THOMSON'S

PLANE & SPHERICAL

TRIGONOMETRY

ELEMENTS

OF

PLANE AND SPHERICAL

TRIGONOMETRY,

WITH THE

FIRST PRINCIPLES

OF

ANALYTIC GEOMETRY.

BY JAMES THOMSON, LL.D.

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF GLASGOW.

FOURTH EDITION,

WITH VARIOUS ADDITIONS AND IMPROVEMENTS.

LONDON:
SIMMS AND M'INTYRE,
ALDINE CHAMBERS, PATERNOSTER-ROW;
AND DONEGALL-STREET, BELFAST.

1844.

ADVERTISEMENT.

THE first edition of this work was intended chiefly as a Text-Book for the use of the students in the BELFAST INSTITUTION, when the Author was Professor of Mathematics in that establishment; and it was therefore written, not as a regular and complete treatise on Trigonometry, but as an outline to be filled up, and illustrated orally in the Lectures. It was received, however, by other readers in a more favourable manner than the Author could have anticipated from its nature and form; and, in consequence of this, he has been induced, in the subsequent editions, to make various alterations, and, it is hoped, improvements. The investigations, though still concise, are given at such length as to be easily understood by readers of ordinary talents and attainments; and various interesting additions are introduced, some from the best recent works on the subject, and others that have occurred to the Author himself. Of the latter kind are the improvements in the numerical resolution of triangles, established in Nos. 57, 58, 73, and 74, and exemplified in Nos. 62, 63, and 110; which, with the modes of operation previously known, seem to render the subject as simple and easy as can be desired. In the present edition, also, a scholium of considerable interest will be found at the end of the third section: and the last four pages of the ninth section contain some curious propositions in Spherical Geometry, most of which the Author believes to be new.

It has been everywhere the aim of the Author, to comprise in a small compass much useful and interesting matter; and, whatever may be the imperfections of the work, he trusts that the person who shall make himself well acquainted with what it contains, will find it easy to acquire a knowledge of all that is yet known in Trigonometry, and to apply it in Astronomy, and other branches of science.

In the present edition, the sines, tangents, &c. are defined as mere numbers or ratios. This mode of representing them has been in use for some time in the University of Cambridge; and it is attended with considerable advantage, particularly in the application of Trigonometry in Natural Philosophy. Should any persons prefer the common mode, they may have recourse to the Note at the end of the volume.

Glasgow College, July, 1844.

CONTENTS.

ELEMENTS OF TRIGONOMETRY.*

I.—GENERAL PRINCIPLES.

1. Each of the four parts into which a circle is divided by two diameters intersecting each other at right angles, is called a *quadrant.* If one of the four right angles be divided into 90 equal parts by radii of the circle, each of the parts is called a *degree.* The parts into which these radii divide the arc of the quadrant are (Euc. III. 27) all equal, and they are therefore called degrees of the circle. A sixtieth part of a degree is called a *minute;* and a sixtieth of a minute, a *second.*† Degrees, minutes, and seconds are denoted by the characters, $^{\circ}$, $'$, $''$.

2. It is proved by writers on geometry, that the circumferences of different circles are proportional to their radii; and that, in the same circle, angles at the centre are proportional to the arcs on which they stand. Hence, if from the vertex of any angle A (*fig.* 1) as centre, two circumferences BCD and B′C′D′, be described, cutting one of the lines forming the angle in B and B′, and the other in C and C′, the ratio of the arc, BC, to its radius, AB, is equal to that of the other arc B′C′ to its radius AB′. This readily follows from the principles stated above: for, since, by No. 1, and Euc. I. 13, cor., all the angles about A amount to 360°, we have, by the second of those principles,

$$\frac{BC}{BCD}=\frac{A}{360^{\circ}}, \text{ and } \frac{B'C'}{B'C'D'}=\frac{A}{360^{\circ}}; \text{ whence } \frac{BC}{BCD}=\frac{B'C'}{B'C'D'}.$$

* Trigonometry, in its primitive meaning, is that branch of mathematical science, which determines certain sides or angles of a triangle from others that are known. It is of two kinds, *Plane* and *Spherical:* the former treating of triangles described on a plane; and the latter, of those on the surface of a sphere. The principles of trigonometry, however, are now of far more general application, furnishing means of investigation in almost every branch of mathematics.

† In some modern French works on mathematics, the *centesimal* division is adopted instead of the *sexagesimal;* the right angle, and consequently the quadrant, being divided into 100 degrees; the degree, into 100 minutes; and the minute, into 100 seconds. This division, however, is likely to fall into disuse.

Also, by the first principle, we have $\frac{BCD}{AB}=\frac{B'C'D'}{AB'}$.

Multiply the members of this equation by those of the preceding, and there will be obtained $\frac{BC}{AB}=\frac{B'C'}{AB'}$, which is the property above stated. Hence, if we assume any radius, and call it r, and if we denote the corresponding arc by s, $\frac{s}{r}$ will be always the same for the same angle, whatever may be the magnitude of r; and since s, and consequently, $\frac{s}{r}$, is proportional to the angle, whatever may be its magnitude, $\frac{s}{r}$ will be a correct measure of any angle whatever.*

3. If an angle be taken from a right angle, or an arc from a quadrant, the remainder is called the *complement* of that arc or angle. From this it follows, that if an angle or arc exceed 90°, its complement is negative.

4. If an angle be taken from two right angles, or an arc from a semicircle, the remainder is called the *supplement* of the angle or arc.

5. The straight line joining the extremities of an arc, is called its *chord*.

6. If from the vertex of any angle A (*fig.* 2, 4, 5, or 6) a circle be described with any radius, cutting the lines forming the angle in two points, B and C, and through one of these points C, a straight line be drawn perpendicular to the other line AB, and cutting it in D; the ratio of the perpendicular CD to the radius AC, that is, the number obtained by dividing the perpendicular by the radius, is called the *sine* of the angle A; and if DB be divided by the radius, the quotient is called the *versed sine* of the same angle. These, for the sake of abbreviation, are written $\sin A$, and $\operatorname{versin} A$, or $\operatorname{vs} A$. Hence, calling the radius r, and multiplying by it, we get $CD = r \sin A$. The sine of an arc is evidently the ratio of half the chord of its double to the radius.

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* Hence, if s and r be equal, the angle becomes 1; and we thus see, that in this mode of measuring angles, the angle which is the unit, is that which has the circular arc on which it stands equal to the radius of the circle. Now, this angle is 57° 17′ 44″·8, or 206264″·8, nearly; as is found by the following analogy: 3·14159265 : 1 : : 180° : 57° 17′ 44″·8; the semicircumference of the circle whose radius is 1, being 3·14159265. (See the Author's Differential and Integral Calculus, page 41.) Hence, if the radius of a circular sector were 12 inches, and its arc 25 inches, we should find its angle to be 119° 21′ 58″·3, by multiplying 57°, &c. by 25, and dividing the product by 12.

7. If through the other point B another perpendicular to the same straight line, AB, be drawn, cutting the other line AC produced in E; the ratio of the perpendicular BE to the radius AB is called the *tangent* of the angle A; and the ratio of the hypotenuse AE to the radius is called the *secant* of the same angle. These, for abbreviation, are written tan A and sec A. Hence, by multiplying by the radius, we get $BE = r \tan A$, and $AE = r \sec A$; so that, by taking along with these the expression found at the end of the last No., we have the three formulas,

$CD = r \sin A \ldots(a)$, $BE = r \tan A \ldots(b)$, $AE = r \sec A \ldots(c)$.

8. The *cosine* of an angle is the sine of its complement. In like manner, the *coversed sine, cotangent,* and *cosecant* of an angle are respectively the versed sine, tangent, and secant of its complement. Hence, since an angle is the complement of its complement, the cosine, cotangent, &c. of the complement of an angle are respectively its sine, tangent, &c. For brevity, the sine, versed sine, tangent, and secant of the complement of an angle A are written cos A, coversin A, or covs A, cot A, and cosec A.*

If now BA be produced to meet the circumference again in F, and the diameter GH be drawn perpendicular to BF, the angle CAG will (No. 3) be the complement of A. Drawing, therefore, CK and GL perpendicular to GH, and producing AC to meet GL in L, it follows from the last two Nos., that if CK, or its equal AD, be divided by the radius AC, the quotient will be the sine of CAG, or the cosine of A; and that, in like manner, if GK, GL, and AL be divided by the radius, the quotients will be covs A, cot A, and cosec A. Hence, by multiplying by the radius r (omitting the coversed sine, as giving a result of no value), we get the following formulas:

$AD = r \cos A \ldots(d)$, $GL = r \cot A \ldots(e)$, $AL = r \operatorname{cosec} A \ldots(f)$.

9. If we regard the line AB as fixed, while AC, commencing its motion from coincidence with AB, revolves about the point A, the angle A will commence from nothing, and, by receiving continual increases, may attain any magnitude, however great. Thus, the revolving line (*fig.* 3) may take the successive positions AC_1, AC_2, AC_3, AC_4, AC_5, &c.; there being evidently no limit to the amount of angular space described by that line, which, like a crank in machi-

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* This notation, and the corresponding ones in Nos. 6 and 7, possess great advantages from their conciseness, and from their suggesting to the mind at once the ideas which the symbols are intended to express.

nery, may be supposed to revolve again and again about A. Hence, A will be one right angle, when AC (*fig.* 2) coincides with AG; two right angles, when it coincides with AF; three, when with AH; four, when having completed a revolution, it again coincides with AB; five, when it falls a second time on AG; and so on.

Now, when the line AC (*fig. 2 and* 4) lies in the first or second right angle, BAG or GAF, the line CD, or $r \sin A$, is on the side of BF which is towards G; but (*fig.* 5 *and* 6) in the third and fourth right angles, FAH and HAB, it falls on the other side of BF. In the first and fourth right angles, the line AD, or $r \cos A$, is a part of AB; but in the second and third, it is a part of AF. In the first and third right angles, the line BE, or $r \tan A$, is on the side of the point B which is towards G; but in the second and fourth, it is on the other side. In the first and third right angles, the line GL, or $r \cot A$, lies on that side of the point G which is towards B; but in the second and fourth, it is on the other side. Lastly, in the first and fourth right angles, the line AE, or $r \sec A$, passes through C, the termination of the arc BC; while, in the other two right angles, it lies on the opposite side of the centre with regard to C: and, in the first and second right angles, the line AL, or $r \operatorname{cosec} A$, passes through C; while, in the third and fourth, it lies on the opposite side of the centre.

10. It will be seen, also, that when, AC coinciding with AB, the angle A is nothing, the line CD is also nothing; but that, when A is a right angle, CD coincides with the radius AG, and is equal to it. In like manner, when A takes the successive values, two, three, and four right angles, CD becomes successively nothing, AH, and nothing. By dividing these, therefore (No. 6), by the radius r, and by carrying out the same principle with regard to angles still larger, we find that the sine of any angle which is nothing, or two right angles, or four right angles, or any even number of right angles, is nothing; and that the sine of any odd number of right angles is 1. In like manner we should find, that the tangent of nothing, or of an even number of right angles, is nothing; and that the secant of any of the same angles is 1. If, again, A be supposed to increase by the approach of AC (*fig.* 2) to coincidence with AG, the lines BE and AE will continually increase, and may be made as large as we please. When, however, AC coincides with AG, it will be parallel to BE, and will therefore never meet it. In this case, BE and AE are said to be infinite; and therefore (No. 7), dividing them by r, we get an infinite quotient; whence it appears, that the tangent and secant of a right angle are both infinite; and the same will evidently be the case

when A is composed of any odd number of right angles. It would appear in a similar manner, that the versed sine of nothing is nothing; of one or three right angles, 1; of two right angles, 2; and of four right angles, nothing; and it is easy to trace similar relations regarding the cotangent, cosecant, and coversed sine.*

11. By eliminating r, and the lines, CD, BE, &c. from the expressions in Nos. 7 and 8, we obtain the following trigonometrical formulas, which are of much importance:

$\cos A \sec A = 1$ ......(1),	$\tan A \cot A = 1$† ....(5),
$\sin A \operatorname{cosec} A = 1$ ...(2),	$\sin^2 A + \cos^2 A = 1$ ......(6),
$\tan A = \dfrac{\sin A}{\cos A}$ ......(3),	$\sec^2 A = 1 + \tan^2 A$ .....(7),
$\cot A = \dfrac{\cos A}{\sin A}$ ......(4),	$\operatorname{cosec}^2 A = 1 + \cot^2 A$ ...(8).

To investigate the first of these, let us take the products of the members of (c) and (d): then, $r^2 \cos A \sec A = AD.AE$. Now, AD.AE is equal to r^2: for, in the similar triangles, ADC, ABE (*fig.* 2, 4, 5, *or* 6), we have AD : AC : : AB : AE; whence, since AB and AC are each equal to r, we get (Euc. VI. 17.) $AD.AE = r^2$. Hence, the expression formerly found becomes $r^2 \cos A \sec A = r^2$; and from this, by dividing by r^2, we get (1). Formula (2) is obtained similarly from (a) and (f), by means of the similar triangles, ADC, AGL. To investigate (3), we might find it very easily by dividing (a) by (d); or we may take the product of formulas (b) and (d), then

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* From what is pointed out above, in connexion with what will be established in No. 11, it will appear that some of the values of the sines, tangents, &c. above mentioned, are positive, and some negative: and the student will have no difficulty in seeing, that if n be any number in the series, 0, 1, 2, 3, &c. $\sin A$ and $\tan A$ will be nothing, when A consists of $2n$ right angles: that $\cos A$ and $\cot A$ will be nothing, when A is $2n+1$ right angles; that $\sin A$ is 1, when A is equal to $4n+1$ right angles: and to -1, when A is $4n+3$ right angles: that $\cos A$ is 1 when A is $4n$ right angles; and -1, when A is $4n+2$ right angles: that $\tan A$ and $\sec A$ are each equal to $+\infty$, when A is equal to $4n+1$ right angles; and to $-\infty$, when A is $4n+3$ right angles, &c.

† Hence we have $\cot A = \dfrac{1}{\tan A}$. Multiply both by any quantity a; then $a \cot A = \dfrac{a}{\tan A}$. From this it appears, that if a quantity be divided by the tangent of an arc, the quotient is the same as the product that would be obtained by multiplying the same quantity by the cotangent of the arc; and it would be shown in a similar manner, that to divide by the cotangent is the same as to multiply by the tangent. It is evident, also, that there is the same mutual relation between the cosine and the secant, and between the sine and the cosecant; and, universally, between *any quantity* and its reciprocal.

$r^2 \cos A \tan A = AD.BE$. Now, from the triangles ADC, ABE, we get AD : DC : : AB : BE; and, consequently, $AD.BE = r.DC = r^2 \sin A$, because, by (*a*), $DC = r \sin A$; and, therefore, $r^2 \cos A \tan A = r^2 \sin A$; whence (3) is obtained by dividing by r^2 and $\cos A$. We find (4) in exactly the same manner from (*a*) and (*f*), by means of the triangles ADC, AGL, in connexion with formula (*d*). The easiest mode of finding (5), is to take the products of the members of (3) and (4). If we now add together the squares of the members of (*a*) and (*d*), we get $r^2 \sin^2 A + r^2 \cos^2 A = AD^2 + CD^2$. But (Euc. I. 47) $AD^2 + CD^2 = AC^2 = r^2$: and therefore $r^2 \sin^2 A + r^2 \cos^2 A = r^2$; whence we get (6) by dividing by r^2. To find (7), we square the members of (*b*), and add to the first of the results AB^2, and to the second, what is equivalent, r^2: then $AB^2 + BE^2$, or (Euc. I. 47) AE^2, or, by formula (*c*), $r^2 \sec^2 A = r^2 + r^2 \tan^2 A$; whence we get (7) by dividing by r^2. In the last place, (8) is found in a similar manner from (*e*), by squaring its members, adding to each AG^2, and applying to the results Euc. I. 47, and formula (*f*).

12. For the purpose of extending the application of analytical formulas, it is often necessary to consider the sines, tangents, &c. of angles which are greater than four right angles. Thus, we may consider the line AC (*fig.* 2 &c.), as having revolved round A once or oftener, and having described the angle BAC besides. In this view it is evident, that in the fifth, ninth, thirteenth, &c. right angles, the sines, cosines, tangents, &c. will be the same as in the first; in the sixth, tenth, &c. the same as in the second; the line AC always occupying the same position after the addition of four right angles.

13. To render the formulas which express the relations of sines, cosines, &c. in the first right angle, applicable in expressing the same relations in the others,* the sine is to be regarded as positive, when

* Thus, in the first right angle, $\text{versin } A = 1 - \cos A$, which formula will hold true also in the second and third right angles, on the supposition, that in them the cosine is negative; and a similar illustration may be given by means of the coversed sine. "Let us lay down, then, this general principle, which is of great utility: *All trigonometrical formulas should be formed for positive angles which do not exceed* 90°; since they will serve equally for angles that are greater than 90°, and for negative angles, by merely making the proper changes on the signs of the trigonometrical quantities."— *Cagnoli, Trigonométrie*, 77. From this conventional mode of expressing difference of position by the use of different signs in the analytical expressions for quantities, much advantage results in extending the application of formulas both in trigonometry, and in other parts of mathematics. Without this artifice, in the instance already given, we should sometimes have $\text{versin } A = 1 - \cos A$, and sometimes, $\text{versin } A = 1 + \cos A$.

CD, the line to which it is proportional, lies on one side of AB, as towards G; but negative, when that line lies on the other side. In like manner, the cosine is to be regarded as positive or negative, accordingly as the line AD, to which it is proportional, is a part of AB or AF. Hence, also, since (No. 11) $\tan A = \frac{\sin A}{\cos A}$, $\cot A = \frac{\cos A}{\sin A} = \frac{1}{\tan A}$, $\sec A = \frac{1}{\cos A}$, and $\operatorname{cosec} A = \frac{1}{\sin A}$, it is easy to trace the mutations of the signs of all these quantities; and it will appear, that the sine and cosecant are positive in the first and second, fifth and sixth, ninth and tenth, &c. right angles, and negative in the others; that the tangent and cotangent are positive in the first, third, fifth, seventh, &c. right angles, and negative in the others; and that the cosine and secant are positive in the first, fourth and fifth, eighth and ninth, &c.;* the tangent and cotangent changing their signs at the end of each right angle; and the sine, cosine, secant, and cosecant, at intervals of two right angles.

14. The angle BAM (*fig.* 2) lying on the opposite side of AB from the angle BAC, may be regarded as negative in relation to BAC taken as positive; and, from considering the sine, cosine, &c. of this negative angle, it will be obvious, that $\sin(-A) = -\sin A$; $\cos(-A) = \cos A$; $\tan(-A) = -\tan A$, &c.

15. If (*fig.* 2)† the angle FAN be made equal to BAC, and NO be drawn perpendicular to BF, it follows (Euc. I. 26), that NO is equal to CD, and AO to AD; the two latter, however, lying in opposite directions. Now, if NO and AO be divided by the radius, the quotients (Nos. 6 and 8) will be the sine and cosine of the angle BAN, which (by construction and No. 4) is the supplement of BAC. Hence, it is evident, that the sines of A and its supplement are equal; and that their cosines are also equal, but have contrary signs.

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* It is easy to see by an inspection of the diagram, that these signs correspond strictly to the positions of the lines CD, AD, BE, and GL, to which the sines, cosines, tangents, and cotangents, are proportional. That the same is the case in relation to the secant, will appear from considering the circle to be described by a revolving radius commencing its motion from AB. In the first, fourth, fifth, &c. quadrants, the line AE, which is proportional to the secant, and this radius, will lie on the same side of the centre; but in the second, third, sixth, &c. on opposite sides. A similar illustration is applicable with regard to the cosecant.

† Everything stated here, as well as in several other cases, will hold equally in figures 4, 5, and 6: the sole difference in the present instance being, that in figures 5 and 6, the supplements would be negative.

If, therefore, π* be put to denote the quotient obtained by dividing the semicircular arc BGF by the radius AB, which quotient (No. 2) is the measure of 180°, we shall have $\sin A = \sin(\pi - A)$; $\cos A = -\cos(\pi - A)$; and consequently, by formula (3), $\tan A = -\tan(\pi - A)$, &c. Hence, also, the sine of one angle of a triangle is equal to the sine of the sum of the other two.

16. Since (Nos. 14 and 15) $\sin A = \pm \sin(\pm A) = \sin(\pi - A)$; by adding to A a multiple of 2π (four right angles), we get (No. 12)

$$\sin A = \pm \sin(2n\pi \pm A) = \sin\{(2n+1)\pi - A\} \ldots\ldots(9).$$

In this, n is any number in the series 0, 1, 2, 3, &c., or 0, —1, —2, &c.: and hence any sine is the sine of an infinite number of angles.

17. In like manner, it would readily appear, that

$$\cos A = \cos(2n\pi \pm A) = -\cos\{(2n+1)\pi - A\} \ldots (10).$$

18. By taking $n = -1$ in the concluding parts of (9) and (10), we get $\sin A = \sin(-\pi - A)$, and $\cos A = -\cos(-\pi - A)$; or (No. 14) $\sin A = -\sin(\pi + A)$, and $\cos A = -\cos(\pi + A)$. Dividing the former by the latter, we have, by (3), $\tan A = \tan(\pi + A)$: whence it appears, that if two angles differ by π, their tangents are equal. By adding, therefore, $n\pi$ to A, and also to —A, we get, by Nos. 12 and 14,

$$\tan A = \tan(n\pi + A) = -\tan(n\pi - A) \ldots\ldots\ldots\ldots\ldots (11).$$

Hence, also, it is plain, that (No. 13)

$$\cot A = \cot(n\pi + A) = -\cot(n\pi - A).$$

19. It follows from Nos. 6, 7, and 8, that the sine, cosine, tangent, cotangent, secant, and cosecant, of an angle, are simply the ratios of the sides compared by pairs, of a rightangled triangle, which has that angle as one of its angles. Thus, if ABC (*fig.* 7) be a triangle right-angled at C, we have at once, by Nos. 6 and 8, AB : AC :: 1 : sin B, and AB : BC :: 1 : cos B; and also $AC = AB \sin B = AB \cos A$, since B is the complement of A. It thus appears, that *the hypotenuse is to either leg, as* 1 *is to the sine of the angle opposite to that leg, or to*

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* For the use of those who are unacquainted with Greek, the following list is given, containing the capital and small letters of the Greek alphabet, with their names, and the letters, or combinations of letters, to which they are respectively equivalent in Latin and English;

Α, α, alpha, *a*; Β, β or ϐ, beta, *b*; Γ, γ, gamma, *g*; Δ, δ, delta, *d*; Ε, ε, epsílon, *e* (short); Ζ, ζ or ϛ, zeta, *z*; Η, η, eta, *e* (long); Θ, θ or ϑ, theta, *th*; Ι, ι, iota, *i*; Κ, κ, kappa, *k*; Λ, λ, lambda, *l*; Μ, μ, mu, *m*; Ν, ν, nu, *n*; Ξ, ξ, xi, *x*; Ο, ο, omícron, *o* (short); Π, π or ϖ, pi, *p*; Ρ, ϱ, rho, *r*; Σ, σ or ς, sigma, *s*; Τ, τ, tau, *t*; Υ, υ, upsilon, *u* or *y*; Φ, φ, phi, *f* or *ph*; Χ, χ, chi, *ch*; Ψ, ψ, psi, *ps*; Ω, ω, ómega, *o* (long).

the cosine of the adjacent angle; and that each leg is equal to the product of the hypotenuse and the sine of the opposite angle, or of the hypotenuse and the cosine of the adjacent angle.

20. We have, in like manner, by No. 7, BC : CA :: 1 : tan B, and BC : BA :: 1 : sec B: that is, *one of the legs is to the other, as the radius is to the tangent of the angle opposite to the latter;* and *one of the legs is to the hypotenuse as the radius is to the secant of the contained angle.*

21. Again, let ABC (*fig.* 8) be any triangle, and CD its perpendicular: then (No. 19) in the rightangled triangles ACD, BCD, CD = AC sin A = BC sin B; whence (Euc. VI. 16) AC : BC :: sin B : sin A; that is, *in any plane triangle, the sides are proportional to the sines of the opposite angles.* Hence, also, $\frac{AC}{BC}=\frac{\sin B}{\sin A}$.

22. One of the most important problems in trigonometry, is that in which it is required *to find the sine of the sum of two angles, in terms of the sines and cosines of the angles themselves.* To investigate this, let BAC and DAC (*fig.* 9), which, for brevity, may be called θ and θ', be the two angles; and through any point C in AC, draw BCD perpendicular to AC, and meeting AB, AD in B and D. Then (No. 19)

$$BC = AB \sin\theta, \quad \text{and } CD = AD \sin\theta';$$

and therefore $BD = AB \sin\theta + AD \sin\theta'$;

$$\text{whence } 1 = \frac{AB}{BD}\sin\theta + \frac{AD}{BD}\sin\theta' \quad \ldots\ldots (g)$$

But (No. 21) $\frac{AB}{BD}=\frac{\sin ADC}{\sin BAD}=\frac{\cos\theta'}{\sin(\theta+\theta')}$,

and $\frac{AD}{BD}=\frac{\sin ABC}{\sin BAD}=\frac{\cos\theta}{\sin(\theta+\theta')}$.

By substituting these in equation (g), and multiplying by $\sin(\theta+\theta')$, we get

$$\sin(\theta+\theta') = \sin\theta\cos\theta' + \cos\theta\sin\theta',$$

the required formula; or putting A for θ and B for θ',

$$\sin(A+B) = \sin A\cos B + \cos A \sin B \quad \ldots\ldots (12)$$

Hence, *to find the sine of the sum of two angles, multiply the sine of each by the cosine of the other, and add the products together.*

From this, taking B negative, since (No. 14) $\sin(-B) = -\sin B$, and $\cos(-B) = \cos B$, we get

$$\sin(A-B) = \sin A \cos B - \cos A \sin B \quad \ldots\ldots (13)$$

Hence, *to find the sine of the difference of two angles, from the pro-*

duct of the sine of the greater and the cosine of the less, take the product of the cosine of the greater and the sine of the less.

23. By No. 3, $\cos(A+B)=\sin(\frac{1}{2}\pi-A-B)=\sin\{(\frac{1}{2}\pi-A)-B\}$. Now, if we take $\frac{1}{2}\pi-A$ as a single angle, it is the complement of A, and we have (No. 8) its sine $=\cos A$, and its cosine $=\sin A$. Hence, therefore (No. 22),

$$\cos(A+B)=\cos A\cos B-\sin A\sin B \qquad (14)^*$$

In this, change the sign of B: then (No. 14)

$$\cos(A-B)=\cos A\cos B+\sin A\sin B \qquad (15)$$

24. Take the sum and difference of (12) and (13)†, and also of (14) and (15): then

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B \qquad (16)$$
$$\sin(A+B)-\sin(A-B)=2\cos A\sin B \qquad (17)$$
$$\cos(A-B)+\cos(A+B)=2\cos A\cos B \qquad (18)$$
$$\cos(A-B)-\cos(A+B)=2\sin A\sin B \qquad (19)$$

25. Let $A+B=S$, and $A-B=D$; then $A=\frac{1}{2}(S+D)$, and

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* This formula may also be investigated in the following manner: Retaining the same construction (*fig.* 9) as in No. 22, draw AE perpendicular to AD, meeting DB produced in E. Then the angle $E=\theta'$, each being the complement of EAC. Hence (No. 19) $EC=EA\cos\theta'$. Then, because (No. 19) $BC=AB\sin\theta$, we have

$$EB=EA\cos\theta'-AB\sin\theta;$$

$$\text{whence } 1=\frac{EA}{EB}\cos\theta'-\frac{AB}{EB}\sin\theta \qquad (h)$$

But $\frac{EA}{EB}=\frac{\sin EBA}{\sin EAB}=\frac{\sin ABC}{\sin EAB}=\frac{\cos\theta}{\cos(\theta+\theta')}$; and $\frac{AB}{EB}=\frac{\sin E}{\sin EAB}=\frac{\sin\theta'}{\cos(\theta+\theta')}$ by Nos. 21 and 15. Hence, by substituting in (h), and multiplying by $\cos(\theta+\theta')$, we get

$$\cos(\theta+\theta')=\cos\theta\cos\theta'-\sin\theta\sin\theta';$$

which is the same as formula (14): and by changing θ' into $-\theta'$, we get (15). It will be readily seen, also, how from (14) and (15), thus established, (12) and (13) might be derived by a process analogous to that employed in No. 23.

Formula (13) might also be derived by putting (in *fig.* 10) $BAC=\theta$, and $DAC=\theta'$; and from it (12), (14), and (15) might all be derived: or, finally, by drawing AE perpendicular to AD, we might investigate (15); and from it the other three might be derived with equal facility.

It may be remarked that when any one of these four formulas is derived from a diagram, it is better to derive the others from it, than again to appeal to first principles, by employing another diagram. Various other modes of investigating these important formulas, have been given by writers on trigonometry.

† That is, put the sum of the first members of equations (12) and (13) equal to the sum of their second members; and likewise the difference of their first members equal to the difference of their second members. Such abbreviated modes of expression may often be used with advantage.

$B=\frac{1}{2}(S-D)$. Substitute these values of A and B, and of their sum and difference, in the last four equations: then, to preserve uniformity of notation, use A and B, instead of S and D, in the formulas thus obtained, and there will finally result

$$\sin A+\sin B=2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) \quad\ldots\ldots\ldots\ldots (20)$$
$$\sin A-\sin B=2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \quad\ldots\ldots\ldots\ldots (21)$$
$$\cos B+\cos A=2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) \quad\ldots\ldots\ldots\ldots (22)$$
$$\cos B-\cos A=2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \quad\ldots\ldots\ldots\ldots (23)$$

26. Divide the second of these by the first, and divide the terms of the second member of the result successively by $2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$ and by $2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$; divide the fourth by the third, and divide the numerator and denominator of the second member of the result, first, by $2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$, and then by $2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$: divide, also, the first by the third, the fourth by the second, the fourth by the first, and the second by the third; then simplify the second members by rejecting the quantities common to the numerators and denominators, and by means of No. 11; and there will be obtained

$$\frac{\sin A-\sin B}{\sin A+\sin B}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{\cot\frac{1}{2}(A+B)}{\cot\frac{1}{2}(A-B)} \quad\ldots\ldots\ldots (24)$$
$$\frac{\cos B-\cos A}{\cos B+\cos A}=\frac{\tan\frac{1}{2}(A+B)}{\cot\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}(A-B)}{\cot\frac{1}{2}(A+B)} \quad\ldots\ldots\ldots (25)$$
$$\frac{\sin A+\sin B}{\cos B+\cos A}=\tan\tfrac{1}{2}(A+B) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$
$$\frac{\cos B-\cos A}{\sin A-\sin B}=\tan\tfrac{1}{2}(A+B) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27)$$
$$\frac{\cos B-\cos A}{\sin A+\sin B}=\tan\tfrac{1}{2}(A-B) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28)$$
$$\frac{\sin A-\sin B}{\cos B+\cos A}=\tan\tfrac{1}{2}(A-B) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29)$$

27. In (20) or (21), and in (22) and (23), let $B=0$; then (No. 10), $\sin B=0$, $\cos B=1$, and we shall have

$$\sin A=2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A \quad\ldots\ldots\ldots\ldots\ldots (30)$$
$$1+\cos A=2\cos^2\tfrac{1}{2}A \quad\ldots\ldots\ldots\ldots\ldots (31)$$
$$1-\cos A=2\sin^2\tfrac{1}{2}A \quad\ldots\ldots\ldots\ldots\ldots (32)$$

28. The following system of formulas is obtained by modifying the numerators of their first members by (30), and the denominators by (20) and (21), and by simplifying the results. Thus, by (30),

$$\sin(A+B)=2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A+B);$$

and by dividing the members of this by those of (20), and simplifying, we find the first of the following:

$$\frac{\sin(A+B)}{\sin A+\sin B}=\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)} \quad\dots\dots\dots\dots\dots\dots (33)$$

$$\frac{\sin(A-B}{\sin A-\sin B}=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)} \quad\dots\dots\dots\dots\dots\dots (34)$$

$$\frac{\sin(A+B)}{\sin A-\sin B}=\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)} \quad\dots\dots\dots\dots\dots\dots (35)$$

$$\frac{\sin(A-B)}{\sin A+\sin B}=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)} \quad\dots\dots\dots\dots\dots\dots (36)^*$$

29. If B be taken equal to A in (16), (17), (18), (19), and (14), the following expressions are obtained, the first and second giving the same value for $\sin 2A$, and the third, fourth, and fifth, giving three different expressions for $\cos 2A$:

$$\sin 2A = 2\sin A\cos A \quad\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (37)$$

$$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \cos^2 A - \sin^2 A \quad\dots\dots (38)$$

30. To investigate some of the most useful properties of the tangents of angles, in addition to those already given, divide (12) by (14), (13) by (15), (14) by (12), and (15) by (13); then divide the numerators and denominators of the second members of the first two equations thus obtained, by $\cos A \cos B$, and those of the last two by $\sin A \sin B$; and, since (No. 11) $\frac{\sin A}{\cos A}=\tan A$, and $\frac{\cos A}{\sin A}=\cot A$, there will result

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B} \quad\dots\dots\dots\dots\dots\dots (39)$$

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B} \quad\dots\dots\dots\dots\dots\dots (40)\dagger$$

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* The following system would be found by using as denominators the sum and difference of the cosines, and modifying them by (22) and (23):

$$\frac{\sin(A+B)}{\cos B+\cos A}=\frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)} \quad\dots\dots\dots\dots\dots\dots (1)$$

$$\frac{\sin(A-B)}{\cos B-\cos A}=\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)} \quad\dots\dots\dots\dots\dots\dots (2)$$

$$\frac{\sin(A+B)}{\cos B-\cos A}=\frac{\cos\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)} \quad\dots\dots\dots\dots\dots\dots (3)$$

$$\frac{\sin(A-B)}{\cos B+\cos A}=\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)} \quad\dots\dots\dots\dots\dots\dots (4)$$

† Formulas (40) and (42) are easily derived from (39) and (41), by changing B into $-B$.

$$\cot(A+B)=\frac{\cot A\cot B-1}{\cot B+\cot A} \quad \text{.................. (41)*}$$

$$\cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A} \quad \text{.................. (42)}$$

31. The tangent and cotangent of the sum of three angles, A, B, C, may be readily derived from the first and third of the preceding formulas, by regarding $A+B$ as a single angle. In this way we have, first,

$$\tan(A+B+C)=\frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C};$$

and then, by substituting for $\tan(A+B)$ its equal, according to (39), and multiplying the numerator and denominator of the result by $1-\tan A\tan B$, we obtain

$$\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan A\tan C-\tan B\tan C} \text{..(43)}$$

By a similar process we should find, that

$$\cot(A+B+C)=\frac{\cot A\cot B\cot C-\cot A-\cot B-\cot C}{\cot A\cot B+\cot A\cot C+\cot B\cot C-1} \text{..(44)}$$

It is easy to see, that formulas for the tangent and cotangent of the sum of four or more angles, might be derived in a similar manner.

32. Take $B=A$ in (39) and (41), and $B=C=A$ in (43) and (44); then

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A} \quad \text{........................ (45)}$$

$$\cot 2A=\frac{\cot^2 A-1}{2\cot A} \quad \text{........................ (46)}$$

$$\tan 3A=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A} \quad \text{............... (47)}$$

$$\cot 3A=\frac{\cot^3 A-3\cot A}{3\cot^2 A-1} \quad \text{................. (48)}$$

33. By dividing (32) by (30), we obtain

$$\frac{1-\cos A}{\sin A}=\tan\tfrac{1}{2}A \quad \text{......................... (49)}$$

Hence, also (No. 11),

$$\operatorname{cosec} A-\cot A=\tan\tfrac{1}{2}A \quad \text{.................. (50)}$$

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* This may be obtained by taking the reciprocals of the members of (39), and multiplying the numerator and denominator of the second member by $\cot A\cot B$. In a similar manner (42) might be derived from (40).

by multiplying which by cos A, there will be obtained, by (16),

$$\sin(54^0+A)+\sin(54^0-A)-\sin(18^0+A)-\sin(18^0-A)=\cos A \ldots (72)$$

We should also have a similar formula by multiplying by sin A, and applying (19).

48. Two formulas giving the cosine and sine of half an angle may be thus investigated: $(\cos A+\sin A)^2=\cos^2 A+2\cos A\sin A+\sin^2 A =1+\sin 2A$, by (37), and by No. 11; and therefore $\cos A+\sin A=\sqrt{(1+\sin 2A)}$. In like manner, we should find $\cos A-\sin A=\sqrt{(1-\sin 2A)}$; and by taking half the sum and half the difference of these, we obtain

$$\cos A=\tfrac{1}{2}\sqrt{(1+\sin 2A)}+\tfrac{1}{2}\sqrt{(1-\sin 2A)} \ \ldots\ldots\ldots (73)$$

$$\sin A=\tfrac{1}{2}\sqrt{(1+\sin 2A)}-\tfrac{1}{2}\sqrt{(1-\sin 2A)} \ \ldots\ldots\ldots (74)^*$$

49. Formulas might also be investigated for the secants and cosecants of the sum and difference of angles, &c.; and we might investigate many more respecting sines and tangents. As much has been done, however, as is consistent with the nature of the present publication; and the person who shall make himself well acquainted with the mode of investigating the formulas that have been here given, will find it easy to derive others. We shall now proceed, therefore, to the *resolution of plane triangles*, or to investigate rules and formulas for calculating certain sides or angles of plane triangles, when others are given.

II.—RESOLUTION OF PLANE TRIANGLES.

50. To assist in effecting calculations in trigonometry, tables have been constructed, containing the sines, tangents, and secants of all the angles, up to 90°, which differ by small equal intervals, usually of one minute; and, in trigonometrical computations, instead of the common numbers, the logarithms not only of the numbers expressing the lengths of the sides of figures, but also those of the sines, tangents, and secants of angles, are almost always employed, since the invention of those remarkable numbers. The theory of logarithms, and the method of computing tables † of them, and of sines, tangents,

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* Let the student consider for what angles the radicals, in this formula and the last, are to be taken positive, and for what negative.

† To employ those tables without being acquainted with their nature and

and secants, are given in the Treatise on the Differential and Integral Calculus, by the Author of this work. It may suffice here to say, that the *logarithms* of two or more numbers, are other numbers whose sum is the logarithm of the product of the numbers to which they belong. Hence it follows, that the difference of the logarithms of two numbers is the logarithm of the quotient obtained by dividing one of the numbers by the other; that the logarithm of the square of a number is double of the logarithm of that number; and the logarithm of its square root, half its logarithm.

51. It is easy to see (Euc. I. 4, 8, and 26) that, with the exception mentioned in No. 53, a plane triangle is determined, when, of its sides and angles, any three, except the three angles, are given. Hence we may divide the resolution of plane triangles into *three cases:*

I. When a side and the opposite angle, and either another side, or another angle, are given;

II. When two sides and the contained angle are given; and,

III. When the three sides are given.

52. The *first case* is resolved on the principle, that (No. 20) the sides are proportional to the sines of the opposite angles.

53. When, in this case, two unequal sides, and the angle opposite to the less, are given, the angle opposite to the greater may (No. 15) be either that which is found in the table of sines, or its supplement, unless it be known from the nature of the problem, whether it is acute or obtuse. This will readily appear from constructing the triangle ABC (*fig.* 11) having the angle B acute, and the side AC less than AB; as it will be seen, that if from A as centre an arc be described, with AC as radius, it will cut BD in two points C and C′, either of which may be taken as the extremity of AC, and the two angles ACB and AC′B are evidently supplements of each other, the triangle CAC′ being isosceles.* Since the sine of an angle and the sine of its supplement are equal, a similar ambiguity would always exist when the quantity to be found is a sine, were it not removed by the nature of the triangle, or by some other circumstance.

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construction, is not strictly scientific. The student, however, will learn their theory and construction with greater ease, when he shall have had more experience in mathematical investigations: and those who prefer the more scientific mode, may have recourse to the Treatise on the Differential and Integral Calculus. In all the books of tables, the method of using them is explained.

* Should the computation give the angle opposite to the greater side equal to 90°, there would be no ambiguity. In this case the arc CC′ would *touch* BD. Should the value of sin C be greater than the radius the solution would be

54. The method which is generally best adapted for resolving the *second case*, may be thus investigated. Let A, B, C be the angles of a triangle, and a, b, c* the sides respectively opposite to them. Then (No. 21) $a : b :: \sin A : \sin B$; whence †

$$\frac{a-b}{a+b}=\frac{\sin A-\sin B}{\sin A+\sin B}; \text{ or, by (24),}$$

$$\frac{a-b}{a+b}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} \quad \text{.......................} (75)$$

and therefore

$$a+b : a-b :: \tan\tfrac{1}{2}(A+B) : \tan\tfrac{1}{2}(A-B);$$

in which analogy the third term, as well as the first and second, is given, since (Euc. I. 32) $A+B=180^0-C$. Hence, therefore, the angles A and B will become known, half their sum and half their difference being known.

55. The angles being found, the third side may be calculated by No. 52. In practice, however, it is generally better to use one of the following analogies: ‡

$$\cos\tfrac{1}{2}(A-B) : \cos\tfrac{1}{2}(A+B) :: a+b : c \quad \text{..........} (76)$$

$$\sin\tfrac{1}{2}(A-B) : \sin\tfrac{1}{2}(A+B) :: a-b : c \quad \text{..........} (77)$$

---

impossible, the data being inconsistent with one another, and having no triangle answering to them, the arc CC′ neither cutting nor touching BD.

* To avoid ambiguity in the use of this very convenient notation, A, B, C may be read *angle* A, *angle* B, *angle* C; and a, b, c, *side* a, *side* b, *side* c.

† For $a+b : a-b :: \sin A+\sin B : \sin A-\sin B$, and consequently

$$\frac{a-b}{a+b}=\frac{\sin A-\sin B}{\sin A+\sin B}.$$

‡ These formulas are given in the 13th page of Thacker's Miscellany, published in 1743, and their use in resolving this case of plane trigonometry, was first pointed out by the late Professor Wallace of Edinburgh, in the Edinburgh Transactions for 1823.

The following geometrical proofs of formulas 75, 76, and 77, may perhaps be preferred by some to the proofs already given.

