THOMSON'S
DIFFERENTIAL & INTEGRAL
CALCULUS.

AN INTRODUCTION

TO THE

DIFFERENTIAL AND INTEGRAL CALCULUS:

WITH

AN APPENDIX,

ILLUSTRATIVE OF THE THEORY OF CURVES
AND OTHER SUBJECTS.

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PREFACE.

THE following publication is intended for the use of persons commencing the study of the Differential and Integral Calculus. It has, therefore, been the aim of the author to render the investigations as simple and easy as possible; and he has illustrated the subjects of inquiry by numerous Examples. Most of the Sections, also, contain a number of questions without solutions, for exercising the student on the principles established in the work, and for training his mind to habits of thought, and cultivating his powers of investigation.

For reasons assigned in the APPENDIX, the Method of Limits has been adopted as the basis of the work, instead of Lagrange's Method, which was employed in the former edition. Besides this fundamental change, a great portion of the volume has been re-written; and the whole work, it is hoped, has been very materially improved.

The APPENDIX exhibits the elements and some of the leading properties of the Conic Sections, and of many other Curves, several of which are referred to in the preceding parts of the work. It contains, also, short articles on the Convergence of Series, on Definite Integrals, and on the principal methods of investigation that have been employed by mathematicians in the higher branches of science. These will be useful to persons who have but limited access to books, and they may occasionally save time and trouble to others.

As the cost of the former edition was felt by many individuals to be an objection to the use of the book, the present edition is published in a different form, and at a reduced price. The amount of matter which the work contains is much larger than would the size of the volume would seem to indicate; and it will be found to be sufficient for the majority of students. Persons, however, who shall have made themselves acquainted with what is here given, will find it easy to read more extensively on subjects which the limits of the present work have made it necessary to treat briefly, and will be able to acquire an acquaintance with others which have been altogether omitted; and, for this purpose, they may have recourse to one or more of the works mentioned in the Appendix, such as the treatises of Cournot, Duhamel, and De Morgan. It may be remarked also, that the student who wishes to prosecute the study of mathematical science extensively, will find the most interesting and valuable applications of the Differential and Integral Calculus in the modern works on Mechanics, Physical Astronomy, and other branches of Natural Philosophy. The study of these subjects, indeed,—supplying, as it does, a large amount of valuable knowledge, and displaying in the most striking manner the wonderful triumphs of modern science,—ought to be kept in view as a most important sequel to the study of pure mathematics.

Glasgow College, Oct. 2, 1848.

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APPENDIX.

INTRODUCTION

TO THE

DIFFERENTIAL AND INTEGRAL CALCULUS.

I.—ELEMENTARY PRINCIPLES AND ILLUSTRATIONS.

1. THE quantities which are considered in the Differential and Integral Calculus are either *constant* or *variable:* a quantity being *constant*, in any particular investigation, if throughout that investigation it retain always the same magnitude; while one which admits of an unlimited number of values is *variable.*

Thus, if a circle of given magnitude be considered, its radius, diameter, and circumference are all constant quantities; but if a radius, revolving like the spoke of a wheel, make angles with another radius which remains fixed, those angles, and their sines, cosines, tangents, &c. are evidently variable. If it were required, however, to describe a circle touching a given straight line, and passing through a given point, the radius of the circle would evidently be a variable magnitude, as there might be an infinite number of such circles, having radii of different magnitudes. When quantities are denoted by small letters, constant ones are usually represented by the earlier letters, a, b, c, &c. and those that are variable by the later ones, u, x, y, &c.

2. If the value of a quantity depend on that of another which is variable, the former is said to be a *function* of the latter. A quantity may also be a function of two or more others which are independent of one another.

Thus, the sine, the cosine, the tangent, &c. of a variable angle are functions of that angle; and the numerical value of the area of a rectangle, being the product of its adjacent sides, is a function of those sides. A function of one or more quantities is generally, but not always variable. Thus, the logarithm of a number varies with the number; but all radii of the same circle are equal, whatever angles they may make with a fixed radius.

The characters f, F, φ, and some others, are employed to denote that one quantity is a function of one or more others. Thus, the expressions, $u=fx$, and $u=f(x,y)$, denote, that in the first, u is a function of x, and in the second of x and y.

3. Functions are distinguished into *algebraic* and *transcendental.* An *algebraic function* is one that may be expressed in a finite number of terms, produced by subjecting a variable to some of the elementary operations of algebra,—addition, subtraction, multiplication, division, involution, and evolution; but if the function be formed otherwise, it is *transcendental.* Of the former kind are $u=ax^3-bx^{\frac{2}{3}}+cx^{-\frac{1}{2}}$; while $u=a^x$, $u=a\log x$, and $u=a\tan x$, are transcendental.

A function is said to be *implicit* when some operation is necessary for exhibiting its value in terms of the variable, or variables, on which it depends; otherwise, it is *explicit.* Thus, in the equation, $u^2-xu+a=0$, u is an implicit function of x; but when, by resolving the equation, we get $u=\frac{1}{2}\{x\pm\sqrt{(x^2-4a)}\}$, u is then an explicit function of x.

A *variable* is said to be *continuous* when in passing from any assigned magnitude to another it passes through all the intermediate ones. In like manner, if a quantity vary continuously, a *function* of it is said to be *continuous* between two certain values, if in passing from one of them to the other, it must pass through all the intermediate ones, increasing or diminishing by quantities as small as we please. Thus, an angle may be made to vary continuously, increasing or diminishing by minutes or seconds, or the minutest fractions of seconds; and it is plain that its sine, cosine, and versed sine, all vary continuously, as it so varies. While its tangent, however, is generally continuous, such as when the angle varies from zero to 30°, from 40° to 80°, &c.; yet it is discontinuous at 90°, passing at once from a positive value, infinitely great, to a like negative one.

4. The elementary principles of the differential calculus may be illustrated by considering the changes produced on some simple functions in consequence of changes in the variables on which they depend. In doing this, we may assume

u to denote the function of the variable x, and u' to represent the same function of $x+h$; so that $u=fx$, and $u'=f(x+h)$. Then will $u'-u$ and h be the respective and simultaneous changes made on the function and the variable; and their relation will be exhibited in the simplest manner, and in the way suited to the nature and objects of the differential calculus, by dividing the former by the latter. Thus, if $u=ax^2$, and, consequently, $u'=a(x+h)^2=ax^2+2axh+ah^2$,

$$\text{we get } u'-u=2axh+ah^2 \ldots\ldots\ldots (a_1),$$

$$\text{and } \frac{u'-u}{h}=2ax+ah \ldots\ldots\ldots\ldots\ldots (b_1).$$

If, again, $u=ax^3$, we have

$$u'=a(x+h)^3=ax^3+3ax^2h+3axh^2+ah^3,$$

and, therefore,

$$u'-u=3ax^2h+3axh^2+ah^3 \ldots\ldots\ldots (a_2),$$

$$\text{and } \frac{u'-u}{h}=3ax^2+3axh+ah^2 \ldots\ldots\ldots (b_2).$$

So, likewise, if $u=\frac{1}{x}$, and consequently $u'=\frac{1}{x+h}$,

$$\text{or, by division, } u'=\frac{1}{x}-\frac{h}{x^2}+\frac{h^2}{x^3+x^2h};$$

$$\text{we have } u'-u=-\frac{h}{x^2}+\frac{h^2}{x^3+x^2h} \ldots\ldots (a_3),$$

$$\text{and } \frac{u'-u}{h}=-\frac{1}{x^2}+\frac{h}{x^3+x^2h} \ldots\ldots\ldots\ldots (b_3).$$

In all these instances, x receives the increment h; and the expressions marked (a_1), (a_2), and (a_3), are respectively the corresponding changes produced on the several functions. All the second members of the equations marked (b_1), &c. have the remarkable property that their first terms are independent of the increment h, while that quantity is contained, as a multiplier, in all their other terms: and, consequently, if h be perpetually diminished down towards zero, those second members will tend to become simply their first terms; and, by that diminution, each of them may be made to differ from its first term by a quantity as small as we please—by a quantity, in fact, smaller than anything that can be assigned. These first terms, therefore, are the *limits* to which the second members, and consequently the first, tend when h, the increment of the variable, approaches more and more to evanescence. In the second example, for instance, the limit to which the quotient obtained by dividing $u'-u$, or $a(x+h)^3-ax^3$, by h, approaches, as h is diminished towards zero, is $3ax^2$. By the continued diminution of h, the numerator and denominator of the left hand member will also suffer diminution, and may be made as small as we please. The fraction itself, however, though it likewise diminishes in magnitude, does not suffer unlimited diminution, but tends to take, as its ultimate value or limit, its first term $3ax^2$.*

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* The student will be assisted in forming correct notions in reference to the important subject of limits by the following remarks and illustrations.

The magnitude of a fraction depends, not on the *actual*, but on the *comparative* magnitudes of its terms. Thus, the value of the fraction $\frac{3000000}{4000000}$ is just the same as that of the fraction $\frac{3}{4}$, whose terms are severally only millionth parts of those of the other; nor would the value be changed, were the terms taken only the millionth parts of 3 and 4, the terms of the latter. Hence, therefore, though in the case of any proposed function, h and $u'-u$ may be made smaller and smaller, so as to become as nearly evanescent as we please, it does not follow that the fraction obtained by dividing the latter by the former will become evanescent. It may be constant like the numerical fraction we have just considered, and as would be the case, if the function were $u=ax$; it may be diminishing towards a limit, as in the functions in the text; or it may increase towards a limit; or, as we shall afterwards see, the ratio of one evanescent quantity to another may be nothing or infinite: the relation always depending on the nature of the particular function or class of functions under consideration. The student may illustrate the subject by substituting particular numbers for a, x, and h, in the functions considered above or in others. Thus, if in the second of the functions in the text, we take $a=1$ and $x=10$, we shall have $3ax^2=300$; and the smaller h is taken, the more nearly will the second member of equation (b_2), and consequently the first, approach to this value; being 331, when $h=1$; 303·01, when $h=0·1$; 300·030001, when $h=0·001$, and so on; the numerator, as well as the denominator, of the first member suffering, at the same time, rapid diminutions; being 331 when $h=1$; 30·301 when $h=0·1$; and 0·300030001 when $h=0·001$.

Trigonometrical functions afford good illustrations of the nature of limits. As an example, let it be required to find the limit of the ratio of the tangent to the sine of an angle x, when that angle is continually diminishing towards zero. Here the fraction expressing the ratio is $\frac{\tan x}{\sin x}$, the terms of which may be reduced to any degree of minuteness by the diminution of x. The ratio, however, though it will suffer diminution, will not be reduced to zero, but will continually tend to unity as its limit. To show this, we may multiply the numerator and denominator by cos x, and divide the results by sin x, and (TRIG. No. 11), we shall have

$$\frac{\tan x}{\sin x}=\frac{1}{\cos x};$$

the second member of which (and consequently, its equivalent, the first) tends continually to unity as its limit, as x is diminished, cos x becoming more and more nearly equal to unity. Hence it appears, that the tangent of a very small angle, such as a minute, a second, or the thousandth part of a second, will contain its sine once with an extremely small remainder, that remainder diminishing as x diminishes, and vanishing when it vanishes; while it is plain, that an angle might be found (nearly equal to 90°), the tangent of which would contain its sine

5. In the foregoing examples, the quantities marked (a_1) &c. found by taking the functions of x from those of $x+h$ —that is, by taking their *differences*—are called the

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any assigned number of times, however great. The reader will do well to illustrate this and some of the following examples by means of diagrams.

To find, again, the limit of the ratio of the versed sine of x to its sine, when x becomes evanescent, we have (TRIG. No. 33)

$$\frac{\text{ver sin } x}{\sin x}, \text{ or } \frac{1-\cos x}{\sin x} = \tan \tfrac{1}{2} x,$$

which, the tangent of an evanescent angle being evanescent, tends to vanish as x approaches zero; thus showing that the sine of a second or any other very small angle is many times greater than the versed sine of that angle, and that by making the angle sufficiently small, we can make the quotient obtained by dividing the versed sine by the sine as nearly equal to nothing as we please. In nearly the same manner it would be shown, that if x tend to evanescence, the ratio of its tangent to its versed sine tends to become infinite.

Similar triangles afford geometrical instances of unvarying ratios; as the ratio of two sides of the smallest triangle that could be made or conceived, is the same as that of the homologous sides of an equiangular triangle, however great.

The following examples, though not bearing so directly on the elementary principles of the differential calculus as those already given, will be useful in enlarging the ideas of the student in reference to the subject of limits.

Let P and Q represent quantities which receive successive contemporaneous values according, in each instance, to a determinate law; and, in each of the examples, let the successive values of P be those in the first line; the corresponding values of Q those standing respectively below them in the second; and the values of the ratio of P to Q those which occupy the corresponding positions in the third. As a first example we may take the following:—

$$\text{P} \ldots\ldots\ 3,\ 5,\ 7,\ 9,\ \ldots\ldots,\ 2n+1:$$
$$\text{Q} \ldots\ldots\ 1,\ 4,\ 7,\ 10,\ \ldots\ldots,\ 3n-2;$$
$$\frac{\text{P}}{\text{Q}} \ldots\ldots\ 3,\ \tfrac{5}{4},\ 1,\ \tfrac{9}{10},\ \ldots\ldots,\ \frac{2n+1}{3n-2}.$$

Here n being put to denote the rank or order (first, second, &c.) of the terms in counting from the beginning, it is plain, that the general values of P and Q will be $2n+1$ and $3n-2$, which are placed after the dots denoting continuation. Then, as is indicated, the numbers in the third line are got by dividing those in the first by the corresponding ones in the second. The last, or general term in this line is found, by division, to be equivalent to $\frac{2}{3}+\frac{7}{3}\frac{1}{3n-2}$; a quantity which (ALG. § 127) will become more and more nearly equal to $\frac{2}{3}$, as n increases, and which may be made to differ from that fraction by a quantity smaller than anything that can be assigned. Accordingly, $\frac{2}{3}$ is the limit to which the ratio of P to Q approaches, as n and consequently P and Q are continually increased.

The following example, in which the series are descending ones, is of a similar kind:—

$$\text{P} \ldots\ldots\ \tfrac{1}{9},\ \tfrac{1}{25},\ \tfrac{1}{49},\ \tfrac{1}{81},\ \ldots\ldots\ \frac{1}{(2n+1)^2};$$
$$\text{Q} \ldots\ldots\ \tfrac{1}{6},\ \tfrac{1}{15},\ \tfrac{1}{28},\ \tfrac{1}{45},\ \ldots\ldots\ \frac{1}{1+2+\ldots\ldots+2n+1}=\frac{1}{(n+1)\ (2n+1)};$$
$$\frac{\text{P}}{\text{Q}} \ldots\ldots\ \tfrac{2}{3},\ \tfrac{3}{5},\ \tfrac{4}{7},\ \tfrac{5}{9},\ \ldots\ldots\ \frac{n+1}{2n+1}.$$

Here the denominators in the successive values of P are the squares of the odd numbers, 3, 5, 7, &c., or $(2n+1)^2$. It is easy to see, also, that the first denomi-

differences, or the *finite differences*, and sometimes the *increments*,* of the original functions: and whenever, as in (a_1) and (a_2), the differences are expressed in terms containing successively h, h^2, &c. as multipliers, *the first term of the difference* is called the *differential* of the original function. Thus, $2axh$ is the differential of ax^2, and $3ax^2h$ that of ax^3.

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nator of Q is the sum of 1, 2, and 3; the next, this increased by 4 and 5, &c.; and that the general denominator is $1+2+3+4+\ldots\ldots+2n+1$: which series, consisting of $2n+1$ terms, has (ALG. 133) $(n+1)(2n+1)$ as its sum; and thus the expression after the dots is obtained. The last term in the third line is readily shown to be equal to $\frac{1}{2}+\frac{1}{2}\frac{1}{2n+1}$; a quantity which (ALG. 127) will become more and more nearly equal to $\frac{1}{2}$ as n becomes greater. Accordingly $\frac{1}{2}$ is the limit to which the ratio of P to Q approaches, as n continually increases, and as therefore P and Q continually diminish.

The values of the ratio of P to Q, as obtained above, are apparently detached or discontinuous, n having been taken as a whole number. By taking it fractional, however, in the general expression for the ratio, the values may be made continuous; and they may be represented by the ordinates of a continuous curve, having the values of n as its abscissas, and those of the ratios of P to Q as its ordinates. Thus, if $n=\frac{1}{2}$, the value of the ratio would be $\frac{3}{4}$; while if $n=2\frac{1}{3}$, it would be $\frac{10}{17}$. It may be remarked also, though not exactly to our present purpose, that n may have negative as well as positive values in this and similar cases.

The following example exhibits two ascending series which have the ratio of P to Q tending to the fixed limit 2, as n is continually increased.

$$P \ldots\ldots 4,\ 9,\ 16,\ \ldots\ldots\ldots,(n+1)^2\ ;$$
$$Q \ldots\ldots 4,\ 7,\ 11,\ \ldots\ldots\ldots,1+2+3+\ldots\ldots+n+n+2\ ;$$
$$\frac{P}{Q}\ldots\ldots 1,\ 1\tfrac{2}{7},\ 1\tfrac{5}{11},\ \ldots\ldots,\frac{2(n^2+2n+1)}{n^2+3n+4}.$$

The sum of the series in the second line is found to be $\frac{1}{2}(n^2+3n+4)$ by adding $n+2$ to the sum of $1+2+3+\ldots\ldots+n$: and the ultimate or limiting value of the general expression in the third line is readily found by dividing the numerator and denominator by n^2, and (ALG. 127) taking n infinite in the result.

The student may find it useful to perform the following exercises:—

(1.) Prove that the ratio of the secant of an evanescent angle to its tangent is infinite; but that when the angle becomes 90°, the ratio is that of equality.

(2.) Prove that the ratio of $\sin x \sec x$ to $\tan x \cos x$, that of $\sin x \cos x$ to $\tan x \sec x$, and that of $\sec x - 1$ to ver $\sin x$, are each, ultimately, that of equality, when x is diminished without limit.

(3.) The ratio of $\sin 2x$ to $\tan x$ is less than 2, but it tends to that value as its limit, when x is continually diminished towards zero: while the ratio of $\tan 2x$ to $\tan x$ tends to the same limit, but is always greater than it.

(4.) If P and Q take the successive values, $\frac{1}{2}, \frac{1}{10}, \frac{1}{24}, \frac{1}{50}, \frac{1}{82}, \ldots\ldots$, and $\frac{1}{8}, \frac{1}{15}, \frac{1}{24}, \frac{1}{35}, \frac{1}{48}, \ldots\ldots$, the law of succession of which is easily discovered; prove that the ratio of P to Q is always greater than $\frac{1}{4}$, but that it approaches that value as its limit, when the number of terms is indefinitely increased.

(5.) If P and Q take the successive values, $1^2, 3^2, 5^2, 7^2, \ldots\ldots$, and $1^2, 2^2, 3^2, 4^2, \ldots\ldots$; prove that the ratio of P to Q is always less than 4, but that it tends to that value as its limit, as P and Q increase.

* The *difference* may sometimes, as in the third function in No. 4, be a *decrement*. Still, however, by an extension of the original meaning of the term, we may call it an increment; just as, by an advantageous extension of meaning, we call $x^{\frac{1}{2}}$ and x^{-3} powers of x. See ALGEBRA 46 and 96. In this case, in fact, the decrement will fall to be regarded as a *negative increment*.

In the simple function, $u = x$, we have $u' = x + h$, and consequently $u' - u = h$; so that h is at once the difference and the differential of x.

6. According to the notation now generally employed, the differential of a quantity is denoted by prefixing to it the letter d, the initial of the word differential. Thus the differential of u is expressed by du: and according to the same notation, in connexion with the conclusion of the last No. we have $h = dx$. In conformity, therefore, with this system, we have in the first of the examples in No. 4, du or $d(ax^2) = 2axdx$, and in the second du or $d(ax^3) = 3ax^2dx$.

In these expressions, $2ax$ and $3ax^2$, the multipliers of dx, are called the *differential coefficients* of the functions from which they are derived. These, it will be recollected, are the limits of the quotients obtained by dividing the differences of the functions by those of the variables in No. 4; and, *on this principle in the present work, the differential calculus will be founded;* the following definition being made the basis of the whole subject:—

If a variable quantity x be increased by a quantity h, and if any function of the former be taken from the same function of the latter, and the remainder be divided by h, the limit to which the quotient so obtained will continually approach, and from which it may be made to differ by a quantity less than anything that can be assigned, is called the differential coefficient of the function of x; and the product of this by dx is called the differential of that function.

7. From the definition in the last No. and from the fraction marked b_3 in No. 4, we find that $d\left(\frac{a}{x}\right) = -\frac{adx}{x^2}$.

II.—DIFFERENTIATION OF ALGEBRAIC FUNCTIONS.

8. In conformity with the definition in No. 6, we may now proceed to the differentiation of algebraic functions; and, first, let us differentiate

$$u = fx + Fx - \varphi x + a,$$

where f, F, and φ, and consequently u, denote functions of x. Then, by changing x into $x + h$ we obtain

$$u' = f(x+h) + F(x+h) - \varphi(x+h) + a;$$

whence, by the subtraction of the original function, and by dividing by h, we get

$$\frac{u'-u}{h}=\frac{f(x+h)-fx}{h}+\frac{F(x+h)-Fx}{h}-\frac{\varphi(x+h)-\varphi x}{h}$$

If in this, h be continually diminished towards zero, we shall have ultimately, according to No. 6,

$$\frac{du}{dx}=\frac{dfx}{dx}+\frac{dFx}{dx}-\frac{d\varphi x}{dx};$$

and, therefore, du, or $d\,(fx+Fx-\varphi x+a)$
$$=dfx+dFx-d\varphi x.$$

Hence it appears, that, *to differentiate a quantity composed of several functions of the same variable connected by addition or subtraction, we are to find the differentials of the several functions, and connect them by the signs belonging to those functions; and that constant quantities so connected, disappear in the differential.*

9. We may next differentiate x^n, the most important of all the functions concerned in the differential calculus. In this function, n may be a positive whole number; a positive fraction; or a negative number, whole or fractional; and hence the investigation presents three cases.

