EVOLUTION:

OR

THE POWER, AND OPERATION

OF NUMBERS,

IN THE STATEMENT, THE CALCULATION, THE DISTRIBUTION, AND THE ARRANGEMENT

OF

QUANTITIES,

LINEAR, SUPERFICIAL, AND SOLID.

By THOMAS SMITH,

Author of "Lessons on Arithmetic," and "The Chairman, and Speaker's Guide."

Not harsh, and crabbed, as dull fools suppose;
But musical as is Apollo's lute.

Milton.

LONDON:

LONGMAN, REES, ORME, BROWN, GREEN, AND LONGMAN,

PATERNOSTER ROW.

MDCCCXXXV.

PRICE FIVE SHILLINGS.

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LIVERPOOL:
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PREFACE,

IN A CONVERSATION BETWEEN A FRIEND AND THE AUTHOR.

FRIEND.—And who, do you think, wishes to know anything about Evolution and Involution; how many persons are there, think you, to whom such matters can ever be of practical use?

AUTHOR.—But few, comparatively, I am aware. But all men wish to learn to think, to reason; and I have selected this subject, have availed myself of the rich vein of materials which it affords, for the purpose of winning the mind into a delightful and profitable exercise of its reasoning powers.

FRIEND.—You deem this subject, then, a favourable one to the accomplishment of such a purpose?

AUTHOR.—I am not singular, as you are aware, in deeming the study of Mathematical learning favourable to the improvement of the mind. This study having long been recommended, by some of the most esteemed authorities in affairs of this nature.

FRIEND.—This appears strange from you! You, who, in your "Lessons on Arithmetic," so earnestly caution your Pupil against the pursuit of this very study of Mathematics.

AUTHOR.—Appear strange, it may, if you do not attend to the distinction, which, in the passage to which you allude, I have, as I think, pretty clearly, drawn, between the useful, and the useless; between what you must allow me to term, the substantial, and the merely abstract. Against the pursuit of abstract mathematics, and, indeed, against abstractions of all sorts, I would caution the student. It is not philosophy, but dreaming. I have, more than once in my life, in my desire to improve myself by the study of this science, entered upon the confines of abstraction, but have withdrawn, on finding that, like indulgence in mere reverie, the mind was sensibly sinking under the attempt.

FRIEND.—Then you do not indulge in abstractions, in this work?

AUTHOR.—In this, nor in any. Philosophy consists, or it merits not the name; philosophy, like common sense, consists in a love of knowledge that is useful. It is a knowledge of substantial things; of Nature, in her various regions; and of the Laws by which She is governed. It is true, that learning, once deemed abstract, speculations once deemed useless, have not infrequently been found to have useful applications. So that we must not be very rigid in our abstinence from the dreams of science. However, I have here followed no abstractions. I have traced the operations of numbers to the verge, I have stopped at the boundary which stands between numbers as they relate, or as they refer, to matter, and the abstract quantities of Mathematicians. In short, I have offered to the consideration of the reader, figures of real and tangible quantities, as the subject of every operation and inquiry.

FRIEND.—Your chief purpose, you say, in the work, is to win the mind to a pleasing, and profitable exercise of its powers?

AUTHOR.—And thereby to strengthen and improve it. It is not for me to say how far I think that I have succeeded in my purpose, nor to speak of any other quality of the work. I found this branch of the science of numbers replete with matter, itself interesting and useful, but far more valuable, as a medium for the exhibition and illustration of effective methods of analysis, and of reasoning; valuable as a means of training the mind, without the irksomeness of dry and repulsive rules, into a vigorous and healthful exercise, and into a taste for the Principles of science.

Besides these, however, I presume to hope that the work will be useful in other ways. I have intended to make the Book complete, by treating of those few rules of Arithmetic to which, on quitting their schools, gentlemen, and professional men, can be supposed to have occasion to refer. And then, I presume, also, that the Book will prove to be a short and easy introduction to Mensuration, Surveying, and Gauging.

EVOLUTION, &c.

INTRODUCTION.

FAMILIARLY acquainted with the first four rules of arithmetic, the very rudiments of the most ordinary education, the enquiring student will proceed without difficulty into the study of this more engaging department of the science of numbers. And if, in those four rules we include the addition, subtraction, multiplication, and division of compound numbers, a knowledge of these, joined to the few rules developed in the following pages, will most probably comprehend all the arithmetic that a gentleman or man of liberal pursuits may require.

The TERMS, and the SIGNS or marks employed throughout the work, are defined or explained as they severally arise. But, for obvious reasons, a list of them in a tabular form is desirable. Such a list will be found at the end of the work. And to make it more useful, some few other terms and marks, immediately connected with the object of the work, and used by other authors, will be comprehended in the Table. And this will be followed by Tables of English weights and measures.

CONTENTS.

OF

PROGRESSION AND PROPORTION,

AND OF

THE RATIOS OF NUMBERS.

1. PROPORTION and RATIOS are, perhaps, the most beautiful and interesting, as they likewise constitute the most useful branch of the science of numbers; most beautiful for the harmony and the order which they exhibit, and most useful in the frequency of their application to affairs of business and of civil life. With regard to progression, it is a very simple matter; a sort of notation, merely, of proportional numbers.

2. Progression is advancement; and the word is applied to the manner in which any series of numbers advance or increase by any sort of regular and orderly progress; as do 2, 6, 10, 14, 18; or 5, 7, 9, 11, 13; or 1, 2, 3, 4, 5; each of which lines, or series, of figures increases, as you see, at each step, by the repeated addition, in the first, of *four*, in the second, of *two*, and in the third, of *one*. And the word progression is, also, applied, although not quite so properly, to series of numbers which, in a mode equally regular, *decrease* as they proceed, as do 18, 14, 10, 6, 2. And, whilst the former may be called *increasing* series, those of the latter description are called *decreasing* series.

3. But there is another mode of progression, a mode in which the increase is produced, not by the repeated addition of one number, but by a repeated multiplication by one number, as in the following series, 2, 8, 32, 128, 512, and so on; the number with which we multiply, in this instance, being four.

4. Now these two modes of progression have, as different things ought to have, different names. That mode which is produced by *Addition*, is called *Arithmetical Progression;* whilst that which is produced by *Multiplication*, is called *Geometrical Progression.* The reasons for adopting *these* two names are not very obvious; and to inquire into those reasons would lead us out of our course. But it is necessary thus to distinguish the two modes. With regard to the numbers, or, as they are called, TERMS, forming both the modes of progression, they have several very curious, and some of them very valuable properties; on some of which properties we may with advantage observe.

5. In the first place, with regard to any series of numbers increasing, or decreasing, in arithmetical progression, thus, 4, 6, 8, or 8, 6, 4, it is to be observed, that the first and the last number in each series, added together, are equal to twice the middle number. And the reason of this, as you cannot fail to see, is, that as the numbers increase, or decrease, by even, or equal steps, so the number on one side is as much *more*, as that on the other side is *less*, than the middle number; so that, put the two together, and they make twice the middle number: and this is true of every such series, whether the steps by which numbers increase or decrease be great or small.

6. But this, which is true with regard to series such as the above, consisting of three numbers, is likewise true of any other regular series of numbers or terms, however extensive the series, and whether the increase, or the decrease, at each step, as I have just stated, be great or small. For, let us suppose a series of terms consisting of a thousand; or rather, in order that we may have *a middle* term, let the series consist of a thousand and one. Is it not evident, that as there will be five hundred steps of increase on one hand of the middle term, and five hundred of

decrease on the other, every step being equal; is it not evident, that what is lost on one hand is gained on the other; and that if we add the last term on each hand together, we shall have, as in the above short series of three terms, just twice the middle term? And, on this principle it is, that in measuring the trunks of trees, as timber, the measurement as to thickness, is taken by *girthing* them around the middle, that is, at an equal distance from each end. And this is regarded as the true measurement, or as it is called, the average thickness; seeing that, whatever the trunk may loose by tapering towards one end, is gained by its increase towards the other.

7. Again, that which is true of the two extreme terms of a series of numbers, whether that series be long or short, is true of any other two terms, taken one on either hand, an equal number of steps from the middle term. And this is true of all such pairs of terms, for the reason above stated; namely, that as much being gained on the one hand, at every step, as is lost by every step on the other, any two terms taken at an equal number of steps from the middle term, will be equal to twice the middle term; and equal too, for the same reason, to any two other terms taken in like manner, however near, or however distant from the middle term.

8. Yet, again: take a line of four terms; thus, 5, 6, 7, 8. Now here is not *one*, but two middle terms; and the extreme terms being each equally distant from the middle terms, whatever one extreme falls short of the two middle terms added together, is made up by the other extreme; so that, *the two extremes are equal to the two means;* for, by this name, "*means*," are these middle terms called. And this, which is true with regard to this series, is, for the same reason, true with regard to those of any other series; and true, also, with regard to any other two terms of any

other series, such terms being taken thus at eqnal distance from the two middle terms.

9. A consequence of the relationship thus existing between a series of numbers of this description is this; that if we be informed of three, of almost any three particulars respecting such series, we can tell all the other particulars respecting it. As, for example, if we have the first and the last terms, and the sum of the whole, we can then tell the number of terms, and the rate of increase at each step; or, as it is called, the rate of progression. Again, having the first term, the rate of progression, and the number of terms, we can tell the last term, and the sum of all the terms. But here is quite sufficient for our present purpose, in arithmetical progression. Now, therefore, for the other, and more important kind of progression; that is, when the increase is produced by a repeated multiplication by one number, which, as before stated, is called geometrical progression; a mode of progression, which produces numbers bearing towards each other a relationship very different from that which is produced by the successive addition of the same number to each successive term. And, for the purpose of more distinctly marking this difference, let us here set down a series of terms in each of the modes of progression. And, further, in order that the difference may be most distinctly seen, let us, in each series begin with the same number, and let the increase be made in each by the use of the number three: thus,

5, 8, 11, 14, 17, 20, 23, Arithmetical Progr.

5, 15, 45 135, 405, 1215, 3645, Geometrical Progr.

10. Now, the vast increase which the latter series makes, compared with the former, is not the point on which I have to remark, but, in the first place, the difference in the relationship which the several

terms bear one towards another. On looking at the first, or arithmetical series, you can scarcely say that any one of the terms is twice as many, three times as many, four times, or, indeed any number of times as many as any other term; in short, no one term, scarcely will help to produce, nor can it be the product of any of the others; or, if occasionally, such a relationship do exist between any of the terms, it is quite accidental. Far otherwise, however, is it with the terms which form the other series; amongst these, each term is a *factor* of all that come after it, and a *product* of each and of all that go before it; so that the whole series, from the first to the last, however long it may be, is linked together by a kind of proportion, one term being exactly three times, nine times, or some other number of times, greater, or less, than any of the others. And these, with several other interesting and very useful qualities, belong, not peculiarly to the series of terms that I have here laid before you, but to any other similar series, whatever their length, whatever the terms, or whatever the number by which, at each step, they may be increased.

11. Series of numbers increasing by geometrical progression afford problems similar to those of which we have spoken in the other mode of progression; that is to say, having any three of the properties of any series; as the first, the last, the middle term, or terms, the sum of the whole, the rate of increase, the number of terms; having any three of these properties, we can find the rest. But, again, these are not the important points at which I am aiming; the great matter, those points for which, almost alone, I have introduced this branch of arithmetic into my book, are now to be treated of.

12. On reverting to the geometrical series of terms before given, you see, that whilst 5 multiplied by 3

increases to 15, 135 multiplied, also, by 3, increases to 405. Now this 405, though so much more than 135, bears the same proportion, bears exactly the same proportion to this latter number, as does 15 to 5; that is to say, each of the larger terms, or numbers, is three times as large as its preceding term; and, of course, the contrary, or in more scientific phrase, the *converse* of this proposition is correct; that is, each of the smaller of these terms is one-third of that which immediately succeeds it; and the *proportion* between the respective terms subsists in this manner.

13. Now, a consequence of these relative proportions amongst numbers is this, that, if you have a number, for which you would find what may be called a *relation*, in *another number*, and if your wish is, that the relationship between the two numbers shall be of the same degree; that is, that it shall be equally near as the relationship subsisting between certain other two numbers, which you have already before you: a consequence of these proportions amongst numbers is this, that you are hereby enabled, with great ease, neatness, and truth, to find out such relation. And this matter, simple as it may yet appear, is the greatest ornament, and may, perhaps, be justly pronounced to be the most valuable application of the art of arithmetic. Of the application hereafter; at present we must direct our attention somewhat more to the nature of these proportions, and to the mode of discovering, and of forming them.

14. Each series of numbers, whether in arithmetical, or in geometrical progression, has the same properties as any other series increasing in like manner, so that it is immaterial what numbers we take as subjects for our experiments and observations. For the present we will take the geometrical series before given, and here repeated: that is,

5, 15, 45, 135, 405, 1215, 3645.

You observe, here are seven terms, of which 135 is the middle term. Now, amongst the properties of these numbers we find these: that the product of the middle term multiplied into itself, is equal to the product of the two extreme terms; equal, also, to that arising from the multiplication of the two next terms on either side of it; these are 45 and 405; and equal, also, to the product of the two terms next again to them, that is 15 + 1215; and, in short, to whatever length the series were extended, the same property would subsist amongst the terms; that is to say, that the middle term, multiplied into itself, would produce the same sum as would be produced by the multiplication of any two terms in the series, such terms being taken thus, in a sort of pairs, one on each side, at an equal distance from the middle term. And, further, be it observed, that as this is the case with every series, consisting of an *odd* number of terms, the same properties exist, and the same relationship also exists, in series that have an even number of terms, and in which, consequently, we find two in the middle: that is to say, the multiplication of these two terms produces, in like manner, a sum equal to that produced by any other two terms in the series, such two terms being taken on either hand, and at equal distance from the two middle terms.

15. The next property, on which I have to remark, in numbers of this description, is this; I have before stated, that any two adjoining terms, in any regular series, bear the same proportion towards each other, as do any other two terms, similarly situated; that is to say, turning again to the series on which we have been observing, 5 and 15 bear the same proportion towards each other, as do 15 and 45; as 45 and 135; as 135 and 405; and so on, through the series. Nor is this sort of relationship confined to terms immediately adjoining each other, but exists in like manner, between those taken at

similar distances from each other, whatever the distance may be: thus, if we take every other term, 5 and 45 bear the same proportion towards each other, as do 15 and 135; and as do 45 and 405, and so on; or if you take the terms at a yet greater distance from each other, as 15 and 405, which bear the same proportion towards each other as do 5 and 135; as 45 and 1215, and so on. And thus it is; and thus will it always be in every geometrical series of numbers.

16. Here I find it expedient to state, that numbers employed for these purposes have their proper names. Generally we call them *terms*, merely; when we take a *pair*, or, as they are called by some, a *couplet* of terms, and would distinguish the first term from the last, we call the *first* the *antecedent;* that is, the *before-going*, and that which *comes after* is called the *consequent;* thus, in the two couplets last cited, that is, 5 and 135, and 45 and 1215, the terms 5 and 45 are *antecedents*, and 135 and 1215 are to be called *consequent terms*. And be it further observed, that as these two couplets are said to be *proportionals*, as the latter is *compared* to that which goes before it, so this may be conveniently called the *antecedent proportional*, and the other the *consequent proportional*.

17. In all these cases on which we have been observing, you will not have failed to perceive, that the proportion subsisting between the corresponding terms is of this nature; that, whilst in every instance, the *antecedent* terms in each couplet bear the same proportion towards each other as do their respective *consequents*, so, also, in the terms stated, you will see, that the *consequents* are either three times, nine times, or twenty-seven times, as large as their respective *antecedents:* as, to turn our attention again to the two couplets last cited, whilst 135 is twenty-seven times as much as 5, so 1215 is, also, twenty-seven

times as much as 45. And this *twenty-seven* being the *rate* of increase, is called the RATE, or more learnedly, the RATIO, of the proportion subsisting between the two couplets of terms. Again, the proportion may be spoken of in the contrary mode, the terms being reversed, thus, 135 and 5 bear the same proportion towards each other as do 1215 to 45; or 1215 is to 45 as 135 is to 5; in which cases, 27 is the ratio of *decrease*. As to the ratio, whether of increase, or of decrease, it might be ever so large, or ever so small; it might be a million times, or only a millionth part as great; or it might be so much less; still, in proportional terms and couplets, all the properties of which I have spoken must and will exist.

18. To save the *words* which I have hitherto employed in stating these proportional numbers, and for the purpose, likewise, of presenting such statements more quickly, and more distinctly to the eye, arithmeticians are in the practice of employing a few points, which you will find it useful to be familiar with, and to which, therefore, I now call your attention.—On referring to the last paragraph, you find the following passage "135 and 5 bear the same proportion towards each other as do 1215 to 45; or 1215 is to 45 as 135 is to 5."—Now, by the use of the points which I am about to describe, all this is presented to the eye more readily, and quite as clearly in this manner.—
135 : 5 : : 1215 : 45; or, 1215 : 45 : : 135 : 5.

To explain the points:

: *The two points mean* "IS TO;" *or* "IS IN THE SAME PROPORTION."

: : *The four dots, or points, mean,* "AS," *or* "SO IS;" *or may be read in any other words descriptive of that sort of relationship which subsists between couplets of numbers that are proportionals.*

So the foregoing proportional terms, translating the dots into words, are read thus: 135 *is to* 5, *as* 1215 *is to* 45; and so on. And this mode of stating such propositions, with the dots, we must henceforth generally employ.

19. Hitherto I have spoken of these proportional terms, only as of a regularly-continued series; that is to say, as in the series given, beginning with 5, and continuing regularly to increase by a ratio of 3. Nothing, however, of this sort is required to form proportional numbers: numbers or terms may continue to increase, or to decrease, in that continued, and uninterrupted order, in which case they are called, *continued proportionals:* or they may be interrupted, or broken, thus, 15-45 : : 405-1215; or, 2-14 : : 21-147; in which case they are called, *interrupted proportionals.*

20. I have shown, in paragraphs 12, and 17, that proportion in numbers consists in the increase, or decrease, amongst them being at the same rate, or in the same ratio; and in paragraph 13, I stated, that the mode of discovering numbers, which shall bear certain required ratios towards each other, may be justly pronounced to be the most valuable application of the art of arithmetic. Let us, then, proceed attentively to the consideration of this mode, in order to become masters of a process of so much importance.

21. To make our experiments on the numbers, or terms, already cited; 15 is to 45 as 405 is to 1215. Now, what proportion does 15 bear towards 45? We remember, indeed, that the former term is one-third of the latter; but, suppose we knew nothing of this, how ought we to proceed in order to discover the proportion, or ratio, between the two numbers? The method is, to *divide* the larger by the smaller, and the *quotient* is the *ratio.* And, were it our

object to find a number that should bear the same proportion to some other number as 45 does to 15, the course we have to take is, to *multiply* that other number by the ratio, which is 3, and the product is the number sought: as, for instance, let the number for which we would find a similiar relation be 405; this, multiplied by 3, produces 1215: but, did we want a number in *decreasing* proportion; did we want a number which should bear the same proportion to 405 as 15 does to 45, then we *divide* the 405 by the ratio, and the *quotient*, 135, is the number we would discover. So, then, in the first case, above stated, 15 : 45 : : 405 : 1215; and, in the second, 45 : 15 : : 405 : 135.

22. These, however, as before stated, are not the only points of resemblance, or of proportion, subsisting between numbers of this description. In the above we have only shown, that the two consequent terms bear a like proportion to their two antecedents; as, that 45 : 15 : : 1215 : 405; whereas 45 is, also, to 1215, as 15 is to 405; that is to say, the two antecedents bear the same proportion towards each other, as do the two consequents. So you will find it to be in every case. But to prove it in this instance, divide the larger antecedent term by the smaller, thus, $1215 \div 45 = 27$; which 27 you will find, also, is the quotient, when the larger consequent is divided by the smaller.

23. Finding, as we thus easily do, the rate of increase, or of decrease, between any two numbers, and finding that this rate, or ratio, is, in proportional numbers, *the measure*, between the antecedent terms and their consequents, we shall, as you will find, have no difficulty in discovering any proportional numbers that we may require.

24. Now of the four terms, 15, 45, 405, and 1215,

suppose that we had only the three first, and that it were our wish to find the fourth, which term bears the same proportion to the third, as the second does to the first. The thing we have first to do is, to discover the ratio between the first and second terms; in order to which, as before shown, we divide the larger by the smaller, and this gives us the ratio 3, with which, by multiplying the third term, we produce the fourth; or, let the three terms be these, 405, 1215, 15; and let it be our wish to find a fourth, which shall bear the same relation to the 15 as 1215 does to 405. We divide and multiply as before, and the fourth term is produced. And in this manner, having *two* numbers, or *two* quantities of any kind, bearing a certain proportion towards each other, and *a third*, to which we would find a number or quantity that should bear a like proportion; in this manner do we proceed, and thus easily may we find the number we require; That is to say, thus may we proceed, when the smaller of the first and second terms will divide the larger without leaving a remainder, as in the cases we have thus far tried. But, observe, this is not always, nay, this is seldom the case; and it is never a thing to be calculated on. So that the proper mode is, to proceed in a method that will be clear, whatever the terms may be with which we have to work: and here is that manner of proceeding.

25. In our experiment on the three terms, 15, 45, and 405, in order to find a fourth, we first sought the ratio of the second to the first, and then produced the fourth term by multiplying the third by that ratio. That ratio was 3. But, instead of an *integer*, with which we can so readily multiply any number, suppose it were a mixed number, or a fraction, even a very minute and long fraction, as it might very well happen to be; suppose this were the case, how much more complex and difficult would be the process of multiplication!

26. To avoid this difficulty, to avoid the entanglement of a ratio of this sort, we *defer* the process of division, from which process only can the fraction arise, and instead of the *division*, we perform the *multiplication first*, and the *division afterwards* ; and the result is the same. For example, in this instance of the three terms 15, 45, and 405. It is the same thing if, instead of multiplying 405 by the ratio 3, which is one fifteenth part of 45, the same thing if we first multiply this term by the 45, and then, having thus made it fifteen times too large, divide it by this 15; that is to say, we have the same result if we multiply the second and third terms together, and divide the product by the first. AND THIS IS THE RULE ; this, when the terms are properly placed, this MULTIPLYING THE SECOND AND THE THIRD TERMS TOGETHER, AND DIVIDING THE PRODUCT BY THE FIRST, avoids all the difficulties arising from the occurrence of fractions in the course of the process, and gives us, in all cases, any proportional terms we may require. This is the RULE of PROPORTION, commonly called the RULE of THREE ; and, in their admiration of it, and in testimony of their sense of its great value, the learned of former times bestowed on it the name of GOLDEN RULE ; a title which on account of its extensive usefulness it richly merits.

OF FRACTIONS:

THEIR NATURE, MODE OF STATEMENT, &c. &c.

27. This word, FRACTION, like many other of the terms used in our art, comes from the Latin: and it is useful, occasionally, to attend to the derivation of a word, because it oftentimes not only gives us a clear insight into its meaning, but, also, fixes that meaning permanently in the mind. FRACTION, comes from the Latin word *fractus*, that is, broken; or, in other words, a part. And, as it is used to describe a part, merely, of any thing that may be the subject of consideration, so, if we be speaking of certain weights, as three ounces and a quarter; (that is, a quarter of an ounce,) this quarter, being but *a part*, is called a *fraction*: do we speak of seven pounds and a quarter, then, this "quarter" is also a *fraction*: only observe, that, meaning, as it would, a quarter of a pound, so it would be called a fraction of a pound, whilst the other means a fraction of an ounce.

28. Thus, PARTS of any thing, whether of weights, of measures, of money, or of periods of time; parts, whether large, or small; as halves, quarters, eighths, sixteenths; or, in short, any portion short of a whole, is a fraction; and the treatment, or the working of these parts in numbers, is called the working of fractions; whilst, in order to distinguish them from these FRACTIONS, the numbers of which we have heretofore treated, are called INTEGERS, or WHOLE NUMBERS.

29. Now, as to the working of fractions, every body knows, that four quarters make a whole; that three thirds, that two halves, that five fifths, or six sixths; every body knows that each of these make

a whole: and it requires very little more knowledge to enable us to say, that three halves make one and a half; that five quarters make one and a quarter; that seven quarters make one and three quarters; that nine quarters make two and a quarter; and so on. Well, but all these are examples, though very simple examples, certainly, of the ADDITION OF FRACTIONS.

30. To give a few examples of the SUBTRACTION of FRACTIONS: every one knows that if we take one half from three quarters, one quarter will remain; that if we take a half from five quarters, then three quarters remain; and it requires but very little more knowledge to say, that if one half be taken from five eighths, then one eighth will remain; a quarter from seven eighths, five eighths remain.

31. As to the MULTIPLICATION OF FRACTIONS, every one knows, that six halves make three wholes; and, that twelve quarters come to the same thing; that fifteen quarters make three and three quarters, and that seven thirds make two and a third.

32. For examples of Division; who does not know, that if we divide three quarters by two, we have, for each portion, three eighths; that, if we divide seven halves by two, that then we have, as the result, seven quarters, or one and three quarters; that nine tenths, divided by three, would give us three tenths, for each portion. And this is a simple exhibition of the DIVISION OF FRACTIONS.

33. To complete the view, I think it just now requisite to take of this subject of Fractions, attend to the mode of stating them, and to some other particulars respecting them, as here described.

34. FRACTIONS, as I have stated, are PARTS: and,

as I have likewise stated, parts of every size, as, one half, one third, one fourth, three fourths, four fifths, seven eighths, or, in short, any other conceivable quantity, either small or large, as one thousandth part; or, as nine hundred and ninety-nine such parts, each of these quantities is a Fraction. Now, then, as to the mode of statement; if you look attentively at the words in which the above fractions are described, you see that there are TWO TERMS employed in each of them; that is, *one*, and, *half*; then, *one*, and, *third*; then come, *three*, and, *fourths*; and so on. And just so are these several quantities expressed in figures; only with this slight addition, that a small line is drawn between the two figures; and, with this further observation, that the two figures be written, not after one another, thus, 1-3, but that they be written smaller than your other figures, and the first of them over the other, thus, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{1000}$, $\frac{990}{1000}$. And, in this manner may every arithmetical fraction be expressed.

35. There being, however, no figure called *half*, *one half* is expressed thus, $\frac{1}{2}$; or, a half being two quarters, it may, very correctly, be thus expressed $\frac{2}{4}$.

36. Some of the other Fractions recited above are thus written, three fourths, four fifths, seven eighths. Now, for example, to observe on the last of these, that is to say, on the $\frac{7}{8}$. Each of the figures by which it is expressed fulfils a certain office, and has its appropriate name: a quantity of any kind, as a yard, a pound, or an hour, may be divided into any number of parts: and in this Fraction, it is stated to be divided into eight parts; and the figure 8 *denominates*, that is to say *names* what is the value of the parts spoken of; it denominates these parts, and is, therefore, called the DENOMINATOR. And, now, what

number of these eighths are they that constitute the quantity described by this fraction ? The other figure tells us this; the figure 7 gives us this number, and it is, therefore, called the *numberer;* or, from the Latin, NUMERATOR. And, so these are the terms, these are the uses of the two figures employed to describe every fraction; that is to say, NUMERATOR, which describes the *number* of parts, and DENOMINATOR, which denotes *the quantity* of those parts.

37. Another point to be observed here, is that Arithmetical Fractions, speaking as they do of parts of any thing, are always understood to be speaking of EQUAL PARTS. But a thing may be divided into UNEQUAL and irregular parts; as, for instance, a pound weight may be broken into eight parts, each different from all the rest in quantity. And, if a person were to speak of three, five, or seven parts of this sort, we should have no certain, no clear comprehension of the quantity he might mean; and certainty, and clearness, are the very life and soul of Arithmetic. This clearness, this certainty is attained, by its being always intended, and always understood, that the parts spoken of in any fraction, are all equal, one with another.

38. As I have stated, a quantity may be divided into any conceivable number of parts; and it is the proper office of a Fraction to speak of some quantity less than the whole, even though it be but one thousandth, or one millionth part less. However, it is sometimes convenient to speak of things, and even to write them down otherwise; as, instead of saying, a penny halfpenny, we say three half-pennies. Drapers talk of their cloths being four or five quarters wide, of blankets, and counterpanes, and table cloths, being twelve, fourteen, fifteen, &c. quarters square; meaning, quarters of the yard. And this mode of speaking and of writing may be useful in other

affairs. Now, although we do not write down three half-pence thus, $\frac{3}{2}$, but do it thus, $1\frac{1}{2}$d., that is, one penny and a half, yet Drapers write the quantities we have spoken of thus, $\frac{4}{4}, \frac{5}{4}, \frac{12}{4}, \frac{14}{4}, \frac{15}{4}$. And this, as stated above, is more convenient to them, being, not only conformable to their mode of speaking, but also shorter than writing, as they otherwise would have to do, in the two latter cases, $3\frac{1}{2}$ yds. $3\frac{3}{4}$ yds.

39. On looking at the Fractions written down in the last paragraph; on looking at $\frac{4}{4}, \frac{5}{4}, \frac{12}{4}, \frac{14}{4}, \frac{15}{4}$, you again observe, that these express something *more* than a part, and all of them, except the first, something more than a whole, although, in the outset of the same paragraph, I have stated, that it is the proper office of a Fraction to speak of some quantity *less* than a whole. And so it is. And in conformity therewith, when a fraction is expressive of a whole quantity, or of anything more than a whole, which quantities it is frequently convenient to express in this manner, when a fraction thus expresses more than a part, it is called an IMPROPER FRACTION. So Fractions, properly speaking, are those numbers which describe less than a whole; and in which, therefore, the numerator is less than the denominator: whilst improper Fractions are those which describe as much as, or more than a whole; and in which, therefore, the numerator is equal to, or greater than the denominator.

40. One other particular remains to be observed on before we dismiss this article on the nature, &c. of Fractions; and this is, to impress on your mind, the propriety of confining yourself to the use of the smallest figures or terms, by which you can express the quantity you speak of. Three-sixths, for instance, are one half; six-eighths, are merely three quarters,

as are nine-twelfths, and fifteen-twentieths. Do you, therefore, always, in such cases, say and write down, *one-half* $\frac{1}{2}$, and *three-fourths* $\frac{3}{4}$; leaving it to others to talk about, and to write $\frac{3}{6}$ *three-sixths*, $\frac{6}{8}$ *six-eighths*, $\frac{9}{12}$ *nine-twelfths*, and so forth; leave this mode of speaking and of writing to people who talk of *three parts*, and so on, without telling us what is the size, what the *quantity* of the *parts* about which they are babbling.