Let ABC (*fig.* 12) be any plane triangle having a greater than b; and from C as centre, with a as radius, describe a circle meeting CA produced in D and E, and BA produced in F; join DB, BE, and CF, and draw EG parallel to AB, and meeting DB produced in G. Then, because DC and CE are each equal to a, DA is equal to $a+b$, and AE to $a-b$. Also (Euc. I. 32), $DCB=A+B$, and (Euc. III. 20) $DEB=\frac{1}{2}(A+B)$. Again (Euc. I. 32), $A=ACF+F=ACF+B$ (Euc. I. 5), and consequently $ACF=A-B$; and (Euc. III. 20) ABE, or its equal, $BEG=\frac{1}{2}(A-B)$. Now, to EB, taken as unity, since (Euc. III. 31) DBE is a right angle, BD is the tangent of DEB, or $\tan\frac{1}{2}(A+B)$, and BG the tangent of BEG, or $\tan\frac{1}{2}(A-B)$; and (Euc. VI. 2) DA : AE :: DB : BG; that is, $a+b : a-b :: \tan\frac{1}{2}(A+B) : \tan\frac{1}{2}(A-B)$, which is the same as (75). Again, DBA is the complement of ABE, or $\frac{1}{2}(A-B)$; and D the complement of BED, or $\frac{1}{2}(A+B)$. But, in the triangles, ABD, ABE, These are easily proved in the following manner. Since (No. 21)

$$\frac{\sin A}{\sin C}=\frac{a}{c},\ \text{and}\ \frac{\sin B}{\sin C}=\frac{b}{c},$$

we have, by addition and subtraction, and by substituting (No. 15), sin (A+B) for sin C,

$$\frac{\sin A+\sin B}{\sin (A+B)}=\frac{a+b}{c},\ \text{and}\ \frac{\sin A-\sin B}{\sin (A+B)}=\frac{a-b}{c};$$

or, by (33) and (35),

$$\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}=\frac{a+b}{c},\ \text{and}\ \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}=\frac{a-b}{c}.$$

It may be remarked, that if the latter of the two formulas just found, be divided by the former, the quotient will be (75): and thus we have a second and an easy mode of investigating that formula.

56. In the *third case,* the three sides being given, to find one of the angles, suppose A (*fig.* 8); from C (either of the other angles) draw CD perpendicular to the opposite side; then (Euc. I. 47) $a^2=CD^2+DB^2$. But $CD^2=b^2-AD^2$, and $DB^2=(c-AD)^2=c^2-2c\times AD+AD^2=c^2-2bc\cos A+AD^2$ (No. 19); and substituting these in the foregoing equation, and contracting, we obtain

$$a^2=b^2+c^2-2bc\cos A \ldots\ldots\ldots\ldots\ldots\ldots (78)$$

Hence we have $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$;— a formula by means of which A may be determined arithmetically, though in general not easily, as it is not adapted to computation by logarithms. To find a more convenient formula, we get from (78), by transposition,

$$2bc\cos A=b^2+c^2-a^2.$$

Subtract the members of this from $2bc$, and also add them to it: then

$$2bc(1-\cos A)=a^2-(b^2-2bc+c^2)=a^2-(b-c)^2,\ \text{and}$$
$$2bc(1+\cos A)=b^2+2bc+c^2-a^2=(b+c)^2-a^2;$$

or, by (32) and (31), and because the difference of the squares of two quantities is equal to the product of their sum and difference,

$$4bc\sin^2\tfrac{1}{2}A=(a-b+c)(a+b-c),\ \text{and}$$
$$4bc\cos^2\tfrac{1}{2}A=(a+b+c)(b+c-a).$$

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we have (No. 21) sin DBA : sin D :: DA : AB, and sin ABE : sin AEB :: AE : AB; that is, $\cos\frac{1}{2}(A-B) : \cos\frac{1}{2}(A+B) :: a+b : c$, and $\sin\frac{1}{2}(A-B) : \sin\frac{1}{2}(A+B) :: a-b : c$; which are (76) and (77).

By putting $2s=a+b+c$, these become

$$4bc\sin^2\tfrac{1}{2}A=2(s-b).2(s-c), \text{ and}$$
$$4bc\cos^2\tfrac{1}{2}A=2s.2(s-a).*$$

Hence, by dividing by $4bc$, and extracting the square root, we get the two formulas,

$$\sin\tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{bc}} \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (79)$$

$$\cos\tfrac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}} \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (80)$$

57. Divide (79) by (80); then, by (3),

$$\tan\tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (81)\dagger$$

In like manner we should have

$$\tan\tfrac{1}{2}B=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \text{ and } \tan\tfrac{1}{2}C=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

Divide the members of these by those of (81); then

$$\frac{\tan\frac{1}{2}B}{\tan\frac{1}{2}A}=\frac{s-a}{s-b}, \text{ and } \frac{\tan\frac{1}{2}C}{\tan\frac{1}{2}A}=\frac{s-a}{s-c} \quad\ldots\ldots\ldots\ldots \quad (82)$$

58. By multiplying the values of $\tan\frac{1}{2}B$ and $\tan\frac{1}{2}C$ in the last No. by the value of $\tan\frac{1}{2}A$ in (81), we get

$$\tan\tfrac{1}{2}A\,\tan\tfrac{1}{2}B=\frac{s-c}{s}, \text{ and } \tan\tfrac{1}{2}A\,\tan\tfrac{1}{2}C=\frac{s-b}{s} \quad\ldots \quad (83)$$

59. Take twice the product of (79) and (80): then (30)

$$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}\ddagger \quad\ldots\ldots\ldots\ldots \quad (84)$$

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* For since $2s=a+b+c$, by subtracting $2b$ we obtain $2s-2b=a-b+c$; and similar remainders would be obtained by subtracting $2a$ and $2c$.

† Formulas 79, 80, and 81 were discovered by William Purser of Dublin, probably about the year 1632, or soon after. See Wallace's Geometrical Theorems, page 1.

‡ Multiply both members of this formula by b; then $b\sin A$, or, by No. 19 (*fig.* 8), the perpendicular $CD=\frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$. If this be multiplied by $\frac{1}{2}c$, we obtain for the area of the triangle, $\sqrt{s(s-a)(s-b)(s-c)}$, which proves the common but important rule for finding the area, when the three sides are given. If this be divided by s, half the perimeter, we find the radius of the inscribed circle $=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.

The third case may also be resolved by drawing the perpendicular CD (*fig.* 8), and thus forming the two rightangled triangles ACD, BCD: for (Euc. I. 47) $AC^2-AD^2=BC^2-BD^2$; whence, by transposition, we have $BD^2-AD^2=$

EXAMPLES OF THE RESOLUTION OF PLANE TRIANGLES.

60. Given $a=13$ yards, $b=15$ yards, and $A=53^0\ 8'$; to find the remaining parts of the triangle.

As a	13	1·11394
: b	15	1·17609
: : sin A	$53^0\ 8'$	9·90311
: sin B	$67^0\ 23'$, or $112^0\ 37'$;	9·96526

this being (No. 53) the doubtful case.

Hence (Euc. I. 32)
$C=59^0\ 29'$, or $14^0\ 15'$.

As sin A	$53^0\ 8'$	9·90311
: sin C	$59^0\ 29'$	9·93525
: : a	13	1·11394
: c	14	1·14608

By taking $C=14^0\ 15'$, and A and a the same as in the preceding analogy, we find $c=4$.*

In these operations, to find the fourth term (No. 50), the second and third terms are added together, and the first is taken from the sum. This may be done more easily by adding together the second and third terms, and the complement of the first to 10, or what remains after subtracting its right-hand figure from 10, and all the rest from 9, which may be done by inspection, as we proceed with the addition. Thus, in the second, we have 4 and 5 are 9, and 9 are 18; then 1 and 9 are 10, and 2 are 12, and 8 are 20, &c. When we use this method, we must reject 10 from the result. It is still easier, however, when the quantity to be subtracted is a sine, to use instead of it the cosecant diminished by 10 in its index, and then to add all the quantities together. The reason of this is evident from the second note to No. 11. In like manner, when a cosine is to be subtracted, we may add the secant diminished by 10; and when a tangent or cotangent is to be subtracted, we may add in the first case the cotangent, in the second, the tangent, subtracting 10, either at first, or from the final result.

The following method of resolving this case is perhaps preferable to

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BC^2-AC^2, or (Euc. II. 5, cor.) $(BD+AD)(BD-AD)=(BC+AC)(BC-AC)$. Now, when the perpendicular falls within the triangle, $BD+AD=AB$; otherwise (A being obtuse) $BD-AD=AB$: in either case, therefore, one of these factors is given, being equal to the base, and the other will be determined by dividing $(BC+AC)(BC-AC)$ by the one which is given, or by converting the equation (by Euc. VI. 16) into an analogy having the given base for its first term. Hence the segments of the base will be known, and then each of the rightangled triangles will be resolved by the first case.

* These results may be thus obtained by the use of natural sines and numbers; as $a : b :: \sin A : \sin B$; that is, as 13 : 15 : : ·80003 : ·92311 = sine of $67^0\ 23'$, or $112^0\ 37'$; whence C, as before, $=59^0\ 29'$, or $14^0\ 15'$. Using the first of these, we have $\sin A : \sin C :: a : c$; that is, ·80003 : ·86148 : : 13 : 14; while, by taking $C=14^0\ 15'$, we should find, in a similar manner, $c=4$.

that which is given above. From log sin A take log a, and the remainder, 9·90311—1·11394, is 8·78917; add this to log b=1·17609, and the sum 9·96526 is log sin B as before. Then, C being found, from the logarithmic sines of its values take successively the same quantity 8·78917, and the remainders are the logarithms of the two values of c. The reason of this is obvious, since, in the first analogy, the first and third terms are a and sin A; and in the second they are the same quantities in a reversed order.

61. Given a=57·38 miles; b=42·6 miles, and C=56° 45′; to resolve the triangle.

By (75).

As $a+b$	99·98	1·99991
: $a-b$	14·78	1·16967
: : $\tan\frac{1}{2}(A+B)$	61° 37′½	10·26750
: $\tan\frac{1}{2}(A-B)$	15° 18′½	9·43726

Hence, by adding and subtracting the half sum and half difference, we find A=76° 56′, and B=46° 19′.

Then, by (76).

As $\cos\frac{1}{2}(A-B)$	15° 18′½	9·98431
: $\cos\frac{1}{2}(A+B)$	61° 37′½	9·67691
: : $a+b$	99·98	1·99991
: c	49·26	1·69251

This might also be found by either of the following analogies, (77), and No. 52: $\sin\frac{1}{2}(A-B) : \sin\frac{1}{2}(A+B) :: a-b : c$, and $\sin A : \sin C :: a : c$.

If C be a right angle, the solution is effected more easily by means of No. 20, than by the foregoing method, as we have simply $a : b ::$ radius : tan B.

62. Given a=679, b=537, and c=429; to find the angles.

Here, by adding the three sides together, we obtain 1645, the half of which, 822·5, is s. Then, by taking from this the three sides successively, we find $s-a$=143·5, $s-b$=285·5, and $s-c$=393·5.* The rest of the work, the subtraction in the first part of which may be performed in the manner pointed out in No. 60, is as follows:

s	822·5	2·91514	subt.
$s-a$	143·5	2·15685	
$s-b$	285·5	2·45561	
$s-c$	393·5	2·59494	
		2)19·97856	
$\tan\frac{1}{2}A$	44° 17′½	9·98928	
A=88° 35′			

$\tan\frac{1}{2}A$		9·98928	add.
$\log(s-a)$		2·15685	
		12·14613	subt.
$\log(s-b)$		2·45561	
$\tan\frac{1}{2}B$	26° 7′½	9·69052	
B=52° 15′			
		12·14613	subt.
$\log(s-c)$		2·59494	
$\tan\frac{1}{2}C$	19° 35′	9·55119	
C=39° 10′.			

* As a check on this part of the operation it may be remarked, that the sum of the three remainders is equal to the half sum s.

Here the first part of the operation proceeds according to (81). In the second and third, according to (82), $\log(s-a)$ is added to $\log\tan\frac{1}{2}A$, both of which are found before; and from the sum the logarithms of $s-b$ and $s-c$ are taken successively to find $\log\tan\frac{1}{2}B$ and $\log\tan\frac{1}{2}C$. The sum of the angles thus found is exactly 180°: and thus, by calculating all the three angles, and taking their sum, we have always a certain means of determining the correctness of the operation.

63. By supplying the radius in (83) and dividing by $\tan\frac{1}{2}A$, we get

$$\tan\tfrac{1}{2}B=\frac{R^2(s-c)}{s\tan\frac{1}{2}A},\ \text{and}\ \tan\tfrac{1}{2}C=\frac{R^2(s-b)}{s\tan\frac{1}{2}A}.$$

These expressions afford an exceedingly easy method of finding $\tan\frac{1}{2}B$ and $\tan\frac{1}{2}C$, after $\tan\frac{1}{2}A$ has been determined; nothing more being necessary for the logarithmic solution than to add 20 successively to the logarithms of $s-c$ and $s-b$, and from the results to take the sum of the logarithm of s, and the logarithmic tangent of $\frac{1}{2}A$.

As an example, let $a=113{\cdot}3$, $b=618$, and $c=628{\cdot}3$. Here we have $s=679{\cdot}8$, $s-a=566{\cdot}5$, $s-b=61{\cdot}8$, and $s-c=51{\cdot}5$; and the logarithmic computation will stand as follows:

s	679·8	2·832381	subt.	$\tan\frac{1}{2}A$		8·958607	add
$s-a$	566·5	2·753200		$\log s$		2·832381	
$s-b$	61·8	1·790988					
$s-c$	51·5	1·711807				11·790988	subt.
		2)17·917214		$\log(s-c)+20$		21·711807	
$\tan\frac{1}{2}A$	$5^{\circ}\ 11'\frac{1}{2}$	8·958607		$\tan\frac{1}{2}B$	$39^{\circ}\ 48'\frac{1}{2}$	9·920819	
$A=10^{\circ}\ 23'$.				$B=79^{\circ}\ 37'$			
				$\log(s-b)+20$		21·790988	subt.
						11·790988	
				$\tan\frac{1}{2}C$	45°	10·000000	
				$C=90^{\circ}$.			

64. No easier or better solution for this case can be desired, or perhaps found, than is afforded by either of the foregoing methods; the taking of only four logarithms from the tables being necessary in the entire operation by either of the methods. It may be resolved, however, by means of No. 56 or 59, or of the note to No. 59; and the learner, for the sake of comparison and of practice, would find it useful to resolve a triangle in all the ways here pointed out. The method in No. 59, besides being tedious, fails in determining whether the angle is acute or obtuse, and is therefore useless in practice. No. 56 affords simple and easy solutions.

65. Rightangled triangles are resolved by the first case, except when

the legs are given; and then the resolution is most easily effected by the method pointed out in No. 20. These are more easily resolved than oblique-angled triangles, as the radius may always be one of the terms.

66. As applications of plane trigonometry, we may consider some of the simplest and most useful cases of the determination of heights and distances. The height of an *accessible* object AB (*fig.* 13), such as a tree, a spire, &c. may be found by assuming a station C on the same horizontal plane with the base B, and measuring with a line, chain, &c. the distance BC; and with a quadrant, theodolite, &c. the angle ACB, called *the angle of elevation.* There will then be given the right angle B, the angle C, and the base BC, to find the height AB; and that will be computed by means of either of the following analogies; cos C : sin C : : BC : AB, or radius : tan C : : BC : AB.

67. The data necessary for determining the height of an *inaccessible* object AB (*fig.* 14), may be found by measuring the distance between two objects C and D, which are on the same horizontal plane, and in the same straight line with the base of the object; and by measuring the angles of elevation, ACB, ADB. Then, as sin CAD (=ACB—ADB, by Euc. I. 32) : sin ADB : : CD : AC; and radius : sin ACB : : AC : AB, the height required. In the computation, the logarithm of AC will be found by the first analogy, and may be used in the second without finding AC itself.* By combining the two analogies, we have $\log AB = \log CD + \log \sin ACB + \log \sin ADB - \log \sin(ACB - ADB) - 10$. When the surface on which the measurement is made is not horizontal, its inclination must be measured, and must be employed in the calculation; and in all cases, when the angle of elevation of the summit of the object above the horizontal plane, passing through the eye, is observed, the height of the eye must be added to the final result.

68. The following is one of the most useful cases of the measurement of distances. Let A and B (*fig.* 15) be two objects whose distance is to be found, and let the base CD, in the same plane, be measured. Measure also the angles ACB, BCD, BDA, and ADC.

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* It follows from No. 19, that $BC = AB \cot C$, and $BD = AB \cot D$. Taking the difference of these we obtain $CD = AB (\cot D - \cot C)$, and consequently

$$AB = \frac{CD}{\cot D - \cot C};$$

a formula which gives an easy method of computing AB by means of the table of natural tangents.

Then (by case I.) in the triangle ACD, calculate AD; and in the triangle BCD, calculate BD: lastly, in the triangle ADB (case II.), calculate AB. The operation may be verified by computing AC and BC, and thence AB. The other common cases of the measurement of distances will be found in the following exercises, and will present no difficulty.

EXERCISES IN PLANE TRIGONOMETRY.

1. Given $A=90°$, $B=37° 52'$, $a=170{\cdot}6$; required b and c. *Answ.* $b=104{\cdot}72$, $c=134{\cdot}68$.
2. $A=90°$, $B=49° 18'$, $c=789$. *Answ.* $a=1210$, $b=917{\cdot}3$.
3. $A=90°$, $a=157{\cdot}8$, $b=100$. *Answ.* $B=39° 19'\tfrac{1}{2}$, $c=122{\cdot}07$.
4. $A=90°$, $b=784{\cdot}3$, $c=940$. *Answ.* $B=39° 51'$, $a=1224{\cdot}2$.
5. $A=68° 23'$, $B=62° 40'$, $a=5000$. *Answ.* $b=4777{\cdot}8$, $c=4055{\cdot}9$.
6. $A=45°$, $a=64{\cdot}3$, $b=57$. *Answ.* $B=38° 49'$, $c=90{\cdot}4$.
7. $A=45°$, $a=57$, $b=64{\cdot}3$. *Answ.* $B=52° 54'\tfrac{1}{2}$, $c=79{\cdot}844$; or $B=127° 5'\tfrac{1}{2}$, $c=11{\cdot}091$.
8. $A=17° 18'$, $b=1376$, $c=149$. *Answ.* $B=160° 38'\tfrac{1}{2}$, $a=1234{\cdot}1$.
9. $a=384$, $b=512$, $c=201$. *Answ.* $A=41° 6'$, $B=118° 46'$.

Ex. 10. Required the breadth AB (*fig.* 8) of a lake, the distance from A to a station C being 24·36 perches, and the angles A and C being 91° 32′ and 69° 18′ respectively. *Answ.* AB=69·408 perches.

Ex. 11. Required the distance between two houses A and B (*fig.* 8), on the opposite sides of a hill; the distance from A to a point C, from which both are visible, being 168 perches, from B to C 212 perches, and the angle ACB= 34° 48′. *Answ.* 121·14 perches.

Ex. 12. Suppose a base AB (*fig.* 15) of 24·36 chains to be measured for ascertaining the distance between two houses C and D beyond a river, and suppose the angles CAD, DAB, DBC, and CBA to be 64° 38′, 51° 12′, 70° 44′, and 49° 50′, respectively: required the distance CD. *Answ.* 132·93 chains.

III.—THEORY OF SPHERICAL TRIGONOMETRY.

69. A *spherical triangle* is a part of the surface of a sphere bounded by arcs of three *great circles;* that is, of three circles whose planes pass through the centre of the sphere. Those arcs are the *sides* of the triangle; and any of its *angles* is the same as the inclination of the planes of the sides which contain that angle.

In what follows, unless the contrary is specified, a spherical triangle will be understood as being the *smaller* of the two parts into which the surface of the sphere is divided by the three *smaller* arcs joining, by pairs, three points on the surface, and which are not on the same great circle.* In such a triangle, therefore, each side is less than a semicircle.

70. To investigate the fundamental formula in spherical trigonometry, let ABC† (*fig.* 16) be a spherical triangle, and S the centre of the sphere; and let the sides opposite to the several angles A, B, C, be denoted by the corresponding small letters, *a*, *b*, *c*. In the planes, ASB, ASC, draw AD, AE, each perpendicular to AS, and

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* If through two points on the surface of a sphere, which are not diametrically opposite, a great circle be described, the points may be regarded as being *joined* by either the less or the greater of the two arcs into which the circle is divided at the points; and, hence, we have one reason for the limitation in the text. Another reason is, that while this limitation simplifies the theory, it excludes no arc or angle which it is ever necessary to consider in the practical application of spherical trigonometry. Besides this, if the triangle mentioned in the text be determined, everything that is excluded by the limitation flows from that triangle, without any new investigation.—See the scholium at the end of this section.

The following remarks and illustrations will assist the student in understanding the theory of spherical trigonometry.

1. If from any point in the line which is the common section of two planes, and which (Euc. XI. 3) is a straight line, two perpendiculars to that line be drawn, one in each plane, the angle of inclination of the planes is the same as the angle contained by these perpendiculars. Hence it is obvious, that a spherical angle is the same as the inclination of the tangents of the containing sides. It may also be remarked, that the planes of the sides form at the centre of the sphere a triedral solid angle, the inclinations of whose planes are the same as the angles of the triangle; and the plane angles made by the radii, or common sections of the planes, are measured by the sides of the triangle.

2. Every section of a sphere by a plane is a circle. For, if the plane pass through the centre,—since, by the definition of the sphere, all its radii are equal,—the section is obviously a circle. But if it do not pass through the centre, a perpendicular to it from the centre will cut it in a point equally distant (Euc. I. 47) from all points of the boundary of the section: and that boundary is therefore the circumference of a circle. A circle whose centre is not the centre of the sphere is called a *small* or a *less circle*.

3. Since the planes of all great circles pass through the centre, the common section of any two is a diameter; and hence all great circles bisect one another.

4. If a straight line be drawn perpendicular to any circle of the sphere through its centre, it cuts the surface in two points called the *poles* of that circle. Hence (Euc. I. 47) either pole of a circle is equally distant from all points of its circumference; also, if great circles be drawn through the poles, the arcs of them between the circle and either pole are (Euc. III. 28) all equal; and, in case of a great circle, each of these is the arc of a quadrant.

† The learner, to assist his conception, may readily make a figure of a convenient form in the following manner: On a piece of pasteboard, describe, with a radius of 2 or 3 inches, an arc of about 190°, marking the centre with S, and each extremity with A, and draw tangents at the extremities; then divide the arc into three portions, AB, BC, and CA, of about 54°, 72°, and 64°, respectively; draw the radii SB and SC, and produce them to meet the tangents already drawn in D and E, and join DE. Then, let the pasteboard be cut half through in the lines SD and SE, and the parts ASD, ASE be turned up on the opposite side, till the two radii AS coincide, and the diagram will be finished; S being the centre of the sphere, and ABC the triangle. The sides here employed are chosen conformably to the note to No. 13.

let them meet SB, SC produced in D and E. Now, if r be put to denote the radius of the sphere, we have (No. 1) the following expressions for the angles at S:

$$BSC=\frac{a}{r},\quad ASC=\frac{b}{r},\quad \text{and}\quad ASB=\frac{c}{r}.$$

In these, we may evidently take $r=1$; that is, we may take the radius of the sphere, as the unit to which all the other magnitudes, the arcs, sines, tangents, &c. shall be referred. On this supposition, we shall have $BSC=a$, $ASC=b$, and $ASB=c$. Hence, also, since $r=1$, we shall have (No. 7) $AD=\tan c$, $AE=\tan b$, $SD=\sec c$, and $SE=\sec b$. But (78) in the triangle DAE,

$$DE^2=EA^2+AD^2-2\,EA.AD\cos DAE, \text{ or}$$
$$DE^2=\tan^2 b+\tan^2 c-2\tan b\tan c\cos A.$$

In a similar manner we obtain from the triangle DSE,

$$DE^2=\sec^2 b+\sec^2 c-2\sec b\sec c\cos a;\text{ or (7)}$$
$$DE^2=2+\tan^2 b+\tan^2 c-2\sec b\sec c\cos a.$$

By equalling these values of DE^2, rejecting the common quantities, transposing, and dividing by 2, we find

$$\sec b\sec c\cos a=\tan b\tan c\cos A+1;\text{ or (1) and (3)}$$
$$\frac{\cos a}{\cos b\cos c}=\frac{\sin b\sin c\cos A}{\cos b\cos c}+1;$$

whence, by multiplying by $\cos b\cos c$, we obtain

$$\cos a=\sin b\sin c\cos A+\cos b\cos c.$$

Hence it appears, that *the cosine of one side* (and it may obviously be *any side*) *of a spherical triangle, is equal to the continual product of the cosine of the opposite angle and the sines of the other sides, together with the product of the cosines of those sides.* We have, therefore, the three following formulas, which exhibit the relation between the three sides and any of the angles, and from which all others relative to spherical triangles may be derived:

$$\cos a=\cos A\sin b\sin c+\cos b\cos c\text{*} \quad\ldots\ldots\ldots\ldots\ldots (85)$$
$$\cos b=\cos B\sin a\sin c+\cos a\cos c \quad\ldots\ldots\ldots\ldots\ldots\ldots (86)$$
$$\cos c=\cos C\sin a\sin b+\cos a\cos b \quad\ldots\ldots\ldots\ldots\ldots\ldots (87)$$

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* 1. From any of these formulas we may infer, that, with the limitation in No. 69, *one side is less than the sum, and greater than the difference, of the other two.* Thus, (14) $\cos(b+c)=\cos b\cos c-\sin b\sin c$, which is always less than $\cos b\cos c+\cos A\sin b\sin c$, the value of $\cos a$, since $\cos A$ cannot be -1, every angle being less than two right angles. But the less arc has the greater cosine, if the arc be less than a semicircle: therefore $b+c>a$. From each of these take b, then $c>a-b$.

71. From the first of these, by transposing $\cos b \cos c$, and dividing by $\sin b \sin c$, we obtain

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c};$$

a formula, which, by means of natural sines, would enable us to find an angle, when the three sides are given.

72. To obtain formulas fitted for logarithmic computation, we get from (85), by transposition,

$$\cos A \sin b \sin c = \cos a - \cos b \cos c.$$

Subtract the members of this from $\sin b \sin c$, and likewise add them to it: then

$$(1 - \cos A) \sin b \sin c = \cos b \cos c + \sin b \sin c - \cos a, \text{ and}$$
$$(1 + \cos A) \sin b \sin c = \cos a - (\cos b \cos c - \sin b \sin c);$$

or, by (32) and (31), and by (15) and (14),

$$2 \sin^2 \tfrac{1}{2} A \sin b \sin c = \cos (b - c) - \cos a, \text{ and}$$
$$2 \cos^2 \tfrac{1}{2} A \sin b \sin c = \cos a - \cos (b + c).$$

By modifying the second members of these by (23), and halving the results, we obtain

$$\sin^2 \tfrac{1}{2} A \sin b \sin c = \sin \tfrac{1}{2} (a - b + c) \sin \tfrac{1}{2} (a + b - c), \text{ and}$$
$$\cos^2 \tfrac{1}{2} A \sin b \sin c = \sin \tfrac{1}{2} (a + b + c) \sin \tfrac{1}{2} (b + c - a).$$

In these, put $a + b + c = 2s$, as in No. 56: then, by dividing by $\sin b \sin c$, and extracting the square root, we get

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}} \text{ .... (88)}$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}} \text{ .... (89)}$$

73. Divide (88) by (89), and there will result, by (3),

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2. Hence, we may prove that *the perimeter is less than* 360°, *or* 2π. For let b and c (*fig.* 17) be produced to meet in D; then $a < DB + DC$; add b and c; then $a + b + c < ABD + ACD$: but ABD, ACD, are (No. 69, note 3) each equal to 180°; therefore $a + b + c < 360°$.

3. If a and b be each $= \frac{1}{2}\pi$, and consequently $\sin a = \sin b = 1$, and $\cos a = \cos b = 0$, we shall have, from (85), (86), and (87), $A = B = 90°$, and $c = C$. Hence it appears that *all great circles passing through the pole of a great circle, are perpendicular to that circle;* and *the arc of a great circle intercepted between two great circles passing through its pole, is the measure of the angle contained by those circles.*

4. From formulas (85) and (86), or those derived from them, it is easy to show, that $B = A$, when $b = a$, or *that the angles at the base of an isosceles triangle are equal;* as, by taking a and b equal, the values of $\cos A$ and $\cos B$ would become equal.

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}} \text{ .... } (90)$$

In a similar manner we should find

$$\tan \tfrac{1}{2} B = \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s \sin(s-b)}}, \text{ and } \tan \tfrac{1}{2} C = \sqrt{\frac{\sin(s-a)(s-b}{\sin s \sin(s-c)}}.$$

Dividing these by (90), we get

$$\frac{\tan \frac{1}{2} B}{\tan \frac{1}{2} A} = \frac{\sin(s-a)^*}{\sin(s-b)}, \text{ and } \frac{\tan \frac{1}{2} C}{\tan \frac{1}{2} A} = \frac{\sin(s-a)}{\sin(s-c)} \text{ ...... } (91)$$

74. By multiplying the values of $\tan \frac{1}{2} B$ and $\tan \frac{1}{2} C$ in the last No. by the value of $\tan \frac{1}{2} A$, we get

$$\tan \tfrac{1}{2} A \tan \tfrac{1}{2} B = \frac{\sin(s-c)}{\sin s}, \text{ and } \tan \tfrac{1}{2} A \tan \tfrac{1}{2} C = \frac{\sin(s-b)}{\sin s} \text{ ... } (92)$$

75. Lastly, by taking twice the product of (88) and (89), we obtain by (30), and by some slight reductions,

$$\sin A = \frac{2\sqrt{\sin s \sin(s-a)\sin(s-b)\sin(s-c)}}{\sin b \sin c} \text{ ...... } (93)$$

76. Dividing (93) by $\sin a$, we obtain

$$\frac{\sin A}{\sin a} = \frac{2\sqrt{\sin s \sin(s-a)\sin(s-b)\sin(s-c)}}{\sin a \sin b \sin c}.$$

Now the second member of this equation is symmetrical in respect to a, b, and c, since they all enter into it in exactly the same manner. It is therefore equivalent also to $\frac{\sin B}{\sin b}$, or $\frac{\sin C}{\sin c}$†; and hence,

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \text{ .... } (94)$$

Hence, $\sin a : \sin A :: \sin b : \sin B :: \sin c : \sin C$;
that is, *the sines of the sides of a spherical triangle are proportional to the sines of the opposite angles.* Hence, also, by multiplying extremes and means, we get

$$\sin A \sin b = \sin B \sin a$$
$$\sin A \sin c = \sin C \sin a$$
$$\sin B \sin c = \sin C \sin b$$

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* In this, if $a > b$, the denominator is evidently greater than the numerator, and since in the first quadrant the greater angle has the greater tangent, it follows that $\frac{1}{2} A$ is $> \frac{1}{2} B$, and consequently that A is also $>$ B. It appears, therefore, that *the greater side has the greater angle opposite to it.*

† This would also appear by finding, as in (No. 75), expressions for $\sin B$ and $\sin C$, and dividing the first by $\sin b$, and the second by $\sin c$. The formulas marked (94) are sometimes called *the Formulas of the Four Sines.*

77. From (85), (86), (87), other important formulas may be obtained by elimination. Thus, to eliminate $\sin c$ and $\cos c$ from (85) and (87), multiply the latter by $\cos b$, and substitute the second member of the result in the former: then, by transposing $\cos a \cos^2 b$, the first member becomes $\cos a - \cos a \cos^2 b$, or (6) $\cos a \sin^2 b$; and, after dividing by $\sin b$, there is obtained

$$\cos a \sin b = \cos A \sin c + \cos C \sin a \cos b.$$

Divide all the terms of this by $\sin a$, and for $\frac{\sin c}{\sin a}$ substitute (94) $\frac{\sin C}{\sin A}$; then (4)

$$\cot a \sin b = \cot A \sin C + \cos C \cos b.$$

Hence, if we call a the *first* side, and b the *second*, it appears that, *if the cotangent of the first side be multiplied into the sine of the second, the product is equal to the cotangent of the angle opposite to the first into the sine of the contained angle, together with the cosine of the contained angle into the cosine of the second side.* Then, taking successively, as first side and second, a and c, b and a, b and c, c and a, and c and b, and employing this theorem, we complete the following system:

$$\cot a \sin b = \cot A \sin C + \cos C \cos b \quad \ldots\ldots\ldots\ldots (95)$$
$$\cot a \sin c = \cot A \sin B + \cos B \cos c \quad \ldots\ldots\ldots\ldots (96)$$
$$\cot b \sin a = \cot B \sin C + \cos C \cos a \quad \ldots\ldots\ldots\ldots (97)$$
$$\cot b \sin c = \cot B \sin A + \cos A \cos c \quad \ldots\ldots\ldots\ldots (98)$$
$$\cot c \sin a = \cot C \sin B + \cos B \cos a \quad \ldots\ldots\ldots\ldots (99)$$
$$\cot c \sin b = \cot C \sin A + \cos A \cos b \quad \ldots\ldots\ldots\ldots (100)$$

These formulas exhibit the relations between a side and the opposite angle, and another side and angle not opposite to one another; and they solve the problem in which two sides and the contained angle are given to find the other angles, and that in which a side and the adjacent angles are given to find the other sides, the required part in each case being found by means of its cotangent; but they are not fitted for computation by logarithms.

78. By farther elimination, other formulas may be derived from the foregoing. Thus, to eliminate b from (95) and (97), multiply the former by $\sin a$, and the latter by $\sin b \cos C$, modifying the products by (4): then

$$\cos a \sin b = \sin a \cot A \sin C + \sin a \cos b \cos C, \text{ and}$$
$$\sin a \cos b \cos C = \sin b \cot B \sin C \cos C + \cos a \sin b \cos^2 C.$$

In the former of these, substitute for $\sin a \cos b \cos C$, its equal in the latter; transpose the last term of the resulting equation; and then, by substituting in the first member $\sin^2 C$ for $1-\cos^2 C$, and by dividing by $\sin C$, we obtain

$$\cos a \sin b \sin C = \cot A \sin a + \cot B \cos C \sin b.$$

In this the term $\cot A \sin a$ is (4) equivalent to $\frac{\cos A \sin a}{\sin A}$, and (94) this is equivalent to $\frac{\cos A \sin b}{\sin B}$. Substituting this in the last equation, dividing the result by $\sin b$, and multiplying the quotient by $\sin B$, we get

$\cos a \sin B \sin C = \cos A + \cos B \cos C$; or, by transposition,
$\cos A = \cos a \sin B \sin C - \cos B \cos C$.

Hence it appears, that *the cosine of an angle is equal to the continual product of the cosine of the opposite side and the sines of the other angles, wanting the product of the cosines of those angles.* We have, therefore, the following formulas, which exhibit the relations between the three angles and each of the sides:

$$\cos A = \cos a \sin B \sin C - \cos B \cos C \quad \ldots\ldots\ldots\ldots\ldots\ldots (101)$$
$$\cos B = \cos b \sin A \sin C - \cos A \cos C \quad \ldots\ldots\ldots\ldots\ldots\ldots (102)$$
$$\cos C = \cos c \sin A \sin B - \cos A \cos B \quad \ldots\ldots\ldots\ldots\ldots\ldots (103)$$

79. On comparing these with formulas (85), (86), and (87), we observe a close resemblance, sides being merely changed for the opposite angles, and angles for the opposite sides, and one of the terms having the contrary sign. These latter equations, indeed, are the same as would be obtained by substituting in the others, $\pi - A$, $\pi - B$, $\pi - C$, for a, b, c; and consequently $\pi - a$, $\pi - b$, $\pi - c$, for A, B, C. Thus, (85) would become, by this substitution, $\cos(\pi - A) = \cos(\pi - a)\sin(\pi - B)\sin(\pi - C) + \cos(\pi - B)\cos(\pi - C)$; or, (No. 15) $-\cos A = -\cos a \sin B \sin C - \cos B \times (-\cos C)$; which, by having the signs of its terms changed, will become the same as (101). Hence, therefore, since equations (85), (86), (87), may be regarded as containing all the properties of spherical triangles, it is evident that a formula expressing any relation of the sides and angles being established, a corresponding one will be obtained by writing sides for angles, and angles for sides, and prefixing the sign *minus* to all the cosines; or, which is the same, by applying the formula to a triangle which has for sides the supplements of the arcs which mea-

sure the angles of the proposed triangle, and for angles those which are measured by the supplements of the sides of the proposed one. This triangle is called the *supplementary* or *polar triangle.**

80. Equations (101), (102), (103), enable us to find, by natural sines and cosines, the sides of a triangle, when the angles are given. Thus, from the first of them we derive $\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}$.

81. Formulas adapted to computation by logarithms may be derived from the same system by processes similar to those employed in Nos. 72, 73, 74, and 75. Thus, we get from (101), by transposition,

$$\cos a \sin B \sin C = \cos A + \cos B \cos C.$$

Subtract the members of this from sin B sin C, and likewise add them to it: then

$$(1 - \cos a) \sin B \sin C = - \cos A - (\cos B \cos C - \sin B \sin C), \text{ and}$$
$$(1 + \cos a) \sin B \sin C = \cos A + \cos B \cos C + \sin B \sin C;$$

or, by (32) and (31), and by (14) and (15),

$$2 \sin^2 \tfrac{1}{2} a \sin B \sin C = - \cos A - \cos (B + C), \text{ and}$$
$$2 \cos^2 \tfrac{1}{2} a \sin B \sin C = \cos A + \cos (B - C).$$

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* Let ABC (*fig.* 18) be a spherical triangle, and from A, B, and C, as poles, let arcs of great circles be described intersecting each other in A′, B′, and C′; A′B′C′ is the polar triangle. For, A being the pole of B′C′, the arcs AB′, AD, AE, &c. are each equal to 90°. For the like reason, CB′, CF, &c. are each equal to 90°; and therefore, since B′A, B′C, are each equal to 90°, B′ is the pole of the arc GACE. Hence, B′E=90°; and it would be shown in a similar manner, that C′D=90°. Now, DE+B′D=B′E=90°, and DE+EC′=DC′=90°; whence, by addition, DE+B′D+DE+EC′, or DE+B′C′=180°. But (page 28, note 3) DE=A; therefore B′C′=π—A: and in a similar manner it might be shown that A′C′=π—B, and A′B′=π—C; and also that BC=π—A′, AC=π—B′, and AB=π—C′.

Any great circle divides the surface of the sphere into two equal parts; another intersecting the first, divides the surface into four *lunes*, the opposite ones of which are equal; and a third great circle intersecting both the former, not in the points in which they intersect each other, divides the surface into eight triangles, the four of which on the one hemisphere are respectively equal to the four similarly situated on the other. Hence, if the circles of which A′B′, B′C′, and A′C′ are arcs, were completed, they would form eight triangles. Of these, however, only A′B′C′, and the corresponding triangle on the other hemisphere, have the property mentioned in the text; each of the others having two sides the *same* as two angles of ABC, and two angles the *same* as two of its sides; while only a side and the opposite angle of the one are *supplementary* to an angle and the opposite side of the other.

The easiest mode of employing the principle above explained, which reduces the cases of spherical triangles to half their number, is, not to consider the polar triangle, but to use sides for opposite angles, and angles for opposite sides, and to prefix the sign *minus* to the cosines; or, when the halves of angles or sides occur, to use π—A for a, π—a for A, &c.

By modifying the second members of these by (22), and halving the results we obtain

$$\sin^2\tfrac{1}{2}a \sin B \sin C = -\cos\tfrac{1}{2}(A+B+C)\cos\tfrac{1}{2}(B+C-A), \text{ and}$$
$$\cos^2\tfrac{1}{2}a \sin B \sin C = \cos\tfrac{1}{2}(A-B+C)\cos\tfrac{1}{2}(A+B-C).$$

Hence, by putting $A+B+C=2S$, as in No. 56; by dividing by $\sin B \sin C$, and extracting the square root, we get

$$\sin\tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos(S-A)^*}{\sin B \sin C}} \quad\ldots\ldots\ldots\ldots (104)$$

$$\cos\tfrac{1}{2}a = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B \sin C}} \quad\ldots\ldots\ldots\ldots (105)$$

82. By dividing (104) by (105), we find

$$\tan\tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos(S-A)}{\cos(S-B)\cos(S-C)}} \quad\ldots\ldots\ldots\ldots (106)$$

Also, by taking twice their product, we obtain

$$\sin a = \frac{2\sqrt{-\cos S \cos(S-A)\cos(S-B)\cos(S-C)}\dagger}{\sin B \sin C} . (107)$$

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* Since (No. 13) $\sin B$ and $\sin C$ are positive, the numerator of the second member of this formula must also be positive, as otherwise the value of $\sin\frac{1}{2}a$ would be imaginary. Now, the numerator will be positive only when $\cos S$ and $\cos(S-A)$ have contrary signs. Of the quantities S and S—A, therefore, one, namely S—A, the less, must (No. 13) be less than 90°; while S must be between 90° and 270°. By doubling these, we find that 2 S, or *the sum of the three angles is greater than two right angles, and less than six;* and that 2 S—2 A, or B+C—A, that is, *the excess of the sum of two angles above the third is less than two right angles.* The remainder, S—A, or $\frac{1}{2}(B+C-A)$, may be negative, since (No. 14) its cosine would still be positive; whence it appears, that *one angle may be greater than the sum of the other two,* which is also the case in plane triangles. Some of these conclusions might be easily derived from considering the polar triangle in connexion with the notes to No. 70.

† The last four formulas may be derived from (89), (88), (90), and (93), by means of the supplementary triangle. Thus, by the substitution of $\pi-a$ for A, $\pi-A$ for a, &c. (89) will become

$$\cos\tfrac{1}{2}(\pi-a) = \sqrt{\frac{\sin\frac{1}{2}\{3\pi-(A+B+C)\}\sin\frac{1}{2}\{\pi-(B+C-A)\}}{\sin(\pi-B)\sin(\pi-C)}};$$

which, by contraction, will become the same as (104).

