + \cos^2\nu = 1.$$

NOTE XII.

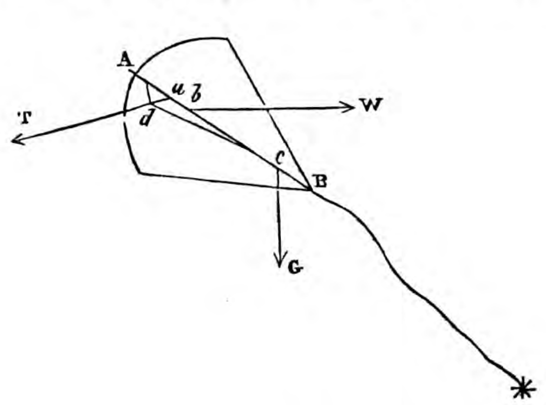
It must not be supposed that the line OR represents both the moment of the couple and the magnitude of the force.

We shall now apply the Statical principles which we have been discussing to the explanation of a few familiar phenomena.

EXPLANATION OF A FEW FAMILIAR PHENOMENA.

THE KITE.

If we conceive the area of the kite to be composed of an indefinite number of indefinitely small equal areas, each of these areas will sustain an indefinitely small equal horizontal force due to the action of the wind upon it. This system of indefinitely small equal parallel forces will have a *finite* resultant W acting at b, the centre of parallel forces. (This point b we may just observe will be the

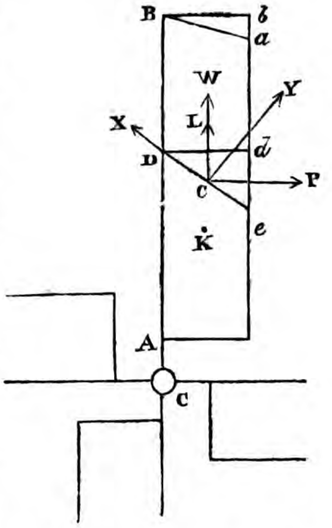


centre of gravity of the area of the kite). The weight of the kite and its tail, which we will call G, will act vertically at some lower point C (which will be the centre of gravity of the whole). The only remaining force will be the tension of the kite-string T. This string is not directly attached to the kite, but to another loose string upon which it slides. This is in order that the point of application of the tension T may accommodate itself to the varying action of the wind, &c. However, we are considering the case of exact equilibrium, and this will not be disturbed if we suppose the point d to be rigidly connected with the kite at a, Tda being a straight line. The force T may therefore be applied at a. The kite will then be in equilibrium under the three forces T, W, and G, acting at the points a, b, c, in the line AB, which divides the kite symmetrically.

By resolving these forces into components respectively perpendicular to and along the line AB we shall see how this equilibrium is perfectly possible. It is manifest that TaA must be an *acute* angle, and that a must be *above* b and c.

EXPLANATION OF THE ACTION OF THE SAILS OF A WINDMILL.

The sail of a windmill is placed with one edge CB in a plane perpendicular to the direction of the wind. Let BbA be this plane. C the axis of rotation. Then the frame-work upon which the sail is spread, will not be a perfect plane, but will be a winding surface the sections of which by planes perpendicular to the line AB will be straight lines, making angles with the vertical plane BbA, which gradually decrease as we recede from the centre. The reason of this we shall explain presently. Let us consider the action of the wind upon one of these sections De. Conceive that along De are ranged an indefinite number of indefinitely small equal areas; each of these areas will be acted upon by an indefinitely small equal horizontal force arising from the action of the wind. This system of parallel forces will have a single resultant W acting perpendicularly to the vertical plane BbA and at the middle point c of the line De. Resolve this force W into two others, X, Y, of which X acts in the direction cX, and produces no effect, and Y in a direction cY perpendicular to De. Again, resolve this force Y into two others, L and P, of which L acts in the direction cW, and P in a direction cP, perpendicular to cW, and parallel to the vertical plane BbA. (It will be observed that these forces L and W are both indefinitely small forces, although indefinitely greater than any one of the forces acting upon one of the small areas, for these are indefinitely small quantities of the *second* order). Now, treating each of the indefinite number of sections or strings of small areas, which can be made parallel to CB in a similar manner, we shall find that the system of parallel forces P will have a single *finite* resultant R acting in a parallel direction to the rest at some point K of the sail. This force may be replaced by an equal parallel force acting at the fixed axis C, and therefore producing no effect, and by a couple acting in a vertical plane and applied, if we choose, to the line CB, and communicating rotation to the sail; for we may