It has, I find, become expedient to state, that these Articles on Fractions, both vulgar and decimal, together with that on Proportion, and the lesson on Duodecimals; it has here become expedient to state, that these Articles have not altogether been written for this work, otherwise I should not, in the examples and illustrations, have employed quantities in money, but rather quantities such as those ennumerated on the title of the Book; these Articles, with some alterations, are from a small work which I published some time ago, entitled "Lessons on Arithmetic." They contribute to the completion of my design in this work, and it would be a superfluous labour to make an alteration where no good could arise.

OF FRACTIONS,

THEIR ORIGIN, REDUCTION, ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION.

41. Fractions present themselves under various forms. They sometimes appear amongst the terms of a question or proposition; but in ordinary concerns of business, saving the simple fractions, halves, quarters, and eighths, they seldom present themselves except as remainders, in operations of division. To give a brief and simple instance; have we to divide £ 109. 10s. 2d. by 6; we have, for the quotient £ 18. 5s. 0¼d., and a remainder of two farthings yet undivided, the sixth of which we must annex to the quotient, before the division be complete. Now, how to do this, is a point in fractions; this remainder is but 2 *fourths*, and how shall we divide it by 6? The fact is, we cannot so divide it: but we can do something else, equally efficient; we can express 2 sixths of a farthing as we have just expressed the fourth of a penny, by writing it after the farthing thus $\frac{1}{4}\frac{2}{6}$. But, bear in mind, that although this may be done, it is not the proper method of stating such a matter, for we have now *two* of those broken quantities which arithmeticians very wisely seek to avoid; and, *two fractions*, with *different denominators*; and to complete the difficulty, one of these fractions is *a fraction of the other!* a thing never to be permitted to stand. A single fraction we cannot avoid, in such a case; but a single one is nothing; it is the *two*; and especially, the having *one* of them *a fraction of the other*, that would make an arithmetician start from the work. So, how to avoid it is the question.

Very easily! Having reduced the two-pence into farthings, and finding, that although you have it thus reduced, to the *lowest denomination*, your divisor will not *evenly* divide it, you express the quotient of 2 pence divided by 6, thus, $\frac{2}{6}$d. or, rather $\frac{1}{3}$d. and then you have, as the exact quotient £ 18. 5s. 0$\frac{1}{3}$d., which, like all quotients, if multiplied by the divisor, will reproduce the dividend.

42. This is saying a great deal about a trifle, it may be said. But I am not saying all this about the third of a penny; it is on the origin, and on the treatment of arithmetical fractions, that I am speaking. And, besides laying a solid foundation for more important matters, we have, in this trifling instance seen, not only how fractions commonly originate, and what is to be done with them in certain cases, but, likewise, what is never to be done with them in any case: a kind of learning which is never to be despised.

43. These fractions will occur. You can seldom divide any sum, except those that are fabricated or selected, for the purpose of being evenly divided, without these troublesome guests. They will intrude in reckonings in real business. And, as it is for real business, that I would prepare the reader, I must teach him to deal with these troublesome visitors, called fractions, which present themselves in various shapes; even the same value appearing under an endless variety of forms. As, for example, *a half* is expressed by any fraction, the *denominator* of which is *twice the* numerator; and a *third* is described by any fraction, the *denominator* of which is *thrice its numerator*, let these denominators and numerators be what they may.

44. Did fractions occur, only as halves, quarters,

fifths, eighths, and so on; did they appear with only one figure for a numerator, and one for a denominator, we should easily manage them; but as they arise from the operations of division, and as the remainder left on our hands after such an operation becomes the numerator of a fraction of which the divisor is the denominator; and as, in cases in which the divisor is a large number, the remainder may be large also, so we sometimes find ourselves encumbered with a fraction, a thing of small value, but described in two long lines of figures.

45. But the same value, as I have said, is often expressed by a short fraction, as by a long one; as, for example, $\frac{1}{5}$ describes as much as $\frac{2357}{11785}$: for in both these expressions, the numerator is *one fifth* of the denominator; and the quantity described by each is, consequently, a fifth, merely, of one whole. But, how much more convenient is the shorter, than the longer expression! Hence, in all cases, to employ the shorter expression, as inculcated in paragraph 40, is a settled law amongst arithmeticians.

46. Fractions are sometimes to be added together, and sometimes to be subtracted from each other. For instance, suppose that, after some operation in long division, the divisor being 181200, we find ourselves with a remainder of 144960; which gives us this fraction $\frac{144960}{181200}$; and then suppose also, that by another operation we have this fraction $\frac{6040}{30200}$, and that we have occasion, as may frequently be the case, to add these two fractions together; that is, in fact, to ascertain how much the two broken quantities expressed by these figures will amount to, when put together

47. Now, how to add these two fractions together,

is the question. To do so with them in their present form is impracticable. A description of them in *words*, would be more puzzling than are the figures, so let us, by way of illustration, use some fractions of a simpler form, but presenting a similar difficulty; let it be that we have to add together $\frac{4}{5}$ and $\frac{3}{15}$. Four fifths we could add to two fifths, or to three fifths, or in short, to any number of *fifths*, for it would but be the *adding* together of so many *fifths*. But to attempt to add *fifths* to *thirds*, to *fourths*, to *fifteenths*, or to *any other parts than fifths*, would be an incongruity; that is to say, to add *fractions* together of *different denominations*, is impracticable. Yet *quantities*, however different in size, may be added together, and the smaller may be subtracted from the larger; but how are the fractions, descriptive of these quantities, to be added or subtracted? Thus it is to be done. They are to be brought into *like denominations;* that is to say, *the parts*, of which two or more fractions speak, are to be brought, or *reduced, to the same size, or same value;* and then you find no incongruity in adding *those parts* together, as $\frac{1}{5}$ and $\frac{4}{5}$ make $\frac{5}{5}$ or one whole. Nor anything difficult in subtracting the smaller from the larger, and having $\frac{3}{5}$ as the difference. And yet these fractions $\frac{1}{5}$ and $\frac{4}{5}$ are in value, not only the same as $\frac{3}{15}$ and $\frac{4}{5}$, but the same precisely as the *two long* fractions stated in the last paragraph.

48. Thus, having fractions of *one and the same denomination*, we can work them together. In addition, we add the numerators together, retaining the denominator; which belonging in common to the two fractions, is called *the common denominator*. And in *subtraction*, also, as in whole numbers, we subtract the smaller number from the larger; that is, the smaller numerator from the larger, and to the

difference, we put the common denominator. But to bring fractions of different denominators thus to have a common denominator. This is called

REDUCTION OF FRACTIONS.

49. This is every thing, in these numbers. As to their addition and subtraction, you have just seen, that these are merely the addition and subtraction of *numerators*; and the multiplication and division of them are much the same; and so far as matters of business require, they are quite as simple. It is the *reduction* of fractions that is every thing; the reduction of them in *two* ways, and for two different purposes. *First*, there is the bringing of fractions of *different denominations* into the *same* denomination, in order, as you have seen, to prepare them for addition and subtraction: and *second*, there is the reduction of them, from long and inconvenient numbers or terms, to their shortest and most compact form.

50 A fraction expresses a part, or parts, of a whole; and its numerator always bears the same proportion to its denominator, as the part described by the fraction bears to the whole thing spoken of. This point needs no illustration here. The numerator, therefore, and the denominator of a fraction, bear a proportion towards each other. But the proportion of numbers is not altered, as shown in paragraphs 12 and 13, by their being multiplied, or divided, *provided* that they be multiplied, or divided, *by the same* number. As, if we multiply both the terms of the fraction, $\frac{1}{2}$, by 8, it remains unaltered in its value; it being then $\frac{8}{16}$; the numerator half of the denominator; so, if we divide these two terms by 4, we have $\frac{2}{4}$, which expresses the same thing. And thus it is with any fraction, whether the figures by which it is described be few or many. This point, of the value of a fraction being unaltered by the *equal* division or multipli-

cation of its numerator and denominator; this point being established we are led very easily to the two methods of *reducing fractions.*

51. And first, as to the reduction of fractions to the *same denomination.* Let it be that we would thus reduce $\frac{1}{2}$ and $\frac{2}{6}$. Beginning with the first of them, let us multiply both its terms by 6; we have $\frac{6}{12}$; and now let us take the other, that is the $\frac{2}{6}$ and multiply both its terms by 2; we have $\frac{4}{12}$. The fractions are now both of *one denomination*, and their value is unaltered. And how has this been accomplished? Look at the terms before, and after they were reduced, and you will see that it has been accomplished by multiplying both the terms of each fraction by the *denominator of its neighbour.* And thus it is, that any two fractions, without altering their value, are to be reduced to the same denomination; that is to say, by multiplying the two terms, that is, the numerator and denominator of each fraction, by the denominator of the other.

$$\frac{1}{2}\ \frac{2}{6} : \frac{6}{12}\ \frac{4}{12}$$

52. Nor is it with *two* fractions merely that this method is to be pursued. The same treatment will reduce any number of fractions to a common denomination; as, for example, in order to keep the matter as simple as possible, let us add one other small fraction to the former; let us take these three to be reduced to a common denominator $\frac{1}{2}\ \frac{2}{6}\ \frac{3}{5}$.

Now this is the method of stating such an operation with clearness. And, when you have the fractions thus reduced, if you have done it for the purpose of adding them together, which is almost the only purpose for which you can have thus to reduce *three* or more fractions, you state them as you see here below; the numerators in a line, with the sign of addition between them, and below the line, and about the middle, you

For the new denominator.

$$2 \times 6 \times 5 = 60$$

For the numerators.

$$1 \times 6 \times 5 = 30$$
$$2 \times 2 \times 5 = 20$$
$$3 \times 2 \times 6 = 36$$

$$\frac{30 + 20 + 36}{60} = \frac{86}{60}$$

write the *common denominator*, *once only*, however many the fractions may be. And then if you would bring the whole into *one* fraction, it is done, as you see, by adding the line of numerators together, and by writing the *common* denominator, in the proper form, under the sum of them.

53. As to the PRINCIPLE of this reduction, it has already been adverted to; but to state it more explicitly. For a common denominator, we *multiply all the denominators* together; that is, we multiply the denominator of the *first* fraction, by those of the *second* and *third*; and were there yet more, we should go on, thus multiplying them. Having done this, having thus multiplied the denominator by certain numbers, *in order to keep the value of the fraction unaltered*, we multiply the numerator by *the same* numbers. Now these numbers are, *the denominators of the other fractions*; and thus it is, that multiplying all the denominators together for a common denominator, and the numerator *of each* fraction by the denominators *of the others*, brings any number of fractions into *one*, or *common*, denomination, *without altering their value.*

PROPOSITIONS, &c.

(1) Reduce $\frac{7}{8}$ $\frac{2}{5}$ and $\frac{4}{7}$ to one denomination.

(2) Reduce $\frac{12}{17}$ $\frac{3}{5}$ and $\frac{2}{9}$ to a common denomination; and state the sum of them.

(3) What is the sum of $\frac{13}{15}$ $\frac{7}{9}$ $\frac{3}{8}$ and $\frac{2}{5}$?

You will, of course, first reduce these fractions to a common denominator; and then adding the numerators together, will place the sum of them, in due form, over that denominator.

(4) Add these fractions together, $\frac{135}{271}$ $\frac{72}{90}$ $\frac{31}{43}$.

(5) Reduce $\frac{23}{30}$ and $\frac{3}{8}$, and subtract the smaller from the larger.

(6) What is the difference between $\frac{17}{20}$ and $\frac{43}{97}$?

(7) Subtract $\frac{25}{62}$ from $\frac{19}{20}$.

(8) Subtract $\frac{33}{70}$ from $\frac{33}{47}$.

54. As for the *reduction* of fractions, from long and inconvenient terms, such as you will find on your hands after each of the foregoing operations, it is done thus. We have fully established the principle, that the division of the terms of a fraction, *provided that both terms be divided by the same number*, makes no alteration in its value. On this principle, then, it is, that the terms of fractions are reduced. We divide those terms, when they will *evenly* divide, until we bring them to their *shortest numbers.* Some terms, or numbers, will not divide by any other number; such terms are called *primary*, or *primes.* And, when both the terms of a fraction, or indeed, when *the larger term* of a fraction is a *prime number*, there is no reducing that fraction. For instance, here is a fraction, the *larger* term of which is of this description $\frac{32}{223}$. As for the smaller of its terms, no number whatever will divide more easily; but because the larger will not divide, this fraction cannot be reduced: But make this stubborn number the *smaller term* of a fraction, and let it be one of the *submultiples* of the larger term, and then no fraction will reduce better. For let the fraction be $\frac{223}{669}$; and seeing that the smaller of these terms is *one third* of the larger, $\frac{1}{3}$ is the proper representative of this long fraction.

55. When the numerator of a fraction will thus evenly divide its denominator, we deem it fortunate; for it brings the denominator at once to its lowest possible term, and makes the numerator only a single unit; $\frac{3}{27}$ so reduced, is $\frac{1}{9}$. And were the terms

inverted thus, $\frac{27}{3}$, that is, were it an *improper fraction*, dividing the larger term by the smaller gives us $\frac{9}{1}$, or rather 9; $\frac{27}{135}$ becomes $\frac{1}{5}$, and $\frac{2781}{11124}=\frac{1}{4}$.

56. In fractions such as the shorter of these, we can see at a glance, what numbers will divide them; and a little consideration will sometimes enable us to discover, almost without trial, what number will divide the terms of a long fraction. But it would not do to rely thus on our sagacity; we must have a *rule;* and, accordingly, we have one, by which, *to a certainty*, we can find the best, that is *the largest* divisor for the two terms of a fraction: or find, also to a certainty, if the terms cannot be divided. *The rule* is this; *Divide the larger term by the smaller*, and if it leave a remainder, bring down the smaller term, that is the last divisor, and divide it by the remainder; and if, again, this leave a remainder, bring down the former remainder, that is, again, the last divisor, and divide it with the last remainder; and so you proceed, *dividing the larger term by the smaller, and the last divisor, by the the last remainder, until you have no remainder left;* (for to this it will come) *and your last divisor, which is also your last remainder, will divide both terms of your fraction:* that is to say, it will divide them if it be a number that has the power to divide: but if the process bring you down to a *single unit*, which has no power of division in it, then you must conclude that your fraction stands already in its shortest terms. A few examples will be useful, therefore I annex them.

(1) Reduce $\frac{3781}{15124}$ to its lowest terms.

```
3781 ) 15124 ( 4
       15124
       .....
```

Thus the numerator is found 4 times in the denominator; that is to say, the numerator is only one fourth of the denominator, $\frac{1}{4}$, therefore, is the reduced fraction.

(2) Reduce $\frac{36240}{181200}$ to its lowest terms.

$$36240)\,181200\,(\tfrac{1}{5}$$
$$\underline{181200}$$
$$\cdots\cdots$$

Or it may be stated as follows, and then the meaning is somewhat more obvious.

$$36240\left\{\begin{array}{l})\ \underline{36240}\ (1\\ )\ 181200\ (5\end{array}\right.$$

(3) Reduce $\frac{3624}{19932}$ to its lowest terms.

$$\begin{array}{l}3624)\ 19932\ (5\\ \quad\ \underline{18120}\\ \qquad 1812)\ 3624\ (2\\ \qquad\qquad\ \underline{3624}\\ \qquad\qquad\ \cdots\end{array}$$

$$1812\left\{\begin{array}{l})\ \underline{3624}\ (\,\underline{2}\\ )\ 19932\ (11\end{array}\right.$$
$$\begin{array}{l}\underline{1812}\\ \ 1812\\ \ \underline{1812}\\ \ \cdots\end{array}$$

In this third example we have, after the first division, a remainder of 1812, with which, according to THE RULE, as laid down in the last paragraph, we had to divide the former divisor. Having done this, and having no remainder, this 1812 stands our *last* remainder and *last* divisor; and it will, therefore, according to the Rule, divide both terms of the fraction, and reduce them to their smallest expression. As here shown.

57. With regard to the PRINCIPLE on which this *common divisor for two numbers* is discovered, it is this. Taking the *third* of the foregoing cases for our remarks: In 19932 we find, that 3624 is contained 5 times, with a remainder of 1812. This *remainder* we afterwards find is *half* of the *divisor*; but that divisor was found 5 times in the larger number and this remainder over: then *this remainder*, which proves to be half of the divisor, will be found *ten times* in the large number, *and once over*; ten and one are eleven; therefore this remainder is to be found *eleven times in the larger number*; and being the half, that is, being found twice in the smaller term, without leaving a remainder, it is a common divisor for both terms, and applied to those terms, it reduces the fraction as you here see repeated.

$$1812\left\{\begin{array}{l})\ \underline{3624}\ (\,\underline{2}\\ )\ 19932\ (11\end{array}\right.$$

EXAMPLES CONTINUED.

(4) Reduce $\frac{3624}{20838}$ to its lowest terms.

```
3624 ) 20838 ( 5                          906 ) 3624 ( 4
       18120                                  ) 20838 ( 23
        2718 ) 3624 ( 1                         1812
               2718                             2718
                906 ) 2718 ( 3                  2718
                      2718                      ....
                      ....
```

(5) Reduce $\frac{3626}{19942}$ to its lowest terms.

```
3626 ) 19942 ( 5                      2 ) 3626  ( 1813
       18130                            ) 19942 ( 9971
        1812 ) 3626 ( 2
               3624
               ...2 ) 1812
                       906
```

Here, our last remainder is so small as *two*. This, therefore, is the only number with which the terms of this fraction can be divided. And this example shows, that it is desirable to have a *large* remainder, wherewith to reduce the fraction.

(6) Reduce $\frac{3625}{19937}$ to its smallest terms.

```
3625 ) 19937 ( 5
       18125
        1812 ) 3625 ( 2
               3624
               ...1
```

Here our last remainder is *one*. In the example just before we saw that it is desirable to have a *large* remainder, seeing that 2 is of so little efficacy in reducing the terms of a long fraction. But here we have only *one* for a remainder, a number of no efficacy whatever. This *one*, as stated in THE RULE, is an assurance to us, that there is no number by which both terms of this fraction can be divided; an assurance, therefore, that the fraction is already in its lowest terms.

58. On the MULTIPLICATION, and the division OF FRACTIONS I have to add a few words. The *proportion* which the numerator of a fraction bears to the denominator being the index of its value, is it not manifest that to *alter* that proportion, will *alter* the value of the fraction? Now a fraction expressive of a *half*, is any fraction the numerator of

which is *half* the denominator. Well, then, suppose we have to multiply such a fraction by any number, say by 3; is it not plain that this multiplication, that is to say, this *increase of the value* of the fraction will be effected, either by *multiplying* the *numerator*, or by dividing the denominator by the 3; for either of these *so alter the proportion* between the terms of the fraction, as to make it express three times its former value? And Division of Fractions is, of course, merely the *reverse* of this. *Divide* the *numerator*, or *multiply* the *denominator*, and the operation is effected.

59. Yes; we can multiply fractions, and we can divide fractions. And, indeed, he who has not frequent occasion to do both of these must have but little to do with arithmetic. But, strange to say, Mathematicians, as their books show, men of the greatest eminence in the science, have taken a fancy to multiply, as they call it, and to divide, BY FRACTIONS!

60. You cannot multiply by a unit, nor divide by a unit; and this you cannot do because it is so small, so powerless. How, then, can you multiply, and divide by something yet smaller than a unit; how do these things by a fraction!

61. Am I referred to the cases in which, in duodecimals, the square foot appears to be multiplied by its fractional parts of inches, twelfths of inches, and so on; am I referred to these? And then, again, to the Involution of numbers, the raising of powers, in which fractions appear to be employed as multipliers,—am I referred to these cases? There it is that the fallacy lurks; and it is high time to dislodge it. The explosion of this long-prevalent fallacy, as a mere occasion of exercising the reason, is well worthy of the time it will require.

F

62. Multiplication is a *repetition*, of a number, or of a quantity of some description. Augmentation, increase of quantity, is likewise required; but there must be a repetition, or there is no multiplication. Hence *one*, which is no repetition, is no multiplication; and the unit one is no multiplier. Two signifies repetition. ONE AND A HALF? The half is not a *repetition*, and therefore there is no power of multiplication in it; nor is there such power in any other fraction. Let it, for instance, be, that we would multiply fifteen yards, fifteen feet, fifteen pounds, or fifteen of any article, by 1½. We may indeed, as is done here in the margin, make a pretence of multiplying by the 1½. But no man does it so. Nor is it, as is abundantly obvious, anything save an *addition* of the fifteen halves, as shown in the second statement.

15
1½
7½
15
22½

15
7½
22½

63. It is the same in all other cases of increase, or augmentation by fractions combined with whole numbers. There is nothing of the nature of multiplication in it. It is whole numbers only, that have the power of multiplication. And there the line is drawn. INTEGERS have power; FRACTIONS have none in the way of multiplication.

15
15½
7½
75
15
232½

64. Still, am I referred to the cases in which fractions of a foot are employed as multipliers? How consistent, how harmonious, how powerful is truth; and how triumphant the answer! So far from having the power of producing a repetition, an augmentation, a multiplication; so far are fractions from possessing this power, that feet multiplied—as it is found convenient to term the process—feet multiplied by inches become inches; feet multiplied by parts become parts; and multiplied by twelfths of parts, twelfths of parts do feet become. But, although, as I have said, we find it *convenient* thus to "term the process" of multiplication, where parts or fractions are concerned, it is not in these, more than in other cases, a multiplication BY *the parts*, but in reality, a multiplication OF the parts *by the larger or integral* denomination. There are the length and the breadth of some superfice to be multiplied together, these two dimensions may each contain feet, inches, and parts: For the convenience of the operation we place, as in other cases of multiplication, the figures descriptive of one of these dimensions under those which describe the other, and so proceed in the operation. But in this, as in all cases of multiplication, it is a mere matter of convenience to call this figure or that the multiplier; the fact is, the larger denomination is the multiplier, whether it stand above or below; and this is demonstrated in the five paragraphs preceding 117, in which individual paragraph the whole process of the matter is summed up.

65. Again, however, am I referred to the Involution of numbers, as furnishing instances of multiplication by fractions? Here, again, how complete is the answer. Look at several instances of such involution, commencing with paragraph 155, throughout which paragraphs the operation, in fractions, both vulgar and decimal, is duly investigated, and made palpable to the sight, as well as to the understanding; and see, if you please, paragraph 173, with which this discussion is concluded.

66. "Yes," as I said in the outset of this enquiry, "we can multiply fractions, and we can divide fractions." But it is pure nonsense to talk of multiplying or of dividing BY or WITH fractions: for, as to the question of *dividing* by fractions, it is now unnecessary to enter into that, seeing that division is a mere inversion of multiplication, and that the

explosion of the erroneous notions as to one of the processes is the explosion of the other.

67. Erroneous such notions are; but not merely erroneous. They are pernicious, as is all error; and most pernicious when it presents itself in the imposing garb of science; and in this, too, of mathematical, the most imperious in its authority of all the sciences! I must not attempt to enumerate the authors who have countenanced this error, but I believe it has been taught by all the writers on Arithmetic, from Leonard Euler down to Dr. Dionisius Lardner, and to Augustus de Morgan of the London University. Suffice, however, for our purpose, the first of these authors, so celebrated for his researches, and his luminous expositions of mathematical science. This distinguished man, whose works are pronounced by his English biographer to have been "the admiration and glory of Europe," Euler, in the English edition of 1797 of his work on Algebra, has the following passage, paragraph 111. "The number 100 divided by $\frac{1}{2}$ will give 200."—200 what? the sensible reader will ask. But he tells us not that they are 200 *halves*. On the contrary, we are left to receive them as 200 integers. And then, in continuation, Euler proceeds; "and 1000 divided by $\frac{1}{3}$ will give 3000. Further, if it be required to divide 1 by $\frac{1}{1000}$, the quotient would be 1000; and dividing 1 by $\frac{1}{100000}$, the quotient is 100000." And then he gravely adds, "This enables us to conceive that, when any number is divided by 0, the result must be a number infinitely great; for even the division of 1 by the small fraction of $\frac{1}{1000000000}$, gives for the quotient 1000000000."—The mind must sink, must become impaired and disordered when, abandoning the use of its senses, abandoning its healthful exercise in the contemplation of the realities around us, it devotes itself to the futile attempt to "conceive" such baseless, such phantom, such nonsensical speculations.

68. Need I say a word in justification of this somewhat elaborate enquiry into a point, in some respects of minute importance? There has been the wholesome exercise of the mind in the course of the enquiry; and the result is a clear and determinate idea of the nature and of the process of working in fractional quantities.

OF DECIMALS:

THEIR NATURE, USES, AND MODE OF STATEMENT.

69. These also are Fractions, or parts of whole numbers. And the difference between these and the fractions treated of in the foregoing lesson consists in this; that, whereas those express parts, as halves, thirds, fourth, fifths, and so on, in *figures* closely conformable to the *words* by which we describe the same parts, Decimals always describe the parts of which they treat, as *tenths*, *hundredths*, *thousandths*, and so on; that is, indeed, as TENTHS: and, hence their name, which comes from the latin *decem*, ten; or from *decimare*, to divide or separate into tenths. Again; the Fractions treated of before, being noted down in figures so conformable to the words by which they are ordinarily described, and being, therefore, more readily applicable to ordinary affairs, they, when we would distinguish them from the fractions of which we are now to treat, are called **VULGAR FRACTIONS**; whilst these, for the purpose of distinction, are called **DECIMAL FRACTIONS**.

70. Decimal Fractions; or, as we will now call them, Decimals, admit, in most cases, of a much simpler and readier mode of statement, and of a much simpler and readier mode of being worked, than do vulgar fractions: and they are, in such cases, on these accounts, to be preferred.

71. The purposes, moreover, for which they are more particularly useful, are those nicer, and more extensive processes of calculation, to which the astronomer, the geographer, the engineer, the surveyor, and the chemist, have frequent occasion to resort. The notice, which, in this work, I shall think proper to take of Decimals, will serve to prepare persons for any of these professions; will serve the man of business, who may occasionally employ them in his reckonings; and will be further useful, as a means of explaining some few points on other matters, of which I shall hereafter treat.

72. With regard to the greater simplicity of statement, of which I have spoken above, it consists in this, that whilst, in the notation of vulgar fractions, we have to write down two lines of figures, the one descriptive of a numerator, and the other of a denominator, in the notation of a decimal it is sufficient to write down the numerator only, the *parts* enumerated thereby being, as I have said, always *tenths*, or *ten tenths*, or *ten times ten tenths*, or so on. In short, the denominator of a decimal being always *understood*, it is unnecessary to write it down; the value being expressed by the numerator alone.

73. There is yet, however, to be noticed, something in the situation, and in the manner of writing down the figures expressive of a decimal; and I go into the notice of these particulars with great pleasure, because they serve to exemplify still further, the nice order, and the great beauty of the science of numbers.

74. In the notation of whole numbers, as the learner knows, the figures placed next to the right hand represent units, those next before them representing tens, and so proceeding, they rise in value, ten-fold at every step: and, now, mark the neatness and beauty with which PARTS of these numbers are expressed by decimals. Take a figure, or line of figures descriptive of whole numbers, make a point, a full stop, after the figure which occupies the unit's place, and after this point, that is, lower down even than units, in the scale of places, write down the numerator of your fraction; that is to say, so write it down if it be a decimal; which is a fraction, the denominator of which is ten, or some mutiple of ten. And the numerator so written below the unit's place, however large and numerous the figures of which it may be composed, represents something less than a whole; it represents merely a fraction; and it represents this decimally.

75. For example. Let the number, with its fraction expressed in the ordinary mode be this $2517 \frac{7}{10}$; the same is expressed thus, 2517.7, the fraction being stated as a decimal. Where the fraction $\frac{17}{100}$, it would be thus, 2517.17. Where it $\frac{176}{1000}$, it would stand thus, 2517.176. And as the value of any decimal figure is determined by its approach to, or its distance from, the place of units, so ciphers placed after the units, and in the higher places of decimals, and thereby driving the significant figures lower down, ciphers so placed diminish the value of such significant figures. As, to recur, for instance, to the first of the foregoing examples, had the decimal .7 stood thus .07, instead of *seven tenths* $\frac{7}{10}$, it would have been only *seven hundredths*, $\frac{7}{100}$. And the second example, instead of being *seventeen hundredths*, would, if written thus, with a cipher before

it, .017, be reduced to only *seventeen thousandths*; for so these decimals are called: and, when written as vulgar fractions, they are thus written $\frac{17}{1000}$. However, and do you be careful to mark it; important as the cipher is in decimals, when placed *above* any of the significant figures, it would be wholly unimportant were it to occur at the end of, or *below* such figures. Between such figures it fulfils its office, and has its effect, as .107, which is $\frac{107}{1000}$; but below them, thus .170, it would signify nothing, but be a mere incumbrance, and ought not to be suffered to remain.

76. As you may thus turn certain vulgar fractions into decimals, by simply discarding the denominator, and writing the numerator after, or below the unit's place, so the contrary change is to be effected, and decimals may be turned into vulgar fractions, by drawing a line underneath the figures thereof, and under this, writing the requisite denominator; which denominator, as before stated, must always be 10, or some multiple of 10; and as to which of these it is to be, that is determined, in the manner which I am now about to describe.

77. A fraction, you know, is a part; and properly speaking, a fraction is always but a part; that is, something short of a whole. To describe such a part you were shown, in paragraph 39, that the denominator must always be something more than the numerator. Now, on converting a decimal into a vulgar fraction, you have to affix a suitable denominator thereto, and if you be confined, as you must be, in this case, to the use of *ten*, or of some *multiple of ten*, you affix the proper denominator at once, by writing a unit, followed by just as many ciphers as there are figures in the numerator. This is done in the foregoing examples, in which the

decimal .7, is expressed $\frac{7}{10}$; .07, thus $\frac{7}{100}$; the decimal .17, thus $\frac{17}{100}$; and .107 and .176, thus, $\frac{107}{1000}$, and $\frac{176}{1000}$. AND THIS IS THE RULE to be followed in affixing the denominator to a decimal; that is to say, you annex to a unit the same number of ciphers as there are figures in the decimal.