By finding expressions, by (106), for $\tan\frac{1}{2}b$ and $\tan\frac{1}{2}c$, and by first dividing them, and then multiplying them, separately, by (106), we should get

$$\frac{\tan\frac{1}{2}b}{\tan\frac{1}{2}a} = \frac{\cos(S-B)}{\cos(S-A)} \quad\ldots\ldots\ldots\ldots (a)$$

$$\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}a} = \frac{\cos(S-C)}{\cos(S-A)} \quad\ldots\ldots\ldots\ldots (b)$$

$$\tan\tfrac{1}{2}a \tan\tfrac{1}{2}b = \frac{-\cos S}{\cos(S-C)} \quad\ldots\ldots\ldots\ldots (c)$$

83. We may now proceed to investigate four remarkable expressions, known by the name of *Napier's analogies*, from their having been discovered by Baron Napier, the inventor of logarithms. To effect this, take the members of (101) from those of (102), and there will remain

$$\cos B - \cos A = \sin C (\cos b \sin A - \cos a \sin B) + \cos C (\cos B - \cos A);$$

or, by transposing the last term,

$$(1 - \cos C)(\cos B - \cos A) = \sin C (\cos b \sin A - \cos a \sin B).$$

By modifying this by (32) and (30), and dividing by $2\sin^2\frac{1}{2}C$, we obtain

$$\cos B - \cos A = \cot\tfrac{1}{2}C\ (\cos b \sin A - \cos a \sin B).$$

Dividing this successively by $\sin A - \sin B$ and $\sin A + \sin B$, we get, by (27) and (28),

$$\tan\tfrac{1}{2}(A+B) = \cot\tfrac{1}{2}C.\frac{\cos b \sin A - \cos a \sin B}{\sin A - \sin B}, \text{ and}$$

$$\tan\tfrac{1}{2}(A-B) = \cot\tfrac{1}{2}C.\frac{\cos b \sin A - \cos a \sin B}{\sin A + \sin B}.$$

Multiply the numerators and denominators by $\sin c$; in the results, in both the numerators and denominators, for $\sin A \sin c$ and $\sin B \sin c$, substitute (No. 76) $\sin a \sin C$ and $\sin b \sin C$; and divide the numerators and denominators by $\sin C$: then

$$\tan\tfrac{1}{2}(A+B) = \cot\tfrac{1}{2}C.\frac{\sin a \cos b - \cos a \sin b}{\sin a - \sin b}, \text{ and}$$

$$\tan\tfrac{1}{2}(A-B) = \cot\tfrac{1}{2}C.\frac{\sin a \cos b - \cos a \sin b}{\sin a + \sin b}.$$

Modify the numerators by (13), and to the results apply (34) and (36): then

$$\tan\tfrac{1}{2}(A+B) = \cot\tfrac{1}{2}C.\frac{\cos\frac{1}{2}(a-b)^*}{\cos\frac{1}{2}(a+b)} \quad \ldots\ldots\ldots\ldots\ (108)$$

$$\tan\tfrac{1}{2}(A-B) = \cot\tfrac{1}{2}C.\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)} \quad \ldots\ldots\ldots\ldots\ (109)$$

$$\tan\tfrac{1}{2}a \tan\tfrac{1}{2}c = \frac{-\cos S}{\cos(S-B)} \quad \ldots\ldots\ldots\ldots\ldots\ (d)$$

These formulas correspond to (91) and (92), but they are of little practical importance.

* From this formula we may infer, that *half the sum of two sides of a spherical triangle, and half the sum of the opposite angles, are of the same* species, *that is, are either each less or each greater than 90°.* For (No. 69) any side of

By taking the difference of the members of (86) and (85), and by a process similar to the foregoing,* we should obtain

$$\tan\tfrac{1}{2}(a+b)=\tan\tfrac{1}{2}c.\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)} \text{ .... } (110)$$

$$\tan\tfrac{1}{2}(a-b)=\tan\tfrac{1}{2}c.\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)} \text{ .... } (111)\dagger$$

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a triangle, and consequently the difference of any two sides, being less than a semicircle, and any angle being less than two right angles, $\cos\frac{1}{2}(a-b)$ and $\cot\frac{1}{2}C$ must (No. 13) both be positive: and from this it follows that $\tan\frac{1}{2}(A+B)$ and $\cos\frac{1}{2}(a+b)$ must have the same sign, which can take place only when $\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ are both greater or both less than 90°. It is also evident, that *if the sum of two sides a and b be 180°, the sum of the opposite angles is the same.* For, in that case, $\cos\frac{1}{2}(a+b)=\cos 90°=0$, which renders the second member, and consequently the first infinite; so that (No. 10) the first member is the tangent of 90°.

* The only difference is, that sides and angles are mutually interchanged; and that, after the transposition, there is $1+\cos c$, instead of $1-\cos C$.

† The foregoing investigation, which the author believes to be new, is very direct and simple, employing only the common elementary formulas. The following, which is taken in substance from a work on trigonometry by Mr. Luby of Dublin, possesses much elegance, but it is more tedious from the number and variety of the reductions and preparatory processes which it involves.

By (88) and (89) we have

$$\text{I. } \sin\tfrac{1}{2}A=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}} \qquad \text{IV. } \cos\tfrac{1}{2}A=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}$$

$$\text{II. } \sin\tfrac{1}{2}B=\sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin a\sin c}} \qquad \text{V. } \cos\tfrac{1}{2}B=\sqrt{\frac{\sin s\sin(s-b)}{\sin a\sin c}}$$

$$\text{III. } \sin\tfrac{1}{2}C=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}} \qquad \text{VI. } \cos\tfrac{1}{2}C=\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}$$

Taking the products of I. and V., of IV. and II., of IV. and V., and of I. and II., and applying VI. to the first and second products, and III. to the third and fourth, we obtain

$$\text{VII. } \sin\tfrac{1}{2}A\cos\tfrac{1}{2}B=\frac{\sin(s-b)}{\sin c}.\cos\tfrac{1}{2}C \qquad \text{IX. } \cos\tfrac{1}{2}A\cos\tfrac{1}{2}B=\frac{\sin s}{\sin c}.\sin\tfrac{1}{2}C$$

$$\text{VIII. } \cos\tfrac{1}{2}A\sin\tfrac{1}{2}B=\frac{\sin(s-a)}{\sin c}.\cos\tfrac{1}{2}C \qquad \text{X. } \sin\tfrac{1}{2}A\sin\tfrac{1}{2}B=\frac{\sin(s-c)}{\sin c}.\sin\tfrac{1}{2}C.$$

Take the sum and difference of VII. and VIII., and of IX. and X., and modify the first members by (12), (13), (15), and (14), and the numerators of the second members by (20) and (21), using $\frac{1}{2}(a+b+c)$ for s, $\frac{1}{2}(a+b-c)$ for $s-c$, &c. and $2\sin\frac{1}{2}c\cos\frac{1}{2}c$ for $\sin c$, and there will arise

$$\text{XI. } \sin\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}C}{\cos\frac{1}{2}c}.\cos\tfrac{1}{2}(a-b) \qquad \text{XIII. } \cos\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}C}{\sin\frac{1}{2}c}.\sin\tfrac{1}{2}(a+b)$$

$$\text{XII. } \sin\tfrac{1}{2}(A-B)=\frac{\cos\frac{1}{2}C}{\sin\frac{1}{2}c}.\sin\tfrac{1}{2}(a-b) \qquad \text{XIV. } \cos\tfrac{1}{2}(A+B)=\frac{\sin\frac{1}{2}C}{\cos\frac{1}{2}c}.\cos\tfrac{1}{2}(a+b)$$

Now, (108) and (109) will be obtained by dividing XI. by XIV., and XII. by XIII.; and if XIII. be divided by XIV., and XII. by XI., (110) and (111) will be found by multiplying the quotients by $\tan\frac{1}{2}c$.

84. The formulas that have been investigated in the foregoing articles furnish the means of resolving all the elementary cases of spherical triangles. When applied to rightangled triangles, they take, in general, a simpler form, some of the terms vanishing in consequence of containing the cosine or cotangent of the right angle.

Thus, if C be a right angle, we have (No. 10) $\cos C = 0$, $\cot C = 0$, and $\sin C = 1$; and in this case (97) will become $\cot b \sin a = \cot B$, or, by multiplying by $\tan b$ (5), $\sin a = \tan b \cot B$; while, from the first and third parts of (94), we derive $\sin a = \sin c \sin A$.

From (103), by transposition, and by dividing by $\sin A \sin B$, we obtain $\cos c = \cot A \cot B$; and from (87) we derive at once $\cos c = \cos a \cos b$.

From (100), also, by dividing by $\cos b$, we obtain $\cos A = \tan b \cot c$; and (101) gives $\cos A = \cos a \sin B$.

These results may be conveniently arranged in the following form:

$$\sin a = \tan b \cot B = \sin c \sin A \qquad (112)$$
$$\cos c = \cot A \cot B = \cos a \cos b \qquad (113)$$
$$\cos A = \tan b \cot c = \cos a \sin B \qquad (114)$$

85. From these formulas, by supplying the radius, we have the following theorems, which are sufficient for the resolution of all the elementary cases of rightangled spherical triangles.

I. The rectangle under the radius and the sine of one of the legs, is equal to the rectangle under the cotangent of the adjacent oblique angle and the tangent of the other leg, or to the rectangle under the sines of the opposite angle and the hypotenuse.

II. The rectangle under the radius and the cosine of the hypotenuse, is equal to the rectangle under the cotangents of the oblique angles, or to the rectangle under the cosines of the legs.

III. The rectangle under the radius and the cosine of one of the oblique angles, is equal to the rectangle under the tangent of the adjacent leg and the cotangent of the hypotenuse, or to the rectangle under the cosine of the opposite leg and the sine of the other oblique angle.

86. Several useful formulas respecting oblique-angled triangles may

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Another investigation might be had by means of (90) and (106), and of the formulas $\tan \frac{1}{2}(A \pm B) = \frac{\tan \frac{1}{2}A \pm \tan \frac{1}{2}B}{1 \mp \tan \frac{1}{2}A \tan \frac{1}{2}B}$, (39) and (40); and other investigations will be found in the works on trigonometry.

be obtained by drawing their perpendiculars, and applying the formulas last found. Thus, let AD (*fig.* 20) be perpendicular to BC. Then, putting φ to represent CD, we have $BD = a - \varphi$; and (No. 85, III.)

$$\cos C = \cot b \tan \varphi; \text{ whence (5)}$$

$$\tan \varphi = \tan b \cos C \text{ .... (115)}$$

87. By considering b, φ, and AD, in one of the rightangled triangles, and c, $a - \varphi$, and AD, in the other, we have (No. 85, II.) $\cos b = \cos \varphi \cos AD$, and $\cos c = \cos (a - \varphi) \cos AD$; by dividing the latter of which by the former, we obtain

$$\frac{\cos c}{\cos b} = \frac{\cos (a - \varphi)}{\cos \varphi} \text{ .... (116)}$$

Hence it appears that *the cosines of the segments intercepted between the perpendicular and the extremities of the base are proportional to the cosines of the adjacent sides of the triangle.*

88. If we now consider φ, C, and AD, in the one triangle, and $a - \varphi$, B, and AD, in the other, we get (No. 85, I.) $\sin \varphi = \cot C \tan AD$, and $\sin (a - \varphi) = \cot B \tan AD$; whence, by dividing the former by the latter, we obtain

$$\frac{\sin \varphi}{\sin (a - \varphi)} = \frac{\cot C}{\cot B};$$

or (5), by multiplying the numerator and denominator of the second member by $\tan B \tan C$,

$$\frac{\sin \varphi}{\sin (a - \varphi)} = \frac{\tan B}{\tan C} \text{ .... (117)}$$

Hence, *the sines of the segments of the base are reciprocally proportional to the tangents of the adjacent angles.*

89. By denoting the angle CAD by Φ, and consequently BAD by $A - \Phi$, and by considering b, C, and Φ, we have (No. 85, II.)

$$\cos b = \cot C \cot \Phi,$$

or (5) by multiplying by $\tan \Phi$, and dividing by $\cos b$,

$$\tan \Phi = \frac{\cot C}{\cos b} \text{ .... (118)}$$

90. Now, from C, Φ, and AD, in the one triangle, and B, $A - \Phi$, and AD, in the other, we obtain $\cos C = \sin \Phi \cos AD$, and $\cos B = \sin (A - \Phi) \cos AD$; whence, by division,

$$\frac{\cos B}{\cos C} = \frac{\sin (A - \Phi)}{\sin \Phi} \text{ .... (119)}$$

Hence, *the sines of the angles, contained by the perpendicular and the sides, are proportional to the cosines of the angles at the base.*

91. By like processes, and by considering, first, b, Φ, AD, and c, $A-\Phi$, AD; and then φ, Φ, AD, and $a-\varphi$, $A-\Phi$, AD, we obtain

$$\frac{\cos\Phi}{\cos(A-\Phi)}=\frac{\tan c}{\tan b} \quad \text{..............} \quad (120)$$

$$\frac{\tan\varphi}{\tan(a-\varphi)}=\frac{\tan\Phi}{\tan(A-\Phi)} \quad \text{......} \quad (121)$$

Hence, *the cosines of the angles, contained by the perpendicular and the sides, are reciprocally proportional to the tangents of the sides;* and *the tangents of those angles are proportional to the tangents of the segments of the base.*

SCHOLIUM.

As was stated in No. 69, a spherical triangle has thus far been understood as being the *smaller* of the two parts into which the surface of a sphere is divided by the three *smaller* arcs, which join, by pairs, three points on the surface, and not on the same great circle. In strictness, however, this view of the nature of a spherical triangle, though convenient in practice, is too limited. As an arc of a great circle (No. 12) may be increased without limit, so a side of a spherical triangle may exceed, not only a semicircle, but even an entire circle of the sphere. In like manner, if the half of any great circle continue fixed, while another semicircle on the same diameter begins to revolve about that diameter from coincidence, and thus to make greater and greater angles with the fixed one, as there is evidently no limit to the angular magnitude thus generated, it is plain that the spherical angle formed by the two semicircles, may exceed two right angles, or any assigned magnitude whatever. It is unnecessary, however, to consider sides that exceed a complete circle, as the extreme points of any such side will occupy the same positions, as when it is diminished by a circle: and, for a similar reason, it is unnecessary to consider any angle greater than four right angles.

To enter into a minute consideration of the views thus opened up, would be unsuitable to our present limits; but the reader who wishes to prosecute the subject will find a paper by the author of this work in the London and Edinburgh Philosophical Magazine for January, 1837, in which the subject is discussed at some length. It may be proper, however, to give here a few illustrations of the subject, and a few of the results established in the paper referred to.

1. If in a spherical triangle, the angle A, and the sides b and c containing it, be given, and if we make on the surface an angle equal to A, and the arcs AC and AB equal respectively to b and c, we determine the points B and C; and since these points may be joined by either of the parts of the great circle passing through them, there will be *two* triangles, each answering to the data, and differing in area by the surface of a hemisphere; and they will be such, that if the third side of one be denoted by a, that of the other will be $2\pi-a$ This agrees exactly with formula (85), which gives the cosine of the third side

by means of A, b, c; and (10) $\cos a$ is the same as $\cos (2\pi - a)$. The remaining angles would be found by means of (98) and (100), and would thus have two values as they ought: and like results would be obtained by means of Napier's analogies, or any of the other modes of solving the third case.

2. If the three sides a, b, c, be given, and if we take three points A, B, C, such that $BC = a$, $AC = b$, and $AB = c$, either of the two parts into which the surface of the sphere is divided by AB, BC, and AC, will be a spherical triangle, having its three sides of the given magnitudes, and consequently agreeing with the conditions of the question. It is plain, also, that if A, B, and C, be the angles of one of these triangles, the angles of the other will be $2\pi - A$, $2\pi - B$, and $2\pi - C$. This result agrees with that which is obtained by any of the modes of solving the first case. Thus, by (85), the angle opposite to a would be determined by its cosine, and would therefore be A or $2\pi - A$. So likewise, since an angle and its supplement have the same sine, (88) will give $\frac{1}{2}A$ for the one triangle, and $\pi - \frac{1}{2}A$ for the other; the doubles of which agree with the result of (85). The same results would also be obtained from (88), by considering, that in consequence of the extraction of the square root, $\sin \frac{1}{2}A$ may be either positive or negative, answering to which we should have either $\frac{1}{2}A$ or $-\frac{1}{2}A$. By doubling these we get A and $-A$; to the latter of which if we add 2π, we get, by (9), A and $2\pi - A$, as before. We have thus an explanation of the meaning of the double sign $\pm$ before the square root in this formula; and a like explanation may be given in every similar case.

3. When the side a, and the adjacent angles B and C are given, if we make the arc $BC = a$, and through its extremities draw BA, CA making with it angles respectively equal to B and C, the arcs BA, CA will again intersect in the point A′ diametrically opposite to A; and if b and c be put to denote the remaining sides CA, BA, drawn to the first intersection, those drawn to the other point A′, will be $b + \pi$ and $c + \pi$; results which (No. 18) agree with the values of b and c, that would be obtained from (97) and (99). The figure A′BC is not indeed a *triangle*, in the ordinary meaning of the term, as two of its sides intersect each other in a point between the vertex and the base. Still, however, it is to be regarded, equally with ABC as answering the conditions of the proposed problem; which, expressed in other words, is simply, to find the lengths of the arcs drawn from B and C to the point, or points, of intersection of those arcs. It will readily appear, also, that if A be the remaining angle in the one triangle, the corresponding angle in the other will be $2\pi - A$; and this agrees with what would be obtained from (101).

4. Viewing triangles in this extended sense, we shall see, that, contrary to what is stated in almost all the books on trigonometry, each side of a spherical triangle is *not* necessarily less than a semicircle: that any side is *not* necessarily less than the sum, or greater than the difference of the other two: that the sum of the three sides is *not* always less than the circumference of a great circle: that the sum of the three angles is greater than two right angles, and less than *ten** (not than *six*) right angles: and that the *greater* side may be opposite to the *less* angle.

* That is, if every angle, according to what is stated at the beginning of this scholium, be taken less than four right angles.

IV.—RESOLUTION OF SPHERICAL TRIANGLES.

92. Of the three sides and the three angles of a spherical triangle, if any three be given, it will appear presently, that we are able to compute the remaining three by means of the principles established in the preceding section. This important problem presents the six following cases:

I. When there are given the three sides.
II. The three angles.
III. Two sides and the contained angle.
IV. A side and the adjacent angles.
V. Two sides and an angle opposite to one of them.
VI. Two angles and a side opposite to one of them.*

93. *The first case, in which the three sides are given*, may be resolved, by means of logarithms,† by any of the formulas, (88), (89), (90). To exemplify this, let us supply the radius in the first of these, and we shall obtain

$$\sin\tfrac{1}{2}A=\sqrt{\frac{\sin(s-b)\,\sin(s-c)\,r^2}{\sin b\,\sin c}};$$

or, as it may be expressed,

$$\sin\tfrac{1}{2}A=\sqrt{\left\{\frac{r}{\sin b}\cdot\frac{r}{\sin c}\cdot\sin(s-b)\,\sin(s-c)\right\}}.$$

By taking the logarithms of both members of this, we find (No. 50)

$$\log\sin\tfrac{1}{2}A=\tfrac{1}{2}\{10-\log\sin b+10-\log\sin c+\log\sin(s-b)+\log\sin(s-c)\}‡ \quad (122)$$

In a similar manner, we find, from (89) and (90),

$$\log\cos\tfrac{1}{2}A=\tfrac{1}{2}\{10-\log\sin b+10-\log\sin c+\log\sin s+\log\sin(s-a)\} \;\ldots\ldots\; (123)$$

$$\log\tan\tfrac{1}{2}A=\tfrac{1}{2}\{10-\log\sin s+10-\log\sin(s-a)+\log\sin(s-b)+\log\sin(s-c)\} \quad (124)$$

This last formula, expressed in words, gives the following theorem:

1. Add the three sides together, and take half the sum. 2. From

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After reading what is given in section V. regarding the areas of spherical triangles, the student will see, that the principles here explained will illustrate Llhuillier's formula, and others in that section regarding the spherical excess.

* Both the fifth and sixth cases might be comprehended in one, the same as the first case of plane triangles. By means of the polar triangle, also, the six cases may be reduced to three, the same as in plane trigonometry.

† No mention is made here, or in the following cases, of any solutions except those by means of logarithms; the solutions by means of natural sines or tangents being now scarcely ever employed.

‡ It is evident from (No. 11), that, P being any arc or angle, we may use log cosec P — 10 instead of 10 — log sin P, and log sec P — 10 instead of 10 — log cos P.

the half sum subtract successively the side opposite to the required angle, and the sides containing it. 3. Find, from the tables, the logarithmic sines of the half sum and the three remainders. 4. Add together the sines of the last two remainders, and the complements to 10 of the sines of the half sum and the first remainder. 5. Take half the sum, and find in the table of logarithmic tangents the angle answering to it. 6. The double of this will be the required angle.

The student will find it easy and useful to express (122) and (123) in like manner in words.

A formula, for the same purpose, may be derived from (93); but it is of scarcely any use in practice; because, besides other inconveniences, it fails in determining whether the angle is acute or obtuse.

94. When one of the angles has been found by (124), the others are derived from it by means of (91), which gives

$$\log\tan\tfrac{1}{2}B=\log\tan\tfrac{1}{2}A+\log\sin(s-a)-\log\sin(s-b)\ldots(125)$$

$$\log\tan\tfrac{1}{2}C=\log\tan\tfrac{1}{2}A+\log\sin(s-a)-\log\sin(s-c)\ldots(126)$$

From this it appears, that *to the tangent of* $\frac{1}{2}A$, *previously found, we are to add the sine of* $s-a$, *and from the sum to subtract successively the sines of* $s-b$ *and* $s-c$, *to find the tangents of* $\frac{1}{2}B$ *and* $\frac{1}{2}C$. This operation is extremely simple and easy, the sines of $s-a$, $s-b$, and $s-c$, having been all taken out of the tables in the finding of A.

In like manner we have from (92),

$$\log\cot\tfrac{1}{2}B=\log\sin s+\log\tan\tfrac{1}{2}A-\log\sin(s-c)\ldots\ldots\ldots(127)$$

$$\log\cot\tfrac{1}{2}C=\log\sin s+\log\tan\tfrac{1}{2}A-\log\sin(s-b)\ldots\ldots\ldots(128)$$

These formulas give the solution with equal facility.

When only one angle is required, it may be found perhaps rather more easily by (122) or (123), than by (124). Of these two formulas, (122) is preferable when the angle is small, and (123) when it is large, as the half angles can thus be found with more accuracy from the common tables.*

It may be farther remarked, that when one angle has been found, the rest may be computed by the formulas of the Four Sines, though not nearly so easily as by (125) and (126), or by (127) and (128).

95. *The second case, in which the three angles are given,* may be

* Thus, in tables carried out to five places of decimals, 9·99998 appears as the sines of 89° 24′, of 89° 31′, and of all the intermediate angles; and the calculator would have no means, by such tables, of knowing which he should prefer. Besides, even when the *minutes* might be determined, the seconds, or other fractional parts of minutes, could not be computed with any tolerable degree of accuracy by means of proportional parts, even with better tables, in the cases referred to.

resolved by means of the formulas in Nos. 81 and 82; and the remarks in the preceding No. are applicable, with slight modifications, in reference to this case.

96. *In the third case, in which two sides and the contained angle are given,* the other angles may be found by (108) and (109). Thus, taking the logarithms of the members of those equations, we have

$$\log\tan\tfrac{1}{2}(A+B)=\log\cot\tfrac{1}{2}C+\log\cos\tfrac{1}{2}(a-b)-\log\cos\tfrac{1}{2}(a+b)^{*}\ldots(129)$$
$$\log\tan\tfrac{1}{2}(A-B)=\log\cot\tfrac{1}{2}C+\log\sin\tfrac{1}{2}(a-b)-\log\sin\tfrac{1}{2}(a+b)\ldots(130)$$

Half the sum and half the difference of the angles being thus found, and thence the angles themselves, the remaining side may be computed by the rule of the Four Sines; or, in a preferable manner, by finding $\frac{1}{2}c$ by means of (110) or (111).

97. If, with these data, *the third side only be required,* as is often so in the actual application of spherical trigonometry, the solution will be effected with more ease by employing an auxiliary arc, according to (115) and (116). Thus, by supplying the radius and taking the logarithms, we obtain from these formulas,

$$\log\tan\varphi=\log\tan b+\log\cos C-10\ \ldots\ldots\ldots\ldots\ (131)$$
$$\log\cos c=\log\cos b+\log\cos(a-\varphi)-\log\cos\varphi\ \ldots\ (132)$$

It is evident that a and b might be mutually interchanged in these formulas, since they bear the same relation to the contained angle.

98. *In the fourth case, in which a side and the adjacent angles are given,* the remaining sides may be found by means of the following formulas derived from (110) and (111):

$$\log\tan\tfrac{1}{2}(a+b)=\log\tan\tfrac{1}{2}c+\log\cos\tfrac{1}{2}(A-B)-\log\cos\tfrac{1}{2}(A+B).\ (133)$$
$$\log\tan\tfrac{1}{2}(a-b)=\log\tan\tfrac{1}{2}c+\log\sin\tfrac{1}{2}(A-B)-\log\sin\tfrac{1}{2}(A+B).\ (134)$$

The sides being thus determined, the remaining angle may be found by the rule of the Four Sines, or, in a preferable manner, by means of (108) or (109).

99. Should only the third angle be required, it is most easily found by means of the following formulas derived from (118) and (119), the given parts being marked b, A, and C:

$$\log\tan\Phi=\log\cot C+10-\log\cos b\ \ldots\ldots\ldots\ldots\ (135)$$
$$\log\cos B=\log\cos C+\log\sin(A-\Phi)-\log\sin\Phi\ldots\ (136)$$

100. *In the fifth case, in which two sides a and b, and an angle,* A, *opposite to one of them, are given,* the angle B opposite to the other, is found by the following formula derived from (94):

$$\log\sin B=\log\sin A+\log\sin b-\log\sin a\ \ldots\ldots\ldots\ (137)$$

B being thus found, c and C will be determined by the following formulas, derived from (133) and (129) by transposition:

$$\log\tan\tfrac{1}{2}c = \log\tan\tfrac{1}{2}(a+b) + \log\cos\tfrac{1}{2}(A+B) - \log\cos\tfrac{1}{2}(A-B) \quad (138)$$
$$\log\cot\tfrac{1}{2}C = \log\tan\tfrac{1}{2}(A+B) + \log\cos\tfrac{1}{2}(a+b) - \log\cos\tfrac{1}{2}(a-b) \quad (139)$$

It is evident that (134) and (130) would give formulas answering the same purpose.

101. The species of B, when not doubtful,* is known by the following rule:

When the sum of the given sides is less than 180°, the angle opposite to the less side is acute; but when the sum of those sides exceeds 180°, the angle opposite to the greater side is obtuse; and, lastly, if the sum of those sides be 180°, the sum of the opposite angles is the same. When these principles fail in determining the species of the angle, it is doubtful.†

102. The third side c might also be found by the following for-

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* It may be proper to remark here, that the required part, either in plane or spherical trigonometry, is never doubtful, except when it is found by means of its sine. To understand the reason of this, it is only necessary to consider, that if the required part be determined by means of its cosine, it must (No. 13) be in the first quadrant if the cosine be positive, but in the second, if it be negative: and the same holds when it is found by means of its tangent or cotangent.

† This rule, which is taken in substance from Cagnoli, is derived from the first note to No. 83, and from the note to No. 73. Thus, by the former note, if the sum of the sides be less than 180°, the sum of the opposite angles is also less than the same, and, therefore, at least the less of them is less than 90°; but, by the latter note, the less is opposite to the less side, which establishes the first part of the rule. The proof of the second part is exactly similar, and the third part follows at once from the first of the notes referred to.

The nature of this case, when doubtful, will be illustrated by figures 21 and 22; the angles A and A′ in the former being acute, and in the latter obtuse. In the former figure, if the side a be less than either b or its supplement CA′, a small circle described at a distance from C equal to a would cut ABB′A′ in two points B and B′, and therefore there are two triangles, ABC, AB′C, either of which answers the conditions of the question. It will appear, from the other figure, that a like ambiguity will exist when a is greater than either b or its supplement. It is also plain, that in the former figure a might be so small, and in the latter so large, that the solution would be impossible, there being no triangle answering to the data. It appears, therefore, that *when the given angle is acute, there is ambiguity only when the opposite side is less than either the other side or its supplement;* and that, *when the given angle is obtuse, there is ambiguity only when the opposite side is greater than either the other side or its supplement. When there is no ambiguity, the species of the required angle may be known by the principle, that the greater angle is opposite to the greater side.* Some may, perhaps, prefer these principles to the rule given in the text. It may also be remarked, that in each figure, if the small circle merely touch ABB′A′, the triangle will be rightangled, and there will be no ambiguity.

mulas, derived from (115) and (116), by changing C into A, a into c, and c into a:

$$\log \tan \varphi = \log \tan b + \log \cos A - 10 \quad \ldots\ldots\ldots\ldots\ldots\ldots (140)$$

$$\log \cos (c - \varphi)^* = \log \cos a + \log \cos \varphi - \log \cos b \quad \ldots. (141)$$

The latter gives the difference of c and φ; and c will be the sum or difference of φ and $c-\varphi$, accordingly as (No. 101) A and B are of the same or of different species. If the species of B be doubtful, either the sum or difference may be taken.

103. The angle C, contained by the given sides, might also be found by the following formulas, derived from Nos. 89 and 91 by making the necessary interchange of letters:

$$\log \tan \Phi = \log \cot A + 10 - \log \cos b \quad \ldots\ldots\ldots\ldots\ldots\ldots (142)$$

$$\log \cos (C - \Phi) = \log \tan b + \log \cos \Phi - \log \tan a \quad \ldots (143)$$

When A and B are of the same species, C is the sum of Φ and $C-\Phi$; otherwise, it is their difference.

104. *In the sixth case, in which two angles,* A *and* B, *and a side, a, opposite to one of them, are given,* the side, b, opposite to the other, is found thus, by (94):

$$\log \sin b = \log \sin a + \log \sin B - \log \sin A \quad \ldots\ldots\ldots\ldots (144)$$

Having thus found b, we may find c and C by (138) and (139).

The species of b, when not doubtful, will be known by the following rule:

When the sum of the given angles is less than 180°, the side opposite to the less is less than 90°; but, if the sum of those angles exceed 180°, the side opposite to the greater angle is greater than 90°; and, lastly, if the sum of the given angles be 180°, the sum of the opposite sides is the same.†

105. The angle C might also be obtained in the following manner, by (118) and (119), after the necessary changes in the letters:

$$\log \tan \Phi = \log \cot B + 10 - \log \cos a \quad \ldots\ldots\ldots\ldots\ldots\ldots (145)$$

$$\log \sin (C - \Phi) = \log \sin \Phi + \log \cos A - \log \cos B \quad \ldots (146)$$

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* Or $\varphi - c$, when φ is greater than c; and the same is to be observed throughout this page and the next. In like manner, instead of $C-\Phi$, we are to use $\Phi - C$, when Φ is greater than C.

† This rule is also taken in substance from Cagnoli, and is founded on the same principles as the rule given in No. 101.

If CB (*fig.* 21 and 22) be equal to the given side, A, and consequently A′ to the opposite angle, and ABC to the other given angle; then, if from C two equal arcs, CB and CB′, can be drawn to ABA′, either of the triangles ABC, A′B′C will answer to the data of the problem, the angles ABC, A′B′C being equal in consequence of the equality of CB, CB′; and thus we have an illustration of the nature of this case, when doubtful, similar to that given of the fifth case in the note to No. 101.

Φ and $C-\Phi$ are of the same or of different species, accordingly as (No. 104) a and b are; also, the sum or difference of Φ and $C-\Phi$ is to be taken, accordingly as A and B are of the same or of different species.

106. The side c, adjacent to the given angles, may be found thus, from (115) and (117), after duly interchanging the letters:

$$\log\tan\varphi = \log\tan a + \log\cos B - 10 \;\ldots\ldots\ldots\ldots\ldots\ldots (147)$$
$$\log\sin(c-\varphi) = \log\sin\varphi + \log\tan B - \log\tan A \;\ldots\; (148)$$

The arcs φ and $c-\varphi$ are of the same species or not, accordingly as a and b are; and c will be their sum or difference, accordingly as A and B are of the same or different species.

107. The resolution of rightangled triangles is effected much more easily by means of formulas (112), (113), (114), than by the principles that have been just established. If we consider these formulas in connexion with the triangle ABC (*fig.* 19) rightangled at C, neglecting the right angle, and attending only to the five remaining parts (the three sides and the oblique angles), we shall perceive, that any of these parts that stands in the first column, in these equations, is situated, in the triangle, in the middle between the two parts in the second column, and that it is separated by these parts from those in the third column. It will also be perceived, that there are, in the first column, the *sine* of a leg and the *cosines* of the other parts; in the second, the *tangent* of a leg and the *cotangents* of the other parts; and in the third, the *cosines* of the legs and the *sines* of the other parts: and hence, supplying the radius in the first column, we may form the following general rule, which will serve for the resolution of all the cases of rightangled spherical triangles:

I. Of the required part and the two given parts of the triangle, let that which is either adjacent to the other two, or is separated from them on each side by an intervening part, be called the *mean;* and if the other two be separated from the mean, let them be called *remote parts;* otherwise, let them be called *adjacent parts.*

II. Then, with the exception mentioned below, the rectangle under the radius and the cosine of the mean, is equal to the rectangle under the cotangents of the adjacent parts, or to the rectangle under the sines of the remote parts.

III. *Exception.* When a leg is one of the parts, its sine, tangent, and cosine, must be used respectively instead of its cosine, cotangent, and sine.*

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* These rules are, *in effect,* the same as those generally known by the name of *Napier's Rules for the Circular Parts;* but they are expressed somewhat

108. It will appear, from No. 104, that when the parts given are a leg and the opposite angle, the results are doubtful. In other cases, the species of the required part is determined either by means of the sign (*plus* or *minus*) prefixed to its cosine, tangent, or cotangent; or by the principle (deducible also from No. 104), that a leg and its opposite angle are of the same species.

109. A triangle which has one of its sides a quadrant, is called a *quadrantal triangle*, or, by some, though not very properly, a *rectilateral triangle*. The general formulas, when applied to triangles of this kind, become as much simplified as in their application to rightangled triangles, and a rule similar to that given in No. 107 might be readily formed for their resolution. As such triangles, however, do not frequently present themselves, and as the use of them may be always avoided, it is unnecessary to give a special rule; as it will be sufficient, in any particular case, to obtain, in a rightangled triangle, a formula expressing the relation of the parts *opposite* to the two given parts and the one required; and then to change, in the formula thus obtained, sides for the opposite angles, and angles for the opposite sides; and to prefix the sign *minus* to the cosines, tangents, and cotangents. Thus, c being a quadrant, if A and B were given to find what has been called the *hypotenusal angle*, C, we should have, by considering the three parts, a, b, and c, in a triangle rightangled at C, $\cos c = \cos a \cos b$; whence, by the changes above mentioned, we obtain, in the quadrantal triangle, $-\cos C = \cos A \cos B$, or $\cos C = -\cos A \cos B$. The reason of this is manifest from No. 79.* It may be remarked, also, that whenever a quadrantal triangle occurs in an investigation, there is some rightangled triangle which may be

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differently, and will perhaps be found rather easier in practice, as they give the required formulas more directly. Delambre, in the valuable article on spherical trigonometry, in the first volume of his "*Astronomie Théorique et Pratique*," discountenances the use of Napier's Rules. Those given above are free from some of the objections which he offers against those commonly employed; and it is probable that few, except experienced calculators, would wish to forego the use of so very convenient a help to the memory. The mere beginner will perhaps derive assistance in discovering, in any particular case, which are the mean, the adjacent, and the remote parts, by writing, at nearly equal intervals round the circumference of a circle, the several parts, a, b, A, c, and B.

In Napier's Rules in their common form, the circular parts are, the legs, and the complements of the hypotenuse and the oblique angles: then, 1, "The rectangle of the radius and the sine of the mean is equal to the rectangle of the tangents of the adjacent parts; and, 2, to the rectangle of the cosines of the remote parts."

* The following formulas, which may be derived from (112), (113), (114),

employed instead of it, and which will give the same results that would be obtained by means of the quadrantal one.

EXAMPLES OF THE RESOLUTION OF SPHERICAL TRIANGLES.

110. Given $a=100^\circ$, $b=37^\circ\ 18'$, and $c=62^\circ\ 46'$; to find the angles.

By adding the three sides together, we obtain $200^\circ\ 4'$, the half of which, $100^\circ\ 2'$, is s. Then, taking from this, first a, then b, then c, we find $s-a=2'$, $s-b=62^\circ\ 44'$, and $s-c=37^\circ\ 16'$; and the angle A is found by (124) in the following manner:

$\sin s$	$100^\circ\ 2'$	9·99331	subt.
$\sin(s-a)$	2	6·76476	subt.
$\sin(s-b)$	62 44	9·94884	
$\sin(s-c)$	37 16	9.78213	
		2)22·97290	
$\tan\frac{1}{2}A$	$88^\circ\ 7'\ 53''$	11·48645	
	$A=176^\circ\ 15'\ 46''$		

Then, according to (125) and (126), we have $\tan\frac{1}{2}A+\sin(s-a)=11{\cdot}48645+6{\cdot}76476=18{\cdot}25121$; and taking from this, first, $\sin(s-b)=9{\cdot}94884$, and, secondly, $\sin(s-c)=9{\cdot}78213$, we find $\tan\frac{1}{2}B=8{\cdot}30237$, and $\tan\frac{1}{2}C=8{\cdot}46908$: whence, by the tables, we get $\frac{1}{2}B=1^\circ\ 8'\ 57''$, and $\frac{1}{2}C=1^\circ\ 41'\ 13''$, the doubles of which are $B=2^\circ\ 17'\ 54''$, and $C=3^\circ\ 22'\ 26''$.

In the first part of the foregoing operation, as in many other computations in trigonometry, the method of performing addition and subtraction at a single operation, in the manner pointed out in No. 60, may be employed with advantage.

As a check on the preparatory part of the process, it may be observed, that, as in plane triangles, the sum of the three remainders, $s-a$, $s-b$, $s-c$, is equal to the half sum s.

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in the manner above mentioned, or from the general formulas of spherical trigonometry, by taking $c=90^\circ$, will resolve all the cases of quadrantal triangles:

$$\sin A = \tan B \cot b = \sin C \sin a$$
$$\cos C = -\cot a \cot b = -\cos A \cos B$$
$$\cos a = -\tan B \cot C = \cos A \sin b$$

The following are the computations of A by (122) and (123):

By (122).				By (123).			
sin b	37° 18′	9·78246	subt.	sin b	37° 18′	9·78246	subt.
sin c	62 46	9·94898		sin c	62 46	9·94898	
sin(s—b)	62 44	9·94884		sin s	100 2	9·99331	
sin(s—c)	37 16	9·78213		sin(s—a)	2	6·76476	
		2)19·99953				2)17·02663	
sin $\frac{1}{2}$A	88° 7′	9·99976		cos $\frac{1}{2}$A	88° 7′ 53″	8·51331	
A=176° 14′				A=176° 15′ 46″*			

111. Given A=139° 27′, B=53° 39′, and C=34° 5′; to find the sides.

Here we have S=113° 35′$\frac{1}{2}$, S—A=—25° 51′$\frac{1}{2}$, S—B=59° 56′$\frac{1}{2}$, S—C=79° 30′$\frac{1}{2}$; and by means of No. 81 or 82, we find a= 126° 34′, b=95° 46′, and c=43° 49′.

112. Given A=42° 43′, b=45° 51′, and c=141° 17′; to resolve the triangle. Here, by taking the sum and difference of c and b, and halving them, we get $\frac{1}{2}(c+b)$=93° 34′, and $\frac{1}{2}(c-b)$=47° 43′: and the half of A is 21° 21′$\frac{1}{2}$. Then, by (129) and (130),

$$\log\tan\tfrac{1}{2}(C+B)=\log\cot\tfrac{1}{2}A+\log\cos\tfrac{1}{2}(c-b)-\log\cos\tfrac{1}{2}(c+b), \text{ and}$$
$$\log\tan\tfrac{1}{2}(C-B)=\log\cot\tfrac{1}{2}A+\log\sin\tfrac{1}{2}(c-b)-\log\sin\tfrac{1}{2}(c+b);$$

whence, by performing the actual operation, we find $\frac{1}{2}$(C+B)= 92° 4′$\frac{1}{4}$,† and $\frac{1}{2}$ (C—B)=62° 11′$\frac{1}{4}$; and by taking the sum and difference of these, we get C=154° 15′$\frac{1}{2}$, and B=29° 53′.

113. To find the side a, we have, by (131) and (132),

$$\log\tan\varphi=\log\tan b+\log\cos A-10, \text{ and}$$
$$\log\cos a=\log\cos b+\log\cos(c-\varphi)-\log\cos\varphi.$$

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* In this example, as A is very obtuse, the use of (123) is much preferable (No. 94) to that of (122); as the former enables us, even by tables carried to only five places of decimals, to find, by proportional parts, the result true to seconds; while the other does not give it, by such tables, true even to the nearest minute, there being at least two sines such that we should not know which to prefer. In what follows, the answers will generally be given true only to the nearest minute.

† In the equation from which this is obtained, $\frac{1}{2}$A and $\frac{1}{2}(c-b)$ being each less than 90°, the cotangent of the former, and the cosine of the latter, are (No. 13) both positive; but $\frac{1}{2}(c+b)$ being greater than 90°, its cosine is negative. Hence, since the product of two positive quantities is divided by a negative one, the quotient, tan$\frac{1}{2}$(C+B), must be negative, and therefore (No. 13) $\frac{1}{2}$(C+B) is greater than 90°. For a similar reason, in No. 113, a in both methods, and φ in the second, are to be taken in the second quadrant, and are therefore the supplements of the values given by the tables. It may be stated as a general principle, that an odd number of negative multipliers or divisors, or of both, gives a negative result; while in other cases the result is positive.

From the first of these we find $\varphi=37^\circ\ 7'\frac{1}{4}$, by subtracting which from c we get $c-\varphi=104^\circ\ 9'\frac{3}{4}$. We then obtain from the second $a=102^\circ\ 20'\frac{1}{2}$.

We should also have from (131) and (132),

$$\log\tan\varphi=\log\tan c+\log\cos A-10, \text{ and}$$
$$\log\cos a=\log\cos c+\log\cos(b-\varphi)-\log\cos\varphi;$$

from which we should find $\varphi=149^\circ\ 30'\frac{1}{4}$, $b-\varphi=-103^\circ\ 39'\frac{1}{4}$,* and $a=102^\circ\ 20'\frac{1}{2}$, as before; these variations thus verifying one another.

When B and C are determined, as in No. 112, by Napier's analogies, a may be computed (No. 104) by either of the following formulas:

$$\log\sin a=\log\sin A+\log\sin b-\log\sin B$$
$$\log\sin a=\log\sin A+\log\sin c-\log\sin C.$$

These formulas fail, however, in determining the species of a.† It is better, therefore, to use one of the following, derived from No. 98:

$$\log\tan\tfrac{1}{2}a=\log\tan\tfrac{1}{2}(c+b)+\log\cos\tfrac{1}{2}(C+B)-\log\cos\tfrac{1}{2}(C-B)$$
$$\log\tan\tfrac{1}{2}a=\log\tan\tfrac{1}{2}(c-b)+\log\sin\tfrac{1}{2}(C+B)-\log\sin\tfrac{1}{2}(C-B).$$

114. Given $a=46^\circ\ 16'$, $b=73^\circ\ 8'$, $A=37^\circ\ 54'$; to resolve the triangle.

Here we have (137) $\log\sin B=\log\sin b+\log\sin A-\log\sin a$; whence $B=54^\circ\ 27'$, or $125^\circ\ 33'$, the rule in No. 101 failing to determine the species of B, the sum of the given sides being less than 180°, and the angle opposite to the less being the *given* angle. Now, from (138) and (139), if B be taken $=54^\circ\ 27'$, we find $c=100^\circ\ 16'$, and $C=123^\circ\ 13'$; but if $B=125^\circ\ 33'$, we find $c=37^\circ\ 41'$, and $C=31^\circ\ 19'$.