To proceed with the first case, let us assume, as usual, $u=x^n$ and $u'=(x+h)^n$. Now the first term of the development of $(x+h)^n$ is x^n, and the second $nx^{n-1}h$; and the others are respectively the products of h^2, h^3, h^4, &c. by quantities which are independent of h. To prove this, when n is a positive integer,* we find by multiplication, that

$(x+h)^2=x^2+2xh+h^2$, and $(x+h)^3=x^3+3x^2h+3xh^2+h^3$,

in each of which the property holds. Now, if it hold for any power, such as the m^{th}, so that

$$(x+h)^m=x^m+mx^{m-1}h+\mathrm{A}h^2+\mathrm{B}h^3+, \text{ \&c.}$$

where A, B, &c. are independent of h, we can show that it will hold for the next higher power, simply by multiplying by $x+h$, as we thus get

$$(x+h)^{m+1}=x^{m+1}+(m+1)x^mh+(\mathrm{A}x+mx^{m-1})h^2, \text{ \&c.}$$

which is still of the same form, the index being now $m+1$.

* It is true universally.—See ALGEBRA, 209.

Since the property holds, therefore, as we have seen, for the third power, it must hold for the fourth; and, holding for the fourth, it must hold for the fifth; and so on continually. We have therefore

$$(x+h)^n, \text{ or } u' = x^n + nx^{n-1}h + Ah^2 + Bh^3 +, \&c.$$

Hence, by subtracting $u = x^n$, and dividing by h, we get

$$\frac{u' - u}{h} = nx^{n-1} + Ah + Bh^2 +, \&c.$$

from which, if h be continually diminished towards zero, we obtain by No. 6,

$$\frac{du}{dx} = nx^{n-1}, \text{ and therefore } du \text{ or } d(x^n) = nx^{n-1}dx,$$

the required differential: and it thus appears, that when u is a positive integer, the differential of x^n is the continued product of the index n, the next lower power of the variable x, and the differential of x.

If the index n be a positive fraction, let it be represented by $\frac{p}{q}$, p and q being whole positive numbers. We have then $u = x^{\frac{p}{q}}$; and from this, by involving both members to the q^{th} power, we get $u^q = x^p$. If this be differentiated according to the principle just established, there will result $qu^{q-1}du = px^{p-1}dx$. Now from $u = x^{\frac{p}{q}}$, we obtain by involution (ALG. 95), $u^{q-1} = x^{p-\frac{p}{q}}$: and the substitution of this in the last equation gives $qx^{p-\frac{p}{q}}du = px^{p-1}dx$. Hence, by dividing (ALG. 105) by the coefficient of du, and substituting n for $\frac{p}{q}$, we get du, or $d(x^n) = nx^{n-1}dx$, the same as in the former case.

Lastly, if the index be negative, let us assume $x^{-n} = y^n$; and we have from this (ALG. 95) $x^{-1} = y$; so that y^n is a power of the reciprocal of x, having the positive index n. Hence, by the preceding part of this No. we have $d(x^{-n}) = ny^{n-1}dy$. Now, since $y = x^{-1}$, we have (ALG. 95) $y^{n-1} = x^{-n+1}$; also by No. 7, taking $a = 1$, we get $dy = -x^{-2}dx$: and by the substitution of these in the preceding differential, we obtain

$$d(x^{-n}) = nx^{-n+1} \times -x^{-2}dx = -nx^{-n-1}dx;$$

a result which agrees with those in the first and second cases, the index being negative.

Hence, therefore, *to differentiate any power of a variable quantity having its index constant, multiply by the index, diminish the index by unity, and multiply by the differential of the variable.*

It is easy to see, that the same rule will be applicable, when the index is a surd; as the value of the surd may be approximated as nearly as we please in a rational fraction.

10. *In differentiating the product of a function by a constant quantity, the function is to be differentiated, and the multiplier retained.* The proof is easily effected by assuming $u = ay$, and as usual, $u' = ay'$, y and y', being like functions of x and $x + h$. Hence we get $u' - u = a(y' - y)$; and from this, by dividing by h, passing to the limit, and multiplying by dx, we obtain $du = ady$.

11. The differential of yz, the product of two functions of x, is readily obtained in the following manner:—Let the members of the identity,

$$y^2 + 2yz + z^2 = (y + z)^2,$$

be differentiated according to Nos. 9 and 10; then,

$$2ydy + 2d(yz) + 2zdz = 2(y + z)(dy + dz) = \\ 2ydy + 2ydz + 2zdy + 2zdz.$$

Hence, by rejecting the quantities common to both members, and dividing by 2, we get

$$d(yz) = ydz + zdy.$$

Again, let $yz = v$, and multiply by t, another function of x; then $tyz = tv$; and by what we have just seen, $d(tyz)$ or $d(tv) = vdt + tdv$, and dv or $d(yz) = zdy + ydz$. Then, by substituting in the foregoing expression for $d(tyz)$, these values of v and dv, we get

$$d(tyz) = yzdt + tzdy + tydz.$$

By putting $tyz = v$, and multiplying by w, another function of x, we should find, by a process exactly similar, that

$$d(wtyz) = tyzdw + wyzdt + wtzdy + wtydz.$$

In all these instances we see, that *the differential of the product of the several functions of the variable is equal to the sum of the several products obtained by multiplying the differential of each factor by all the other factors; and* *this principle holds universally, whatever may be the number of the factors,* as would appear by continuing the same process of investigation.

12. To find the differential of a fraction having its numerator y and its denominator z functions of x, we may assume

$$u = \frac{y}{z}, \text{ or } u = yz^{-1}.$$

Hence, (Nos. 11 and 9) $du = z^{-1}dy - yz^{-2}dz$. From this, by simple and obvious modifications, we get the three following forms of the required differential:

$$d\left(\frac{y}{z}\right) = \frac{dy}{z} - \frac{ydz}{z^2}, \qquad d\left(\frac{y}{z}\right) = \frac{zdy - ydz}{z^2},$$

$$d\left(\frac{y}{z}\right) = \frac{y}{z}\left(\frac{dy}{y} - \frac{dz}{z}\right).$$

The following rule is the expression in words of the second of these, and it is the rule which is commonly employed in practice: *To find the differential of a fraction whose terms are functions of the same variable, multiply the differential of the numerator by the denominator, and the differential of the denominator by the numerator; take the latter product from the former, and divide the remainder by the square of the denominator.*

13. We have now investigated the methods of differentiating algebraic functions of every kind; and what has been done may be followed up by an investigation of the binomial theorem of Newton,* which in its nature is purely algebraical. Preparatory to the investigation, as it is here given, it is necessary to establish the following proposition:—

The differential coefficient of a function of a binomial, such as $f(x + y)$, is the same, whichever of the parts, x and y, is supposed to vary. To prove this, suppose x or y to receive the increment h: then $x + y$ will become $x + y + h$. Put $u' = f(x + y + h)$; and by taking the difference in the usual way, and dividing by h, there is obtained

* This most important and valuable theorem may be investigated without the aid of the differential calculus, though not so easily. (See ALG. pp. 202—205). Its investigation here will afford a useful example of the application of the principles that have been established, and will form an easy introduction to the investigation of Taylor's theorem, of which this theorem is a particular case. It will be used with advantage also in the next section.

$$\frac{u'-u}{h}=\frac{f(x+y+h)-f(x+y)}{h}.$$

Now, if h be continually diminished towards zero, the first member (No. 6) will tend to become $\frac{du}{dx}$ or $\frac{du}{dy}$, according as h is regarded as the increment of x or of y, while the second member tends to the same limit in each case; and that limit (No. 6) is the differential coefficient; and therefore the two differential coefficients are identical.

14. Now to investigate the binomial theorem, if we put $u=(x+h)^n$, we have (No. 9)

$$u=x^n+nx^{n-1}h+Ah^2+Bh^3+Ch^4+, \&c. \ldots\ldots (a);$$

and it only remains that we determine the coefficients, A, B, C, &c. which, as we saw, are independent of h, and depend solely on n and x; and which therefore will be variable if x be variable, and constant if it be constant. Let us now differentiate equation (a) on the supposition that x is variable and h constant, and divide by dx; and again, let us differentiate the same regarding h as variable and x as constant, and divide by dh. In the first of these operations, the powers of h will (No. 10) be simply retained as constant multipliers; and the forms of A, B, &c. being unknown, their differentials can only be indicated by prefixing the letter d. In the second x^n (No. 8) will disappear and nx^{n-1}, A, B, &c. will (No. 10) be retained as constant multipliers. In this way we shall obtain

$$\frac{du}{dx}=nx^{n-1}+n(n-1)x^{n-2}h+\frac{dA}{dx}h^2+\frac{dB}{dx}h^3+, \&c. \ (b);$$

$$\frac{du}{dh}=nx^{n-1}+\quad 2Ah \quad +3Bh^2+4Ch^3+, \&c. \ (c).$$

Now (No. 13) these must be identical, and they will plainly be such, if values be assigned to A, B, C, &c. which will render the coefficients of the like powers of h equal. (See ALG. Chap. XI.) To accomplish this, we must make

$$2A=n(n-1)x^{n-2}, \text{ whence } A=\frac{n}{1}\cdot\frac{n-1}{2}x^{n-2}.$$

By differentiating this, and dividing by dx, we get

$$\frac{dA}{dx}=\frac{n}{1}\cdot\frac{n-1}{2}(n-2)x^{n-3},$$

which is the coefficient of h^2 in (b), and must therefore be equal to 3B, the coefficient of the same in (c). Dividing, therefore, by 3, we find

$$B=\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}x^{n-3}.$$

From this, in like manner, by differentiating, and dividing by dx and by 4, we should get C; and the succeeding coefficients would be found in a similar manner by carrying out the series to a greater number of terms. This, however, is unnecessary, as the law of formation is evident; and by substituting in (a) the values of A, B, C, &c. we get as the required development, u, or

$$(x+h)^n=x^n+nx^{n-1}h+\frac{n}{1}.\frac{n-1}{2}x^{n-2}h^2+\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}x^{n-3}h^3$$
$$+\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}x^{n-4}h^4+\text{ \&c.}$$

15. By dividing each of the terms of the foregoing development after the first by the term immediately preceding it, we get $n\frac{h}{x}$, $\frac{1}{2}(n-1)\frac{h}{x}$, $\frac{1}{3}(n-2)\frac{h}{x}$ &c.; and therefore, conversely, if the first term be multiplied by the first of these quotients, the product will be the second; if the second be multiplied by the next quotient, the product will be the third term, and so on. Hence, if the terms be called T_1, T_2, T_3, &c. the formula may be written thus:

$$(x+h)^n=x^n+nT_1\frac{h}{x}+\frac{n-1}{2}T_2\frac{h}{x}+\frac{n-2}{3}T_3\frac{h}{x}+\text{ \&c.:}$$

a form which is convenient in practice.

EXAMPLES OF THE DIFFERENTIATION OF ALGEBRAIC FUNCTIONS, AND OF THE USE OF THE BINOMIAL THEOREM.

16. REQUIRED the differential of $u=a+bx-c\sqrt[3]{x}-\frac{a}{x}$.

Here (No. 8) we are to differentiate the terms separately, and the first term will disappear. The differential of bx is (No. 10) bdx. The third term may be put under the form $-cx^{\frac{1}{3}}$; the differential of which (No. 9) is

$-\frac{1}{3}cx^{-\frac{2}{3}}dx$, or $-\frac{c\,dx}{3x^{\frac{2}{3}}}$. The differential of the last term (No. 9) is $\frac{adx}{x^2}$. Hence (No. 8) $du=bdx-\frac{c\,dx}{3x^{\frac{2}{3}}}+\frac{adx}{x^2}$; and, by dividing by dx, we get the differential coefficient,

$$\frac{du}{dx}=b-\frac{c}{3x^{\frac{2}{3}}}+\frac{a}{x^2}$$

17. Let $u=a\sqrt[3]{x^2}-\frac{b}{x\sqrt[4]{x}}+\frac{c}{x^5}$. This may be written

$$u=ax^{\frac{2}{3}}-bx^{-1-\frac{1}{4}}+cx^{-5}, \quad \text{or } u=ax^{\frac{2}{3}}-bx^{-\frac{5}{4}}+cx^{-5}.$$

Hence, by differentiating the several terms, we get (No. 9)

$$du=\tfrac{2}{3}ax^{\frac{2}{3}-1}dx+\tfrac{5}{4}bx^{-\frac{5}{4}-1}dx-5cx^{-5-1}dx;$$

or, by contraction, and by taking the quantities with negative indices to the denominators,

$$du=\frac{2adx}{3x^{\frac{1}{3}}}+\frac{5bdx}{4x^{\frac{9}{4}}}-\frac{5cdx}{x^6}.$$

18. Let $u=\frac{x+a}{x-a}$. Here we may put $x+a=z$, and $x-a=z'$; by differentiating which, we get $dx=dz$, and $dx=dz'$. This substitution changes the proposed function into $u=\frac{z}{z'}$, the differential of which (12) is $du=\frac{z'dz-zdz'}{z'^2}$. In this, substitute for z, z', dz, and dz', their several values and there will result $du=\frac{(x-a)\,dx-(x+a)\,dx}{(x-a)^2}$; or, by contraction, $du=-\frac{2a\,dx}{(x-a)^2}$. This might have been obtained simply by means of the rule for differentiating a fraction, without the introduction of z and z'. Such a substitution, however, when some of the quantities are not monomials, and especially if they be complicated, often facilitates the operation. It also fits the proposed quantities for the direct application of the general rules already obtained.

19. Let $u=\left(a+bx^{\frac{3}{5}}+\frac{c}{x\sqrt{x}}\right)^n$. Here put $z=a+bx^{\frac{3}{5}}+\frac{c}{x\sqrt{x}}$: then $u=z^n$; and by differentiating both these, we

get $dz=\frac{3bdx}{5x^{\frac{2}{5}}}-\frac{3cdx}{2x^{\frac{5}{2}}}$, and $du=nz^{n-1}dz$. In the latter, substitute the values of z and dz: then,

$$du=n\left(a+bx^{\frac{3}{5}}+\frac{c}{x\sqrt{x}}\right)^{n-1}\times\left(\frac{3bdx}{5x^{\frac{2}{5}}}-\frac{3cdx}{2x^{\frac{5}{2}}}\right).$$

20. To show the method of employing the binomial theorem, let us take the following examples; and, first, let it be required to find the sixth power of $x+h$. Here we have $n=6$, $\frac{n-1}{2}=\frac{5}{2}$, $\frac{n-2}{3}=\frac{4}{3}$, $\frac{n-3}{4}=\frac{3}{4}$, $\frac{n-4}{5}=\frac{2}{5}$, $\frac{n-5}{6}=\frac{1}{6}$, and $\frac{n-6}{7}=0$. Hence, (No. 14) we get,

$$(x+h)^6=x^6+6x^5h+15x^4h^2+20x^3h^3+15x^2h^4+6xh^5+h^6.$$

21. To expand $(a^3+z)^{\frac{2}{3}}$, that is, to extract the cube root of the square of a^3+z, we have $x=a^3$, $h=z$, and $n=\frac{2}{3}$. Then $x^n=(a^3)^{\frac{2}{3}}=a^2$, $x^{n-1}=(a^3)^{-\frac{1}{3}}=a^{-1}=\frac{1}{a}$, $x^{n-2}=(a^3)^{-\frac{4}{3}}=a^{-4}=\frac{1}{a^4}$, &c. Also, $\frac{n-1}{2}=-\frac{1}{6}$, $\frac{n-2}{3}=-\frac{4}{9}$, $\frac{n-3}{4}=-\frac{7}{12}$, &c. Substituting these values in the formula (No. 14), we get

$$(a^3+z)^{\frac{2}{3}}=a^2+\frac{2z}{3a}-\frac{2.1z^2}{3.6a^4}+\frac{2.1.4}{3.6.9}\cdot\frac{z^3}{a^7}-\frac{2.1.4.7}{3.6.9.12}\cdot\frac{z^4}{a^{10}}+$$

&c. where the law of the series is evident.

22. Required a series equivalent to $\frac{a}{\sqrt{(a^2-x)}}$, or its equal $a(a^2-x)^{-\frac{1}{2}}$. Here we must expand $(a^2-x)^{-\frac{1}{2}}$, and multiply the result by a. In doing this, we are to substitute a^2 for x, and $-x$ for h in the binomial theorem. We have, also, $n=-\frac{1}{2}$, $n-1=-\frac{3}{2}$, $n-2=-\frac{5}{2}$, &c.; and, therefore, $x^n=(a^2)^{-\frac{1}{2}}=a^{-1}=\frac{1}{a}$, $x^{n-1}=(a^2)^{-\frac{3}{2}}=a^{-3}=\frac{1}{a^3}$, &c.; and the coefficients n, $\frac{n-1}{2}$, $\frac{n-2}{3}$, &c. are equal

to $-\frac{1}{2}$, $-\frac{3}{4}$, $-\frac{5}{6}$, $-\frac{7}{8}$, &c. Substituting these, and multiplying by a, we get

$$\frac{a}{\sqrt{(a^2-x)}}=1+\frac{1}{2}\cdot\frac{x}{a^2}+\frac{1.3}{2.4}\cdot\frac{x^2}{a^4}+\frac{1.3.5}{2.4.6}\cdot\frac{x^3}{a^6}+\text{ \&c.}$$

EXERCISES IN THE DIFFERENTIATION OF ALGEBRAIC FUNCTIONS.

1. LET $u=ax^{\frac{7}{3}}$. *Ans.* $du=\frac{7ax^{\frac{4}{3}}dx}{3}$.

2. Let $u=\frac{a}{x^{\frac{7}{3}}}$. *Ans.* $du=-\frac{7adx}{3x^{\frac{10}{3}}}$.

3. Let $u=\frac{a-x}{a+x}$. *Ans.* $du=-\frac{2adx}{(a+x)^2}$.

4. Let $u=x(a+x)(a^2+x^2)$.
Ans. $du=(a^3+2a^2x+3ax^2+4x^3)dx$;
or, if $a=1$, $du=(1+2x+3x^2+4x^3)dx$.

5. Let $u=\frac{a+x}{a'+x}$. *Ans.* $du=\frac{(a'-a)dx}{(a'+x)^2}$.

6. Let $u=\sqrt{\frac{1+x}{1-x}}$. *Ans.* $du=\frac{dx}{(1-x)\sqrt{(1-x^2)}}$.

7. Let $u=(a-x)\sqrt{(a+x)}$. *Ans.* $du=-\frac{1}{2}\cdot\frac{(a+3x)dx}{\sqrt{(a+x)}}$.

8. Let $u=(a-x)\sqrt{(a^2+x^2)}$.
Ans. $du=-\frac{(a^2-ax+2x^2)dx}{\sqrt{(a^2+x^2)}}$.

9. Let $u=(a^2-x^2)\sqrt{(a+x)}$.
Ans. $du=\frac{1}{2}(a-5x)\sqrt{(a+x)}dx$.

10. Let $u=(a^2+x^2)\sqrt{(a-x)}$.
Ans. $du=-\frac{(a^2-4ax+5x^2)dx}{2\sqrt{(a-x)}}$.

11. Let $u=\frac{\sqrt{(a+x)}}{a-x}$. *Ans.* $du=\frac{(3a+x)dx}{2(a-x)^2\sqrt{(a+x)}}$.

12. Let $u=\frac{a+x}{\sqrt{(a-x)}}$. *Ans.* $du=\frac{(3a-x)dx}{2(a-x)^{\frac{3}{2}}}$.

13. Let $u=\frac{a-x}{(a^3+x^3)^{\frac{1}{3}}}$ *Ans.* $du=-\frac{(a^3+ax^2)dx}{(a^3+x^3)^{\frac{4}{3}}}$.

14. Let $u=\frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}}$.*

$$Ans.\ du=-\frac{dx}{x^2\sqrt{(1-x^2)}}-\frac{dx}{x^2}.$$

15. The sum of the infinite series, $1+x+x^2+x^3+$ &c. is $\frac{1}{1-x}$, when x is less than unity. Differentiate these, and divide the results by dx.† *Ans.* $1+2x+3x^2+4x^3+$ &c. $=\frac{1}{(1-x)^2}$.

16. Differentiate the answer of the last exercise, and divide by dx. *Ans.* $2+2.3x+3.4x^2+4.5x^3+$ &c. $=\frac{2}{(1-x)^3}$.

III.—DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS.

23. Of transcendental functions there are two great classes; first, the *logarithmic*, including their inverse, the *exponential;* and secondly, the *trigonometrical.*

24. The logarithms of numbers are other numbers depending on them, and characterized by the property, that *the sum of the logarithms of any two numbers is equal to the logarithm of their product.* Thus, $\log b+\log c=\log (bc)$.

Hence also, since $b=\frac{b}{c}.c$, it follows, that

$$\log b=\log\frac{b}{c}+\log c;\ \text{whence}\ \log b-\log c=\log\frac{b}{c}:$$

that is, *if the logarithm of any number be taken from that of any other, the remainder is the logarithm of the quotient obtained by dividing the latter by the former:* and hence it follows, that in place of the difference of two logarithms, we may write the logarithm of the quotient of the corresponding

* By multiplying the numerator and denominator by $\sqrt{(1+x)}+\sqrt{(1-x)}$, and dividing the products by 2, we get $\frac{1+\sqrt{(1-x^2)}}{x}$, which is easily differentiated.