78. Now, as to the value of a decimal. Turning to paragraph 74, you are reminded, that in the notation of whole numbers, at each step which any figure is advanced *above* the unit's place, its value becomes increased tenfold; and in neat accordance with this principle do figures, as they take their stations *below* this place, decrease at each step, to just one tenth of their former value. Hence it is that .7 is seven *tenths*; .17 are seventeen *hundredths*; .107 a hundred and seven *thousandths*, and so on; the value of each number being only one tenth, one hundredth, or one thousandth of what it would be, were the figure describing it written in the unit's place.

79. In paragraph 69, having reminded you that vulgar fractions describe all manner of parts, in figures conformable to the words by which they are ordinarily described, I proceed to state that decimals always describe such parts as tenths, and so on. And thus are halves, quarters, thirds, and in short, all manner of parts described by decimals. The figure 5, written after what is termed the decimal point, thus .5, expresses a half; for written as a vulgar fraction it stands thus, $\frac{5}{10}$. A quarter of *ten*, which is two and a half, we cannot express in figures, otherwise than thus, $2\frac{1}{2}$, which will not serve for a decimal, but a quarter of a hundred, that is 25, being a whole number, will serve the purpose; so, be it a quarter of what it may, a quarter, in decimals, is thus expressed .25; that is, twenty-five hundredths.

Three quarters, of course, then, are to be expressed .75: and if you describe these as vulgar fractions, they stand thus $\frac{25}{100}$, $\frac{75}{100}$. And here, be it observed, is shown THE REASON OF THE RULE laid down towards the close of paragraph 77; which rule is this, that on changing a decimal into a vulgar fraction, the denominator shall be a unit, followed by just as many ciphers as there are figures in the decimal.

80. Halves, and quarters, as expressed by decimals, we have thus described; and now for other parts. A tenth is, of course, expressed thus .1; a fifth, thus .2; the first of these being a tenth, and the other a fifth part of ten. A twentieth, thus .05; a fiftieth, thus .02: for of these, the first is the twentieth, and the latter the fiftieth, part of a hundred.

81. But, how may we express a third, a sixth, a seventh, an eighth, or a ninth? These are questions of moment. The fact is, that as neither 10, nor any of its multiples can he evenly and completely divided by 3, 6, 7, or 9, so neither can these parts be *exactly* expressed by decimal figures. However, though they cannot be expressed with *perfect exactness*, we can come within a thousandth part, a millionth, or in short, we can make as near an approach to the exact quantity as we may please to make. And thus it is done. Suppose we would have the decimal for $\frac{1}{3}$, we take the 1, that is the numerator of the fraction, and annexing thereto one or more ciphers, as 10, 100, 1000, 10000, then divide by the denominator 3, and the nearer we would come to the exact third, the more ciphers do we annex. Now, you know, that when you divide 10 by 3, you have a remainder of one; thus you come at once to the third within one tenth; add another cipher, making the numerator 100, and divide again, and again have you one left; but this One is only a

hundredth; another cipher would bring you within a thousandth part of the perfect third; and thus may you make as near an approach as you please to the quantity sought. Now let it be that we would come within a ten thousandth part: annexing four ciphers to the numerator, and dividing by the 3, it would stand thus, and the quotient, consisting of four threes, is the decimal expression of a third; but, indeed, two figures are generally deemed sufficient for the expression of any decimal.

$$3\,)\,\underline{10000}$$
$$\quad 3333$$

The decimal for a ninth, is a line of units; that for a sixth, is a unit, followed by one or more of the figure 6; a seventh, if you would come within a millionth part, you will find to be .142857. But, as I have stated, two decimal figures are generally deemed sufficient for all practical purposes; so .14 expresses a seventh.

82. Now the decimals of which I have thus far spoken are such as, when expressed as vulgar fractions, have for their numerator a unit merely; and those spoken of in the last paragraph, as a third, a sixth, a seventh, and a ninth, have also this feature, that divide the numerator as often as you please the same figure recurs. To go on thus dividing, coming again and again to the same point, is like travelling in a circle, and from this circumstance it is, that decimals of this description are called CIRCULATING, or RECURRING decimals: of distinctions of this sort it is not necessary to speak further.

83. With regard, however, to Fractions, the numerators of which are other than units, you arrive at the decimal expression of them by just the same process as that which I have described in paragraph 81; that is, you take the numerator of the fraction to be turned into a decimal for a dividend, and adding thereto as many ciphers as you choose, you divide by the denominator, and the quotient will be the

decimal. As for example: Let the decimal expressive of $\frac{3}{8}$ be required. It is done thus. Is it required to find the decimal of $\frac{5}{9}$? Thus it is done.—

$$8\,)\,\underline{3000}\quad .375$$

$$9\,)\,\underline{5000}\quad .555$$

Again: suppose that we would have the decimal expresssive of $\frac{7}{18}$. Instead of working by Long Division, the better mode will be to divide by some submultiples or factors of the denominator 18; and this, in other cases, where it can be done, will be the better mode. Let us divide, then, by the sub-multiples 6 and 3; and it will stand thus.—And here it may be worth our observation, that whilst, in the first of these three examples, we come, after three divisions only, to the perfect decimal expression sought for, there being no remainder after those three divisions, the latter eludes our search, by presenting a continued repetition of the same remainder. But this is a matter of little, or of no practical importance: we have already arrived at something less than one eight hundredth part of the exact quantity, and were it the fraction of a hundred weight that we were thus calculating, we are within about two ounces of the precise quantity; and were this not near enough, we come within a dram and a half by only one division more.

$$6\,)\,\underline{70000}$$
$$3\,)\,\underline{11666}$$
$$.3888$$

84. One other particular relating to decimals remains here to be noticed; and this, whilst it will yet further exemplify their nature, shows the ready usefulness with which they present themselves, on certain occasions.

85. The learner already knows, that when we have to divide a sum by 10, by 100, or in short, by any multiple of 10, the operation is at once effected by a simple cutting-off, on the right hand of the dividend, of so many figures as there are ciphers in

the divisor; as an example, let us take 628513, from which, when you divide by 10, you cut off the 3; when by 100, you cut off the 13; and had you to divide by 1000, you would have to cut off the three last figures, thus 628·513. Now, observe, these figures, so cut off, are decimals; we are dividing by a thousand, we find in the dividend 628 thousand, and a remainder of 513; that is to say, a five hundred and thirteenth part of a thousand; and, written as a vulgar fraction, this remainder would stand thus, $\frac{513}{1000}$.

86. Decimals, of course, like other numbers, may be added together, subtracted from each other, multiplied, and divided; and it is the ease and readiness with which, by the use of these numbers, parts of all sorts can be reckoned, that constitute the great value of decimals. A description of these processes will be but a light matter.

OF DECIMALS:

OF THEIR ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION.

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87. With any taste for order and simplicity, and exactness of statement, it is impossible to think of this mode of expressing, and of working fractional numbers without pleasure. And on sitting down to treat of the working of decimals, and to conclude what remains to be said on the subject, I abstain with difficulty from launching out into a new and eulo-

gistic description of these numbers. Our next step will be their Addition, Subtraction, Multiplication, and Division

88. With regard to these operations. They are all of them performed precisely the same as in whole numbers, except that in stating the lines of figures for addition, and for subtraction, we are to range them, not as in whole numbers, the last *figures* on the right hand, in each line, right over each other; but the decimal points, being the index of value, the figures are to be very carefully written so that these points all fall *right under each other*; as you will see that they do in each of the following examples.

89. Adding up these figures in the usual manner, you see that those in the column of the highest value amount to 17; that is, *seventeen tenths* of a unit. Now, *seventeen tenths* are *ten tenths* and *seven tenths*; *ten tenths* are *one whole*; and as such, it is set down above the decimal point. And thus it is with these numbers. You have no *improper fractions*. As soon as, by addition or multiplication, you have parts *sufficient* to make a whole number, those parts, without any of the troublesome reduction required in vulgar fractions, quietly take their station on the other side of the decimal point, as whole numbers; and are thus in readiness, either to remain, or for any further operation. In this second example you see the *whole* number arising from the addition of the decimals carried on, and added to the integers. In short it is altogether simple addition, only that we have the decimal point to be preserved in its proper place.

.99
.753
.014
1.757

7.325
19.753
12.39
39.468

90. As to SUBTRACTION; it is nothing! You write the numbers to be subtracted, underneath those from which they are to be taken, taking care to keep the

decimal points right under each other; and then work as in simple subtractiion. See two or three examples.

.835	15.327	1732.1693
.619	9.74315	598.782516
.216	5.58385	1133.386784

91. In MULTIPLICATION we pay no regard to this matter, of ranging the decimal *point* of the multiplier *under* that of the multiplicand, but write these two terms under each other, as in simple multiplication; and as you see them stand in the annexed example. Thus multiplying, we find the product as in whole numbers. Which, having done, we count the *number* of *decimals* that there are in *both* the *multiplicand* and *multiplier*, and then, this number, of both together, we mark off for decimals in the product, and the figures to the left of the point are of course integers.

```
      75.5943
        32.16
    ---------
      4535658
      755943
     1511886
    2267829
  -----------
  2431.112688
```

92. However, there may be no integers. If there be none in the multiplicand nor none in the multiplier, there will be none in the product. Take, for example, the decimal quantities, without the integers, employed in the foregoing instance, and work them alone; And what is the result? In this result we have only *five* figures; whilst, in the multiplicand and the multiplier, together, there are *six* figures. How, then, are we to fulfil the direction in our rule, which direction requires us to mark off, for decimals, after the process of multiplication, the same number of figures in the product, as there are in *both* the multiplicand and the multiplier; how, in a case of this kind, are we to fulfil this direction? The rule is, to *affix a cipher or ciphers*, sufficient to

```
  .5943
    .16
  -----
  35658
  5943
 ------
.095088
```

make up the number of figures, and then to set the decimal point. But, to *which end* of the line of decimal figures is this addition of cipher or ciphers to be made? This is a question of moment. And it is thus determined. Ciphers following, that is standing to the right-hand of decimals, have no meaning, are mere incumbrances, as is shown in paragraph 75; these additional ciphers, therefore, as they have an office to fulfil, must of necessity take their place on the *left-hand;* thus lowering the value of the decimal product; as is shown in the example.

93. But this multiplication of one decimal by another decimal; this multiplication, as we must needs at present call it, of mere fractions, whether vulgar or decimal, is a mere trifling with figures, having no application to any end whatever, being of no use whatever; and being, in fact, as, in paragraphs 59 to 68, I have proved, not a multiplication; this mere playing with figures has no appropriate name; because indeed, that the process is useless, and deserves no name.

94. It is true, however, that in the Involution of numbers, there may be use, and there is meaning, in the multiplication of the two terms of a fraction into themselves; the numerator into itself, and also the denominator, expressed, as in vulgar fractions, or, as in decimals, understood; there may be use, and there is meaning, in this operation. But this is not a multiplication by fractions, it is not a multiplication of one fraction by another, but a separate *involution* of the two terms of a fraction, each into itself. See this interesting and beautiful process described and illustrated in the article on Involution.

95. For this purpose of involution we shall hereafter have occasion to resort to this method of working a small decimal. To which end let us, then, involve

the decimal .4 into itself, once, and again a second time; the result appears to be .64; for 4 × 4 × 4 = 64. But this result consists of *two* figures only, whilst *three* figures have been involved in its production. To complete the process we must, then, in conformity with the rule just cited, prefix a cipher to these two figures, making the decimal .064. Which, as will be shown in paragraph 172, is the true and rational expression of the quantity.

96. DIVISION, I scarcely need observe, is the reverse of *multiplication*. It is calculated to *undo*, exactly, that which is effected by multiplication. As, for example, suppose we would *undo* the work in a foregoing example; that is, suppose we would divide 2431.112688 by 32.16. We write down these two sums, and divide just as we do in simple division; only, with regard to the *decimal point;* having found the quotient to be 755943, we observe this rule in *fixing* that point in its *proper place;* we count the number of decimal figures in the *dividend*, we see how many *more* there are in it, than there are in the *divisor*, and *this difference* is the number to be cut off for decimals in the quotient. And this proceeding, as you will observe, restores to us the original multiplicand; and proves, not only that both operations are right, but that *the rules* by which we work them are right, likewise.

97. We must not, however, omit to observe here, that cases arise, in which the quotient is so small, that there may not be decimal figures so many in the quotient as there are in the dividend over and above those in the divisor. When a case of this kind occurs, you put, *before* the figures of the quotient, *ciphers*, sufficient to make up the required number, and then you set the decimal point.

98. Here I might finish on this rule of division of

decimals, but that there yet remains this point to be noticed. It is not, as you know, the *quantity of figures*, that determines exactly the *value* of any number, but the individual value of each figure, and the station in which it stands. The figure 9 expresses a higher number than 8, when spoken of without regard to station, but place 9 *below* the decimal point, and 8 above it, and 8 is then the higher number. In like manner 2.9756, is a number of less value than is 3.12; and, as a smaller number will always divide a larger, we might have to divide these three figures by the five. How, then, are we to act in a case like this. To divide the smaller number of figures by the larger, without some *abatement* of the number in the larger, or some *increase* of the smaller, is manifestly impracticable. One of these courses must, then, be adopted, and it must be one which will *not alter* the relative *value* of the two numbers. Now what method is there of doing this? Look at the *two last sentences* in paragraph 75, and there you see, that you may put any number of ciphers *below*, that is *after* the figures of a *decimal*, *without altering its value*. Adopt this course, in this instance, with your short dividend, and having completed your division, you fix the decimal point as before instructed.

99. Such ease in working, such simplicity of statement, such readiness for every operation, as we find in decimal numbers, cannot fail to suggest to every mind engaged on the subject, this question: How happens it, that this neat and beautiful method of stating *parts* of whole numbers, has not entirely supplanted the method by vulgar fractions? The answer to this question leads us to the last rule that remains to be spoken of in these numbers. Decimals have not supplanted vulgar fractions, *because remainders*, whence almost alone come fractions, always present themselves in the usual operations of arithmetic, as the *nominators* of vulgar fractions, the

divisor being the *denominator*. Remainders, after division, thus present themselves; and they require, before they can be stated, or worked as decimals, to be *reduced* to these numbers, an operation which it is not always worth our while to perform.

100. This reduction of vulgar fractions to decimals, is already treated of in paragraphs 79, 80, and 81, but as, besides its utility, this is a very interesting process, another and yet clearer exhibition of the principle on which it is done may here follow, with considerable advantage.

REDUCTION

OF VULGAR FRACTIONS TO DECIMALS.

101. A vulgar fraction is a quantity described by *two* numbers, namely, a numerator and a denominator; and the quantity so described bearing the same proportion to a unit or whole, as the numerator bears to the denominator, we shall have that proportion, which is the value of the fraction, and have it, too, in its most compact form, if the fraction be such that we can divide the numerator by the denominator; that is to say, if it be an improper fraction, and the numerator be some multiple of the denominator; see paragraphs 54 & 55. But this lucky facility will prove of but rare occurrence. And proper fractions, too, have always larger denominators than numerators, so that we cannot divide the latter by the former. For instance, $\frac{2}{5}$ is a proper fraction. But we can no more divide the numerator of this fraction by the denominator—the 2 by the 5—than we can di-

vide two gallons of spirit of wine into two or more equal quantities with a five gallon measure.

102. However, let us think a little of this; let us consider it. Is it indeed *impossible* to divide two gallons of spirit into two, into four, or into any other number of equal parts; is it impossible so to divide two gallons of spirit with a five gallon measure?

103. Spirit of wine, and it is this sort of spirit that I have fixed on, for the reason which will now appear; spirit of wine may be *increased* in *bulk*, just as decimal figures may be increased in number, *without an alteration in value;* spirit may be thus increased by the addition of water; and, again, by distillation, the spirit may be drawn from the water, as the significant figures of a decimal may be withdrawn from ciphers, without causing an alteration in its value. So suppose we increase the bulk of the spirit in this manner, suppose we increase it to ten gallons. We can now measure it, and divide it with the five gallon measure. Again, to come a little nearer to our purpose, suppose we thus increase the bulk of the two gallons of spirit to twenty, just as we increase the number two to twenty, when we would decimally divide it; then, how easily do we divide it with the five gallon measure. The quotient will then be 4. But *four what?* Is the question. The answer is, *four tenths.* We increased the 2 *tenfold*, by the addition of a cipher, an increase, however, which in decimals adds nothing to the value of a number; we thus made the *two* into *two tenths*, when it became divisible by the five, and gave us a quotient of 4, that is *four tenths;* and the figure 4, which, placed next after the decimal point, means *four tenths*, is the *decimal* expression of $\frac{2}{5}$. This, as the terms import, is *decimating* the fraction; it is reducing it to *tenths;* and then finding the number of tenths which constitute its value, and *that number* is

the *decimal expression* thereof. Let us, in this method, find the decimal of $\frac{4}{5}$.—4, reduced to tenths, is 40; which, divided by the denominator gives .8; which is the decimal expression of $\frac{4}{5}$. Now try $\frac{1}{4}$. Reduce the numerator to tenths; it is 10; which will not divide evenly by the denominator 4, so reduce it to *tenths* of *tenths*; that is, to *hundredths*. Done by adding another cipher merely, and then 100 ÷ 4 = 25; which .25 is the decimal expression of $\frac{1}{4}$.

104. This is THE RULE, and here is the PRINCIPLE of the rule made manifest, for reducing *vulgar* fractions to decimals. One or two other examples will make all familiar.

(1) Reduce $\frac{1}{2}$ to a decimal.

$$2\,)\,\underline{10}$$
$$.5$$

(2) What is the decimal $\frac{2}{3}$?

$$3\,)\,\underline{2000}$$
$$.666, \text{ a recurring, or circulating decimal.}$$

(3) Reduce $\frac{7}{11}$ to a decimal.

$$11\,)\,\underline{70000}$$
$$.6363, \text{ another recurring decimal?}$$

(4) What is the decimal expression of $\frac{11}{7}$?

$$7\,)\,\underline{11000000000}$$
$$1.571428571$$

105. Thus, in many cases, will the figures continue to recur, or circulate, as we have termed it. To prevent this it is usual, on proposing to reduce any fraction to a decimal, to name, or limit the number of figures to which the reduction shall be carried. In ordinary calculations two or three figures are deemed sufficient; the first bringing us within a hundredth, and the second within a thousandth, of

the exact quantity; nay, sometimes a single figure suffices, indicating, as it does, the quantity within a tenth part. How desirable it is thus to limit the number of figures to which a decimal shall run is shown in the last example—question 4—in which we have proceeded until we have nine figures, giving us the precise quantity within a thousand millioneth part; the discovery of which quantity we might pursue in vain, seeing that, in the three last, the same figures are recurring as those with which we set out; and they would, of course, continue to recur. So arithmeticians limit the number of figures thus,

(5) Find the decimal of $\frac{5}{7}$, to the third figure.

```
7 ) 5000
   .714
```

(6) What is the decimal expression of $\frac{3576}{15724}$, to the third figure?

```
15724 ) 35760 ( 227  the decimal.
        31448
        43120
        31448
       116820
       110068
       ......
```

(7) What is the decimal expression of $\frac{3}{4}$, $\frac{1}{8}$ and $\frac{3}{32}$, to three figures?

These three fractions reduced, according to paragraphs 51 and 52, to one denomination, are $\frac{992}{1024}$, which, divided by a common divisor, found according to the rule in paragraphs 54, 55 and 56 namely 32, and thus reduced to its lowest terms, is equal to $\frac{31}{32}$. And now we proceed to reduce this to a decimal.

```
32 ) 310 ( 968  the decimal
     288
      220
      192
       280
       256
       ...
```

DUODECIMALS,

TWELFTHS, OR CROSS MULTIPLICATION;

Being Calculations of Measurements in Feet, Inches, &c.

106. Duodecimals are used to calculate quantities; quantity in *superficial extent*, such as the measurement of boards, of the work of painters, glaziers, and plaisterers; and quantity in *bulk*, as in a piece of timber, or a block of stone. All which things are usually measured by the foot of 12 *inches;* which inches, for nice purposes, are each divided into twelfths, called parts; and these parts may be divided again into twelfths; and hence the name, and the use of duodecimals, or twelfths.

107. As an example of the calculation of superficial quantity. Here is a wall 8 *feet* long, and 6 *feet* high. And we would know how many feet, that is to say, how many squares, measuring a foot each way, there are in the whole. The wall is 8 feet long: let us divide the length into 8 equal parts, as you see in figure 1. But it is, also, 6 feet high. Let us, then, taking the same figure, divide the *height* thereof into six equal parts, and ruling the divisions *across*, as we ruled the former downwards, we have it divided as Fig. 2.—Now count the squares; there are 48, which number is the *product* of 8 multiplied by 6; that is to say, the *product* of the *length* of the wall multiplied by the *height*. Knowing this, knowing that the length, multiplied by the height or breadth, will give the superficial contents, we have, in future, no occasion, when

Fig. 1.

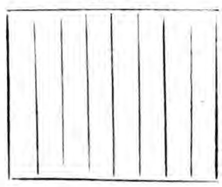
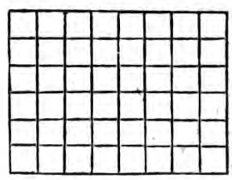


Fig. 2.



we would know the number of square feet in anything, to divide the thing thus into squares, and then to count them; having the number of feet in length, or indeed, the number in yards either, and the number in breadth, we have only to multiply these two numbers together, and the product is the number of square feet, or square yards, or square miles, just as the denominations may be.

108. Again, suppose that it were a solid body, such as a block of stone, measuring in length and breadth just what we have stated, and in thickness, 6 feet. Dividing this thickness into layers of one foot thick each, and then cutting each layer across, in the lines marked in figure 2, would give us 48 small blocks in each of the 6 layers, that is, 6 times 48 blocks, which make 288; which number of blocks is the product of the length, breadth, and thickness, multiplied successively into each other.

109. But suppose we come to *inches*. Let us take one of these small blocks, measuring a foot on every side, and called, therefore a cubic foot. It is 12 inches long, 12 inches broad, and the same in thickness. Now, how many solid, or cubic inches are there in such a block? We need not, now, cross it with lines, in order to ascertain this; for multiplying these three dimensions into each other gives us the number of solid inches. And, to finish this matter of *cubic* measurement, in order that we may afterwards proceed without interruption in the consideration of that of superfices, let us take as many of these blocks, of one cubic foot each, as will make a cubic yard. Laying the blocks together, we must have, for the first layer, three of them each way, that is, 9 blocks; these will make a layer of due length and breadth, and *one foot thick*; but we are to have it *3 feet* thick, so that we must have 3 times 9 blocks, which are 27, which is the number of *cubic feet* in a *cubic* yard.

110. Thus are there *square* measure, and *solid*, or *cubic* measure. *Square* measure applies only to *surfaces*, and the superficial quantity in any thing is found, by multiplying the *length* into the *breadth;* whilst the quantity in *cubes* is found, by multiplying the *length*, *breadth*, and *thickness* into each other. And the quantity thus found, whether the thing measured be square or cube, is called the *contents* of that thing. To return to the measurement of superfices.

111. For our first experiment, let us take a surface 9 *feet* 7 *inches* long, and 8 *feet* 3 *inches* broad; and let the contents be to be found in feet. Now in this, as in all other proceedings, it is desirable that we have a clear and distinct idea of the matter of which we are treating. So let us fix, very clearly and steadfastly in our minds, *what quantity* of surface is meaned to be described by each of these denominations, *feet*, *inches*, and *parts*. As to the quantity, a *foot*, here it means, as before explained, *a foot square;* and because we are about to ascertain the contents in *feet*, a foot is the INTEGER, or WHOLE NUMBER, of which *inches* are 12*ths*, and *parts*, 12*ths* of 12*ths*.

112. But, now, mark well. We have determined that the *square foot* shall be the integer, or whole number. What, then, is an inch; is it a *square inch?* By no means. It must be twelve square inches; that is to say, it must be a space, or slip one inch broad, and one foot long; it must, in short, be the *twelfth* of the foot; whilst one square inch would be only the 144th part of a square foot. The whole matter stands thus. The *foot* is, in all cases, composed of *twelve inches*. But, then, there are different sorts of feet. There is the foot-length, or *running*, or *linear foot*, there is the *square foot*, and then, again, there is the *cubic foot*. Now, *one inch;* that

is, the mere *length* of an inch, is the *twelfth* of a *running foot;* but, if you look at it, you will see that the twelfth of a square foot must be, as I have said above, a space or slip one inch broad, and one *foot* long, which is no less than twelve square inches; and, then, for *an inch* in a *cubic foot*, is that a slip, formed of twelve cubic inches? Not it, indeed. Such a slip would be, not the twelfth, but the hundred and forty fourth part of a cubic foot. The twelfth of a *cubic foot*, that is to say, an inch in such a foot, must be, as you will see when you reflect on it, a stratum, or layer of inch cubes, containing twelve one way, and twelve the other; that is, 144 cubic inches. Thus, then, in linear or *running measure*, the twelfth of a foot is *one inch;* in *square measure*, this twelfth is *twelve inches;* that is, *twelve square inches;* in *cubic measure*, the twelfth is *one hundred and forty-four inches;* that is, 144 *cubic inches:* of which inches there are 1728 in the foot. So much for feet and inches; and now for *parts*.

113. WHAT IS A PART? It is the twelfth of an inch. So, in running measure, twelve parts make an inch; but not so in square measure; nor so in cubic measure. Were an inch the integer, or whole number in which we were reckoning, then all that has been said in the last paragraph respecting the inch, would serve for *the part*, of which we are now treating; that is to say, 12 parts would be one inch in running measure, 144 parts in square measure, and twelve times this, or 1728 parts would make one inch in cubic measure: that is to say, such would be the parts were *an inch* the integer, or whole number in which we were calculating.

114. But we have determined, in the last paragraph but one, that THE FOOT shall be the *integer;*

so our question now is, *what is a part*; what are the *contents* of a part, in measurements in which a foot is the integer? In running measure, that is, in mere length, the part is still but the twelfth of the inch: but—and here comes the great distinction—in square measure, the foot being the integer, the part must be a *foot long*, but only the twelfth of an inch broad. However, a slip of this kind, although called a part, contains no less than 144 *square* parts; and it would require twelve times twelve of them to make a square foot. And, then, for *a part* in *cubic* measure: What is *this part*, a foot being the integer? Call to mind what such a thing must be. It will be a small layer or stratum, the twelfth of an inch thick. Twelve times twelve of these would be required, laid one on the other, to form the cubic foot. But what would this layer, what would *this part* be? It will be the twelfth of an inch *thick*, but *in breadth* it will be twelve inches, that is, 144 parts, and in *length* the same; 144 × 144 is 20736, that is to say, in cubic measure, a foot being the integer, one twelfth part of an inch contains 20736 cubic parts. Thus have we, I trust, a definite, and what may be termed, a tangible idea of the quantities in which we are about to make calculations.

115. However, with respect to this quantity called PART in square measure, another observation remains to be made. In the foregoing paragraph I have described this *part* as a foot long and the twelfth of an inch broad, and as containing 144 square parts.—This, of course, you bear in mind, is the quantity of *a part*, in all calculations in which the foot is the integer.—Thus, then, 144 *square parts* constitute *one part*, and such parts generally present themselves in the form that I have described; that is, as long slips, a foot long and the twelfth of an inch broad.

116. For example. Here is a Diagram, on a scale of about an inch to a foot. I intend the larger square to represent a foot; then come the inches, and then the parts, as I have described them in long slips. But, and this is the observation to which I have now to call your attention,—there are other forms besides these; there are some *small squares*, and some *short slips*, formed by the extension, and the crossing of the lines by which the inches and the parts are marked out. The inches and the parts are complete without this extension; they are complete, as shown in this figure 2. So, what are the small squares, and the slips, which are omitted in this figure, but which appear in the foregoing diagram, and which are requisite to complete the figure; what are these; and what are they called? They take their names, as every thing in this way ought to do, from the quantities they contain. Of the small squares, the larger, which are mere extensions of the inch slips, and which are, therefore, an inch square;—these are counted as *parts*, seeing that they are each but the twelfth of one of the inch slips; seeing, also, that they each contain 144 square parts, which, as before shown in the measure we are speaking of, is the contents of *a part*. Next in size come the short slips. They are the breadth of a part, but being only one inch, that is the twelfth of a foot long, they are called the *twelfths* of *parts*. Lastly come the smallest squares. What are these? They are each the breadth of a part, but being only the twelfth of an inch long, that is, a twelfth of the quantities last

Fig. 1.

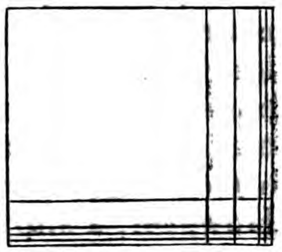


Fig. 2.

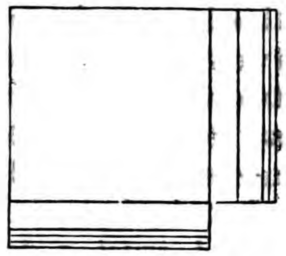
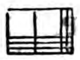


Fig. 3.



described, they are the *twelfths* of *twelfths* of *parts.* Thus, then, at length, is the view completed. We know not only the *contents*, but the *forms*, likewise, in which these various quantities severally present themselves.

117. Thus, then, it is; a square inch requires to be *extended*, or *multiplied*, twelve-fold, when, becoming as long as a foot, and one twelfth the breadth, it becomes the *twelfth* of a *foot;* that is to say, it becomes an *inch* in this sort of measure. In like manner it is with a *part.* A PART multiplied by, that is, extended to the length of a *foot*, is a *part;* multiplied by an inch, therefore, it is only the *twelfth* of a *part;* and *parts* multiplied together are, of course, one remove *yet lower* in the scale of denominations; they are only the *12ths* of *12ths* of *parts.* Thus, *feet* multiplied by *feet*, are *feet; inches* multiplied by *feet*, are *inches;* and *parts* multiplied by feet are parts: and so on to the twelfths and the twelfths of twelfths of parts; and to yet more minute divisions, where the subject requires it. Every quantity is to be considered as running the length of the foot, or integral number, and when so extended takes its denomination from its breadth.

118. However, "twelfths," "twelfths of twelfths," and such denominations, are unwieldy and inconvenient, so arithmeticians have adopted some compact terms and marks, wherewith to express these small quantities. The DENOMINATIONS run thus; The names and marks have been determined by their several distances from the *foot. Inches* being the *first* remove from the foot are called *primes*, that is, *firsts*, and the mark over them is a single comma ('). Those which we have called *parts*, which is the name generally given by workmen, being in the *second* station, are called *seconds*, and have two commas ("); and the next have three

commas (''') and are called thirds. You may proceed to *fourths* (''''), and so on. And they are thus affixed to the figures 29 ft. 5' 9'' 7''' 2''''.