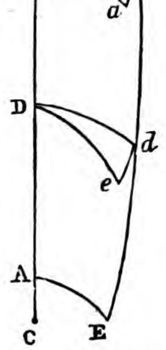
suppose the centre of its arm applied at C, and then its action will be quite obvious.

Next let us explain why the lines De as they recede from the axis should make decreasing angles with the vertical plane.

In order to this, let us endeavour to construct a system of sails such that, upon a suitable angular velocity being given to them, the wind may pass them without being in the least arrested in its course, or disturbed in any manner, and consequently without producing any pressure at all upon the sails.

We shall suppose of course that there is no friction upon the axis.

With the centre C and radii CB, CD, CA, describe equal arcs of circles Bb, Dd, AE. And through E, d, b, draw the curve line Edb such that any other arc equal to any one of the former, and with centre C may have its extremities on the line AB, and the curve Eb. (The polar equation to this curve will be $r\theta = a$, a being the length of an arc, C the pole, CB the initial line.)



The sail will wind as before in front of the vertical plane BbA; and it is the law of this winding which it is our object to discover.

Let Ba, De, be sections of the sail by cylinders whose axes are horizontal and pass through C; they will of course cut the vertical plane in the arcs Bb, Dd; draw ba, de horizontal. Then the figures Bba, Dde, are to be, when flattened out, right-angled triangles of equal base a, and of heights ba, de respectively.

Now, since the sail is to move with such a velocity as that the wind is to produce no pressure upon it in passing, it will follow that whilst the wind is describing the horizontal distance ba, the sail must turn through the angle whose circular measure is $\dfrac{a}{CB}$. For the same reason, whilst the wind is describing the horizontal space de the sail must turn through an angle whose circular measure is $\dfrac{a}{CD}$. Hence we obtain the proportion

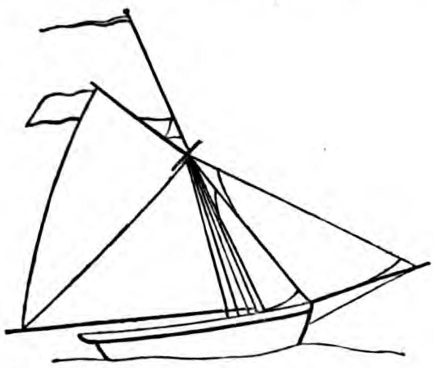
$$de : ba = \frac{a}{CD} : \frac{a}{CB} = CB : CD.$$

And this proportion completely determines the law of the construction of the sail.

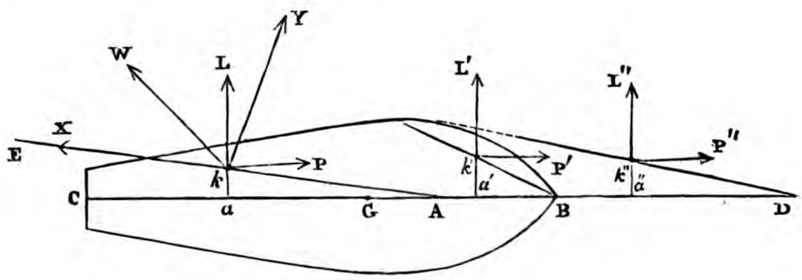
We now see why the sails of a windmill ought to wind, for it is of course an object to get them as nearly perfect as possible.

WHY A SHIP SAILS TO WINDWARD.

What is meant is this, that when the wind is from the N.W. for instance, the vessel will sail in a northerly direction. When sailing to windward the sails are all made as flat as possible and placed with their planes at a small angle to the keel. A cutter when going to windward would present very much the appearance of the figure.