† This exercise, and the following, afford each an instance in which, when the sum of one series is known, other series may be derived from it, the sums of which will also be known: and there are numberless other instances of a similar kind.

numbers, and conversely. It follows, also, since $a \times 1 = a$, that $\log a + \log 1 = \log a$; and, therefore, $\log 1 = 0$.*

25. Since, if n be a whole positive number, $x^n = x.x.x.....$ we have $\log(x^n) = \log x + \log x + \log x + \log x + \ldots\ldots\ldots,$ or $\log(x^n) = n \log x$.

Let, again, $y = x^{\frac{m}{n}}$, and consequently $y^n = x^m$, m and n being positive integers, and we have, by what we have just seen, $n \log y = m \log x$; whence $\log y$ or $\log(x^{\frac{m}{n}}) = \frac{m}{n} \log x$.

Lastly, let $y = x^{-n}$; and consequently $yx^n = 1$, and (No. 24) $\log y + n \log x = 0$. Hence,

$$\log y = \log(x^{-n}) = -n \log x.$$

From these three results, it appears that *the logarithm of any power of a number is found by multiplying the logarithm of the number by the index of the power.*†

26. Hence, if a be a positive number, of which, accordingly, every power will be real, and if we have $a^y = x$; then, by the last No. $y \log a = \log x$: and hence, conversely, if we have $y \log a = \log x$, we may derive from it at once the inverse expression, $a^y = x$. Of expressions of this kind, the most interesting is that in which a is the *base* of the system of logarithms, that is, *the number whose logarithm in that system is unity.* In that case, while we have still $a^y = x$, we have simply $y = \log x$. Hence, as x may be any positive number, and y its logarithm, we arrive at the conclusion, that *the logarithm of a number is the index of the power to which the base of the system must be raised to produce that number.* Thus, in the common logarithms, in which 10, the radix of the decimal system of notation, is the base, we have $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, &c.; where 0, 1, 2, 3, &c. are the logarithms of 1, 10, 100, 1000, &c.; and it is plain, that between these there may be innumerable fractional indices or logarithms corres-

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* It follows from what has been stated, that if $bc = 1$, $\log b + \log c = 0$, and therefore $\log b = -\log c$; and it thus appears that the logarithm of a number and the logarithm of its reciprocal are the same but with contrary signs. It is plain also, that the logarithm of a fraction between 0 and 1 is negative.

† It is easy to see from this, that the logarithm of an infinite number is infinite. Thus, in the common system, we have $\log(10^n) = n$; and the number 10^n cannot be infinite, unless the index or logarithm n be so likewise. It follows also, since the quotient obtained by dividing unity by a quantity infinitely great, is zero, that the logarithm of 0 is $-\infty$. The views given in this note and the preceding one point out no logarithms for negative numbers. We shall see hereafter, that such numbers have not real but imaginary logarithms.

ponding to the numbers between 0 and 10, 10 and 100, 100 and 1000, &c.

27. We may now proceed to investigate the differential of $\log x$; and, as usual, let us assume $u=\log x$ and $u'=\log(x+h)$; whence we have

$$\frac{u'-u}{h}=\frac{\log(x+h)-\log x}{h}=\frac{(1+hx^{-1})}{h},$$

the latter form being obtained by means of No. 24. The limit to which each of these equivalent quantities tends when h is evanescent, will be readily found from the last of them, by putting $hx^{-1}=v$, and consequently $h=xv$; an assumption on which it is plain that h and v will be evanescent simultaneously. By this means we get

$$\frac{\log(1+hx^{-1})}{h}=\frac{\log(1+v)}{xv}=\frac{1}{x}\cdot\frac{\log(1+v)}{v}=\frac{1}{x}\cdot\log(1+v)^{\frac{1}{v}}.$$

To find the limit of this, when v is evanescent, let us expand $(1+v)^{\frac{1}{v}}$ by the binomial theorem. In doing this, by means of No 14, we have $x=1$, $h=v$, and $n=\frac{1}{v}$; and therefore $(1+v)^{\frac{1}{v}}=$

$$1+\frac{1}{v}.v+\frac{1}{v}\cdot\frac{1}{2}\left(\frac{1}{v}-1\right)v^2+\frac{1}{v}\cdot\frac{1}{2}\left(\frac{1}{v}-1\right)\cdot\frac{1}{3}\left(\frac{1}{v}-2\right)v^3+\ \&c.;$$

which, by easy modifications, becomes

$$(1+v)^{\frac{1}{v}}=1+1+\frac{1-v}{1\cdot 2}+\frac{(1-v)(1-2v)}{1\cdot 2\cdot 3}+\ \&c.$$

the development required, and the law of continuation is evident. Now when v is continually diminished towards zero, the second member of this tends to become

$$1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\ \&c.;$$

and it may plainly be made to differ from this by as small a quantity as we please. The sum of all the terms of this series after the first and second is less than unity; each of them after the third being less than the corresponding terms of the infinite series $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+$, &c. the sum of which (ALG. 138) is $\frac{1}{2}$. Hence the sum of the entire series is a fixed number between 2 and 3. This number may be denoted by ε, and its value by the actual addition of a sufficient number of its terms,* would be found to be 2·718281828459......We have therefore, the limit of

$$\frac{1}{x}\log(1+v)^{\frac{1}{v}},\ \text{or}\ \frac{u'-u}{h},\ \text{or}\ \frac{\log(x+h)-\log x}{h}=\frac{1}{x}\log\varepsilon.$$

Hence, (No. 6) $\frac{du}{dx}=\frac{1}{x}\log\varepsilon$; and du, or

$$d\log x=\frac{dx}{x}\log\varepsilon=\mathrm{M}\frac{dx}{x},$$

if M be assumed to denote $\log\varepsilon$. The number M is called the *modulus* of the particular system of logarithms that may be under consideration.

28. The modulus M may have any value whatever. If it be equal to unity, the logarithms are called *Neperian.*† These, which are generally used in investigations, will be denoted in what follows, by *log;* while other logarithms may be represented by *l*. Hence, also, we have the differential of the Neperian logarithm of $x=\frac{dx}{x}$.

29. The theorem found in No. 27 leads to the method of differentiating an *exponential function*, such as a^x; that is, *a power whose index is variable.* Thus, let $u=a^x$, and (No. 24) we have $\log u=x\log a$. Hence (by No. 28),

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* Thus by working for only six places of decimals, we get

First term	....	=1·000000
Second	....	=1·000000
Third	(=half the second)	= ·500000
Fourth	(=one third of the third)	= ·166667, nearly
Fifth	(=one fourth of the fourth)	= ·041667, nearly
Sixth	(=one fifth of the fifth)	= ·008333
Seventh	(=one sixth of the sixth	= ·001389, nearly
Eighth	(=one seventh of the seventh)	= ·000198
Ninth	(=one eighth of the eighth)	= ·000025, nearly
Tenth	(=one ninth of the ninth)	= ·000003, nearly
	Sum= ϵ	=2·718282, nearly.

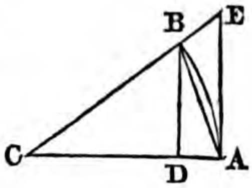
The first ten figures of this remarkable number are easily recollected from the circumstance, that in our common arithmetical notation, the figures 1828 occur twice in succession.

† The Neperian logarithms are so called from Baron Neper or Napier, of Merchiston, near Edinburgh, the inventor of logarithms, as they were the kind which he investigated. These are also frequently called *hyperbolic logarithms*, because they may be derived from the quadrature of the spaces comprehended between the equilateral hyperbola and its asymptotes. This name is improper, however; as logarithms of every kind may be derived from the hyperbola, the change in the angle of the asymptotes, or the proportional enlargement or diminution in any way of the spaces referred to, producing the different kinds. The Neperian logarithms have also been sometimes called *natural* ones. Other details respecting logarithms will be given hereafter.

we obtain $\frac{du}{u} = dx \log a$; and, therefore, $du = u\,dx \log a$, or $da^x = a^x dx \log a$; which, in words, gives the following rule:—*To differentiate a variable power of a constant quantity, multiply that power by the differential of the index, and the result by the Neperian logarithm of the constant quantity.* Hence, if a become ε, the base of the Neperian logarithms, we have $d\varepsilon^x = \varepsilon^x dx$, and consequently $\frac{d\varepsilon^x}{dx} = \varepsilon^x$.

30. We may now proceed to the differentiation of trigonometrical functions; and it may be stated as a lemma, that *when an arc of a given circle is less than a quadrant, it is greater than its sine and less than its tangent.**

To illustrate this, let AB be such an arc of a circle of which the centre is C, and let BD, BA, and AE, be its sine, chord, and tangent. Now, according to an obvious axiom, a straight line is the shortest distance between two points; and therefore the chord BA is less than the arc BA. But (EUCLID I. 19) the sine BD is less than the chord BA; and therefore, *a fortiori*, the sine is less than the arc.



Again, if we suppose the angle ACB to be divided into equal parts by radii, these radii will (EUCLID, VI. 33, and III. 26) divide the sector ACB into equal sectors, the arcs or bases of which will also be equal. Now, if the number of these sectors be indefinitely increased, their arcs may each be made as small and as little different from straight lines as we please; and each of the small sectors will tend to become a rectilineal triangle, the area of which (EUCLID I. 46, cor. 6) will be half the product of its base by its perpendicular, the radius. Hence, the area of the whole sector will be equal to half the product of the radius and the arc AB, this arc being the sum of all those bases. But the area of the triangle ACE is half

* The radius is here assumed to be unity. If it be not, the sine and tangent are not BD and AE, but the quotients obtained by dividing the numbers expressing the lengths of these lines by the number expressing the length of the radius. —See TRIG. Nos. 6 and 7.

the product of the radius and AE; and this area being greater than that of the sector, AE must be greater than the arc AB.

31. A consequence of the preceding lemma is, that *when an arc is continually diminished towards zero, the ratio of it and its sine* (and likewise that of it and its tangent) *tends to the ratio of equality as its limit.* To prove this, we have (from TRIG. No. 11) $\frac{\sin x}{\tan x} = \cos x$. Now, as x is diminished, $\cos x$ becomes more and more nearly equal to unity; and it has this as its limit, when x is evanescent. The ratio, therefore, of $\sin x$ to $\tan x$, when x is continually diminished, tends to unity as its limit: and the arc being always of intermediate magnitude between its sine and tangent, must, *a fortiori*, tend to have equality as the limit of its ratio to each, when it is diminished towards zero.

32. Let us now assume $u = \sin x$, and $u' = \sin(x+h)$;

then, $$\frac{u'-u}{h} = \frac{\sin(x+h) - \sin x}{h}.$$

Now, since (TRIG. No. 25) $\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$, the foregoing, by substituting $x+h$ for A and x for B, will become

$$\frac{u'-u}{h} = \frac{2\cos(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h} = \cos(x+\tfrac{1}{2}h).\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}.$$

The limit of this when h is diminished towards zero, will plainly have $\cos x$ for its first factor; and, by the last No. its other factor, $\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}$, will have unity as its limit. Hence, (No. 6) we shall have $\frac{du}{dx} = \cos x$, and therefore, du or

$$d\sin x = \cos x\,dx.$$

33. Since $\cos x = \sin(\frac{1}{2}\pi - x)$, we have (by the last No.) $d\cos x = \cos(\frac{1}{2}\pi - x)\,d(\frac{1}{2}\pi - x)$. But $\cos(\frac{1}{2}\pi - x) = \sin x$, and $d(\frac{1}{2}\pi - x) = -dx$, π being constant; by substituting which, in the foregoing expression, we get $d\cos x = -\sin x dx$.

The same might also be obtained by a process similar to that which was employed in the last No.

34. Since versin $x = 1 - \cos x$, we get

$$d\,\text{versin}\,x = -d\cos x, \quad \text{or (No. 33)} \quad d\,\text{versin}\,x = \sin x\,dx.$$

Also, because coversin $x = 1 - \sin x$, we find, in a similar manner, $d\,\text{coversin}\,x = -\cos x\,dx$.

35. To differentiate $\tan x$, we have (TRIG. No. 11)

$$\tan x \cos x = \sin x; \text{ whence (No. 11)}$$

$$\cos x\,d\tan x + \tan x\,d\cos x = d\sin x;$$

or, by substituting for $d\sin x$ and $d\cos x$ their equals, according to Nos. 32 and 33,

$$\cos x\,d\tan x - \tan x \sin x\,dx = \cos x\,dx.$$

From this, by transposing the second term, and dividing by $\cos x$, we obtain

$$d\tan x = dx + \tan^2 x\,dx, \quad \text{or } d\tan x = (1 + \tan^2 x)\,dx.$$

Since (TRIG. No. 11) $1 + \tan^2 x = \sec^2 x$, and $\sec^2 x = \dfrac{1}{\cos^2 x}$, this differential may also be put under the forms,

$$d\tan x = \sec^2 x\,dx, \quad \text{and } d\tan x = \frac{dx}{\cos^2 x}.$$

By substituting, in these expressions, $\frac{1}{2}\pi - x$ for x, we obtain

$$d\cot x = -(1 + \cot^2 x)\,dx = -\text{cosec}^2 x\,dx = -\frac{dx}{\sin^2 x}.$$

36. Since (TRIG. No. 11) $\sec x \cos x = 1$, we get (Nos. 11 and 33) $\cos x\,d\sec x - \sec x \sin x\,dx = 0$, whence, transposing, and dividing by $\cos x$, we find $d\sec x = \sec x \tan x\,dx$. This (TRIG. No. 11) may also be put under the forms,

$$d\sec x = \sec x \sqrt{(\sec^2 x - 1)}\,dx, \text{ and } d\sec x = \frac{\sin x\,dx}{\cos^2 x}.$$

By substituting $\frac{1}{2}\pi - x$ for x in these formulas, we obtain,

$$d\,\text{cosec}\,x = -\text{cosec}\,x \cot x\,dx$$

$$= -\text{cosec}\,x \sqrt{(\text{cosec}^2 x - 1)}\,dx = -\frac{\cos x\,dx}{\sin^2 x}.$$

37. In all the foregoing investigations regarding trigonometrical functions, the arc has been taken as the independent variable, and the sine, tangent, &c. as functions of it. It is often advantageous, however, to regard an arc as a function of its sine, tangent, &c.; and we may now proceed to derive from the preceding results some of the most

useful of the expressions that arise from this view of the subject.

A very convenient notation for expressing these and other *inverse* functions, as they have been called, has been proposed by Sir John Herschel. According to this notation, if $y = \sin x$, the inverse notation will be $x = \sin^{-1} y$: and while according to the first expression, y is pointed out as being the sine of x, the second expresses the same idea in an inverted order as it were, denoting that x is the arc whose sine is y. In like manner, if $y = \cot x$, then $x = \cot^{-1} y$, the first equation expressing that y is the cotangent of the arc x, and the second that x is the arc whose cotangent is y. So likewise if $y = \log x$, we should have $x = \log^{-1} y$; the first denoting that y is the logarithm of the number x, and the second that x is the number whose logarithm is y. In like manner, universally, if $y = fx$, the inverse function is $x = f^{-1} y$; the former expressing the idea that y is a function of the quantity x, and the latter that x is the quantity of which y is that function.

38. Adopting this notation, we may now proceed to find expressions for the differential of an arc in conformity with the views stated above. If we assume, then, $\sin^{-1} x' = x$, and consequently $x' = \sin x$, we have (No. 32) $dx' = \cos x dx$. But (from TRIG. No. 11) we have $\cos x = \sqrt{(1 - \sin^2 x)}$ or, by our notation, $\cos x = \sqrt{(1 - x'^2)}$. Hence $dx' = \sqrt{(1 - x'^2)} dx$, and therefore

$$dx \text{ or } d \sin^{-1} x' = \frac{dx'}{\sqrt{(1 - x'^2)}}; \text{ or } d \sin^{-1} x = \frac{dx}{\sqrt{(1 - x^2)}},$$

if, to preserve uniformity of notation, we drop the accents, which are no longer necessary. This expression shows, that if x be any quantity, not exceeding unity, and if its differential be divided by $\sqrt{(1 - x^2)}$, the quotient is the differential of the arc which has x as its sine.

39. If again, $\text{versin}^{-1} x' = x$, and consequently $x' = \text{versin}\, x$, we have (No. 34) $dx' = \sin x dx$. Now (TRIG. No. 11) $\sin^2 x = 1 - \cos^2 x = 1 - (1 - \text{versin}\, x)^2 = 2 \,\text{versin}\, x - \text{versin}^2 x = 2x' - x'^2$; and therefore, by substitution,

$$dx' = \sqrt{(2x' - x'^2)} dx:$$

whence dx or $d \,\text{versin}^{-1} x' = \dfrac{dx'}{\sqrt{(2x' - x'^2)}}$; where likewise, as in all such cases, the accents may be dropped.

40. In the third place, let $\tan^{-1}x' = x$, so that $x' = \tan x$: then (No. 35) $dx' = (1 + \tan^2 x)dx = (1 + x'^2)\, d\tan^{-1}x'$;

whence $$d\tan^{-1}x = \frac{dx}{1+x^2}.$$

41. Lastly, let $\sec^{-1}x' = x$; and consequently $x' = \sec x$. Hence (No. 36), we have $dx' = \sec x \sqrt{(\sec^2 x - 1)}dx$, or, by substitution, $dx' = x'\sqrt{(x'^2 - 1)}d\sec^{-1}x'$; and, consequently, $$d\sec^{-1}x = \frac{dx}{x\sqrt{(x^2-1)}}.$$

42. From the four formulas last obtained, others which are sometimes preferable, may be derived by changing x into $\frac{x}{a}$, or $a^{-1}x$. By this means, the first will become,

$$d\sin^{-1}\frac{x}{a} = \frac{d(a^{-1}x)}{\sqrt{(1-a^{-2}x^2)}} \quad \text{or,} \quad d\sin^{-1}\frac{x}{a} = \frac{dx}{\sqrt{(a^2-x^2)}},$$

by multiplying the terms of the fraction in the second member by a.

By processes exactly similar, the remaining three become

$$d\,\text{versin}^{-1}\frac{x}{a} = \frac{dx}{\sqrt{(2ax-x^2)}}; \qquad d\tan^{-1}\frac{x}{a} = \frac{a\,dx}{a^2+x^2};$$

$$\text{and } d\sec^{-1}\frac{x}{a} = \frac{a\,dx}{x\sqrt{(x^2-a^2)}};$$

the sole difference in the investigations being, that the terms of the latter fractions in the third and fourth require to be multiplied by a^2 instead of a. It is plain that the expressions now found comprehend the former, becoming the same when $a = 1$.

The corresponding expressions for $d\cos^{-1}x$, $d\cot^{-1}x$, &c. might be found in a similar way. They are of no value, however; being merely the same as those found above, with their signs changed.

EXAMPLES OF THE DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS.

43. Required the differential of $\log\frac{x}{\sqrt{(a^2+x^2)}}$. Denoting this by u, we have (Nos. 24 & 25) $u = \log x - \frac{1}{2}\log(a^2+x^2)$;

whence (No. 28) $du = \frac{dx}{x} - \frac{x\,dx}{a^2+x^2}$; or, by performing the actual subtraction, $du = \frac{a^2dx}{x(a^2+x^2)}$.

44. Let $u = \log\{(a+x)^m(b-x)^n\}$. This may be expressed, $u = m\log(a+x) + n\log(b-x)$; whence (No. 28) $du = \frac{m\,dx}{a+x} - \frac{n\,dx}{b-x}$.

45. Let $u = x^y$. Here (No. 25) $\log u = y\log x$; whence (Nos. 11 and 28) $\frac{du}{u} = \frac{y\,dx}{x} + \log x\,dy$. Multiply the first member by u, and the second by its equal x^y; then,

$$du = yx^{y-1}dx + x^y\log x\,dy,$$

which is the required differential. If $x = a$, a constant quantity, this becomes $du = a^y\log a\,dy$, the same as the result in No. 29.

46. Let $u = x^{y^v}$. Here, putting $y^v = z$, we have $u = x^z$; whence, as in the last No. we find

$$du = zx^{z-1}dx + x^z dz\log x.$$

Again, from $z = y^v$, we get

$$dz = vy^{v-1}dy + y^v dv\log y;$$

the substitution of which for dz, and of y^v for z, in the value of du, gives

$$du = y^v x^{y^v-1}dx + x^{y^v}vy^{v-1}dy\log x + x^{y^v}y^v dv\log x\log y;$$

or, as it may also be expressed,

$$du = x^{y^v}y^v\left(\frac{dx}{x} + \frac{v\,dy\log x}{y} + dv\log x\log y\right).$$

47. Let $u = \sin^3 x$. Here (Nos. 9 and 32)

$$du = 3\sin^2 x\,d\sin x = 3\sin^2 x\cos x\,dx;$$

which, if we substitute $1-\cos^2 x$ for $\sin^2 x$, becomes

$$du = 3(\cos x - \cos^3 x)dx.$$

48. If $u = (\sin x)^x$, we have (No. 25) $\log u = x\log\sin x$; and, differentiating this (by Nos. 11, 28, and 32), we obtain

$$\frac{du}{u} = \log\sin x\,dx + \frac{x\,d\sin x}{\sin x}, \text{ or } \frac{du}{u} = \log\sin x\,dx + x\cot x\,dx.$$

Hence, by multiplying the first member by u, and the second by $(\sin x)^x$, we obtain

$$du = (\sin x)^x (\log \sin x + x \cot x) dx.$$

49. Since (TRIG. No. 217) $\sin x + \sin 2x + \sin 3x \ldots\ldots + \sin nx = \frac{\cos \frac{1}{2}x - \cos (n + \frac{1}{2})x}{2 \sin \frac{1}{2}x}$, we obtain, by differentiating, dividing by dx, and making some reductions,

$$\cos x + 2\cos 2x + 3\cos 3x \ldots\ldots\ldots + n\cos nx$$

$$= -\frac{1}{4\sin^2 \frac{1}{2}x} + \frac{(n+\frac{1}{2})\sin\frac{1}{2}x \sin(n+\frac{1}{2})x + \frac{1}{2}\cos\frac{1}{2}x \cos(n+\frac{1}{2})x}{2\sin^2 \frac{1}{2}x}$$

$$= \frac{n \sin (n + \frac{1}{2})x}{2 \sin \frac{1}{2}x} - \frac{\sin^2 \frac{1}{2}nx}{2\sin^2 \frac{1}{2}x}.$$

By performing a similar operation on the result here obtained, we should find the sum of the series,

$$\sin x + 2^2 \sin 2x + 3^2 \sin 3x + \text{\&c.};$$

and thus we might proceed as far as we please. The sums, however, would be very complicated.