119. To return, now, to our case, as stated in the last paragraph but one. Writing down the dimensions, the different denominations right under each other, as follows, you begin to multiply by the feet in the lower line; saying, "8 times 7 are 56." But, now, what are these 56? They are the product of inches multiplied by feet; they are, therefore, *inches*; divided by 12 they give 4 *feet* 8 *inches*; the inches being set down in their proper place, the feet are carried forward. Then, multiplying the feet, saying, "8 times 9 are 72, and 4 carried are 76." And now to multiply by the 3 inches. *Inches* you have seen, multiplied together, produce *parts*; so 3 times 7 = 21 *parts* = 1 *in.* 9 *pts.* Proceeding next to multiply the 9 feet by the 3 inches, we have 27, which, with *one* carried from the last, make 28 inches; which are 2 ft. 4 in. and which, written down, as you see, and the two lines of product added together, we have 79 *ft.* 0 *in.* 9 *pts.* as the product of 9 *ft.* 7 *in.* by 8 *ft.* 3 *in.*

ft.	*in.*	
9	7	
8	3	
76	8	
2	4	9
79	0	9

120. To sum up the process in the form of a Rule: Having properly stated the dimensions to be multiplied together, you begin the work by multiplying the lowest denomination in the upper line, by the highest in the lower line. And, from this circumstance, it is, that the name of CROSS MULTIPLICATION comes; Beginning thus, and bearing in mind that *parts* multiplied by feet produce parts, you divide the produce of such parts by 12, set down the remainder, *as parts*, and carry the quotient forward as inches; then, multiplying the inches by the feet, you have *inches*, which divided by 12, are carried onwards to

the feet. The *feet* are multiplied together, like any other integers, and we set down the product, with the addition of anything that may have been carried. So much for the Multiplication *by* FEET.

121. And, for the Multiplication by INCHES. Multiplying the parts by inches give, as you recollect, *twelfths of parts;* the product here, therefore, you must divide by 12, set down the *remainder, one station below the parts*, and carry on the quotient as parts; then, multiplying the *inches* together, you have parts for the product, which added to whatever may have been carried, and divided by 12, give inches, which inches, added to the product of the feet, and divided by 12, produce feet; and this finishes the multiplication by *inches.*

122. Then, to multiply by PARTS, if you go to such minute dimensions, which you scarcely will, unless you be computing the value of plates of fine glass, or cloth of gold, or some such costly article: if you multiply by these small parts, you proceed on the same principle. On this principle, *parts* multiplied by *parts;* produce *twelfths* of *twelfths* of *parts;* and these will be a denomination *two* places *below parts.* All this might be made palpable, to the eye, by a diagram, which it was my design to give, but I relinquish the design on finding, that in order to show the *minute* divisions, and the larger ones likewise, such a figure must be inconveniently large. So I rely a little on the imagination of the ingenious student, who will be at no loss to understand the matter, after the description just given.

123. Another proposition, a little nice in its measurements, will assist yet further to clear up this matter, and to confirm right impressions on the mind of the learner. Let us, then, see what are the superficial contents of a Plate of Glass, measuring

7 feet 5 in. 8 pts. in length, and *5 ft. 3 in. 7 pts.* in breadth.

124. Worked in the method we have just learned, the question is answered as you see in the first of these operations. But in order to show, beyond all question, that it is correct, let us, discarding this process of multiplying parts by inches, and inches by parts, and so on, from which alone arises any difficulty that may yet hang about the matter; let us, discarding this process of multiplication, and writing all the dimensions down, repeating them, just as when they are repeated by that process; and just, indeed, as if we had the *several spaces* described by the figures, actually spread out before us, in order that we might place them together in rows, according to their several denominations, and then sum them up; let us thus set down these several dimensions, and *add* them together, and we thereby see, not only that the result is the same as that produced by the other operation, but we see, also, the value, that is to say, the *extent* of every space indicated by each figure.

BY MULTIPLICATION.

ft.	*in.*	*pts.*		
7	5	8		
5	3	7		
37	4	4		
1	10	5	0	
	4	4	3	8
39	7	1	3	8

BY ADDITION.

ft.	*in.*	″	‴	⁗	
7	5	8	·	·	The upper line five times by the feet.
7	5	8	·	·	
7	5	8	·	·	
7	5	8	·	·	
7	5	8	·	·	
·	7	5	8	·	3 times by the inches.
·	7	5	8	·	
·	7	5	8	·	
·	·	7	5	8	7 times by the parts.
·	·	7	5	8	
·	·	7	5	8	
·	·	7	5	8	
·	·	7	5	8	
·	·	7	5	8	
·	·	7	5	8	
39	7	1	3	8	

125. One point more, and I conclude the lesson. For the sake of simplicity, I have used as examples, cases, in each of which there is but *one* figure in each denomination, whether in the multiplicand, or in the multiplier. For example, I have shown how to multiply 9 *feet* 7 *in.* by 8 *ft.* and 7 *ft.* 5 *in.* 8 *parts* by 5 *feet;* that is to say, I have treated of *short multiplication* in this rule. There must, of course, be cases, in which the number of feet with which we have to multiply is expressed by *two* or *more* figures, requiring, in consequence, something like *long* multiplication. For example, let it be that we have to multiply 153 *ft.* 2 *in.* by 25 *ft.* 9 *in.* Writing these multiplicand and multiplier, down in the usual manner, you would stop when you come to do the work, seeing that the product, in multiplying by the 25, must be stated in *two lines*. And then come 153 *feet* to be multiplied by 9 *inches*, the product of which, being *inches*, must be written, not in the place of inches, at once, but *beside* the chief working, and when reduced into feet, by being divided by 12, may come into its proper station, as you see it done in the *first* of these two examples. But the better mode of doing it is this, as shown in the second example, in which the multiplication by the 25 *feet* is done like a common sum in compound multiplication; and for the 9 *inches*, which are *three quarters of a foot*, we take *aliquot parts* of the multiplicand, as in *practice*, that is to say, *a half*, and then the *half of a half;* and these written in their proper places, and the whole added together, give us the product of the two dimensions proposed in this example

(1st.)

	ft.	*in.*	
	153 ·	2	
	25 ·	9	
	765 ··	10	
	3063 ··	4	
	114 ··	10 ·	6
	3944 ··	0 ··	6

	ft.	*in.*	
	153 .	2	
		9	
12)	1378 ·	6	
	114 ·	10 ·	6

(2nd.)

½)	153 ·	2	
		5×5=25	
	765 .	10	
		5	
	3829 ·	2	
½)	76 ·	7	
	38 ·	3 ·	6
	3944 ·	0 ·	6

OF INVOLUTION AND EVOLUTION.

OF THE SQUARE, &c.

126. A square is a figure, a flat surface or space, of any size, bounded by four straight lines, each line of equal length, and having four corners or angles, each of which, also, are equal; that is to say, no angle shall be closer and sharper, nor more broad and open than are its fellows: such figure is called a square.

127. However, we shall find it useful to go systematically into a description of this figure, the square; and into descriptions of one or two other geometrical figures that are closely related to it. To this end we must first observe, that

128. Lines, by which geometrical figures, are marked out, or bounded, may be straight, waved or curved. But it is with straight lines only, that we have here to deal. Again,

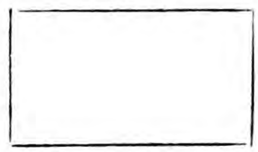
129. Lines may be drawn, or may lie, at a uniform distance from each other, as do the edges of a board that is nicely dressed of an equal breadth, and an equal thickness; or they may approach each other, towards one end, and diverge, or separate towards the other, as do the two sides of a wedge.

130. Lines which are drawn, or which lie, at a uniform distance from each other, whether they be two or more, whether they be curved or waved, or straight; lines running along thus are said to be *parallel;* and any geometrical figure formed by four straight lines or sides, each of which sides is parallel to its opposite side, any figure of this description, whether its length and its breadth be equal or not,

takes its name from its paralled sides, and is called a PARALLELOGRAM.

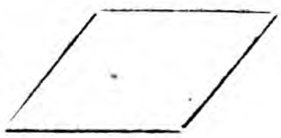
131. A square, therefore, its opposite sides being parallel, is a parallelogram

Fig. 1.



132. But so, also, is this figure 1, in the margin, the opposite sides being parallel, and the ends parallel; and so is figure 2, the opposite sides of which, also, are parallel; both are parallelograms; but neither of them is a square.

Fig. 2.



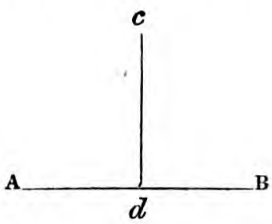
133. The first of these figures is not a square, because its sides are not of equal length. Its sides, that is to say, its opposite sides are parallel, and its angles, being all equal, are those of a square, but the four lines which form it are not of equal length; and for this reason it is not a square. Figure 2, which is called a rhombus, is not a square, because its angles are not equal. Its four sides are of equal length, and the opposite sides lie parallel to each other, but its angles are not those of a square.

134. However, these figures have names; names derived as we shall see, from the nature of their lines and angles. And, with these names we must become familiar, seeing that by them we shall be able to understand one another, and to describe figures without being required to draw them.

135. Observe, then; Any figure that is bounded by straight lines, such lines being of *equal length;* any such figure is called *equilateral*, that is *equal-sided*. Fig. 2, as we have seen, is a parallelogram, and being bounded by four lines of equal length, it is called an *equilateral parallelogram*.

136. Figure 1 is not bounded by lines of equal length, it is what we familiarly call an oblong; but its proper name, as we have seen, is parallelogram. But as this name might be applied to it, were it longer or shorter, broader or narrower, we do not, we cannot distinguish it from other parallelograms by its lines, which may be any length. But its *angles* are uniform, so we distinguish it by its angles; and the angles derive their name thus.

137. Take a straight line of any length, as A B—let us call this the *base*, that is, the *bottom* line—from some part of it draw another straight line, and let it be drawn perpendicular, or upright from the point from which it starts; that is, let it lean neither to the right nor to the left, but stand directly upright on the first line, as *c d.* Now mark. On each side of this second line, just where it meets or touches the first line, on each side is an angle; which angles, being formed by the meeting of an *upright* line with the base line, are called *right* angles.



Now, the parallelogram Fig. 1, has its angles of this description. Its length and breadth may be various, but its sides being perpendicular to each other, or forming *right* angles, it is called a *right angled* parallelogram.

138. And this is the kind of angle which is essential to the formation of the square, and from which, consequently, it takes part of its name. We have before seen that the square is a parallelogram, we have also seen that it is *equilateral*, and we now see that it is *right-angled.* Its correct description, therefore; those terms which distinguish it clearly from all other geometrical figures are these, an *equilateral right-angled parallelogram.*

139. And now for the Arithmetic of the subject. Observe that, in treating of duodecimals we have seen, in paragraph 107, that having the length and the breadth of any figure, of any piece of land, of timber, marble, glass, or other thing; that having the length and the breadth of any right-angled figure; by multiplying this length and this breadth together, we find the quantity or superficial contents.—Here, for example, is a piece of land lying in a square form, being 25 yards long, and 25 yards broad, and we would know the number of yards in the piece. This number, as we have seen in the paragraph just referred to, is found by a mere multiplication of the length by the breadth. And when, as in this case, the length and the breadth are the same, then is the form in which the surface lies, a square. And the multiplication in question, that is to say, the multiplication into itself of the number by which the side of the square is described, is called, SQUARING that number. Thus 25 × 25 is 625; which latter number is called the square of 25. The process has, however, another name than this of squaring, it is called the INVOLUTION of the number: of which Involution we may now proceed to treat.

OF INVOLUTION;

ITS NATURE; AND ESPECIALLY OF THE INVOLUTION OF FRACTIONAL QUANTITIES.

140. To involve is to roll up, to fold up. Involution comes from the Latin word *involutio*, the act of folding. And the word has been adopted by arithmeticians to express the multiplication, or rolling up, as it were, of a number into itself. An operation, also, by which the contents or quantity of a square

space, or of a cube, are to be ascertained. In the case of a square, for example, which being merely a superficies or space, has only length and breadth, a single involution, that is, a multiplication of what may be termed the length into the breadth, just as in the case of any other right-angled figure gives its contents: See paragraph 139.

141. It is when figures, or bodies have thus only what may be termed *one* dimension, that they can be regarded as subjects for involution. If the length, and the breadth, or the thickness, be of different dimensions, then are such figures, by this circumstance, taken out of this rule of Involution.

142. The multiplication, then, of a number into itself is called *involving* that number. A square has length and breadth, merely; the length and the breadth are the same, are, in quantity, identical; and the multiplying together of these two dimensions is called *squaring* the number: as $4 \times 4 = 16$.

143. To length and breadth the cube adds the *third* property of thickness; which thickness, to constitute a cube, must be the same as the length, the same as the breadth: and the multiplication, or involution of these three dimensions into each other, giving us, as shown in paragraphs 108 and 109, the contents of the cube, is called *cubing* the number.

144. Again, the squaring of a number, that is to say, the multiplication of the two dimensions of length and breadth is, in Involution, called *raising* that number to its *second power*. And another involution, that is, a cubing of the number, is termed *raising it to its third* power; thus $4 \times 4 \times 4 = 64$.

145. Further than this, higher than this *third power*, to other dimensions than length, breadth and

thickness, we cannot, with reference to actual form and substance, proceed. Nevertheless, we can multiply numbers together, we can carry on the involution of a number, raising it to yet higher powers; and this is constantly done by mathematicians, for various purposes

146. As the squaring of a number, or the first involution of it is called raising it to its *second* power; and the cubing of it is raising it to its *third* power, so, another involution raises it to what is termed its *fourth*, another to its *fifth* power, and so on.

147. To express, or to describe these powers with brevity, mathematicians have adopted a very neat and simple method; a method which, in many instances without the irksome labour of an actual involution of a long number, serves the same purpose; thus, had we, in the statement of some proposition, or of some process, to put down the cube number of 573, it is done by a simple annexation of a small figure 3, thus 573^3: which signifies that the number is to be taken as being raised to its *third* power; and it is understood to mean 188132517. The same number with the figure 2, thus 573^2, signifies 328329; which is the second power, or square of the number in question. And so of smaller numbers; thus 5^2 means the second power of 5; that is 25; 5^4 means 625; that is the *fourth* power of 5.

148. With regard to this involution of numbers, this raising of them to various powers, it must be a very simple affair when the number to be raised is altogether a whole number; a mere multiplication of the most ordinary kind accomplishes it. But when that number consists in whole or in part of a fraction, whether vulgar or decimal, then are some little additional knowledge and care required.

149. If it be a vulgar Fraction, it is raised by a multiplication of its two terms into themselves, respectively, just as we raise other numbers; thus, $\frac{2}{5}$ is raised to its second power by multiplying its numerator by 2, and its denominator by 5, which brings it to $\frac{4}{25}$. Another involution raises it to its third power, which is $\frac{8}{125}$.

150. If the number to be raised be a mixed number, that is, if it consist of a whole number and of a fraction or fractions, then should it be reduced or brought into an improper fraction, the nature of which is described in paragraph 39. For example: suppose we have to raise $1\frac{1}{2}$ to its third power, or in other words, to its cube; one and a half, expressed fractionally is three halves, that is $\frac{3}{2}$, which raised, as just directed, to its second power is $\frac{9}{4}$, to its third power it is $\frac{27}{8}$.

151. Thus it is that quantities are raised, when treated as vulgar fractions. But it is often more eligible to treat the fractional parts as decimals, the working of which is generally more simple.

152. When it is determined thus to treat a fractional quantity, the thing first to be done is, of course, to bring it into a decimal, as taught in paragraphs 101, 102 and 103, and then to raise it to the required power. For instance, let us thus treat the small quantity employed in the foregoing paragraph, that is $1\frac{1}{2}$.

$1\frac{1}{2} = 1.5$, and $1.5 \times 1.5 = 2.25$, and $2.25 \times 1.5 = 3.375$.

But mathematicians would state it more briefly; thus $(1\frac{1}{2})^2 = (1.5)^2 = 2.25$; and $(1.5)^3 = 3.375$.

153. In order, yet more fully to exhibit the process, let us take a larger number, let us raise $125\frac{3}{5}$

to its second power; and then to its third power. And, first, let us treat it as a vulgar fraction.

$125\frac{3}{5} = \frac{628}{5} \times \frac{628}{5} = \frac{394384}{25}$ The second power, or square.

$\frac{394384}{25} \times \frac{628}{5} = \frac{247673152}{125}$ The third power, or cube.

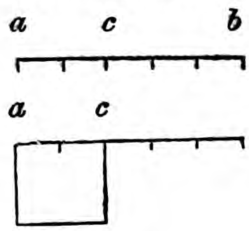
Treated as a decimal,

$125\frac{3}{5} = 125.6 \times 125.6 = 15785.36$. The second power.

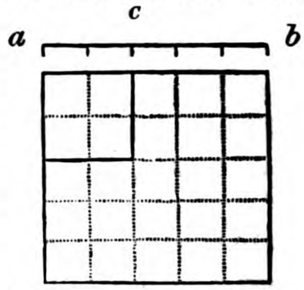
$15785.36 \times 125.6 = 1982641.216$. The third power.

154. However, these latter quantities form improper fractions, let us carry the enquiry into the involution of a proper fraction. We spoke, a few paragraphs back, of raising the fraction $\frac{2}{5}$ to its second, and then to its third power; let us take this fraction, thus involve it, and closely mark the process.

155. What is it that is produced by this involution of $\frac{2}{5}$? The product of the first involution, in figures, is $\frac{4}{25}$. But in MATTER; in QUANTITY of *matter*, what is it? This, of course, depends on the thing we determine to fix on as the unit, or integral quantity. This unit may be a foot, a yard, or an inch; or any other such quantity. Let it be a foot. And let this foot be represented by the line *a b* divided into five equal parts. Now it is manifest that two of these parts as *a c*, are $\frac{2}{5}$. Let us, then, as we proposed, square this quantity. Geometrically it is done, here in the margin. Arithmetically it is done, as before shown, by involving the numerator for a new numerator, and the denominator for a new denominator. And, now, mark the beauty of this process!



The numerator, that is the 2, described two of these five parts, the denominator 5 described the whole line. When squared these two numbers become $\frac{4}{25}$. And look at their respective geometrical representations, as here laid down in the fig. in the margin. The square of two fifths of the line *a b* gives us 4 small squares, of which squares there are 25 in the square of the whole line: So the square of $\frac{2}{5}$ is $\frac{4}{25}$.



156. Now, to trace the involution of this fraction to its third power; that is, to its cube.—To the length and the breadth of the square, the cube joins thickness; thickness the same as the length, the same as the breadth; and we have determined that in this case, these dimensions shall be *one foot*.

157. It is of *fifths* that we are about to treat, that is, of fifths of the length, of the breadth, or of the thickness; it is, supposing the cube to be divided into five equal layers, or strata, by four parallel cuts across it, as in fig. 1; these further divided by four cuts, transverse of the first, as in fig. 2; and then again divided by four other perpendicular and transverse lines, as in fig. 3, presenting, on whatever side we view it, the line by which its size is described as divided into fifths; it is of fifths of this description that we are now speaking.

Fig. 1.

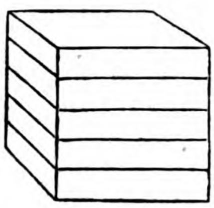


Fig. 2.

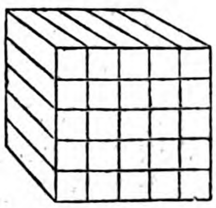
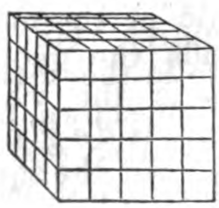
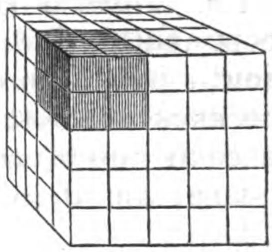


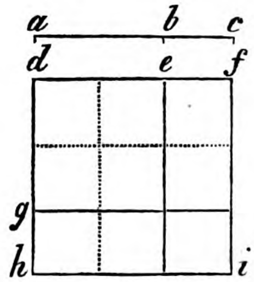
Fig. 3.



158. It is true, that the cube of a *fifth* of this description is but the 125th part, that is, *the fifth of the fifth of the fifth* of the whole cubic foot. This is true. But the cube, like the square, has its size described by the linear extent of each, or of any one of its sides, and, as the cube of a foot measures one foot over each of its sides, so the cube of the fifth of a foot measures the fifth of that denomination, as exhibited in the case before us. And our present purpose is to ascertain the quantity of matter comprised in a cube, the dimension of every side of which shall be *two* of these *fifths*.



159. In the annexed figure, which is divided like the former into 125 equal cubes; in this figure, a cube *two fifths* is marked and distinguished from the other parts by being shaded. This cube of two fifths is seen to contain, or to be formed of 8 of the 125 small cubes; that is to say, it is $\frac{8}{125}$. The arithmetical process stands thus, $\frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} = \frac{8}{125}$. So admirably does this process and statement accord with the material or geometrical representation.



160. Suppose, now, that we have to raise $1\frac{1}{2}$ to its second power. Let the $1\frac{1}{2}$ be represented by the line *a b c*, and its square by the figure *d f h i*. Now the number $1\frac{1}{2}$, which is $\frac{3}{2}$, when squared, is $\frac{9}{4}$. This is the arithmetic of the matter. And how stands the geometrical figure? *a b* is the whole number, its square is *d e g*, divided into quarters. But the square of the whole figure, *d f h i*, comprises nine of these quarters, that is $\frac{9}{4}$, which, as we

have seen above, is the arithmetical expression of the square of $1\frac{1}{2}$. And, now, to raise this number to its third power, that is to its cube.

161. Let the two figures which are placed below represent a cube, and let the line E F G, on which it is raised, be $1\frac{1}{2}$ ft. long; that is E F, one foot, and F G, half a foot.—The two figures are designed to represent the same cube in different views.—Now, let this cube be divided, as marked by the dotted lines; that is to say, separate the cube of *the foot* from the additions made to it by the half foot. And, what is the result? The cube of the foot appears in fig. 1, and is marked *a*,—in fig. 2 it is not to be seen, another side of the figure being presented to view,—*b b b*, in each of the figures, are three square pieces of one foot each, and half a foot thick; *c c c*, in each figure, are three prisms, each half a foot square; and *d*, in each of the figures, is a single cube of half a foot.

Fig. 1.

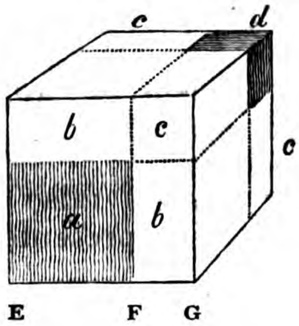
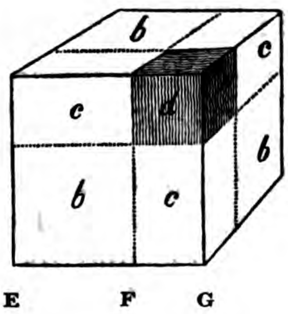


Fig. 2.



162. Now to enquire, what is the quantity of matter in one of these cubes? As the gauge, or denominator of this matter, let us take the half-foot cube, seeing that all the other parts, or pieces of the cube are commensurable with it.

163. First, then, as to the contents of the foot cube. In it, of these half-foot cubes, there are eight; for two halves in length by two in breadth, give four,

and these, involved into the depth, which, also, is two, give eight, the number of half-foot cubes in one cube of a foot.

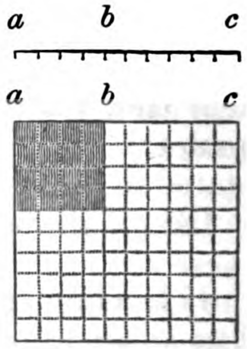
164. Second. The three square pieces, or parallelopipeds, as they are termed, are each half a foot thick. And being a foot square, they are, each of them, half of a cubic foot. The whole foot gives us eight of our gauge, these pieces, then, contain each four, and the three contain twelve of that gauge.

165. Third come the three prisms, marked *c c c*, these are each half a foot square, and one foot long, they contain, therefore, each two cubic half feet; that is, six of our denominator in the three. And, though last and least, not to be forgotten, comes our gauge, or denominator itself, the single cube of half a foot.

166. Let us now add these several quantities together. In the foot cube we found 8, in the three parallelopipeds 12, in the three prisms 6, these make 26, to which add the small cube itself, and we have 27 of the denominator we have chosen. Now, our whole number is the foot, the *cube foot;* this foot, divided into eighths, gives us cubes of half a foot; the cube of a foot and a half contains, as we have found, 27 of these, that is 27 eighths.—Such is the geometrical calculation. And, now, what says the arithmetical process. Turning to paragraph 150, we see that $1\frac{1}{2}$ raised to its third power is $\frac{27}{8}$.

167. Hitherto, however, we have not traced the process of involution in decimal numbers. It is, in itself, so simple an affair, that a single example may suffice, and for this example let us take one of the quantities that we have already treated as a vulgar fraction; let us take $\frac{2}{5}$ and treat it as a decimal.

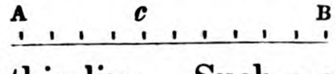
And first as to the square. Arithmetically the statement and process will stand thus, $\frac{2}{5} = .4$; that is $\frac{4}{10}$. The square of the decimal $.4$ is $.16$; that is $\frac{16}{100}$. So much for the arithmetical process. Now for the geometrical.



168. Take a line of any length, as *a b c*. Let this line represent *the whole* or *integral* number. Decimate it, that is, divide it into tenths. Now what can be more simple, or more clear; four of these decimal parts are $\frac{4}{10}$. And if we square the whole line, and then divide that square into ten equal parts each way, forming thereby ten-times-ten, that is 100 small squares, as is seen in the figure, then will the square of *a b*, that is, of four tenths of the whole line, be found to contain sixteen of the hundred; that is, according to the arithmetical expression, $\frac{16}{100}$ of the small squares.

169. And, now for the Cube of this decimal fraction. Arithmetically the matter stands thus $.4 \times .4 = .16$; and $.16 \times .4 = .064$—see rule in paragraphs 91 and 92. Observe, now, this decimal, stated with its denominator, would be $\frac{.064}{1000}$; see paragraphs 76 and 77; or, rather, now that we drop the decimal form, it is $\frac{64}{1000}$; and bear in mind that the quantity involved, when stated in like manner, is $\frac{4}{10}$.

170. Let the line A B be 10 inches, and let A *c* be four of those inches. Now imagine a cube raised



on the whole of this line. Such a cube would contain 1000 cubic inches; for $10 \times 10 \times 10 = 1000$. Then imagine—the figure itself, to answer the purpose, must be inconveniently large—so imagine a cube raised on the part A *c*, that is, on 4 inches; such a cube would contain 64 cubic inches. And if considered as being raised within the larger cube, if considered as being formed of a part, that is, as being a fraction of the cube of 1000 inch cubes, then is this cube of 4, which comprises 64 inch cubes, sixty-four thousandths of the whole cube. And this agrees with the arithmetical expression of the quantity, as stated in the foregoing paragraph. Such, so simple, is the principle of involution of decimal quantities.

171. Having dismissed this subject of Involution, I may be permitted, as it can here be done with great advantage, to say a few words on two other points, one of which has been reserved until now.

172. In paragraph 95 we involved the decimals 4 . 4 . 4, and the result was .064. Now, why is this cipher before the two significant figures? It is according to rule laid down and illustrated in paragraphs 91 and 92; but, WHY THIS RULE? We have just seen the WHY. The rule is, that in a multiplication in which decimals are factors, on the conclusion of the process, there shall be the same number of decimal figures in the product as there are in the factors multiplied together, and that, if, as in this case, of the multiplication of three decimal figures, there be one, or indeed any number of figures short, we shall make up such deficiency by prefixing a cipher or ciphers. Now, the REASON, that is to say, THE PRINCIPLE OF THIS RULE is become apparent; $.4 \times .4 \times .4$ without this rule, produces .64; that is $\frac{64}{100}$, if we place its denominator under it according to rule, paragraph 77. But we have seen; in fact, we have it demonstrated in the foregoing process of the

involution of these three numbers, that the result is not sixty-four hundredths, but only a tenth of that quantity, that is to say, $\frac{64}{1000}$, the decimal expression of which is, and must be, .064.

173. Need another word be said in order to show, in order to prove, the entire dissimilarity between this, which can only, with propriety, be called, THE INVOLUTION OF FRACTIONAL NUMBERS, and any process which can be termed multiplication. We have before argued the case, in the 59th and a few of the succeeding paragraphs, but here the matter has become so clear, the foregoing diagrams and reasoning so palpably show that it is NOT A MULTIPLICATION, as the word *multiplication* is used in any other case whatever, but that it is clearly an arrangement, merely, of those quantities that are expressed by the figures and numbers with which the operation is performed; an arrangement of those quantities in the form of square, or of cube: that it is no augmentation, nor a diminution neither; but a separate *squaring*, or *cubing*, of the two terms of the fraction; that it is, in fact, their INVOLUTION merely: that all the incongruous notions about quantities, or numbers, being diminished when multiplied by a fraction, and increased when divided by a fraction, that all these notions ought to be dissipated. And, further, is it become manifest, that this multiplication, by the two terms of a fraction, can be of no use, can never have any meaning, save in a case of Involution; that is to say, save in a case in which the two terms of a fraction are to be involved for the purpose of ascertaining the proportion which the fractional part bears to the whole or integral quantity contemplated in the given case.

OF EVOLUTION;

OR,

THE EXTRACTION OF THE ROOTS OF NUMBERS.

174. In the foregoing lesson we have seen that, to involve, is to roll, to fold up; so, to *evolve* is to unfold. And as the term Involution is applied to that process of arithmetic by which a number is raised to its several powers, that is, to its second power, or square; to its third, or cube, and so forth; so to evolve is to retrace this process, and thereby to discover the number which has been raised, or which is susceptible of being raised to a given number. And the term EVOLUTION is applied to the process by which such number may be discovered; and the number so discovered, is called THE ROOT of the given number: for example, let the given number be 16, which is the second power, or square of 4; or let it be 64, which is the cube, or third power of the same number. In these cases, 4 is called *the root* of the given numbers; it is the square root of 16, and the cube root of 64; and the discovery of it is called Evolution; or the Extraction of the Root of those numbers.

OF THE EXTRACTION

OF THE

SQUARE ROOT.