Now let us proceed to the explanation of this really extraordinary phenomenon; which we may here remark does not depend at all upon the rudder, for if the sails were in *perfect* trim, the vessel would sail to windward with its keel always parallel to itself, without any rudder at all.



Since the ship is a free body acted upon by forces, the reader in order to the full comprehension of what is to follow must be informed what the action of a couple really is; although *the proof* of this forms no part of Statics. The effect then is this, to communicate rotation to the body about an axis passing through the centre of gravity and perpendicular to the plane of the couple.

Let kW be the direction of the wind. If we conceive the sail AE to be composed of an indefinite number of indefinitely small equal areas, each of these areas will be acted upon by an indefinitely small equal force, due to the action of the wind, and parallel to kW. This system of equal parallel forces will have a single finite resultant

W acting at some point k, not far from the centre of the line AE. This force W may be decomposed into two others, X, Y, the force X acting in the line AE and producing no effect, and the force Y acting in the line kY perpendicular to AE. This force Y may be decomposed into two others, L, P, L perpendicular to the keel and P parallel to it.

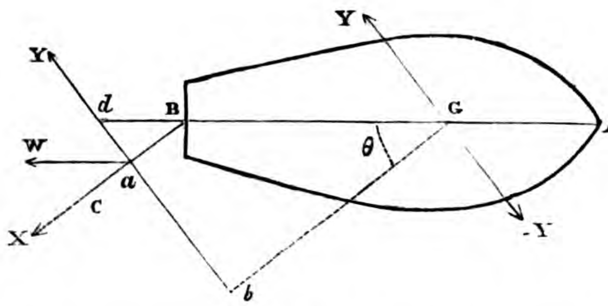
Similarly we find for the other two sails the forces L', P', L'', P''.

Now let us first discuss what the effect of the forces P, P', P'' will be upon the vessel; replacing them by equal parallel forces at G the centre of gravity, and by necessary couples we get a force $P + P' + P''$ acting at G and tending to urge the vessel forwards in the direction of the keel, and a resultant couple whose moment is $Pka + P'k'a' + P''k''a''$ tending to make the vessel turn head to wind. Next let us consider the action of the forces L, L', L''; replacing them as before by equal parallel forces at the centre of gravity and by couples, we get a parallel force $L + L' + L''$ acting at G perpendicular to the keel; and a resultant couple whose moment is $L\,.\,Ga - L'\,.\,Ga' - L''\,.\,Ga''$. On account of the great size of the sail AE as compared with the other sails, the effect of this couple, as well as the other, will be in general to make the vessel turn head to wind; this tendency, which most ships possess, is called "carrying weather-helm;" it can only be counteracted by the rudder, but is thought in general an advantage, as it makes the vessel more manageable in case of a sudden squall. Finally, let us consider the action of the force $L + L' + L''$ acting at G, in a direction perpendicular to the keel; this force will manifestly have to encounter the whole opposition of the water against the side of the vessel, it will therefore produce but little effect; still it will produce *some*, and this is called "making leeway;" the less leeway a vessel makes the better.

TO EXPLAIN THE ACTION OF THE RUDDER.

When the rudder is held at an angle with the keel, the water presses against it with a finite force W suppose, acting parallel to the keel and at a point a, which will be the centre of the rudder BC. This force W may be decomposed into two others, X, Y, X acting in the line BC and producing no effect, and Y acting in the line aY

perpendicular to BC; replacing this force Y by an equal and parallel force at G, the centre of gravity, and by the couple whose



moment is $Y \,.\, Gb$, we see what the effect of this couple will be in turning the ship.

The reader may now take the two following problems.

1. To find at what angle θ the rudder must be inclined to the keel in order that its effect in turning the ship may be a maximum.

Answer. $\cos\theta = \frac{\sqrt{3b^2 + a^2} - a}{3b}$ a being half the length of the rudder, and b being equal to GB.

2. To explain why a vessel at anchor in a tide way should obey her helm within certain limits, the same as if she were free and moving through the water.

EXAMPLES.

A FEW PRELIMINARY REMARKS UPON REACTIONS, TENSIONS, &c.