50. Since (by TRIG. No. 27) $\sin x = \cos \frac{1}{2}x \times 2 \sin \frac{1}{2}x$, $\sin \frac{1}{2}x = \cos \frac{1}{4}x \times 2 \sin \frac{1}{4}x$, $\sin \frac{1}{4}x = \cos \frac{1}{8}x \times 2 \sin \frac{1}{8}x$, &c. it follows, by successive substitutions, that

$$\sin x = \cos \tfrac{1}{2}x \cos \tfrac{1}{4}x \cos \tfrac{1}{8}x \cos \tfrac{1}{16}x \ldots\ldots \cos \frac{x}{2^n} \times 2^n \sin \frac{x}{2^n},$$

n denoting the number of the cosines. Taking the logarithms of both members, we obtain $\log \sin x = \log \cos \frac{1}{2}x + \log \cos \frac{1}{4}x + \log \cos \frac{1}{8}x + \ldots + \log \cos \frac{x}{2^n} + \log \sin \frac{x}{2^n} + n \log 2$; whence, by differentiating and dividing by dx, we get

$$\cot x = -\tfrac{1}{2} \tan \tfrac{1}{2}x - \tfrac{1}{4} \tan \tfrac{1}{4}x - \tfrac{1}{8} \tan \tfrac{1}{8}x - \ldots\ldots$$
$$-\frac{1}{2^n} \tan \frac{x}{2^n} + \frac{1}{2^n} \cot \frac{x}{2^n}.$$

From this we obtain, by transposition,

$$\tfrac{1}{2} \tan \tfrac{1}{2}x + \tfrac{1}{4} \tan \tfrac{1}{4}x + \tfrac{1}{8} \tan \tfrac{1}{8}x + \ldots\ldots$$
$$+ \frac{1}{2^n} \tan \frac{x}{2^n} = \frac{1}{2^n} \cot \frac{x}{2^n} - \cot x,$$

which is the sum of the series proposed (TRIG. page 112, Ex. 70). If the differential of this equation be multiplied by 4, and divided by dx, the result will be the same as that obtained (TRIG. No. 219); and many other series, and their sums, might be derived from those in Sections IX. and X. of the same work.

51. Let $u=(a+x)^m(a-x)^n(a-bx)^p$. This is not a transcendental function; but it may be differentiated, perhaps most easily, by means of logarithms. Thus (Nos. 24 and 25) we have

$$\log u = m\log(a+x)+n\log(a-x)+p\log(a-bx).$$

Then, by differentiating this, and multiplying the first member by u, and the second by its equal, we obtain

$$du=(a+x)^m(a-x)^n(a-bx)^p\left(\frac{m\,dx}{a+x}-\frac{n\,dx}{a-x}-\frac{bp\,dx}{a-bx}\right),$$

which might be modified by performing the actual multiplication. The use of logarithms is equally advantageous when there are variable divisors.

52. Let $u=\sin^{-1}\frac{x^2}{a^2}$. While the differentials of functions of this kind may be easily obtained by means of No. 38, &c.; they may be readily found by recurring to the original direct functions. Thus, in the present instance, we have $\sin u=\frac{x^2}{a^2}$; and therefore, $\cos u\,du=\frac{2x\,dx}{a^2}$. But

$$\cos u=\sqrt{(1-\sin^2 u)}=\sqrt{\left(1-\frac{x^4}{a^4}\right)}=\frac{\sqrt{(a^4-x^4)}}{a^2}:$$

and by substituting this in the foregoing result for $\cos u$, and dividing by it, we get $du=\frac{2x\,dx}{\sqrt{(a^4-x^4)}}$.

53. Let $u=\log\{(x^2\pm a^2)^{\frac{1}{2}}+x\}$. Then

$$du=\frac{(x^2\pm a^2)^{-\frac{1}{2}}x\,dx+dx}{(x^2\pm a^2)^{\frac{1}{2}}+x};$$

whence, by multiplying the numerator and denominator by $(x^2\pm a^2)^{\frac{1}{2}}$, and by dividing the terms of the resulting fraction by $(x^2\pm a^2)^{\frac{1}{2}}+x$, we obtain $du=\frac{dx}{\sqrt{(x^2\pm a^2)}}$.

54. If $u=\log\{(2ax+x^2)^{\frac{1}{2}}+a+x\}$, we may put it under the form,

$$u=\log\{[(a+x)^2-a^2]^{\frac{1}{2}}+u+x\}, \text{ or}$$
$$u=\log\{(x'^2-a^2)^{\frac{1}{2}}+x'\},$$

by putting $a+x=x'$. Then (No. 53),

$$du=\sqrt{\frac{dx'}{x'^2-a^2}}=\frac{dx}{\sqrt{(2ax+x^2)}},$$

by restoring the value of x'. This might also be readily obtained by direct differentiation.

55. If $u=\log\dfrac{x-a}{x+a}$, or $u=\log(x-a)-\log(x+a)$, we get

$$du=\frac{dx}{x-a}-\frac{dx}{x+a}; \text{ or } du=\frac{2adx}{x^2-a^2},$$

by actual subtraction.

56. Let again, $u=\log\dfrac{x}{(a^2-x^2)^{\frac{1}{2}}+a}$, or

$$u=\log x-\log\{(a^2-x^2)^{\frac{1}{2}}+a\}.$$

Here the work will be simplified by assuming $y^2=a^2-x_2$, which gives $x=(a^2-y^2)^{\frac{1}{2}}$, and $ydy=-xdx$; and we get $u=\frac{1}{2}\log(a^2-y^2)-\log(a+y)$, and, therefore,

$$du=-\frac{ydy}{a^2-y^2}-\frac{dy}{a+y}.$$

Hence, by multiplying the terms of the latter fraction by $a-y$, and contracting, we obtain

$$du=-\frac{ady}{a^2-y^2}=-\frac{aydy}{(a^2-y^2)y};$$

or, by what precedes, $du=\dfrac{axdx}{x^2\sqrt{(a^2-x^2)}}=\dfrac{adx}{x\sqrt{(a^2-x^2)}}$.

It would be shown in a similar manner, that the differential of $\log\dfrac{x}{a+\sqrt{(a^2+x^2)}}$ is $\dfrac{adx}{x\sqrt{(a^2+x^2)}}$; and therefore the two results may be combined, by means of the double sign, in the one formula,

$$d\log\frac{x}{a+\sqrt{(a^2\pm x^2)}}=\frac{adx}{x\sqrt{(a^2\pm x^2)}}.$$

EXERCISES IN THE DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS.

1. Differentiate $u = x \log x$. *Ans.* $du = dx(1 + \log x)$.
2. .... $u = \frac{\log x}{x}$. *Ans.* $du = \frac{dx(1 - \log x)}{x^2}$.
3. .... $u = \frac{x}{\log x}$. *Ans.* $du = \frac{dx(\log x - 1)}{(\log x)^2}$.
4. .... $u = \log \frac{\sqrt{(1+x)} + \sqrt{(1-x)}}{\sqrt{(1+x)} - \sqrt{(1-x)}}$.*

   Ans. $du = -\frac{dx}{x\sqrt{(1-x^2)}}$.
5. .... $u = \varepsilon^x(x-1)$. *Ans.* $du = \varepsilon^x x\, dx$.
6. .... $u = \log \log x$, or $u = \log^2 x$.†

   Ans. $du = \frac{dx}{x \log x}$.
7. .... $u = \sin 3x$. *Ans.* $du = 3 \cos 3x\, dx$.
8. .... $u = \cos nx$. *Ans.* $du = -n \sin nx\, dx$.
9. .... $u = \sin^n x$. *Ans.* $du = n \sin^{n-1} x \cos x\, dx$.
10. .... $u = \sin^{-1} \frac{1-x^2}{1+x^2}$. *Ans.* $du = -\frac{2dx}{1+x^2}$.
11. .... $u = \sin^{-1} \frac{x}{\sqrt{(1+x^2)}}$. *Ans.* $du = \frac{dx}{1+x^2}$.
12. .... $u = \tan^{-1} \frac{2x}{1-x^2}$. *Ans.* $du = \frac{2dx}{1+x^2}$.
13. .... $u = \log \sqrt{\frac{1+\sin x}{1-\sin x}}$.

    Ans. $du = \frac{dx}{\cos x}$.

IV.—ELEMENTS OF THE INTEGRAL CALCULUS.‡

57. The object of the differential Calculus, as we have seen, is to derive from a proposed function another peculiar

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* See Exercise 14, page 23.

† Either of these expressions means the logarithm of $\log x$.

‡ In this edition it has been thought advisable not to keep the differential and the integral calculus distinct, as was done in the former edition, and as has been very generally done by writers on the subject. These two great and kindred branches of analysis materially assist and illustrate each other; and De Morgan, Cournot, and some other writers, introduce with propriety and advantage, the elementary principles of integration long before they have finished the theory and applications of the differential calculus.

one called its differential. The inverse problem, *the finding of the primitive function, when its differential is given, is the object of the integral calculus.* In this great branch of analysis, the primitive function, in reference to its differential, is called the *integral* of that differential, and the process of finding the integral is termed *integration.* As the letter d is employed to indicate the differential of a quantity, so the long italic $\int$* is used to signify the integral of a differential. Thus, $\int x^n dx$ denotes the integral of $x^n dx$, that is, the function of which $x^n dx$ is the differential.

58. Since (No. 8), in differentiating a function, any constant quantity, connected with the variable part only by addition or subtraction disappears, it follows conversely, that, in assigning the integral, a constant quantity should be annexed. This may be represented by the letter C. Thus, since the differential of $x^n + C$ is $nx^{n-1}dx$, we have, conversely, $\int nx^{n-1}dx = x^n + C$; where, according to the object in view, in any particular case, C may be either nothing or a positive or negative quantity. The method of discovering the value of C, to answer the end in view, will be exemplified hereafter. It also follows (No. 10), that a constant coefficient in a differential will remain in the integral.

59. In the differential calculus every result is obtained by means of direct investigations founded on the consideration of the relations which arise from a change produced on a function by a change in the value of the variable on which it depends. In the integral calculus the results are not found by any such direct process, but are obtained simply by reversing the processes of the differential calculus. In this way, the elementary formulas of differentiation give an equal number of elementary ones for the integral calculus. These we may now proceed to establish; commencing with the differential of a power, which furnishes the principle of integration that is more

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* This character is the initial letter of the word *sum*, and its synonymes, in several languages. It was employed by the early writers on the integral calculus, because, according to the idea of Leibnitz, they supposed the primitive function to be the sum of an infinite number of infinitely small increments. For the same reason, also, they called that function the *integral* of the proposed differential. It is plain, that according to the notation explained in page 30, d^{-1} might be used instead of $\int$. The latter, however, is more convenient in practice.

frequently employed than any other. Now, since, as we saw in the last No. $d(x^n+C)=nx^{n-1}dx$, and consequently $\int nx^{n-1}dx=x^n+C$, we have the following theorem (evidently the reverse of the rule in No. 9), by means of which the second member of the last equation would be derived from the first:

To integrate a differential in which the differential of a variable is multiplied by a power of that variable, the power having a constant index; divide by the differential of the variable, add unity to the index, divide by the index thus increased, and annex the constant quantity.

Hence, also, we have $\int x^n dx=\frac{x^{n+1}}{n+1}+C$.*

60. We may extend the principle above obtained, to the integration of the more general formula, $(ax^n+b)^m x^{n-1}dx$. To effect this, put $ax^n+b=x'$; then, by differentiating, and dividing by na, we get $x^{n-1}dx=\frac{dx'}{na}$, and the proposed differential becomes $\frac{x'^m dx'}{na}$; the integral of which (No. 59) is $\frac{x'^{m+1}}{na(m+1)}$. Hence, by restoring the value of x', we obtain, for the required integral,

$$\int(ax^n+b)^m\, x^{n-1}dx=\frac{(ax^n+b)^{m+1}}{na(m+1)}+C.$$

If $b=0$, and a and n each equal to 1, this would become the same as the formula at the end of the last No. This principle is applicable when the index $(n-1)$ of the power of x without the vinculum, is less by unity than (n) the index of the power within it. It fails, however, when $m=-1$, or $n=0$; and so does the formula in No. 59, when $n=-1$. In these cases the integral is obtained by means of the principle established in the next No.

61. Since (No. 28) $d\log(x+a)=\frac{dx}{x+a}$, we have, con-

* The constant quantity C may be put under the same form as the quantity with which it is connected. Thus, since its value is arbitrary, let us suppose it to be such as to make the integral vanish when $x=b$, and we shall have $0=\frac{b^{n+1}}{n+1}+C$, which gives $C=-\frac{b^{n+1}}{n+1}$; and by the substitution of this for C in the integral, we find $\int x^n dx=\frac{x^{n+1}-b^{n+1}}{n+1}$.

versely, $\int \frac{dx}{x+a} = \log(x+a) + C$; whence it appears that, *when the numerator of a differential is the differential of its denominator, the integral is the Neperian logarithm of the denominator.**

62. Since (No. 29), $a^x dx \log a = da^x$, we have, by dividing by $\log a$, and integrating, $\int a^x dx = \frac{a^x}{\log a} + C$. Hence, also, $\int \varepsilon^x dx = \varepsilon^x + C$.

63. Since (No. 33) $d\cos x = -\sin x\, dx$, by substituting nx for x, we get $d\cos nx = -n\sin nx\, dx$; whence, conversely, $\int \sin nx\, dx = -\frac{1}{n}\cos nx + C$. We should obtain (from No. 32), in a similar manner, $\int \cos nx\, dx = \frac{1}{n}\sin nx + C$.

64. It was proved in No. 38, that $d\sin^{-1}x = \frac{dx}{\sqrt{(1-x^2)}}$; and (No. 42), that $d\sin^{-1}\frac{x}{a} = \frac{dx}{\sqrt{(a^2-x^2)}}$.

From these expressions we obtain

$$\int \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1}x, \quad \text{and} \int \frac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1}\frac{x}{a}.$$

65. By No. 39, we have $d\,\text{versin}^{-1}x = \frac{dx}{\sqrt{(2x-x^2)}}$;

whence, on similar principles, we have

$$\int \frac{dx}{\sqrt{(2x-x^2)}} = \text{versin}^{-1}x, \text{ and} \int \frac{dx}{\sqrt{(2ax-x^2)}} = \text{versin}^{-1}\frac{x}{a}.$$

66. Since (No. 40) $d\tan^{-1}x = \frac{dx}{1+x^2}$, and (No. 41)

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* If, for the sake of simplicity, $a=0$, the proposed differential might be put under the form, $x^{-1}dx$, the integral of which (No. 59) is $\frac{x^0}{0} + C$; an expression which, in a practical point of view, is useless. Putting it, however, under the form pointed out in the note to No. 59, we get $\frac{x^0 - b^0}{0} = \frac{0}{0}$, which is what $\frac{x^z - b^z}{z}$ becomes when $z=0$, and x constant. Now, the value of this, as will be shown hereafter, is $\log x - \log b$, which agrees with the result in the text, $\log b$ being the constant quantity.

$d\sec^{-1}x = \dfrac{dx}{x\sqrt{(x^2-1)}}$, we obtain, on like principles,

$$\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}, \qquad \text{and} \int\frac{dx}{x\sqrt{(x^2-a^2)}}=\frac{1}{a}\sec^{-1}\frac{x}{a}.$$

67. From the four results established in Nos. 53, 54, 55, and 56, we obtain the four following formulas, which are analogous to those found in Nos. 64, 65, and 66.

$$\int\frac{dx}{\sqrt{(x^2\pm a^2)}}=\log\{\sqrt{(x^2\pm a^2)}+x\},$$

$$\int\frac{dx}{\sqrt{(2ax+x^2)}}=\log\{\sqrt{(2ax+x^2)}+a+x\};$$

$$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\frac{x-a}{x+a};$$

$$\int\frac{dx}{x\sqrt{(a^2\pm x^2)}}=\frac{1}{a}\log\frac{x}{\sqrt{(a^2\pm x^2)}+a}.$$

68. We saw (No. 11) that $d(uv)=udv+vdu$, u and v being functions of x; whence $uv=\int udv+\int vdu$, or by transposition,

$$\int udv=uv-\int vdu.$$

This indicates a process of integration by means of which one part (uv) of an integral ($\int udv$) is obtained, and the finding of the remaining part is made to depend on the integration of another differential ($-vdu$). This method of obtaining integrals is, therefore, called *integration by parts:* and it is much employed. If we change u into u^{-1}, we obtain

$$\int\frac{dv}{u}=\frac{v}{u}+\int\frac{v}{u}\frac{du}{u},$$

a formula which is often preferable to the foregoing.

69. As the formulas that have now been obtained, will be often referred to hereafter, they are collected into the following table, so that they may be easily found when required. The constant quantity C is omitted both here and in many other instances. In all applications, however, of the integral calculus, it must be carefully attended to in one way or another.

TABLE OF ELEMENTARY INTEGRALS.

$\int (ax^n+b)^m x^{n-1} dx = \frac{(ax^n+b)^{m+1}}{na(m+1)}$ .... A

$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}$, and $\int x^n dx = \frac{x^{n+1}}{n+1}$ .... A_2

$\int \frac{dfx}{fx} = \log fx$, and $\int \frac{dx}{x+a} = \log(x+a)$...B

$\int a^x dx = \frac{a^x}{\log a}$, and $\int \varepsilon^x dx = \varepsilon^x$ C

$\int \sin nx dx = -\frac{1}{n}\cos nx$, and $\int \sin x dx = -\cos x$... D

$\int \cos nx dx = \frac{1}{n}\sin nx$, and $\int \cos x dx = \sin x$ .... D_2

$\int \frac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1}\frac{x}{a}$, and $\int \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1}x$... E

$\int \frac{dx}{\sqrt{(x^2 \pm a^1}} = \log\{(\sqrt{(x^2 \pm a^2)}+x\}$........................ E_2

$\int \frac{dx}{\sqrt{(2ax-x^2)}} = \text{versin}^{-1}\frac{x}{a}$, and $\int \frac{dx}{\sqrt{(2x-x^2)}} = \text{versin}^{-1}x$. F

$\int \frac{dx}{\sqrt{(2ax+x^2)}} = \log\{\sqrt{(2ax+x^2)}+a+x\}$ .... F_2

$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$, and $\int \frac{dx}{1+x^2} = \tan^{-1}x$ .... G

$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}$ .... G_2

$\int \frac{dx}{x\sqrt{(x^2-a^2)}} = \frac{1}{a}\sec^{-1}\frac{x}{a}$, and $\int \frac{dx}{x\sqrt{(x^2-1)}} = \sec^{-1}x$...H

$\int \frac{dx}{x\sqrt{(a^2 \pm x^2)}} = \frac{1}{a}\log\frac{x}{\sqrt{(a^2 \pm x^2)}+a}$ .... H_2

$\int u dv = uv - \int v du$ .... K

$\int \frac{dv}{u} = \frac{v}{u} + \int \frac{v}{u}\frac{du}{u}$ .... K_2

ELEMENTARY EXAMPLES.

70. REQUIRED the integral of $\frac{2dx}{5x^{\frac{4}{3}}}$. This may be put under the form $\frac{2}{5}x^{-\frac{4}{3}}dx$; and hence (page 41, A_2) we get $-3.\frac{2}{5}x^{-\frac{1}{3}}$, or $-\frac{6}{5x^{\frac{1}{3}}}$.

71. Required the integral of $\frac{3x^4dx}{x^5 \pm a}$. Here, since the differential of the denominator is $5x^4dx$, by multiplying the numerator and denominator by the coefficient 5, we may put the given differential under the form, $\frac{3}{5}.\frac{5x^4dx}{x^5 \pm a}$; the integral of which (page 41, B) is $\frac{3}{5}\log(x^5 \pm a)$, or, as (No. 25) it may be expressed, $\log(x^5 \pm a)^{\frac{3}{5}}$.

72. Required the integral of $\frac{2dx}{\sqrt{(2-3x^2)}}$. By dividing the numerator and denominator of this differential by $\sqrt{3}$, since $\frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$, it may be expressed in the form,

$\frac{2}{3}\sqrt{3}.\frac{dx}{\sqrt{(\frac{2}{3}-x^2)}}$. By comparing this with p. 41, E, we have $a^2 = \frac{2}{3}$; and the required integral is

$$\frac{2}{3}\sqrt{3}.\sin^{-1}\frac{x}{\sqrt{\frac{2}{3}}} = \frac{2}{3}\sqrt{3}.\sin^{-1}x\sqrt{\tfrac{3}{2}}.$$

73. If the proposed differential be $\frac{dx}{3+4x^2}$, by dividing the numerator and denominator by 4, we get $\frac{1}{4}.\frac{dx}{\frac{3}{4}+x^2}$. Comparing this with page 41, G, we have $a^2 = \frac{3}{4}$, and the required integral is $\frac{1}{4}.\frac{1}{\sqrt{\frac{3}{4}}}\tan^{-1}\frac{x}{\sqrt{\frac{3}{4}}}$, or $\frac{1}{6}\sqrt{3}.\tan^{-1}\frac{2}{3}x\sqrt{3}$.

74. Required the integral of $\log x\,dx$. By comparing this with formula K, page 41, we have $u = \log x$, and $dv = dx$; and hence we get $du = \frac{dx}{x}$, and $v = x$; by substituting which in that formula, we obtain $\int \log x\,dx = x\log x - \int\frac{x\,dx}{x}$, or $x\log x - x$, the required integral.