175. The square roots of certain even numbers, as of 100, or of 400, is seen at once. We see that $10 \times 10 = 100$, and that $20 \times 20 = 400$. Again, the root of 144 is seen at once to be 12; as may be the root of any square number yet smaller than this. But in real calculations, for purposes of business, we

cannot have numbers thus fitted to our hands; and to discover the roots of numbers as they arise, is quite another matter. Take, for trial, a small number, take 361, which has a whole number for its root; or take one yet a little larger than any of those of which we have yet spoken, take 1369, which, also, is the square of a whole number.

176. When you have, with all your previous knowledge of arithmetic, tried to discover the roots of these numbers, *without* THE RULE; you will feel the value, and indeed the necessity of that rule by which such operations are effected.

177. To commence, then, in the simplest form, our inquiry into the process and principle of this rule.—Let it be that we have to find the square root of the number *four;* let it be that we have to find what would be the dimension of each side of a square slab of marble, containing four square feet.

178. We see what the root of this number is, instantly, and almost without reflection; we see that had we *four* pieces of marble, each a foot square, so to arrange that they would form a square, we must place them as is indicated by the first figure in the margin, forming a square the side of which would be two feet. Now this *two* is the *root* of *four*. Were they *nine* pieces that we had to place thus in a square, you will see that they must be placed as in figure 2, in three rows, three in a row. And hence *three* is the root of *nine*. Were there sixteen pieces, a very little consideration leads us to arrange them in four

Fig. 1.

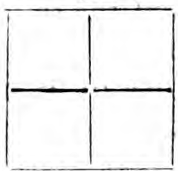


Fig. 2.

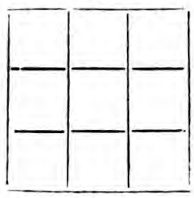
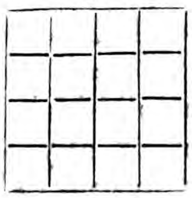


Fig. 3.



rows of four each, as in fig. 3. So *four* is the root of *sixteen*. But it is not, as I have before intimated, for small numbers such as these, that the learning contained in this rule is required, but for large numbers, for numbers that will not be so easily divided; numbers, the roots of which we cannot conjecture, nor for every unit in which we can, as in the three foregoing instances, be provided with a square piece. There are, for example, four paragraphs back, the square numbers 361, and 1369.

179. Now ordinary men cannot conjecture the root of either of these numbers; nor is it desirable that any one should waste his faculties in such an employment; neither is it to be expected that we should have 361, and still less 1369 little squares, for the purpose of arranging in rows, of forming a square, and thereby, of finding out the root: and if this be not to be expected in these, which are comparatively, small numbers, what is to be said of large numbers; what of thousands of millions, of billions, what of yet larger numbers; and, more difficult and unmanagable still, what is to be said of numbers the roots of which sink into fractions; or which of themselves consist in part, or in whole, of fractions; roots of such numbers being frequently required.

180. In short, another process is necessary; a process, not of conjecture, nor such as a child might practise, in some simple cases, were it furnished with the requisite number of little squares, but a process capable of mastering any number; a process worthy of the man and the scholar; and such process is furnished by our rule.

181. To proceed in the wise and sure way, that is, step-by-step, let us take for our next experiment the smaller of the two square numbers of which we have just spoken; however, we must first have recourse to the Rule.

THE RULE,

182. With the explanations that I deem requisite may be thus laid down, in nine articles. Recollecting what has been said on the nature of the square root of any number; recollecting, and always bearing it in mind, that this root is a number which multiplied into itself will produce the square, or original number; that the ROOT, therefore, of any number, is such a *divisor* of that number as will give a quotient exactly the same as the divisor; recollecting this, you will see that the root is to be extracted by a process of DIVISION; a process, however, by which you have to find, not merely, as in ordinary division, what is the *quotient*, but, likewise, what the *divisor*; recollecting this, you will perceive, that the number on which you have to operate is to be fairly written out, as you write out your dividend in a case of long division. And then comes the question—if such number be expressed by several figures—How much of your dividend are you to divide at a time? To determine this point, and properly to prepare your square number for the whole operation, is the object of the first article of the rule.

ARTICLE I.—If the given number, the root of which is to be extracted, consist, in whole or in part of vulgar fractions, these fractions must be reduced to decimals. Which, being done, and the figures clearly written out, you proceed to mark them off into portions, or periods, of two figures each, by making a distinct and legible dot over every second figure, beginning, however, not with the second, but with the first on the right hand; that is to say, with the figure in the unit's place; then proceeding to the left, placing a dot over every other figure; and, after this, if there be decimal figures, marking them off in like manner; as is shown in the following example.

$$6\dot{2}7\dot{0}4\dot{8}1\dot{3}5\dot{9}.2\dot{5}0\dot{3}7$$

Observe, however, with regard to decimals, that if the figures be not *even* in number, as is the case in the example just given, that is to say, if you have a single figure left, an *odd* one, you complete the period, or couple, by annexing a cipher, as in the example thus re-stated, and completed.

$$6\dot{2}7\dot{0}4\dot{8}1\dot{3}5\dot{9}.2\dot{5}0\dot{3}7\dot{0}$$

NOTE.—These dots give you the required portions, on which you are successively to proceed with your work. For instance: Beginning, as usual in division, with the figures on the left, you take the *first* period, which in this case is 62, and so proceed with the other four periods of integers, and then with the three periods of decimals, bringing them successively down for division, just as in ordinary long division you bring down the several single figures of your dividend. But mark; although you have, in the instance just given, *two* figures for your first operation, that is to say, 62, such will not always be the case. To annex a cipher, as in the case of decimals, that is, to prefix it, would be nothing, so we take the odd, or single figure, as we find it, which, in this case, if the 6 were withdrawn, would be 2, the root of which, as the first step, we proceed to extract. Be it, however, for various reasons, remembered, that this 2 is expressive of no contemptible number. Standing in the place in which we find it, it expresses no less than two hundred millions; a sum nearly three times as great as is that expressed by all the other figures in the series.

ARTICLE II.—Find the square root of the first period, and write it down, as the first figure of your quotient, in the usual place, and in the manner of a quotient in long division.

NOTE.—This finding of the root is to be done either by reflection, by a trial with your pencil, or by refering to the Table of square numbers and their roots, as you will find that Table at the end of the Articles of the Rule.

Article III.—Square this quotient number, that is, multiply it into itself, write the product under its period, as you do the product of the divisor and quotient in long division, draw a line underneath, and subtract it from the period, writing the difference below, as in long division.

Note.—This statement will duly illustrate the mode of performing the process laid down in the first, second, and third Articles of the Rule.

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6270481359.250370 ( 7
49
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13
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Article IV.—Bring down the next period of figures in your dividend, as you do the single figure in long division, annexing it, in like manner, to the remainder left in the former operation. And the number thus formed is the portion next to be divided.

Article V.—Double the quotient already found, for a new divisor, write down that double to the left of the number now to be divided, as you do your divisor in long division; only, observe, that you are to leave space for another figure or two between them. Then find out the number of times that this divisor is contained—not in the whole of its dividend, but in all save the last figure, write down that number of times in the vacant space you have just been required to leave, and annex the same figure to your quotient.

Article VI.—Work, now, just as in ordinary long division; that is, multiply your divisor by the new number in your quotient, write the product under its dividend; subtract, as before, and bring down the difference, if any, to the next period, for a new dividend.

NOTE.—The statement and working of the same sum, as directed in the fourth, fifth, and sixth Articles of the Rule, is here exhibited.

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           . . . . . .  . . .
          6270481359.250370 ( 791
          49 . . .
     149) 1370 . .
          1341 . .
   1581 )  2948 .
           1581 .
 1582 . . ) 136713
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ARTICLE VII.—Thus do you proceed, bringing down the several portions, or periods of your dividend, annexing them to your remainders, finding what number of times your quotient, when doubled, is to be found in your successive dividends, annexing that number to both your quotient and your divisor, and so proceeding until you have brought down and divided the whole of the dividend or square number. And then your quotient will be the square root required.

ARTICLE VIII.—Observe, moreover, that if, in the course of the process, after bringing down and annexing a period of figures to your remainder, you find that your dividend is not sufficient to contain your divisor a single time; that is to say, that, exclusive of the last figure, which you know, by the fifth article of the rule, you are here to reserve; if you find that your dividend, exclusive of its last figure, is not sufficient to contain your divisor a single time, then, as in ordinary long division, you annex a cipher to your quotient, and, in this rule, another to its corresponding place in the divisor; and, bringing down the next period, you proceed with your work as before.

NOTE.—The working of the small sum in the margin will duly exhibit this part of the rule.

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       . . .
       93025 ( 305
       9 : :
 605 ) ·3025
        3025
        ....
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ARTICLE IX.—The decimal point is, of course, of the last importance, and its place is thus determined. When the root of any number, consisting of integers and decimals, has been extracted, you count the number of PERIODS in the integers of that number, and then, counting off the *same number of figures* in the root, you mark them for integers, by placing the decimal point after them. This, as you will find, leaves decimal figures in the root, equal in number to the periods of decimals contained in the square number.

	SQUARE NUMBER.		ROOT.
For example,	6270481359.250370	——	79186,371

The Rule and its principle are further developed in succeeding paragraphs, and the process of evolving the square root is recapitulated in a more succinct form in paragraphs 216, 217 and 218.

METHOD OF PROOF.

183. Square the root; that is, multiply it into itself, and, if any remainder were left, add that remainder to the product, and the sum of these, if the work be right, will be the original square number.

TABLE

OF NUMBERS, AND OF THEIR SQUARE ROOTS WITHOUT FRACTIONS.

Nos.	1 to 3,	4 to 8,	9 to 15,	16 to 24,	25 to 35,	36 to 48,
Roots.	1	2	3	4	5	6

Nos.	49 to 63,	64 to 80,	81 to—
Roots.	7	8	9

NOTE.—Nine being the largest number expressed by a single figure, is the largest root that can be required in any single step of the process. For we never, as the learner will recollect, in any sort of division, have to write down in our quotient a higher number than this figure expresses at any single step of the process.

184. Having the Rule thus complete, let us finish the working of the number on which we have been making our experiments and observations; proceeding with the extraction of the root down so low as to three decimal figures, which yielding us the root within something less than a fraction of a thousandth part, is near enough for most purposes. Were it, however, desirable to work yet nearer, it might be done, as in all decimal processes, by annexing ciphers to the remainder, and so continuing the operation.

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                 . . . . .   . . .
              6270481359.250370 ( 79186.371
              49 . . . . . . .
       149 ) 1370 . . . . . .
             1341 . . . . . .
      1581 ) ··2948 . . . . .
               1581 . . . . .
     15828 ) 136713 . . . .
             126624 . . . .
    158366 ) ·1008959 . . .
               950196 . . .
   1583723 ) ·5876325 . .
              4751169 . .
  15837267 ) 112515603 .
             110860869 .
 158372741 ) ··165473470
               158372741
               ··7100729
```

Proof. 79186.371 × 79186.371 + 7100729 = 6270481359.250370

185. Before we had the Rule laid down we were about to inquire into the square root of 361. Let us now proceed with this inquiry, according to the rule.

Having stated the number, and marked it off, we find *one* figure only in our first period, the figure 3. Now the square root of three is something *less* than two and *more* than one. We must not, for many obvious reasons, take the larger number, so the first figure of our quotient will be 1, and we write it down, in its proper place, according to the second article of the rule.

Proceeding according to the third article, to square this quotient number, we find that it has no power of multiplication; it is, in short, its own square. See paragraph 228. Observing this, we have no course

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3̇6̇1 ( 1
1
—
2
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but to write it down, under the first period, which having done, and the subtraction being made, the matter will stand as in the example in the margin.

Having brought down the next period, we proceed, according to the fifth article of the rule, to "double the quotient already found, for a new divisor," and to write it down, in its proper place, according to the instructions in the first clause of that article. The process will then stand thus, space being left for another figure in the divisor.

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    361 ( 1
    1
2  ) 261
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Leaving the unit, or last figure in the dividend, as a reserve, we proceed to inquire, according to the latter clause of the fifth article, how many times this new divisor is contained in our new dividend. Now it is true, that in this case, this divisor is contained 13 times in this dividend; but, for a reason which will appear on a little reflection, we can never, in such a case, rise so high as 10,—so it is, also, in ordinary long division.—For if it would require so large a number as 10 for a quotient, that would prove that the division just before effected had not been duly made; that the remainder was too large; that the foregoing quotient figure ought to be higher, which, however, in this case, as we have seen, it could not be, and therefore, the answer to the question,—How many times are two found in twenty-six? must be something *under ten.* Let us then try the next lower number, let us try what nine will do.

This nine, as you will here see, is the number required. Write it down, according to the rule, in both divisor and quotient, and finish the process, according to the sixth article of the rule, and, as it stands here in the margin: and the result is that 19 is the square root of 361.

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    361 ( 19
    1
29) 261
    261
    ...
```

186. The process is somewhat intricate, and we shall not do amiss by practising on another example or two. In paragraphs 175 and 176, before we had the rule, I suggested the difficulty, if not impracticability of discovering, without that rule, the root of the small number we have just worked, and the yet greater difficulty of discovering that of a number a little larger, namely, 1369. We have seen how very easily, by means of the rule, the square root of 361 is extracted. Let us now take the other number, and see whether its square root can, with the same ease be evolved. It stands in the

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    1369 ( 37
    9 .
67) 469
    469
    ...
```

margin, and, although about four times as large in amount, as the preceding example, it yields, as you see, with equal facility to the operation of the rule.

187. However, the mere operation; the rule, merely, by which the operation is to be performed, is not the thing that I am ambitious to teach. That has been taught by others. We must have the reasons, the principle on which the rule is founded; the principle on which all the operations or rules for extracting the square root of numbers are and must be founded.

188. Let us, for the purpose of developing this principle, begin with the last example; namely, the extracting of the square root of the number 1369.— In paragraph 178 I have spoken of a mode of ascertaining the square root of any number by arranging, in one square, the given number of smaller squares on which the operation is to be made. I spoke of such a mode, for the purpose of illustration, as practicable, but not as a mode ever practised.

189. In conformity with that suggestion, let us now suppose that we have before us the 1369 small squares implied in the proposition; and, that these are, each of them, squares of *one inch*: The question being, what will be the dimension, what the root of a square which shall be formed by, and which shall include the whole of these 1369 small squares?

190. The number is thirteen hundred and sixty-nine, which, in figures may be stated as here in the margin. Now observe the operation of the rule, in reducing these 1369 several squares, into a single square, the root of which, that is to say, one side of which, and of consequence, every side of which is, as the quotient tells us, 37 inches. The first step in the operation is directed to the discovery of the nearest root to the

$$\begin{array}{r} 1300 \\ 69 \\ \hline 1369 \end{array}$$

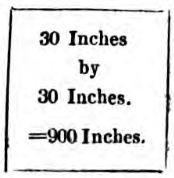
number 13, which 13 is, of course, 13 hundred. The nearest square root of thirteen, as we find is three, the square of which 3, namely 9, being subtracted from the 13, leaves a remainder of 4.

191. Now, let us look well at the *value* of these several numbers; let us not suffer them to remain as mere unsubstantial abstractions; but look well at the things which they here represent.

192. The entire square number, as we have seen, represents 1369 squares, each of one inch. In extracting the root of such a number we have found it expedient to divide it into portions, or periods, as we have called them, and the portion on which we have commenced the operation is the two figures 13, This 13 we find contains, leaving a considerable remainder, it contains the square of 3; that is, 9, and our present purpose is, to ascertain what the quantities are that these several figures represent.

193. In the first place we have ascertained, by marking off into periods, the given square number, that its root will be expressed by *two* figures, the higher of which will therefore stand in the place of, and represent *tens*. This 3, the root of 9, is the first, and higher figure; therefore it represents *thirty;* and its square, 9, consequently represents *nine hundred.* Look at *the places* in which these several figures stand, and you see that such is their respective value.

194. Now these *nine hundred* are nothing save so many of the small squares with which we are dealing; and the *thirty*, which is their square root, tells us that nine hundred square inches will form one square of thirty inches. Let us suppose this square of thirty inches to be represented by the figure in the margin.

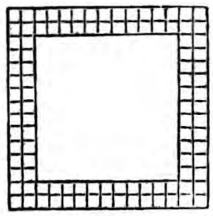


195. But it is the size of a square constructed of *thirteen hundred* and *sixty-nine* that we have to discover. Very well: we have ascertained the size of a square which *nine hundred* will form, and have, thereby, disposed of so many of the number. This much of the process leaves us, as is shown by the statement in the margin, *four hundred and sixty-nine* of the lesser squares still to be operated upon; still to be disposed of.

$$\begin{array}{r} \dot{1}3\dot{6}9\,(3 \\ 9 \\ \hline 469 \end{array}$$

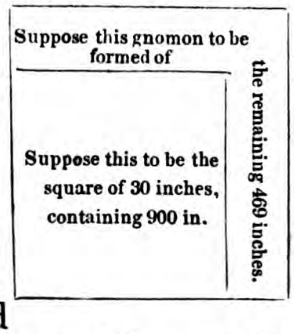
196. The question now before us is, In what manner does the process laid down in the rule effect the disposal of these remaining 469 small squares?

197. Had we the square forms all laid before us, we might proceed to place these small ones *around* the larger one, somewhat in the manner indicated by the figure in the margin, until we had disposed of the 469.



But arithmetic, which has enabled us, in a single minute, and by the mere writing down of a very few figures, to dispose of *nine hundred* of the smaller squares; this arithmetic saves us from so tedious, and so childish a proceeding. Still we have to inquire; In what manner does the process laid down in the rule, dispose of these remaining 469 small squares.

198. There is another mode in which, had we the several squares before us, we might dispose of the 469 small ones, in the enlargement of the larger square. Instead of arranging the small ones all *around*; instead of entirely enveloping the large square with the remaining small ones, as indicated above, the *square form* may be preserved by making the additions *to two of the sides* merely; as shown in the figure annexed to this paragraph.



199. Other modes of actually arranging the materials to the proposed end than these two, there are not. So that, if the arithmetical process be indicative of, or founded on any palpable and visible erection of a real square, that process must proceed in one of these two modes. Let us, then, look closely at the process, and inquire.

200. The arithmetical process standing as here stated in the margin, we are next directed, by the *fifth* and *sixth* articles of the rule, to "double the quotient already found for a new divisor."—*There are, also, some little expedients for shortening the process taught in the article, which expedients may, with advantage be here omitted.*—We are to double this quotient, and to write down its *double* in the usual place of a divisor; then, to find out the number of times that this divisor is contained in the number remaining, to annex the figure representing that number of times, to the quotient, and so to proceed with the work, as in ordinary long division, and as is here shown in the margin. Leaving out, however, as I here do, the shortening expedients, I must call the first figure in the quotient,—not *three*,—but *thirty*—which number it really represents,—I call this first figure thirty, the double of which, as directed, I write down, for the new divisor: then finding that this divisor is seven times in the dividend, I annex the 7 to the quotient, multiply the divisor by it, and subtract the product, which is 420, from the remainder.

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1369 ( 3
   9
 ---
 469
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     1369 ( 37
        9
      ---
 60 ) 469
      420
      ---
       49
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201. Let us now, again, stop to mark what we have done; let us, as we did before, look well at the value of the several numbers which have presented themselves since our former examination; let us see clearly, what it is that they respectively represent.

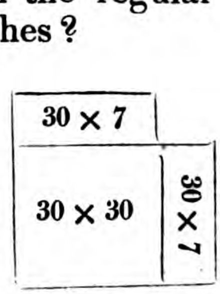
202. The first number is the double of the quotient, namely *sixty:* the second is *seven*, and their product is *four hundred and twenty*. Now it is of square inches that we are treating. And we have multiplied sixty such inches by seven; or, rather, we have multiplied *thirty doubled*, by seven; that is to say, two *thirties* by seven.

203. Now, it is but a part of our business here to observe, that the product of thirty multiplied by seven is 210; our main purpose is, to consider, *what is the size and the form* of the superficies described by the words, *thirty inches by seven.*

204. Our course of reasoning only requires me to state that which is obvious to the reader, namely, that the superficies represented by these two dimensions is a parallelogram, thirty inches long and seven inches broad.

205. But how does the discovery of this contribute to the solution of the question on which we are engaged; in what manner can a parallelogram of this description be applied?

206. It is applied thus,—It will be recollected that, in the first step of the operation, we found *a square of thirty inches.* Here we have a parallelogram of the same length, that is, *thirty inches*, and of the breadth of *seven inches;* nay, we have *two* such parallelograms, and the question is, How can these two parallelograms be applied in the regular enlargement of the square of thirty inches?



207. These parallelograms are to be applied; by the principle of the rule, indeed, they are applied in the manner indicated by the figure in the margin; that is to say, one of them goes to augment the primary square in what may be termed its breadth, and the other adds to its depth.

208. And what, let us now ask; what is the number of the small squares on which we have just been working; what is the number of these squares employed in the formation of these two parallelograms? That number is, as is shown by the process, 420; and this is the third number concerning which we proposed to inquire.

209. Thus have we, by two steps of the process, employed, or *worked up*, as we may say, first 900, and now 420, making, together, 1320 of the small squares, which it was proposed to erect into one larger one; and we have yet remaining, as is likewise shown in *the last statement* of the process, 49 of the small squares; from which arises the question, how can these remaining 49 be disposed of in the erection of the square; in what manner does the rule teach us to apply them to that end?

210. Casting our eyes on the figure, as we left it three paragraphs back, we see a deficiency in it, which must be supplied, in order to complete the square. This deficiency, which is occasioned by want of length in one of the parallelograms to reach over the end of the other, this deficiency is, as is sufficiently obvious, a small square of the breadth of those parallelograms; that is to say, a square of seven inches. We have a remainder of 49, the root of which being 7, is just the quantity required to complete the larger square. It is scarcely necessary to observe, that for this first experiment I have deemed it proper to adopt a number which forms a square without leaving a remainder. In a subsequent analysis of the process on other numbers, it will be our business to mark how *remainders*, how *fractions* are disposed of in the formation of the square.

211. However, although we have now completed the geometrical figure of the square, we have yet to trace the manner in which the arithmetical process treats and applies this remainder of 49 to that end.

212. In paragraph 200, in which we left the process standing with this remainder of 49, in that paragraph I alluded to some expedients for shortening the process, which expedients, for greater simplicity, I deemed it advisable there to omit, and that omission it was which left this remainder on our hands. We there "*doubled the quotient* 30 *for a new divisor*," and so proceeded with the work. Whereas THE RULE DIRECTS US—see ARTICLE the FIFTH—to annex to that *double*, the number of times that it is found in a certain part of the dividend, which number of times being written, also, in the quotient, we are directed, in the *sixth* Article, to proceed with the operation, as in long division.

```
     1369 ( 37
     9
60 ) 469
     420
      49

     1369 ( 37
     9
67 ) 469
     469
     ...
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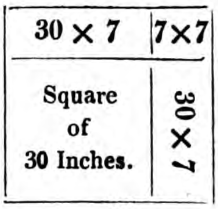
213. This annexation of the number of times that the doubled quotient is found in the dividend, this addition to the new divisor, is the *expedient for shortening the process*, to which I alluded. Omitting this expedient, we had the 49 as a remainder, as it again appears in the first example here in the margin. Whereas, *when we perform the operation* ACCORDING TO THE RULE, that is to say, when we annex the new quotient figure to the new divisor, and so proceed with the division, the whole number is disposed of; there is no remainder. And so the square number 1369 is duly resolved into its root of 37.

214. Is it observed that we *double the former quotient number*, for a new divisor, and do not double the number just found, which number, how-

ever, we annex to it; is this remarked upon; and are we asked, why one number is to be doubled, and not the other; why this divisor is to be composed of a *doubled number* and a *single one*? The answer to this question completes the analysis. And this answer we have if we cast our eyes on the geometrical figure as it stands in the margin, recollecting, at the same time, that our first step in the process gave us the larger square of 30 inches, with 469 yet to be applied to its enlargement, that this enlargement is effected by the addition of one parallelogram of 30 inches by 7, on one side, and by another of the same dimensions on the other, that is to say, by THE DOUBLE of one parallelogram of this size, and by ONE, *only by one*, small *square of* 7. Therefore it is, that we *double* the 30, or *former quotient, for a new divisor*, thereby making it 60; and requiring the 7 *only once*, we *annex* it, making the new divisor 67, complete.



215. Thus have we completed the square; thus have we employed and worked up, in the construction of that square, the number which was given wherewith to form it. And in this construction we have that which we sought, namely, THE PRINCIPLE OF THE RULE. And having this; once knowing how it is that figures are employed in order to reduce a mere aggregate number of smaller squares to one large one; once knowing HOW IT IS that figures are thus employed, we not only proceed like intelligent beings, knowing the meaning of every movement we make, but when, for want of use, or of practice, we forget the rule, we can, without the aid of the book, re-construct that rule, simply by calling to mind the palpable GEOMETRICAL FIGURE, and with it the PRINCIPLE of the operation from which it is scarcely conceivable that it can be dissociated.

216. Another recapitulation or summing up of the process will afford us an opportunity, not only of bringing it within a smaller compass, and of fixing it more clearly in the mind, but will, also, enable us to introduce one or two other circumstances that are requisite to the completion of the rule, the introduction of which on any earlier occasion, would have tended to encumber our progress.

217. The rule, then, directs us, first to take the number, the root of which we have to find, and to mark it off into *periods* of a suitable size for the process; so to mark it off that, when we take the first period, that is, the highest part of the number for the formation of the first and chief square, we shall still, in one period at least that we leave, have a sufficiency to form the two parallelograms and the small square wherewith to complete its enlargement. Having so marked off the given number, we find the root of the first period, which, even when a single unit only, is, because of the place in which it stands, always significant of a larger quantity than that signified by all the other figures; finding the square root of this first period, we are directed to write that root down as the *first and chief figure* of our quotient or *root;* then to subtract *its square* from the period spoken of; then, to the remainder, if there be one, we are to bring down the next period; and, indeed, remainder or not, we bring down the next period wherewith to enlarge the first or chief square. Having done this, having thus what is usually termed a new dividend, we see, whether it be sufficient to supply something more than the double of the quotient, that is to say, whether it will furnish us with the two parallelograms and the square, which, if it will do, we proceed with the division, if it will not, we say *not*, by writing a *nought* or cipher in the quotient, and a corresponding one in the divisor, and then proceed

by bringing down to this inadequate dividend, the next period of figures, if there be yet a period to bring down; but, if there be no such period, we have to deal with the remainder as with a decimal fraction; of which hereafter.

218. However, if, leaving out the last figure, as a reserve for the small square; if, leaving this figure, we find in the new dividend sufficient to take *the double of the quotient*, our next thought is, *how often* will it take that double quotient. In the example on which we have been working, we found that the new dividend would contain the double quotient *seven times*, therefore, it was, that our parallelograms were, in breadth, 7 *inches*, and one small square to complete the corner, was also a square of 7 *inches*. But had this double quotient been found six times only, in that portion of the dividend, *six*, instead of *seven inches* would have been the breadth of the parallelograms and of the little square; or, had that double quotient been found *once* only, then would *one* inch have been the breadth of the parallelograms, and the little square one inch also.

219. Although we have thus accomplished a development of the PRINCIPLE, and shown how nicely the operation and the rule conform to it, yet are there *two* other points incidental to that operation, on which it remains to say a few words. These points are; *First*, The manner in which, when the square number on which we have to operate contains more than *two periods* of figures—for to this extent only have we yet traced the operation—and, *Second*, When we have a remainder insufficient in quantity to give us an *integral* number in the quotient or root. A few words remain to be said on the manner in which, in each of these cases, the quantities are disposed of in the enlargement of the square.

220. First; On the case in which the square number, the root of which is to be extracted, contains *more than two* periods of figures; our analysis having been performed on a number of two periods only. The question is, how does the third; and how does any subsequent period go towards the construction and enlargement of the square; and, in what manner are such quantities disposed of in such construction and enlargement?

221. The answer is very simple, and clear. The square is complete, as we have seen, after the operation on the first period; it is enlarged, and as we have seen, again completed, by the operation on the second period; and, just as the second period goes to its enlargement and completion, so does the third, and so does any succeeding period.—How orderly, how beautiful, and how universally applicable to their respective purposes, are the principles of science!—We have this beauty, this order, this harmony again in the

222. Second, and only remaining point to be considered; namely, the manner in which, when we come to deal with fractional quantities, the manner in which, when we have done with integral numbers, these *parts* fall into their office of yet further contributing to enlarge the square, until it have involved itself in the whole of the quantity proposed to be resolved.

223. Fractions are parts; a fraction is a part of a whole. Now, what is a whole; what is one of the integral quantities with which we have been dealing in the foregoing process; what is it but a square inch?

224. We supposed that we had 1369 square inches to reduce into, or to construct into one square; and

we found that this quantity of superficial measure gave us one square of 37 inches. We have seen the manner in which, by the working of the problem, these several inches were employed in the construction of this larger square; we have seen that the last divisor being found *seven* times in its dividend, gave us the requisite parallelograms and the little square wherewith to enlarge our first square from 30 to 37 inches; that is to say, to add 7 inches to the size of the first square: now, had the number to be evolved, instead of 1369 been 1296, the size of our square would have been 36. Or, had it been 1444, we should have had material wherewith to make the additional parallelograms and the small square *eight*, instead of *seven* inches broad; for 1444 gives a square root of 38. Thus, then, these several square numbers give each a root in whole numbers or integers.

225. But, suppose that, instead of either of these numbers, we had to evolve the root of some number between some two of them; suppose that, instead of 1369 which just gives us a square of 37 inches, we had a number which would give us one of 37½ inches; that is to say—for thus is the quantity expressed decimally—suppose we had 1406.25. How, then, would the half inch be dealt with; how brought, by the arithmetical process, to the enlargement of the first square? The answer is shown in the process here exhibited in the margin, and may be thus explained. As the first square of 30 inches is enlarged, by the second step in the working, to 37 inches, so, by a third step is this 37 enlarged by the decimal .5, that is 5 tenths, or one half. It is enlarged in the same manner as the first square was enlarged by the second step of the process; that is, by the addi-

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      1406.25(37.5
      9
  67) 506
      469
 745) .3725
       3725
       ....
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tion of two parallelograms and one square, the parallelograms being in this case each 37 inches long, and half an inch broad, and the square being simply a square half inch.