Were the angle C the only part required, we should find (142) $\Phi=77^\circ\ 16'\frac{1}{4}$, and (143) the difference of C and $\Phi=45^\circ\ 57'$. Hence, since this example belongs to the doubtful case, C will be either the sum or difference of these, and will be found to be nearly the same as before. In a similar manner c might be found by No. 102.

115. Given $C=90^\circ$, $a=133^\circ\ 4'$, and $A=111^\circ\ 37'$; to find c, b, and B.

Here, in finding the hypotenuse c, a is the mean, and A and c are the remote extremes; and therefore, by No. 107, or formula (112), $r\sin a=\sin A\sin c$, or, by taking the logarithms, and transposing,

$$\log\sin c=\log\sin a+10-\log\sin A,$$

an equation which (No. 104) gives $c=51^\circ\ 48'$, or $128^\circ\ 12'$.

---

* By No. 14, the cosine of this is the same as that of the positive arc $103^\circ\ 39'\frac{1}{4}$. In practice, indeed, in this problem, it is sufficient to employ the difference of b and φ, without attending to its sign.

† It is somewhat curious that the rule given in No. 104 always determines, in the sixth case, the species of the required side, when it is not doubtful; and yet that, as in the present instance, it sometimes fails in the third case, when solved in the method last pointed out, though that case can never be doubtful. This circumstance forms a strong objection to the use of this method, in finding the remaining side, either in the third or fourth case, or in the second.

In finding b, that side is the mean, and a and A are the adjacent extremes. We have, therefore (No. 107), $r \sin b = \tan a \cot A$; whence, by taking the logarithms and transposing, we get

$$\log \sin b = \log \tan a + \log \cot A - 10,$$

an expression which gives $b = 25^\circ\ 5'$, or $154^\circ\ 55'$.

Lastly, in computing B, A is the mean, and a and B are the remote extremes; wherefore (No. 107), $r \cos A = \sin B \cos a$. Hence, by taking the logarithms and transposing, we obtain

$$\log \sin B = \log \cos A + 10 - \log \cos a;$$

whence $B = 32^\circ\ 39'$, or $147^\circ\ 21'$.

EXERCISES IN SPHERICAL TRIGONOMETRY.

No.	Given.	Answers.
1.	a = 56° 17′	A = 62° 32′
	b = 147 33	B = 145 5
	c = 112 48	C = 79 33
2.	A = 47 9	a = 16 24
	B = 136 21	b = 164 36
	C = 88 34	c = 157 22
3.	a = 78 41	A = 133 15
	b = 153 30	B = 160 39
	C = 140 22	c = 120 50
4.	a = 71 45	A = 70 31
	B = 104 5	b = 102 17
	C = 82 18	c = 86 41
5.	a = 136 25	A = 123 19
	c = 125 40	B = 62 6
	C = 100 0	b = 46 48
6.	b = 124 53	a = 155 35½
	c = 31 19	C = 10 19½
	B = 16 26	A = 171 48½
7.	a = 115 28	A = 59 39
	b = 60 29	C = 172 43
	B = 56 17	c = 172 23
		or A = 120 21
		C = 72 52
		c = 88 53
8.	A = 103 16	a = 149 53*
	B = 76 44	c = 164 50
	b = 30 7	C = 149 30
9.	B = 146° 12′	a = 112° 39′
	C = 40 40	c = 43 39
	b = 143 54	A = 60 37
		or a = 9 27
		c = 136 21
		A = 8 56
10.	A = 90 0	a = 91 42
	B = 95 6	b = 95 22½
	C = 71 36	c = 71 31½
11.	a = 53 19	A = 66 40
	b = 35 23	B = 41 32
	C = 90 0	c = 60 51
12.	A = 75 36	c = 31 51
	B = 90 0	b = 68 10½
	a = 64 3	C = 34 38
		or c = 148 9
		b = 111 49½
		C = 145 22
13.	a = 3 0	A = 36 54
	b = 4 0	B = 53 10
	c = 5 0	C = 90 2
14.	a = 30 0	A = 40 39
	b = 40 0	B = 56 52
	c = 50 0	C = 93 41
15.	a = 60 0	A = 56 52
	b = 80 0	B = 72 13
	c = 100 0	C = 107 47

* This may be found by subtraction, according to No. 104, since the sum of the given angles is 180°.

V.—MISCELLANEOUS INVESTIGATIONS.*

116. From the three sides of the spherical triangle ABC (*fig.* 20), the segments of the base made by the perpendicular AD may be determined in the following manner, without calculating the angles: Putting $CD=\varphi$, we have (No. 87) $\cos(a-\varphi) : \cos\varphi :: \cos c : \cos b$; whence we obtain, by composition and division, $\cos(a-\varphi)+\cos\varphi : \cos(a-\varphi)-\cos\varphi :: \cos c+\cos b : \cos c-\cos b$, or

$$\frac{\cos(a-\varphi)-\cos\varphi}{\cos(a-\varphi)+\cos\varphi}=\frac{\cos c-\cos b}{\cos c+\cos b};$$

which, by (25) and the note to (5), becomes

$$\tan\tfrac{1}{2}a\,\tan(\varphi-\tfrac{1}{2}a)=\tan\tfrac{1}{2}(b+c)\,\tan\tfrac{1}{2}(b-c)\ldots\ (149)$$

By converting this into an analogy, we obtain $\tan\frac{1}{2}a : \tan\frac{1}{2}(b+c) :: \tan\frac{1}{2}(b-c) : \tan(\varphi-\frac{1}{2}a)$. The quantity $\varphi-\frac{1}{2}a$ is the distance from the middle point of BC to the point D. Since $\varphi-\frac{1}{2}a$ is found by its tangent, it may (No. 13) be taken, if positive, either in the first or third quadrant, and if negative either in the second or fourth.† The two values thus found will give the two points in which the perpendicular, continued round the sphere, cuts BC similarly continued. The values of the segments will indicate whether either part of the perpendicular falls within the triangle.

The segments of the base being thus determined, the angles of the rightangled triangles ADB, ADC may be calculated; and thus we have another method of solving the first case, in addition to those given in the last Section.

117. Putting the angle $CAD=\Phi$, we have (No. 90), $\sin\Phi : \sin(A-\Phi) :: \cos C : \cos B$; whence, by composition and division, by (24) and (25), and by the note to (5),

$$\cot\tfrac{1}{2}A\tan(\Phi-\tfrac{1}{2}A)=\tan\tfrac{1}{2}(B+C)\tan\tfrac{1}{2}(B-C)\ldots\ (150)$$

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* The theory of the most usual modes of resolving the several cases of spherical trigonometry has now been established. As many examples and exercises have also been given, as will enable the student to effect the necessary computations in any elementary problem in this important branch of science. It may now be proper to subjoin some curious and interesting matter of a miscellaneous kind, which could not with propriety be introduced before. A considerable part of this will often be found practically useful; and the investigations will afford the student exercise in the application of the principles already established. Should his time be limited, however, or should he wish to study only those parts of spherical trigonometry that are necessary in astronomy, he may omit this section altogether.

† When $\tan(\varphi-\frac{1}{2}a)$ is negative, one of the values of $\varphi-\frac{1}{2}a$ might be taken negative, and the other in the second quadrant. This, however, would make no change in the final results.

The quantity $\Phi-\frac{1}{2}A$ is the angle contained by the perpendicular and the line bisecting A; and observations similar to those at the end of the last No. are applicacle with respect to the different values of this angle. We have thus another method of solving the second case.

118. We have also (No. 91) $\cos(A-\Phi) : \cos\Phi :: \tan b : \tan c$; whence, by composition and division,

$$\frac{\cos(A-\Phi)-\cos\Phi}{\cos(A-\Phi)+\cos\Phi}=\frac{\tan b-\tan c}{\tan b+\tan c}.$$

Multiply the numerator and denominator of the second member by $\cos b \cos c$; then, by modifying the result by (13) and (12), and the first member by (25), there is obtained

$$\frac{\tan(\Phi-\frac{1}{2}A)}{\cot\frac{1}{2}A}=\frac{\sin(b-c)}{\sin(b+c)} \dots (151)$$

This result affords an easy means of determining the parts of the angle A; and thence, by the rightangled triangles ABD, ACD, we have another method of solving the third case.

119. It was shown in No. 88 that $\sin\varphi : \sin(a-\varphi) :: \tan B : \tan C$; whence, by a process nearly the same as that employed in the last No. we obtain

$$\frac{\tan(\varphi-\frac{1}{2}a)}{\tan\frac{1}{2}a}=\frac{\sin(B-C)}{\sin(B+C)} \dots (152)$$

This formula affords an additional method of solving the fourth case.

120. Let AD (*fig.* 24) bisect the angle A, and let $CD=b'$, and $BD=c'$. Then (94)

$$\frac{\sin b}{\sin b'}=\frac{\sin ADC}{\sin\frac{1}{2}A}, \text{ and } \frac{\sin c}{\sin c'}=\frac{\sin ADB}{\sin\frac{1}{2}A}.$$

But (No. 15) $\sin ADB=\sin ADC$: wherefore,

$$\frac{\sin b}{\sin b'}=\frac{\sin c}{\sin c'}, \text{ and consequently } \frac{\sin b}{\sin c}=\frac{\sin b'}{\sin c'} \dots (153)$$

This result will be seen to be analogous to Euc. VI. 3; and it will furnish the means of computing the segments when the sides are given: for, by composition and division, and by (24), we have

$$\frac{\tan\frac{1}{2}(b-c)}{\tan\frac{1}{2}(b+c)}=\frac{\tan\frac{1}{2}(b'-c')}{\tan\frac{1}{2}a} \dots (154)$$

121. If the exterior angle formed by producing b through A, were bisected by a great circle cutting the base produced in D′, and i

CD′$=b''$, and BD′$=c''$, we should have by (94), since half the exterior angle is the complement of $\frac{1}{2}$A,

$$\frac{\sin b}{\sin b''}=\frac{\sin \mathrm{D}'}{\cos\frac{1}{2}\mathrm{A}}, \text{ and } \frac{\sin c}{\sin c''}=\frac{\sin \mathrm{D}'}{\cos\frac{1}{2}\mathrm{A}}; \text{ whence,}$$

$$\frac{\sin b}{\sin b''}=\frac{\sin c}{\sin c''}, \text{ and, consequently, } \frac{\sin b}{\sin c}=\frac{\sin b''}{\sin c''} \ldots (155)$$

We should also find, by a process nearly the same as in the last No.

$$\frac{\tan\frac{1}{2}(b-c)}{\tan\frac{1}{2}(b+c)}=\frac{\tan\frac{1}{2}a}{\tan\frac{1}{2}(b''+c'')} \ldots (156)$$

122. Let AD (*fig.* 25) bisect the side BC, and let the angle BAD$=$B′, and CAD$=$C′, Then (94),

$$\frac{\sin c}{\sin\frac{1}{2}a}=\frac{\sin \mathrm{ADB}}{\sin \mathrm{B}'}, \text{ and } \frac{\sin b}{\sin\frac{1}{2}a}=\frac{\sin \mathrm{ADC}}{\sin \mathrm{C}'}.$$

Divide the latter of these by the former, and since (No. 15) sin ADB$=$ sin ADC, we get

$$\frac{\sin b}{\sin c}=\frac{\sin \mathrm{B}'}{\sin \mathrm{C}'} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (157)$$

From this we obtain, as in No. 120,

$$\frac{\tan\frac{1}{2}(b-c)}{\tan\frac{1}{2}(b+c)}=\frac{\tan\frac{1}{2}(\mathrm{B}'-\mathrm{C}')}{\tan\frac{1}{2}\mathrm{A}} \ldots (158)$$

The learner may exercise himself in extending the formulas obtained thus far in this section to the triangles formed by continuing the sides of the triangle ABC till they meet; and, by expressing the sides and angles of these triangles in terms of a, b, c, A, B, C, and π, he will find analogous, and, in several cases, interesting results.

123. Several formulas, remarkable for their symmetry, may be derived from (88), (89), (90), (93), and (104), (105), (106), (107), by finding the corresponding expressions in relation to b, c, B, and C. In investigating these, for the sake of brevity, put

$$\sqrt{\sin s \sin(s-a) \sin(s-b) \sin(s-c)}=n, \text{ and}$$
$$\sqrt{-\cos \mathrm{S} \cos(\mathrm{S}-\mathrm{A}) \cos(\mathrm{S}-\mathrm{B}) \cos(\mathrm{S}-\mathrm{C})}=\mathrm{N}.$$

124. Let now the continual product of the values of $\sin\frac{1}{2}$A, $\sin\frac{1}{2}$B, and $\sin\frac{1}{2}$C (88) be taken, and there will result, by actual extraction of the square root,

$$\sin\tfrac{1}{2}\mathrm{A} \sin\tfrac{1}{2}\mathrm{B} \sin\tfrac{1}{2}\mathrm{C}=\frac{\sin(s-a) \sin(s-b) \sin(s-c)}{\sin a \sin b \sin c} \ldots (159)$$

This, by multiplying the numerator and denominator by $\sin s$, becomes

$$\sin\tfrac{1}{2}A \sin\tfrac{1}{2}B \sin\tfrac{1}{2}C = \frac{n^2}{\sin s \sin a \sin b \sin c} \quad \ldots (160)$$

125. In a similar manner, from the values of $\cos\frac{1}{2}A$, &c. (89) we obtain

$$\cos\tfrac{1}{2}A \cos\tfrac{1}{2}B \cos\tfrac{1}{2}C = \frac{n \sin s}{\sin a \sin b \sin c} \quad \ldots\ldots (161)$$

126. In like manner, by multiplying together $\tan\frac{1}{2}A$, $\tan\frac{1}{2}B$, $\tan\frac{1}{2}C$ (90), and contracting, we find

$$\tan\tfrac{1}{2}A \tan\tfrac{1}{2}B \tan\tfrac{1}{2}C = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin^3 s}} \quad \ldots (162)$$

Multiply the numerator and denominator by $\sqrt{\sin s}$; then,

$$\tan\tfrac{1}{2}A \tan\tfrac{1}{2}B \tan\tfrac{1}{2}C = \frac{n}{\sin^2 s} \quad \ldots\ldots\ldots (163)$$

127. In like manner, from (93), we find

$$\sin A \sin B \sin C = \frac{8n^3}{\sin^2 a \sin^2 b \sin^2 c} \quad \ldots\ldots (164)$$

128. By processes exactly similar, we obtain from (104), (105), (106), and (107),

$$\sin\tfrac{1}{2}a \sin\tfrac{1}{2}b \sin\tfrac{1}{2}c = \frac{-N\cos S}{\sin A \sin B \sin C} \quad \ldots\ldots\ldots (165)$$

$$\cos\tfrac{1}{2}a \cos\tfrac{1}{2}b \cos\tfrac{1}{2}c = \frac{\cos(S-A)\cos(S-B)\cos(S-C)}{\sin A \sin B \sin C} \quad \ldots (166)$$

$$\cos\tfrac{1}{2}a \cos\tfrac{1}{2}b \cos\tfrac{1}{2}c = \frac{N^2}{-\cos S \sin A \sin B \sin C} \quad \ldots (167)$$

$$\tan\tfrac{1}{2}a \tan\tfrac{1}{2}b \tan\tfrac{1}{2}c = \sqrt{\frac{-\cos^3 S}{\cos(S-A)\cos(S-B)\cos(S-C)}} = \frac{\cos^2 S}{N} \quad (168)$$

$$\sin a \sin b \sin c = \frac{8N^3}{\sin^2 A \sin^2 B \sin^2 C} \quad \ldots\ldots\ldots (169)$$

129. From (164) and (169) we find

$$n = \tfrac{1}{2}(\sin^2 a \sin^2 b \sin^2 c \sin A \sin B \sin C)^{\frac{1}{3}} \quad \ldots (170)$$

$$N = \tfrac{1}{2}(\sin^2 A \sin^2 B \sin^2 C \sin a \sin b \sin c)^{\frac{1}{3}} \quad \ldots (171)$$

130. Divide (170) by (171); then,

$$\frac{n}{N} = \left\{\frac{\sin a \sin b \sin c}{\sin A \sin B \sin C}\right\}^{\frac{1}{3}} = \frac{\sin a^*}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \quad \ldots. (172)$$

---

* This expression and the two following, are obtained from the preceding radical, on the principle (94), that

$$\left\{\frac{\sin a \sin b \sin c}{\sin A \sin B \sin C}\right\}^{\frac{1}{3}} = \left\{\frac{\sin^3 a}{\sin^3 A}\right\}^{\frac{1}{3}} \quad \left\{\frac{\sin^3 b}{\sin^3 B}\right\}^{\frac{1}{3}} = \left\{\frac{\sin^3 c}{\sin^3 C}\right\}^{\frac{1}{3}}.$$

Hence, also, $\frac{N}{\sin A}=\frac{n}{\sin a}$, $\frac{N}{\sin B}=\frac{n}{\sin b}$, and $\frac{N}{\sin C}=\frac{n}{\sin c}$...... (173)

131. From (93) and (107), we have

$$\frac{\sin A}{\sin a}=\frac{2n}{\sin a\sin b\sin c}, \text{ and } \frac{\sin a}{\sin A}=\frac{2N}{\sin A\sin B\sin C}:$$

and, by equalling the second member of one of these with the reciprocal of the second member of the other, we get, by multiplying by the denominators,

$$4nN=\sin a\sin b\sin c\sin A\sin B\sin C \text{ .... (174)}$$

132. In this, substitute for sin A sin B sin C its value in (164); and, by contracting, there will arise

$$N=\frac{2n^2}{\sin a\sin b\sin c} \text{ ...... (175)}$$

By a like process we find, by means of (169),

$$n=\frac{2N^2}{\sin A\sin B\sin C} \text{ .... (176)}$$

133. From (167) by substituting $\frac{n^2}{\sin b\sin c}$ for $\frac{N^2}{\sin B\sin C}$, according to No. 130, and by clearing the result of fractions, we obtain

$$-\cos S\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c\sin b\sin c\sin A=n^2.$$

But (93) $\sin b\sin c\sin A=2n$; by substituting which in the first member, and dividing by $2n\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c$, we obtain, by restoring the value of n,

$$-\cos S=\frac{\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}=\frac{n}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c} \text{ (177)}$$

134. By a similar process we should obtain, from (160),

$$\sin s=\frac{\sqrt{-\cos S\cos(S-A)\cos(S-B)\cos(S-C)}}{2\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}=\frac{N}{2\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C} \text{... (178)}$$

135. A curious formula, given by Cagnoli, may be thus investigated: Transpose the first term of the right-hand member of (85); multiply the result by cos A, and to both members of the product add $\sin b\sin c$: then, the first member will contain $\sin b\sin c-\sin b\sin c\cos^2 A$; and, substituting for this its equal, $\sin b\sin c\sin^2 A$, we obtain

$$\cos A\cos a+\sin b\sin c\sin^2 A=\sin b\sin c+\cos b\cos c\cos A \text{ ...... } (z)$$

From (101), also, we find, by a similar process,

$$\cos A\cos a+\sin B\sin C\sin^2 a=\sin B\sin C-\cos B\cos C\cos a.$$

In the second term of this, substitute, according to (94), $\sin b \sin A$ and $\sin c \sin A$, for $\sin B \sin a$ and $\sin C \sin a$, and there will arise

$$\cos A \cos a + \sin b \sin c \sin^2 A = \sin B \sin C - \cos B \cos C \cos a.$$

Hence, by comparing this equation and equation (z), we obtain

$$\sin b \sin c + \cos b \cos c \cos A = \sin B \sin C - \cos B \cos C \cos a^* \ \ldots \ (179)$$

136. By multiplying (85) by $\cos B \cos C$, and (101) by $\cos b \cos c$, and by adding the results together, we obtain, after transposition,

$$\cos a \cos B \cos C - \cos A \cos B \cos C \sin b \sin c = -\cos A \cos b \cos c + \cos a \cos b \cos c \sin B \sin C \ (180)$$

This formula, in common with (179) and other formulas of a similar kind, possesses the property of not being changed by the substitution of $\pi - A$ for a, $\pi - B$ for b, &c. It is the same, therefore, in triangles that are supplementary to each other.

137. If we now multiply (85) by $\cos a \sin B \sin C$, and (101) by $\cos A \sin b \sin c$, and subtract the latter product from the former, we shall have, after dividing by $\sin b \sin c$,

$$\frac{\cos^2 a \sin B \sin C}{\sin b \sin c} - \cos^2 A = \frac{\cos a \cos b \cos c \sin B \sin C}{\sin b \sin c} + \cos A \cos B \cos C.$$

In this, according to (94), substitute $\frac{\sin^2 A}{\sin^2 a}$ for $\frac{\sin B \sin C}{\sin b \sin c}$: then, by writing $1 - \sin^2 a$ for $\cos^2 a$, there will arise (6)

$$\frac{\sin^2 A}{\sin^2 a} - 1 = \frac{\cos a \cos b \cos c \sin^2 A}{\sin^2 a} + \cos A \cos B \cos C\,;$$

or, by substituting $\frac{N^2}{n^2}$ for $\frac{\sin^2 A}{\sin^2 a}$, according to (172), by multiplying by n^2, and, first, by transposition, and then by division,

$$\left.\begin{array}{l} N^2(1 - \cos a \cos b \cos c) = n^2(1 + \cos A \cos B \cos C), \\ \text{and } \dfrac{N^2}{n^2} = \dfrac{1 + \cos A \cos B \cos C}{1 - \cos a \cos b \cos c} \end{array}\right\} \ldots\ldots \ (181)$$

From the former of these we get, by transposition,

$$N^2 - n^2 = N^2 \cos a \cos b \cos c + n^2 \cos A \cos B \cos C.$$

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* This formula contains all the six parts of the spherical triangle, and is remarkable for its symmetry; the one member containing the sines or cosines of three parts, and the other the respective sines or cosines of the parts respectively opposite, and the only difference being in one of the signs. Cagnoli's investigation is founded on nearly the same principles as that given above, but is perhaps, scarcely so simple. Delambre investigates it by finding two expressions for the same line—one by means of three parts of the triangle, and the other by means of the remaining parts. It is scarcely necessary to observe, that two corresponding formulas might be obtained from (86), (87), and (102), (103), or from the preceding by changing the letters. Delambre gives also various other formulas containing each the three sides and the three angles.

Now, by (18), $\cos b\ \cos c=\frac{1}{2}\cos(b+c)+\frac{1}{2}\cos(b-c)$; and multiplying this by $\cos a$, and applying the same formula, we find

$$\cos a\ \cos b\ \cos c=\tfrac{1}{4}\cos(a+b+c)+\tfrac{1}{4}\cos(-a+b+c)+\tfrac{1}{4}\cos(a-b+c)+\tfrac{1}{4}\cos(a+b-c), \text{ or}$$
$$\cos a\ \cos b\ \cos c=\tfrac{1}{4}\cos 2s+\tfrac{1}{4}\cos 2(s-a)+\tfrac{1}{4}\cos 2(s-b)+\tfrac{1}{4}\cos 2(s-c).$$

Introducing this, and the corresponding value of $\cos A\ \cos B\ \cos C$, into the foregoing value of N^2-n^2, dividing the result by N^2 and n^2, and restoring the values of these quantities in the second member, we obtain

$$\frac{1}{n^2}-\frac{1}{N^2}=\frac{1}{4}\left\{\begin{array}{l}\dfrac{\cos 2s+\cos 2(s-a)+\cos 2(s-b)+\cos 2(s-c)}{\sin s\ \sin(s-a)\ \sin(s-b)\ \sin(s-c)}\\ +\dfrac{\cos 2S+\cos 2(S-A)+\cos 2(S-B)+\cos 2(S-C)}{-\cos S\ \cos(S-A)\ \cos(S-B\ \cos(S-C)}\end{array}\right\} \ldots (182)$$

By alternately adding and subtracting unity in the numerators of this formula, we obtain, by (31) and (32),

$$\frac{1}{n^2}-\frac{1}{N^2}=\frac{1}{2}\left\{\begin{array}{l}\dfrac{\cos^2 s-\sin^2(s-a)+\cos^2(s-b)-\sin^2(s-c)}{\sin s\ \sin(s-a)\ \sin(s-b)\ \sin(s-c)}\\ +\dfrac{\cos^2 S-\sin^2(S-A)+\cos^2(S-B)-\sin^2(S-C)}{-\cos S\ \cos(S-A)\ \cos(S-B)\ \cos(S-C)}\end{array}\right\} (183)$$

It is evident that this formula will admit of several variations, according to the parts of the numerators which are increased or diminished by unity; and it would be easy to introduce various other modifications.*

138. The foregoing formulas would furnish other expressions for n and N, and values for $\sin s$, $\cos S$, &c. As the investigations of such expressions, however, present no considerable difficulty, they are left for exercise to the pupil. We may now proceed, therefore, to investigate the methods of determining the inscribed and circumscribed circles; and we shall thus obtain some curious and interesting results.

139. Let D (*fig.* 26) be the pole of a circle inscribed in the spherical triangle ABC, and E, F, G, the points of contact; and let DA, DE, DF, DG, be arcs of great circles. Then the angles at E, F, and G, are right angles; and, since DF is equal to DG, and DA common to the triangles ADF, ADG, the sides AF, AG are (113) equal, as also (112) the angles DAF, DAG. In like manner, it would appear that BE=BG, and CE=CF. Hence, AF+BE+EC is half the sum of the sides, that is, $AF+a=s$; and consequently $AF=s-a$.

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* The formulas in this and the preceding No. the author has never before met with. They are curious, particularly those in the latter No. on account of their complete symmetry.

Now, if DF$=r$, the rightangled triangle AFD gives (112) tan DF$=$ sin AF tan DAF, or, by (90) and No. 123,

$$\tan r=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}=\frac{n}{\sin s}\ \ldots\ldots\ (184)$$

140. Dividing n in this formula by the value of $\sin s$ found from (161), we get

$$\frac{n}{\sin s}\text{ or }\tan r=\frac{n^2}{\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C\sin a\sin b\sin c}.$$

In this, for n^2 substitute its value according to (175); then, by contracting, there will be obtained

$$\tan r=\frac{N}{2\sin\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C},\text{ or }\cot r=\frac{2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}{N}\ ..(185)$$

141. If three circles were described, each touching one of the sides externally and the other two produced, and if r' were put to denote the arc drawn from the pole to the circumference of the one touching a externally, and r'' and r''' the like arcs in those touching b and c externally, it would be found, in nearly the same manner as in No. 139, that

$$\tan r'=\sqrt{\frac{\sin s\sin(s-b)\sin(s-c)}{\sin(s-a)}}{}^{*}=\frac{n}{\sin(s-a)}\ \ldots\ldots\ (186)$$

$$\tan r''=\sqrt{\frac{\sin s\sin(s-a)\sin(s-c)}{\sin(s-b)}}=\frac{n}{\sin(s-b)}\ \ldots\ldots\ (187)$$

$$\tan r'''=\sqrt{\frac{\sin s\sin(s-a)\sin(s-b)}{\sin(s-c)}}=\frac{n}{\sin(s-c)}\ \ldots\ldots\ (188)$$

142. By taking the continual product of the members of (184), (186), (187), and (188), we find the following remarkable symmetrical formula:

$$\tan r\tan r'\tan r''\tan r'''=\sin s\sin(s-a)\sin(s-b)\sin(s-c)=n^2.\ (189)$$

143. From D (*fig.* 27), the pole of the circle described about the triangle ABC, let arcs of great circles be drawn to the several angles: let, also, the arc DE be drawn perpendicular to the side a. Then, the triangles ADB, ADC, and BDC being isosceles, the angles at their bases are equal. Hence, ABD$+$ACD$=$A; and therefore DBC$+$DCB$=$B$+$C$-$A, and DBE$=\frac{1}{2}$(B$+$C$-$A)$=$S$-$A.

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* This formula may be easily found from considering that a, and the continuations of b and c, form a triangle having for sides a, $\pi-b$, and $\pi-c$: as half the sum of these is $\pi-\frac{1}{2}(b+c-a)$ or $\pi-(s-a)$; and the remainders found, by taking from this a, $\pi-b$, and $\pi-c$, are $\pi-s$, $s-c$, and $s-b$: the substitution of which in (184), for s, $s-a$, $s-b$, and $s-c$, gives (186). In a similar manner, also, (187) and (188) might be obtained.

Now, by (114) or No. 85, we have, in the rightangled triangle BED, $\cos DBE = \tan BE \cot BD$; which, by multiplying both members by $\tan BD$, dividing by $\cos DBE$, and putting $BD = R$, gives, by note to (5),

$$\tan R = \frac{\tan BE}{\cos DBE} = \frac{\tan \frac{1}{2} a}{\cos (S - A)}; \text{ or, by (106),}$$

$$\tan R = \sqrt{\frac{-\cos S}{\cos (S-A) \cos (S-B) \cos (S-C)}} = \frac{-\cos S}{N} \quad \dots (190)$$

144. The foregoing formula gives the value of R by means of the three angles. To find it in terms of the sides, divide the value of $-\cos S$ found from (165) by N; then,

$$\frac{-\cos S}{N}, \text{ or } \tan R = \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c \sin A \sin B \sin C}{N^2}.$$

In this, substitute for N^2 its value according to (176), and there will result

$$\tan R = \frac{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c^*}{n}, \text{ or } \cot R = \frac{n}{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}. (191)$$

145. If circles were described about the three triangles formed by continuing the sides, and if R' were put to denote, in the triangle having a for a side, the arc corresponding to R; and R'' and R''' the corresponding arcs in the triangles having b and c as sides, we should have the following formulas analogous to those in No. 141:

$$\tan R' = \sqrt{\frac{\cos (S-A)}{-\cos S \cos (S-B) \cos (S-C)}} = \frac{\cos (S-A)}{N} \quad \dots (192)$$

$$\tan R'' = \sqrt{\frac{\cos (S-B)}{-\cos S \cos (S-A) \cos (A-C)}} = \frac{\cos (S-B)}{N} \quad \dots (193)$$

$$\tan R''' = \sqrt{\frac{\cos (S-C)}{-\cos S \cos (S-A) \cos (S-B)}} = \frac{\cos (S-C)}{N} \quad \dots (194)$$

146. Take the continual product of these and of (190); then,

$$\tan R \tan R' \tan R'' \tan R''' = \frac{1}{N^2}, \text{ or } \cot R \cot R' \cot R'' \cot R''' = N^2 \dots (195)$$

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* If p be the perpendicular from A to BC, we have $\sin p = \sin C \sin b$, or (93) $\sin p = \frac{2n}{\sin a} = \frac{n}{\sin \frac{1}{2} a \cos \frac{1}{2} a}$. Hence, from (191), by multiplying, we get

$$\sin p \tan R = \frac{2 \sin \frac{1}{2} b \sin \frac{1}{2} c}{\cos \frac{1}{2} a}.$$

In a plane triangle, this becomes (by Nos. 162 and 163) $pR = \frac{1}{2} bc$; from which, by doubling, and putting D for 2R, we get $pD = bc$, the same as Euclid, VI. D.

147. Since the sides of the supplementary triangle are $\pi-A$, $\pi-B$, $\pi-C$, by substituting these in (184) instead of a, b, c, we should obtain, in respect to the circle inscribed in that triangle,

$$\tan r=\sqrt{\frac{\cos(S-A)\cos(S-B)\cos(S-C)}{-\cos S}}, \text{ or}$$

$$\cot r=\sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}}.$$

Now, this value of $\cot r$ being the same as that of $\tan R$ (190), it follows that r and R are complements of each other; whence it appears, that *the arc drawn from the pole to the circumference of the circle inscribed in a spherical triangle, is the complement of the arc drawn from the pole to the circumference of the circle described about the polar triangle*, which seems to be a new relation of these triangles.

148. To investigate the method of finding the area of a spherical triangle ABC (*fig.* 28), let the circle of which AB, one of the sides, is an arc, be completed, and let the continuations of AC and BC cut its circumference in D and E, and intersect each other in F, on the other hemisphere. Then, since (page 26, note 3) BCE and CEF are semicircles, BC and EF are equal; and for a similar reason, AC and DF are equal. We have also the angles F and C equal, as they are each the inclination of the planes of AC and BC. Hence the triangles ABC, DEF are equal. Now, the lune, bounded on the surface of a sphere by the halves of two great circles is evidently proportional to the angle at which those circles are inclined. Hence, if H denote the surface of a hemisphere, we have $180^\circ : A :: H :$ lune $ABDC=\frac{H.A}{180^\circ}$. In like manner we should find the lune $BAEC=\frac{H.B}{180^\circ}$, and the lune $CEFD=\frac{H.C}{180^\circ}$. Now, it is evident that the sum of these lunes is equal to the surface of the hemisphere AEDB, and twice that of the triangle ABC; that is, $\frac{H(A+B+C)}{180^\circ}$ $=H+2\,ABC$; whence we readily find $ABC=\frac{\frac{1}{2}H(A+B+C-180^\circ)}{180^\circ}$

Now (Diff. and Int. Calc. No. 288), the surface of a hemisphere is equal to twice the surface of a great circle of the sphere, or to $2\pi r^2$, where r is the radius of the sphere. Putting T, therefore, equal to the area of ABC, we have

$$T=\frac{\pi r^2(A+B+C-180^\circ)}{180^\circ} \quad \ldots\ldots\ldots\ldots (196)$$

Hence $180^\circ : A+B+C-180^\circ :: \pi r^2 :$ area of ABC; and it appears, therefore, that *(in the same sphere) the area of a spherical triangle is proportional to the excess of the sum of its angles above two right angles, or to what is called its* SPHERICAL EXCESS.* It is also plain, that, when the spherical excess or the sum of the three angles is known, the area can be determined from it; and that all triangles on the same sphere which have the same spherical excess, are equal, however dissimilar they may be.

149. By arcs of great circles drawn from any angle of a spherical polygon to all the remote angles, we may divide it into as many triangles as it has sides, wanting two. Hence, if n be the number, and S the sum of its angles, we shall have the sum of the spherical excesses of the triangles $= S-(n-2)\times 180^\circ$; and consequently, putting P to represent the area of the polygon, we have

$$180^\circ : S-(n-2)\times 180^\circ :: \pi r^2 : P \quad \ldots\ldots (197)$$

150. When the data in a spherical triangle are not the three angles, the angles, or their sum, must be found by some of the methods explained in Section IV., before (196) can be applied in finding the area. Thus, if the three sides be given, the angles may be computed by Nos. 93 and 94; or if two sides and the contained angle be given, the half sum, and thence the sum of the remaining angles, may be found by (129). In these two cases, however, which most frequently occur, formulas have been discovered which give the spherical excess without the previous determination of the angles.

Thus, putting S equal to $\frac{1}{2}(A+B+C)$, and E equal to the spherical excess, we have $\frac{1}{2}E = S-90^\circ$, $\sin\frac{1}{2}E = -\cos S$, $\cos\frac{1}{2}E = \sin S$; and $\tan S = -\cot\frac{1}{2}E$; and, since (108) $\tan\frac{1}{2}(A+B) = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}.\cot\frac{1}{2}C$, we have, from (39), by taking $\frac{1}{2}(A+B)$ as one arc, and $\frac{1}{2}C$ as another, and by multiplying the numerator and denominator of the second member by $\cos\frac{1}{2}(a+b)$,

$$\tan\tfrac{1}{2}(A+B+C) = -\cot\tfrac{1}{2}E = \frac{\cos\frac{1}{2}(a-b)\cot\frac{1}{2}C + \cos\frac{1}{2}(a+b)\tan\frac{1}{2}C}{\cos\frac{1}{2}(a+b) - \cos\frac{1}{2}(a-b)}.$$

By multiplying again the numerator and denominator of the second

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* This remarkable and beautiful theorem is ascribed to Albert Girard, a Flemish mathematician. In 1787, upwards of 150 years after its discovery, an important and ingenious application of it was made by General Roy, in correcting the angles observed in the Trigonometrical Survey of Britain;—a fact which proves, with many others, that no principle should be rejected as useless, however long it may have been known merely as a speculative truth.

member of this by $2\sin\frac{1}{2}C\cos\frac{1}{2}C$, and then by modifying the numerator by (31) and (32), and the denominator by (30), we obtain, by changing the signs,

$$\cot\tfrac{1}{2}E=\frac{\cos\frac{1}{2}(a-b)+\cos\frac{1}{2}(a+b)+\{\cos\frac{1}{2}(a-b)-\cos\frac{1}{2}(a+b)\}\cos C}{\{\cos\frac{1}{2}(a-b)-\cos\frac{1}{2}(a+b)\}\sin C}.$$

From this, by modifying the denominator and the first and second terms of the numerator by (25) and (5), and dividing the denominator and the rest of the numerator by their common factor, we obtain

$$\cot\tfrac{1}{2}E=\frac{\cot\frac{1}{2}a\cot\frac{1}{2}b+\cos C^{*}}{\sin C} \quad \ldots\ldots\ldots\ldots\ldots\ (198)$$

151. If the three sides be given, we have (177) —cos S, or

$$\sin\tfrac{1}{2}E=\frac{\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c} \ \ldots\ (199)$$

152. From (198) we may eliminate cos C and sin C. To do this, multiply the first term of the numerator by $2\sin\frac{1}{2}a\cos\frac{1}{2}a.2\sin\frac{1}{2}b\cos\frac{1}{2}b$, and the denominator and the remaining term of the numerator by what is equivalent, $\sin a\sin b$, and there will arise

$$\cot\tfrac{1}{2}E=\frac{2\cos^2\frac{1}{2}a.\,2\cos^2\frac{1}{2}b+\sin a\sin b\cos C}{\sin a\sin b\sin C}.$$

Now, in this, the first term of the numerator is (31) equivalent to $(1+\cos a)(1+\cos b)$; the second, by (87), to $\cos c-\cos a\cos b$; and the denominator, by (93), to $2n$. Substituting these, therefore, performing the actual multiplication, and contracting, we obtain

$$\cot\tfrac{1}{2}E=\frac{1+\cos a+\cos b+\cos c}{2\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}} \ \ldots\ldots\ (200)$$

153. The product of this and (199) is

$$\cos\tfrac{1}{2}E=\frac{1+\cos a+\cos b+\cos c}{4\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}=\frac{\cos^2\frac{1}{2}a+\cos^2\frac{1}{2}b+\cos^2\frac{1}{2}c-1}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}\ ..(201)$$

The second of these expressions is obtained by putting the numerator of the first under the form $1+\cos a+1+\cos b+1+\cos c-2$, and modifying it by (31).

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* This may be derived more easily, though not so conformably to a just analysis, in the following manner: By taking the reciprocals of the members of equation (c) in the note to No. 82, expanding cos(S—C), and performing the actual division by cos C we get

$$\cot\tfrac{1}{2}a\cot\tfrac{1}{2}b=-\cos C-\tan S\sin C:$$

and, by resolving this for —tan S, or its equal $\cot\frac{1}{2}E$, we find (198).

154. By subtracting the members of (201) from unity, and dividing the remainders by the members of (199), we obtain, by (49),

$$\tan\tfrac{1}{4}\mathrm{E}=\frac{1-\cos^2\frac{1}{2}a-\cos^2\frac{1}{2}b-\cos^2\frac{1}{2}c+2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}{\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}}.$$

Now, the numerator of this is the product of the two factors,

$\cos\frac{1}{2}a-\cos\frac{1}{2}b\cos\frac{1}{2}c+\sin\frac{1}{2}b\sin\frac{1}{2}c$, and
$-\cos\frac{1}{2}a+\cos\frac{1}{2}b\cos\frac{1}{2}c+\sin\frac{1}{2}b\sin\frac{1}{2}c$; or (14) and (15)
$\cos\frac{1}{2}a-\cos\frac{1}{2}(b+c)$, and $-\cos\frac{1}{2}a+\cos\frac{1}{2}(b-c)$;

and these again are (23) equivalent to

$2\sin\frac{1}{4}(a+b+c)\sin\frac{1}{4}(-a+b+c)$, and
$2\sin\frac{1}{4}(a-b+c)\sin\frac{1}{4}(a+b-c)$; or
$2\sin\frac{1}{2}s\sin\frac{1}{2}(s-a)$, and $2\sin\frac{1}{2}(s-b)\sin\frac{1}{2}(s-c)$.

Substituting these in the numerator, modifying the denominator by (30), and dividing the numerator by the denominator, we finally obtain

$$\tan\tfrac{1}{4}\mathrm{E}=\sqrt{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)} \quad \ldots. \ (202)$$

This beautiful formula, which gives the spherical excess by means of the three sides, was discovered by Lhuillier of Geneva.

155. If the base and area, or, which is the same, (196) if the base and the sum of the angles of a triangle be given, the locus of the vertex may be found in the following manner. In the circle BCB′C′, (*fig.* 27) take BC equal to the given base a, and BCB′, CBC′, each equal to a semicircle; and make the angles BC′D′, CB′D′ each equal to half the sum of the given angles: then a small circle described through B′ and C′, and having D′ as its pole, will be the locus required; that is, the vertex A may be taken any where on its circumference. For, drawing BAB′, CAC′, and AD′, arcs of great circles, we have (page 26, note) the angles AB′D′, AC′D′ together equal to B′AC′, or its equal A in the triangle ABC; and BB′C, CC′B are respectively equal to B and C. Therefore, wherever A is taken on the circumference of the small circle, the sum of the three angles A, B, C, is equal to the sum of the angles CB′B, AB′D′, BC′C, and AC′D′, which, by construction, are together equal to the given sum of the angles.*

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* This curious proposition was discovered by Lexell, and first appeared in the Petersburgh Acts. The proof given above is more simple and easy than any other that the author has seen. It would be shown in a similar manner, that if the base B′C′, and the area of the triangle AB′C′, be given, the circle described about ABC is the locus of its vertex.

156. It is evident from No. 143, that if the base of a spherical triangle and the difference between the vertical angle and the sum of the other two be given, the locus of the vertex is the circumference of the circle described about the triangle.

157. In resolving rightangled triangles by the methods already explained (Nos. 84 and 85), when the required quantity is to be found by its sine or cosine, if the sine or cosine be nearly equal to the radius, the quantity required cannot be found from the ordinary tables with much accuracy. This practical inconvenience may be obviated in different ways according to circumstances. Thus, from (112), (113), and (114), we have, by (19), (18), and (16),

$$\sin a = \tfrac{1}{2}\{\cos(A \backsim c) - \cos(A+c)\} \quad \ldots\ldots\ldots\ (203)$$
$$\cos c = \tfrac{1}{2}\{\cos(a \backsim b) + \cos(a+b)\} \quad \ldots\ldots\ldots\ (204)$$
$$\cos A = \tfrac{1}{2}\{\sin(a+B) - \sin(a-B)\} \quad \ldots\ldots\ldots\ (205)$$

Any of these will give the required results by addition and subtraction, by means of natural sines and cosines; and tables of these give the results in the circumstances under consideration, with more precision than can be attained by means of the other tables.