75. As another example, let it be required to integrate $\sin^{-1}x\,dx$. Here (page 41, K) we have

$$\int\sin^{-1}x\,dx = x\sin^{-1}x - \int x\,d\sin^{-1}x.$$

By introducing into the last term of this the value (No. 38) of $d\sin^{-1}x$, and integrating the result by A_2, page 41, we get

$$\int\sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{(1-x^2)}.$$

76. Required the integral of $\varepsilon^x x\,dx$. Here, since (No. 29) $d\varepsilon^x = \varepsilon^x dx$, we have, by integration by parts,

$$\int\varepsilon^x x\,dx = \varepsilon^x x - \int\varepsilon^x dx = \varepsilon^x x - \varepsilon^x = \varepsilon^x(x-1).$$

77. Let it be required to integrate $du = \dfrac{x^4dx}{x^2+a^2}$. Here, by division, we get $du = x^2dx - a^2dx + \dfrac{a^4dx}{x^2+a^2}$; the integral of which is readily found to be $\frac{1}{3}x^3 - a^2x + a^3\tan^{-1}\dfrac{x}{a}$.

78. Required the integral of $du = \dfrac{x^5dx}{x^2+a^2}$. This becomes, by division, $du = x^3dx - a^2xdx + \dfrac{a^4x\,dx}{x^2+a^2}$, and the integral is $u = \frac{1}{4}x^4 - \frac{1}{2}a^2x^2 + \frac{1}{2}a^4\log(x^2+a^2)$.

79. If the given differential be $du = \dfrac{x^3dx}{x+a}$, it is changed by division into $du = x^2dx - axdx + a^2dx - \dfrac{a^3dx}{x+a}$; and the integral is $u = \frac{1}{3}x^3 - \frac{1}{2}ax^2 + a^2x - a^3\log(x+a)$.

EXERCISES.

1. $\int 3x^{\frac{1}{2}}dx = 2x^{\frac{3}{2}}$.

2. $\displaystyle\int\frac{3dx}{x^{\frac{1}{2}}} = 6x^{\frac{1}{2}}$.

3. $\displaystyle\int\frac{2}{3}\cdot\frac{dx}{x^4} = -\frac{2}{9x^3}$.

4. $\displaystyle\int\frac{3}{4}\cdot\frac{dx}{x^{\frac{3}{4}}} = 3x^{\frac{1}{4}}$.

5. $\displaystyle\int\frac{ax\,dx}{\sqrt{(1-2x^2)}} = -\frac{1}{2}a\sqrt{(1-2x^2)}$.

6. $\displaystyle\int\frac{x^2dx}{\sqrt[3]{(x^3-a)}} = \frac{1}{2}(x^3-a)^{\frac{2}{3}}$.

7. $\int(1-2x+x^2)dx = x - x^2 + \frac{1}{3}x^3$, or $-\frac{1}{3}(1-x)^3$.

8. $\int \frac{dx}{(a-x)^5} = \frac{1}{4(a-x)^4}$.

9. $\int \frac{dx}{(1+x)^2} = -\frac{1}{1+x}$, or $\frac{x}{1+x}$.*

10. $\int \frac{4x\,dx}{(1-x^2)^2} = \frac{2}{1-x^2}$, or $\frac{1+x^2}{1-x^2}$, or $\frac{2x^2}{1-x^2}$.

11. $\int x\,dx \log x = \frac{1}{2}x^2 \log x - \frac{1}{4}x^2$.

12. $\int \frac{dx}{x}(\log x)^n = \frac{1}{n+1}(\log x)^{n+1}$.

13. $\int \frac{dx}{x \log x} = \log\log x = \log^2 x$.

14. $\int \cos 3x\,dx - \sin 5x\,dx = \frac{1}{3}\sin 3x + \frac{1}{5}\cos 5x$.

15. $\int \frac{dx}{x\sqrt{(3x^2-1)}} = \sec^{-1} x\sqrt{3}$.

16. $\int \frac{dx}{\sqrt{(2x-5x^2)}} = \frac{1}{5}\sqrt{5} \times \text{versin}^{-1} 5x$.

17. $\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \log \sin x$.

18. Integrate $\frac{x^4 dx}{x-a}$, $\frac{a^4 dx}{x-a}$, and their difference.

Ans. $\frac{1}{4}x^4 + \frac{1}{3}ax^3 + \frac{1}{2}a^2x^2 + a^3x + a^4 \log(x-a)$; $a^4 \log(x-a)$; and $\frac{1}{4}x^4 + \frac{1}{3}ax^3 + \frac{1}{2}a^2x^2 + a^3x$.

19. $\int \frac{(x^4+a^4)\,dx}{x^2-a^2} = \frac{1}{3}x^3 + a^2x + a^3 \log \frac{x-a}{x+a}$; and

$$\int \frac{(x^5+a^5)\,dx}{x^2+a^2} = \frac{1}{4}x^4 - \frac{1}{2}a^2x^2 + \frac{1}{2}a^4 \log(x^2+a^2) + a^4 \tan^{-1}\frac{x}{a}.$$

V.—SUCCESSIVE DIFFERENTIATION, TAYLOR'S THEOREM, ETC.

80. If the differential coefficient of a function be variable, it may itself be differentiated, and its differential coefficient, if variable, may in like manner be differentiated; and thus we may proceed as far as we choose, obtaining continually new successive differential coefficients, unless we arrive at one which is constant, and which, therefore, will have no differential. In such differentiations, we regard the differential

* These two values of the integral differ only by the constant quantity 1; since, if it be added to the first, the sum will be the second. A similar remark is applicable in respect to *Ex.* 7 and *Ex.* 10.

of the independent variable x as constant, which amounts to the same as supposing that variable to increase uniformly. We call, also, *the differential of the differential of a function the second differential of the function*, and we denote it by prefixing d^2 to the function. Thus, we write d^2u instead of ddu. The differential of this, again, is called the *third differential* of the original function, and is denoted by prefixing d^3; so that the third differential of u is written d^3u: and thus we may have the fourth, the fifth, and in general, the nth differential, which are denoted by d^4u, d^5u, ... d^nu. It will thus be seen, that the *order* (first, second, ... nth) of any differential, or of its differential coefficient corresponds to the number of times the differentiation has been performed. It may be farther remarked, that for simplicity, the powers of dx are written dx^2, dx^3, ... dx^n, and not $(dx)^2$, $(dx)^3$,...... $(dx)^n$.*

81. To exemplify what has now been said, let

$$u = ax^3 - bx^2 + cx + e. \quad \text{Then,}$$

$$du = 3ax^2dx - 2bxdx + cdx, \qquad \text{and } \frac{du}{dx} = 3ax^2 - 2bx + c;$$

$$\frac{d^2u}{dx} = 6axdx - 2bdx, \qquad \text{and } \frac{du_2}{dx^2} = 6ax - 2b;$$

$$\frac{d^3u}{dx^2} = 6adx, \qquad \text{and } \frac{d^3u}{dx^3} = 6a.$$

Here the third differential coefficient is constant, and the operation terminates, as each of the succeeding differentials and differential coefficients would be zero.

82. As a second example, let

$$u = x^{\frac{5}{2}} + \frac{a}{x} + b, \quad \text{or } u = x^{\frac{5}{2}} + ax^{-1} + b. \quad \text{Then,}$$

$$du = \tfrac{5}{2}x^{\frac{3}{2}}dx - ax^{-2}dx, \qquad \text{and } \frac{du}{dx} = \frac{5}{2}x^{\frac{3}{2}} - ax^{-2};$$

$$\frac{d^2u}{dx} = \frac{3}{2}\cdot\frac{5}{2}x^{\frac{1}{2}}dx + 1.2x^{-3}dx, \quad \text{and } \frac{d^2u}{dx^2} = \frac{3}{2}\cdot\frac{5}{2}x^{\frac{1}{2}} + 1.2x^{-3};$$

---

* The student should impress on his mind the exact and distinctive meanings of the different modes of notation employed in successive differentiations, recollecting that d^nu denotes the result obtained from n successive processes of differentiation, and dx^n, the nth power of dx; while $d(x^n)$ or $d.x^n$ indicates the differential of x^n.

$$\frac{d^3u}{dx^2}=\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\;x^{-\frac{1}{2}}dx-1.2.3x^{-4}dx, \quad \text{and}$$

$$\frac{d^3u}{dx^3}=\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}x^{-\frac{1}{2}}-1.2.3x^{-4};$$

$$\frac{d^4u}{dx^4}=-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}x^{-\frac{3}{2}}+1.2.3.4x^{-5};\ \&c.$$

Here it is plain that the process may be carried on as far as we please; and this, it is evident, will always be the case, except when the indices of x are all positive integers. As is done in the last line in the latter example, the division by dx may be easily and conveniently performed as the work proceeds. It is scarcely necessary to observe, that it is perhaps preferable at the conclusion of such operations, to take quantities with negative indices to their respective denominators.*

83. If in No. 14, we denote x^n by fx, and $(x+h)^n$ by $f(x+h)$, and if we assume A, B, C, &c. to represent the coefficients of h, h^2, h^3, &c. we shall have, for this particular function,

$$f(x+h)=fx+Ah+Bh^2+Ch^3+\&c.$$

Now it is found that *any function whatever* of $x+h$ may be developed in a series of the same form; there being exceptions, however, in some singular cases in which, for *particular* values of x, the development of the function in integral powers of h may be impossible. Let us for the present assume the possibility of the development, without considering the exceptions, and let us determine the precise form of the series in terms of x and h.

Now, if we denote fx by u, and $f(x+h)$ by u', we have, by the principle which we have assumed,

$$u'=u+Ah+Bh^2+Ch^3+Dh^4+\&c.$$

where A, B, C, &c. are functions of x independent of h.

* For gaining practice, the student may work the following exercises:—

(1.) Find the fifth differential coefficient of $u=x^4-x^{-4}$.

Ans. $\frac{d^5u}{dx^5}=4.5.6.7.8\frac{1}{x^9}$.

(2.) Show that the nth differential coefficient of $u=\frac{1}{x}$ is $\frac{1.2.3\ \ldots\ n(-1)^n}{x^{n+1}}$.

(3.) Required the seventh and eighth differential coefficients of $\cos x$.

Ans. $\sin x$, and $\cos x$.

(4.) Required the fifth and sixth differential coefficients of $\sin nx$.

Ans. $n^5\cos nx$, and $-n^6\sin nx$.

(5.) Find the nth differential coefficients of a^x and ϵ^x. *Ans.* $a^x(\log a)^n$, and ϵ^x.

Then, by taking the differential coefficient of this, first on the supposition, that x alone, and secondly, that h alone, is variable, we get, as in No. 14,

$$\frac{du'}{dx}=\frac{du}{dx}+\frac{dA}{dx}\cdot h+\frac{dB}{dx}\cdot h^2+\frac{dC}{dx}\cdot h^3+\&c. \text{ and}$$

$$\frac{du'}{dh}=A+2Bh+3Ch^2+4Dh^3+\&c.$$

Now, by No. 13, these are identical. Hence, by the theory of indeterminate coefficients, we have

$$A=\frac{du}{dx};\quad B=\frac{1}{2}\cdot\frac{dA}{dx};\quad C=\frac{1}{3}\cdot\frac{dB}{dx};\quad D=\frac{1}{4}\cdot\frac{dC}{dx};\ \&c.$$

By differentiating the first of these, we find $dA=\frac{d^2u}{dx}$; and, if half this be divided by dx, there results $\frac{1}{2}\cdot\frac{dA}{dx}=\frac{1}{2}\cdot\frac{d^2u}{dx^2}$, the value of B. Assuming, therefore, $B=\frac{1}{2}\cdot\frac{d^2u}{dx^2}$, we differentiate; and, by dividing one third of the result by dx, we find $\frac{1}{3}\cdot\frac{dB}{dx}=\frac{1}{2.3}\cdot\frac{d^3u}{dx^3}$, which, we have seen above, is the value of C. Putting, therefore, $C=\frac{1}{2.3}\cdot\frac{d^3u}{dx^3}$, differentiating both members, and dividing one fourth of the result by dx, we get $\frac{1}{4}\cdot\frac{dC}{dx}=\frac{1}{2.3.4}\cdot\frac{d^4u}{dx^4}$, the value of D. By like processes, we should find $E=\frac{1}{2.3.4.5}\cdot\frac{d^5u}{dx^5}$, &c.; and the law of relation of the coefficients is obvious. Substituting these values, therefore, for A, B, C, &c. we get

$$u'=u+\frac{du}{dx}\cdot\frac{h}{1}+\frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2}+\frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3}+\frac{d^4u}{dx^4}\cdot\frac{h^4}{1.2.3.4}+\&c.; \text{ or}$$

$$f(x+h)=fx+\frac{dfx}{dx}\cdot\frac{h}{1}+\frac{d^2fx}{dx^2}\cdot\frac{h^2}{1.2}+\frac{d^3fx}{dx^3}\cdot\frac{h^3}{1.2.3}+\&c.^*$$

* We see that the relation of the coefficients of h, h^2, h^3, &c. is such, that each of them is the differential coefficient of the one before it, divided in the second term by 1, in the third by 2, in the fourth by 3, and in the nth by $n-1$.

The nature of the theorem will be illustrated, by applying it to particular functions, such as sine, logarithm, &c. Thus,

$$\sin(x+h)=\sin x+\frac{d\sin x}{dx}\cdot\frac{h}{1}+\frac{d^2\sin x}{dx^2}\cdot\frac{h^2}{1.2}+\frac{d^3\sin x}{dx^3}\cdot\frac{h^3}{1.2.3}+\&c.$$

$$\log(x+h)=\log x+\frac{d\log x}{dx}\cdot\frac{h}{1}+\frac{d^2\log x}{dx^2}\cdot\frac{h^2}{1.2}+\frac{d^3\log x}{dx^3}\cdot\frac{h^3}{1.2.3}+\&c.$$

This important formula is known by the name of *Taylor's Theorem*, because it was discovered by Dr. Brook Taylor, an English mathematician, who published it in 1715. Its great value was overlooked, till it was made the basis of the Differential and Integral Calculus by Lagrange, in 1772.

84. A particular case of this formula is commonly called Maclaurin's theorem, because it was first made generally known by that writer. It had been given previously, however, by Stirling, another Scotch mathematician; and therefore, if a particular case of Taylor's general theorem should be named after any other mathematician, this ought to be called *Stirling's theorem.* To investigate it, let U, U_1, U_2, U_3, &c. be assumed to represent what fx, $\frac{dfx}{dx}$, $\frac{d^2fx}{dx^2}$, $\frac{d^3fx}{dx^3}$, &c. become when $x=0$; and, on this supposition, Taylor's formula will become

$$fh = U + U_1 h + U_2 \frac{h^2}{1.2} + U_3 \frac{h^3}{1.2.3} + \text{\&c.}$$

Now, since x and h are involved in exactly the same way in $f(x+h)$, it is evident that the values of U, U_1, U_2, &c. are the same that we should have obtained, had we supposed h variable and x constant; and since these are plainly independent of the original variable x, and depend merely on the nature of the particular function, it follows that they will be the same, whether x or h is regarded as the variable. Hence it is plain, that in the last development we may substitute x for h, a change which is convenient for keeping up uniformity of notation; and thus we get

$$fx = U + U_1 \frac{x}{1} + U_2 \frac{x^2}{1.2} + U_3 \frac{x^3}{1.2.3} + \text{\&c.};$$

which is Stirling's theorem.*

* The following method of investigating this theorem is the same in principle as that which was employed by Stirling himself. Let u be a function of x which it is required to express in a series, and let us assume

$$u = A + Bx + Cx^2 + Dx^3 + \text{\&c.};$$

where A, B, C, &c. are constant quantities (being independent of x) which it is required to determine. Then, by successive differentiations, we get

$$\frac{du}{dx} = 1B + 2Cx + 3Dx^2 + 4Ex^3 + \text{\&c.}$$

$$\frac{d^2u}{dx^2} = 1.2C + 2.3Dx + 3.4Ex^2 + \text{\&c.}$$

$$\frac{d^3u}{dx^3} = 1.2.3D + 2.3.4Ex + \text{\&c.}$$

85. It is often desirable to be able to determine the limits between which, in a development given by Taylor's theorem, the part succeeding an assigned term is comprehended. To obtain the means of doing this, let us write the formula thus:

$$f(x+h)=fx+\frac{h}{1}f^1x+\frac{h^2}{1.2}f^2x+\dots\dots$$

$$+\frac{h^{n-1}}{1.2.3\dots(n-1)}f^{n-1}x+\mathrm{R};$$

where f^1x, f^2x, ..., $f^{n-1}x$ denote respectively the first, second, ..., $(n-1)^{\text{th}}$ differential coefficients of fx;* and R the sum of all the remaining terms, so that

$$\mathrm{R}=\frac{h^n}{1.2.3\dots n}f^nx+\frac{h^{n+1}}{1.2.3\dots(n+1)}f^{n+1}x+\text{ \&c.; or,}$$

$$\mathrm{R}=\frac{h^n}{1.2.3\dots n}\left(f^nx+\frac{h}{n+1}f^{n+1}x+\text{\&c.}\right)\dots\dots(a).$$

The quantity in the vinculum may be written thus,

$$\mathrm{V}=f^nx+\mathrm{S}_1\dots\dots(b);$$

V being put to denote its entire value, and S_1, the sum of all the terms after the first. Now, by Taylor's theorem,

$$f^n(x+h)=f^nx+\frac{h}{1}f^{n+1}x+\frac{h^2}{1.2}f^{n+2}x+\text{ \&c.; or,}$$

$$f^n(x+h)=f^nx+\mathrm{S}_2\dots\dots(c),$$

if S_2 be put to denote the sum of all the terms after the first. By comparing the component terms of S_1 and S_2, it

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Now by assuming U, U_1, U_2, &c. to denote what the original function u, and the successive differential coefficients become respectively, when $x=0$, and by taking $x=0$ in the several second members, we get $U=A$, $U_1=1B$, $U_2=1.2\,C$, $U_3=1.2.3\,D$, &c.; and we readily find that the assumed development becomes

$$u=\mathrm{U}+\mathrm{U}_1\frac{x}{1}+\mathrm{U}_2\frac{x^2}{1.2}+\mathrm{U}_3\frac{x^3}{1.2.3}+\text{\&c.}$$

as before. This method of investigation may be employed with advantage on other occasions.

* This notation, having the advantage of being *suggestive*, and of being concise, will be used on several occasions hereafter. The functions, f^1x, f^2x, &c. are what Lagrange called the first, second, &c. *derived functions* of x,—names which are very little expressive of the nature of the quantities.

will be seen that these two quantities have the same signs; and also, that in absolute magnitude, S_2 is the greater, its coefficients, h, $\frac{1}{2}h^2$, &c. being severally greater than the corresponding ones in the value of S_1. Hence we may put $S_2 = S_1 + \delta$, where δ will have the same sign as S_1. In this way, we shall have

$$f^n(x+h) = f^n x + S_1 + \delta = V + \delta; \text{ and therefore}$$
$$V = f^n(x+h) - \delta.$$

Comparing this with (b), we see, that if S_1 be positive, V is greater than $f^n x$ by S_1, and less than $f^n(x+h)$ by δ; while, if S_1 be negative, and consequently $-\delta$ positive, V is less than $f^n x$ by the absolute value of S_1, and greater than $f^n(x+h)$ by that of δ: so that in each case it is intermediate in value between the two. Multiplying these limits, therefore, by $\frac{h^n}{1.2.3\ldots n}$, we find that R must lie between the limits,

$$\frac{h^n}{1.2.3\ldots n}f^n x, \text{ and } \frac{h^n}{1.2.3\ldots n}f^n(x+h);$$

and these are the limits required.

From this also, it is evident that there is some quantity θ, between 0 and 1, such that

$$R = \frac{h^n}{1.2.3\ldots n}f^n(x+\theta h).$$

It is to be carefully observed, that throughout this investigation, it has been tacitly assumed, that *the various differential coefficients $f^n x$, &c. are continuous between x and $x+h$;* and it is plain that the conclusion will not hold true, should any of them become infinite between these limits.*

* Should the proposed function fx be of such a nature, that for a particular value, a, of x, either that function or any of its differential coefficients would become infinite, Taylor's theorem would fail, as it ought, in giving a development corresponding to that value of x; and the occurrence of any such infinite quantities would show, that for that value, the corresponding function of $x+h$ could not be developed in ascending powers of h with whole positive indices, which is implied in the hypothesis on which Taylor's theorem is founded. Now it will be readily seen, that such infinite quantities can arise in two cases;—first, when the function contains a power of $x-a$ with a negative index, in which case all the functions, fx, f^1x, f^2x, &c. are infinite, when x takes the particular value a; and secondly, when the function contains a power of $x-a$ with an index of the form $p+q$, where p is zero or a positive integer, and q a positive fraction; in which case, the first p differential coefficients will be finite, and all the rest infinite.

To illustrate the first case, suppose fx to be of the form $\phi x\,(x-a)^{-p}$. Then,

86. If in the development of $f(x+h)$ at the beginning of the last No. we substitute for R the value just found, we shall have as the complete development,

$$f(x+h)=fx+\frac{h}{1}f^1x+\frac{h^2}{1.2}f^2x+\ldots\ldots$$

$$+\frac{h^{n-1}}{1.2.3\ldots(n-1)}f^{n-1}x+\frac{h^n}{1.2\ldots n}f^n(x+\theta h).$$

Should the limits found above for R, show in a particular case that it will tend to become evanescent, when n is indefinitely increased, the value of R may be omitted, and we may write simply

$$f(x+h)=fx+\frac{h}{1}f^1x+\frac{h^2}{1.2}f^2x+\frac{h^3}{1.2.3}f^3x+\text{\&c.};$$

and the limit to which the series in the second member of this tends as its ultimate value, when n is continually increased, is the value of $f(x+h)$. This is the form in which the series is generally exhibited and used.