226. Yet further to illustrate this operation of the rule.—Let the number, the square root of which we are to extract, be 1387.5625. This is a smaller number than the last, and its root, supposing that we are still treating of inches, is 37¼ inches But, although smaller in amount, it has one period of figures more, and consequently requires one operation more, and gives us one figure more in its root. This root, as it will appear, is 37.25.—Now the .25 being expressive of one quarter merely, it may naturally be asked, Is the square of 37 increased, in this case, by the addition of two parallelograms and a square of one quarter of an inch in breadth; or, is it increased first by parallelograms and a square of 2 tenths of an inch; and being thus 37.2, is this square then augmented, in like manner, only by the minute layer, on two of its sides, of 5 hundredths of an inch; that is to say, by the small quantity expressed by the second decimal figure;—In which of these two modes is it that the square of 37 is increased? The answer is, that the increase is made in the latter mode. At each step in the operation by which we add a significant figure to the root, at each of these steps do we make an addition, such as is expressed by that figure, an addition to the size of the square; at each step in the process the square is completed; the next step again leaving it complete, whether the addition made to its dimension be ever so large, or so minute as to be imperceptible to our senses.

227. Take one other example, and let it be a square number, giving us a root of three figures, and without decimals: let this number be 56169. The

working of this example here follows, and with this a diagram, showing the manner in which, by the second, and then by the third step in the process, the additions are made to the first or primary square. And with these is given a statement, in figures, of the several quantities, of the several squares and parallelograms, which quantities added together, and amounting to the given square number, furnish a new, and an additional proof, not merely of the correctness of the process, but likewise of the soundness of the principle which we have developed.

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       . . .
       56169 ( 237
       4
  43 ) 161
       129
 467 )  3269
        3269
        ....
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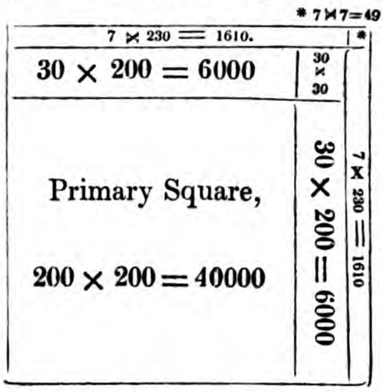
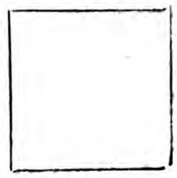


The primary square	200 × 200 =	40000
The first two parallelograms . . {	30 × 200 =	6000
	30 × 200 =	6000
The small square	30 × 30 =	900
The second two parallelograms . {	7 × 230 =	1610
	7 × 230 =	1610
The small square	7 × 7 =	49
		56169

228. In the notes appended to paragraph 185 it is observed that the unit *One has no power of multiplication.* This is a negative proposition, and therefore not directly demonstrable. To this observation it is added, that *One is its own square.* Now this is scarcely more demonstrable than is any self-evident proposition. It is, indeed, but an *identical* proposition; the *root*, and the *square* of *one* being identical. For, what is THE THING spoken of? In this case it is *superficial quantity*; it is one square inch,

one square foot, one square yard, one square mile, or some such space or surface. Now let it be the first of these. Let this figure be a square inch. That it is its own root, and its own square; or in other words, that its *root* and its *square* are *identical* is evident. For let it be, as it is in all other quantities, that *the square is* MORE *than its root*. The supposition is irreconcileable and absurd: and equally absurd would be the supposition that the root of One is *less* than its square. If it were *more*, or if it were *less*, then it would not be *one square inch*, which, by the terms of the proposition it is to be. So the root and the square of ONE is the same thing.



229. Let us take another view of this matter; for it is important. Let it be that we have to extract the square root of $2\frac{1}{4}$. And let the quantity be two and a quarter inches. Our business, then, is to ascertain the size of *a square*, that is, the extent of one side of a square, the contents of which shall be $2\frac{1}{4}$ inches. In order to work the question, we must have the fraction stated as a decimal. So the expression will stand thus, 2.25. Now to extract the square root of this number. The process stands here in the margin. The answer to the question being that 1.5 is the root; that is, that $1\frac{1}{2}$ inch is the size of a square the contents of which are $2\frac{1}{4}$ inches. So much for the process and the statement *in figures*.

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      2.25 ( 1.5
      1
2.5 ) 1.25
      1.25
      . . .
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230. Let us now direct our attention to the material of which we are treating. It is of superficial quantity; the quantity being two and a quarter square inches. Let this quantity be represented by the

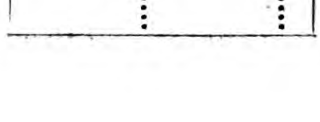
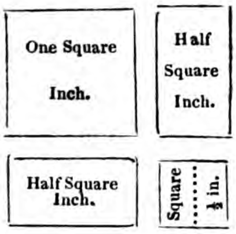


figure in the margin. The arithmetical process gave, as above, for the first in the quotient, the figure 1. This One is an integer. It is *one square inch*. Let it be represented by the square below. Deducted from 2.25 this 1 leaves 1.25 still to be applied to the formation or enlargement of the square. The next step in the process applies this quantity thus. It furnishes us with two parallelograms, each one inch long and five tenths of an inch broad. See the decimal .5 in the quotient, multiplied by 2 in the divisor; and, then, the corner is filled, and the enlarged square completed, by the square half inch, formed of the quarter inch slip, or parallelogram which appears in the first figure; and which is represented by the decimal .25, the square root of which, as seen in the quotient, is .5.



231. In the foregoing process of figures, the decimals are marked off from the integers, at every step, and this has been done in order to show, at every step, the value of every figure. This marking, however, on ordinary occasions, is quite unnecessary. It must of course, when there are decimals, be seen in the *first dividend*, or square number, and again in the *root*. The statement, in short, should stand as here in the margin.

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    2.25 (1.5
    1
25) 125
    125
    ...
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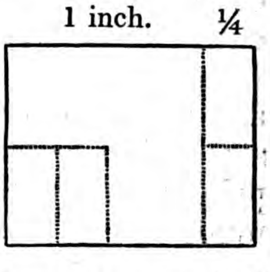
232. Let us not overlook the favourable opportunity, but mark once more, how aptly, and how nicely the arithmetical process applies the given quantity, namely, 2.25, to the formation of the required square. First we have *a square inch*—see the diagrams—to enlarge this square we require two parallelograms, each one inch long, that is, together two inches. Observe this 2, the double of the first

quotient figure, take its place as a divisor. Then comes the question, *What breadth may these two inches be?*—They may be 5, is the answer. Five! But *five what;* we have only 1.25—that is, *one inch and a quarter*, wherewith to form them; so what is this 5? It is the decimal .5; it is five tenths; that is, *one half.* So the two parallelograms being but half an inch broad, although together two inches long, require only one inch, *one square* inch of superficies wherewith to form them.

233. There yet remains the decimal quantity .25 to dispose of, and the small square in the corner is to be supplied. The parallelograms are half an inch broad; the square must be the same breadth; it must, then, be a square half inch.—*A square half inch!*—Why, here is only the decimal quantity .25, that is, a quarter of an inch wherewith to form it! And yet it does appear, from the working, to be sufficient. For the decimal .5, that is, a half, when squared, gives us but .25, that is a quarter.—How is this, that a square *half* should thus seem to be formed out of *a quarter?*—Nothing more natural, nothing more reasonable than this question. It is answered, however, when we call to mind, or look at the things of which we are speaking. Look at a diagram. Let this in the margin represent a square inch and a quarter. It is sufficient, as I have said, to look at it, in order to have the answer. It is not *a square inch and a square quarter* of an inch; but a *square inch* and a *quarter of that inch.* Now *a quarter* of a square inch will be equal to, will make *a square half inch.* It is, in fact, as in the diagram, a parallelogram, one inch long and a quarter of an inch broad, if divided in the middle it makes two parallelograms, each half an inch long and a quarter of an inch broad;



and if placed side by side they form a square half inch, as shown in the diagram; and this square half inch, as is likewise apparent, is one quarter of the square inch. How perfect, how consistent, how admirable is truth!

234. There are other methods, and other expedients than this which I have laid down for extracting the square root; but all those methods and expedients are dependent on the same principle; they are merely other methods, and other expedients, to which the persons who have incurred the labour of learning them, may have become partial, without any rational ground of preference. With respect to this method, which I have adopted, it is, I believe, as simple as is any other, and I have given it the preference, because it appears to me to keep the principle constantly in sight. So much for methods of performing the operation. But, for expedients, of which doubtless there is a variety; for expedients of approximating to the root, the readiest and the best, I imagine, must be by Tables of Logarithms, of which I shall find occasion to speak, when, at the close of the article on the Cube Root, I treat of the extraction of that root by the use of the same Tables. It only remains with us now, with regard to this affair, the extraction of the square root, to say something respecting the uses of the operation, the principle, and the practice of which has engaged so much of our attention.

Of the Utility of this Rule.

235. This, after all, is the important point; a point which ought to be borne in mind in all our undertakings and employments. The utility, however, may be direct, or it may be indirect. There are many direct advantages derivable from a knowledge of the rule. It has numerous practical applications;

besides that, without it there would be a sort of chasm in mathematical science. But the advantages, chiefly, which I contemplate in the study of it, when that study is pursued in the manner in which I have here attempted to conduct it, these advantages are of an indirect nature, and to be looked for in the improvement of the mind; in the pleasing and invigorating exercise which is here required; and which study will, I expect, find its ample reward in an increased relish for truth; and in a conscious improvement of the faculties by which truth is to be discovered.

236. With regard, however, to the direct uses of the rule. I may name, and may treat of, but a few of them; and these, too, must be the more obvious and more general. Architects and Surveyors may, for any thing that I know, follow their occupations without the actual use of the rule; they may accomplish its purposes by the aid of scales, of tables, and of little expedients or contrivances which they learn from each other, or derive from their practice; but the rule, or rather, the principle of the rule, is the source of several of these expedients; and it is satisfactory, and useful, therefore, to be acquainted with that principle. But in Gauging, in Mensuration, in Engineering, both civil and military, and in several of the sciences dependent on calculations, the employment of this rule, or a reference to its principle, is of frequent, and in some of them of constant occurrence. However, as I have intimated, an enumeration of a few of the more general and more obvious uses of this rule must here suffice; and we will treat of them in this order.

FIRST.—To find the mean proportional between two given numbers.

SECOND.—To find the size of a square equal in area to a superficies of any other form.

THIRD.—To find the diameter of a circle, the area or contents being given.

FOURTH.—To find the hypothenuse of a right-angled triangle, the base and perpendicular being given.

FIFTH.—The hypothenuse, and one other side of a right-angled triangle being given, to find the other side.

237. A mean proportional between two numbers, is a number which shall stand between, bearing the same relation towards the smaller of the given numbers, as does the larger towards it. Thus, 3 is the mean proportional between 1 and 9; for this mean, or middle number, is three times as large as the first; which is the proportion borne by the last to the middle; and the three terms, or proportionals would stand thus 1 : 3 : 9. In like manner, 3, 9, 27 are proportionals; 9 being the mean proportional. Again, 8, 32, 128 are proportionals; the mean of which is 32. Now, how to discover this middle number, having only the two extremes. The Rule is, to multiply the two given numbers together, and, of the product to extract the square root; which root will be the mean proportional. See the operation in the margin: to find the mean proportional between 7 and 175.

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    175
      7
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   1225 ( 35
   9
65) 325
    325
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    ...
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238. The size of a square, equal in area to any given superficies, is found by a mere extraction of the square root of the number expressing that superficies.

239. The Diameter of a Circle is the longest right line that can be drawn within its circumference; this line, therefore, passes through the centre, and extends each end to the circumference. A line of this description is one of the measures or gauges of a circle, and, to a square, of which this line might be

one side, the circle, of which it is the diameter, has been found to bear a proportion of somewhat more than three-fourths of a whole: the proportion is generally expressed by the decimal .7854 to a unit. Now, the size of a circle, the area of which shall be equal to a given quantity, is frequently required. The extraction of the square root of the number expressive of any area, will give us the size of its square; so to find the diameter of a circle, the area being given, reduce that area to its due proportion, by dividing it—as we must term it—by the decimal .7854, then extract the square root of the quotient, and that root will be the diameter required. For an example—setting aside small fractions—suppose we would find the diameter of a circle the area of which should comprise 5 feet. We find that 5, that is 5.00000000, divided—or rather reduced by .7854, gives us 6.3278, the square root of which is 2.51; that is a trifle more than 2½ feet.

240. The hypothenuse is the longest side of a right-angled triangle. A right-angled triangle is a triangle having one of its angles a right-angle—see paragraph 137—and, opposite to this angle is the hypothenuse. Now, whatever may be the size of the triangle, or the relative size of its two other sides, the sum of the squares of those two sides will be the square of the hypothenuse. To find the hypothenuse, then, we square the dimensions of the two given sides, add these two squares together, and the square root of that sum will be the dimension of the hypothenuse.

241. This fifth and last theorem, of which I propose to speak, is a mere inference or corollary from the latter. The square of the hypothenuse being equal to the squares of the other two sides of a right angled triangle, if we subtract from the square of the hypothenuse the square of the given side, whatever remains will be the square of the other side.

OF THE CUBE, AND ITS ROOT.

242. The sphere may be as simple, but the cube is more compact. The prism and the pyramid may be equally compact; but, whilst there is no form in which more matter can be comprised within given dimensions than is comprised in the cube, its form is more simple than are those, for the cube, like the sphere, requires but one dimension; its length, its breadth, and its thickness being the same: so that a single line will be the measure, and will apprise us of the quantity comprised in any cube.

243. This single line, which may be the length, or the breadth, or the thickness of the cube, this single line is the Root of the cube; as it is the root of one of its sides, each of which sides presents an exact and true square.

244. The cube, like all other solid figures or bodies, has length, breadth, and thickness; that is, extent or dimension in THREE several directions. Length, necessarily requires *two ends*, breadth, demands *two sides*, as does thickness also require *two sides*. These ends and sides of a cube are all equal; they are, in fact, *six* sides, and these sides are all squares. Thus, then, a cube may be defined—A quantity of matter bounded by six equal squares.

245. However, the cube has length, breadth, and thickness, and these three dimensions being uniform and equal, a line, which is the true measure of one of these dimensions, as before stated, is the Root of the cube. The discovery of this root, the method of discovering or of finding this line, by the use of numbers, and the principle on which the operation proceeds; these are the matters concerning which we are now to inquire.

246. Not, however, that the cube, the root of which is to be found, not that the cube is supposed to be formed, and lying before us. Were it so, the question as to its root would be solved by simply applying a suitable measure or gauge to one of its sides. No; the supposition is, that it is given to us to inquire—*What will, or what would be the size of a cube constructed of a given quantity of materials?* This is really the question in all propositions as to the cube root.

247. The question may not specify anything about *materials*. It may ask, merely, what is the cube root of a given NUMBER; a mere *number*, abstract from any specific quantity or material. And this is the manner in which mathematicians commonly deal with the subject.

248. Still, however, *quantity*, and not *number* merely, is supposed. It is, for instance, proposed to us, to find the cube root of 3375—which root is 15—the numbers are applicable to any description of measure of solid quantity, that is, of bulky quantity; it may be inches, feet, yards; or gallons, pecks, bushels, and so on. But it is not the custom to carry on the calculations immediately in the latter denominations of measures; that is, in gallons, pecks, nor bushels. For measuring, that is to say, for *gauging* vessels, or places containing, or designed to contain articles measured thus, the practice is to make the calculations in *inches*. For example. Did we want to know what would be the dimension of a cubical vessel adapted to contain exactly the above number, that is, 3375 gallons, of liquor measure, we should proceed thus: Finding that each gallon contains—according to the recent enactment of what they have called the "Imperial Gallon"—finding that each gallon contains 277.274 cubic inches, we should multiply this 3375, the number of gallons, by 277.274;

that is to say, we should reduce the gallons to cubical inches; then, extract the cube root of the number of those inches, and thus should we find the required dimension in inches. This root would then be 97.81. So a vessel 97.81 inches each way; that is, a cubical vessel of this dimension, would contain 3375 "*Imperial*" Gallons. This is the manner in which the practical gauger proceeds; and, to this end he has his TABLES, which inform him, on a mere inspection, of the number of cubic, or, as they are familiarly called, of *solid inches* in each of these measures; and of the number of gallons in any given number of cubic inches.

249. So that, although, as mathematicians merely, we deal in abstract numbers, those numbers, in propositions relating to squares and cubes, are always applicable to actual form and matter. After solid measure; that is to say, after the cube, which having length, breadth, and thickness, has, in point of quantity, all that matter can contain, after the cube, containing these *three dimensions*, and called, therefore the THIRD POWER, after this come fourth, fifth, and *higher powers*, with which, for certain purposes, mathematicians have to deal. But all these are mere abstractions. Having no reference to form, nor to substance. And, in the contemplation of them the mind cannot be too sparingly indulged. I know of no other pursuit so well calculated to clear and invigorate the mind, as is the pursuit of this science of numbers, so far as real substances are the objects of calculation, but beyond this, the mind seems to have nothing to sustain it, and I am convinced by observation, and by my own experience, that however it may soar in the pursuit, an habitual indulgence in these abstractions, impairs, and very frequently destroys the mind. Let us now proceed with our inquiry.

250. The question to be resolved is this; In what manner shall we proceed, having a certain quantity of materials proposed for the purpose; in what manner shall we proceed, without actually building up, or constructing a cube; in what manner shall we best, by the use of numbers alone, determine the size of a cube constructed out of a given quantity of materials?

251. Now, the materials must, of course, be of a cubical form, or reducible to this form. We cannot work with incongruous materials. Again, *the parts*, wherewith it is proposed to construct the cube, must be of an uniform size; or, as the first step in our undertaking, they must be reduced to an uniform size. For example; suppose we had to determine what must be the size of a cubical vessel, that would just contain 560 bushels, 3 pecks, 1 gallon, and 3 quarts. We must, in the outset, proceed to bring these several quantities under one denomination. Which denomination should be *the integer*. Did we choose that bushels should be the denomination, then must the pecks, gallon, and quarts be reduced to a decimal, as they are reduced, after being stated as fractions, in proposition (7) subjoined to paragraph 105. These pecks, gallon, and quarts, reduced to the decimal of a bushel, are .968; so the quantity, in bushels, for which we have to find a cubical vessel is 560.968.

252, But we do not extract the cube root of this, *the number of bushels*. Were we to do this, the answer to our inquiry would come out in bushels, and we should be but little nearer to our object; which object must, of course, be to ascertain the dimensions of the required vessel, in feet and inches; or in some such measurement. To attain this end, then, before we extract the root, we reduce the bushels to inches; of course, by multiplying the number of bushels by the number of cubical inches wh ch constitute a bushel. And then, extracting the root of the number produced by that multiplication, we have the required dimension in inches.

253. However, before we proceed to the operation, we must look yet a little nearer at the object we have to attain.

254. Speaking of it arithmetically, the cube is the *third power*. That is to say, it is the product of the length, the breadth, and the thickness of a body bounded by six equal sides; it is the PRODUCT of these THREE *dimensions* multiplied together. That is THE CUBE.

255. But it is THE ROOT of the cube that we have to inquire into; it is *one* of these three dimensions that we have to discover; it is, having the quantity, having the sum, the amount of material contained, or to be contained, within a form of this description, it is the having *this quantity* merely *named* to us, to find out, what must be the dimension of one of the equal sides of the whole quantity.

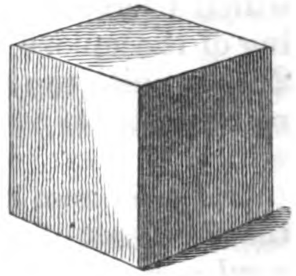
256. There are very small numbers the cube root of which we can see at once, for instance 8, the root of which is 2; for $2 \times 2 \times 2 = 8$. And this, it may be as well to observe, is the *first*, or smallest number that has a *whole* number for its cube root: this is the smallest cube number having another yet smaller whole number for its root. But, and we cannot have a better opportunity than the present of dismissing this point; but, observe; the *unit* ONE may be descriptive of a cube; it is smaller than *eight*, and yet it has a *whole* number for its root; if *one* can be called a *number;* which, in strictness I think it cannot. However, it would be inconvenient, and seldom of any use to make this strict distinction, so

let it pass amongst numbers. The unit ONE, then, has a whole number for its root. That root is identical with itself. It is, in fact, its *own root;* and that which I have said respecting this integer, in speaking of the square, and of *the root* of *one*, in paragraph 228, is quite to the purpose, and I beg the reader to refer to it.

257. We can pronounce, without hesitation, that the cube root of eight is two. *Three* is the next *integral* root; and its cube is twenty-seven. The next cube number of this description is *sixty-four;* and it requires but a trifling effort to trace down its root to *four*. But, when we have to discover the roots of large numbers, and yet more difficult, when we depart from the easy and plain work amongst integers, and become involved in fractions, it is the regular process which the rule prescribes; or the yet higher and more gratifying knowledge of the principle on which the rule must proceed; it is these only that can disengage us. Having these, large numbers or small, fractions or integers, yield alike to our inquiries. Let us then turn to the consideration of the rule, or to that of the principle on which the rule is founded. And first as to THE PRINCIPLE.

258. In paragraph 245 we have seen, that the root of a cube is a line, being the true measure of, that is to say, the length or the breadth of one of its sides; all of which are equal. Now length must have its denomination, it must be feet, inches, yards, furlongs, or something of this description. Dropping the merely abstract numbers, let us give the quantities with which we are dealing a certain denomination, and let us, as clearly and strongly as we can, throughout the whole of this discussion, figure to our minds, where we have not the actual image before us; let us thus figure, clearly as we can, the real, palpable quantities and forms of which we are treat-

ing. Our denomination might be yards, or feet, or inches, or any other convenient quantity; but *let us fix on* INCHES.



259. Now a cubic inch is, of course, a small block, having six equal sides, each of those sides a true square, the dimension, or root of which square is one inch.

260. Let us imagine that we have a number of these small blocks before us; and let it be asked, What will be the size of a cube constructed of eight of them? We see that two in length, two in breadth, and two in height would be the size: so two inches would be the root of such a cube.

261. Again, let us be asked, what would be the root of twenty-seven. Here a little experience in the matter, or the actual construction of a cube with 27 of the small cubes, might be requisite to enable us readily to give an answer. We should soon find, however, that three of the little cubes, every way; that is, three in length, three in breadth, and three in thickness, would be the size of the cube formed of 27. *Sixty-four* of the inch cubes would construct a cube *four inches* over each of its sides; *one hundred and twenty-five* would make one *five inches* over; and thus might we proceed to a certain extent, and thus, indeed, do we proceed, without the aid of the rule, up to the several cube numbers of which 6, 7, 8, and 9 are the roots: so far do we proceed without the rule, and merely by a little reflection, by a trial with the pen, or by a reference to a short table of these few smaller cube numbers and their roots, such a table as accompanies all instructions and rules on this subject, and such as I shall here insert; calling it

A TABLE

OF PRIMARY CUBE NUMBERS, AND THEIR SEVERAL ROOTS.

Cube Numbers	1	8	27	64	125	216	343	512	729
Roots . . .	1	2	3	4	5	6	7	8	9

262. These cube numbers I call *primary*, because we have their several roots at *once*, or at FIRST; those roots being expressed completely and fully by a single figure. It is a name which I must beg the reader to bear in mind, seeing that this, or some equivalent name, wherewith to distinguish these numbers, will prove essentially serviceable throughout this investigation.

263. Thus, then, as shown in the foregoing table, 9, which is the highest unit, is the root of the cube number 729; and thus far, if we confine ourselves to integers, may we proceed without any rule.

264. Thus far, I say, we may, without any peculiar knowledge of rule or of principle, proceed, if we confine ourselves to those which I have termed PRIMARY CUBES. But it is observable that, although in this Table of Primary Cubes, the roots advance by regular steps, 1, 2, 3, 4, and so on, without the lapse of any entire numbers, yet is there room, between the several cube numbers, for other numbers; the lapse, for instance, between the cube of 8 and that of 9 being no less than 216 whole numbers; and that between the cubes of 7 and 8 is 168, whole numbers also.

265. Now these intervening numbers, that is, for example, the numbers between 343 and 512, which are the cubes of 7, and of 8, respectively; all numbers between these will have roots, of course, expressed by the smaller unit, that is to say, by 7 and some fraction. Then comes the primary cube 512, having

the integer 8 for its root; the intervening numbers between this and the cube of 9, that is to say, 513 to 728 have, for their several cube roots, 8 and some fraction.

266. The next entire root is 10, the cube of which is 1000; so that 9 with its fractions, is the root of all numbers between 729 and this cube.

267. Confining our observations to integers, then, the units 1 to 9 are the roots of all cube numbers short of 1000; that is, of all numbers expressed by *three figures*. When we come to the next numbers, that is to 10, we find that its cube has an additional figure; that is to say, four figures.

268. Pursuing our inquiry, we find that, as the first, or smallest root expressed by two figures, that is, as 10 increases in value or amount, *approaching* to *three figures*, the cube numbers increase, *approaching* to *seven figures*: Thus, the cube of 50 is 125,000; the cube of 99 is 970,299; and the cube of 99 with fractions *approaching* towards 100, leads onwards to a million; that is, to *seven figures*, thus 1,000,000 is the cube of 100. Pursuing this course from 100 to 1000, that is, from roots expressed by *three figures* to the verge of those which require *four*, we are carried to the extremity of *nine figures*, passing which extremity we enter upon another division, which have *four figures* for their respective roots, the bounds of which are *twelve figures*. And thus do these numbers proceed; every *three figures* of a *cube number* demanding *one* figure for *its root*. And hence we can, in the very outset of an investigation, tell, *how many figures there will be in the root of any number*.

269. However, although every *three figures* of a cube number demand *one figure* for their root, every

single figure of the root does not require three for its cube. For, casting our eyes over the last paragraph, we see, that although three figures will express the cube, or third power of every number that can be expressed by one figure, yet, that the moment we pass this number, getting into cubes of only four figures, the root will have *an additional figure.* Roots of two figures carry us through the cubes of six figures, when, entering on those expressed by only one figure more, that is, on those of *seven figures,* a *third figure* in the root is demanded. All which may be thus briefly exhibited.

Roots, 1	*to*	10	*to*	100	*to*	1000	*to*	10000
Cubes, 100	„	1,000	„	1,000,000,	„	1,000,000,000	„	1,000,000,000,000

270. This is no barren, no profitless knowledge: this knowledge of *the number of figures* required to express the cube root of any given number; as will be seen when we proceed to the working of the rule. Let us, then, thoroughly understand it.

271. In this last table of cube numbers and their roots, the figures of the cubes are marked off in the ordinary manner of notation, into *periods* of *three figures each,* beginning with those on the left hand, the last of these periods may be, as it happens, *short of three figures;* it may be *one,* or *two,* or *three* figures.

272. Still this last period gives us, as we have seen, even when it consists of the smallest integer, it gives us one figure for its root. *Last period,* I have called it, and last it is, telling and marking off the figures from the unit's place; but *first period* is it, in all other respects; it *comes first* to hand in the working; it must always be *higher in value* than are all the other periods, and *first in importance;* therefore, the moment we have thus marked off the figures into periods, the moment we know which of

the figures it is that constitute the period of the first importance, then ought we to call this portion of figures, whether it consist of one, two, or three figures; then ought we to call it the FIRST PERIOD.

273. Besides the circumstances thus enumerated, by which this first period of figures is exalted, there is another circumstance by which it is yet more emphatically distinguished; this first period will always contain that important portion of the number, called the primary cube.

274. With this preparation we may proceed to the consideration of the method of extracting the cube root of any given number; or, in other words, as we have before seen, of the method of finding what would be the dimension of any cube, formed of a given number of other cubes of a given dimension.

275. Let it be, then, that we have to find, what would be ths size of a cube formed of 79507 cubic inches.

276. We know, in the first instance, that the root of this number, consisting of two periods, will be two figures; and we further know, that as there are in it but five figures, the first, or highest period, will be left with only two figures, that is, with 79... which, however, is 79000.

277. Considering the matter for a moment, or turning to the Table of Primary Cube Numbers, we find that the primary cube in this case is 64, having 4 for its root.

278. However, it is not 64 units, but 64 thousands; nor is the root 4, but 40; for the 4 is to have another figure after it, which will raise it to the place of tens.

279. Thus have we, on the outset, as our primary cube, a cube formed of 64000 inch cubes, and being, in length, or breadth, or height, 40 inches.

280. So far all is simple enough, and we have, if we may use the terms, consumed, or employed 64,000 of the 79 thousand odd inch cubes, leaving a remainder of 15507 inch cubes yet to be disposed of.

281. Now the question comes, how are these 15507 inch cubes to be disposed of; and in what manner are they to be applied to the augmentation of the primary cube?

282. To repeat the operation just passed, that is, to extract the cube root of this remaining quantity, will not tend towards our purpose, for it would give us another cube, and another remainder, on which remainder, if we were to repeat the operation, we should find another, and another cube, each smaller than the former. It is not to a series of cubes, declining in size, that we would reduce the 79507 inch cubes, but to a single cube, which should comprise this number of inch cubes.

283. We have employed 64000 of the given number of cubes, and have, as before stated, 15507, that is, nearly one fourth of the number remaining: so the question continues to present itself, how shall this remainder be disposed of in augmentation, as it must be, of the primary cube?

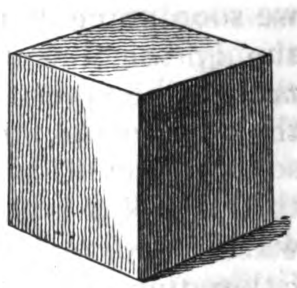
284. Had we, in substance, the primary cube of 40 inches actually before us, and these 15507 inch cubes, there are two modes of employing them in its augmentation; we might, as one of the modes, first form a sort of bed, or stratum, of the inch cubes, a square bed, the size of the primary cube, that is, 40 inches square, and, placing this cube upon it, we

might pile up, as a sort of wall around the four sides, a quantity of these remaining small cubes; and finally, in order to make all the sides equal, place a layer of the small cubes, similar to that below, upon the upper side of the primary cube, thus enveloping it equally on all sides, and so preserving its cubical form. This, I say, had we the materials before us, is one of the modes in which we might proceed to accomplish our object, and, in that accomplishment, to dispose of the given materials.

285. The other mode—and there are but two—of attaining both these ends, is, instead of placing a layer or stratum of the small cubes upon each of the *six sides* of the primary cube, to place these layers upon *three of the sides only;* for instance, upon the top, on one side, and then, on what may be called one end. This method accomplishes the purpose effectually as the other, and is more easy, and more simple.