This inconvenience may also be obviated by first finding a part not required; and then, by means of it and one of the given parts, the required part may be determined. Thus, if the legs were given, to find the hypotenuse, and if the legs, and consequently the hypotenuse, were very small, instead of using the common formula (113) $\cos c = \cos a \cos b$, we might first find A from the equation (112) $\sin b = \cot A \tan a$, and then c from the equation (114) $\cos A = \tan b \cot c$; in both of which the results may be obtained with great accuracy.

158. Some of the formulas may be modified so as to give the final results very easily and accurately by means of tangents. To exemplify this in some of the most useful instances, let A and a be given to find B: then we derive from (114) the analogy, $1 : \sin B :: \cos a : \cos A$; whence, by composition and division,

$$\frac{1-\sin B}{1+\sin B} = \frac{\cos a - \cos A}{\cos a + \cos A}.$$

Now, since $\sin B = \cos(90^\circ - B)$, this becomes, by (32), (31), and (25), and by extracting the square root,

$$\tan(45^\circ - \tfrac{1}{2}B) = \pm\sqrt{\tan\tfrac{1}{2}(A+a)\tan\tfrac{1}{2}(A-a)} \quad \ldots\ (206)$$

The arc thus found may be either positive or negative. Calling it, therefore $\pm\Phi$, we shall have $B = 90^\circ \pm 2\Phi$. The value of B may thus be found with great accuracy, and the formula (206) is adapted to computation by logarithms.

159. We have also by (112), $1 : \sin A :: \sin c : \sin a$; and by processes exactly similar, we get

$$\tan(45^{\circ}-\tfrac{1}{2}c)=\pm\sqrt{\frac{\tan\frac{1}{2}(A-a)}{\tan\frac{1}{2}(A+a)}} \quad \text{......... (207)}$$

$$\tan(45^{\circ}-\tfrac{1}{2}A)=\pm\sqrt{\frac{\tan\frac{1}{2}(c-a)}{\tan\frac{1}{2}(c+a)}} \quad \text{......... (208)}$$

160. From (113) we have $1 : \cos a :: \cos b : \cos c$; whence, by a like process, we get

$$\tan\tfrac{1}{2}a=\sqrt{\tan\tfrac{1}{2}(c+b)\tan\tfrac{1}{2}(c-b)} \quad \text{........... (209)}$$

161. We have likewise, from (112), (113), and (114),

$\sin a=\tan b \cot B$, $\cos c=\cot A \cot B$, and $\cos A=\tan b \cot c$.

Subtracting the members of the first of these from unity, and also adding them to it, we obtain, by dividing the remainders by the sums,

$$\frac{1-\sin a}{1+\sin a}=\frac{1-\tan b \cot B}{1+\tan b \cot B}.$$

The first member of this may be reduced as in former instances; and, by multiplying the numerator and denominator of the second member by $\cos b \sin B$, we obtain by (13) and (12), and by extracting the square root,

$$\tan(45^{\circ}-\tfrac{1}{2}a)=\pm\sqrt{\frac{\sin(B-b)}{\sin(B+b)}} \quad \text{..... (210)}$$

By operations nearly similar, we obtain, from the other two formulas,

$$\tan\tfrac{1}{2}c=\sqrt{\frac{-\cos(A+B)}{\cos(A-B)}} \quad \text{............... (211)}$$

$$\tan\tfrac{1}{2}A=\sqrt{\frac{\sin(c-b)}{\sin(c+b)}} \quad \text{..................... (212)}$$

162. If, while the sides of a spherical triangle retain always the same absolute length, the radius of the sphere continually increase, the triangle will become more and more nearly a plane one, and it would actually become such, if the radius were infinite. In this case, the side and its sine and tangent would coincide and become equal, while the cosine would become equal to the radius, and consequently infinite. The cotangent would also be infinite. From this it follows, that any formula respecting a spherical triangle, which contains the sines or tangents of the sides, or the sines or tangents of their sum, half sum, &c. will also hold respecting a plane triangle, if the terms

sine and tangent, thus used, be omitted, so as to leave simply the side instead of its sine or tangent. In this way, the theorem in No. 76 becomes the same as that in No. 21; and (88), (89), (90), and (93) are reduced to (79), (80), (81), and (84). Since, also, in the plane triangle, $\tan\frac{1}{2}(A+B)=\cot\frac{1}{2}C$, (109) and (151) by slight reductions become the same as (75). The same principle, also, reduces (91) and (92) to (82) and (83), and (110) and (111) to (76) and (77). So, likewise, (117), (121), (149), (152), (153), (154), (155), (156), (157), (158), (159), (160), (161), (162), (164), and many others, all give results which are true respecting plane triangles; and (202) becomes, by a slight reduction, the same as the expression for the area of a plane triangle, given in the note in page 20.

163. The formulas in (112) and (114) become immediately applicable to plane triangles by taking $\cos a$ equal to the radius, and by multiplying in the latter by $\cot c$. By the same means, the first formula in (113) gives $\tan A=\cot B$. The second, by squaring, and by (6), becomes $1-\sin^2 c=1-\sin^2 a-\sin^2 b+\sin^2 a \sin^2 b$. Hence, rejecting 1, changing the signs, and rendering the terms of the same dimensions by dividing the last term by r^2, that term, in case of the plane triangle, will vanish, the radius being infinite, and the formula will be reduced to $c^2=a^2+b^2$, the same as Euclid I. 47.

164. By a like substitution, (85) would become

$$\sqrt{(1-\sin^2 a)}=\cos A \sin b \sin c+\sqrt{(1-\sin^2 b-\sin^2 c+\sin^2 b \sin^2 c)}.$$

The first member of this, by extracting the square root in series, becomes

$$1-\tfrac{1}{2}\sin^2 a-\tfrac{1}{8}\sin^4 a-\&c.;$$

while the second, by a like process, is changed into

$$\cos A \sin b \sin c+1-\tfrac{1}{2}\sin^2 b-\tfrac{1}{2}\sin^2 c+\tfrac{1}{4}\sin^2 b \sin^2 c-\tfrac{1}{8}\sin^4 b-\&c.$$

Hence, by rejecting 1, dividing the terms of four dimensions by r^2, multiplying by 2, and otherwise proceeding as in the last No. we get $a^2=b^2+c^2-2bc\cos A$, the same as (78).

VI.—ASTRONOMICAL AND GEOGRAPHICAL PROBLEMS.

165. SPHERICAL Trigonometry derived its origin from the computations which are necessary in astronomy; and its principal and most important applications are still furnished by the same science. Some of the most useful of these, and some of its applications in geography, are exhibited in the following problems.*

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* For the use of those who may be unacquainted with astronomy, or the mathematical principles of geography, it may be proper to explain some of the terms which most frequently occur in those branches of science.

The astronomer conceives an imaginary sphere of great, though indefinite magnitude, to have the same centre as the earth; and, in all applications of spherical trigonometry, he employs, instead of the real position of any of the heavenly bodies, the point in which the surface of this sphere is cut by a straight line joining the centres of the earth and the body. (1) The extremities of the axis on which this sphere appears to revolve, in consequence of the earth's diurnal rotation, are called the *poles of the celestial sphere*, or *of the heavens;* and (2) the points in which the earth's surface is cut by its axis, are called the *poles of the earth.* (3) The great circle on the imaginary sphere which has the poles of the heavens as its poles, is called the *celestial equator* or the *equinoctial;* and (4) the corresponding great circle on the surface of the earth is called the *terrestrial equator.* Hence it is evident, that the celestial and terrestrial equators are in the same plane. (5) In consequence of the earth's motion in its orbit, the sun appears to move eastward round the heavens, and, in the course of a year, to describe, among the fixed stars, a great circle which is named the *ecliptic.* (6) This circle is inclined to the equator at an angle which is called the *obliquity of the ecliptic*, and which, in the lapse of many centuries, varies within narrow limits. On the first of January, 1838, the mean obliquity was $23^{0}\ 27'\ 37''$, and at present it is diminishing at the very slow rate of $52''$ in a century; not, however, each year by the same quantity; and it increases and diminishes by minute quantities at different times of the year. (7) The points in which the equator and ecliptic intersect each other, are called the *equinoctial points;*—the one at which the sun appears to cross the equator northward, the *vernal equinoctial point*, or the *first point of Aries;* the other, the *autumnal equinoctial point*, or the *first point of Libra.* (8) A *secondary* to a great circle is another great circle perpendicular to it. (9) If a secondary to the equator be drawn through any point of the celestial sphere, the part of it between the point and the equator, is called the *declination* of that point; and (10) the part of the equator extending eastward from the first point of Aries to the secondary, is called the *right ascension* of that point. (11) In like manner, if a secondary to the ecliptic be drawn through any point of the sphere, the part of it between the point and the ecliptic is called the *latitude* of the point; and (12) the part of the ecliptic extending eastward from the first point of Aries to the secondary, is called the *longitude* of the point. This was formerly reckoned in *signs* of 30^{0} each, and of which the ecliptic must therefore contain twelve. It is now reckoned in degrees.

166. Given the obliquity of the ecliptic, and the right ascension and declination of any of the heavenly bodies; to find its latitude and longitude.

In geography, (13) a secondary to the equator passing through any place, is called the *meridian* of that place; and (14) the *longitude of a place* is the smaller of the two parts into which the equator is divided by the *meridian* of the place and the *first meridian*, that is, the meridian from which geographers have agreed to reckon the longitude. The British geographers assume the meridian of the Observatory of Greenwich as the first meridian; but nature points out no particular meridian, in preference to another, for this purpose. (15) The *latitude* of a place is its distance from the equator measured on its meridian. (16) If a plane touch the earth, its intersection with the celestial sphere is called the *sensible horizon* of the point of contact; and (17) a great circle parallel to this is the *true* or *rational horizon* of the same place. (18) Of the poles of the horizon of any place, that which is over the place is called the *zenith*, and the other the *nadir*. (19) The arc of the horizon intercepted between the meridian and a secondary to the horizon passing through any of the heavenly bodies, is the *azimuth* of the body; and (20) the arc of the secondary between the body and the horizon, is its *altitude*. (21) The complement of the azimuth of a body at rising or setting, or its distance from the east point at rising, or from the west point at setting, is often called its *amplitude*. (22) A secondary to the horizon is called a *vertical* or *azimuth circle;* and of such circles, that which is perpendicular to the meridian, and which, consequently, cuts the horizon in the east and west points, is called the *prime vertical*.

Astronomical observations show, that the apparent diurnal motions of the fixed stars in the circles which they appear to describe, are at all times uniform; and hence it follows, that the earth's rotation on its axis is likewise uniform. The sun's apparent diurnal motion is also very nearly uniform, being affected only by a slight inequality arising from his apparent motion in the ecliptic. If we regard his apparent motion as uniform, or rather, if we consider him as moving with his mean or average motion, it is evident that since he appears to describe round the earth a circuit of 360° in twenty-four hours, he must describe 15° each hour, and consequently a degree in four minutes, a minute of a degree in four seconds of time, &c. Now, it is evident that the portions of the equator and of any circle parallel to it, intercepted between two circles passing through the poles, will contain the same number of degrees, each measuring the inclination of the planes of those circles. (23) Such circles, considered with respect to time, are called *hour circles;* and it follows, that the arc of the equator intercepted between any two of them is proportional to the time occupied by a body in passing from one of them to the other in its apparent diurnal revolution. Hence, also, of the arc of the equator and the corresponding time, if either be given, the other will be known, an hour and 15° being equivalent. Degrees, minutes, &c. are most easily reduced to time by multiplying by four, and taking the degrees, minutes, and seconds of the product as minutes, seconds, and thirds of time; and time is readily reduced to degrees, &c. by multiplying it by ten, and adding to the product half of itself; the result, instead of hours, minutes, &c. will be degrees, minutes, &c. (24) The mean period employed by the sun in apparently moving from the meridian to the meridian again, is twenty-four hours, and is called a *mean solar day*. (25) The difference between the time at which the sun actually passes the meridian and the time at which he would pass it, were he to move equably in the equator, or in a parallel to it, is called the *equation of time*. (26) The period that is occupied by a fixed star in passing from the meridian to the meridian again, is 23 hours, 56 minutes, 4·1 seconds, of solar or civil time, and is called a *sidereal day*. This quantity, which is the time of the earth's revolution on its axis, is always the

To resolve this useful problem, let EQ and KL (*fig.* 29) be the equator and ecliptic, P and P′ their north poles, and O the first point of Aries; and let the body S be so situated, that its right ascension, OR, and its longitude, OG, may be each less than 90°; and its declination, SR, and latitude, SG, both north. Then, in the spherical triangle P′PS, P′P is equal to LQ, the obliquity of the ecliptic, each being the complement of PL; $PS=90°-SR=90°-dec.$ or the north polar distance; $P'S=90°-SG=90°-lat.$; P′PS (measured by ER) $=90°+right\ asc.$; and PP′S (measured by GL)$=90°-lon.$: and the resolution will be effected by the third case of spherical trigonometry. Thus, drawing SV perpendicular to PL, and putting φ for PV, we have $PS=90°-dec$, and the angle $SPV=180°-P'PS=90°-R.\ asc.$; whence, in the rightangled triangle SPV, by No. 107, $\cos SPV=\cot PS \tan PV$, or $\sin R.asc.=\tan dec. \tan\varphi$. Then, No. 87, $\cos PV : \cos P'V :: \cos PS : \cos P'S$, or $\cos\varphi : \cos(\varphi+obl.) :: \sin dec. : \sin lat.$ Lastly, the rightangled triangle P′VS gives $\cos SP'V=\cot P'S \tan P'V$, or $\sin lon.=\tan lat. \tan(\varphi+obl.)$. Hence we have the following formulas for resolving this problem:

$$\left.\begin{array}{l}\tan\varphi=\sin R.A.\cot dec.\\ \sin lat.=\sin dec.\cos(\varphi+obl.)\sec\varphi,\\ \sin lon.=\tan lat.\tan(\varphi+obl.)\end{array}\right\}\ldots\ldots (213)$$

These formulas will serve for all positions of S, if, when the latitude and declination are south, they, and consequently their sines, tangents, and cotangents be taken negative; and if it be considered, that, as is evident from the diagram, when either the right ascension or longitude is between 90° and 270°, the other is between the same limits. P′SP, which is called the *angle of position*, is easily calculated.

167. When the obliquity, and the latitude and longitude, are given, to find the declination and right ascension, we have P′P, P′S, and the contained angle, and the resolution of this also is effected by the third case. Thus, putting $\varphi=P'V$, and proceeding as in the last No.

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same; while, as we have already seen, the length of the solar day varies slightly at different times of the year, being, at Christmas, thirty seconds more, and at the middle of September, twenty-one seconds less, than twenty-four hours. The number of sidereal days in the year is evidently *one* more than the number of solar, the sun losing one revolution in a year, in consequence of his apparent motion among the fixed stars.

we find the following formulas, which, with attention to the remarks at the end of that No. will solve the problem in every case:

$$\left.\begin{array}{l}\tan\varphi = \sin lon.\ \cot lat. \\ \sin dec. = \sin lat.\ \cos(\varphi - obl.)\ \sec\varphi, \\ \sin R.A. = \tan dec.\ \tan(\varphi - obl.)\end{array}\right\} \ldots\ldots (214)$$

168. In case of the sun, both these problems become much simplified, as in consequence of his being on the ecliptic, suppose at S′, (*fig.* 27) his latitude is nothing, and the triangle becomes quadrantal. In resolving this case of the problem, we may rather employ the triangle ORS′, in which R is a right angle, O the obliquity, OS′ the longitude, OR the right ascension, and RS′ the declination; and when the obliquity is known, it is sufficient that either the right ascension or declination be given, as, by means of the obliquity and one of these, the other and the longitude can be computed by No. 107. It is also plain, that if both the right ascension and declination be given, the obliquity may be calculated. The following are the formulas obtained by the method alluded to; and, by means of them, any two of the four quantities, the obliquity, the longitude, the right ascension, and the declination, being given, the rest can be found:*

$$\cos obl. = \tan R.A.\ \cot lon. \ldots\ldots (215)$$
$$\sin R.A. = \cot obl.\ \tan dec. \ldots\ldots (216)$$
$$\sin dec. = \sin obl.\ \sin lon. \ldots\ldots (217)$$
$$\cos lon. = \cos R.A.\ \cos dec. \ldots\ldots (218)$$

169. If the right ascensions and declinations of two heavenly bodies, S and T (*fig.* 27) be given, to find their distance asunder; their distances from either pole P, are two sides of a spherical triangle; the

* The right ascension of a body is observed by means of the transit instrument, and of a clock regulated to sidereal time. The transit instrument is a telescope placed on a horizontal axis, and always moving in the plane of the meridian, so that any body, when it can be seen through it, must be on the meridian. The index of the clock points to 0, when the first point of Aries is on the meridian; and therefore, at the instant at which any other body is seen through the transit instrument, the time indicated by the clock, estimated, according to No. 165, note (10) and (23), at 15° to the hour, will give the right ascension in degrees. The altitude of the body taken at the same instant, will give its declination if the latitude of the place be known; since, as will appear hereafter (No. 173), H*n* (*fig.* 28) is the meridian altitude, EH the complement of the latitude, and their difference E*n* the declination. Conversely, if the declination and altitude be known, the latitude can be found. It is evident, also, that if the sun's right ascension and declination be determined by observation at the same instant, the obliquity can be computed by (112).

difference of their right ascensions* is the contained angle SPT; and the third side, which is the required distance, may be computed by the third case.

If the latitudes and longitudes were given, the sides SP′, TP′, would be the distances of the bodies from P′, one of the poles of the ecliptic; and the difference of their longitudes would be the contained angle.†

170. In exactly the same way, the distance, on the arc of a great circle, between two places on the earth (regarded as a sphere), may be determined; their distances from one of the poles being two sides, and the difference of their longitudes the contained angle of a spherical triangle, of which the third side is the distance. The remaining angles of this triangle, or those which the great circle passing through the places makes with their meridians, are called the *angles of position.*

171. Given the distances of a body, X, from two stars, T and S, whose positions are known, to find its right ascension and declination.‡

In the triangle TPS, calculate, by the method shown in No. 169, TS, and the angle PTS. Then, in the triangle TXS, the three sides will be known; whence, by case I., the angle XTS can be found, and thence the angle PTX, which will be the sum or difference of PTS, STX, accordingly as X and P are on opposite sides, or on the same side of TS. Lastly, in the triangle PTX, the sides PT and TX, and the contained angle are known; whence PX, the polar distance, can be computed; as also the angle TPX, the sum or difference of which, and of the right ascension of T, will be the right ascension of X. If a great circle were drawn from P′ to X, the method of finding the latitude and longitude of X would evidently be the same as the preceding.

172. The apparent diurnal motions of the heavenly bodies give origin to a class of problems of much importance in the application of spherical trigonometry. To investigate the method of resolving these, let *fig.* 30 represent a projection of the celestial sphere on the plane

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* If the difference of the right ascensions exceed 180°, the difference between it and 360° will be the contained angle.

† This problem occurs in the computation of several pages in each month of the Nautical Almanac, which exhibit the distances between the moon and the sun, or between the moon and certain fixed stars, for determining the longitude.

‡ This problem is useful in determining the positions of comets, as it frequently happens that they cannot be observed when on the meridian, in consequence of attaining that position in the day-time.

of the circle PHNR, the meridian of the observer; and let P be the north pole; EQ, the equator: PD, PCO, PLO, &c. portions of hour circles; Z the zenith, and HCR the horizon of the observer, R being its northern, H its southern, and C its eastern or western point. Now, from the quadrants ZH and EP, take away ZE, the latitude of the place, and there will remain EH and ZP, each equal to the complement of the latitude. Also, by taking away ZP from the quadrants ZR and PE, we shall have PR=ZE; whence it appears, that the elevation of the pole above the horizon is equal to the latitude. Then, B being the body, BD will be its declination; BP, its north polar distance; BF, its altitude; BZ, its zenith distance, or the complement of its altitude; PZB, or RF, its azimuth from the north, or the supplement of its azimuth from the south; and the angle ZPB, converted into time, will show the interval that must elapse between the body's being in its present position and on the meridian. In this class of problems, therefore, nothing more is necessary than to resolve the triangle PZB according to the data.

173. In several particular problems of this class, the operations become simplified. Thus, if the circle *ns′m*, parallel to the equator, represent the apparent diurnal path of the sun (his declination being supposed not to vary during the time considered in the problem, or, rather, being corrected for any particular time), it is evident that at noon he will be at *n*; at midnight at *m*; at six o'clock (morning or evening, accordingly as the diagram is supposed to represent the hemisphere east or west of the meridian) at *s′*; at rising or setting at *s*; and on the prime vertical, or due east or west, at *w*.

Now, if the sun be at *s*, we may employ the quadrantal triangle Z*s*P,* or either of the rightangled triangles, PR*s*, CL*s*. Thus, in PR*s*, PR is the latitude; P*s* the north polar distance; *s*R the azimuth at rising, or the complement of C*s* the amplitude; and RP*s* (in time) will be the interval between the sun's being at *m* and *s*, or between midnight and rising or setting.

Again, if the sun be at *s′*, we have, in the rightangled triangle ZP*s′* Z*s′*, the complement of his altitude at six o'clock; PZ*s′*, his azimuth from the north at the same time; and P*s′*, the polar distance.

Lastly, if he be at *w*, ZP*w* will show the interval between noon and

* The lines Z*s*, Z*s′*, and P*w*, are omitted in the diagram, to prevent it from appearing confused.

the time of his being east or west, and Zw will be the complement of his altitude at the same time.*

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* If we put l = the latitude; d = the declination; P = the angle ZPs, the supplement of RPs, and consequently the measure of the time between noon and the hour of rising or setting; P′ = ZPw; a = the altitude at six; a' = the altitude when east or west; Z = Rs, the azimuth from the north at rising or setting; Z′ = PZs', the azimuth from the north at six o'clock; and m = the amplitude: we have, from the triangles above mentioned,

1. $\cos P = -\tan l \tan d$;	4. $\cos l = \cot d \cot Z'$;
2. $\sin d = \cos l \cos Z = \cos l \sin m$;	5. $\sin d = \sin l \sin a'$;
3. $\sin a = \sin l \sin d$;	6. $\cos P' = \cot l \tan d$.

In deriving these formulas, the latitude, declination, and amplitude, have been considered north; and consequently, if any of them be south, its sine, tangent, and cotangent, will be negative. In like manner, should $\sin a$ or $\sin a'$ be found negative, a and a', instead of *altitude*, would denote *depression* below the horizon.

EPL, or EL, is called the *semidiurnal arc*, as it measures half the time of the body's continuance above the horizon. A table of semidiurnal arcs, which may be computed by the formula $\cos P = -\tan l \tan d$, is useful in calculating the rising and setting of the heavenly bodies at any particular place.

Formula 1 shows that a body will not descend to the horizon, if the latitude be greater than the complement of the declination, as in that case $\tan l \tan d$ would be greater than 1, and therefore $\cos P$ impossible. In like manner, we infer, from formula 6, that a body cannot be on the prime vertical if its declination exceed the latitude.

As astronomical tables give the declinations, right ascensions, &c. of the *centres* of the heavenly bodies, the method that has been pointed out will determine the time at which the centre of any of them would be on the horizon, were its apparent position not affected by refraction. When the atmosphere is in its mean state, however, refraction causes the heavenly bodies to appear on the horizon, when they are in reality about 33′ below it. Supposing this to be the case, we should find the apparent rising of the centre, by taking ZB = 90° 33′ and resolving the triangle ZPB. If it were required to find at what time the sun's upper limb would appear on the horizon, we must take ZB = 90° 49′ (= 90° 33′ + 16′) the sun's radius being about 16′. In respect to the moon, also, if great accuracy were required, which, however, is seldom necessary, and not easily attainable, ZB must be taken equal to 90° 33′, diminished by her horizontal parallax (which varies from about 54′ to about 61′½, according to the moon's distance), because parallax occasions her to appear lower by its own quantity than her true place, as she would appear at the earth's centre. In computing the time of the moon's rising or setting, however, the changes of her declination and right ascension are much more to be attended to than any of the corrections already mentioned. In the Nautical Almanac, her declination is given at Greenwich for several equal intervals; and hence, the approximate time of her rising or setting being determined by the method already explained, her declination at the time thus found may be obtained with tolerable accuracy by the method of proportional parts, and the process repeated with this declination; and, by a similar method, allowance may be made for the change in her right ascension. A method of approximating the corrections for refraction, semidiameter, and parallax, might be employed, which would be shorter than the method already pointed out, but which it is unnecessary to explain here. No allowance for parallax is necessary, except for the moon; nor for semidiameter, except for the sun or moon.

174. When the latitude of a place, and the altitude and declination of a body, B, are given, we have ZP the complement of the latitude, ZB the zenith distance, and BP the polar distance; and we can thence find the hour angle ZPB, which will show the difference between the times of the body's being at B and on the meridian. We can compute also the angle PZB, which is the azimuth from the north. In this problem, which is useful in regulating time-pieces, and in finding the variation of the compass, it is necessary to correct the observed altitude for parallax, refraction, &c. and the result for the equation of time.

ASTRONOMICAL AND GEOGRAPHICAL EXERCISES.

Ex. 1. If the right ascension and declination of a star be found, by observation, to be 142° 14′, and 35° 17′ S., respectively; what are its latitude and longitude?—*Answ. Lat.* 46° 48′ S.; *lon.* 160° 29′½.

Ex. 2. If the latitude of a star be 1° 12′ N., and its longitude 135° 35′, what are its right ascension and declination?—*Answ. R.A.* = 138° 32′½, *dec.* = 17° 19′¾ N.

Ex. 3. When the sun's longitude is 43° 47′, what are his right ascension and declination?—*Answ. R.A.* = 41° 19′, *dec.* = 15° 59′½ N.

Ex. 4. If, by observation, the sun's declination be found to be 22° 30′, and his right ascension 72° 35′, what is the obliquity of the ecliptic? Find, also, the sun's longitude.—*Answ. Obliquity* = 23° 28′, *very nearly; lon.* = 73° 57′.

Ex. 5. Required the sun's right ascension and longitude, when his declination is 11° 44′ S.—*Answ. R. A.* = 208° 35′, or 331° 25′; *lon.* = 210° 42′½, or 329° 17′½.

Ex. 6. If the right ascension and declination of the moon be respectively 0° 33′, and 5° 19′ S.; and those of the star Regulus 148° 49′, and 13° 10′ N.; what is their distance asunder?—*Answ.* 147° 45′.

Ex. 7. If the sun's longitude be 202° 24′¼, and the moon's latitude and longitude, 4° 54′½ N. and 89° 25′½; what is their distance asunder?—*Answ.* 112° 53′½.

Ex. 8 Required the distance between Belfast, in lat. 54° 36′ N. lon. 5° 54′ W. and Port Jackson, in lat. 33° 51′ S. lon. 149° 52′ E.—*Answ.* 153° 13′¼, *or* 10580 *English miles* (69$\frac{1}{20}$ *to a degree*).

Ex. 9. Suppose the distances of a comet from Aldebaran and Regulus to be observed to be 40° 12′ and 51° 36′ respectively: required its latitude and longitude, the respective latitudes of the two stars being 5° 28′¾ S. and 0° 27′½ N., their longitudes 67° 12′¼ and 147° 15′½, and the comet being south-east of the arc of a great circle joining them.—*Answ. Lat.* $28^\circ\ 0\frac{1}{4}'$ S.; *lon.* $102^\circ\ 19'$.

Ex. 10. Required the times of the sun's rising and setting at Glasgow (latitude $55^\circ\ 52'$ N.) on the longest and shortest days (June 21, and December 21), his declination at those times being $23^\circ\ 28'$. *Answ. Longest day, rising,* $3^h\ 20^m\ 44^s$; *setting,* $8^h\ 39^m\ 16^s$: *shortest day, rising,* $8^h\ 39^m\ 16^s$; *setting,* $3^h\ 20^m\ 44^s$.*

Ex. 11. When the sun's declination is $23^\circ\ 28'$ N., required the times of his rising and setting at Belfast and at Port Jackson: required also his amplitude at each; his azimuth, and his altitude or depression at six o'clock; the time when he is west at each, and the altitude or depression at that time.

Answ. At Belfast, rising, $3^h\ 29\frac{1}{2}^m$; *setting,* $8^h\ 30\frac{1}{2}^m$; *west,* $4^h\ 48\frac{1}{4}^m$;
At Port J. ——— $7^h\ 7\frac{3}{4}^m$; ——— $4^h\ 52\frac{1}{4}^m$; ——— $8^h\ 41\frac{1}{4}^m$;
At Belfast, amplitude, $43^\circ\ 26'$ N.; *azimuth at six,* $75^\circ\ 53'$ *from* N.;
At Port J. ——— 28 39 N.; ——— 70 10 *from* N.;
At Belfast, alt. at six, $18^\circ\ 56'$; *alt. when west,* $29^\circ\ 15'$;
At Port J. depr. ——— 12 49; *depr.* ——— 45 38.

Ex. 12. Given the apparent altitude of the sun's lower limb, at Belfast, April 4, 1823, equal to $29^\circ\ 24'$, at ten minutes past nine in the morning, by a clock; to find the error of the clock.†—*Answ.* $1^m\ 24^s$ *fast.*

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* These answers show the times at which the sun's centre would be on the horizon, if no effect were produced by refraction or parallax, or by the equation of time. Should the time be required at which his centre, affected by the mean refraction, would be on the horizon, his zenith distance must be taken as being $90^\circ\ 33'$; while for finding the time at which his upper limb would be on the horizon, this must be increased by $16'$, his semidiameter: and both computations would be effected in the manner pointed out for the solution of Ex. 13. Easy approximations, however, may be obtained by means of the *variations of triangles.* See *Differential and Integral Calculus, Section* IV.

In finding the rising or setting of a star, from its right ascension in time (increased, if necessary, by 24 hours), take the sun's right ascension; the remainder will be the hour of the star's passing the meridian. From this, take the semidiurnal arc to find the rising, and add it to it for the setting. If great precision were required, the sun's right ascension should be found by proportion, for the times of the star's rising and setting, and the operation repeated. The finding of the moon's rising and setting may be facilitated by means of tables and other contrivances. See Mackay on the Longitude, vol. II.

† The sun's semidiameter (note, page 71) being $16'$, we have $29^\circ\ 24'+16'=29^\circ\ 40'$, the apparent altitude of the centre; corresponding to which, in the tables of refraction and parallax, we find $1'\ 42''$, and $7''$: consequently the true altitude of the centre is $29^\circ\ 40'-1'\ 42''+7''=29^\circ\ 38'\ 25''$. Now, by the Nautical Almanac, the sun's declination, April 3, at noon, is $5^\circ\ 6'\ 39''$, and the next day $5^\circ\ 29'\ 37''$; whence, by proportion, we find the declination, at ten minutes

Ex. 13. Required the beginning of morning and the end of evening twilight at Belfast, on the first of September, the sun's declination being $8^{0}\ 30'$ N.*—*Answ.* $2^{h}\ 46^{m}$ *morn.* and $9^{h}\ 14^{m}$ *even.*

Ex. 14. Suppose two altitudes of the sun observed in the forenoon, in the same place, at the interval of an hour and a half, on the 25th of June, to be $28^{0}\ 40'$ and $39^{0}\ 50'$; required the latitude of the place, and the times of observation, the declination being $23^{0}\ 26'$ N.†—*Answ. Latitude,* $59^{0}\ 17'\frac{1}{4}$ N.; *times of observation,* $7^{h}\ 8^{m}\ 24^{s}$ and $8^{h}\ 38^{m}\ 24^{s}$.

Ex. 15. How much shorter is the distance from Port Jackson to the Bay of Valparaiso on the arc of a great circle than on their common parallel; and what is the highest latitude attained by a ship sailing between them on the arc of a great circle, their latitude being $33^{0}\ 51'$ S. and their difference of longitude $136^{0}\ 10'$‡—*Answ. Highest latitude* $= 60^{0}\ 54'$; *difference of the distances* $= 737{\cdot}6$ *geographical miles.*

Ex. 16. What is the highest latitude attained by a ship sailing on

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past nine, to be $5^{0}\ 26'\ 54''$; and hence (124) the apparent time is found to be $9^{h}\ 5^{m}\ 21^{s}$; and adding $3^{m}\ 15^{s}$, the equation of time, we find the true time to be $9^{h}\ 8^{m}\ 36^{s}$.

* Twilight being supposed to begin and end when the sun is 18^{0} below the horizon, we have the zenith distance 108^{0}, the polar distance $81^{0}\ 30'$, and the colatitude $35^{0}\ 24'$: and the solution will be effected by the first case.

† To illustrate the method of solving this problem, let B and w (*fig.* 28) be any two positions of the sun, and suppose wB, wP, to be joined by arcs of great circles. Then, in the isosceles triangle BPw, compute Bw, and the adjacent angles. This is most easily effected by dividing it into rightangled triangles by an arc of a great circle bisecting the angle P, and consequently the side Bw. Then, the three sides of the triangle BZw being given, let the angle ZBw be computed, and the difference between it and PBw will be ZBP; by means of which, and of the sides containing it, the colatitude ZP will be found by the third case. Practical modes of solving this useful problem will be found in the later works on navigation.

The problem in which the altitudes of two known stars, taken at the same instant, in the same place, are given, to find the latitude and the time of observation, is solved in the same manner, except that, unless the stars have the same declination, the triangle BPw is not isosceles.

‡ In solving this question, there are given, in an isosceles triangle, two sides, each equal to the complement of the latitude, and the contained angle equal to the difference of longitude. A perpendicular to the base, from the opposite angle, will bisect both; and, hence, the resolution will be effected by No. 85 or No. 107. The base is the distance on the arc of a great circle, and the perpendicular is the complement of the highest latitude. The radius of the parallel will obviously be the cosine of the latitude, the radius of the earth being taken as unity. Hence, the distance on the parallel will be found by the analogy, 1 : cos *lat.* : : *diff. lon.* : *dist. on parallel.*

the arc of a great circle from Port Jackson to Cape Horn, their latitudes being 33° 51′ and 55° 58′ S., and their difference of longitude 140° 27′?*—*Answ.* 72° 41′.

Ex. 17. In a given latitude, and on a given day, how may the time be found at which two given stars have the same azimuth?†

Ex. 18. The same being given, how may the time be found at which the stars have the same altitude or depression?‡

Ex. 19. Given the altitudes of two known stars, at the instant when they have the same azimuth; to find the latitude.‖

Ex. 20. Given the difference of the azimuths of two known stars, at the instant when they have the same altitude; to find the latitude.

Ex. 21. To find the latitude at which two given stars have always the same altitude, when two others have the same azimuth.§

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* Here we have two sides (the complements of the latitudes) and the contained angle (the difference of longitude), to find the perpendicular drawn from that angle to the opposite side, that perpendicular being the complement of the required latitude; and the solution may be effected by finding the remaining angles, and then the perpendicular by the rules for resolving rightangled triangles; or, more easily, by finding the parts of the contained angle by (151), and thence (114) the perpendicular.

† S and S′ (*fig.* 31) being the stars, find (case III.) in the triangle SPS′, the angle PS′S; and in PZS′ find (case V.) the angle ZPS′, which will show the difference of time between the star S′ being in its present position and on the meridian.

‡ S and S′ (*fig.* 32) being the stars, bisect SS′ in M, and join MP, MZ. Then, in the triangle SPS′, compute (case III.) SS′ and PSS′; and in SMP, compute (case III.) MP, SPM, and SMP. From this last angle take SMZ= 90°, and there will remain ZMP; and the resolution of the triangle ZPM, by case V. will give ZPM; the difference between which and the angle SPM will be ZPS; and this will give the required time. From this, the method of solving Ex. 20 will be manifest.

‖ This problem, the method of solving which is plain from the note to Ex. 17, affords a mode of finding the latitude of a place. The instant at which the stars have the same azimuth can be ascertained by means of a plumb-line.

§ To show the method of solving this question, let A, B, C, D (*fig.* 31) be the given stars, and suppose great circles to be drawn joining AC, BC, BD, PA, PB, PC, PD, and ZB; suppose also ZE to be drawn perpendicular to AB, and consequently bisecting it. Then, since the stars are given, their right ascensions and declinations are given, and their distances asunder can be computed, as also the angle BCP, by No. 169. In the next place, the sides of the triangles ABC and BCD being given, the angles ABC, BCD will be found by case I.; and thence CBF and FCB, their supplements, will be known. From these two angles, and the adjacent side BC, the sides BF and CF, and the remaining angle F, can be computed by case IV. Then, in the rightangled triangle ZEF, the angle F and the side EF are given, to find ZF: from which, and from CF, CZ will be known. The angle ZCP will also be known from BCP and BCD; and CP being given, the colatitude ZP will be found by case III. This solution would evidently admit of several variations.

Ex. 22. Given the latitudes and longitudes of three places on the earth's surface; to find the latitudes and longitudes of the two places equally distant from them.*

Ex. 23. Given as in the last problem; to find the latitudes and longitudes of the poles of a circle touching the three great circles passing each through two of the places.

Ex. 24. At a given place, to find the greatest azimuth of a given star whose declination is greater than the latitude of the place: to find also the time, on a given day, when the star will have the greatest azimuth, and when, consequently, it will appear to move perpendicularly to the horizon.†

Ex. 25. On what days of the year is the sun on the horizons of Dublin and Pernambuco at the same instant; their respective latitudes being 53° 21′ N., and 8° 13′ S.; and their longitudes 6° 19′ W., and 35° 5′ W.? ‡—*Answ. The sun will set at the same instant at both on the 12th of May and the 1st of August, and will rise at both at the same instant on the 29th of January and the 14th of November, the declination being 18° 6′.*

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* There may be a similar problem respecting stars.

† The required point is that in which a vertical circle touches the parallel of the star's declination, and a circle drawn from the pole to the point is perpendicular to that vertical circle: whence the azimuth, hour, and altitude may all be found by resolving a rightangled triangle.—This exercise will furnish an explanation of the curious fact, that when the sun's declination is greater than the latitude of a place on the same side of the equator, the shadow of an upright object on a horizontal plane goes backward each day during a certain period, which may be computed. In any latitude, indeed, the shadow of a pin perpendicular to a plane will move during a part of each day, contrary to the usual direction, if the plane be so placed that the pin will be directed to a point of the meridian farther from the elevated pole than the complement of the sun's declination.

‡ In solving this problem, it is evident that the difference between the hours of rising or setting at the two places must be equal to the difference in their reckoning of time, that is, to the time equivalent to their difference of longitude. Denoting, therefore, the latitudes of Dublin and Pernambuco by l and l', the angles corresponding to the times from noon to rising or setting by P and P′, and the difference of longitude by D, we have (note 1, page 73) $\cos P = -\tan l \tan dec.$ and $\cos P' = -\tan l' \tan dec.$ Dividing the latter by the former, and converting the quotient into an analogy, we obtain $\cos P' : \cos P :: \tan l' : \tan l$; whence, by a process the same as in No. 118, we obtain the following analogy:

$$\sin(l+l') : -\sin(l-l') :: \cot\tfrac{1}{2}D : \tan\tfrac{1}{2}(P+P').$$

Hence, $\frac{1}{2}$D being half the difference of P and P′, these quantities become known; and thence the declination from the formula $\cos P = -\tan lat. \tan dec.$ It must be observed in the computation that the latitude of Pernambuco is to be taken negative.

Ex. 26. Given the longitude of the ascending node* of Ceres $= 80^\circ$ $54'$, and that of the same node of Pallas $= 172^\circ\ 31'$: given, also, the inclination of the orbit of the former to the ecliptic $= 10^\circ\ 37'\frac{1}{2}$, and that of the latter $= 34^\circ\ 37'$; to find the mutual inclinations of the orbits of Ceres and Pallas.—*Answ.* $36^\circ\ 18'$.

VII.—DIALLING.

175. As a farther application of the principles that have been thus far established, we may now investigate some of the more important parts of the theory of dialling; a branch of science which, notwithstanding the very improved methods of measuring time that modern ingenuity has discovered, is still of some value, and of considerable interest.

Since the sun's apparent diurnal motion is uniform, it is plain that an opaque straight line or wire, occupying, in free space, the same position as the earth's axis, would cast a shadow, the plane of which would revolve uniformly, describing an angular space of 15° in each hour; and the same will hold, without sensible error, respecting any line at the earth's surface parallel to the axis, its distance from the axis being extremely small compared with the sun's distance from the earth. Hence, if HR (*fig.* 30) be a great circle, parallel to the plane of a dial, such as a horizontal dial at the place whose zenith is Z, we have only to find how HR would be divided at different times by the shadow of the axis PO, which would evidently divide the equator EQ, and consequently, the angular space about P, into parts proportional to the times in which they are described. To effect this, we have, in the rightangled triangle PRs, PR $= 90^\circ -$ RQ $=$ the complement of the inclination of the planes of the equator and dial; RPs, the measure in degrees of the time between the instants at which the shadow occupies the positions PQO and PLO; and Rs will be the

* The two opposite points in which the orbit of a planet cuts the ecliptic, are called the *nodes* of its orbit;—the one in which the planet crosses the ecliptic northward, the *ascending* node,—the other, the *descending* one.

measure of the angle at the centre of the dial corresponding to that time; and by the resolution of this triangle, we have

$$R : \sin PR \text{ (or } \cos RQ) :: \tan RPs : \tan Rs.$$

Hence, putting I to denote the inclination of the plane of the dial to that of the equator, P equal to the *horary angle* RP*s*, and H equal to the arc R*s*, or the hour angle on the dial, we have

$$R : \cos I :: \tan P : \tan H \text{ .... } (219)$$

176. In case of a *horizontal dial*, the angle I is evidently equal to the complement of the latitude, and (219) becomes

$$R : \sin lat. :: \tan P : \tan H \text{ ..... } (220)$$

177. In case of a *vertical south* or *north dial*, that is, a dial whose plane is perpendicular to the meridian and horizon, and which, therefore, has its face directed exactly towards the south or north point of the horizon, I is equal to the latitude; and, therefore, (219) becomes

$$R : \cos lat. :: \tan P : \tan H \text{ .... } (221)$$

178. To exemplify the use of (220) in the construction of a horizontal dial for Belfast, by taking the latitude in that formula equal to 54° 36′, and P successively equal to 15°, 30°, 45°, 60°, 75°, and 90°, we find the hour angle for one hour to be 12° 19′, for two 25° 12′, for three 39° 11′, for four 54° 41′, for five 71° 48′, and for six 90°.* Then (*fig.* 32) draw two parallel lines, *ab*, *a′b′*, at a distance asunder, equal to the thickness of the stile, and let them be crossed perpendicularly by 6*cc′*6, the six o'clock hour line. Draw, also, *c* 11, *c* 10, *c* 9, &c. making with *ca* angles respectively equal to 12° 19′, 25° 12′, &c. and draw *c′*1, *c′*2, *c′*3, &c. making equal angles with *c′a′*. The hour lines for the times before six in the morning, and after six in the evening, are found by producing 3*c′*, 4*c′*, 5*c′*, through *c′*; and 7*c*, 8*c*, 9*c*, through *c*. The stile is to be a firm slip of metal, or other substance, alike thick throughout, having its back a plane inclined to the plane of the dial on the northern side, at an angle equal to the latitude, and meeting it in the points *cc′*. The half hours, quarters, or other minuter divisions, if the dial be on so large a scale as to admit them, may be obtained with nearly sufficient accuracy by dividing the angular spaces between the hour lines into equal parts. Should

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* For Glasgow (latitude 55° 52′) the hour angles are 12° 30′, 25° 32′½, 39° 37′½, 55° 6′½, 72° 4′, and 90°.

much accuracy be required, however, the analogy (220) will give the angular spaces with precision by taking P in it successively equal to $7^\circ 30'$, $15^\circ + 7^\circ 30'$, &c. for the half hours; and equal to $3^\circ 45'$, $7^\circ 30' + 3^\circ 45'$, &c. for the quarters. It is plain that the line bounding the dial, which in *fig.* 34 is a circle, may be a square, or any other figure that the maker of the dial may prefer; and the northern side of the stile may be straight or curved, or of any outline whatever.