87. We may now proceed to exemplify the use of Taylor's theorem in the development of functions in series; and, as a first example, we may find a series equivalent to $l(x+h)$. To do this, let us assume $u=lx$; and, by No. 27, and by successive differentiations, we shall obtain the following differential coefficients:—

$$f^1x=\frac{M}{x}=Mx^{-1},\qquad f^2x=-1Mx^{-2},\qquad f^3x=1.2\,Mx^{-3},$$

$$f^4x=-1.2.3\,Mx^{-4},\ \text{\&c.}$$

---

changing x into $a+h$, we get $\phi(a+h)h^{-n}$; which by developing $\phi(a+h)$, and multiplying by h^{-n}, is changed into

$$h^{-n}\phi a+h^{1-n}\phi^1a+\tfrac{1}{2}h^{2-n}\phi^2a+\text{\&c.};$$

showing that in this case the development must have some of the indices of h negative. To illustrate the second, suppose fx to be of the form $\phi(x-a)x^{p+q}$: then, by changing x into $a+h$, expanding, &c. we get

$$\phi(a+h)h^{p+q}=h^{p+q}\phi a+h^{p+q+1}\phi_1a+\text{\&c.};$$

which shows, that in this case the powers of h in the development will be fractional.

As instances, the student may develop $u'=\frac{\log(x+h)}{x+h}$; $u'=(x+h)^{\frac{1}{2}}\sin x$; and the simple binomials $(x+h)^{-1}$ and $(x+h)^{\frac{1}{3}}$, and take $x=0$ in the results.

Other investigations of this important theorem, and additional matter regarding it will be found in various works on the differential calculus, particularly those of De Morgan, Duhamel, and Cournot.

Then, by substituting these in Taylor's formula, and by some obvious modifications, we obtain

$$l(x+h)=lx+\mathrm{M}\left(\frac{h}{x}-\frac{1}{2}\frac{h^2}{x^2}+\frac{1}{3}\frac{h^3}{x^3}-\frac{1}{4}\frac{h^4}{x^4}+\text{\&c.}\right)^{*}\ldots(1).$$

88. If in this we take $x=1$, since (No. 24) $l1=0$, we get

$$l(1+h)=\mathrm{M}(h-\tfrac{1}{2}h^2+\tfrac{1}{3}h^3-\tfrac{1}{4}h^4+\text{\&c.})\ \ldots\ldots\ (2).\dagger$$

This, again, if h be taken negative, is changed into

$$l(1-h)=\mathrm{M}(-h-\tfrac{1}{2}h^2-\tfrac{1}{3}h^3-\tfrac{1}{4}h^4-\text{\&c.});$$

by taking which from (2) we get $l(1+h)-l(1-h)$, or (No. 24)

$$l\frac{1+h}{1-h}=2\,\mathrm{M}(h+\tfrac{1}{3}h^3+\tfrac{1}{5}h^5+\tfrac{1}{7}h^7+\text{\&c.})\ \ldots\ldots\ (3).$$

If in this we put $\frac{1+h}{1-h}=x$, we find, by multiplying by $1-h$, transposing, &c. $h=\frac{x-1}{x+1}$, and the series becomes

$$lx=2\,\mathrm{M}\left\{\frac{x-1}{x+1}+\frac{1}{3}\left(\frac{x-1}{x+1}\right)^3+\frac{1}{5}\left(\frac{x-1}{x+1}\right)^5+\text{\&c.}\right\}\ldots\ldots\ (4).$$

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* This series may be easily obtained by means of the integral calculus, in the following manner:—Since (No. 27)

$$dl(x+h)=\mathrm{M}\frac{dx}{x+h}=\mathrm{M}\frac{1}{x+h}dx,$$

we get by the actual division of 1 by $x+h$, and by multiplying the quotient by M and dx

$$d\log(x+h)=\mathrm{M}(x^{-1}\ dx-x^{-2}\ hdx+x^{-3}\ h^2dx-x^{-4}\ h^3dx+\text{\&c.})$$

Hence, by integrating by means of B and A_2, page 41, we get

$$l(x+h)=lx+\mathrm{M}\left(\frac{h}{x}-\frac{1}{2}\frac{h^2}{x^2}+\frac{1}{3}\frac{h^3}{x^3}-\frac{1}{4}\frac{h^4}{x^4}+\text{\&c.}\right)+\mathrm{C}.$$

To determine C, let $h=0$; then $lx=lx+\mathrm{C}$; whence $\mathrm{C}=0$; and the series becomes the same as that which was determined in the text.

It may be remarked that in the result in the text, we should have (No. 85), neglecting the sign, $\mathrm{R}=\frac{\mathrm{M}h^n}{n(x+\theta h)^n}$; so that the error committed by stopping at the n^{th} term lies between $\frac{\mathrm{M}h^n}{nx^n}$ and $\frac{\mathrm{M}h^n}{n(x+h)^n}$. Hence, for example, if x were 10 and h were 1, the error in deriving the Neperian logarithm of 11 from that of 10, resulting from stopping at the fifth term, would be less than $\frac{1}{5\times10^5}$ (=0·000002) and greater than $\frac{1}{5\times11^5}$ (=0·00000124...).

† This series was first given by Nicolas Mercator or Kaufman in the year 1667, in his *logarithmotechnia;* and from it series (3) was derived by James Gregorie, who published it in his *Exercitationes Geometricæ*, in 1668. Mercator was a native of Holstein in Germany, but he settled in England.

89. Another series may be derived from (3) by putting $\frac{1+h}{1-h}=\frac{x+n}{x}$; from which, after multiplying by $1-h$ and x, transposing, &c. we find $h=\frac{n}{2x+n}$. Then, since (No. 24) $l\frac{x+n}{x}=l(x+n)-lx$, we obtain from (3), after transposing lx,

$$l(x+n)=lx+2\,\mathrm{M}\left\{\frac{n}{2x+n}+\frac{1}{3}\left(\frac{n}{2x+n}\right)^3+\frac{1}{5}\left(\frac{n}{2x+n}\right)^5+\&c.\right\}\ \ldots\ldots (5).$$

If $n=1$, this becomes

$$l(x+1)=lx+2\,\mathrm{M}\left\{\frac{1}{2x+1}+\frac{1}{3}\left(\frac{1}{2x+1}\right)^3+\frac{1}{5}\left(\frac{1}{2x+1}\right)^5+\&c.\right\}\ \ldots\ldots (6).$$

If, in the same series, we take $\frac{1+h}{1-h}=\frac{x^2}{x^2-1}$, which gives $h=\frac{1}{2x^2-1}$, we shall have, since (No. 25) $lx=\frac{1}{2}l(x^2)$, and since $x^2-1=(x+1)(x-1)$,

$$lx=\tfrac{1}{2}\{l(x+1)+l(x-1)\}+\mathrm{M}\left\{\frac{1}{2x^2-1}+\frac{1}{3}\left(\frac{1}{2x^2-1}\right)^3+\ \&c.\right\}\ldots\ldots (7).$$

90. By changing h into $-h$ in (1), and by taking half the sum and half the difference of (1) and the result, we get

$$\frac{l(x+h)+l(x-h)}{2}=lx-\mathrm{M}\left(\frac{1}{2}\frac{h^2}{x^2}+\frac{1}{4}\frac{h^4}{x^4}+\frac{1}{6}\frac{h^6}{x^6}+\&c.\right),$$

and $$\frac{l(x+h)-l(x-h)}{2}=\mathrm{M}\left(\frac{h}{x}+\frac{1}{3}\frac{h^3}{x^3}+\frac{1}{5}\frac{h^5}{x^5}+\&c.\right).$$

Multiply the latter of these by h, and divide the product by $2x$; then by adding to the former the result thus obtained, and by transposition, we get, after some modifications,

$$lx = \frac{l(x+h)+l(x-h)}{2} + \frac{h\{l(x+h)-l(x-h)\}}{4x} +$$

$$M\left(\frac{1}{3.4}\frac{h^4}{x^4} + \frac{2}{5.6}\frac{h^6}{x^6} + \frac{3}{7.8}\frac{h^8}{x^8} + \&c.\right) \ldots\ldots (8).$$

If $h=1$, this becomes simply

$$lx = \frac{l(x+1)+l(x-1)}{2} + \frac{l(x+1)-l(x-1)}{4x} +$$

$$M\left(\frac{1}{3.4}\frac{1}{x^4} + \frac{2}{5.6}\frac{1}{x^6} + \frac{3}{7.8}\frac{1}{x^8} + \&c.\right)^* \ldots\ldots (9).$$

91. The modulus of the Neperian logarithms (No. 28) is unity; and we are now prepared to investigate the method of determining that of any other assigned system. To do this in one of the simplest ways, let us denote for brevity the series in the vinculum in formula (4) by s, then $lx = 2Ms$, and $\log x = 2s$. Dividing the former by the latter, we get $M = \frac{lx}{\log x}$; so that *the modulus of any system of logarithms is obtained by dividing the logarithm of any number in that system by the Neperian logarithm of the same number;* and, conversely, *the logarithm of a number in any system is equal to the product of its Neperian logarithm by the modulus of the system.*

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* To the series here investigated, may be added the following, given, the first by Borda, and the second by Haros:

$$l(x+2) = l(x-2) + 2l(x+1) - 2l(x-1) + 2M\left\{\frac{2}{x^3-3x} + \frac{1}{3}\left(\frac{2}{x^3-3x}\right)^3 + \&c.\right\}:$$

$$l(x+5) = l(x-3) + l(x+3) + l(x-4) + l(x+4) - l(x-5) - 2lx$$

$$-2M\left\{\frac{72}{x^4-25x^2+72} + \frac{1}{3}\left(\frac{72}{x^4-25x^2+72}\right)^3 + \&c.\right\}.$$

The student will readily discover the method of investigating these; and he will find the convergence to be very rapid, when x is considerable. The following are of a similar kind:

$$l(x+3) = l(x+1) + l(x+2) + l(x-3) - l(x-2) - l(x-1) +$$

$$2M\left\{\frac{6}{x^3-7x} + \frac{1}{3}\left(\frac{6}{x^3-7x}\right)^3 + \&c.\right\}.$$

$$l(x+3) = l(x-3) + 3l(x+1) - 3l(x-1) +$$

$$2M\left\{\frac{8x}{x^4-6x^2-3} + \frac{1}{3}\left(\frac{8x}{x^4-6x^2-3}\right)^3 + \&c.\right\}.$$

Other similar series will be found in Lacroix's large work on the Differential and Integral Calculus, vol. I. pages 48, 49, &c.

92. If in the expression for M found in the last No. we take x equal to a, the base of any system of logarithms, we get (No. 26) $M = \frac{1}{\log a}$: which shows, that *the modulus of any system of logarithms is the reciprocal of the Neperian logarithm of the base of that system.* Hence, the modulus of the common logarithms will be found by dividing unity by the Neperian logarithm of 10.

93. Let it next be required to develop a^{x+h} by means of Taylor's theorem. Here we have $fx = a^x$; and by successive differentiations (No. 29) we obtain

$$f^1x = a^x \log a, \quad f^2x = a^x(\log a)^2, \quad f^3x = a^x(\log a)^3, \quad \&c.$$

Then, by substituting these values of fx, f^1x, &c. in Taylor's formula, we get

$$a^{x+h} = a^x + \frac{a^x h \log a}{1} + \frac{a^x h^2 (\log a)^2}{1.2} + \frac{a^x h^3 (\log a)^3}{1.2.3} + \&c.^*$$

From this, either by dividing by a^x, or by taking $x = 0$, we obtain

$$a^h = 1 + \frac{h}{1}\log a + \frac{h^2}{1.2}(\log a)^2 + \frac{h^3}{1.2.3}(\log a)^3 + \&c.:$$

and hence, if a take the particular value ε, we get (No 29)

$$\varepsilon^h = 1 + \frac{h}{1} + \frac{h^2}{1.2} + \frac{h^3}{1.2.3} + \&c.;$$

which, if $h = 1$, becomes

$$\varepsilon = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \&c.,$$

the series already found in No. 27.

94. In developing $\sin(x+h)$, we have $fx = \sin x$; and therefore, (No. 32)

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* In this development we have $R = a^{x+\theta h} h^n (\log a)^n$, and therefore the error arising from stopping at the n^{th} term lies between $\frac{a^x h^n (\log a)^n}{1.2.3\ldots\ldots n}$ and $\frac{a^{x+h} h^n (\log a)^n}{1.2.3\ldots\ldots n}$. When $a = \epsilon$, $h = 1$, and $x = 0$, these limits become $\frac{1}{1.2\ldots\ldots n}$ and $\frac{\epsilon}{1.2\ldots\ldots n}$; and therefore, if only ten terms of the series be employed, so rapid is the convergence, that the error produced by stopping at the tenth term is greater than the quotient obtained by dividing unity by the continual product of 1, 2, 3, &c. up to 10; and less than the same quotient multiplied by 2·71828.....

$f^1x = \cos x$, $f^2x = -\sin x$, $f^3x = -\cos x$, $f^4x = \sin x, \ldots\ldots$, $f^nx = \sin(\frac{1}{2}n\pi + x)$.*

Hence, by Taylor's theorem,

$$\sin(x+h) = \sin x + \frac{h}{1}\cos x - \frac{h^2}{1.2}\sin x - \frac{h^3}{1.2.3}\cos x + \frac{h^4}{1.2.3.4}\sin x + \ldots\ldots + \frac{h^{n-1}}{1.2.3\ldots(n-1)}\sin\{\tfrac{1}{2}(n-1)\pi + x\} + \frac{h^n}{1.2.3\ldots n}\sin(\tfrac{1}{2}n\pi + x + \theta h).$$

Here, by the continual increase of n, the concluding term (the value of R, No. 86) may be made as nearly evanescent as we please; the sine of $\frac{1}{2}n\pi + x + \theta h$ (as well as of every other angle) being always finite, never exceeding the limits 1 and -1, and $\frac{h^n}{1.2.3\ldots n}$ becoming smaller and smaller without limit, when n is increased above h. Hence, this term may be omitted, and the sine of $x+h$ will be the limit which the sum of an infinite number of the terms in the series in the second member, without R, tends to take.

95. If in the series now found, we take $x = 0$, and consequently (TRIG. No. 10) $\sin x = 0$, and $\cos x = 1$, we get

$$\sin h = h - \frac{h^3}{1.2.3} + \frac{h^5}{1.2.3.4.5} - \frac{h^7}{1.2.3.4.5.6.7} + \&c.;$$

and if we differentiate the members of this result, and divide by dh, we obtain the corresponding series,

$$\cos h = 1 - \frac{h^2}{1.2} + \frac{h^4}{1.2.3.4} - \frac{h^6}{1.2.3.4.5.6} + \&c.\dagger$$

96. In several functions the successive differential coefficients of the higher orders become so complicated, that the

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* By trigonometry, $\sin(\frac{1}{2}n\pi + x) = \sin\frac{1}{2}n\pi\cos x + \cos\frac{1}{2}n\pi\sin x$, the first term of which vanishes when n is even, and the second when it is odd; and the expression becomes in the first case $\pm\sin x$, in the second $\pm\cos x$. It will be readily seen, therefore, by taking n successively equal to 1, 2, 3, &c. that $\sin(\frac{1}{2}n\pi + x)$ is a correct general expression for the differential coefficients of $\sin x$.

† This may also be found by taking $fx = \cos x$ in Taylor's theorem, or from the series in No. 94, by taking $x = \frac{1}{2}\pi$; as (Trig. No. 10) in the latter case, $\sin x = 1$, $\cos x = 0$, and $\sin(x+h) = \sin(\frac{1}{2}\pi + h) = \sin(\frac{1}{2}\pi - h) = \cos h$. These series have the curious property of re-producing each other continually by successive differentiations, and by divisions by dx or $-dx$, as may be required. We should see, *a priori*, from Nos. 32 and 33, that this must be the case.

series obtained by means of Taylor's theorem are of little use. This the student would find to be the case, were he to attempt in this way to develop $u = \tan(x+h)$, or $u = \sec(x+h)$, or the inverse functions, $\sin^{-1}(x+h)$, $\tan^{-1}(x+h)$, &c. Of the series, however, which are connected with such functions, those which are of value may often be obtained with facility by means of Stirling's theorem, in connexion with expedients belonging to the particular functions. This will be exemplified by the two following investigations, which lead to results of importance.

97. Let it be required to express $\tan^{-1}x$ in a series in terms of x. Here, putting $fx = \tan^{-1}x$, we have (No. 40.)

$$f^1 x = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \&c.;$$

the series being obtained simply by dividing 1 by $1 + x^2$. Hence we get, by successive differentiations,

$$f^2 x = -2x + 4x^3 - 6x^5 + 8x^7 - \&c.$$
$$f^3 x = -1.2 + 3.4x^2 - 5.6x^4 + 7.8x^6 - \&c.$$
$$f^4 x = 2.3.4x - 4.5.6x^3 + 6.7.8x^5 - \&c.$$
$$f^5 x = 1.2.3.4 - 3.4.5.6x^2 + 5.6.7.8x^4 - \&c.$$

Then, by taking $x = 0$ in fx, f^1x, &c., we find (No. 84) $U = 0$, $U_1 = 1$, $U_2 = 0$, $U_3 = -1.2$, $U_4 = 0$, $U_5 = 1.2.3.4$, &c,: and, by substituting these in Stirling's theorem, we get

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \&c.*$$

98. As another example, let it be required to find the length of a circular arc in terms of its sine. Assuming here, $fx = \sin^{-1}x$, we have (No. 38)

$$f^1 x = \frac{1}{\sqrt{(1-x^2)}} = (1-x^2)^{-\frac{1}{2}} = 1 + \frac{x^2}{2} + \frac{1.3\,x^4}{2.4} + \frac{1.3.5\,x^6}{2.4.6} + \&c.;$$

by development by the binomial theorem. Hence

$$f^2 x = x + \frac{1.3\,x^3}{2} + \frac{1.3.5\,x^5}{2.4} + \&c.$$

$$f^3 x = 1 + \frac{1.3^2\,x^2}{2} + \frac{1.3.5^2\,x^4}{2.4} + \&c.;$$

* This result may be obtained with great ease from the expression found above for $f^1 x$; since, if it be multiplied by dx we get

$$d\tan^{-1} x = dx - x^2\,dx + x^4\,dx - \&c.$$

Hence, by integration, by A_2, page 41, we find,

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \&c. + C:$$

and, by taking $x = 0$ when $\tan^{-1} x = 0$, we find $C = 0$; so that no constant quantity is required; and the result becomes the same that was found above.

and the succeeding functions are found with equal facility. Taking then $x=0$ in these various functions, we get

$$U=0,\quad U_1=1,\ U_2=0,\ U_3=1,\ U_4=0,\ U_5=1^2.2^2,\ \&c.$$

Hence by Stirling's theorem we have

$$\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \&c.*$$

99. The following investigation gives a remarkable series, which expresses the length of a circular arc in terms of its tangent. By No. 40, we have $d\tan^{-1}x = \frac{dx}{1+x^2}$; and integrating this by means of K_2, page 41, we get

$$\tan^{-1}x = \frac{x}{1+x^2} + \int \frac{x}{1+x^2}.\frac{2x\,dx}{1+x^2},$$

$$\text{or } \tan^{-1}x = \frac{x}{1+x^2} + \int \frac{2x^2\,dx}{(1+x^2)^2}.$$

The integral indicated in the last term of this is found by means of the same formula (K^2), to be

$$\frac{2}{3}\frac{x^3}{(1+x^2)^2} + \frac{2.4}{3}\int \frac{x^4\,dx}{(1+x^2)^3}.$$

It is plain that this process may be continued without limit; and, the law of continuation being manifest, we obtain

$$\tan^{-1}x = \frac{x}{1+x^2} + \frac{2}{3}\frac{x^3}{(1+x^2)^2} + \frac{2.4}{3.5}\frac{x^5}{(1+x^2)^3} + \frac{2.4.6}{3.5.7}\frac{x^7}{(1+x^2)^4} + \&c.$$

This is the series proposed to be investigated; and for giving an arc in the first quadrant, it requires the addition of no constant quantity.

When x is a fraction $\frac{p}{q}$, the foregoing series may be exhibited, after some modifications, under the convenient form,

$$\tan^{-1}\frac{p}{q} = \frac{pq}{p^2+q^2}\left\{1 + \frac{2}{3}\frac{p^2}{p^2+q^2} + \frac{2.4}{3.5}\left(\frac{p^2}{p^2+q^2}\right)^2 + \&c.\right\}.$$

* This would also be found with great facility from f^1x as exhibited above in series, by multiplying by dx, and integrating the result.

This series and the two in No. 95, were discovered by Newton, and were published in his *Analysis per Æquationes Numero Terminorum Infinitas*. The series in No. 97, was discovered by James Gregorie, in 1671; and it was afterwards rediscovered by Leibnitz in 1676.