WHEN a smooth straight line presses upon a smooth point, the reaction of the point acts in a direction perpendicular to the line, for there is no reason why it should make an angle with this perpendicular on one side rather than on the other.

Hence it follows that when a smooth curve line presses upon a smooth point, the reaction of the point acts in the direction of a normal to the curve, because, for a very short distance about the point, the curve may be considered as coinciding with the tangent.

When a smooth plane presses upon a smooth point, the reaction of the point acts in a direction perpendicular to the plane, for there is no reason why it should make an angle with this perpendicular on one side rather than on the other.

Hence it follows, that when a smooth curve surface presses upon a smooth point the reaction of the point acts in the direction of a normal to the surface, for the surface may be considered, for a very small space about the point, as coinciding with its tangent plane.

It also follows that, when two smooth curves or surfaces press against one another, the mutual reactions act in the direction of the common normal.

When an inextensible imponderable string is in a state of tension, the tension of the string (that is to say, the resistance which we should experience if we were to take hold of it at any point and try to pull it in the direction of its length) is always in Statics the same at every point, for otherwise motion would ensue.

Example I.

A man and a boy carry a weight of 50 lbs. upon a pole between them; the man is half as strong again as the boy; the pole is 5 ft. long. Find where the weight ought to be placed and what portion of it each has to carry; (the weight of the pole being neglected.)

Answer. The boy carries 20lbs., the man 30 lbs., and the weight should be placed 2 ft. from the man's hand.

Example II.

In the figure of Art. 24, $P = 3$lbs., $R = 7$lbs., $AC = 1$ yard. Find the length CB and the magnitude of Q.

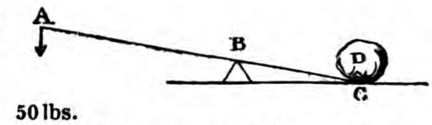
Answer. $CB = 2$ ft. 3 ins., $Q = 4$ lbs.

Example III.

A labourer, whose weight is 150lbs., stands at the point C upon a scaffold board AB, which rests upon two putlocks A and B. $AB = 8$ ft., $BC = 2$ ft. Find how much of his weight each putlock has to support.

Answer. A supports $37\frac{1}{2}$ lbs., B $112\frac{1}{2}$ lbs.

Example IV.

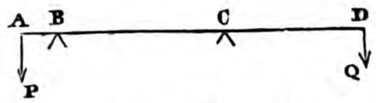


AC is a rigid bar without weight capable of turning about B the edge of a fixed prop or fulcrum; a force of 50 lbs. is applied vertically at A, $AB = 1$ yard, $BC = 8$ inches. What will be the weight of a block of stone D placed at C, which will just counteract the force at A, and what will be the vertical pressure upon the fulcrum B?

Answer. The weight of $D = 225$ lbs., and the pressure on $B = 275$ lbs.

Example V.

Two weights, P and Q, act vertically at the extremities of a rigid imponderable bar AD, which is supported horizontally upon two fixed fulcra B, C. $AB = p$, $BC = a$, $CD = q$. Find the respective pressures upon these fulcra.



Answer. B sustains a weight $= \dfrac{(a+p)P - qQ}{a}$,

C .... $= \dfrac{(a+q)Q - pP}{a}$.

Example VI.

Find where a weight ought to be placed upon any triangular table, in order that each leg may sustain an equal share of the weight; the legs being placed at the corners of the table.

Answer. At the common point of intersection of lines drawn from the angular points to the centres of the opposite sides.

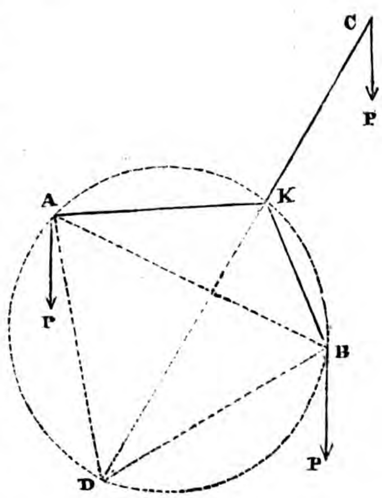
Example VII.