179. A north or south erect dial would be constructed by means of (221) in a manner exactly similar. On the south erect dial it is unnecessary to put any hours, except those between six in the morning and six in the evening, as the sun can never shine on it at other times. According, also, to Ex. 11, page 75, the sun can shine on a north erect dial at Belfast, even at the longest day, only a little more than the time before seven in the morning and after five in the evening; and, therefore, in constructing such a dial, the intermediate hours may be omitted. The angle of the stile is equal to the colatitude.

180. If the dial be a *polar* one, that is, if its plane be perpendicular to that of the equator, the formula (219) fails; but the construction is more simple. Thus, in case of a vertical dial (*fig.* 35) facing the east or west, the hour lines are all parallel, and the stile is erected perpendicularly on the six o'clock hour line, to which its back is parallel. Then, if the height of the stile be assumed as radius, the distances from the six o'clock line to the seven o'clock and five o'clock lines are each, according to the principles explained in No. 175, the tangent of 15°: the distances to the next lines are each the tangent of 30°, &c. Hence, if s be put to denote the height of the stile, and H′ the perpendicular breadth on the dial between the six o'clock line and any other hour line corresponding to P, the distances between the six o'clock line and the others may be computed by this analogy,

$$R : \tan P :: s : H' \quad \ldots\ldots\ldots\ldots\ldots (222)$$

Thus, if the height of the stile were 2 inches, we should have the distance between the six o'clock hour line and the five or seven o'clock one equal to ·5359; the distance between the same and the four or eight o'clock line equal to 1·1547, &c. These distances may be easily determined by construction, without computation, by describing from C as centre, with a radius equal to the height of the stile, an arc AB of 90°, and dividing it into six equal parts. Then, straight lines drawn from C through the several points of division will cut DE in the points 7, 8, 9, &c. The dial is to be placed so as exactly to face the east, and so that a line drawn through C, and making with CA an angle equal to the latitude, may be horizontal; as by this means the back of the stile will be parallel to the earth's axis.

It is plain that the same dial would serve as a west erect one, if the numbers 1, 2, 3, &c. were substituted for 11, 10, 9, &c.

181. Having now seen the method of constructing horizontal dials, and vertical ones facing the four cardinal points of the horizon, we may consider briefly the theory of those dials whose planes occupy other positions. Of these there are two varieties, *vertical declining dials*, and *inclining dials*. A *vertical declining dial* has its plane perpendicular to that of the horizon, and cutting it at a distance from the east and west point called the *declination* of the dial; while an *inclining dial* has its plane oblique to the plane of the horizon, and making with it an angle called the *inclination* of the dial.

182. It is plain from No. 175, that a dial which is truly constructed for any place whatever on the earth's surface, will *divide the time truly* at any other place on its surface, and in any position, provided its stile is perpendicular to the equator. Hence, a horizontal dial, constructed for any given place, will divide the time truly at any other, if it be so placed that its plane and its stile are parallel to their original positions; and, conversely, a dial may be constructed on a declining or inclining plane at a given place, by finding the two places whose horizons are parallel to that plane, or, which is the same, by finding the positions of the poles of the great circle parallel to it; as a horizontal dial for either of those places will indicate the time truly at the given place, except the difference in their reckoning of time arising from their difference of longitude, if they be not on the same meridian.

183. These principles will be illustrated by the following example: If, at Belfast, a great circle cut the horizon 33° 45′ from the south towards the west, and be inclined to it at an angle of 25°, rising above it on the north-western side, it is required to find the latitude and longitude of the place in the northern hemisphere, of which this circle is the horizon.

Let Z (*fig.* 30) be the zenith of Belfast, and B the pole of the given circle, or the zenith of the required place. Then, since the given circle and the horizon are inclined at an angle of 25°, ZB, the distance of their poles, will likewise be 25°. It is also evident, that the angle EZB is the complement of 33° 45′; and therefore BZC is equal to 33° 45′, and PZB to 123° 45′. From these, and from ZP, the co-latitude of Belfast, we find, by case III., BPZ = 26° 8′, PBZ = 37° 8′, and PB = 52° 56′. The last of these quantities is the colatitude of the required place, and consequently its latitude is 37° 4′; also, BPZ is the difference of longitude of Belfast and the required place. Now, by reducing this difference of longitude to time, according to note (part 23), page 68, we obtain $1^h\ 44^m\ 32^s$, the quantity by which the reckoning of time at the place whose zenith is B, is more advanced than the time at Belfast. Hence it appears that, were a horizontal dial constructed for latitude 37° 4′, and erected at Belfast in the position described in the problem, with its stile directed to the pole, it would constantly indicate the time $1^h\ 44^m\ 32^s$ too far advanced; making it appear, for instance, to be twelve o'clock, when it is only $15^m\ 28^s$ past ten. The time, therefore, might be ascertained correctly at Belfast by such a dial, placed so as to be parallel to its original position, by always subtracting the constant quantity $1^h\ 44^m\ 32^s$. This subtraction, however, which would be very inconvenient, may be obviated in the following manner.

184. Draw the parallels (*fig.* 36) *cs*, *c′s′*, making each with *ab* an angle of 37° 8′ (= PBZ, last No.) *ab* being the intersection of the plane of the dial, and a vertical circle perpendicular to it: then the stile is to be erected perpendicularly on the space *scc′s′*, which is therefore called the *substile;* and, by what we saw in the last No., when the shadow falls on this space, it is $15^m\ 28^s$ past ten o'clock. We have now (220) R : sin 37° 4′ : : tan 26° 8′ : tan 16° 28′, the hour angle on the dial between the substile and the twelve o'clock line. Again,

$$\begin{aligned}
R &: \sin 37^\circ\ 4' :: \tan(26^\circ\ 8'-15^\circ) : \tan\ 6^\circ\ 46',\\
R &: \sin 37^\circ\ 4' :: \tan(26^\circ\ 8'+15^\circ) : \tan 27^\circ\ 46',\\
R &: \sin 37^\circ\ 4' :: \tan(26^\circ\ 8'-30^\circ) : \tan(-2^\circ\ 20'),\\
R &: \sin 37^\circ\ 4' :: \tan(26^\circ\ 8'+30^\circ) : \tan 41^\circ\ 46'.
\end{aligned}$$

These are the angles which the hour lines for eleven, one, ten, and two o'clock make respectively with the substile; and the other hour angles are found in a similar manner. The angle of the stile is 37° 4′; and the dial is to be erected in such a manner that, if two plumb-lines be suspended, one above *a* and the other above *b*, a horizontal line passing through them may be directed to a point of the horizon 56° 15′ (= 90°—33° 45′) from the south towards the east, and so that the line *ab* may have an elevation of 25°, at the north-western side.

185. By taking $l = 37^\circ\ 4'$, and $d = 23^\circ\ 28'$ (in the note, page 73), we get $P = 7^h\ 17^m$, which is half the length of the longest day at a place in the latitude of 37° 4′. Subtracting this from $10^h\ 15^m$, and also

adding it, we get $2^h\ 58^m$ and $17^h\ 32^m$. From this it appears that, at Belfast, the sun would cease to shine on this dial on the longest day at thirty-two minutes past five in the evening; and that, if he rose so soon, he would begin to shine on it at two minutes before three in the morning. Hence, as the sun rises that day at half-past three, it is unnecessary to mark any hours on the dial except those between that hour and half-past five or six in the evening.

186. In case of a vertical declining dial, the process is rather more simple, ZB (*fig.* 30) being a quadrant. The simplest of all dials, however, is the *equatorial* or *equinoctial* one, so called from its plane being parallel to that of the equator. It will appear, from the slightest consideration, that each hour line will make, with the one next to it, an angle of 15°, and consequently that the positions of the hour lines will be obtained by dividing each quadrant into six equal parts. It is manifest, also, that the stile will be a pin perpendicular to the plane of the dial; and that, during the six summer months, the sun will shine on the upper side of the dial, and, during the rest of the year, on the other side.

187. The determination of the meridian line is necessary to enable us to give to a dial its proper position at any particular place. This may be effected by marking the position of the shadow of an object perpendicular to a horizontal plane at a particular instant, and at the same time measuring the sun's altitude by a quadrant or other instrument. Then, in the triangle ZBP (*fig.* 30) there are given the three sides to find the angle PZB, the azimuth from the north; and one of the lines drawn on the horizontal plane, making with the direction of the shadow an angle equal to the computed one, will be the meridian line.

The same may also be effected by bisecting the angle contained by the shadows of a perpendicular object, on the same day, at the two times when it has the same altitude. If this method be employed, the observations should be made about the solstices.

It may be observed, in conclusion on this subject, that, to find the true time by means of a sun dial, the time which it indicates must always be corrected for the *equation of time*, the table of which is given in almanacs, and various other publications.*

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* The foregoing section exhibits the general principles on which the construction of the commoner kinds of dials depends. Dials may also be described on curve surfaces; and there are various other modes of determining time by means of shadows, several of which are ingenious and interesting. For information

VIII.—MULTIPLE ARCS.

188. By squaring $\cos A + \sin A\sqrt{-1}$, we obtain

$$(\cos A + \sin A\sqrt{-1})^2 = \cos^2 A - \sin^2 A + 2\cos A \sin A\sqrt{-1};$$

or, by the application of (38) and (37),

$$(\cos A + \sin A\sqrt{-1})^2 = \cos 2A + \sin 2A\sqrt{-1}.$$

Multiply by $\cos A + \sin A\sqrt{-1}$, and modify the second member of the result by (14) and (12); then,

$$(\cos A + \sin A\sqrt{-1})^3 = \cos 3A + \sin 3A\sqrt{-1}.$$

By successive multiplications by $\cos A + \sin A\sqrt{-1}$, and by means of (14) and (12), we should find that, n being any whole positive number,

$$(\cos A + \sin A\sqrt{-1})^n = \cos nA + \sin nA\sqrt{-1} \ldots (223)$$

It will appear hereafter (Nos. 190 and 191) that this formula holds true when n is *any* number, whole or fractional, positive or negative.

189. By taking A negative, the last formula becomes (No. 14)

$$(\cos A - \sin A\sqrt{-1})^n = \cos nA - \sin nA\sqrt{-1} \ldots (224)$$

This result might also be obtained from $\cos A - \sin A\sqrt{-1}$, by a process nearly the same as that in No. 188; and it will appear hereafter (Nos. 190 and 191), that this formula will also hold when n is any number, either whole or fractional.

190. If $\cos nA + \sin nA\sqrt{-1}$ be multiplied by $\cos nA - \sin nA\sqrt{-1}$, the product is $\cos^2 nA + \sin^2 nA$; which (6) is $= 1$. Hence, by division, we get

$$\frac{1}{\cos nA + \sin nA\sqrt{-1}} = \cos nA - \sin nA\sqrt{-1};$$

or, by substituting for the denominator its equal in (223),

$$\frac{1}{(\cos A + \sin A\sqrt{-1})^n} = \cos nA - \sin nA\sqrt{-1}, \text{ or}$$

$$(\cos A + \sin A\sqrt{-1})^{-n} = \cos nA - \sin nA\sqrt{-1}.$$

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respecting these contrivances, as well as many other particulars connected with the subject, should the student feel a wish to devote time to a study that is more curious than useful, recourse may be had to the treatises written expressly on this branch of science.

Now (No. 14), $\cos n\,A = \cos(-n\,A)$, and $\sin n\,A = -\sin(-n\,A)$. The last formula, therefore, may be written

$$(\cos A + \sin A\sqrt{-1})^{-n} = \cos(-n\,A) + \sin(-n\,A)\sqrt{-1}\,;$$

a formula which proves that (223) is true when n is a negative integer, as well as when it is a positive one.

191. The same formula is also true when n is a rational fraction. To prove this, it follows, from (223), that p and q being any whole numbers, positive or negative,

$$\left(\cos\frac{p}{q}A + \sin\frac{p}{q}A\sqrt{-1}\right)^q = \cos p\,A + \sin p\,A\sqrt{-1}$$
$$= (\cos A + \sin A\sqrt{-1})_p\,;$$

whence, by extraction, and by taking the second member first,

$$(\cos A + \sin A\sqrt{-1})^{\frac{p}{q}} = \cos\frac{p}{q}A + \sin\frac{p}{q}A\sqrt{-1}\,;$$

a formula which, if n be taken instead of the fractional index, is the same as (223).

It would be shown, in a similar manner, that (224) is true, while n is rational, whether it is whole or fractional, positive or negative.

If n be a surd, or any other irrational number, we can find a fraction differing from it by a quantity less than anything that can be assigned; and hence (223) and (224) are proved to be true for any real value whatever of n. These two important formulas were discovered by De Moivre.

192. The formulas (223) and (224) are true without any modification, when n is a whole number. To show how they are true, and how they give the different values of the left-hand members, when n is a fraction, we must substitute for $\cos A$ and $\sin A$ their equals, by (10) and (9), $\cos(2\,n'\pi + A)$ and $\sin(2\,n'\pi + A)$, n' being an integer. By this means, the first members remaining unchanged, we obtain

$$(\cos A + \sin A\sqrt{-1})^n = \cos(2\,nn'\pi + n\,A) + \sin(2\,nn'\pi + n\,A)\sqrt{-1} \;\ldots\ldots\; (225)$$
$$(\cos A - \sin A\sqrt{-1})^n = \cos(2\,nn'\pi + n\,A) - \sin(2\,nn'\pi + n\,A)\sqrt{-1} \;\ldots\ldots\; (226)$$

These formulas are true for every value whatever of n. If it be an integer, the second members become simply $\cos n\,A \pm \sin n\,A\sqrt{-1}$, and the formulas are therefore reduced to (223) and (224). If n be a fraction, the left-hand members will have, by the theory of equations, as many values as there are units in the denominator of n; and these values will be found by taking n' successively equal to 0, 1, 2, 3, &c. When, in these successive substitutions, n' is at length taken equal to a denominator of n, the product nn' will be a whole number, and (No. 12) the resulting value of the second member is the same as when $n'=0$; so that the right-hand member will thus, as it ought, have just as many values as the left.

193. To illustrate the foregoing remarks by an example, let $n=\frac{3}{4}$; then (225) becomes

$$(\cos A+\sin A\sqrt{-1})^{\frac{3}{4}}=\cos(\tfrac{3}{2}n'\pi+\tfrac{3}{4}A)+\sin(\tfrac{3}{2}n'\pi+\tfrac{3}{4}A)\sqrt{-1};$$

and by taking n' successively equal to 0, 1, 2, and 3, we find the only four values of the second, and, consequently, of the first member to be

$$\cos\tfrac{3}{4}A+\sin\tfrac{3}{4}A\sqrt{-1},$$
$$\cos(270^\circ+\tfrac{3}{4}A)+\sin(270^\circ+\tfrac{3}{4}A)\sqrt{-1},$$
$$\cos(540^\circ+\tfrac{3}{4}A)+\sin(540^\circ+\tfrac{3}{4}A)\sqrt{-1}, \text{ and}$$
$$\cos(810^\circ+\tfrac{3}{4}A)+\sin(810^\circ+\tfrac{3}{4}A)\sqrt{-1};$$

or, by contraction, by means of No. 12, and formulas (12) and (14),

$$\cos\tfrac{3}{4}A+\sin\tfrac{3}{4}A\sqrt{-1},\quad \sin\tfrac{3}{4}A-\cos\tfrac{3}{4}A\sqrt{-1},$$
$$-\cos\tfrac{3}{4}A-\sin\tfrac{3}{4}A\sqrt{-1},\quad \text{and } -\sin\tfrac{3}{4}A+\cos\tfrac{3}{4}A\sqrt{-1}.$$

Were n' taken equal to 4, 5, 6, &c. we should have the same series of values recurring perpetually.

194. By taking half the sum and half the difference of (223) and (224), and dividing the latter by $\sqrt{-1}$, we obtain

$$\cos nA=\tfrac{1}{2}\{(\cos A+\sin A\sqrt{-1})^n+(\cos A-\sin A\sqrt{-1})^n\}\ldots\ldots (227)$$

$$\sin nA=\frac{1}{2\sqrt{-1}}\{(\cos A+\sin A\sqrt{-1})^n-(\cos A-\sin A\sqrt{-1})^n\}(228)$$

195. By expanding the parts composing the second members of these, by the binomial theorem, and contracting the results, the imaginary expressions disappear, and we obtain the following interesting formulas for the cosine and sine of a multiple arc:

$$\cos nA=\cos^n A-\frac{n(n-1)}{1.2}\cos^{n-2}A\sin^2 A$$
$$+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\cos^{n-4}A\sin^4 A-\&c.\ldots\ldots (229)$$

$$\sin nA=n\cos^{n-1}A\sin A-\frac{n(n-1)(n-2)}{1.2.3}\cos^{n-3}A\sin^3 A$$
$$+\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}\cos^{n-5}A\sin^5 A-\&c.\ldots.. (230)$$

When n is a whole positive number, some one of the factors $n-1$, $n-2$, &c. will vanish by being $n-n$, and the series will consist of a finite number of terms; but if n be fractional, the series will be infinite.

196. By taking n successively equal to 2, 3, 4, &c. in these, and substituting $1-\cos^2 A$ for $\sin^2 A$ in the first, and $1-\sin^2 A$ for $\cos^2 A$ in the second, we obtain the following systems of the cosines and sines of multiple arcs:

$$\left.\begin{aligned}
\cos 2A &= 2\cos^2 A - 1 \\
\cos 3A &= 4\cos^3 A - 3\cos A \\
\cos 4A &= 8\cos^4 A - 8\cos^2 A + 1 \\
\cos 5A &= 16\cos^5 A - 20\cos^3 A + 5\cos A \\
\cos 6A &= 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1 \\
&\&c. \quad \&c. \quad \&c.
\end{aligned}\right\} \ldots\ldots\ldots (231)$$

$$\left.\begin{aligned}
\sin 2A &= 2\sin A\cos A \\
\sin 3A &= 3\sin A - 4\sin^3 A \\
\sin 4A &= (4\sin A - 8\sin^3 A)\cos A \\
\sin 5A &= 5\sin A - 20\sin^3 A + 16\sin^5 A \\
\sin 6A &= (6\sin A - 32\sin^3 A + 32\sin^5 A)\cos A \\
&\&c. \quad \&c. \quad \&c.
\end{aligned}\right\} \ldots\ldots\ldots (232)$$

197. We may now investigate the method of transforming a power of the cosine or sine of an angle into an expression composed of sines or cosines of its multiples,—an important problem which is the converse of the one investigated in No. 195. To effect this, assume

$$\cos A + \sin A\sqrt{-1} = z, \quad \text{and } \cos A - \sin A\sqrt{-1} = v.$$

Adding these together, we get $2\cos A = z + v$; whence, by the binomial theorem,

$$2^n \cos^n A = z^n + nz^{n-1}v + \frac{n(n-1)}{1.2} z^{n-2}v^2 + \frac{n(n-1)(n-2)}{1.2.3} z^{n-3}v^3 + \&c.$$

or, as it may be expressed,

$$2^n \cos^n A = z^n + nz^{n-2}zv + \frac{n(n-1)}{1.2} z^{n-4}z^2v^2 + \frac{n(n-1)(n-2)}{1.2.3} z^{n-6}z^3v^3 + \&c.$$

But, by No. 190, $zv=1$, and consequently its powers, z^2v^2, &c. are equal to the same. These factors, therefore, will disappear in the foregoing development. We have also (223)

$$z^n = \cos nA + \sin nA\sqrt{-1},$$
$$z^{n-2} = \cos(n-2)A + \sin(n-2)A\sqrt{-1},$$
$$\&c. \quad \&c. \quad \&c.$$

Then, by substituting these for their equals, and by separating the real and imaginary parts, we obtain $2^n \cos^n A$

$$=\left\{\begin{array}{l}\cos nA + n\cos(n-2)A + \frac{n(n-1)}{1.2}\cos(n-4)A + \frac{n(n-1)(n-2)}{1.2.3}\cos(n-6)A + \&c. \\ + \left\{\sin nA + n\sin(n-2)A + \frac{n(n-1)}{1.2}\sin(n-4)A + \frac{n(n-1)(n-2)}{1.2.3}\sin(n-6)A + \&c.\right\}\sqrt{-1}\end{array}\right\} \quad (233)$$

198. This formula is general, being, like those from which it is derived, equally applicable, whether n is integral or fractional. When n is a whole positive number, the second part of the series vanishes, and the expression becomes simply the following, which will consist of $n+1$ terms, the first and last of which are equal, as also those equally distant from them:

$$2^n \cos^n A = \cos nA + n\cos(n-2)A + \frac{n(n-1)}{1.2}\cos(n-4)A + \frac{n(n-1)(n-2)}{1.2.3}\cos(n-6)A + \&c. \ldots\ldots (234)$$

To illustrate this, it is plain that, in both parts of the series, the coefficients of all the terms following the first $n+1$ terms would vanish, in consequence of containing the factor $n-n$. It will also be seen that the numerator of the last coefficient will be $n(n-1)\ldots\ldots 3.2.1$, and its denominator $1.2.3\ldots\ldots(n-1)n$, so that the numerator and denominator being equal, the coefficient itself will be unity, the same as the coefficient of the first term; while, in the coefficient of the last term but one, the numerator will be $n(n-1)\ldots\ldots 3.2$, and the denominator $1.2.3\ldots\ldots(n-1)$, so that the coefficient itself becomes simply n, the same as that of the second term: and a like illustration is applicable in respect to the other terms. It is also easy to see that the arc in the last term is $-nA$, that in the preceding term $-(n-2)A$, &c. the negatives of the first, second, &c. terms; and, since, as we have already seen, the coefficients of these terms are equal, it follows (No. 14) that, in the second part of the series, the first and last terms, the second and the last but one, &c. mutually destroy one another; and, when n is even, the arc in the middle term, and consequently its sine, will become nothing. From this it appears that, when n is a whole positive number, the coefficient of $\sqrt{-1}$, becomes nothing; and, therefore, the second part of the series disappears.

199. By taking n successively equal to 2, 3, 4, &c. in (234),* and halving the several results, we obtain the following system:

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* In deriving these, we may avail ourselves of contractions arising from the considerations contained in the last No. Thus, when n is odd, and conse-

$$\left.\begin{array}{l} 2\cos^2 A = \cos 2A + 1 \\ 4\cos^3 A = \cos 3A + 3\cos A \\ 8\cos^4 A = \cos 4A + 4\cos 2A + 3 \\ 16\cos^5 A = \cos 5A + 5\cos 3A + 10\cos A \\ 32\cos^6 A = \cos 6A + 6\cos 4A + 15\cos 2A + 10 \\ \quad \&c. \qquad \&c. \qquad \&c. \end{array}\right\} \ldots\ldots\ldots (235)$$

200. Take the difference of the formulas at the beginning of No. 197, and multiply by $-\sqrt{-1}$; then, $2\sin A = (z-v)(-\sqrt{-1})$. Expanding this by the binomial theorem, and proceeding as in No. 197, we find $2^n \sin^n A$

$$= \left\{ \begin{array}{l} \cos nA - n\cos(n-2)A + \frac{n(n-1)}{1.2}\cos(n-4)A - \&c. \\ + \left\{ \sin nA - n\sin(n-2)A + \frac{n(n-1)}{1.2}\sin(n-4)A - \&c. \right\} \sqrt{-1} \end{array} \right\} \times (-\sqrt{-1})^n \ldots (236)$$

201. We may now consider this formula when n is a whole positive number,* and we shall find that it takes different forms according to the form of n. It would appear, as in No. 198, that when n is even, the second part of the series vanishes, the coefficient of $\sqrt{-1}$ becoming nothing. In this case, also, if n be 4, 8, 12, &c. or, in general, if it be of the form $4m$, m being a whole number, the multiplier $(-1\sqrt{-1})^n$ will become simply 1; but, if n be 2, 6, 10, &c. or $4m+2$, the multiplier will become -1. On this supposition, therefore (236) will be reduced to the following, in which the upper sign is to be used when $n=4m$; and the lower, when $n=4m+2$:

$$2^n \sin^n A = \pm \left\{ \cos nA - n\cos(n-2)A + \frac{n(n-1)}{1.2}\cos(n-4)A - \&c. \right\}. (237)$$

202. If n be an odd number, it would appear, in nearly the same manner, that the first part of (236) would vanish, and that the formula would be converted into the following, in which the upper

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quently the number of terms even, the second members will be found simply by substituting n in the general formula (234), and taking all the terms in which the arc is positive; and, in addition to this, when n is even, the last term, which is a number, is to be halved. The same is to be observed in respect to (237) and (238).

* When n is a fraction, (233) and (236) give the several roots or values of $2^n \cos^n A$ and $2^n \sin^n A$; but the examination of this part of the theory is not sufficiently elementary to be given here. The subject of this Section has been lately examined by the French mathematicians, Poisson and Poinsot, and freed from errors which had escaped the observation of former writers.

sign is to be used when $n=4m+1$, and the lower when $n=4m+3$:

$$2^n \sin^n A = \pm \left\{ \sin nA - n \sin(n-2)A + \frac{n(n-1)}{1.2} \sin(n-4)A - \&c. \right\} \quad (238)$$

203. By taking n equal to 2, 3, 4, &c. we obtain from the last two formulas the following:

$$\left.\begin{aligned}
2\sin^2 A &= -\cos 2A + 1 \\
4\sin^3 A &= -\sin 3A + 3\sin A \\
8\sin^4 A &= \cos 4A - 4\cos 2A + 3 \\
16\sin^5 A &= \sin 5A - 5\sin 3A + 10\sin A \\
32\sin^6 A &= -\cos 6A + 6\cos 4A - 15\cos 2A + 10 \\
\&c. \quad & \quad \&c. \quad \&c.
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (239)$$

These formulas might also be obtained, very simply and easily, from formulas (235), by substituting 90^0-A for A.

204. By dividing (230) by (229), and again (229) by (230), and then by dividing the numerators and denominators of the second members respectively by $\cos^n A$ and $\sin^n A$, we get

$$\tan nA = \frac{n\tan A - \frac{n(n-1)(n-2)}{1.2.3}\tan^3 A + \&c.}{1 - \frac{n(n-1)}{1.2}\tan^2 A + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\tan^4 A - \&c.} \quad \ldots\ldots (240)$$

$$\cot nA = \frac{\cot^n A - \frac{n(n-1)}{1.2}\cot^{n-2} A + \&c.}{n\cot^{n-1} A - \frac{n(n-1)(n-2)}{1.2.3}\cot^{n-3} A + \&c.} \quad \ldots\ldots\ldots\ldots (241)$$

IX.—MISCELLANEOUS PROPOSITIONS.*

205. PROVE that the sum of the tangents of the three angles of a plane triangle is equal to their product.

This follows at once from Euc. I. 32., and from (43); as, in case of a plane triangle, the first member of this equation vanishes, which

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* This Section will be found to contain, in small compass, much useful matter that could not be given at greater length, without swelling this work beyond its intended limits. By studying it with care, the student will not only be presented with useful exercise on the preceding part of the work, but he will find himself enabled to apply with success, the principles already established, in such cases as may turn up in the farther prosecution of his studies.

can take place only when the numerator of the second member also vanishes; that is, when $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

It is evident, also, that, if the sum of three arcs or angles be any multiple of 180°, the product of their tangents is equal to their sum.

We should find, in like manner, from (44), that the sum of the cotangents of three arcs is equal to their product, when the sum of the arcs is $(n+\frac{1}{2})\pi$, n being any whole number.

206. Given the base, the vertical angle, and the sum of the sides of a plane triangle; to find the sides.

Since (Euc. I. 32) half the sum of two angles of a triangle is the complement of half the remaining angle, we have, from (76),

$$c : a+b :: \sin\tfrac{1}{2}C : \cos\tfrac{1}{2}(A-B);$$

whence the remaining angles will be known, and thence the remaining sides.

We should have, in like manner, from (77),

$$c : a-b :: \cos\tfrac{1}{2}C : \sin\tfrac{1}{2}(A-B);$$

an analogy which solves the problem in which the base, the vertical angle, and the difference of the sides, are given. The same formulas may also be adapted to solve the problems in which the base, the difference of the angles at the base, and either the sum or difference of the sides are given; and they solve, without modification, the problems in which there are given two angles and the sum or difference of the opposite sides.

207. Given the base of a plane triangle, one of the angles at the base, and the difference of the other sides; to resolve the triangle.

Here, taking c as base, and taking half the difference, and half the sum of it and $a-b$, we find $s-a$ and $s-b$; and if A be the given angle, we have (82)

$$s-b : s-a :: \tan\tfrac{1}{2}A : \tan\tfrac{1}{2}B.$$

208. Given the base of a plane triangle, one of the angles at the base, and the sum of the other sides; to resolve the triangle.

This will be solved by means of formula (83).

209. Given the angles and the perimeter of a plane triangle; to resolve it.

By dividing the value of $\sin\frac{1}{2}C$, according to (79), by the product of the values of $\cos\frac{1}{2}A$ and $\cos\frac{1}{2}B$, according to (80), and converting the product into an analogy, we obtain

$$\cos\tfrac{1}{2}A \cos\tfrac{1}{2}B : \sin\tfrac{1}{2}C :: s : c.$$

210. Given two sides of a plane triangle, and the difference of the opposite angles; to resolve the triangle.

By (75), $a-b : a+b :: \tan\frac{1}{2}(A-B) : \tan\frac{1}{2}(A+B)$.

211. Given the times at which the sun sets and is west on the same day, at a particular place; to find the latitude of the place, and the sun's declination.

To solve this, taking formulas 1 and 6 in the note, page 73, divide the first of them by the second, and also find their product: then, if the signs be changed, the square roots of the results will be the tangents of the latitude and declination.

212. Given the time of the sun's setting, and his altitude at six o'clock on the same day, and at the same place; to find the latitude and the declination.

This question will be easily solved by dividing equation 3 by equation 1 in the same note; as the sum and difference of equation 3 and the quotient will become, by (14), (15), (31), and (32),

$$\cos(l+d) = -\frac{2\sin a\cos^2\frac{1}{2}P}{\cos P}, \text{ and } \cos(l \backsim d) = -\frac{2\sin a\sin^2\frac{1}{2}P}{\cos P}.$$

213. Given the azimuth at setting, and the altitude at six o'clock; to find as before.

By equalling the values of $\sin d$ in the second and third of the same equations, we find, by an easy process,

$$\sin 2l = \frac{2\sin a}{\cos Z}.$$

214. Given the azimuths of the sun at setting and at six o'clock; to find as before.

Multiply the second of the same equations by the fourth, divide the result by $\cos l$, and multiply by $\sin d$: then, by writing $1-\cos^2 d$ for $\sin^2 d$, there will arise

$$1-\cos^2 d = \cos Z \cot Z' \cos d;$$

from which the value of $\cos d$ will be found by the resolution of a quadratic.

215. Given the sun's meridian altitude, and his altitude at six o'clock, at the same place, and on the same day; to find the latitude and declination.

Let a'' and a be the respective altitudes: then it will be seen from *fig.* 30, that $En = Hn + EZ - ZH$, or $d = a'' + l - 90^0$; whence $\sin d =$

$-\cos(a''+l)$. By multiplying this by $\sin l$, and comparing the product with equation 3, note, page 73, we obtain

$$\sin l \cos(a''+l)) = -\sin a, \text{ or (17)}$$
$$\sin(a''+2l) - \sin a'' = -2\sin a.$$

From this $a''+2l$, and consequently l itself, will be known.

216. Given the sun's declination, and the interval between the times at which he is west and sets; to find the latitude.

Let T be the given time; then $P-P'=T$, and (15)

$$\cos P \cos P' + \sin P \sin P' = \cos T.$$

We have also, by taking the product of equations 1 and 6, in the note, page 73, $\cos P \cos P' = -\tan^2 d$. From the double of this take the former, and there will remain

$$\cos P \cos P' - \sin P \sin P' \text{ or (14) } \cos(P+P') = -\cot T - 2\tan^2 d.$$

Having thus the sum and difference of P and P′, we can find these angles; and, from either of them and the declination, the latitude can be found by the note already referred to.

217. Required the sum of n terms of the two series',

$$\sin\varphi + \sin 2\varphi + \sin 3\varphi + \ldots\ldots\ldots\ldots + \sin n\varphi, \text{ and}$$
$$\cos\varphi + \cos 2\varphi + \cos 3\varphi + \ldots\ldots\ldots\ldots + \cos n\varphi.$$

To sum these and similar series', in which the arcs are equidifferent, *multiply by twice the sine of half the common difference, modify the products by* (19) *and* (17) *contract the results, and divide by twice the sine of half the common difference.* Thus, to sum the first of these, put

$$s = \sin\varphi + \sin 2\varphi + \sin 3\varphi + \ldots + \sin(n-1)\varphi + \sin n\varphi.$$

Then, multiplying by $2\sin\frac{1}{2}\varphi$, applying (19), and separating the positive and negative terms, we obtain $2s\sin\frac{1}{2}\varphi$

$$= \begin{cases} \cos\frac{1}{2}\varphi + \cos\frac{3}{2}\varphi + \ldots + \cos(n-\frac{3}{2}\varphi) + \cos(n-\frac{1}{2})\varphi \\ \quad -\cos\frac{3}{2}\varphi - \cos\frac{5}{2}\varphi + \ldots\ldots - \cos(n-\frac{1}{2}\varphi) - \cos(n+\frac{1}{2})\varphi. \end{cases}$$

Now, it is evident, that the second member becomes, by contraction, $\cos\frac{1}{2}\varphi - \cos(n+\frac{1}{2})\varphi$, or (23), $2\sin\frac{1}{2}n\varphi \sin\frac{1}{2}(n+1)\varphi$. Dividing, therefore, by $2\sin\frac{1}{2}\varphi$, we get

$$s = \frac{\sin\frac{1}{2}n\varphi \sin\frac{1}{2}(n+1)\varphi}{\sin\frac{1}{2}\varphi},$$

which is the required sum; and, by a process exactly similar, we should find for the other series,

$$s=\frac{\sin\frac{1}{2}n\varphi\,\cos\frac{1}{2}(n+1)\varphi^*}{\sin\frac{1}{2}\varphi}.$$

218. Required the sum of n terms of the series,

$$\operatorname{cosec} A+\operatorname{cosec} 2A+\operatorname{cosec} 4A+\operatorname{cosec} 8A+\ldots+\operatorname{cosec} 2^{n-1}A.$$

Here, by successive applications of (53), we obtain

$$\operatorname{cosec} A=\cot\tfrac{1}{2}A-\cot A,$$
$$\operatorname{cosec} 2A=\cot A-\cot 2A,$$
$$\ldots\ldots\ldots\ldots\quad\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\operatorname{cosec} 2^{n-1}A=\cot 2^{n-2}A-\cot 2^{n-1}A;$$

and by summing these, we find, in consequence of all the terms of the second members destroying one another, except the first in the first equation, and the last in the second,

$$s=\cot\tfrac{1}{2}A-\cot 2^{n-1}A.$$

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* Dividing the former sum by the latter, we obtain

$$\frac{\sin\varphi+\sin 2\varphi+\sin 3\varphi+\ldots\ldots\ldots+\sin n\varphi}{\cos\varphi+\cos 2\varphi+\cos 3\varphi+\ldots\ldots\ldots+\cos n\varphi}=\tan\tfrac{1}{2}(n+1)\varphi.$$

From the first of the foregoing sums it would appear, by taking $\varphi=1'$, that the sum of the sines of all the minutes in the quadrant is $=\frac{\sin 45^\circ\times\sin(45^\circ\,0'\,30'')}{\sin 30''}$ $=3438{\cdot}2468$; and, in a similar manner, the sum of the cosines of the minutes in the quadrant would be found to be $3437{\cdot}2468$, being less than the foregoing by the radius.

In the sums found above, it may be remarked, that the arc in the denominator is half the common difference; and, of those in the numerator, one is n times half the common difference, while the other is half the sum of the arcs in the first and last terms. The sum of the arcs themselves is $\frac{1}{2}n(n+1)\varphi$, or, as it may be expressed. $\frac{\frac{1}{2}n\varphi.\,\frac{1}{2}(n+1)\varphi}{\frac{1}{2}\varphi}$, an expression analogous to the first of those found above.

To find the sum of an *infinite* number of terms of series', such as those in the text, Cagnoli modifies the numerator of the sum by (19) or (17), and rejects in the result the term which contains the infinite arc; as, he says, the sine or cosine of such an arc is unassignable. In this way, the sums in the text would become $\frac{1}{2}\cot\frac{1}{2}\varphi$, and $-\frac{1}{2}$. This is an error, however; as the sine or cosine of an arc indefinitely increased is unassignable, not from its minuteness, but from its perpetually changing its magnitude as the arc varies; and hence the sum of the infinite series in both cases, is indeterminate, as there is no limit to which it approximates.

The sum of m terms of the series, $\sin\varphi+2^n\sin 2\varphi+3^n\sin 3\varphi+4^n\sin 4\varphi+$ &c. or of any similar series, either of sines or cosines, when n is a whole number, would be found by multiplying $n+1$ times by $\sin\frac{1}{2}\varphi$, and each time applying (19) or (17); or, more generally, the same method is applicable in all cases,

219. Required the sum of n terms, and the sum of an infinite number of terms, of the series,

$$\sec^2 A+\tfrac{1}{4}\sec^2\tfrac{1}{2}A+\tfrac{1}{16}\sec^2\tfrac{1}{4}A+\tfrac{1}{64}\sec^2\tfrac{1}{8}A+\ \&c.*$$

To sum this, we have $\sec^2 x=\dfrac{1}{\cos^2 x}=\dfrac{\sin^2 x}{\sin^2 x\cos^2 x}=\dfrac{1-\cos^2 x}{\sin^2 x\cos^2 x}=$

$$\frac{1}{\sin^2 x\cos^2 x}-\frac{\cos^2 x}{\sin^2 x\cos^2 x}=\frac{4}{4\sin^2 x\cos^2 x}-\frac{1}{\sin^2 x}=\frac{4}{\sin^2 2x}-\frac{1}{\sin^2 x}.$$

Taking x in this formula successively equal to A, $\frac{1}{2}$A, $\frac{1}{4}$A, &c. and dividing the second result by 4, the third by 16, &c. we obtain

$$\sec^2 A=\frac{4}{\sin^2 2A}-\frac{1}{\sin^2 A},$$

$$\tfrac{1}{4}\sec^2\tfrac{1}{2}A=\frac{1}{\sin^2 A}-\frac{1}{4\sin^2\frac{1}{2}A},$$

$$\tfrac{1}{16}\sec^2\tfrac{1}{4}A=\frac{1}{4\sin^2\frac{1}{2}A}-\frac{1}{16\sin^2\frac{1}{4}A},$$

....

$$\frac{1}{4^{n-1}}\sec^2\frac{A}{2^{n-1}}=\frac{1}{4^{n-2}\sin^2\frac{A}{2^{n-2}}}-\frac{1}{4^{n-1}\sin^2\frac{A}{2^{n-1}}}:$$

and adding these together, we get

$$s=\frac{4}{\sin^2 2A}-\frac{1}{4^{n-1}\sin^2\frac{A}{2^{n-1}}}.$$

Now if n, the number of terms, become infinite, the arc in the denominator of the last term will become infinitely small, and therefore (No. 162) equal to its sine. Using, therefore, the square of the arc instead of the square of its sine, and observing that 4^{n-1} is equal

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when the arcs are equidifferent, and the coefficients are such that some of the orders of successive differences vanish; such as when the coefficients are figurate numbers, or the powers of equidifferent numbers.

It may be proper to remark, that if in the two series' considered above, and in their sums, ϕ be changed into $\pi-\phi$, we should obtain the sums of the two,

$$\sin\phi-\sin 2\phi+\sin 3\phi-\sin 4\phi+\ \ldots\ \pm\sin n\phi,\text{ and}$$
$$\cos\phi-\cos 2\phi+\cos 3\phi-\sin 4\phi+\ \ldots\ \pm\cos n\phi.$$

* A series similar to this is proposed for summation in Luby's Trigonometry, page 92, No. 16; but both the result, and the principles on which it is obtained, are incorrect.

to the square of 2^{n-1}, if we put s' for the sum of the infinite series, we get

$$s'=\frac{4}{\sin^2 2A}-\frac{1}{A^2}.$$

Hence, $\frac{1}{A^2}=\frac{4}{\sin^2 2A}-s'=4\operatorname{cosec}^2 2A-s'$, a formula which gives the square of the reciprocal of an arc, and thence the arc itself, by means of the cosecant of its double, and of the secants of itself, its half, its fourth, &c. Thus, if $A=45^0=\frac{1}{4}\pi$, we get, since cosec $90^0=1$,

$$\frac{1}{(\frac{1}{4}\pi)^2}=4-(\sec^2 45^0+\tfrac{1}{4}\sec^2 22^0\ 30'+\tfrac{1}{16}\sec^2 11^0\ 15'+\ \&c.)$$

Now, from (38), we have $1+\cos 2A=2\cos^2 A$; whence, by dividing both members by $1+\cos 2A$, and $\cos^2 A$, we get $\sec^2 A=\frac{2}{1+\cos 2A}$; a formula by means of which, and of (73) and (74), the square of the secant of half an arc may be computed. Thus, by taking A successively equal to 45^0, 22^0, $30'$, &c. we should find the values of the cosines of these arcs, and thence $\sec^2 45^0$, $\sec^2 22^0\ 30'$, $\sec^2 11^0\ 15'$, $\sec^2 5^0\ 37'\frac{1}{2}$, &c.; by employing which in the foregoing formula, we should be enabled to find the value of π. As, however, this important number may be more easily calculated by other means, it is unnecessary to give the computation here; but it may serve as an exercise to the student.