By putting t_1, t_2, t_3, &c., to denote the successive terms of the last series, and k to denote the fraction $\frac{p^2}{p^2+q^2}$, we get the following expression, which answers best for the purposes of computation:—

$$\tan^{-1}\frac{p}{q}=\frac{pq}{p^2+q^2}+\frac{2}{3}kt_1+\frac{4}{5}kt_2+\frac{6}{7}kt_3+\&c.^*$$

100. Two remarkable formulas, establishing a connexion between the two great classes of transcendental functions, were discovered by Euler. These may be thus investigated:—

Assume $h=x\sqrt{-1}$; then $h^2=-x^2$, $h^3=-x^3\sqrt{-1}$,

$$h^4=x^4,\quad h^5=x^5\sqrt{-1},\quad h^6=-x^6,\ \&c.;$$

the law of continuation being manifest. Let these values of h and its powers be substituted in the development for ε^h (No. 93), and it will become

$$\varepsilon^{x\sqrt{-1}}=1+\frac{x\sqrt{-1}}{1}-\frac{x^2}{1.2}-\frac{x^3\sqrt{-1}}{1.2.3}+\frac{x^4}{1.2.3.4}+\frac{x^5\sqrt{-1}}{1.2.3.4.5}-\&c.;$$

and from this, by changing the sign of x, we obtain

$$\varepsilon^{-x\sqrt{-1}}=1-\frac{x\sqrt{-1}}{1}-\frac{x^2}{1.2}+\frac{x^3\sqrt{-1}}{1.2.3}+\frac{x^4}{1.2.3.4}-\frac{x^5\sqrt{-1}}{1.2.3.4.5}-\&c.$$

From these series, by taking their sum and difference we get

$$\varepsilon^{x\sqrt{-1}}+\varepsilon^{-x\sqrt{-1}}=2\left(1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\&c.\right),$$

$$\varepsilon^{x\sqrt{-1}}-\varepsilon^{-x\sqrt{-1}}=2\sqrt{-1}\left(x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\&c.\right).$$

* The first series in No. 99, was given, with two investigations, in a paper by Euler, in the *Nova Acta* of the Petersburg Academy for 1793. The investigation here given, which is different from either of Euler's, and which is very simple, was published in the Edinburgh Transactions for 1838. See the end of p. 64.

Now, (No. 95) the series in the vinculums, are respectively the values of $\cos x$ and $\sin x$; and therefore

$$\varepsilon^{x\sqrt{-1}} + \varepsilon^{-x\sqrt{-1}} = 2\cos x \ldots\ldots\ldots\ldots\ldots(1);$$

$$\varepsilon^{x\sqrt{-1}} - \varepsilon^{-x\sqrt{-1}} = 2\sqrt{-1}.\,\sin x \ldots\ldots(2);$$

which are the formulas referred to.

101. By taking half the sum and half the difference of the members of these equations, we get

$$\varepsilon^{x\sqrt{-1}} = \cos x + \sqrt{-1}.\sin x \ \ldots\ldots\ldots\ldots(3);$$

$$\varepsilon^{-x\sqrt{-1}} = \cos x - \sqrt{-1}.\sin x \ldots\ldots\ldots\ldots(4).$$

102. The formulas obtained in the last two Nos. are often employed with much advantage as instruments of investigation. The following may suffice here as instances.

On the members of the two formulas in the last No. perform two distinct operations:—first, raise them to their n^{th} powers, and, secondly, change x into nx. Then, in the results, the left hand members will be the same ($\varepsilon^{nx\sqrt{-1}}$ and $\varepsilon^{-nx\sqrt{-1}}$) in both cases; and, therefore, by making the other members equal, we get

$$(\cos x + \sqrt{-1}.\sin x)^n = \cos nx + \sqrt{-1}.\sin nx, \text{ and}$$
$$(\cos x - \sqrt{-1}.\sin x)^n = \cos nx - \sqrt{-1}.\sin nx.$$

These formulas were discovered, on other principles, by De Moivre. For a different investigation, see TRIG. Sect. VIII.

103. By dividing the first of the formulas in No. 101 by the second, we get, after dividing the numerator and denominator of the second member by $\cos x$,

$$\varepsilon^{2x\sqrt{-1}} = \frac{1 + \tan x\sqrt{-1}}{1 - \tan x\sqrt{-1}}.$$

Now, (No. 25) the Neperian logarithm of the first member of this is $2x\sqrt{-1}$; and the series expressing the Neperian logarithm of the second member is found by taking $\tan x\sqrt{-1} = h$, and $M = 1$, in series (3), No. 88. Hence, putting these logarithms equal to one another, and dividing by $2\sqrt{-1}$, we obtain

$$x = \tan x - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - \tfrac{1}{7}\tan^7 x + \&c.;$$

the same formula that was obtained by very different means in No. 97.

104. Since (No. 100) $2\cos\frac{1}{2}x = \varepsilon^{\frac{1}{2}x\sqrt{-1}} + \varepsilon^{-\frac{1}{2}x\sqrt{-1}} = \varepsilon^{-\frac{1}{2}x\sqrt{-1}} + \varepsilon^{\frac{1}{2}x\sqrt{-1}}$, we shall have, by No. 88, the two following developments:

$\log 2 + \log\cos\frac{1}{2}x = \frac{1}{2}x\sqrt{-1} + \varepsilon^{-x\sqrt{-1}} - \frac{1}{2}\varepsilon^{-2x\sqrt{-1}} + \frac{1}{3}\varepsilon^{-3x\sqrt{-1}} -$ &c. and

$\log 2 + \log\cos\frac{1}{2}x = -\frac{1}{2}x\sqrt{-1} + \varepsilon^{x\sqrt{-1}} - \frac{1}{2}\varepsilon^{2x\sqrt{-1}} + \frac{1}{3}\varepsilon^{3x\sqrt{-1}} -$ &c.

By taking half the sum and half the difference of these we get

$\log 2 + \log\cos\frac{1}{2}x = \frac{1}{2}(\varepsilon^{x\sqrt{-1}} + \varepsilon^{-x\sqrt{-1}}) - \frac{1}{2}.\frac{1}{2}(\varepsilon^{2x\sqrt{-1}} + \varepsilon^{-2x\sqrt{-1}}) + \frac{1}{3}.\frac{1}{2}(\varepsilon^{3x\sqrt{-1}} + \varepsilon^{-3x\sqrt{-1}}) -$ &c.

and $0 = x\sqrt{-1} - \frac{1}{2}(\varepsilon^{x\sqrt{-1}} - \varepsilon^{-x\sqrt{-1}}) + \frac{1}{2}.\frac{1}{2}(\varepsilon^{2x\sqrt{-1}} - \varepsilon^{-2x\sqrt{-1}}) - \frac{1}{3}.\frac{1}{2}(\varepsilon^{3x\sqrt{-1}} - \varepsilon^{-3x\sqrt{-1}}) +$ &c.

Let the first of these be modified by formula (1), and the second by formula (2) in No. 100: then, after dividing the second result by $\sqrt{-1}$, and by transposition, we obtain

$$\log 2 + \log\cos\tfrac{1}{2}x = \cos x - \tfrac{1}{2}\cos 2x + \tfrac{1}{3}\cos 3x - \tfrac{1}{4}\cos 4x + \text{\&c.} \ldots (1),$$
$$\text{and } \tfrac{1}{2}x = \sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x + \text{\&c.} \ldots (2).$$

If in these two formulas we change x into $\pi - x$, and change the signs in the first, we obtain

$$-\log 2 - \log\sin\tfrac{1}{2}x = \cos x + \tfrac{1}{2}\cos 2x + \tfrac{1}{3}\cos 3x + \text{\&c.} \ldots (3),$$
$$\tfrac{1}{2}(\pi - x) = \sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \text{\&c.} \ldots (4).*$$

105. From (3) No. 101, by changing x successively into

* Regarding series such as the four curious ones above obtained, information will be found in various recent works, particularly in Cournot's "Traité Elémentaire de la Théorie des Fonctions et du Calcul Infinitésimal," Liv. II. Chap. IV. and Liv. V. Chap. XII.; and in De Morgan's "Differential and Integral Calculus," Chap. XII. It would be inconsistent with the nature of the present work to enter at any length into minute details on such matters. It may be stated, however, that the series in the text are periodic, giving the same sums or values, when we use $2\pi + x$, or in general $2n\pi + x$ (n being a whole number), instead of x in the second members. Thus, in the second, for instance, while as long as x is between $-\pi$ and π, the sum of the series in the second member will be equal to half the arc x; yet, if x be between π and 2π, the sum will be $\frac{1}{2}(\pi - x)$ as in formula (4); and if instead of x, $2n\pi + x$ be taken, the sum will be $n\pi + \frac{1}{2}x$. The student will obtain particular formulas of some interest by taking $x = 0$, $x = \pi$, $x = \frac{1}{2}\pi$, $x = \frac{1}{4}\pi$, $x = \frac{1}{3}\pi$, $x = \frac{1}{6}\pi$, &c. In this way he will obtain from (1),

$$\log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \text{\&c.}$$

the same that would be obtained from formula (1) p. 52, by taking x and h, each equal to 1; and from the second formula found above,

$$\tfrac{1}{4}\pi = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \text{\&c.};$$

which is the same that we get from No. 97, by taking $x = \frac{1}{2}\pi$.

$2n\pi$ and $(2n+1)\pi$, n being a whole number, we get (TRIG. Nos. 17 and 16, taking $A=0$)

$$\varepsilon^{2n\pi\sqrt{-1}}=1, \text{ and } \varepsilon^{(2n+1)\pi\sqrt{-1}}=-1:$$

and from this, by taking the logarithms, we obtain

$$\log 1=2n\pi\sqrt{-1}, \text{ and } \log(-1)=(2n+1)\pi\sqrt{-1}.*$$

Hence, since n may have as many values as we please, it follows that log 1 has an infinite number of values,—all of them imaginary, however, except zero, which is found by taking $n=0$. In like manner it appears, that -1 has also an infinite number of logarithms, but that they are all imaginary. It follows also, since $a=a\times 1$, and $-a=a\times -1$, that

$$\log a=\log a+2n\pi\sqrt{-1}, \text{ and } \log(-a)=\log a+(2n+1)\pi\sqrt{-1},$$

a being any positive whole number. Hence it appears, that every positive number has an infinite number of logarithms, but that they are all imaginary except the ordinary arithmetical one; and that every negative number has an infinite number of logarithms, all imaginary.†

106. The computation of logarithms and other transcendental numbers is now of no great practical importance, as tables of them have long since been computed and published, which are quite sufficient for all the calculations that occur in the applications of science in its present state. Still, however, it is unscientific in the student to employ these or other numbers, without knowing how they are obtained: and, therefore, it may now be proper to advert briefly to some of the ways in which they may be computed.

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* If in this we take $n=0$, and divide by $\sqrt{-1}$, we get the remarkable expression $\pi=\frac{\log(-1)}{\sqrt{-1}}$. Several other curious expressions may be derived from the formulas in Nos. 100 and 101. Thus, for instance, from (3), by taking $x=\frac{1}{2}\pi$ we get $\varepsilon^{\frac{1}{2}\pi\sqrt{-1}}=\sqrt{-1}$; whence by involving both members to the power whose index is $2\sqrt{-1}$, we obtain $\sqrt{-1}^{\,2\sqrt{-1}}=\epsilon^{-\pi}$; or (No. 93)

$$\sqrt{-1}^{\,2\sqrt{-1}}=1-\pi+\frac{\pi^2}{1.2}-\frac{\pi^3}{1.2.3}+\frac{\pi^4}{1.2.3.4}-\&c.$$

† It may be proper to state, that the logarithm of an infinite positive number is infinite and is positive. This is plain from the expression (No. 25) $l a^n=n$, in which n may evidently be taken as great as we please. Since also (ALG. 127) $\frac{1}{\infty}=0$, we have $l0=-\infty$; that is, the logarithm of zero is likewise infinitely great in absolute magnitude, but is negative.

All whole numbers are either prime or composite; a *prime* number being that which is not produced by the multiplication of other integers, while a *composite* one is the product of two or more such factors. Thus, 2, 3, 5, 7, 11, &c. are primes; while 4, 6, 8, 9, 10, 12, &c. are composite. Now, by the nature of logarithms (No. 24), the logarithm of a composite number will be found by taking the sum of the logarithms of its factors. Thus, $l6 = l2 + l3$; $l35 = l5 + l7$, &c. Hence, the logarithms of the composite numbers will be obtained by mere addition, if those of the primes be known. The logarithms of the latter are easily computed by means of series, particularly those marked (4), (5), (6), (7), (8), and (9) in pages 52, 53, and 54.

Thus, if in (4) we take $x = 2$, or in (6) $x = 1$, we get

$$l2 = 2\,\mathrm{M}\left\{\tfrac{1}{3} + \tfrac{1}{3}(\tfrac{1}{3})^3 + \tfrac{1}{5}(\tfrac{1}{3})^5 + \tfrac{1}{7}(\tfrac{1}{3})^7 + \&c.\right\};*$$

whence the logarithm of 2 is readily obtained, and (No. 25)

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* To exemplify the use of this series in computing the Neperian logarithm of 2, let $\mathrm{M} = 1$, and the series becomes

$$\log 2 = 2\left\{\tfrac{1}{3} + \tfrac{1}{3}(\tfrac{1}{3})^3 + \tfrac{1}{5}(\tfrac{1}{3})^5 + \tfrac{1}{7}(\tfrac{1}{3})^7 + \tfrac{1}{9}(\tfrac{1}{3})^9 + \&c.\right\}.$$

Now the first term of this series is $\frac{2}{3}$; and the other terms without the coefficients, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, &c. are each derived from the one before it, by multiplying by $\frac{1}{9}$, the square of $\frac{1}{3}$, or, which is the same, by dividing by 9: the computation therefore will stand as in the margin, the ciphers at the commencement of the decimals being omitted for simplicity. The numbers in the last column are all taken so as to have their last figures as

$\frac{2}{3}$ =	·66666667	....	·66666667—
$2(\frac{1}{3})^3$ =	7407407	($\frac{1}{3}$ of this) ...	2469136—
$2(\frac{1}{3})^5$ =	823045	($\frac{1}{5}$ ——) ...	164609
$2(\frac{1}{3})^7$ =	91449	($\frac{1}{7}$ ——) ...	13064
$2(\frac{1}{3})^9$ =	10161	($\frac{1}{9}$ ——) ...	1129
$2(\frac{1}{3})^{11}$ =	1129	($\frac{1}{11}$ ——) ...	103—
$2(\frac{1}{3})^{13}$ =	125	($\frac{1}{13}$ ——) ...	10—
$2(\frac{1}{3})^{15}$ =	14	($\frac{1}{15}$ ——) ...	1—
		Log 2 =	0·69314719

nearly as possible true, and the sign — is put after those that are too great. The number of them so marked in the present instance being greater than that of the others, the final result may be expected to be too great, and it is so, as the last figure ought to be 8; the value found by carrying out the decimals to more places, and the series to more terms, being 0·6931471805599.... In general, in computing the sums of converging series by the addition of their terms reduced to decimals, the correctness of the last figure or two cannot be depended on; and therefore the decimals should be carried out to two or more figures beyond the number of places for which it is intended that the result should be true.

by doubling this, trebling it, &c. we should get the logarithms of 4, 8, 16, and the other powers of 2.

To find the logarithm of 3, the next prime number, we might take $x = 3$ in (4), and we should thus obtain

$$l3 = 2\,\mathrm{M}\left\{\tfrac{1}{2} + \tfrac{1}{3}\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{5}\left(\tfrac{1}{2}\right)^5 + \&c.\right\};$$

The computation of log 3 according to the second method indicated in the text, is as in the margin. By either of the methods pointed out in the text or by others, we should readily find the Neperian logarithm of 5 to be 1·6094379; and by adding to this 0·6931472, the logarithm of 2, we get 2·3025851, the Neperian logarithm of 10.

$\frac{2}{5}$ =	0·40000000,	....	0·40000000
$2(\frac{1}{5})^3$ =	1600000,	($\frac{1}{3}$ of this)	533333
$2(\frac{1}{5})^5$ =	64000,	($\frac{1}{5}$ ——)	12800
$2(\frac{1}{5})^7$ =	2560,	($\frac{1}{7}$ ——)	366
$2(\frac{1}{5})^9$ =	102,	($\frac{1}{9}$ ——)	11
			0·40546510
		Log 2 =	·69314718
		Log 3 =	1·09861228

The easier method of computing the common logarithms referred to in the text (No. 106) will be best illustrated by the computation by means of it of some common logarithm, suppose that of 2. The work for this arising from the series

$$l2 = 2\,\mathrm{M}\left\{\tfrac{1}{3} + \tfrac{1}{3}\left(\tfrac{1}{3}\right)^3 + \tfrac{1}{5}\left(\tfrac{1}{3}\right)^5 + \tfrac{1}{7}\left(\tfrac{1}{3}\right)^7 + \&c.\right\},$$

will stand as follows, the double of the modulus being 0·86858896:

$\frac{1}{3} \times 2\,\mathrm{M}$ =	0·28952965	....	0·28952965
$(\frac{1}{3})^3 \times 2\,\mathrm{M}$ =	3216996......	($\frac{1}{3}$ of this)	1072332
$(\frac{1}{3})^5 \times 2\,\mathrm{M}$ =	357444......	($\frac{1}{5}$ ——)	71489—
$(\frac{1}{3})^7 \times 2\,\mathrm{M}$ =	39716......	($\frac{1}{7}$ ——)	5674—
$(\frac{1}{3})^9 \times 2\,\mathrm{M}$ =	4413......	($\frac{1}{9}$ ——)	490
$(\frac{1}{3})^{11} \times 2\,\mathrm{M}$ =	490......	($\frac{1}{11}$ ——)	45—
$(\frac{1}{3})^{13} \times 2\,\mathrm{M}$ =	54......	($\frac{1}{13}$ ——)	4
		Common logarithm of 2 =	0·30102999

It may be remarked, in conclusion, in reference to the computation of logarithms, that, of all the series in the text, the most convergent are those marked (8) and (9). These first appeared in the Transactions of the Royal Society of Edinburgh for 1838, in a paper by the author of the present work. For the computation of a table of logarithms, formula (9) affords remarkable facilities, even, giving the logarithms of the larger numbers without the use of any term in the series in the vinculum. Thus, as is remarked in the paper referred to, the logarithm of any whole number from 40 upwards would be obtained by means of this formula, true for seven or more places of decimals, merely by means of the logarithms of the two numbers immediately preceding and following it, without employing any of the terms in the last vinculum, and consequently *without any trouble with the modulus*. It is shown as an instance in the same place, that in this way the logarithm of 61 is obtained true to eight places of decimals from the logarithms of 60 and 62, without employing any terms of the series.

and the computation would not be laborious; but it would be lengthened on account of the convergence being slow, and its being necessary on this account to compute a large number of terms. This inconvenience will be avoided by taking $x = 2$ in formula (6), as we thus get

$$l3 = l2 + 2M\{\tfrac{1}{5} + \tfrac{1}{3}(\tfrac{1}{5})^3 + \tfrac{1}{5}(\tfrac{1}{5})^5 + \&c.\};$$

a series of rapid convergence. By taking, also, $x = 3$ in (7), we should have

$$l3 = \tfrac{1}{2}(l4 + l2) + M\{\tfrac{1}{17} + \tfrac{1}{3}(\tfrac{1}{17})^3 + \&c.\};$$

a series of very rapid convergence. It would not answer very well, however, in practice, as the number 17 is not easily managed in computation.

The logarithms of 2 and 3 being thus known, we should derive from them, without having recourse to series, the logarithms of 6, 12, 18, &c.; and in general, those of $2^m \times 3^n$. The logarithm of 5 may be easily obtained by taking x equal to 4 in (6), or equal to 5 in (7): and in similar ways the logarithms of 7, 11, 13, and the other prime numbers may be readily computed.

107. By taking $M = 1$ in the computations indicated in the last No. we should obtain the Neperian logarithms of 2, 3, 5, 10, &c.; and (No. 92) the important number, the modulus of the common logarithms would be found, by dividing 1 by the Neperian logarithm of 10, to be 0·434294481903, &c. Then (No. 91) the common logarithms would be obtained by multiplying the Neperian by this modulus; an operation which would be greatly facilitated by making a table of the products of that modulus by 2, 3, &c. up to 9. A much easier method, however, is given in the note to No. 106.

108. For computing sines, tangents, and other trigonometrical functions, the first thing to be done is to determine π, the remarkable number which expresses the length of the semi-circumference of a circle whose radius is unity. For effecting this, we must have the means of computing the length of some arc of which we know the exact relation to π, such as its fourth part, its sixth, or the like. This we may accomplish in the following manner, by means of the formula in No. 98. An arc of 30° is one sixth of one of 180°; and its sine, being half the chord of 60°, is $\frac{1}{2}$. Tak-

ing, therefore, $x=\frac{1}{2}$, and $\sin^{-1}x=\frac{1}{6}\pi$ in the series in that No. we get

$$\tfrac{1}{6}\pi=\frac{1}{2}+\frac{1^2(\frac{1}{2})^3}{1.2.3}+\frac{1^2.3^2(\frac{1}{2})^5}{1.2.3.4.5}+\frac{1^2.3^2.5^2(\frac{1}{2})^7}{1.2.3.4.5.6.7}+\&c.\text{; or,}$$

$$\tfrac{1}{6}\pi=\frac{1}{2}+\frac{\frac{1}{4}t_1}{2.3}+\frac{\frac{1}{4}\times 3^2t_2}{4.5}+\frac{\frac{1}{4}\times 5^2t_3}{6.7}+\frac{\frac{1}{4}\times 7^2t_4}{8.9}+\&c.:$$

where t_1, t_2, t_3,...... denote respectively the first, second, third,...... terms. The multipliers attached to these are found by dividing the second term by the first, the third by the second, &c.; and $\frac{1}{4}$, the square of $\frac{1}{2}$, is a factor in each. Hence, by multiplying by 6, we obtain

$$\pi=3+\frac{\frac{1}{4}t_1}{2.3}+\frac{\frac{1}{4}\times 3^2t_2}{4.5}+\frac{\frac{1}{4}\times 5^2t_3}{6.7}+\frac{\frac{1}{4}\times 7^2t_4}{8.9}+\&c.;$$

where t_1, t_2, &c. denote the first, second, &c. terms of the newly obtained series. By means of this series π may be easily computed to eight or ten places of figures. For finding it true for many figures, however, the labour would be great, both because larger numbers would require to be employed and because the rate of convergence would be diminished, each term tending to become more and more nearly one fourth of the one preceding it.

The value of π, however, is computed with much more facility, by means of the principles established in No. 99, as will appear from the note below.