Suppose that in the last problem, the legs have to sustain portions of the weight respectively proportional to the lengths of the sides opposite to them. Where ought the weight to be placed in this case?

Answer. At the centre of the inscribed circle.

Example VIII.

A, B, C are perfectly smooth points situated in a vertical plane, over each of them a weight P is suspended by means of an inextensible imponderable string, at the extremities of which the weight is attached; these three strings have their other extremities knotted together at K. Find the position of the point K when there is equilibrium.

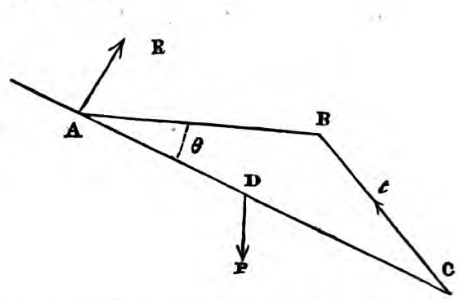


Answer. Join AB and on it describe the equilateral triangle ABD; about this triangle draw a circumscribing circle. Join CD cutting the circle in K. K is the point required.

It ought to be observed, however, that the two points which we join ought to be such, that the remaining point may be without the circle, and also that the line drawn from it to the circle may cut the small segment on the chord joined.

Example IX.

A and B are two points in the same horizontal line, CA is a rigid imponderable bar of indefinite length resting on the smooth point A, and having its extremity C attached to the point B by means of an inextensible imponderable string BC, A, B, C are in a vertical plane; at the point D a weight P acts vertically; $AB = BC = CD = a$. Find the position of the rod when there is equilibrium, and also the tension of the string.



Let R = reaction of point A, t = tension of string, $AD = x$, $BAC = \theta$. Then resolving P and t into their components perpendicular to, and in the line CA, we get

$$t \cos\theta = P \sin\theta, \;\ldots\ldots\ldots\; (1)$$

and observing that R is equal and opposite to the resultant of those two components which are perpendicular to the line CA we get

$$t \sin\theta\,(a + x) = Px \cos\theta, \;\ldots\ldots\ldots\; (2) \text{ Art. 25.}$$

Finally, by geometry we have

$$a + x = 2a \cos\theta, \;\ldots\ldots\ldots\; (3).$$

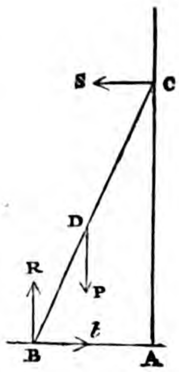
These three equations are all we require for the determination of t, θ, x.

Answer. $$\theta = \cos^{-1}\left(\frac{1 + \sqrt{33}}{8}\right),$$

$$t = \frac{\sqrt{30 + 2\sqrt{33}}}{1 + \sqrt{33}} P.$$

EXAMPLE X.

BC is a ladder (whose weight is neglected) which rests against a smooth vertical wall *AC*, and with its lower end *B* upon a smooth surface of ice. In order to prevent it from slipping there is a rope *BA* attaching it to the wall. A man whose weight is *P* has ascended to a position *D*. Find the tension of the string; and also the position of *D* when this tension is a maximum.

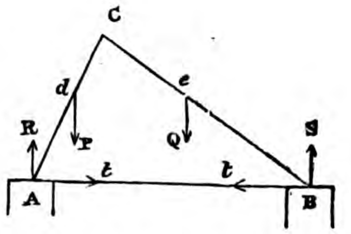


Answer. If $AB = a$, $AC = b$, $BD = c$.

The tension of the string will equal $P \cdot \dfrac{ac}{b\sqrt{a^2 + b^2}}$. From which it follows that the tension will increase as the man ascends, and will be a maximum when he reaches the top of the ladder. At this point the tension will equal $P \cdot \dfrac{a}{b}$.

EXAMPLE XI.