220. If the logarithms of two numbers, a and b, be given, the logarithms of their sum and difference, and of the sum and difference of their squares, may be obtained in the following manner, without finding the numbers themselves.

Put $a+b$ under the form $a(1+\frac{b}{a})$; and put $\frac{b}{a}=\tan^2\varrho$, so that, supplying the radius, taking the logarithms, and halving, we shall have $\log\tan\varrho=\frac{1}{2}(20+\log b-\log a)$. By this means, we obtain $a+b=a(1+\tan^2\varrho)=a\sec^2\varrho$; whence, by supplying the radius, and taking the logarithms, we get $\log(a+b)=\log a+2\log\sec\varrho-20$.

Again, putting $a-b$ under the form $a\left(1-\frac{b}{a}\right)$, assume $\frac{b}{a}=\sin^2\varrho$; then, $a-b=a(1-\sin^2\varrho)=a\cos^2\varrho$. Hence, therefore, find ϱ from the equation $\log\sin\varrho=\frac{1}{2}(20+\log b-\log a)$; then $\log(a-b)=\log a+2\log\cos\varrho-20$.

By a similar process, we should find that, if $\log\tan\varrho=10+\log b$

$-\log a$, then $\log(a^2+b^2)=2(\log a+\log\sec\varrho-10)$; and if $\log\sin\varrho =10+\log b-\log a$, then $\log(a^2-b^2)=2(\log a+\log\cos\varrho-10)$.

The same results might also be obtained in other modes. Thus, putting $a+b$ under the same form as before, and assuming $\frac{b}{a}=\cos\varrho$, we get $a+b=a(1+\cos\varrho)=2a\cos^2\frac{1}{2}\varrho$. Hence, therefore, find ϱ from the equation $\log\cos\varrho=10+\log b-\log a$; and then will $\log(a+b) =\log 2+\log a+2\log\cos\frac{1}{2}\varrho-20$. In using this method, the greater number must be put $=a$, as otherwise $\cos\varrho$ would be greater than the radius.

The principle employed in this No.—that of introducing an auxiliary arc—is often useful in adapting formulas for computation by means of logarithms. On the same principle, we might derive the results contained in Section III. from No. 86 up to the scholium in page 38; and the following Nos. afford other examples of its use.

221. Quadratic equations may be solved, by means of trigonometrical quantities, in a way which is sometimes easier in practice than the common method. Thus, if the proposed equation be $ax^2+bx =c$, by multiplying by $4a$, adding b^2 to both members of the product, extracting the square root, transposing b, and dividing by $2a$, we obtain

$$x=\frac{-b\pm\sqrt{(b^2+4ac)}}{2a},\text{ or } x=\frac{b}{2a}\left\{-1\pm\sqrt{\left(1+\frac{4ac}{b^2}\right)}\right\}.$$

Now, accordingly as $4ac$ is positive or negative, let us assume

$$\frac{4ac}{b^2}=\tan^2\varrho,\text{ or }\frac{4ac}{b^2}=\sin^2\varrho;\text{ and we shall have}$$

$$x=\frac{b}{2a}(-1\pm\sec\varrho),\text{ and } x=\frac{b}{2a}(-1\pm\cos\varrho).$$

But $-1\pm\sec\varrho=-\frac{\cos\varrho\mp1}{\cos\varrho}$; the two values of which, by (32) and (31), are

$$\frac{2\sin^2\frac{1}{2}\varrho}{\cos\varrho}\text{ and }-\frac{2\cos^2\frac{1}{2}\varrho}{\cos\varrho}.$$

In the first case, therefore, putting x' and x'' to denote the roots, we shall have $x'=\frac{b}{a}\cdot\frac{\sin^2\frac{1}{2}\varrho}{\cos\varrho}$, and $x''=-\frac{b}{a}\cdot\frac{\cos^2\frac{1}{2}\varrho}{\cos\varrho}$; and dividing the latter by the former, and multiplying the result by x', we obtain $x''=-x'\cot^2\frac{1}{2}\varrho$.

In the second case, $-1+\cos\varrho=-2\sin^2\frac{1}{2}\varrho$, and $-1-\cos\varrho=$

$-2\cos^2\frac{1}{2}\varrho$; and therefore $x' = -\frac{b}{a}.\sin^2\frac{1}{2}\varrho$, and $x'' = -\frac{b}{a}.\cos^2\frac{1}{2}\varrho$; or $x'' = x'\cot^2\frac{1}{2}\varrho$.

Hence, to resolve the equation $ax^2 + bx = c$:

I. If a and c have the same sign, find ϱ from the equation,

$$\left.\begin{array}{l}\log\tan\varrho = \frac{1}{2}(20 + \log 4 + \log a + \log c - 2\log b);\text{ then,}\\ \log x' = \log b + 2\log\sin\frac{1}{2}\varrho - \log a - \log\cos\varrho - 10,\text{ and}\\ \log x'' = \log x' + 2\log\cot\frac{1}{2}\varrho - 20.\end{array}\right\}\dots(242)$$

The signs of these roots are contrary.

II. If a and c have contrary signs, find ϱ from the equation,

$$\left.\begin{array}{l}\log\sin\varrho = \frac{1}{2}(20 + \log 4 + \log a + \log c - 2\log b);\text{ then,}\\ \log x' = \log b + 2\log\sin\frac{1}{2}\varrho - \log a - 20,\text{ and}\\ \log x'' = \log x' + 2\log\cot\frac{1}{2}\varrho - 20.\end{array}\right\}\dots(243)$$

The latter of these cases is impossible, when the data are such as to render $\sin\varrho$ greater than the radius; that is, when $4ac$ is greater than b^2.

These methods of solution may be employed with advantage when the coefficients are large numbers, or fractions with large numerators or denominators.

222. Cubic equations may also be solved on similar principles. Thus, let the proposed equation be $x^3 + ax = b$. By substituting $y + z$ for x, this is changed into

$$y^3 + z^3 + 3yz(y+z) + a(y+z) = b;$$

which, by taking $3yz = -a$, becomes $y^3 + z^3 = b$.
From the square of this, take four times the cube of yz, found from the equation $3yz = -a$, and the square root of the remainder will be the value of $y^3 - z^3$. Then, taking the cube roots of half the sum and half the difference of this, and of $y^3 + z^3 = b$, and adding them together, we find

$$x = \sqrt[3]{\left\{\tfrac{1}{2}b + \tfrac{1}{2}\sqrt{\left(b^2 + \frac{4a^3}{27}\right)}\right\}} + \sqrt[3]{\left\{\tfrac{1}{2}b - \tfrac{1}{2}\sqrt{\left(b^2 + \frac{4a^3}{27}\right)}\right\}}.$$

Now, be positive, put $\frac{4a^3}{27b^2} = \tan^2\varphi$, which gives $\frac{1}{2}b = \cot\varphi\sqrt{\frac{a^3}{27}}$; then,

$$x = \sqrt[3]{\{\tfrac{1}{2}b(1 + \sec\varphi)\}} + \sqrt[3]{\{\tfrac{1}{2}b(1 - \sec\varphi)\}};$$

or, by substituting its equal for $\frac{1}{2}b$, and by an easy reduction,

$$x = \sqrt{\tfrac{1}{3}a} \times (\sqrt[3]{\cot\tfrac{1}{2}\varphi} - \sqrt[3]{\tan\tfrac{1}{2}\varphi}).$$

Put $\sqrt[3]{\tan\frac{1}{2}\varphi}=\tan\varrho$, and consequently $\sqrt[3]{\cot\frac{1}{2}\varphi}=\cot\varrho$; then (51)

$$x=2\cot 2\varrho\sqrt{\tfrac{1}{3}a};\text{* where } \tan\varphi=\frac{2ar}{3b}\sqrt{\tfrac{1}{3}a}, \text{ and } \tan\varrho=\sqrt[3]{r^2\tan\tfrac{1}{2}\varphi}. \quad (244)$$

If in this case b be negative, the value of x will be the negative of the foregoing; but every thing else will be the same as above.

223. If a be negative, this method fails. In that case, assume $\sin^2\varphi=\frac{4a^3}{27b^2}$, and $\tan\varrho$ as before. Then, by a process similar to the foregoing, and by using at the conclusion the formula found by adding together (50) and (53), we obtain

$$x=2\,\text{cosec}\,2\varrho\sqrt{\tfrac{1}{3}a}; \text{ where } \sin\varphi=\frac{2ar}{3b}\sqrt{\tfrac{1}{3}a}, \text{ and } \tan\varrho=\sqrt[3]{r^2\tan\tfrac{1}{2}\varphi}. \quad (245)$$

If in this case b be negative, the value of x, as in the last No. will be the negative of the foregoing; but the method of computation will be the same.

224. Even this method also fails, if $4a^3$ be greater than $27b^2$, as $\cos\varphi$ would be imaginary, and the equation would belong to what has been called the *irreducible case.* To investigate the method of solution in this case, we have the equation $x^3-ax=b$, where the coefficient of x is negative, and b either positive or negative. Now, using $\frac{1}{3}z$ instead of A in the second of formulas (231), supplying a radius r, and dividing by 4, we obtain $\cos^3\frac{1}{3}z-\frac{3}{4}r^2\cos\frac{1}{3}z=\frac{1}{4}r^2\cos z$. Comparing this with the proposed equation, we have

$$x=\cos\tfrac{1}{3}z, \qquad a=\tfrac{3}{4}r^2 \qquad b=\tfrac{1}{4}r^2\cos z.$$

The second of these gives $r=2\sqrt{\frac{1}{3}a}$; while, by using this value for r, we find, from the third, $\cos z=\frac{3b}{a}$; and it is obvious, that if the radius were equal to the value of r above mentioned, we should have $x=\cos\frac{1}{3}z$. To find the value of x, therefore, to the radius 1, we divide the value of $\cos z$, and multiply that of x, by the foregoing value of r, and we thus obtain

$$\cos z=\frac{3b}{a}\sqrt{\frac{3}{4a}}, \text{ or } \cos z=\frac{3b}{2a\sqrt{\frac{1}{3}a}}, \text{ and } x=\frac{\cos\frac{1}{3}z\times 2\sqrt{\frac{1}{3}a}}{r}.$$

Now, since (No. 17) $\cos z$ is the same as $\cos(z+360^\circ)$, or $\cos(z+720^\circ)$,

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* In the actual computation here and elsewhere, the student will have no difficulty in supplying the radius r, when necessary.

we may use one third of each of these arcs as well as $\frac{1}{3}z$. Hence, therefore, since (Nos. 15 and 14) $\cos(\frac{1}{3}z+120^\circ)=-\cos(\frac{1}{3}z-60^\circ)$, and $\cos(\frac{1}{3}z+240^\circ)=-\cos(\frac{1}{3}z+60^\circ)$, we have the following values of x, which are the three roots of the equation:

$$\left.\begin{array}{l} x=\dfrac{\cos\frac{1}{3}z\times 2\sqrt{\frac{1}{3}a}}{r}, \quad x=-\dfrac{\cos(\frac{1}{3}z\backsim 60^\circ)\times 2\sqrt{\frac{1}{3}a}}{r}, \\ x=-\dfrac{\cos(\frac{1}{3}z+60^\circ)\times 2\sqrt{\frac{1}{3}a}}{r}; \text{ where } \cos z=\dfrac{3br}{2a\sqrt{\frac{1}{3}a}} \end{array}\right\} \ldots (246)$$

When b is negative, the signs of these roots are to be changed.

225. To find series' expressing the sine and cosine of an arc in terms of the arc itself, divide both members of (229) and (230) by $\cos^n A$, and the terms in the second members, exclusive of the coefficients, will become

$$1,\ \tan^2 A,\ \tan^4 A,\ \&c.;\ \text{and}$$
$$\tan A,\ \tan^3 A,\ \tan^5 A,\ \&c.$$

In the results thus obtained, let $nA=x$, and consequently $n=\frac{x}{A}$; then retaining the denominators of the left-hand members unchanged, we shall have, instead of their numerators, $\cos x$ and $\sin x$; while the right-hand members will become

$$1-\frac{x(x-A)\tan^2 A}{1.2\,A^2}+\frac{x(x-A)(x-2A)(x-3A)\tan^4 A}{1.2.3.4\,A^4}-\&c.;\ \text{and}$$
$$\frac{x\tan A}{A}-\frac{x(x-A)(x-2A)\tan^3 A}{1.2.3\,A^3}+\&c.$$

Let, now, $A=0$; then, $\frac{\tan A}{A}=1$*, $\cos^n A=1$ (No. 10); and, A disappearing in the coefficients, we shall have

$$\cos x=1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\frac{x^6}{1.2.3.4.5.6}+\&c.\ \ldots (247)$$
$$\sin x=x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\&c.\ \ldots\ldots\ldots\ldots\ldots (248)$$

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* That the ratio of a diminishing arc and its tangent tends to become that of equality as its limit, will appear as follows. The area of the triangle ACH (*fig.* 1) is equal to $\frac{1}{2}$CA.AH; and the area of the sector ACF is equal to $\frac{1}{2}$CA.AF, as may be shown by resolving it into an infinite number of infinitely small sectors by radii, which sectors may be regarded as triangles. Hence, CAH : CAF :: $\frac{1}{2}$CA.AH : $\frac{1}{2}$CA.AF :: AH : AF. But CAH > CAF; therefore, AH > AF; that is, $\tan A > A$, or, by dividing by A, $\frac{\tan A}{A}>1$. Now, since a straight line is the shortest distance between two

These series' were discovered by Newton; and they enable us to compute the sine and cosine of any arc, the length of which is given, to the radius 1.

226. Given, in a survey, the angular elevations of two points above the plane of the horizon, and their angular distance asunder; to find the *horizontal angle*, or angular distance of the projections of the given points on the plane of the horizon by perpendiculars.

This angle is evidently the same as the vertical angle of a spherical triangle, of which the measured distance is the base, and the complements of the elevations the sides, as would appear by producing the perpendiculars till they meet in the zenith. The calculation, therefore, is effected by the first case of spherical trigonometry. When the elevations are small, the observed angle may be reduced to the horizontal one by an easy approximation; which, with several others of a similar kind, may be investigated most easily by means of the differential calculus.

227. The following interesting theorem was discovered by Legendre:

If there be a spherical triangle, the sides of which are very small compared with the radius of the sphere, it is very nearly equivalent to a plane triangle which has its sides equal to the sides of the proposed triangle, and its angles equal to the angles of the same diminished respectively by one third of the spherical excess. (See No. 148.)

To prove this, we have, from (85),

$$\cos A \sin b \sin c = \cos a - \cos b \cos c.$$

This, by applying (247) and (248) to all the sines and cosines contained in it, except cos A, and by rejecting all quantities rising above four dimensions, will become $bc \cos A\{1-\frac{1}{6}(b^2+c^2)$

$$=1-\tfrac{1}{2}a^2+\tfrac{1}{24}a^4-1+\tfrac{1}{2}(b^2+c^2)-\tfrac{1}{24}(b^4+c^4)-\tfrac{1}{4}b^2c^2$$
$$=\tfrac{1}{2}(b^2+c^2-a^2)-\tfrac{1}{24}(b^4+c^4-a^4)-\tfrac{1}{4}b^2c^2.$$

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points, and since the sine of an arc is half the chord of twice the same arc, we have $\sin A < A$, or $\frac{\sin A}{A} < 1$; and, therefore, dividing by cos A, we get $\frac{\tan A}{A} < \frac{1}{\cos A}$. Hence, $\frac{\tan A}{A}$ is of a magnitude intermediate between 1 and $\frac{1}{\cos A}$, the latter of which (No. 10) tends continually to 1 as its limit, when A is diminished indefinitely; and hence, when the arc is infinitely small, the ratio of it and its tangent will differ in an infinitely small degree from that of equality; so that, when the arc is in a vanishing state, it and its tangent are to be regarded as equal.

By contracting this, dividing by the coefficient of cos A, and performing the actual division by $1-\frac{1}{6}(b^2+c^2)$ we obtain

$$\cos A=\frac{b^2+c^2-a^2}{2bc}+\frac{a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2}{24bc}.$$

Now, if A′ be the angle of a rectilineal triangle, the sides of which are equal in length to the arcs, a, b, c, the first part of this is equivalent (78) to cos A′; and, by taking the square of this value of cos A′ from unity, we find the second part to be equal to $-\frac{bc\sin^2 A'}{6}$; so that

$$\cos A=\cos A'-\frac{bc\sin^2 A'}{6}.$$

If $A=A'+x$, we get $\cos A=\cos A'\cos x-\sin A'\sin x$; which, by applying (247) and (248), and rejecting the second and higher powers of x, becomes $\cos A=\cos A'-x\sin A'$. Equalling this and the former value of cos A, and rejecting cos A′, we get $x=\frac{bc\sin A'}{6}$; and, therefore,

$$A=A'+\frac{bc\sin A'}{6}=A'+\frac{\frac{1}{2}bc\sin A'}{3}.$$

Now, according to the note, page 20, $\frac{1}{2}bc\sin A'$ is the area of the rectilineal triangle whose sides are a, b, c; and this will not differ sensibly from the area of the spherical triangle, which, by No. 148, is proportional to the spherical excess. Putting, therefore, the excess equal to E, we have $A=A'+\frac{1}{3}E$, and it might be shown, in a similar manner, that $B=B'+\frac{1}{3}E$, and $C=C'+\frac{1}{3}E$.

By the introduction of the radius r, the proof would be rendered apparently more strict. The final conclusion, however, is the same; and the proof is somewhat more simple, as given above.*

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* By means of this theorem, a spherical triangle, whose sides do not exceed a degree or two, may be resolved as a plane one by taking from each of its angles one third of the spherical excess. Hence, if the angles and one side be determined by measurement, the excess will be known; and the other sides will be computed by the first case of plane trigonometry.

This theorem, as well as those in Nos. 226 and 228, is useful in *geodetical* operations, that is, in surveys of kingdoms or other large portions of the earth's surface, in measuring the lengths of degrees, or in determining the relative positions of noted places. It would be inconsistent with the nature of this work to give minute details respecting this important branch of practical science. At the same time, it may be interesting to the student to have some idea of the general nature of such operations.

228. If, in a spherical triangle, two sides b and c and the contained angle A be given, and if A′ be put to denote the angle contained by the chords of b and c, then

$$\cos A' = \cos A \cos\tfrac{1}{2}b \cos\tfrac{1}{2}c + \sin\tfrac{1}{2}b \sin\tfrac{1}{2}c.$$

For (38) $\cos a = 1 - 2\sin^2\tfrac{1}{2}a = 1 - \tfrac{1}{2}k^2$, $\cos b = 1 - \tfrac{1}{2}k'^2$, and $\cos c = 1 - \tfrac{1}{2}k''^2$, k, k', and k'', being the chords of a, b, and c: also (37) $\sin b = 2\sin\tfrac{1}{2}b \cos\tfrac{1}{2}b = k' \cos\tfrac{1}{2}b$, and $\sin c = k'' \cos\tfrac{1}{2}c$. The substitution of these in (85) gives, after rejecting 1 in each member, transposing $-\tfrac{1}{2}k'^2$ and $-\tfrac{1}{2}k''^2$, and dividing by $k'k''$,

$$\frac{k'^2 + k''^2 - k^2}{2k'k''} = \cos A \cos\tfrac{1}{2}b \cos\tfrac{1}{2}c + \tfrac{1}{4}k'k''.$$

But (78) the first member of this is equivalent to $\cos A'$; and, by our assumption, $\tfrac{1}{4}k'k'' = \sin\tfrac{1}{2}b \sin\tfrac{1}{2}c$; whence,

$$\cos A' = \cos A \cos\tfrac{1}{2}b \cos\tfrac{1}{2}c + \sin\tfrac{1}{2}b \sin\tfrac{1}{2}c \ldots \quad (249)$$

If, by this formula, all the three angles of the triangle formed by the chords, be computed, their sum should be 180°; and this affords a test of the accuracy both of the observations and of the computa-

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Let us suppose, then, that the earth is an exact sphere, or rather that the observations made on its surface, varied as it is, are reduced, by No. 226, to what they would be on a sphere. Then, if signals, such as spires, towers, poles erected for the purpose, or other objects, be assumed at as great a distance asunder as will admit of distinct and accurate observations, with telescopes of considerable power attached to the instruments used in measuring the angles, the country will be divided into a series of *primary triangles;* and if any side of any one of these be measured, the remaining sides of all of them may be computed by Legendre's theorem. By means exactly similar, each of these triangles is resolved into a number of others, called *secondary triangles;* and thus the positions of towns, and other remarkable objects, are determined.

The length of the *base*, or line measured, which is an arc of a great circle, must be determined with extreme accuracy; as an error in measuring it would affect the entire survey. For checking the measurements and the computations, it is proper to measure some other line at a considerable distance from the first; as the comparison of its measured and computed lengths will be a test of the accuracy of the intermediate operations. Such a line is called a *base of verification.* The measurement of a base is one of the principal difficulties in the survey, chiefly on account of the inequalities of the surface, and the variations in the lengths of the measuring instrument, arising from the change of temperature. On this account, the base is assumed on as flat a portion of country as can be obtained; and the chain, or other measuring instrument is constructed with extreme care. The degree of accuracy obtained by these means, is very remarkable. Thus, in the Trigonometrical Survey of England, the difference between the measured and computed lengths of a base of verification on Romney Marsh, nearly five miles and a half in length, was only about two feet; though the series of triangles, connecting it with the original base on Hounslow Heath, extended over a space of eighty miles.

tion. Delambre has given another formula instead of the foregoing, and has computed tables to facilitate the calculations.

229. Let AD$=q$ (*fig.* 37) be an arc of a great circle drawn from the vertex, to any point in the base, of the spherical triangle ABC, and let the segments of the base CD, DB be respectively denoted by b', c'. Then, by (85), and (No. 15),

$$\cos b=\cos \text{ADC} \sin b' \sin q+\cos b' \cos q \quad \text{..........} \quad (250)$$

$$\text{and } \cos c=-\cos \text{ADC} \sin c' \sin q+\cos c' \cos q \quad \text{.......} \quad (251)$$

These two equations contain six quantities, b, c, b', c', q, and ADB. If, however, any one of them be given, or if one of them be a function of one or two others,—that is, if it depend upon them, so that it may be determined by means of them, the number of independent quantities will be reduced to five. By eliminating one of these between the two equations, an equation will be obtained containing only four quantities, and the resolution of it will give any one of them in terms of the other three. It is only in a few cases, however, that formulas of elegance or value will be thus obtained. The following are some of the more interesting.

230. Let $b'=c'=\frac{1}{2}a$. Then, by adding (250) and (251), we get

$$\cos b+\cos c=2\cos \tfrac{1}{2}a \cos q\text{; or (22)}$$

$$\cos \tfrac{1}{2}(b+c) \cos \tfrac{1}{2}(b-c)=\cos \tfrac{1}{2}a \cos q \quad \text{......} \quad (252)$$

Hence $\cos \frac{1}{2}a : \cos \frac{1}{2}(b+c) :: \cos \frac{1}{2}(b-c) : \cos \text{AD}$; whence we can find AD, the line bisecting the base, when the three sides are given.

Square both members of (252): then by (6), &c.

$$\sin^2 \tfrac{1}{2}(b+c)+\sin^2 \tfrac{1}{2}(b-c)-\sin^2 \tfrac{1}{2}(b+c) \sin^2 \tfrac{1}{2}(b-c)=\sin^2 \tfrac{1}{2}a+\sin^2 q-\sin^2 \tfrac{1}{2} a \sin^2 q.$$

By dividing the last terms of these equals by r^2, &c.; as in No. 163, we have, in a plane triangle,

$$\tfrac{1}{4}(b+c)^2+\tfrac{1}{4}(b-c)^2=\tfrac{1}{4}a^2+q^2\text{; or, by doubling, \&c.}$$

$$b^2+c^2=\tfrac{1}{2}a^2+2q^2;$$

—a well known theorem. (Euc. II. A.)

231. Let, again, $b=c$: then, by taking the difference of (250) and (251), and by transposition, we get

$$\cos \text{ADC} \sin q(\sin b'+\sin c')=\cos q(\cos c'-\cos b').$$

Divide by $\cos q$ and $\sin b'+\sin c'$; then, by (28),

$$\cos \text{ADC} \tan q=\tan \tfrac{1}{2}(b'-c') \quad \text{.........} \quad (253)$$

This formula might also be obtained by No. 107, from the right angled triangle contained by the perpendicular of the isosceles triangle by q, and by the part of the base intercepted between them.

232. If $q=\frac{1}{2}\pi$, we have, by (250) and (251),

$$\cos b=\cos ADC \sin b', \text{ and } \cos c=-\cos ADC \sin c'.$$

Hence $\frac{\cos c-\cos b}{\cos c+\cos b}=-\frac{\sin b'+\sin c'}{\sin b'-\sin c'}$; or, (25) and (24),

$$\frac{\tan\frac{1}{2}(b-c)}{\cot\frac{1}{2}(b+c)}=-\frac{\tan\frac{1}{2}a}{\tan\frac{1}{2}(b'-c')} \text{ ...... (254)}$$

This formula affords the means of finding the segments of the base made by either of the quadrantal arcs drawn from the vertex.

By taking $ADC=90^{\circ}$, and dividing the difference of (250) and (251) by their sum, we should get formula (149).

233. Putting now the angle $BAD=B'$, and $CAD=C'$, we have by (101) and No. 15,

$$\left.\begin{array}{l}\cos B=\cos q \sin B' \sin ADC+\cos B' \cos ADC \\ \cos C=\cos q \sin C' \sin ADC-\cos C' \cos ADC\end{array}\right\} \text{... (255)}$$

Like (250) and (251), these formulas contain six quantities, and will give different formulas according to the values or relations of those quantities.

234. Thus, let $B'=C'=\frac{1}{2}A$. Then, by taking the first from the second, and by (23),

$$\sin\tfrac{1}{2}(B+C) \sin\tfrac{1}{2}(B-C)=-\cos\tfrac{1}{2}A \cos ADC \text{ .... (256)}$$

Hence, $\cos\frac{1}{2}A : \sin\frac{1}{2}(B+C) :: \sin\frac{1}{2}(B-C) : -\cos ADC$, or $\cos ADB$. We have thus the means of computing the angles which the arc bisecting the vertical angle makes with the base.

235. Let us now take $B=C$; and, by subtracting, transposing, &c., we get

$$\cos q \sin ADC(\sin B'-\sin C')=-\cos ADC(\cos C'+\cos B').$$

Hence, by resolving this for $\cos q$, and by (29),

$$\cos q=-\cot ADC \cot\tfrac{1}{2}(B'-C') \text{ ... (257)}$$

This might be obtained by means of No. 107, from the rightangled triangle mentioned in No. 231.

236. By taking $q=\frac{1}{2}\pi$, and dividing the difference of formulas (255) by their sum, we get, by (25),

$$\frac{\tan\frac{1}{2}(B+C)}{\cot\frac{1}{2}(B-C)}=\frac{\cot\frac{1}{2}(B'-C')}{\tan\frac{1}{2}A} \text{.... (258)}$$

By taking $ADC=90^{\circ}$, we should get formula (150).

237. Let $DBCD'$ and $DB'C'D'$ (*fig.* 38) be the halves of two great

circles intersecting each other in D and D′; and through any point A let great circles be drawn cutting them in B, B′, C, C′: join also AD. Then, putting AB$=c$, AB′$=c'$, AC$=b$, and AC′$=b'$, we have, by (95),

$$\cot c \sin AD = \cot ADB \sin DAB + \cos DAB \cos AD,$$
$$\cot c' \sin AD = \cot ADB' \sin DAB + \cos DAB \cos AD,$$
$$\cot b \sin AD = \cot ADB \sin DAC + \cos DAC \cos AD,$$
$$\cot b' \sin AD = \cot ADB' \sin DAC + \cos DAC \cos AD.$$

Divide the difference of the first and second of these by the difference of the third and fourth: then

$$\frac{\cot c' - \cot c}{\cot b' - \cot b} = \frac{\sin DAB}{\sin DAC} \quad \ldots\ldots\ (259)$$

Multiply the numerators of this by $\sin c \sin c'$, and the denominators by $\sin b \sin b'$: then, by (13),

$$\frac{\sin (c - c')}{\sin (b - b')} = \frac{\sin DAB \sin c \sin c'}{\sin DAC \sin b \sin b'}.$$

Now, by the formulas of the four sines, in the triangles DAB, DAC,

$\sin DAB \sin c = \sin ADB \sin DB$, and $\sin DAC \sin b = \sin ADB \sin DC$:

and by substituting these in the last, and contracting, we obtain, since CD′$=\pi-$DC,

$$\frac{\sin (c - c')}{\sin (b - b')} = \frac{\sin DB \sin c'}{\sin DC \sin b'} = \frac{\sin DB \sin c'}{\sin CD' \sin b'} \quad \ldots\ldots\ (260)$$

Hence, when DB$=$CD′, we get the curious and interesting formula,

$$\frac{\sin (c - c')}{\sin (b - b')} = \frac{\sin c'}{\sin b'}; \text{ that is } \frac{\sin BB'}{\sin CC'} = \frac{\sin AB'}{\sin AC'} \quad \ldots\ (261)$$

From this we have the analogy,

$$\sin AB' : \sin B'B : \sin AC' : \sin C'C;$$

which, when the radius of the sphere is infinite, becomes

$$AB' : B'B :: AC' : C'C,$$

the same as Euclid, VI. 2.

Formulas of some interest would be derived from (260), and (261), by taking b and c each $=90^\circ$; by taking $b'=c'$; by taking $b-b'=c-c'$, &c.

238. Let BEC (*figs.* 39 and 40) be a circle (great or small) on the

surface of a sphere, and through any point A on the surface, draw great circles cutting it in B, C, D, and E; then

$$\tan\tfrac{1}{2}AC \tan\tfrac{1}{2}AB = \tan\tfrac{1}{2}AE \tan\tfrac{1}{2}AD \ldots\ldots (262)$$

To prove this, take P the pole of BEC, and join PA, PC, PE; draw also PF perpendicular to BC, and PG to DE. Then, by (116),

$$\frac{\cos FC}{\cos FA} = \frac{\cos PC}{\cos PA}, \text{ and } \frac{\cos GE}{\cos GA} = \frac{\cos PE}{\cos PA} = \frac{\cos PC}{\cos PA}.$$

Equalling the first members of these, we get, by composition and division, and by (25), the expression given above.

This formula corresponds to Euc. III. 35 and 36. It gives those propositions, in fact, by taking the radius of the sphere infinite, quadrupling the results, and making one of the circles drawn from the external point touch the given circle.

239. Let BAC (*fig.* 20) be a right angle, and let AD be perpendicular to BC. Then (No. 107)

$$\sin AD = \tan BD \cot BAD, \text{ and } \sin AD = \tan DC \cot CAD.$$

By taking the product of these, since $BAD = \frac{1}{2}\pi - CAD$, we get

$$\sin^2 AD = \tan BD \tan DC \ldots\ldots\ldots (263)$$

Again, (No. 107),

$$\cos C = \tan AC \cot BC, \text{ and } \cos C = \tan DC \cot AC.$$

Hence, by equalling the second members, and multiplying by tan AC tan BC, we get

$$\tan^2 AC = \tan BC \tan DC \ldots\ldots\ldots\ldots (264)$$

When the radius of the sphere is infinite, these two formulas become the same as the corollary to Euclid, VI. 8.

X.—QUESTIONS FOR EXERCISE.

Ex. 1...4. PROVE the truth of the following formulas:

(1) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B)\sin(A-B)$;

(2) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B)\cos(A-B)$;

(3) $\tan^2 A - \tan^2 B = \dfrac{\sin(A+B)\sin(A-B)}{\cos^2 A \cos^2 B}$;

(4) $\cot^2 B - \cot^2 A = \dfrac{\sin(A+B)\sin(A-B)}{\sin^2 A \sin^2 B}$.

Ex. 5...18. Prove also the following:

(5) $\sin 9^\circ = \frac{1}{4}\sqrt{(3+\sqrt{5})} - \frac{1}{4}\sqrt{(5-\sqrt{5})}$;

(6) $\sin 81^\circ = \frac{1}{4}\sqrt{(3+\sqrt{5})} + \frac{1}{4}\sqrt{(5-\sqrt{5})}$;

(7) $\sin 22^\circ\ 30' = \frac{1}{2}\sqrt{(2-\sqrt{2})}$;

(8) $\sin 67^\circ\ 30' = \frac{1}{2}\sqrt{(2+\sqrt{2})}$;

(9) $\sin 15^\circ = \frac{1}{2}\sqrt{(2-\sqrt{3})}$;

(10) $\sin 75^\circ = \frac{1}{2}\sqrt{(2+\sqrt{3})}$;

(11) $\tan 15^\circ = 2-\sqrt{3}$;

(12) $\tan 75^\circ = 2+\sqrt{3}$;

(13) $\tan 18^\circ = \sqrt{(1-\frac{2}{5}\sqrt{5})}$;

(14) $\tan 72^\circ = \sqrt{(5+2\sqrt{5})}$;

(15) $\tan 36^\circ = \sqrt{(5-2\sqrt{5})}$;

(16) $\tan 54^\circ = \sqrt{(1+\frac{2}{5}\sqrt{5})}$;

(17) $\sin 3^\circ = \frac{1}{8}(-1+\sqrt{5})\sqrt{(2+\sqrt{3})} - \frac{1}{8}\sqrt{(10+2\sqrt{5})}\sqrt{(2-\sqrt{3})}$;

(18) $\sin 87^\circ = \frac{1}{8}(-1+\sqrt{5})\sqrt{(2-\sqrt{3})} + \frac{1}{8}\sqrt{(10+2\sqrt{5})}\sqrt{(2+\sqrt{3})}$.

Ex. 19. In a plane triangle, prove the following formulas:

1. $\tan\frac{1}{2}A \tan\frac{1}{2}B \tan\frac{1}{2}C = \dfrac{r}{s}$;

2. $\text{versin}\, A = \dfrac{2(s-b)(s-c)}{bc}$;

3. $\dfrac{b+c}{a+b+c} = \dfrac{\cos\frac{1}{2}(B-C)}{2\cos\frac{1}{2}B\cos\frac{1}{2}C} = \dfrac{1+\tan\frac{1}{2}B\tan\frac{1}{2}C}{2}$;

4. $\dfrac{b+c}{b+c-a} = \dfrac{\cos\frac{1}{2}(B-C)}{2\sin\frac{1}{2}B\sin\frac{1}{2}C} = \dfrac{1+\cot\frac{1}{2}B\cot\frac{1}{2}C}{2}$:

r being the radius of the inscribed circle, and s half the perimeter,

Ex. 20. In a plane triangle, if $C=60^\circ$, $c^2=a^2+b^2-ab$; but if $C=120^\circ$, $c^2=a^2+b^2+ab$. Required the proof.

Ex. 21. Prove that, in a plane triangle, the straight line drawn from A to the centre of the inscribed circle is $= b^{\frac{1}{2}} c^{\frac{1}{2}} \sqrt{\dfrac{s-a}{s}}$.

22. If equilateral triangles be described externally on the three sides of any plane triangle, the square of the straight line joining the centres of any two of these triangles is equal to $\frac{1}{6}(a^2+b^2+c^2)+\frac{2}{3}A\sqrt{3}$, where A is the area of the given triangle. Required the proof.

Ex. 23. Prove that, in a spherical triangle,

$$\tan\tfrac{1}{2}(a+b) : \tan\tfrac{1}{2}(a-b) :: \tan\tfrac{1}{2}(A+B) : \tan\tfrac{1}{2}(A-B);$$

and show from this how the problems may be solved, in which two sides and the sum or difference of the opposite angles, or two angles and the sum or difference of the opposite sides, are given.

Ex. 24. In a spherical triangle, if the sum of the three sides be 180°, the sine of half any angle is a mean proportional between the cotangents of the containing sides; but if the sum of the three angles be 360°, the cosine of half any side is a mean proportional between the cotangents of the adjacent angles. Required the proofs.

Ex. 25...36. Required the method of resolving a right-angled spherical triangle, from any of the following data:—(25, 26) The hypotenuse and the sum or difference of the legs; (27, 28) The hypotenuse and the sum or difference of the adjacent angles; (29, 30) An angle and the sum or difference of the opposite leg and the hypotenuse; (31, 32) An angle and the sum or difference of the adjacent leg and the hypotenuse; (33, 34) A leg and the sum or difference of the other leg and the hypotenuse; (35, 36) A leg and the sum or difference of the adjacent angle and the other leg.*

Ex. 37...40. Given the base and the vertical angle of a spherical triangle; given, also (37, 38) the sum or difference of the other sides; and (39, 40) the sum or difference of the other angles; to resolve the triangle.†

Ex. 41. Given the altitudes of the sun at six o'clock and when east or west, on the same day, and at the same place; to find the latitude and declination.

Ex. 42...51. Required the method of finding the latitude and declination from any of the following data, according to the notation adopted in the note, page 73:

(42) a, Z'; (44) a', Z'; (46) Z, P; (48) a', P; (50) P, Z';
(43) a', Z; (45) a', P'; (47) Z', P'; (49) a, P'; (51) P', Z.

Ex. 52...56. Given the sun's meridian altitude, and (52) his altitude when west; (53) the time when west; (54) his azimuth at six

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* Solutions of all these, and several similar questions may be obtained from (203) to (212), inclusive.

† These questions will be solved by means of XI, XII, XIII, XIV, in the note, page 35. Questions similar to these may be solved by means of (108), (109), (110), and (111); and also of (149), (158), and the intermediate formulas.

o'clock; (55) the time of rising or setting; and (56) the amplitude: to find, in each case, the latitude and declination.

Ex. 57. Given the interval between the times at which the sun rises and is east, at a place whose latitude is known; to find his declination.

Ex. 58. Given the angle contained by two hour lines (such as the three and four o'clock ones) on a horizontal dial; to find the latitude.

Ex. 59. In what latitude will the angle contained by the five and six o'clock lines on a horizontal dial, be double of the angle contained by the twelve and one o'clock lines?—*Answ.* $44^\circ\ 0'\frac{1}{2}$.

Ex. 60. In what latitude are the hour lines for ten and five o'clock, on a south vertical dial, perpendicular to each other?—*Answ.* $47^\circ\ 3'\frac{1}{2}$.

Ex. 61. To find the latitude at which, on a given day, the hour angle on a horizontal dial, at the time when the sun is east or west, will be of a given magnitude.

Ex. 62. On a horizontal dial for latitude $54^\circ\ 36'$, what two hour angles differ by 15°, while the corresponding times differ by an hour? *Answ.—The times are* $2^h\ 41\frac{1}{4}^m$ *and* $3^h\ 41\frac{1}{4}^m$.

Ex. 63. In latitude $54^\circ\ 36'$, for what time are the hour angles, on a horizontal, and on a south vertical dial, complements of each other? *Answ.* $3^h\ 42^m\ 1^s$.

Ex. 64. Required the declination of a star, which in latitude $54^\circ\ 36'$ N. would shine on a north vertical dial during half the time of its continuance above the horizon.—*Answ.* $27^\circ\ 23'$ N.

Ex. 65. Investigate the following formulas; m denoting the number of terms in the second member before the last or fractional term:

$$\frac{\cos n\mathrm{A}}{2\cos\mathrm{A}}=\cos(n-1)\mathrm{A}-\cos(n-3)\mathrm{A}+\cos(n-5)\mathrm{A}-\ldots\pm\frac{\cos(n-2m)\mathrm{A}}{2\cos\mathrm{A}};$$

$$\frac{\cos n\mathrm{A}}{2\sin\mathrm{A}}=-\sin(n-1)\mathrm{A}-\sin(n-3)\mathrm{A}-\sin(n-5)\mathrm{A}-\ldots+\frac{\cos(n-2m)\mathrm{A}}{2\sin\mathrm{A}};$$

$$\frac{\sin n\mathrm{A}}{2\cos\mathrm{A}}=\sin(n-1)\mathrm{A}-\sin(n-3)\mathrm{A}+\sin(n-5)\mathrm{A}-\ldots\pm\frac{\sin(n-2m)\mathrm{A}}{2\cos\mathrm{A}};$$

$$\frac{\sin n\mathrm{A}}{2\sin\mathrm{A}}=\cos(n-1)\mathrm{A}+\cos(n-3)\mathrm{A}+\cos(n-5)\mathrm{A}+\ldots+\frac{\sin(n-2m)\mathrm{A}}{2\sin\mathrm{A}}.$$

Ex. 66. Prove that the sums of n terms of the series',

$\sin m\varphi+\sin(m+r)\varphi+\sin(m+2r)\varphi+\sin(m+3r)\varphi+$ &c., and

$\cos m\varphi+\cos(m+r)\varphi+\cos(m+2r)\varphi+\cos(m+3r)\varphi+$ &c.,

are, respectively,

$$\frac{\sin\frac{1}{2}nr\varphi\sin\{m+\frac{1}{2}(n-1)r\}\varphi}{\sin\frac{1}{2}r\varphi},\ \text{and}\ \frac{\sin\frac{1}{2}nr\varphi\cos\{m+\frac{1}{2}(n-1)r\}\varphi}{\sin\frac{1}{2}r\varphi}.$$

Ex. 67. Prove that

$$\frac{\sin\varphi+\sin 3\varphi+\sin 5\varphi+\sin 7\varphi+\ \&c.}{\cos\varphi+\cos 3\varphi+\cos 5\varphi+\cos 7\varphi+\ \&c.}=\tan n\varphi,$$

n being the number of terms in either the numerator or denominator.

Ex. 68. Prove that the sum of the infinite series,

$$\sin\frac{m-1}{m}\text{A}\cos\frac{m+1}{m}\text{A}+\sin\frac{m-1}{m^2}\text{A}\cos\frac{m+1}{m^2}\text{A}+\sin\frac{m-1}{m^3}\text{A}\cos\frac{m+1}{m^3}\text{A}+\ \&c.$$

is $\sin\text{A}\cos\text{A}$;* and that the sum of n terms of the same series is

$$\sin\frac{m^n-1}{m^n}\text{A}\cos\frac{m^n+1}{m^n}\text{A}.$$

Ex. 69. Prove that the sum of n terms of the series,

$$\sin\text{A}\sin\text{A}+\sin 2\text{A}\sin 2^2\text{A}+\sin 3\text{A}\sin 3^2\text{A}+\ \&c.$$

is $\frac{1}{2}\text{versin}\,n(n+1)\text{A}$.

Ex. 70. Prove that the sum of n terms, and of an infinite number of terms, of the series,

$$\tfrac{1}{2}\tan\tfrac{1}{2}\text{A}+\tfrac{1}{4}\tan\tfrac{1}{4}\text{A}+\tfrac{1}{8}\tan\tfrac{1}{8}\text{A}+\ \&c.$$

are respectively $\frac{1}{2^n}\cot\frac{1}{2^n}\text{A}-\cot\text{A}$, and $\frac{1}{\text{A}}-\cot\text{A}$.