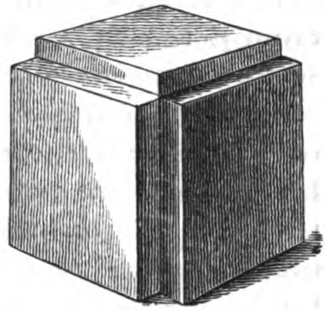
286. "More easy, and more simple:" These ought to determine our choice between the two modes; so let us proceed upon the latter, the more simple and more easy mode.

287. We have a cube of 40 inches, and a parcel of inch cubes to be added to it. We have determined that the best method of making the addition to the first or primary cube, is to place the small cubes *in strata on three of its sides*, in such manner, of course, as will preserve its cubical form.



288. The primary cube is 40 inches. A layer for one of its sides, that is to say, 40 inches square, will require 1600 of the inch cubes. Three of these layers;

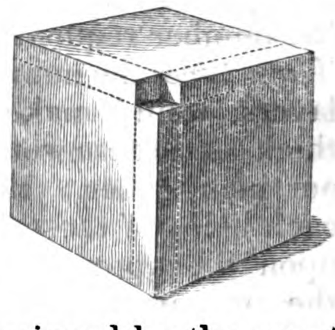
that is, one for each of the three sides, will require 4800 of the inch cubes. But we have 15507 of them. That is more than *three times*—but not four times—the number requisite to form three complete layers. *Three times* the number required to form three single layers, would make these layers each three inches thick, and these would consume, or require, 14400 of our inch cubes. Let us, then, place three layers, each being of these dimensions, upon the appropriate sides of the primary cube; and, then, how will the matter stand; the figure will not be a cube, but will be of the form represented in the margin; that is, a cube covered on three of its sides, with strata or layers, of its own dimension, and sufficient, therefore, to cover, each its proper side of the cube; but, extending no further, the ends of the layers remain uncovered, and there is, consequently, a deficiency; the cube being incomplete.



289. Directing our attention to this deficiency, we soon perceive that three long pieces, or prisms, each the length of the primary cube, that is, 40 inches, and of the thickness of the layers, that is, 3 inches; we soon perceive that three of these would go far to supply the deficiencies.—*Three inches* is the *thickness* of the layers. The pieces required to supply the deficiencies must, therefore, be each three inches square, and 40 inches long. Three inches square, that is 3×3 is 9, which multiplied by the length, which is 40, make 360 inches; and this is the quantity required to form *one* of these prisms. But we want *three* of them. Three times 360 are 1080. For which we have a sufficiency in our remainder.

290. And do these complete the cube? Do these

three pieces, of three inches square and forty inches long, supply the entire deficiencies which we observed in the last figure? They do not. Added to the layers which were before placed on the primary cube, they make those layers—which were before 40 inches — 43 inches square, save and except the small deficiency, a sort of knotch, observable in the figure in the margin, which deficiency is occasioned by the want of length in some one of the three prisms, to cover the ends of the two others, and thereby to complete the cube. The ends of these prisms are each three inches square: a cube of three inches would, therefore, cover them, and thus is the cube completed: a cube of 43 inches.



291. Let us now review the work we have done. And, *first*, as to the quantity of the small cubes employed in the construction of this cube which we have just completed.

292. Turning back to paragraph 275, and tracing the matter through the process, we see, that in the construction of this cube, there has been employed, of the inch cubes,

For the primary cube . .	40^3	= . .	64000
For the three layers . .	$40^2 \times 3 \times 3$ . .	= . .	14400
For the three long pieces	$40 \times 3^2 \times 3$ . .	= . .	1080
For the small cube . . .	3^3	=	27
	Total . . .		79507

293. And this is the number, the cube root of which we proposed to extract. This is the number of inch cubes which we proposed to erect into a single cube. Or, rather, we proposed to inquire, what would be the dimension of a cube constructed of 79507 inch

cubes; which number was, of course, adopted, on this occasion, because, amongst other reasons, it is *a cube number*, yielding a root free from fractions.

294. And, now that we have added together the quantity of small cubes employed in the construction of this larger one, let us trace the process by which it is thus arithmetically constructed.

295. Turning back to the commencement of this process, that is, to paragraph 276, we see, that having divided, or marked off the given cube number into periods, the next thing we did was to form, as the basis of the operation, a cube of 40 inches; and this we called the primary cube.

296. The next step we took was, to enlarge this cube, by placing on three of its sides a layer, one on each of the three sides; which layers were each three inches thick, and just the size requisite to cover each its side of the primary cube: that is, 40 inches square. And this was done—see paragraph 288—by squaring the dimension of the primary cube, which squaring provides a single layer for one of its sides; thus, $40 \times 40 = 1600$; then by tripling this square, in order to provide layers for two other of its sides, thereby regularly enlarging the primary cube, by making it longer, broader, and deeper; and then, finding that we had materials for the purpose, our next step was an increase of the thickness of these layers, a threefold increase of that thickness, thereby making them each three inches thick. The cube, however, being not yet complete,

297. We proceeded—see paragragh 289—to complete it by the addition of three square prisms, each prism the length of the primary cube, and the thickness of the square layers, that is, 40 inches

long, and three inches square, and this we accomplished by multiplying the square of three by 40, *for one of these prisms;* and then, in order to provide *three* of *them*, by *tripling* the product.

298. Finally, to complete the cube ; to carry out one of these square prisms from 40 to 43 inches, which forty three inches is now becoming the size of our cube, we provide a small cube of 3 inches — see paragraph 290 — and thus is the process finished. Now, to sum up this process into the form of a rule.

RULE

FOR EXTRACTING THE CUBE ROOT.

ARTICLE I.—Prepare the given number for the operation, as you are instructed to prepare a number for the extraction of its square root, in ARTICLE I. page 84; that is to say, if there be a fraction, in the common or vulgar form, reduce it to a decimal, and then count, and mark off the figures, as there taught, into periods; but, with this difference, that each period, when complete, instead of two figures, as in that rule, shall, in this, contain *three figures*; as instanced in the following line of figures.

$$\dot{1}72\dot{0}53\dot{6}19\dot{4}8.6\dot{1}47\dot{0}0$$

ARTICLE II.—Find the primary, that is the greatest, cube root of the first, or highest period of figures, write down this root, on the right hand of the line of figures, as you write your first quotient figure in a process of long division, then write the cube of that root under its proper period, and subtract, as in ordinary long division.

Article III.—To any remainder that you may have, bring down, and annex, the next period of three figures; and call the number thus formed, the resolvend; expressing, as it will, the quantity next to be resolved.

Note on Articles II. and III.—The first period, in the number stated above, is 17 .... the cube root of which, confining ourselves to a whole number, is 2, of which 8 is the cube, which cube, subtracted from the period 17, leaves a remainder 9; and our resolvend will be 9205.

Article IV.—We are now about to place a second figure in the quotient, thereby raising the first, which indicates the primary root, to the place of tens. And of this value, must we, throughout this step of the process, consider it. In the case under consideration, then, our first quotient figure signifies twenty; THREE TIMES THE SQUARE of which, we are now, according to the rule, to take as a divisor; and to inquire, how often it is contained in our resolvend.

Note.—The first figure in the quotient, or primary root, in the example before us, is 2, now signifying 20; the square of which, namely 400, tripled, gives us a divisor 1200. On inquiring how often this divisor is contained in the resolvend, care must be taken to make a considerable abatement from the apparent number of times: For instance, our *resolvend* being 9205, the apparent number of times which it would contain the divisor 1200, is 7. Whereas, on *a trial*—which should be always made by the side of the main statement—5 will prove to be the figure to be next placed in the quotient.

Article V.—Having thus determined how often the divisor is contained in the resolvend, write it down, as the second figure of the quotient or root.

Article VI.—Take, now, THE DIVISOR, as found by Article IV. that is to say, the quotient augmented tenfold, squared, and then tripled;

multiply this divisor by the new quotient number; to the product add, first, the primary root, still augmented tenfold, multiplied by the square of the new quotient number, and then tripled; and, second, the cube of this new quotient number; add these three quantities together, call their amount the *subtrahend*, write it underneath the *resolvend*, and subtract it therefrom, bringing down the remainder, as in ordinary long division.

NOTE.—It will, generally, be found advisable to make the calculations just now directed, beside the main statement, and to bring them into that statement when their accuracy and fitness are ascertained. The process so far as taught by the foregoing six articles, is here exhibited.

20	20		$\dot{1}7\dot{2}05\dot{3}6\dot{1}94\dot{8}.61\dot{4}70\dot{0}$ (25
20	25 = 5^2		8 .
400	500	$20^2 \times 3 =$ 1200	) 9205. . resolvend.
3	3	$1200 \times 5 =$ . .	6000
1200	1500	$20 \times 5^2 \times 3 =$ . .	1500
		$5^3 =$ . .	125
			7625 subtrahend.
			1580

ARTICLE VII.—To the remainder, if there be one, bring down, as before, the next period of figures, if there be such period. Then, treating the two figures of the quotient as expressive of hundreds and tens, seeing that, by the annexation of another figure, we are about to raise them to those ranks; treating these figures thus, proceed as directed in the fourth, fifth, and sixth articles of the rule; as just laid down: and thus, until the whole of the given number be brought down and evolved, thus do we proceed, repeating the operation, as instructed in those articles; and as exhibited in the example on which we have been operating; to which example, as resumed and completed at the close of these articles, we may with advantage refer.

ARTICLE VIII.—If, when the whole of the periods have been brought down, and evolved, there be a remainder, and it be your object to come yet nearer to the exact root; if such be the case, then, to the remainder, annex *a period* of ciphers—that is to say, *three ciphers*—and proceed in the work, as before.

NOTE.—The fixing of the decimal point, as in all operations of arithmetic, is of the greatest moment; and its place is thus determined. The first thing we have to do, on proceeding to evolve the root of a number, is, as directed in Article I., to mark off such number into periods, or portions, for the several steps of the operation. Each of these periods gives us one figure in the root, and these figures will be integers, or decimals, just as the periods from which they have been evolved were integers or decimals. So that there will be the same number of integral figures in the root, and the same number of decimals also, as there were periods of those numbers respectively in the number resolved. The rule, therefore, for fixing the decimal point will stand thus:—

ARTICLE IX.—Count, in your given number, the number of *periods* that you have in whole numbers; and, counting *the same number of figures* in the root, fix the decimal point. There will then remain, for decimals, figures sufficient to correspond in number with the periods of decimals, whether those periods appeared in the given number as it stood in the outset, or were, in whole or in part, added in the course of the operation.

299. Here follows the entire working of the Problem on which we have been making our observations; namely, the evolving of the root of 17205361948.6147. I have, of course, omitted the *trials*, and such subsidiary parts of the operation; stating only those that it may be useful to retain; and these, also, I have stated in the manner which is generally practised; and which is, therefore, found to be most eligible.

	17205361948.614700 (2581.59
$2^3 =$	.8
$(20)^2 \times 3 = 1200$	9205
$1200 \times 5 =$	6000
$5^2 \times 20 \times 3 =$	1500
$5^3 =$	125
First Subtrahend	7625
$(250)^2 \times 3 = 187500$	1580361
$187500 \times 8 =$	1500000
$8^2 \times 250 \times 3 =$	48000
$8^3 =$	512
Second Subtrahend	1548512
$(2580)^2 \times 3 = 19969200$	31849948
$19969200 \times 1 =$	19969200
$1^2 \times 2580 \times 3 =$	7740
$1^3 =$	1
Third Subtrahend	19976941
$(25810)^2 \times 3 = 1998468300$	11873007614
$1998468300 \times 5 =$	9992341500
$5^2 \times 25810 \times 3 =$	1935750
$5^3 =$	125
Fourth Subtrahend	9994277375
$(258150)^2 \times 3 = 199924267500$	1878730239700
$199924267500 \times 9 =$	1799318407500
$9^2 \times 258150 \times 3 =$	62730450
$9^3 =$	729
Fifth Subtrahend	1799381138679
Remainder	79349101021

300. One point, relating to this method of extracting the cube root, yet remains, as worthy of our observation. On referring to the process, as recapitulated in 288, and the three following paragraphs, we see that the first step gave us a sort of a *nucleus* for our further proceeding; it gave us what we have termed a *primary cube*, the due augmentation of which is the business of each succeeding step of the process. That primary cube —in the instance before us—is of the dimension of 40 inches. The second step of the process gave us layers, or strata, for three of its sides, by which it became augmented to a cube of 43 inches. And

there the operation terminated; that being the complete root of the number proposed to be evolved. But how, it may be asked, in the evolution of a number containing *more than two periods* of figures, *more than two steps* in its process; how do subsequent operations; how do a third and a fourth step proceed towards the same end; namely, the employment of the given materials in the enlargement of the cube: and especially, how are remainders, how are fractions employed towards this end? The answer to these questions is the point to which I have now to invite the reader's attention.

301. Just as, in the second step of the operation, we employ the matter of our resolvend in the enlargement of the primary cube, just in the same manner does each subsequent operation employ its materials, whether those materials be integral or fractional quantities. On the termination of each step of the process, by which a figure is placed in the root, the cube is completed; is perfected. For example, let us cast our eyes on the statement and operation appended to the last paragraph but one; an example which furnishes us with ample scope for the observations we have here to make. The first figure in the root is 2, which, standing in the place of thousands, represents 2000; which two thousand is the *primary cube.* We determined, in order that we might have some definite quantity in view, that the number there set down for evolution, should represent so many cubic inches, so that here we have, for our primary cube, a cube of 2000 inches. The second step in the operation gives us, as the second figure in our quotient or root, the figure 5, representing 500, and adding to the dimension of the primary cube, that number of inches. We have now, therefore, a cube of 2500 inches; having employed in its construction, first 8,000,000,000, and in the second step 7,625,000,000 inches.

302. The third step in the operation proceeds to enlarge this cube of 2500 inches, by adding, as the third figure in our root shows, 80 inches to its dimension; an addition which, as the subtrahend of the third operation shows, employs a further sum of 1548,512,000,—so that we have now a cube of 2580 inches, to which, our cube becoming so large, and our material so considerably diminished, the fourth step in the operation makes an addition of only one inch; which inch, however, seeing that layers are required to extend over three of the sides of this large cube, demands no less than 19,976,941 inch cubes; as the subtrahend of the fourth step of the process shows. So much for *whole numbers*, as far as regards this example. And now for the fractions.

303. After the foregoing operations, after the fourth step in the process, we find ourselves with a remainder—*see the statement*—of 11,873,007, which is little more than half the quantity;—*again see the statement*—half the quantity that was required to enlarge our cube by a single additional inch. To this remainder, then, we bring down, or annex, a period of decimals, and so proceed with the operation. By the annexation of three figures, our remainder is greatly increased in extent, although not in value, and the fifth step of our process gives us 5, as the first decimal figure; that is five-tenths. I need scarcely observe that this adds half an inch to the size of our cube. The last integral addition, or layer, was one inch, employing, as we have seen, nearly twenty millions of the inch cubes, here is the decimal .5—*that is, one half*—requiring and employing, as the statement evidences 9,994,277.375, nearly ten millions, and by this step, strata, or layers, each half an inch thick, are, as in the foregoing processes, added to three of the sides of the growing cube.

304. However, accurately to finish the matter, let us inquire, of what are these *fractional strata* formed; Are they formed of *half inch cubes;* that is, of cubes of the dimensions of half an inch? By no means; they are nine millions odd of *half cubes;*—that is to say, so many *halves of inch cubes;* save and except—*and mark this distinction*—save and except the three small prisms, required, as shown in paragraph 289, to complete the corners, and the little cube required to close the *notch*, as I have been obliged to term it. The three prisms, each of which must be 2581.5 inches long, will be formed of so many smaller prisms, half an inch square, but, of course, an inch long each: and the little cube required to fill up the notch, and so to perfect the large cube, will be indeed a cube of half an inch. Observe *the quantity* of our material required to construct these several additions to the large cube; look at the foregoing operation and statement. The three prisms require, of the half-inch-square-one-inch-long prisms, of these small prisms, the three long prisms require no less than 7743. But four of these small prisms would be required to form a cubic inch, divide 7743 by 4, and we have 1935.75, which is—*see the statement*—the quantity arrived at by the other process. And then for "*the little cube;*" What is the decimal expression of a cubic half inch, what save .125, as we find it in the statement?

305. Such is the manner in which fractional quantities are wrought up, in the evolution of the cube root. There is another decimal figure in the root, namely, 9, that is, nine hundredths; which, further, is nine hundredths of an inch, merely. It would be tedious, and cannot now be of use, to trace this small fraction, through its process, as we have traced the other. Suffice it to say, that by this quantity, three strata, or layers, are placed on

so many sides of the cube, regularly increasing its size, as in the foregoing instances; but increasing it only by this minute quantity of nine hundredths of an inch.

306. There are other methods, besides this which we have just seen, of extracting the cube root, but so far as there is any principle, any science, in those methods, they conform to this, and will deviate from it merely in some unimportant and trifling expedients. It is, at best, a somewhat intricate affair, and when the number to be evolved is large, the operation will always be one of some labour. Of these difficulties, I hesitate not to aver, that no man will ever divest the operation. We have ascended to THE PRINCIPLE, and have developed that principle. A more lucid development may be effected. But in this, as in all other branches of science, nothing that is of essential importance, save its uses and its application, remains to be discovered, after the attainment of the principle.

307. I shall not, therefore, occupy the time of the reader by detailing the little varieties of method, by which different arithmeticians choose to conduct the operation. But there is one method, one expedient, by which the roots of numbers are to be discovered, that is worthy of being mentioned: that method is the use of Logarithmic Tables. And on this, I propose, in the sequel, to say a few words. Our next object will be a—

METHOD OF PROOF.

308. The rational, more obvious, and most certain method of effecting this, must be an involution, a re-folding, of the root which has been evolved. When the number is a long one, the labour may be considerable; but, as I have said, it is the most

certain method of proof; and if it be worth our while to extract a root, it will be worth while to prove whether our work be, or be not, correct. Re-involve the root, then; raising it to the power from which it has been evolved; and, if the work be all correct, the original number will be restored. However, there may have been *a remainder*, not evolved, as in the instance which I shall here subjoin. Such remainder must, of course, be added to the product of the root; and, in making this addition, great care must be taken to place it in its proper situation. The following example, of the entire working of the two processes of evolution, and of re-involution, will, on a careful inspection, resolve any doubt, any uncertainty, that may remain on the mind of the reader, with regard to any part of either process.

Proposition.—Extract the cube root of 935799.75, to the second decimal figure: And then prove the truth of the work, by re-involving the root.

$$\dot{9}35\dot{7}99.75\dot{0}\ (27.81$$
$$9^3 = 729$$
$$90^2 \times 3 = 24300)\ 206799$$
$$24300 \times 7 = 170100$$
$$90 \times 7^2 \times 3 = 13230$$
$$7^3 = 343$$
$$183673$$
$$970^2 \times 3 = 2822700)\ 23126750$$
$$2822700 \times 8 = 22581600$$
$$970 \times 8^2 \times 3 = 186240$$
$$8^3 = 512$$
$$22768352$$
$$9780^2 \times 3 = 286945200)\ 358398000$$
$$286945200 \times 1 = 286945200$$
$$9780 \times 1 \times 3 = 29340$$
$$1^3 = 1$$
$$286974541$$
$$71423459$$

97.81
97.81
9781
78248
68467
88029
9566.7961
97 81
95667961
765343688
669675727
861011649
935728.326541
71.423459
935799.750000

USES OF THIS RULE.

309. To ascertain the dimensions of vessels, and the quantities contained in solid bodies, in certain cases, or under certain conditions; their relative dimensions or quantities; and to find proportionals between given numbers; these are amongst the uses of this Rule. And, with a few examples, they may be thus briefly enumerated and explained.

FIRST.—To find the dimension of a cube, that shall be equal in solid quantity to any given solid of another form.

310. The *quantity* of the *other form*, whether sphere, cone, prism, cylinder, or other body, being given, all that is requisite in this case, is to extract the cube root of that quantity, which root will, of course, be the side, or dimension, of the required cube; in feet, inches, or in whatever denomination of measure the given quantity may be stated.

SECOND.—Having the dimensions of one solid body, or vessel, to find those of another body, or vessel, of a similar form, that shall, in its capacity, or contents, bear any required proportion to the given body.

311. You have a packing case, or vessel, which will, as you have found, contain a certain quantity of goods; and you would have another such case, of a similar form, to contain 2½ times the quantity of the same kind of goods. For example. Let the given case be 32 inches long; 25 inches broad; and 16 inches deep. How shall we proceed to find the dimensions of the required case; that is, a case which shall contain 2½ times as many goods of the same kind?

312. Involve the several dimensions, raising them to their third power, or cube; and multiply those cubes, each separately, by the difference between the given case, and the required case, which in this instance is 2½; or expressed as a decimal 2.5. Then, extract the cube root of each of the products; and those roots will be the dimensions of a case, or vessel, of the same proportions, but capable of containing 2.5; that is, twice and five-tenths as much:

$32 \times 32 \times 32 \times 2.5 = 81920.0 \quad \sqrt[3]{(81920.0)} = 43.4$

$25 \times 25 \times 25 \times 2.5 = 39062.5 \quad \sqrt[3]{(39062.5)} = 33.9$

$16 \times 16 \times 16 \times 2.5 = 10240.0 \quad \sqrt[3]{(10240.0)} = 21.6$

Thus a vessel, or case, 43.4 inches long, 33.9 inches broad, and 21.6 deep, will contain 2½ times as much as will the given case or vessel.

The two propositions which follow come under the same article of the rule. They are very well calculated to show its usefulness: to save the time that would be required to work new ones. I take them from the "*Arithmetical Collections*" of A. and J. Birks, of 1766.

Suppose the length of a ship's keel be 125 feet, the breadth, on the midship beam, 25 feet, and the depth of the hold 15 feet; required the dimensions of another ship, of a similar form, that shall carry three times the burthen.

$125 \times 125 \times 125 \times 3 = 5859375$

$25 \times 25 \times 25 \times 3 = 46875$

$15 \times 15 \times 15 \times 3 = 10125$

The cube root of	5859375	is	180.28, length of keel
. . . .	46875	,,	36.05, . . beam
. . . .	10125	,,	21. 6, depth of hold.

Or, suppose it were required to build a vessel capable of carrying half the burthen of that described in the foregoing proposition.

Here, instead of *multiplying* by the difference of capacity between the given vessel, and that required to be constructed, we *divide* the cubes of the several dimensions by that difference; or, rather, we *reduce* them, by that difference, thus:

$$125 \times 125 \times 125 \div 2 = 976562.5$$
$$25 \times 25 \times 25 \div 2 = 7812.5$$
$$15 \times 15 \times 15 \div 2 = 1687.5$$

Stated with the signs, instead of words:—

$$\sqrt[3]{(976562.5)} = 99.202$$
$$\sqrt[3]{(7812.5)} = 19.84$$
$$\sqrt[3]{(1687.5)} = 11.906$$

THIRD.—Having one, suppose the chief dimension, of a vessel, and its capacity both given, to find the corresponding dimension of a vessel of the same proportions, but of another specified, or prescribed capacity. Example:—

If a vessel of 100 tons burthen be 44 feet long at the keel, what length shall be the keel of a vessel, having the same proportions, that shall be capable of carrying 220 tons?

313. Here we have to raise the given dimension to its cube, to multiply that cube by the *capacity*—in this case 220—of the proposed vessel; then, to divide the product by the capacity of the given vessel; and, of the quotient, to extract the cube root; which root will be the dimension required. Thus:— (*The Calculations I take from Birks.*)

$$44 \times 44 \times 44 \times 220 = 18740480$$
$$100\,)\,18740480\,(\,187404.8$$

The cube root of 187404.8 is 57.229592.

But this decimal is carried much too far. We can never want the dimension within the *millionth part of a foot*. It must be quite sufficient to say that the root is 57.229; which is the length for the keel of the required vessel.

FOURTH. To find mean proportionals.

314. We have seen the method of finding a mean proportional between two given numbers;—

that is to say, a *single* proportional. Which is done chiefly by the extraction of the square root. And the example given is the finding of a mean proportional between 7 and 175; which mean is 35: the terms then standing thus:—7, 35, 175. But suppose we had to find *two* mean proportional numbers between these two extremes; or say, between 7 and 189,—for, between 7 and 175 we should, to no purpose, entangle ourselves amongst fractions;—suppose, then, that we would find *two* mean proportionals between 7 and 189. It is done thus. Divide the larger number by the smaller. Then, of the quotient, extract the cube root; which root will be THE RATIO of the proportional terms. So, if we multiply the smaller of the two terms by this root, we have one of the mean proportionals; and, if, again, we multiply this mean term by the same ratio, we have the second proportional; and the terms will then stand thus: 7 : 21 : : 63 : 189. The ratio, here, is 3. And it is, of course, the same thing, if, instead of *multiplying* the smaller of the terms, we *divide* the larger by the ratio; thus $189 \div 3 = 63$; and $63 \div 3 = 21$. Which 63 and 21 are the mean proportionals. This, like all the doctrines concerning ratios, and proportions, is a very beautiful, simple, and interesting branch of the science of numbers.

OF BIQUADRATE, AND OTHER ROOTS.

315. These, as I have intimated, in paragraph 145, have no reference to actual form, nor substance. Matter cannot have more than the three dimensions of length, breadth, and thickness. But here, in the *biquadrate*, merely, is there an assumption of something having *four* dimensions; for it means the *two-fold* or *double square;* it means length, and breadth, and thickness, *and some other dimension;* of which, therefore, we should deceive ourselves, did we pretend to have any conception. However, these imaginary quantities are generally treated of, in works on arithmetic; these higher, or lower—I really cannot tell which to call them—but these Roots, and Powers, "*which shape have none;*" they are generally treated of, in such works. And, as a few words may suffice, we will bestow those few words on them.

316. The Biquadrate Root, then, as its name imports, is the double, or two-fold, square root. It may be extracted, by dividing, by marking off the given number, into periods of four figures each; and so proceeding to evolve; somewhat in the manner in which we have worked out the cube, and the square roots. But this is an intricate, and laborious process; and whilst time can be applied to any useful purpose, this can be no commendable application of it. The direct and simple method of extracting the biquadrate root, is to extract the square root of the given number; and, then, of that root, to extract the square root.

317. As, in the biquadrate, we have the double-square, or square squared, so we must have the *cube squared*, and this is called the Sursolid. After that which is stated in the last paragraph, it can only be requisite to say, that although there are other methods of extracting the root of this power, the obvious and

more eligible mode is, first to extract the cube root, and, then, of that root to extract the square root. And so with regard to the extraction of the Roots of any other powers. Finally, we may observe, that as the extraction of the square root is called the extraction of the *second power*; and that of the cube, the *third power*; so the *biquadrate* is termed the *fourth*; the sursolid is called the *fifth*; and then comes the *sixth power*, called the SQUARE CUBED; and so on to an useless extent.

ON TABLES OF LOGARITHMS:

SOMETHING OF THEIR USES.

318. These tables are a very ingenious invention, for the purpose of saving the very irksome labour of multiplying, and of dividing, in cases in which large numbers are to be dealt with. It appears that they were not wholly unknown to the ancients; but the knowledge of them, and their construction, was very incomplete, until about the year 1614, when John Napier, baron of Merchiston, in Scotland, published his improvements and discoveries with regard to these tables, of which further improvements have since been effected by other able Mathematicians.

319. With respect to the principle on which these Tables are constructed, being one of considerable intricacy, and forming no part of my plan in this book, I shall not attempt to enter upon it. But I will here insert a short table of these numbers; that is to say, a Table of natural numbers, as they are termed, beginning with one, and rising to two hundred; and the Logarithms of these numbers. After which, we shall see something of the manner of using them.

TABLE OF LOGARITHMS OF NUMBERS FROM 1 TO 200.

No.	*Logarithm.*	*No.*	*Logarithm.*	*No.*	*Logarithm.*	*No.*	*Logarithm.*
1	0,000000	51	1,707570	101	2,004321	151	2,178977
2	0,301030	52	1,716003	102	2,008600	152	2,181844
3	0,477121	53	1,724276	103	2,012837	153	2,184691
4	0,602060	54	1,732394	104	2,017033	154	2 187521
5	0,698970	55	1,740363	105	2,021189	155	2.190332
6	0,778151	56	1,748188	106	2,025306	156	2,193125
7	0,845098	57	1,755875	107	2,029384	157	2,195900
8	0,903090	58	1,763428	108	2,033424	158	2,198657
9	0,954243	59	1,770852	109	2,037426	159	2,201397
10	1.000000	60	1.778151	110	2,041393	160	2.204120
11	1,041393	61	1,785330	111	2,045323	161	2,206826
12	1,079181	62	1,792392	112	2,049218	162	2,209515
13	1,113943	63	1,799341	113	2,053078	163	2,212188
14	1,146128	64	1,806180	114	2,056905	164	2,214844
15	1,176091	65	1,812913	115	2,060698	165	2,217484
16	1,204120	66	1,819544	116	2,064458	166	2,220108
17	1.230449	67	1,826075	117	2,068186	167	2,222716
18	1,255273	68	1,832509	118	2,071882	168	2,225309
19	1,278754	69	1,838849	119	2,075547	169	2,227887
20	1,301030	70	1,845098	120	2,079181	170	2,230449
21	1,322219	71	1,851258	121	2,082785	171	2.232996
22	1,342423	72	1,857332	122	2,086360	172	2,235528
23	1,361728	73	1,863323	123	2,089905	173	2,238046
24	1,380211	74	1,869232	124	2,093422	174	2,240547
25	1,397940	75	1,875061	125	2,096910	175	2,243038
26	1,414973	76	1,880814	126	2,100371	176	2,245513
27	1,431364	77	1,886491	127	2,103804	177	2,247973
28	1,447158	78	1,892095	128	2,107210	178	2,250420
29	1,462398	79	1,897627	129	2,110590	179	2,252853
30	1,477121	80	1,903090	130	2,113943	180	2,255273
31	1,491362	81	1,908485	131	2,117271	181	2,257679
32	1,505150	82	1,913814	132	2,120574	182	2,260071
33	1,518514	83	1,919078	133	2,123852	183	2,262451
34	1,531479	84	1,924279	134	2,127105	184	2,264818
35	1,544068	85	1,929419	135	2,130334	185	2,267172
36	1,556303	86	1,934498	136	2,133539	186	2,269513
37	1,568202	87	1,939519	137	2,136721	187	2,271842
38	1,579784	88	1,944483	138	2,139879	188	2,274158
39	1,591065	89	1,949390	139	2,143015	189	2,276462
40	1.602060	90	1,954243	140	2,146128	190	2.278754
41	1,612784	91	1,959041	141	2,149219	191	2,281033
42	1,623249	92	1,963788	142	2,152288	192	2,283301
43	1,633468	93	1,968483	143	2,155336	193	2,285557
44	1,643453	94	1,973128	144	2,158362	194	2,287802
45	1,653213	95	1,977724	145	2,161368	195	2,290035
46	1,662758	96	1,982271	146	2,164353	196	2,292256
47	1,672098	97	1,986772	147	2,167317	197	2,294466
48	1,681241	98	1.991226	148	2,170262	198	2,296665
49	1,690196	99	1,995635	149	2,173186	199	2,298853
50	1,698970	100	2,000000	150	2,176091	200	2,301030

320. Thus are the tables divided into columns, of which there are four on the page; that is, four columns, each containing—*first*, a column with the natural numbers rising from 1 to 200, by the advance of a single unit at each step; and, *second*, a column of Logarithms of these numbers, each opposite to its appropriate number. Thus the logarithm of 2 is the decimal .301030; that of 12 is 1.079181; and so on.