AC, *CB* are two rigid imponderable lines, connected together by a hinge at *C*, and standing in a vertical plane with their extremities *A*, *B* upon the smooth tops of two walls. *A* and *B* are in the same horizontal line and are connected together by a string *AB* inextensible and without weight; *ACB* is a right angle; at *d*, *e*, the middle points of *AC*, *BC*, weights *P* and *Q* which are respectively proportional to the lengths *AC*, *BC*, act vertically. Find the tension of the string.



Answer. If $AC = b$, $BC = a$, $P = \mu b$, $Q = \mu a$, then the tension of the string will equal $\mu \cdot \dfrac{ab}{2} \cdot \dfrac{a + b}{a^2 + b^2}$.

NOTE. The best way of setting about this problem is this.

First, remove *BC* and supply its place by a suitable force at *C*. The magnitude and direction of this force are both unknown. Next replace each of the forces R, P, t by equal and parallel forces acting at C, and by the necessary couples resulting from this translation. By equating to zero, the resultant of these couples, we get an equation. By next removing AC, and treating BC in a similar manner, we get another equation. Finally, by supposing the hinge-joint to become rigid, which will not disturb the equilibrium, we get the third equation $R + S = P + Q$. Eliminating R and S between these three equations we get a resulting equation in which t is expressed in terms of known quantities.

Example XII.

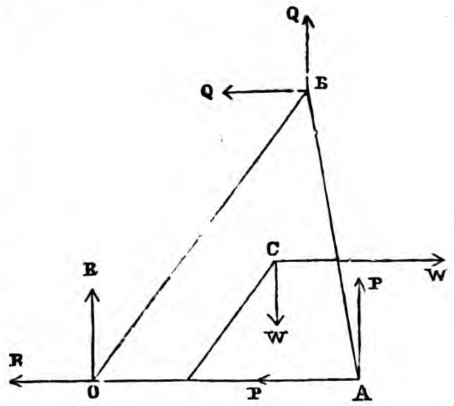
Find the smallest force of an integral number of lbs., which acting horizontally at the axle will be sufficient to pull a coach wheel 5 ft. in diameter over an obstacle 5 inches high. The total pressure on the axle of the wheel being 500 lbs.

Answer. 332 lbs.

Example XIII.

A weight is placed anywhere upon any triangular table. Find what portion of it each leg has to sustain.

Let OAB be the table; $OA = a$, $OB = b$; and let P, Q, R be the portions of the weight W, sustained respectively by the legs A, B, O; also, considering OA, OB as oblique axes of x and y, let C the point of application of the weight be the point (x, y).



Now all these parallel forces act vertically; but we may turn them, if we choose, through a right angle into the horizontal plane of the table without disturbing the equilibrium of the points of application O, A, B, C. Let us suppose this done, and in such a manner as that the forces may be all parallel to OA. Then P may be transferred directly to O; and Q and W may be replaced by equal parallel forces at O

and by couples whose moments are respectively proportional to Qb and Wy. Since these couples are to have a resultant zero, we get the equation $Qb = Wy$.

Similarly, by turning the forces parallel to OB we get $Pa = Wx$. We have also $P + Q + R = W$; these three equations give

$$P = W \cdot \frac{x}{a}, \quad Q = W \cdot \frac{y}{b}, \quad R = W\left\{1 - \left(\frac{x}{a} + \frac{y}{b}\right)\right\}.$$

It is to be observed that 1 is always greater than the sum of the ratios $\frac{x}{a} + \frac{y}{b}$, for the sum of these ratios would only equal 1 if the point C were to fall upon the line AB, in which case R would vanish as it ought to do.

When there are more than three legs the problem becomes indeterminate, because we can only get three equations.

A FEW CONCLUDING OBSERVATIONS.

In working out a Mathematical Problem, there is no better check upon the accuracy of the work than the careful observation of the *dimensions* of the quantities involved. If cosines, tangents, or any other trigonometrical ratios, come out of one, two, or of *any* dimensions, we conclude at once that our work is wrong, and immediately retrace our steps to find out the error, as these ratios should of course be *numerical;* or should forces appear as lines, areas, or volumes, we at once conclude that there is an error, and we do not proceed with the work until we have discovered it.