Ex. 71. Prove that the sum of the series,

$$\sin 2\text{A}\sin 2\text{A}+\sin 4\text{A}\sin 5\text{A}+\sin 6\text{A}\sin 10\text{A}+\sin 8\text{A}\sin 17\text{A}+\ \&c.$$

is $\cos^2\frac{1}{2}\text{A}-\cos(n^2+n+\frac{1}{2})\text{A}\cos(n+\frac{1}{2})\text{A}$.

Ex. 72. Prove that the sum of $\tan\text{A}+2\tan 2\text{A}+4\tan 4\text{A}+8\tan 8\text{A}+\ \&c.$, to n terms, is $\cot\text{A}-2^n\cot 2^n\text{A}$.

Ex. 73. Prove that the sum of n terms of the series,

$$2\sin\frac{\text{A}}{2}\sin^2\frac{\text{A}}{2^2}+2^2\sin\frac{\text{A}}{2^2}\sin^2\frac{\text{A}}{2^3}+2^3\sin\frac{\text{A}}{2^3}\sin^2\frac{\text{A}}{2^4}+2^4\sin\frac{\text{A}}{2^4}\sin^2\frac{\text{A}}{2^5}+\ \&c.$$

is $\frac{1}{2}(2^n\sin\frac{\text{A}}{2^n}-\sin\text{A})$; and that the sum of an infinite number of terms is $\frac{1}{2}(\text{A}-\sin\text{A})$.

Ex. 74. Prove that

$$\sin x=\cos\tfrac{1}{2}x\cos\tfrac{1}{4}x\cos\tfrac{1}{8}x\cos\tfrac{1}{16}x\ldots\ldots\cos\frac{x}{2^n}\times 2^n\sin\frac{x}{2^n},$$

n denoting the number of the cosines.

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* Since, whatever m is, the sum of the infinite series is $\sin\text{A}\cos\text{A}$, or $\frac{1}{2}\sin 2\text{A}$, by giving various values in m we may have as many infinite series' as we please, the sums of which will be all equal.

Ex. 75. If A′ be the angle contained by the chords of b and c, two sides of a spherical triangle ABC, prove that

$$\cos A' = \frac{1+\cos a - \cos b - \cos c}{4\sin\frac{1}{2}b\,\sin\frac{1}{2}c}.$$

Ex. 76. Prove that, on the same supposition,

$$\cos A' = \frac{\sin B\cos(S-B)+\sin C\cos(S-C)-\sin A\cos(S-A)}{2\sqrt{\sin B\sin C\cos(S-B)\cos(S-C)}}.$$

Ex. 77. Prove, that in an equilateral spherical triangle, $\cos A = \frac{\cos a}{2\cos^2\frac{1}{2}a}$; that $\cos a = \frac{\cos A}{2\sin^2\frac{1}{2}A}$; and that $2\cos\frac{1}{2}a\sin\frac{1}{2}A = 1$.

Ex. 78. In a rightangled spherical triangle, if the hypotenuse c be double of the leg a, prove that $\sin A = \frac{1}{2}\sec a$; but if a be double of c, prove that $\sin A = 2\cos\frac{1}{2}a$.

Ex. 79. When the sun's longitude increases m times as fast as his declination, prove that

$$\sin d = \sqrt{\frac{(m\sin\omega+1)(m\sin\omega-1)}{(m+1)(m-1)}};$$

where d is the declination, and ω the obliquity.

Ex. 80. Divide the surface of a sphere into two equilateral triangles, having their areas in the ratio of m to n.—*Answ. In one of the triangles each angle will be* $\frac{5m+n}{m+n}\times 60^\circ$; *in the other* $\frac{m+5n}{m+n}\times 60^\circ$.

Ex. 81. From a given spherical lune, contained by two great circles, to cut two isosceles triangles, so that the remaining quadrilateral may have its sides equal.—*Answ.* $\cot AB = \sin\frac{1}{2}A$; *where* AB *is one of the equal sides of the isosceles triangles, and* A *the angle of the lune.*

Ex. 82. Required the angle contained by two great circles forming a lune, which is such that it may be divided, by two great circles, into three equal parts, two of which are equilateral triangles.—*Answ.* $77^\circ\ 8'\frac{1}{7}$.

Ex. 83. To draw B′C′, an arc of a great circle, dividing a given spherical triangle ABC, in the ratio of m to n, and so that, in the triangles ABC, AB′C′, B : B′ : : C : C′.—*Answ.* $B' = B - \frac{m-n}{m}\cdot\frac{B}{B+C}(A+B+C-180^\circ)$.

XI.—ELEMENTS OF ANALYTIC GEOMETRY.*

240. In analytic geometry, the position of a point in a plane is generally determined either by the lengths of two straight lines drawn from it parallel to two lines given in position, and terminated by those lines; or by the length of a line drawn from it to a fixed point, and the angle which that line makes with a fixed line passing through the same point. Thus, if the straight lines AB, CD, (*fig.* 41) intersecting in O, be given in position, and P be any point in the same plane, and if PF, PE be drawn parallel to AB and CD, it is evident, that if PF and PE were given, the position of P would be determined, as it would only be necessary to make OE, OF respectively equal to them, and to complete the parallelogram. In the second method, if the straight line OB be given in position, the position of P will be determined, if the angle BOP, and the straight line OP, be given; as it is only necessary to make each of them of the given magnitude.

241. In the first method, PF and PE are called the *co-ordinates* of the point P; or if only one of them, as PE, be drawn, it is called the *ordinate*, and OE, equal to FP, the *abscissa:* AB and CD are called the *axes of the co-ordinates;*—AB, the *axis of the abscissas;* and CD, the *axis of the ordinates:* O is called the *origin* of the co-ordinates, and the co-ordinates are said to be *rectangular*, when the axes cut one another perpendicularly;—otherwise, they are *oblique.* In the second method, O is called the *pole*, OB the *fixed axis*, and OP

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* The object of analytic geometry is to effect investigations in geometry by means of algebraic operations. Algebra has been long employed in geometrical inquiries, particularly since equations were employed by Descartes to express the properties of curves. It is only in the hands of some late eminent writers, however, that this combination of algebra and geometry has acquired that regular form, and that degree of importance in the higher departments of geometry, which entitle it to be regarded as a separate branch of science. Some writers term it the *application of algebra to geometry;* and in one or two works on the subject it is called, though not very properly, *algebraic geometry.* By whatever name it may be designated, however, it is carefully to be distinguished from the ancient *geometrical analysis*, in which, with great elegance, but with much less generality and power, geometrical inquiries are conducted by means of geometry itself, without any assistance from algebra. What is here given is a mere sketch of the first principles, as applied to the straight line; and it may serve as an introduction to works expressly on the subject.

the *radius vector:* and OP and the angle BOP (or the circular arc which measures it) are sometimes called the *polar co-ordinates.*

242. The position of a point is expressed algebraically by means of an equation, from which, if one of the co-ordinates be given, the other can be determined; as it shows their relation, being expressed in terms of them, and of one or more other quantities.* Thus, suppose BOC to be a right angle, and the straight line OP to be given $=a$; then, if $OE=x$, and $EP=y$, we have (Euc. I. 47) $y=\pm\sqrt{(a^2-x^2)}$; whence, if x as well as a be given, y, and consequently the position of P will be determined. If, however, while a continues always the same, x should vary in magnitude, the equation would become indeterminate, and y might have any value that would result from taking x of a magnitude not greater than a, nor less than $-a$; consequently the point P might have an indefinite number of positions. Still, however, its positions are circumscribed; as, while the values of x are confined within the limits above mentioned, y, for each value of x, can have only two values, one positive and the other negative. From these views it is easy to conceive, that, as x may change its value by insensible and continuous variations, there is a *line* in which P will always be found. We see, in fact, that this line, since in all positions OP is of a fixed magnitude, is the circumference of a circle, whose centre is O, and radius OP. In like manner, in every equation containing two indeterminate quantities, if these quantities represent lines, the intersection of the co-ordinates is always found in a line, the nature and position of which depend on the equation. The line thus determined is called the *locus* of the point of intersection, or the *locus* of the equation; and the equation, on the contrary, is called the *equation* of the line, or locus.

Lines are distinguished into orders, according to the degree of their equations in reference to their co-ordinates; a line being of the first order when its equation is of the first degree, of the second order when its equation is of the second degree, &c.

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* Quantities which have always the same value, such as the radius of the same circle, are called *constant,* and are denoted by the first letters of the alphabet, a, b, c, &c.; but those which may change in magnitude, such as the sine, cosine, tangent, &c. of a variable arc, or the co-ordinates of any line, are called *variable,* and are denoted by the last letters of the alphabet. The abscissa and ordinate are usually denoted by x and y respectively, and the radius vector may be denoted by v, and the angle which it makes with the fixed axis, by φ.

243. We may now proceed to consider the nature and circumstances of a line of the first order, which is the locus of

$$y = ax + b \ldots\ldots\ldots (265)$$

a general equation of the first degree;* P (*fig.* 41) being any point in the line; and OE $= x$, and EP $= y$, the co-ordinates of that point. Hence we have $ax = y - b$, or if OG $= b$, $ax =$ OF $-$ OG, or $a \times$ FP $=$ FG; and consequently FP : FG :: 1 : a; whence it appears, that wherever P is taken in the line which is the locus of the equation, the ratio of FP to FG is constant;—a property which can hold only when P is in an indefinite *straight* line drawn through G and P, in which case all the triangles GFP would be similar: that line, therefore, is the locus of the equation; and hence we infer, that *the only line of the first order is the straight line.*

244. If $x = 0$ in (265), $y = b$, and if $y = 0$, $x = -\frac{b}{a}$; whence a and b being given, OG and OH, and consequently the points G and H in which the line meets the axes, will become known. G will be in OC or OD, accordingly as b is positive or negative; and H will be in OA or OB, accordingly as a and b have the same or contrary signs. If $b = 0$, G coincides with O, and consequently $y = ax$ is the equation of a straight line passing through the origin of the co-ordinates.†

245. Let the angle BOC, made by the axes, $= \omega$, and FPG or PHE, that which PH makes with the axis of the abscissas, $= \theta$: then

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* In this, and in all *general* equations, the coefficients a, b, &c. may be either positive or negative, though the sign $+$ is prefixed to them. In all geometrical inquiries, also, they must be such as to make the dimensions of the several terms the same. Thus, if b represent a line, ax must represent a line also; and as x represents a line, a must represent a number, or the ratio of two geometrical magnitudes of the same kind. (See the first note to No. 11).—For the sake of brevity, the line whose equation is $y = ax + b$, may be called the line $y = ax + b$; and the point of which the co-ordinates are x and y, may be called the point x, y. It may be remarked, that the equation $y = ax + b$ is in reality not less general than $Ay = Bx + C$; as the latter becomes the same as the former, when all its terms are divided by A, and b and a are substituted for the absolute term and the coefficient of x in the result.

† The student will find it useful to apply the principles established in the text, here and in what follows, in the performance of exercises such as the following.

Exercises. Draw the straight lines whose equations are as follows:

1. $y = 2x + 3$.	3. $y = -2x + 3$.	5. $3y + x = 6$.
2. $y = 2x - 3$.	4. $y = -2x - 3$.	6. $y + 1 = x$.

(Euc. I. 32) CGP$=$CFP$-$FPG$=\omega-\theta$, the angle which PH makes with the axis of the ordinates. Now it was shown (No. 243), that FP : FG, or OH : OG :: 1 : a; or (No. 21) $\sin(\omega-\theta)$: $\sin\theta$ :: 1 : a. Hence the coefficient $a=\frac{\sin\theta}{\sin(\omega-\theta)}$; and we may express the equation of the line thus:

$$y=\frac{\sin\theta}{\sin(\omega-\theta)}x+b, \text{ or } y\sin(\omega-\theta)=x\sin\theta+b\sin(\omega-\theta) \ldots (266)$$

If ω be a right angle, $\sin(\omega-\theta)$ becomes $\cos\theta$, and consequently $a=\tan\theta$, and the equation is changed into

$$y=x\tan\theta+b \ldots\ldots\ldots\ldots (267)$$

246. *In finding the equation of a straight line which will satisfy certain conditions, the values of the constants a and b are to be determined according to the given conditions.* Thus, to find the equation of a line passing through two points whose co-ordinates are x', y', and x'', y'', we have, at that point, by the general equation (265),

$$y'=ax'+b \ldots\ldots\ldots (p), \quad \text{and } y''=ax''+b \ldots\ldots\ldots (q);$$

where a and b are to be found in terms of x', y', x'', and y'': and, by any of the common methods of elimination, we should find $a=\frac{y'-y''}{x'-x''}$ and $b=\frac{x'y''-x''y'}{x'-x''}$: the substitution of which in the general equation (265) gives

$$y=\frac{y'-y''}{x'-x''}x+\frac{x'y''-x''y'}{x'-x''},$$

the equation required. The following method, however, is rather more simple: Subtract (q) from (p); the remainder will give as before, $a=\frac{y'-y''}{x'-x''}$. Take the difference of (p) and (265), substitute in it this value of a, and there will result $y-y'=(x-x')\frac{y'-y''}{x'-x''}$; or, when cleared of fractions,

$$(y-y')(x'-x'')=(x-x')(y'-y'')^{*} \ldots\ldots\ldots\ldots (268)$$

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* As an example of the use of this, suppose the co-ordinates to be rectangular, and $x'=2$, $y'=4$, $x''=-2$, and $y''=1$. Substitute these in (268); then $(y-4)(2+2)-(x-2)(4-1)=0$, or, by contracting, &c. $y-\frac{3}{4}x=2\frac{1}{2}$, the equation of the line to be drawn. Hence, when $y=0$, we have $x=-3\frac{1}{3}$;

the equation required, which might be easily reduced to the form found above. If the line were required to pass through only one point, x', y', we should have, by subtracting (p) from (265),

$$y-y'=a(x-x')^{*} \ldots\ldots\ldots\ldots (269)$$

This, which is the equation required, is indeterminate, a being unknown; and consequently the line may have an indefinite number of positions, as is also evident from the nature of the problem.

247. If $y=ax+b$, and $y=a'x+b'$, be the equations of two lines, the co-ordinates x' and y' of their point of intersection, will be determined by taking in each equation x equal to x', and y equal to y', and then determining the values of x' and y' from the two equations. We should thus find

$$x'=-\frac{b-b'}{a-a'}, \text{ and } y'=\frac{ab'-a'b}{a-a'}\dagger \ldots\ldots\ldots\ldots (270)$$

248. To find, for rectangular co-ordinates, the equation of a straight line passing through a given point x', y' (L, *fig.* 42) and making a given angle θ' (HPK) with a given straight line (HGP) whose equation is $y=ax+b$: let the required equation be $y=a'x+b'$.

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and when $x=0$, $y=2\frac{1}{2}$. To trace the required line, therefore, draw (*fig.* 42) AB and OC at right angles to each other, and in AO take OH$=3\frac{1}{3}$, that is, equal to three times the line assumed as the lineal measure, and a third of the same, and in OC take OG$=2\frac{1}{2}$; the straight line passing through G and H will be the locus required. The coefficient $-\frac{3}{4}$ is the tangent of $143^\circ 8'$, the angle which HG makes with HA; and by means of this also we might be assisted in determining the position of the required locus.

Exercises. Required the equations of the straight lines passing through points whose co-ordinates are as follows:

1. $x'=2, y'=3$; $x''=1, y''=2$.
2. $x'=2, y'=-1$; $x''=-1, y''=-1$.
3. $x'=0, y'=4$; $x''=3, y''=1$.
4. $x'=1, y'=-2$; $x''=-3, y''=4$.

* In like manner, if y'' be another point, we have $y-y''=a(x-x'')$; and dividing (269) by this, we get $\frac{y-y'}{y-y''}=\frac{x-x'}{x-x''}$, another form of (268).

† *Exercises.* Show which of the lines represented by the following pairs of equations, intersect, and which do not: and, when they do intersect, find the co-ordinates of their points of intersection:

1. $\begin{cases} y=x+1, \\ y=2x+1. \end{cases}$
2. $\begin{cases} 2y=x+1, \\ y=-2x+1 \end{cases}$
3. $\begin{cases} y=2x-3, \\ y=3x-2. \end{cases}$
4. $\begin{cases} 2y=x+2, \\ 6y=3x-1. \end{cases}$
5. $\begin{cases} 3y=2x-1, \\ 2x=3y. \end{cases}$
6. $\begin{cases} 2x=y, \\ 3x=-2y. \end{cases}$

For the point x', y', this will become $y'=a'x'+b'$; whence, by subtraction,

$$y-y'=a'(x-x') \text{ .... } (p)$$

Now (No. 245) $a=\tan\theta$, and $a'=\tan \text{LKO}=\tan(\theta-\theta')$. Hence, by (40), and by substituting a for $\tan\theta$, we get $a'=\frac{a-\tan\theta'}{1+a\tan\theta'}$; and this changes (p) into

$$y-y'=\frac{a-\tan\theta'}{1+a\tan\theta'}(x-x')^{*} \text{ .... } (271)$$

From this, which is the required equation for *any* value of θ', we derive *the equation of a line passing through a given point x', y', and parallel to a given line, $y=ax+b$,* simply by taking $\theta'=0$, and we thus get

$$y-y'=a(x-x') \text{ .... } (272)$$

From the same equation, by multiplying the numerator and denominator of the fraction in the second member by $\cos\theta'$, and then taking $\theta'=\frac{1}{2}\pi$, we find, for *the equation of a straight line passing through a given point, x', y', and perpendicular to a given straight line $y=ax+b$,*

$$y-y'=-\frac{1}{a}(x-x') \text{ .... } (273)$$

To find the equation of a straight line passing through the origin, and making a given angle θ' with a given straight line, $y=ax+b$, take x', y', each $=0$ in (271), and the required equation is found to be

$$y=\frac{a-\tan\theta'}{1+a\tan\theta'}.x \text{ .... } (274)$$

By the same means we find from (273) the equation of a straight line passing through the origin, and cutting $y=ax+b$ perpendicularly, to be

$$y=-\frac{x}{a} \text{ .... } (275)$$

If $\theta'=\theta=0$, the line is parallel to the axis AB. In that case (271) becomes simply $y-y'=0$, a being $=0$; and x will be indeterminate.

Again, from (271), $x-x'=\frac{1+a\tan\theta'}{a-\tan\theta'}(y-y')$. In this, change

---

* By changing x', y', into x'', y'', and dividing (271) by the result, we get $\frac{y-y'}{y-y''}=\frac{x-x'}{x-x''}$, another form of (268), the same as in the note to (269).

θ' into $-(90^\circ-\theta)$: then $\tan\theta'=-\cot\theta=-\frac{1}{a}$; and LPK being then parallel to the axis OC, we have for its equation $x-x'=0$, y being indeterminate.*

249. If it be required to draw a line through x', y', making a given angle θ' with the axis AB, we have, by (269) and No. 245,

$$y-y'=\tan\theta'(x-x') \quad\dots\dots\dots\dots\dots \quad (276)$$

250. For rectangular co-ordinates, the length of a line joining two points, x, y, and x', y', is (Euc. I. 47) $\sqrt{\{(x-x')^2+(y-y')^2\}}$.

If x', y', one of the points, be the origin, this will become simply

$$\sqrt{(x^2+y^2)},\text{ or } x\sqrt{(1+a^2)},\text{ because (No. 244) } y=ax.$$

251. To find the length of a perpendicular drawn from a given point x', y', to a given line whose equation is

$$y=ax+b \quad\dots\dots\dots\dots \quad (p)$$

By (273) the equation of the perpendicular to that line from the point x', y' is

$$y-y'=-\frac{1}{a}(x-x') \quad\dots\dots\dots\dots\dots \quad (q)$$

From (p) take $y'=y'+ax'-ax'$; then

$$y-y'=a(x-x')+b-y'+ax'$$

Equalling the second members of this and (q), and transposing, we get

$$\left(a+\frac{1}{a}\right)(x-x')=y'-ax'-b,$$

or, multiplying by a, and dividing by $1+a^2$,

$$x-x'=\frac{a}{1+a^2}(y'-ax'-b).$$

If we divide this by $-a$, and compare the result with (q), we obtain

$$y-y'=-\frac{1}{1+a^2}(y'-ax'-b)$$

By taking the sum of the squares of these values of $x-x'$ and $y-y'$,

---

* *Exercises.*—1. What is the equation of a straight line drawn through a point whose co-ordinates are $x'=4$ and $y'=-1$, and making an angle of 45°, with a straight line whose equation is $x+y=1$?

2. Find the equation of a straight line passing through the origin of the co-ordinates, and perpendicular to the line $x-y=1$.

3. What is the equation of a straight line passing through the point whose co-ordinates are $x'=5$, and $y'=0$, and parallel to the line $x+y+1=0$?

and extracting the square root, we get, after a slight modification, the required length equal to $\frac{y'-ax'-b}{\sqrt{(1+a^2)}}$.*

252. As an application of these principles, let it be required to discover whether the three perpendiculars AD, BE, CF, (*fig.* 43) drawn from the three angles of a triangle ABC to the opposite sides, intersect in the same point. To facilitate this investigation, let BC be assumed as the axis of the abscissas, and a perpendicular to it through B, as the axis of the ordinates: then, assuming $BC=x''$, and the co-ordinates of $A=x', y'$, we have those of $B=0, 0$, and of

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* It may be convenient, for the sake of reference, to bring together the results thus far obtained, which may be regarded as the elementary principles of analytic geometry. We have, therefore, the following table:

1. The equation of a straight line passing through the origin of the co-ordinates is $y=ax$. (No. 244.)
2. The coefficient $a=\frac{\sin\theta}{\sin(\omega-\theta)}$; or, if $\omega=90^\circ$, $a=\tan\theta$. (No. 245).
3. Another form of the equation of a straight line is, by No. 245, $y\sin(\omega-\theta)=x\sin\theta+b\sin(\omega-\theta)$; or, if $\omega=90^\circ$, $y=x\tan\theta+b$.
4. The equation of a line passing through the points, x', y', and x'', y'', is $(y-y')(x'-x'')=(x-x')(y'-y'')$. (No. 246.)
5. The equation of a line passing through one point x', y', is $y-y'=a(x-x')$. (No. 246).
6. The co-ordinates x', y', of the intersection of two lines $y=ax+b$, and $y=a'x+b'$, are $x'=-\frac{b-b'}{a-a'}$, and $y'=\frac{ab'-a'b}{a-a'}$. (No. 247.)
7. x', y', being a given point, and $y=ax+b$ the equation of a given line for rectangular co-ordinates, the equation of a line making with that line a given angle θ', is $y-y'=\frac{a-\tan\theta'}{1+a\tan\theta'}(x-x')$. (No. 248.)
8. The equation of a line from the point x', y', perpendicular to the line $y=ax+b$, ω being $=90^\circ$, is $y-y'=-\frac{1}{a}(x-x')$. (No. 248.)
9. The equation of a line parallel to $y=ax+b$, and passing through the point x', y', ω being $=90^\circ$, is $y-y'=a(x-x')$. (No. 248.)
10. The equation of a line through x', y', parallel to the axis AB (*fig.* 33), is $y-y'=0$; and that of one parallel to CD is $x-x'=0$. (No. 248).
11. The equation of a line passing through x', y', and making a given angle θ' with the axis AB, is $y-y'=\tan\theta'(x-x')$. (No. 249.)
12. When $\omega=90^\circ$, the length of a line joining the points x', y', and x'', y'', is $\sqrt{\{(x'-x'')^2+(y'-y'')^2\}}$. (No. 250.)
13. When $\omega=90^\circ$, the length of a perpendicular from the point x', y', to the line $y=ax+b$, is $\frac{y'-ax'-b}{\sqrt{(1+a^2)}}$. (No. 251.)

$C=x''$, 0. Then (No. 251, note 4) the equation of AB is $(y-y')\,x=(x-x')y'$, or by contraction, $yx'=xy'$; whence $y=\frac{y'x}{x'}$: also (same note, 8) the equation of CF is $y=-\frac{x'}{y'}(x-x'')$. In a similar manner we should find the equation of BE to be $y=\frac{x''-x'}{y'}x$. Now, (by No. 251, note 6) we find from these equations of BE and CF, that the abscissa of the point of intersection of these lines is $=x'$: and this being the abscissa of the point A, it follows that these perpendiculars intersect one another in the remaining one AD.

253. As a second example, let AD, BE, CF (*fig.* 44) be drawn from the angles, and bisecting the sides. Then, retaining the same notation, we have obviously the co-ordinates of $D=\frac{1}{2}x''$, 0; of $E=\frac{1}{2}(x'+x''), \frac{1}{2}y'$; and of $F=\frac{1}{2}x'$, $\frac{1}{2}y'$; whence (No. 251, note 4) we find the equation of AD to be $y(x'-\frac{1}{2}x'')=(x-\frac{1}{2}x'')y'$; of BE, $y(x'+x'')=xy'$; and of CF, $y(\frac{1}{2}x'-x'')=\frac{1}{2}y'(x-x'')$. Now by taking x and y in the first of these equal to x and y in the second, we readily find $y=\frac{1}{3}y'$; and $x=\frac{1}{3}(x'+x'')$; and also, by treating the second and third equations in the same manner, we still find $y=\frac{1}{3}y'$; and $x=\frac{1}{3}(x'+x'')$: whence it appears that the three lines pass through the same point, since the co-ordinates of the intersection of AD and BE are the same as those of the intersection of BE and CF. It is also obvious, since $y=\frac{1}{3}y'$, that $DO=\frac{1}{3}DA$, O being the point of intersection; and consequently the lines divide each other in the ratio of 1 : 2.

254. As another example, let it be required to determine whether the three perpendiculars drawn through D, E, and F, (*fig.* 45) the points of bisection of the sides BC, AC, and AB, pass through the same point. Here, retaining the same notation, we have (No. 251, note 4) the equations of the three sides BC, BA, and AC, $y=0$, $x'y=xy'$, and $y(x'-x'')=(x-x'')y'$; and (same note, 8) from these equations, and from the co-ordinates of D, E, and F, we find the equations of the perpendiculars through D, F, and E, to be

$$x-\tfrac{1}{2}x''=0,$$
$$(y-\tfrac{1}{2}y')y'=-(x-\tfrac{1}{2}x')x', \text{ and}$$
$$(y-\tfrac{1}{2}y')y'=(x''-x')(x-\tfrac{1}{2}x'-\tfrac{1}{2}x'').$$

Then, by taking x the same in the first and second, and also in the first and third, and by taking y also the same, we find the abscissas of the intersections of the perpendicular through D with those through

F and E, each equal to $\frac{1}{2}x''$, and the ordinates of the same each equal to

$$\tfrac{1}{2}y' - \frac{x'}{2y'}(x''-x'), \text{ or } \frac{y'^2 - x'x'' + x'^2}{2y'};$$

whence it appears that the intersections coincide, or that the perpendiculars all pass through the same point. Let this point be O; and, if we take the sum of the squares of BD ($\frac{1}{2}x''$) and DO $\left(\frac{y'^2 + x'^2 - x'x''}{2y'}\right)$, we find

$$BO^2 = \frac{(y'^2 + x'^2 - x'x'')^2 + x''^2 y'^2}{4y'^2}.$$

Now, if the three sides be put $=a, b, c$, we have

$x''=a$, $x'^2+y'^2=c^2$, and $(x''-x')^2+y'^2=b^2$, or
$x''^2-2x''x'+x'^2+y'^2=b^2$, or $a^2-2ax'+c^2=b^2$. Hence,

$$BO^2 = \frac{(c^2-ax')^2+a^2y'^2}{4y'^2} = \frac{c^4-2ac^2x'+a^2x'^2+a^2y'^2}{4y'^2}$$

$$= \frac{c^4-2ac^2x'+a^2x'^2+a^2(c^2-x'^2)}{4y'^2} = \frac{c^4-2ac^2x'+a^2c^2}{4y'^2}$$

$$= \frac{(c^2-2ax'+a^2)c^2}{4y'^2} = \frac{b^2c^2}{4y'^2};$$

and consequently $BO = \frac{bc^*}{2y'} = \frac{abc}{2ay'} = \frac{b^2}{4A}$,

where A denotes the area of the triangle. This expression being alike related to the three sides, it is evident that the same would be found for the line drawn from O to either of the other angles. Hence O is the centre of the circumscribed circle; and it appears that the radius of that circle may be found by dividing the continual product of the three sides by four times the area.

255. The equation of a line referred to one system of co-ordinates being given, it is often of use to find its equation in relation to another system. To effect this, let $y=ax+b$ (*fig.* 46) be the equation of a line in relation to the axes OB, OC, x and y being the co-ordinates of any point P in the line, and the angle BOC being $=\omega$;

---

* This agrees with what is found in the note to No. 144.

and let it be required to find its equation in relation to the axes O′B′, O′C′, which are so situated with respect to OB, OC, that OE″ being parallel to OC, OE″$=x'$ and E″O′$=y'$; and that O′S being parallel to OB, the angle B′O′S is $=\theta$, and C′O′S$=\theta'$. Then, representing O′E′, and E′P, the co-ordinates of P in relation to the new axes, by X and Y, and drawing E′R parallel to OB, and E′Q to OC, we have EP or

$$y = \text{ES} + \text{SR} + \text{RP} = \text{E}''\text{O}' + \text{QE}' + \text{RP} \ldots\ldots (p)$$

$$\text{and } x = \text{OE}'' + \text{O}'\text{Q} + \text{E}'\text{R} \ldots\ldots (q)$$

But O′E″$=y'$, and OE″$=x'$: and we have also in the triangles O′QE′, E′RP,

$\sin \text{O}'\text{QE}' : \sin \text{QO}'\text{E}' :: \text{O}'\text{E}' : \text{QE}'$, or $\sin\omega : \sin\theta :: \text{X} : \text{QE}'$;

$\sin \text{O}'\text{QE}' : \sin \text{O}'\text{E}'\text{Q} :: \text{O}'\text{E}' : \text{O}'\text{Q}$, or $\sin\omega : \sin(\omega-\theta) :: \text{X} : \text{O}'\text{Q}$;

$\sin \text{E}'\text{RP} : \sin \text{RE}'\text{P} :: \text{E}'\text{P} : \text{RP}$, or $\sin\omega : \sin\theta' :: \text{Y} : \text{RP}$;

$\sin \text{E}'\text{RP} : \sin \text{E}'\text{PR} :: \text{E}'\text{P} : \text{E}'\text{R}$, or $\sin\omega : \sin(\omega-\theta') :: \text{Y} : \text{E}'\text{R}$.

By means of these analogies we find the values of QE′, O′Q, &c.: by substituting these again, along with the values of O′E″ and OE″, in (p) and (q), we obtain the values of y and x: and lastly, by substituting these values of y and x in the equation $y=ax+b$, we get, for the required equation,

$$y' + \frac{\text{X}\sin\theta + \text{Y}\sin\theta'}{\sin\omega} = ax' + a.\frac{\text{X}\sin(\omega-\theta) + \text{Y}\sin(\omega-\theta')}{\sin\omega} + b .. (277)$$

This general equation becomes simplified in particular cases. Thus, if $\omega = 90^\circ$, the denominators disappear. If O′ be in OB, $y'=0$; if in OC, $x'=0$; and if O′ and O coincide, x' and y' are each equal to 0. Also, if either of the new axes coincide with the corresponding original one, or be parallel to it, we shall have the angle θ, or $\omega - \theta'$ $=0$.

256. If the co-ordinates be rectangular, and P (*fig.* 41) a point whose co-ordinates are x and y, we have OP, or, (see note, page 100) $v=\sqrt{(x^2+y^2)}$; OE $=$ OP cos POE, or $x = v\cos\varphi$; and, in like manner, $y=v\sin\varphi$. Hence, if the origin be taken as pole, the polar equation of the straight line whose equation to rectangular co-ordinates is $y=ax+b$, will become $v\sin\varphi = av\cos\varphi + b$. In like manner, the equation of any line referred to rectangular co-ordinates, may be transformed to one for the same line for polar co-ordinates.

From the foregoing expressions, since $v=\sqrt{(x^2+y^2)}$, we get, also, $\cos\varphi=\frac{x}{\sqrt{(x^2+y^2)}}$, and $\sin\varphi=\frac{y}{\sqrt{(x^2+y^2)}}$; whence $\tan\varphi=\frac{y}{x}$, $\cot\varphi=\frac{x}{y}$, &c., expressions which are of use in finding the equation for rectangular co-ordinates from that for polar co-ordinates.

257. If the pole and the origin of the co-ordinates be different points, let the co-ordinates of the pole be x' and y'; and it would appear, in almost the same manner as in the last No., that

$$v=\sqrt{\{(x-x')^2+(y-y')^2\}},$$
$$x=v\cos\varphi+x', \quad \text{and } y=v\sin\varphi+y'.$$

258. If the fixed axis be oblique to the axis of the abscissas, making with it a given angle θ, it is easy to see, that the expressions in the last two Nos. will still hold true, if φ be changed into $\varphi+\theta$.

NOTE.

If any persons should prefer the common mode of defining sines, tangents, &c. as lines, instead of mere numbers or ratios, the following, from the former editions, may be taken instead of Nos. 2, 6, and 7.

"2. If a circle be described from the vertex of an angle as centre, the arc of it, intercepted between the lines forming the angle, is called *the measure of the angle*;‡ and the angle is said to be an angle of as many degrees, minutes, &c. as there are in the arc.

"6. The straight line drawn from one extremity of an arc, perpendicular to the diameter, passing through the other extremity, is called the *sine* of that arc, or of the angle measured by it; and the part of the diameter intercepted between the sine and the arc is called the *versed sine* of the arc or angle. Hence, the sine of an arc is half the chord of its double.

"7. If a straight line touch a circle at one extremity of an arc, the part of it intercepted between the point of contact and the diameter passing through the other extremity, is called the *tangent* of the arc, or of the angle which it measures; and the straight line drawn from the centre to the remote extremity of the tangent, is called the *secant* of the arc or angle."

With the same view, omit the second paragraph of No. 8; and in No. 9, omit the multiplier r in $r\sin A$, $r\cos A$, &c. In No. 10, omit throughout the division by r, or, which comes to the same, take $r=1$. Instead of the first paragraph of No. 11, take the following:

"Putting the arc AF=A, and the radius =1, we have, from the similarity of the triangles, CGF, CAH, and CDK, 1°. $\cos A : 1 :: 1 : \sec A$; 2°, $\sin A : 1 :: \operatorname{cosec} A$; 3°, $\cos A : \sin A :: 1 : \tan A$; 4°, $\sin A : \cos A :: 1 : \cot A$; 5°, $\tan A : 1 :: 1 : \cot A$; &c." Then, formulas 6, 7, and 8, will follow at once from Euc. I. 47.

In No. 13, the expression "the line to which it is proportional," which occurs twice, may be omitted: and in No. 15, let NO and AO, without division by the radius, be taken as the sine and cosine of the angle BAN.

The commencing lines of Nos. 19 and 20 may be changed into the following:

19. "Let ABC (*fig.* 7) be a triangle rightangled at C; from B as centre, with any radius, describe the arc DE, and draw its sine DF. Then (Euc. VI. 4) AB : AC :: DB : DF, and AB : BC :: DB : BF; that is, AB : AC :: 1 : sin B, and AB : BC :: 1 : cos B. Hence it appears, that," &c.

20. "Draw also EG the tangent of B. Then, by similar triangles, BC : CA :: BE : EG, and BC : BA :: BE ; BG, that is," &c.

A few other changes of a similar kind may be necessary, but none of them will present any difficulty.

In Spherical Trigonometry, the change from the common method has not been made, as it would be attended with little advantage; but should the student wish to make it, he will find no difficulty in doing so.

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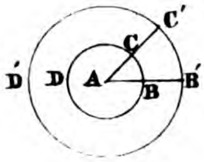

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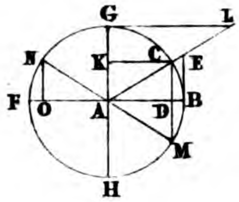

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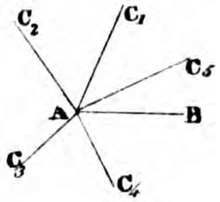

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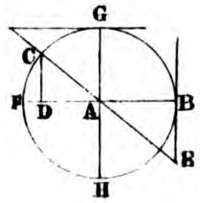

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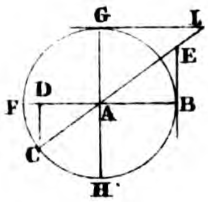

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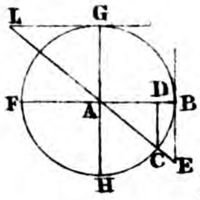

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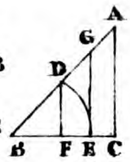

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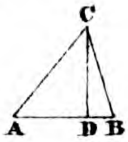

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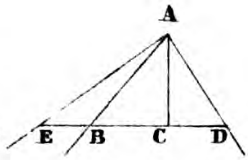

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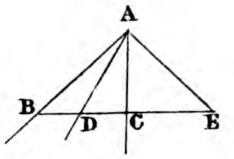

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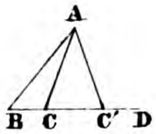

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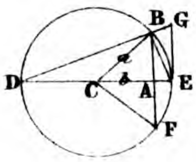

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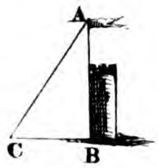

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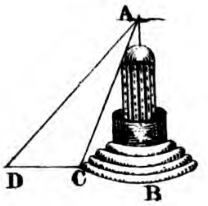

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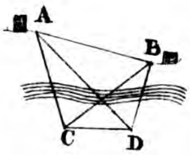

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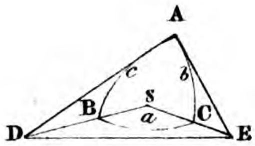

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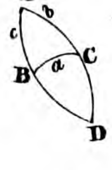

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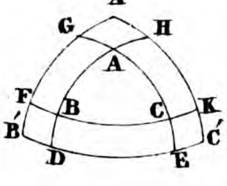

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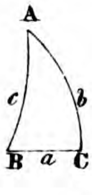

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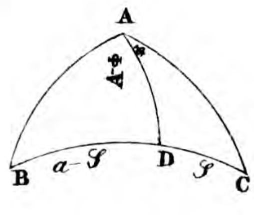

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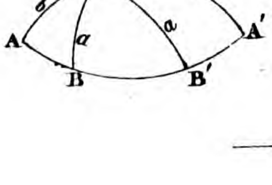

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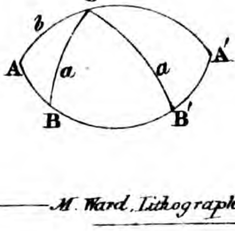

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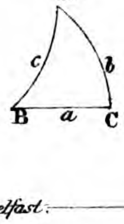

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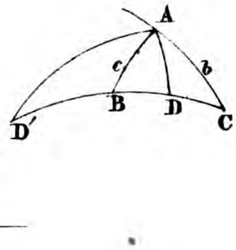

M. Ward, Lithographer, Belfast.

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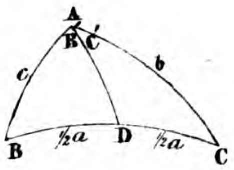

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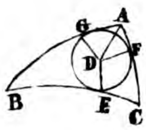

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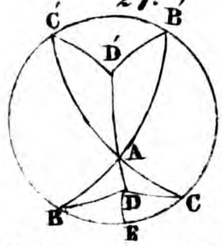

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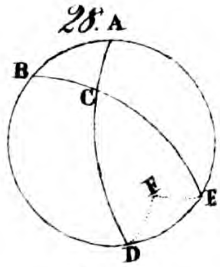

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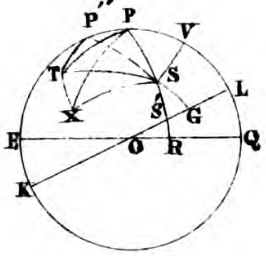

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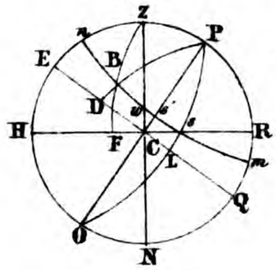

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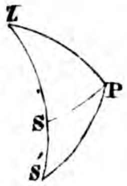

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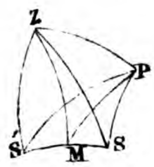

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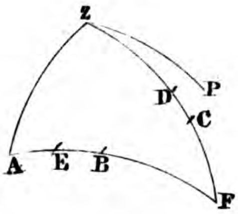

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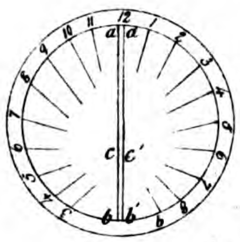

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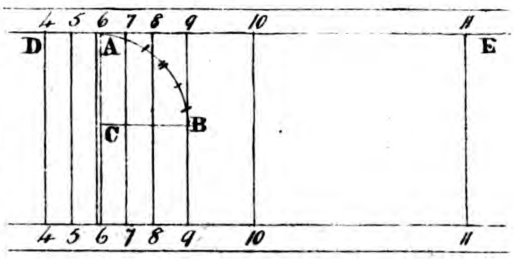

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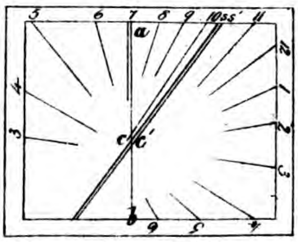

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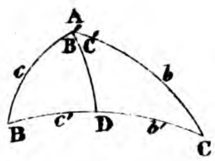

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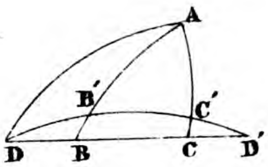

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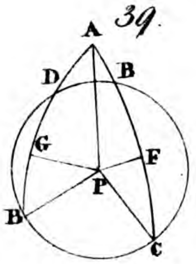

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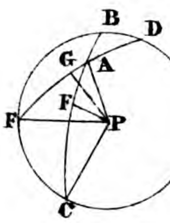

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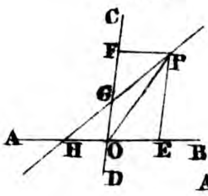

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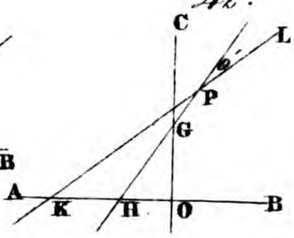

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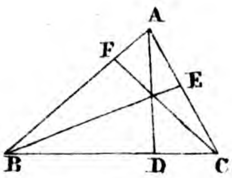

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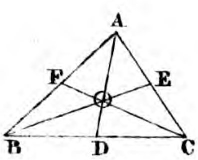

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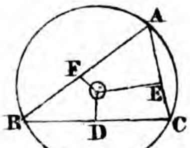

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