321. Now, for the application of these tables, to the purposes of ordinary multiplication and division. The natural numbers, as we have observed, rise by a single unit at each step. And the advance of the Logarithm is made by a ratio which augments them in this wise, that whilst the natural number, whatever it may be, advances to its square, or second power, the Logarithm simply doubles itself. Thus, as appears in the Table, whilst 10 advances to 100, the Logarithm, which is 1, merely rises by a minute decimal increase at each step, to 2, which is the logarithm of 100. The logarithm of 9 is .954243; that of 9 times 9 is the double of this; that is 1.908485. See the logarithm of 81.

322. Again, it is not merely when a number is thus to be multiplied into itself, that the logarithm serves as an index to the product. For, let it be that we would multiply 10 and 12 together. The logarithms of these two numbers are 1, and 1.079181, which. added together, give 2.079181. Look at the Table, find this logarithm; observe its number; it is 120. Add the logarithms of 13 and 14 together; the sum is 2.260071; opposite to which logarithm, in the Table, stands 182; which is the product of 13 × 14. Thus does ADDITION, mere addition, a very simple process compared with multiplication, furnish us, by the aid of these Tables, with THE PRODUCT of two numbers.

323. It is true, that *addition* in these instances, saves neither time nor attention, but rather the contrary; it being easier to multiply 10 and 12; or 13 and 14, together, than to make the addition of their logarithms. But this arises from the limited extent of this mere specimen of logarithmic Tables. The numbers, as I before stated, rise no higher, in this scrap of a Table, than 200; so that I am confined within a narrow compass. In Tables of due extent, the numbers might advance to millions, and, in fact, to any amount, and we should find their product with nearly the same ease as that with which we find the product of these trifling numbers.

324. Moreover; it is not to the multiplication of *two* numbers together, that we are confined, when we use Tables of logarithms. Three, four, or in short, any number of factors may be thus dealt with. Add their logarithms together, find the sum of these amongst the logarithms in the Tables, and, adjoining to it, in the column of natural numbers, stands the product. Such is the case, as I have stated, when the Tables are of due extent. However, there are limits to the most extended of Tables. And when we have factors, and products, which overstep those limits, the results must be attained by double processes; so that it is no longer so very simple an affair. So much for multiplication by Logarithms, and now for its converse; now for division,

325. As the *addition* of the logarithms of two or more numbers, furnishes us with the logarithm of their product, so the SUBTRACTION of the logarithm of one number from that of another, gives us the QUOTIENT of the latter *divided* by the former; that is to say, is gives us the logarithm of the quotient. And thus may a long and laborious process of division be effected, by the simple means of subtraction. As an example;—*I am still confined within the nar-*

row limits of our Table.—Suppose we would divide 195 by 13. Look for the logarithm of the dividend 195; it is 2.290035; from this *subtract* the logarithm of the divisor, which is 1.113943; the difference is 1.176092, which we *find* to be the logarithm of 15, as nearly as can be, so 15 is the quotient, on a division of 195 by 13. And thus might any number, within the limits of our Tables, be divided by any smaller number, merely by subtracting the logarithm of the divisor from that of the dividend, and then finding the difference.

326. It appears, however, in the latter instance, that we do not find, in our small table, a logarithm precisely the same, as the difference; it is 1.176091, instead of 1.176092. And thus it will occasionally be; we shall not find precisely the same figures in the logarithm; but it directs us to the proper quotient; it answers its purpose. And that is all that we can require. Thus it will be, occasionally, in all Tables of Logarithms. And, although these Tables serve in numerous, nay, in countless instances, to abridge the labour of making extensive calculations, yet have the Tables their limits. The most extensive of them hitherto published, or ever likely to be published, not furnishing those who use them with any, not with every number that they may require, at once, and by a simple, by a mere inspection. With the volumes of these Tables there are always instructions, though not always very clear instructions, as to the manner of using them, and the process, in many instances, is one of no small difficulty, even to those who understand them well, but who are not in the constant practice of using them. It is, even to persons of this description, often a matter of considerable labour; and then, too, there is, on consulting such piles of figures, no small danger of casting the eye on a wrong one. So that, although it may be desirable to know just this much of them, in order

that we may not feel ourselves humbled in the presence of mere calculators, I would by no means advise any one, whose occupation does not lie in calculating the situations and distances of planets and of stars, to have anything further to do with Tables of Logarithms. We must, however, before we dismiss them, show the manner of extracting the roots of numbers by their aid.

327. The Root of a number, is that number which, involved into itself produces the given number. It is a very simple matter to *involve* a number; a mere multiplication of the number by itself. But, as we have seen, it is a process of some intricacy, requiring great care and attention, to evolve a number, that number being one of considerable magnitude. It is, as we have seen in paragraph 182, to find, not merely *the quotient*, but *the divisor*, also.

328. Now here come Tables of Logarithms to furnish us with these divisors and quotients, almost on a mere inspection. We have just seen—paragraphs 321 and 322—that if we take the logarithm of a number, and double it, that is, *multiply it by* 2, we have, in this double, the index, or logarithm of the *second power* of the given number. Thus, the logarithm of 9 is .954243; multiplied by 2 we have 1.908486, which is the logarithm of the square, or second power of 9, Now, if a *multiplication* by 2 furnish us, as it does, with the logarithm of the square of a number, the *division* by 2 will reverse the process, and so furnish us with the logarithm, that is to say, with the *index or guide to*, the square root of that number. And so, of course, it does. Divide the logarithm of 81 by 2, and we have that of 9, which is the square root. Thus it is in small numbers, and thus would it be in numbers of any magnitude. THE DIVISION OF THE LOGARITHM OF ANY NUMBER BY TWO GIVES US THE LOGARITHM OF THE SQUARE ROOT OF THAT NUMBER.

329. And, as the division of the logarithm of a number by 2, gives us the logarithm of the square root of that number; just so will its DIVISION BY THREE GIVE US THE LOGARITHM OF THE CUBE ROOT.

330. All this appears very simple and easy, naturally calling forth the question—" If the roots of numbers can be thus easily evolved, why trouble us with the intricate, and labourious method which you have just been teaching? I answer, in the first place, that I have not written this work, nor this branch of the work, for the purpose of teaching the extraction of the roots of numbers. This I hope it does teach, and the PRINCIPLES, also, on which the processes proceed; and so the work may be useful in that respect. But this has been with me a very subordinate consideration; little more than an adventitious circumstance. The nature of the human mind, its propensities, its capabilities, its powers; these have been, with me, long a favourite object of study and contemplation. And I have written this work chiefly, almost entirely, as I state in the preface, for the purpose of leading that mind, in those who may be pleased to read the work, to a pleasing, an orderly, and a vigorous exercise of its powers: aware, as I am, that when thus trained, the mind may choose its subject; and may exult and revel in its attainments, and in the consciousness of its improved capacity and power.

331. To return, however, to our subject, namely, the uses of logarithmic Tables, and so to conclude the work. It would be a waste of time, indeed, to extract the roots of numbers by the processes before taught, if those roots could be found thus in tables. The roots of small numbers may be thus found, and so might those of large, were there tables of due extent. But these are not to be had; nor are they desirable. Already it makes the head dizzy to look

at a volume of the tables now constructed; and we must have additional volume upon volume; a work of indescribable, and of thankless labour, and of great cost too, ere we should have tables to furnish us thus, *at once*, with the roots of numbers, even to the amount of that which we have evolved on page 132. With the aid of, perhaps any tables now in print, a number of that magnitude would require, in the extraction of its root, so many references, some statements, and so much vigilance, in order to avoid error; that, as I have before stated, unless done by a person in the constant practice of using the tables, however well he may understand their uses, the difficulty, and labour of the operation are not thereby abated. In all the operations of arithmetic, the regular process, the ordinary rule, founded on the principle of the operation, is the safer course; and the shorter also, unless we be in the habitual practice of some expedient; without which practice, expedients, like mental calculations, will but waste our time, and betray us into uncertainty and error.

TABLES OF SIGNS, TERMS, &c.

USED IN ARITHMETIC.

OF SIGNS.

+ The mark of ADDITION, called PLUS; which, in the Latin, signifies MORE. When placed between two numbers, it means the first number with the addition of the second: that is to say, it means both numbers taken together. Thus $4 + 5$ are equal to 9; $4 + 5 + 2$ are 11.

— A short horizontal line is the mark of SUBTRACTION, signifying LESS. It is called MINUS, the Latin word for LESS. Its use and meaning are the reverse of the former sign, thus, $9 - 5$ is 4; and $11 - 2 - 5$ is 4.

× The sign of MULTIPLICATION. It is a small cross, but differs from that for Addition, being two transverse sloping lines; like the cross appropriated to St. Andrew. It has not a name, but it signifies that any numbers, between which it may be placed, are to be multiplied, or regarded as multiplied, into each other, as 8×9 are 72. $5 \times 3 \times 7$ are 105. To translate this mark into a word, we say "*into.*"

÷ The sign of DIVISION; without a name: but used for the word "*by,*" meaning, when placed between two numbers, that the larger is to be, or to be considered as being, divided by the smaller. Thus $32 \div 4$ is 8; $84 \div 7$ is 12.

= The sign of EQUALITY, and used instead of the words "*equal to.*" Thus $4 + 5 = 9$; $4 + 5 + 2 = 11$; $9 - 4 = 5$; $5 \times 3 \times 7 = 105$.

: Two dots, placed thus, are used as THE SIGN OF PROPORTION; and, placed between two numbers, they invite our attention to the proportion which the first bears to the second. They are to be read thus, 8 : 16; that is, 8 *is to* 16; or 8 *is in the proportion to* 16.

:: Four dots, or points, are the second mark of comparison, or proportion, between numbers; meaning "*as,*" or "*so is.*" They are not used except to follow the two dots just described in the last definition, and then they are placed and read thus, 8 : 16 :: 3 : 6. *Eight is to* 16 *as* 3 *is to* 6; or 8 *is in the same proportion to* 16, *as* 3 *is to* 6.

8^2, 8^3, 8^4. Small figures placed thus are called **Exponents.** They announce to us *the power* to which the number they may be annexed to, is to be, or is to be regarded as being raised. See paragraphs 144 to 147, both inclusive.

$\surd$ $\sqrt[3]{}$ $\sqrt[4]{}$. This $\surd$ is called the *radical sign*, that is, the *sign* of *the root*. Placed *before* a number, it means the same thing as does the small exponent, when placed after it, as just explained. But its use in this manner has, of late, been discontinued; the the *exponent* being preferred. The use of this sign is now confined to express, *not the power* of the number to which it is annexed, but the contrary; it means *the root;* and, to express this meaning, it *follows* the number, thus 8 $\surd$. The *sign alone* means the *square root;* with the small exponent $\sqrt[3]{}$ it means the *cube root;* $\sqrt[4]{}$ means the *biquadrate root*, &c.

$\overline{}|$ Called the *Vinculum;* used to bind, to connect two or more numbers together. Thus: do we desire to express the third power of these three numbers $5 + 2 + 7$ the *vinculum* may be used thus; $\overline{5 + 2 + 7}|^3$ and then they are all affected by the exponent: whereas, without the *vinculum*, written thus, $5 + 2 + 7^3$, the last figure, only, would be affected by the exponent; the statement would signify 5 and 2 and the cube of 7 only.

$(5 + 2 + 7.)^3$ This, however, between two *braces*, is the more recent method of binding or connecting two or more numbers; and is, I believe, superceding the use of the vinculum.

OF TERMS.

Aliquot Parts.—These *parts* have a very suitable name. We use them as we use *factors*, and *submultiples;* that is, for the purpose of avoiding *long multiplication*, and *long division;* on occasions in which it is more convenient to work with two, or more, small numbers, rather than with one large one. But, and mark the neatness of the distinction; *these parts* with which we work in the Rule of Practice, differ from those, in this respect; that whilst, in the use of factors and submultiples we care not how many we have to work with; because, if there be many they are so much the smaller, and therefore, more easily worked; whilst this is the case with those parts, the contrary is the case as to the parts used in Practice; for here, simplicity, and ease, and clearness of statement, are all promoted by working our sums by as *few parts* as possible, and hence comes the name *aliquot;* which is a Latin word signifying *some few;* so, *some few parts*, which are the things we ought to work with, in Practice, are called **Aliquot Parts.** *(Lessons on Arithmetic.)*

Components, Composite Numbers.—As the numbers which, *multiplied* together, producing a certain sum, are called the FACTORS of that sum; so numbers *added* together, and thereby *composing* a sum, are called the COMPONENTS of that sum: and, as these numbers are called its components, so the sum itself, when required to be distinguished by a term, is called, A COMPOSITE NUMBER. So that the use of these Terms is, to distinguish numbers when employed for one purpose, from numbers employed for another purpose: and the use of the *numbers, themselves*, lies here; suppose you have to multiply, or to divide, a certain sum by several small numbers, as by these, 6, 9, 6; instead of making three operations, you add the numbers together, and multiply, or divide by the sum of them, which is 21. And, if you have to speak of the manner of the operation, you say, that you multiply, or divide, by 21, the *composite* number of the *components* 6, 9, 6.

(Lessons on Arithmetic.)

Cube Number.—A number, the product of any whole or integral number, twice multiplied into itself: as 8, which is the cube of 2; 27, the cube of 3; 64 the cube of 4; 1000 the cube of 10; and so on. The more learned definition of a cube number is, *the third power of an integral number*.—(*See Surd.*)

Digit.—The word comes from the Latin, *digitus;* that is, *finger*. Savages, who have no better mode, they say, *count* by their *fingers;* and in this manner, say the learned, our ancestors counted, before they had acquired a knowledge of numerical figures, to which figures it appears, that they, very naturally, and it must be allowed, very excusably in them, transferred the word. But is this any reason, I ask, for us to defile our language with the heterogeneous and barbarous term, even were they our own immediate ancestors, instead of the ancestors of the ancient Greeks, or Romans, who had first so applied it? That its use is not necessary, in order to describe either the meaning or the uses of numbers, has, I hope, been made to appear in these pages; through which I have used, when speaking of the *forms* by which we express numbers, the word *figure;* and, when speaking of the *numbers themselves*, I have used *the word itself;* to be sure. And this proper use of words, gives clearness, simplicity, and certainty to writing. Whilst the darkness, and confusion which have been spread over this subject of Arithmetic, must be the inevitable consequence of a casual and *senseless* use of the terms *number*, *digit*, and *figure*.

(Lessons on Arithmetic.)

Dividend.—The Number divided, or to be divided.

Divisor.—The Number with which we divide.

FACTORS.—Numbers which, multiplied together, *have produced* another number: Having produced such number, they are called Factors; the factors of that number.—*See Multiple, and Submultiple.*

FIGURES.—The FORMS by which Numbers are expressed; Applied, also, to other forms, as diagrams.—(*See term Digit.*)

MULTIPLE.—A Number having, in certain other numbers, relations called its submultiples. The relationship which the multiple bears towards its submultiples, is of this nature, that they are *capable of producing it; that they are capable of becoming its factors.* For example, the numbers, that is, a due portion of the numbers, 2, 3, 4, 6, 8, and 12, are thus *capable* of producing the number 24. They are, therefore, SUBMULTIPLES of this number; whilst this number is called a MULTIPLE of them. This relationship between numbers means, merely, that the smaller *may* be found in the larger; and that the larger *may* be produced by a multiplication of a due portion of the smaller numbers, without there being either surplus or deficiency. The terms multiple and submultiple properly belong to numbers bearing this relationship towards each other, *only before* the latter be multiplied, and *before they have produced* the former. When the *production* has taken place, the numbers change their names; the *multiple* becomes a *product*, and the *submultiples* become *its factors.*

PRIMES, OR PRIMARY NUMBERS.—It is not all numbers that can be *evenly* divided. Numbers of this kind, as well as those which may be evenly divided, have a name; and as these are termed multiples, so the *odd* and *indivisible* numbers are distinguished by the name PRIMES, or PRIMARY NUMBERS: that is, they are *primary* or *original*, numbers, and not *producible* by the multiplication of other numbers; as 3, 5, 7, 9, 13, 17, &c.

PRODUCT.—The number *produced* by the multiplication together of two or more numbers.

QUOTIENT.—The result of a division of one number by another; the number of times that the larger number, the dividend, contains the divisor.

RESOLVEND.—A number to be divided, or evolved.

SUBMULTIPLE.—One of two or more numbers that are *capable*, when multiplied together, of producing a given number.—(*See Multiple.*)

SUBTRAHEND.—The *lower*, *under*, *smaller*, number, in a case of subtraction. A term useful in operations of extraction of roots, but not required in any of the earlier operations of arithmetic.

SQUARE NUMBER.—A number, the product of any whole or integral number, multiplied into itself; as 4, which is the square of 3; 16 of 4; 100 of 10; and so on. Its more scientific definition is, *the second power of any integral number*.

SURD; or SURD NUMBER.—Terms, as used in arithmetic, on which I do not find that the learned are very well agreed. The word *surd*, is the Latin for *deaf*. And a deaf number, or Surd, is a number the root of which you cannot extract without having fractions. Some seem to apply it to numbers only which, on the evolution of their roots, fall into recurring, or endless fractions. However, it is a term, in contradistinction to square and cube numbers; as before defined.

TABLES

OF WEIGHTS, AND MEASURES,

Used in England; with the marks by which each of them is commonly distinguished.

AVOIRDUPOISE WEIGHT.

dr. drachm
oz. ounce
lb. pound
qr. quarter of a
cwt. hundred weight
T. ton

16 drams	make	1 ounce
16 ounces	..	1 pound
14 pounds	..	1 stone
28 pounds	..	1 quarter
4 quarters	..	1 hund. wt
20 hun. wt.	..	1 ton

TROY WEIGHT.

gr. grain
dwt. penny weight
oz. ounce

24 grs.	make	1 penny wt.
20 dwts.	..	1 ounce
12 ounces	..	1 pound

APOTHECARIES' WEIGHT.

gr. grain | ʒ dram
℈ scruple | ℥ ounce

20 grains	make	1 scruple
3 scruples	..	1 dram
8 drams	..	1 ounce
12 ounces	..	1 pound

WOOL WEIGHT.

7 pounds	make	1 clove
2 cloves	..	1 stone
2 stones	..	1 tod
6½ tods	..	1 wey
2 weys	..	1 sack
12 sacks	..	1 last

ALE AND BEER.

2 pints	make	1 quart
4 quarts	..	1 gallon
9 gallons	..	1 firkin
2 firkins	..	1 kilderkin
2 kilders.	..	1 barrel
1½ barrel	..	1 hogshead
2 barrels	.	1 puncheon
3 barrels	..	1 butt

WINE MEASURE.

2 pints	make	1 quart
4 quarts	..	1 gallon
10 gallons	..	1 anker
63 gallons	..	1 hogshead
2 hhds.	..	1 pipe
2 pipes	..	1 tun
42 gallons	.	1 tierce

DRY MEASURE.

2 gallons	make	1 peck
4 pecks	..	1 bushel
4 bushels	..	1 sack
8 bushels	..	1 quarter
4 quarters	..	1 chaldron
10 quarters	..	1 last

SOLID MEASURE.

1728 in.	make	1 solid foot
27 ft.	..	1 yard

LONG MEASURE.

3 barley corns	1 inch
12 inches	1 foot
3 feet or 36 in.	1 yard
6 feet	1 fathom
5½ yards	1 pole
40 poles	1 furlong
8 furlongs	1 mile
3 miles	1 league
69½ miles	1 degree

LAND MEASURE.

9 feet	make	1 yard
30¼ yards	..	1 pole
40 ploes	..	1 rood
4 roods	..	1 acre

COAL MEASURE.

3 bushels	..	1 sack
36 bushels	..	chaldron

CLOTH MEASURE.

2¼ inches	..	1 nail
4 nails	..	1 quarter
4 quarters	..	1 yard
3 quarters	..	1 Flemish ell
5 quarters	..	1 English ell
6 quarters	..	1 French ell

TIME.

60 seconds	..	1 minute
60 minutes	..	1 hour
24 hours	..	1 day
7 days	..	1 week
4 weeks	..	1 month

PARTICULAR MEASURES OF LENGTH.

A Nail is	2¼	Inches	used for measuring cloths of all kinds.
A Quarter	4	Nails	
A Yard	4	Quarters	
An Ell	5	Quarters	
A Hand	4	Inches; used for the height of horses.	
A Fathom	6	Feet; used in measuring depths.	
A Link	7	Inches; 92-hundredths	used in Land Measure.
A Chain	100	Links	
An Acre, 10 Chains, square			

MEASURE OF SURFACE.

144 Square Inches make 1 Square Foot	40 Perches .... 1 Rood
9 Square Feet .... 1 Square Yard	4 Roods, or 160 Perches 1 Acre
30¼ Square Yards ..... 1 Perch or Rod	640 Acres .... 1 Square Mile

MEASURES OF SOLIDITY AND CAPACITY.

DIVISION I.—SOLIDITY.

1728 Cubic Inches make .... 1 Cubic Foot
27 Cubic Feet .... 1 Cubic Yard

Division II.

Imperial Measure of CAPACITY for all liquids, and for all dry goods, except such as are comprised in the third Division :—

4 Gills make..	1 Pint	= $34\frac{2}{3}$	cubic inches, nearly
2 Pints	1 Quart	= $69\frac{1}{3}$	——
4 Quarts	1 Gallon	= $277\frac{1}{4}$	——
2 Gallons	1 Peck	= $554\frac{1}{2}$	——
8 Gallons	1 Bushel	= $2218\frac{1}{5}$	——
8 Bushels	1 Quarter	= $10\frac{1}{4}$	cubic feet, nearly
5 Quarters....	1 Load	= $51\frac{1}{3}$	——

Division III.

Imperial Measure of CAPACITY for coals, culm, lime, fish, potatoes, fruit, and other goods, commonly sold by *heaped measure* :—

2 Gallons make..	1 Peck	= 704	cubic inches, nearly
8 Gallons	1 Bushel	= $2815\frac{1}{2}$	——
3 Bushels	1 Sack	= $4\frac{8}{9}$	cubic feet, nearly
12 Sacks	1 Chaldron	= $58\frac{2}{3}$	——

The Goods are to be heaped up in the form of a cone, to a height above the rim of the measure, of at least three-fourths of its depth. The outside diameter of measures, used for heaped goods are to be at least double the depth, consequently not less than the following dimensions :—

Bushel......	$19\frac{1}{2}$ inches	Gallon	$9\frac{3}{4}$ inches
Half-Bushel..	$15\frac{1}{2}$ ——	Half-Gallon..	$7\frac{3}{4}$ ——
Peck........	$12\frac{1}{4}$ ——		

The Imperial Measure was established by Act 5 Geo. IV. c. 74. Before that time, there were four different measures of capacity used in England :—1st. For wine, spirits, cider, oils, milk, &c.; this was one-sixth less than the Imperial measure. 2nd. For malt liquor, this was one fifty-ninth part greater than the Imperial measure. 3rd. For corn and all other dry goods not heaped, this was one thirty-third part less than the Imperial Measure. 4th. For coals, which did not differ sensibly from the Imperial Measure.

The Imperial Gallon contains exactly 10 lbs. avoirdupoise of pure water; consequently the pint will hold $1\frac{1}{4}$ lb., and the bushel 80 lbs.

PARTICULAR WEIGHTS.

		cwt.	*qrs.*	*lbs.*	
8 Pounds make..	1 Stone				used for Meat.
14 Pounds	1 Stone	= 0	0	14	used in the Wool Trade.
2 Stone	1 Tod	= 0	1	0	
$6\frac{1}{2}$ Tod.........	1 Wey	= 1	2	14	
2 Weys.........	1 Sack	= 3	1	0	
12 Sacks.........	1 Last	= 39	0	0	

MISCELLANEOUS.

12 Dozen make a Gross
A Weigh is 256lbs.
12 Barrels make a Last
A Quire of Paper is 24 Sheets
A Ream of Paper is 20 Quires
A Bundle of Paper is 2 Reams
A Bale of Paper is 10 Reams
A Roll of Parchment, or Vellum, is five dozen, or 60 Skins
A Dicker of Hides is 10 Skins
A Last of Hides is 20 Dickers
A Dicker of Gloves is 10 dozen pairs
A Firkin of Butter is 56lbs.
A Firkin of Soap is 64lbs.
A Tierce of Rice is about 5 cwt.
A Hogshead of Tobacco is from 9 to 10 cwt.
A Barrel of Gunpowder is 1 cwt.
A Pack of Wool is 240lbs.
20 Stones of Flour make a Sack
A Load of Timber, unhewed, is 40 feet
A Load of Bricks, 500 in number
A Load of Tiles, 1000 in number
A Load of Hay, in London, is nearly 18 cwt.
A Load of Straw, 36 Trusses, of 36lbs. each
A Chaldron of Coals, in London, is 36 bushels
A Chaldron of Coals, in Newcastle, is 53 cwt.
A Cart of Coals, in Scotland, is 12 cwt.
A Deal of Coals, in Scotland, is 23 cwt.
A Grain of Gold is worth about 2*d.*
An ounce of fine Silver is worth from 5*s.* to 5*s.* 6*d.*

DIVISIONS OF THE YEAR.

The time in which the Earth performs one complete journey round the Sun, being divided into 12 equal portions, called months, gives, for each month, 30 *days*, 10½ *hours*, within a few seconds; and these months, from their refernce to the Sun, are called SOLAR MONTHS, from *Sol*, the Latin word for *Sun*. In like manner is the same portion of time measured by the motion of the Moon; and the divisions arising therefrom are called LUNAR MONTHS; from the name of the planet, which in Latin is *Luna*. These divisions may both be considered as *natural*, or as *astronomical*, months. But for *legal* purposes, and for purposes of *business*, a somewhat different mode of dividing the year prevails.

This other division of the year, is into what are called civil, or Calendar months; *civil*, because they are the divisions adopted by the *civil* governments of Europe; and *Calendar*, from a custom of the ancient Romans, who called certain days of each month the calends of that month. The Calendar months are, of course, the months distinguished by the names January, February, &c.

ROMAN NUMERALS.

The ancient Romans, who enjoyed not the benefit of the beautiful system of notation, which the modern world derived from the Arabians; the Romans employed certain of their Capital letters, wherewith to describe numbers; and, as this mode is still useful, for particular purposes, and still used, it may be well to describe it. It will be sufficient to write some of these numerals, and to place opposite to each, the Arabic figure of the same value.

I......1	XI 11	XL 40	C............ 100
II.....2	XII......12	XLI.........41	CL 150
III...3	XIII.....13	XLV.......45	CC.......... 200
IV ..4	XIV.....14	L............50	CCC. 300
V.....5	XV......15	LV 55	CCCC...... 400
VI....6	XVI.....16	LX..........60	D............ 500
VII..7	XX......20	LXV........65	DC.......... 600
VIII 8	XXI 21	LXX. 70	DCC 700
IX ...9	XXX...30	LXXX.....80	M............1000
X...10	XXXIII..33	LC...........90	MD.........1500

FINIS.

"All the Books on this subject that I had ever seen, were so bad, so destitute of every thing calculated to lead the mind into a knowledge of the matter, so void of principles, and so evidently tending to puzzle and disgust the learner, by their sententious, and crabbed, and quaint, and almost hieroglyphical definitions, that I, at one time had the intention of writing a little work on the subject myself. It was put off from one cause or another; but a little work on the subject has been written and published by Mr. Thomas Smith, of Liverpool. The author has great ability, and a perfect knowledge of his subject. It is a Book of principles; and any young person of common capacity, will learn more from it in a week, than from all the other books, that I ever saw on the subject, in a twelvemonth."
(Cobbett's Advice to Young Men, p. 315.*)*

"Mr. Smith leaves nothing, in the outset, to be guessed at by the learner; he does not explain the rules in language which, to the early pupil, is itself a mystery; but he uses clear, home-spun words, and points out and explains, step by step, the gradual workings of the system, in all its developments. We have been much pleased with the pervading simplicity, and the arrangement of the contents of the volume; and cheerfully recommend it to the consideration of those to whom Mr. S. more immediately addresses it." *(Ballot, Aug.* 14, 1831.*)*

"This is a shrewed, clever little book, well adapted to its object. Figures, to be sure, have a stubborn and matter-of-fact character, which in some cases is peculiarly unsightly and uninviting, but the author of this treatise, Mr. Thomas Smith, has contrived to invest the science of numbers with very considerable amusement, and great plainness of illustration." *(Mirror, No.*464.*)*

"Mr. Smith's "Lessons on arithmetic," is a book of Principles, it does not direct, it really teaches; it unfolds all the little intricacies of the art; it shows the reader the reason of every thing he is instructed to do; it does not lead him blindly along, but makes him familiar with the path he is pursuing; it makes him feel the firmness of the ground on which he treads. In what it embraces, and in what omits, it is calculated, not to display learning, but to communicate knowledge; one part is made to rise upon, and to grow out of another, in a manner which shows the mind of the author to have been fully master of his subject; which makes this very intricate subject clear and interesting to the least informed, and at the same time must delight the truly learned as a fine example of that *lucidus ordo*, that bright arrangement, to which the most exact of critics refers the greatest charm even of works of fiction. It is, in short, a book of recondite principles, clearly and gracefully developed."
(Manchester Advertiser, Feb. 26, 1831.*)*

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