By way of illustration, we would refer the reader to the answers to the Examples in the present work. Let us take, for instance, our first problem on the Rudder. Here we assert that $\cos\theta = \frac{\sqrt{3b^2 + a^2} - a}{3b}$; which is of no dimensions, or numerical, as it ought to be, for the following reasons. $3b^2 + a^2$ is the sum of two areas, and is therefore of two dimensions;

the square root of this is consequently a line, and of the same dimensions (viz. one) as a; hence the numerator is a line, (for the sum or difference of two homogeneous quantities is of the same dimensions as either of them) and so is the denominator $3b$, the cosine is therefore a *number* as it ought to be.

On entering upon any new *Physical* subject in particular, the Student is advised most strongly to investigate at once the dimensions of any quantity whose definition is presented to him before proceeding another step in the subject.

END OF PART I.

Characteristics of the Greek Philosophers.

Socrates and Plato.

By the Rev. J. P. POTTER, M.A., late of Oriel Coll., Oxford.

Dedicated, by Permission, to The Lord Bishop of Lincoln.

Foolscap Octavo, 4*s*. 6*d*.

Travels in the Track of the Ten Thousand Greeks;

A Geographical and Descriptive Account of the Expedition of Cyrus and of the Retreat of the Ten Thousand.

By W. F. AINSWORTH, Surgeon to the Euphrates Expedition.

Post Octavo, with Map, 7*s*. 6*d*.

The New Cratylus;

Or, Contributions towards a more accurate Knowledge of the Greek Language.

By the Rev. J. W. DONALDSON.

. Octavo, 17*s*.

The Public Economy of Athens.

By Professor BŒCKH.

Translated by G. C. LEWIS, M.A., late Student of Christ Church.

New Edition, Revised. Octavo, 18*s*.

The Speeches of Demosthenes,

Against Aphobus and Onetor; translated, with Notes explanatory of the Athenian Laws and Institutions.

By C. R. KENNEDY, M.A., Fellow of Trinity College, Cambridge.

Post Octavo, 9*s*.

Theocritus.

Codicum Manuscriptorum Ope Recensuit et Emendavit CHRISTOPHORUS WORDSWORTH, S.T.P., nuper Scholæ Harroviensis Magister.

Octavo, 13*s*. 6*d*., boards.

Pindar's Epinician Odes,

And the Fragments of his Lost Compositions, Revised and Explained; with copious Notes and Indices.

By J. W. DONALDSON, B.D., Head Master of the Bury School.

Octavo, 16*s*.

The Aulularia of Plautus,

The Menæchmei of Plautus,

With Glossaries and Notes.

By JAMES HILDYARD, B.D., Fellow and Tutor of Christ's Coll., Cambridge. Octavo, 7*s*. 6*d*. each.

The Æneid of Virgil.

With Anthon's Notes, Edited

By J. R. MAJOR, D.D., Head Master of King's College School,

Foolscap Octavo, 7*s*. 6*d*., strongly bound.

Classical Texts,

Carefully Revised, from the best Editions.

A Series of very elegantly printed brochures of 'Classical Texts,' adapted either for the desk or the pocket.—*Spectator*.

Cicero de Amicitia. 1*s*.
Cicero de Senectute. 1*s*.
Cicero pro Plancio. 1*s*.
Ciceronis Oratio Philippica Secunda. 1*s*.
Taciti Germania. 1*s*.
Taciti Agricola. 1*s*.
Virgilii Georgica. 1*s*. 6*d*.
Ovidii Fasti. 2*s*.
Platonis Phædo. 2*s*.

Hebrew Grammar.

By the late CHRISTOPHER LEO, of Cambridge.

Octavo, 12*s*. 6*d*.

The Psalms in Hebrew,

with a Critical, Exegetical, and Philological Commentary.

By G. PHILLIPS, B.D., Fellow and Tutor of Queens' College, Cambridge.

Two Volumes, Octavo, 32*s*.

A Practical Arabic Grammar.

By DUNCAN STEWART.

Octavo, 16*s*.

The Elements of Syriac Grammar.

By G. PHILLIPS, B.D., Fellow and Tutor of Queens' College, Cambridge.

Second Edition, with Additions. Octavo, 10*s*.