The value of this number, at the expense of much time and labour uselessly employed, has been computed to upwards of 200 places of figures. The first sixteen are 3·14159,26535,89793; and these are sufficient for all useful purposes.*

* The computation of π for a few figures, by means of the series given in the text, will stand as in the margin. In this process t_2 is obtained by taking one fourth of t_1 and dividing it by 6 (= 2.3); t_3 by taking one fourth of t_2, multiplying it by 9 (= 3^2), and dividing the product by 20 (= 4.5), &c. according to the series. The result found is true in all its figures except the last. It is plain that were the decimals carried out to more figures, and the work to more terms, the labour would be greatly increased.

$t_1=$	3·0000000
$t_2=$	1250000
$t_3=$	140625
$t_4=$	20926
$t_5=$	3560
$t_6=$	655
$t_7=$	127
$t_8=$	26
$t_9=$	5
$t_{10}=$	1
$\pi=$	3·1415925

To show the great facilities afforded by the principles established in No. 99, for

109. We may now consider the mode of computing tables of sines, tangents, &c.; which we are enabled to do by means of the principles established in this Section. If A be an arc expressed in degrees and parts of a degree, we

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calculating the value of π, it may not be improper to subjoin some illustrations taken from the paper referred to in the note in p. 64.

Even in the very unfavourable case in which $x=1$, and the arc $=45^\circ$, we should have from the first series above found, by multiplying by 4,

$$\pi=4\left\{\frac{1}{2}+\frac{2}{3}\left(\frac{1}{2}\right)^2+\frac{2.4}{3.5}\left(\frac{1}{2}\right)^3+\&c.\right\};$$

— a series such that less than the first twenty terms of it would give the circumference true for six places of decimals; while thousands of terms of the series,

$$\pi=4\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\&c.\right),$$

derived from Mercator's series given in No. 97, would be required to effect the same object. "In the actual computation, however, of the circumference to a great degree of accuracy, the latter of the series found above is applied with most advantage in connexion with the curious and elegant principle first employed by Machin, and afterwards extended by Euler—that of finding arcs whose tangents are rational, and are small known fractions, and the sum or difference of which, or of their multiples, is a known part of the circumference. Such arcs are innumerable; and by taking them sufficiently small, any degree of convergence whatever may be obtained. Rapidity of convergence, however, is far from being the sole important consideration. The convergence may be very great, and yet the fraction k may be of such a form as to render the computation laborious and difficult. No arc, indeed, answers well, unless p^2+q^2 be of the form $10^m\times 2^{-n}$, m and n being whole positive numbers; and even of arcs having this property many are in other respects inconvenient. Of a great number of tangents which I have tried, those which seem to answer best are $\frac{1}{3}$, $\frac{2}{11}$, $\frac{1}{7}$, and $\frac{3}{79}$, which give respectively for the value of k, 0·1, 0·032, 0·02, and 0·00144; and, since it is easy to show that $3\tan^{-1}\frac{1}{3}-\tan^{-1}\frac{2}{11}=\frac{1}{4}\pi$, we get, by quadrupling,

$$\pi=12\tan^{-1}\tfrac{1}{3}-4\tan^{-1}\tfrac{2}{11}.$$

In a similar manner, it would appear that

$$\pi=8\tan^{-1}\tfrac{1}{3}+4\tan^{-1}\tfrac{1}{7};\qquad \pi=10\tan^{-1}\tfrac{1}{3}-2\tan^{-1}\tfrac{3}{79};$$

$$\pi=8\tan^{-1}\tfrac{2}{11}+12\tan^{-1}\tfrac{1}{7};\qquad \pi=20\tan^{-1}\tfrac{2}{11}-12\tan^{-1}\tfrac{3}{79};$$

$$\text{and } \pi=20\tan^{-1}\tfrac{1}{7}+8\tan^{-1}\tfrac{3}{79}.$$

"By means of any of these six formulas in connexion with the second series in the text, the value of π may be computed with great facility and despatch. As an example, let us take the last of them; and putting successively in that series $p=1$, $q=7$; and $p=3$, $q=79$, we get

$$\pi=\left\{\begin{array}{l}2{\cdot}8+\frac{2}{3}t_1\times 0{\cdot}02+\frac{4}{5}t_2\times 0{\cdot}02+\frac{6}{7}t_3\times 0{\cdot}02+\&c.\\ +\,0{\cdot}30336+\frac{2}{3}t'_1\times 0{\cdot}00144+\frac{4}{5}t'_2\times 0{\cdot}00144,\ \&c.\end{array}\right\}.$$

Hence, by carrying the decimals out to twelve places, the computation will stand thus:

First Series.

$t_1=$	2·800000000000
$t_2=$	37333333333
$t_3=$	597333333
$t_4=$	10240000
$t_5=$	182044
$t_6=$	3310
$t_7=$	61
$20\tan^{-1}\frac{1}{7}=$	2·837941092081

Second Series.

$t_1=$	0·303360000000
$t_2=$	291225600
$t_3=$	335492
$t_4=$	414
$8\tan^{-1}\frac{3}{79}=$	0·303651561506
$20\tan^{-1}\frac{1}{7}=$	2·837941092081
$\pi=$	3·141592653587

have 180° to A, as 3·14159, &c. to the length of the arc.* Then, by the substitution of this in the series in No. 95, the sine and cosine of the arc will be obtained: and, to find the tangent divide the sine by the cosine; to find the cotangent, divide the cosine by the sine, or 1 by the tangent; to find the secant, divide 1 by the cosine; and to find the cosecant, divide 1 by the sine.

110. The labour of computation may be abridged by

" This value of π is true in all its figures except the last. The computation of the terms is effected with great ease. Thus, in the first series, t_2 is found by doubling t_1, subtracting from the result one third of itself, and rejecting the last two figures; t_3 by doubling t_2, taking from the result one fifth of itself, and rejecting two figures, and so on: while, in the second series, t_2 is found by multiplying t_1 by 144 (which is easy on account of the repetition of the figure 4), by taking from the result one third of itself, and rejecting five figures; and in both the computations various arithmetical contractions will suggest themselves as the work proceeds."

The tangents proper to be tried are readily found, and it is easy to combine the arcs corresponding to them to get π. Thus, if we assume one of them as being $\frac{1}{2}$, for which k is of the proper form, being $\frac{1}{5}$ or $10^1 \times 2^{-1}$, we find (TRIG. No. 30) that whatever $\tan^{-1}\frac{1}{2}$ may be, the tangent of the difference between that arc and $\frac{1}{4}\pi$ is $\frac{1}{3}$; and therefore $\pi = 4(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3})$. Again (TRIG. No. 32) if the tangent of an arc be $\frac{1}{3}$, the tangent of its double is $\frac{3}{4}$, and therefore (TRIG. No. 30) the difference between $\frac{1}{4}\pi$ and $2\tan^{-1}\frac{1}{3}$ is $\tan^{-1}\frac{1}{7}$; whence $\pi = 8\tan^{-1}\frac{1}{3} + 4\tan^{-1}\frac{1}{7}$. As a third example, if we assume $\tan\theta = \frac{1}{3}$, we readily find that $\tan 5\theta = -\frac{79}{3}$ or $\cot(\frac{1}{2}\pi - 5\theta) = -\frac{79}{3}$. Hence, we have $\tan(\frac{1}{2}\pi - 5\theta) = -\frac{3}{79}$, and therefore, $\tan(5\theta - \frac{1}{2}\pi) = \frac{3}{79}$; whence $5\theta - \frac{1}{2}\pi = \tan^{-1}\frac{3}{79}$, or $5\tan^{-1}\frac{1}{3} - \frac{1}{2}\pi = \tan^{-1}\frac{3}{79}$; and therefore by transposition, and by doubling, we get $\pi = 10\tan^{-1}\frac{1}{3} - 2\tan^{-1}\frac{3}{79}$.

It may be remarked in conclusion, that the first series in No. 99 may be reduced (ALG. Chap. XII.) to the curious continued fraction $\tan^{-1}x =$

$$\cfrac{x}{1+x^2 - \cfrac{1.2\,x^2}{3 - \cfrac{1.2\,x^2}{5(1+x^2) - \cfrac{3.4\,x^2}{7 - \cfrac{3.4\,x^2}{9(1+x^2) - \cfrac{5.6\,x^2}{11 - \cfrac{5.6\,x^2}{13(1+x^2) - \&c.}}}}}}}$$

* The finding of the lengths of arcs will be facilitated by the following table, showing the lengths of various arcs to the radius 1:

1′	·00029089	9′	·00261799	1°	·01745329	10°	·17453293
2	·00058178	10″	·00290888	2	·03490659	20	·34906585
3	·00087266	15	·00436332	3	·05235988	30	·52359878
4	·00116355	20	·00581776	4	·06981317	40	·69813170
5	·00145444	30	·00872665	5	·08726646	50	·87266463
6	·00174533	40	·01163553	6	·10471976	60	1·04719755
7	·00203622	45	·01308997	7	·12217305	70	1·22173048
8	·00232711	50	·01454441	8	·13962634	80	1·39626340
				9	·15707963	90	1·57079633

As an example of the use of this table, let it be required to find the length of an arc of 23° 28′. To do this, we take from the table the lengths of 20°, 3°, 20′, and 8′; and, by adding them together, we get the required length = ·34906585 + ·05235988 + ·00581776 + ·00232711 = ·40957060.

several artifices, some of which are the following. Of the three quantities,

$\sin A \cos B - \cos A \sin B$, $\sin A$, and $\sin A \cos B + \cos A \sin B$,

the first and last (Trig. No. 22, page 9) are equal to $\sin(A-B)$ and $\sin(A+B)$. Take the first from the second, and the second from the third; then, take the second remainder from the first, and there will be obtained $2 \sin A (1 - \cos B)$, or $2 \sin A$ versin B. Hence, to find $\sin(A+B)$, to $\sin A$ add the difference of $\sin A$ and $\sin(A-B)$, and from the sum take the product of $\sin A$ and twice versin B. If B be taken a constant quantity, such as $1'$, this theorem will render the computation very simple and easy.

111. When the sines have been calculated up to $30°$, those that follow up to $60°$ may be computed by means of the formula,

$$\sin(30° + B) = \sin(30° - B) + \sin B \sqrt{3},$$

found from formula (17), Trig. page 10, by taking $A = 30°$, and consequently (Trig. No. 41) $\cos A = \frac{1}{2}\sqrt{3}$. Thus, we should have $\sin 37° = \sin 23° + \sin 7° \times \sqrt{3}$.

112. When the sines up to $60°$ have been found, the rest may be obtained merely by addition, by means of the following formula (Trig. No. 46): $\sin(60° + A) = \sin(60° - A) + \sin A$.

The formula (Trig. No. 47),

$$\sin(54° + A) + \sin(54° - A) - \sin(18° + A) - \sin(18° - A) = \cos A,$$

affords an easy means of trying the accuracy of the results obtained.

113. When the tangents are computed up to $45°$, the rest may be found by addition by means of the formula (Trig. No. 39),

$$\tan(45° + A) = \tan(45° - A) + 2 \tan 2A.$$

Thus, if $A = 5°$, we have

$$\tan 50° = \tan 40° + 2 \tan 10°.$$

114. The entire table of secants may be found by addition, by means of the formula (Trig. No. 35), $\sec A = \tan A + \tan(45° - \frac{1}{2}A)$.

115. With respect to the logarithmic sines, tangents, and

secants, series may be found which will give them directly. No series, however, that have yet been discovered for this purpose, are of much use in practice; and it is in general preferable, after the natural sines, tangents, and secants, have been computed, to find their logarithms. This may be done by means of proportional parts, or, with more correctness, by series (5), No. 89.* The radius for the logarithmic sines, &c. is taken, for ease in calculation, as 10,000,000,000, or such that its index may be 10; and thus the index is, in every case, greater by 10 than it would be for the radius 1.

116. It is evident, from the nature of logarithms, and from Trig. No. 11, that the logarithmic tangent will be found by adding 10 to the logarithmic sine, and subtracting the logarithmic cosine from the sum; and that the logarithmic secant will be found by subtracting the logarithmic cosine from 20.

EXERCISES.

1. From ·69314718, the Neperian logarithm of 2, find that of 128; and thence, by series (5), No. 89, that of 131.

   Answ. 4·8520303, and 4·8751973.

2. Compute the sine and cosine, natural and logarithmic, of $8^\circ\ 6'$, true for nine places of decimals.

   Answ. Nat. sine = ·140901232; logarithmic sine = 9·148914790; nat. cosine = ·990023658; log. cosine = 9·995645573.

3. Find what arc is equal in length to the radius, and compute its sine, cosine, and tangent.

   Answ. The arc is $57^\circ\ 17'\ 44''\cdot 8$, or 206264·8 seconds; its sine is 0·8414710, its cosine 0·5403023, and its tangent 1·5574077.

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* Thus, the natural tangent of 13° is ·2308682. Then, by Hutton's tables, the logarithm of 23086, without the index, is ·3633487; and by the *differences* given in those tables, the proportional part for 80 is 151, and that for 2 is 4. The sum of these three is ·3633642, which, with the index 9 prefixed, is the logarithmic tangent of 13°, true except the last figure, which should be 1. The required result might also be found from the series above referred to, by taking $x = 2308600$ and $n = 82$; and, for the degree of accuracy here required, only the first term of the series is necessary.

In the actual computation of tables, several abbreviations, particularly by the method of differences, may be employed; and the calculator may avail himself of various artifices and checks, which it would be inconsistent with the nature of this work to explain.

4. Given the common logarithm of $900 = 2 \cdot 954242509439$, and that of $902 = 2 \cdot 955206537542$, to find that of 901, by means of formula (9), No. 90.
Answ. 2·954724790979.

5. Compute ε^2, by means of the series for ε^x.
Answ. 7·389056.

VI.—ON SINGULAR VALUES OF FUNCTIONS.

117. While, in general, the values of explicit functions can be determined by carrying out, in the ordinary manner, the operations indicated, yet it not unfrequently happens, that such a value may be assigned to the variable as shall render it impossible to ascertain the value of a function in this way. Thus, if $u = \frac{\frac{1}{4}\pi^2 - x^2 \sin x}{\frac{1}{2}\pi - x}$, the value of u when x is given (suppose 20°, 50°, &c.) would in every case except one, be determined by finding, in the manner that will be shown hereafter, the length of x, and substituting it and the natural sine of x in the second member. The case of exception is that in which $x = \frac{1}{2}\pi$, which gives u equal to $\frac{0}{0}$, the value of which cannot be determined in the ordinary way. Any such value is termed a *singular* one, and it is plainly the limit to which the value of the function tends, as the variable approaches more and more nearly to the value which makes the function take the singular value. We may now investigate the method of finding the values of singular expressions, commencing with those of the form $\frac{0}{0}$.

118. Let $\frac{fx}{\varphi x}$ be a fraction, the value of which can be computed by ordinary means, except when x takes the particular value a, but which in that case becomes $\frac{0}{0}$, the numerator and denominator both vanishing. To get the value in this case expressed in a different and a manageable form, we may change x into $x + h$, and having obtained developments of the numerator and denominator, by means

of Taylor's theorem, we shall revert to the singular value by taking $fx = 0$, $\varphi x = 0$, $x = a$, and $h = 0$. In this way we first obtain

$$\frac{f(x+h)}{\varphi(x+h)} = \frac{fx + hf^1x + \frac{1}{6}h^2f^2x + \frac{1}{24}h^3f^3x + \&c.}{\varphi x + h\varphi^1x + \frac{1}{6}h^2\varphi^2x + \frac{1}{24}h^3\varphi^3x + \&c.}$$

Then, by the hypothesis, if x be taken equal to a, the first terms fx and φx vanish. Omitting them, therefore, dividing the remaining terms by h, and in the result taking $h = 0$, we get simply $\frac{f^1a}{\varphi^1a}$, the value of the fraction.

Hence it appears, that to find the required value, we are to use instead of the terms of the original fraction, their differential coefficients, and in them to take $x = a$. Should the result thus obtained be itself $\frac{0}{0}$, it is plain that it may be treated in a similar manner: and thus we may proceed by successive differentiations, till a fraction is found which has either one of its terms, or both of them, of finite magnitude.

119. To exemplify the use of the principle now established, let us resume the fraction $\frac{\frac{1}{4}\pi^2 - x^2\sin x}{\frac{1}{2}\pi - x}$, which, as we have seen, becomes $\frac{0}{0}$, when $x = \frac{1}{2}\pi$. To determine its value in this particular case, we take the differential coefficients of the numerator and the denominator, which are $-2x\sin x - x^2\cos x$ and -1; and, by dividing the former by the latter, and taking $x = \frac{1}{2}\pi$ in the result, we get simply π, the required value.

120. It is plain (from No. 85) that the principle established in No. 118 will hold true only for such values of x as do not render any of the differential coefficients of fx and φx infinite. When such coefficients occur, we may substitute $a + h$ for x, and if necessary, develop the terms of the fraction so obtained, in ascending powers of h: then, by dividing by the lowest power of h in the terms of the result, and taking h equal to nothing in what is thus obtained, we shall find the required value.

Thus, for instance, the fraction $\frac{(x-a)^{\frac{3}{2}}}{(x^2-a^2)^{\frac{4}{3}}}$, becomes $\frac{0}{0}$ when $x=a$, and so does its first differential coefficient; but all its succeeding differential coefficients become infinite on the same supposition. To find its value, then, let x be changed into $a+h$. The fraction will thus become

$$\frac{h^{\frac{3}{2}}}{(2ah+h^2)^{\frac{4}{3}}}=\frac{h^{\frac{3}{2}}}{h^{\frac{4}{3}}(2a+h)^{\frac{4}{3}}}=\frac{h^{\frac{1}{6}}}{(2a+h)^{\frac{4}{3}}};$$

a quantity which becomes nothing when $h=0$; and therefore the value of the original fraction tends to zero as its limit, when x tends to become equal to a.

121. Should a fraction take the form $\frac{\infty}{\infty}$, we may divide each of its terms by the product of both. By this means we get an equivalent fraction having for numerator the reciprocal of the given denominator, and for denominator the reciprocal of the given numerator; and as it will then be of the form $\frac{0}{0}$, its value may be found by one or other of the modes pointed out above. It will be more easily obtained, however, when the differential calculus is admissible, by simply using, as in No. 118, instead of the given numerator and denominator, their differential coefficients. To prove that this may be done, let $u=\frac{fx}{\varphi x}$. Then, by dividing both terms by $fx\varphi x$, and taking the differential coefficients of the quotients, we get $-\frac{\varphi^1 x}{(\varphi x)^2}$, and $-\frac{f^1 x}{(fx)^2}$; and, by No. 118, if the former be divided by the latter, the quotient must be the value of u, that is, $\frac{\varphi^1 x}{f^1 x}.\frac{(fx)^2}{(\varphi x)^2}=u$. If in this we substitute u^2 for its equivalent, the second factor of the first member, and if we multiply the members of the result by $f^1 x$, and divide the products by u and $\varphi^1 x$, there will be obtained $u=\frac{f^1 x}{\varphi^1 x}$; which proves the rule.

122. The singular form, $0\times\infty$, such as $\sin x \operatorname{cosec} nx$, when $x=0$, will be reduced to the form $\frac{0}{0}$ by taking the

first factor as the numerator, and the reciprocal of the second as the denominator of a fraction; or, to the form $\frac{\infty}{\infty}$, by taking the second factor as the numerator, and the reciprocal of the first as the denominator.

123. When singular forms of an exponential kind, such as 0^0, 0^∞, 1^∞, or $1^{-\infty}$, occur, the logarithms of the general expressions may be taken, and the solution will then be obtained by some of the methods already given. Thus, if $u=x^x$, which becomes 0^0, when $x=0$, be proposed, we have $\log u = x \log x$, which becomes $0 \times -\infty$, when $x=0$. This is changed into $\frac{-\infty}{\infty}$, by taking $\log x$ as numerator, and x^{-1} as denominator; and, by means of No. 121, we find $\log u = 0$; and therefore we have u itself, or $x^x=1$, when $x=0$.

EXERCISES.

Prove the following results:

1. $\dfrac{x^5-a^5}{x^3-a^3}=\dfrac{5}{3}a^2$, when $x=a$.

2. $\dfrac{6x^2-17x+12}{8x^2-18x+9}=\dfrac{1}{6}$, when $x=\dfrac{3}{2}$.

3. $\dfrac{\varepsilon^x-\varepsilon^{-x}}{\log(1+x)}=2$, when $x=0$.

4. $\dfrac{a^x-b^x}{x}=\log\dfrac{a}{b}$, when $x=0$.

5. $\dfrac{a^{\sin x}-a}{\log \sin x}=a\log a$, when $x=\frac{1}{2}\pi$.

6. $\dfrac{1}{2^n}\cot\dfrac{A}{2^n}=\dfrac{1}{A}$, when $n=\infty$.

7. $\dfrac{x^2\tan x}{1+\tan x}=(\frac{1}{2}\pi)^2$, when $x=\frac{1}{2}\pi$.

8. $\dfrac{\sec x}{x}-\cot x$* $=0$, when $x=0$.

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* By actual subtraction this may be put under the form,

$$\frac{\sin x \sec x - x\cos x}{x \sin x}, \text{ or } \frac{\tan x - x\cos x}{x\sin x};$$

which becomes $\frac{0}{0}$, when $x=0$: and the like may be done in other cases in which a quantity becomes $\infty - \infty$.

9. $\dfrac{x-\sqrt{(2x^2-a^2)}}{2x-\sqrt{(5x^2-a^2)}}=2$, when $x=a$.

10. $\dfrac{a^x}{x}=\infty$, when $x=\infty$, if $a>1$.

11. $\dfrac{\log x}{x}=0$, if $x=\infty$.

12. $x\log x=0$, if $x=0$.

13. $x\,\varepsilon^{\frac{1}{x}}=\infty$, if $x=0$.

14. $x^{\frac{1}{x}}=1$, if $x=\infty$.

15. $(\cos nx)^{\frac{1}{x}}=1$, if $x=0$.

16. $(1+x)^{\frac{1}{x}}=\varepsilon$, if $x=0$.

VII.—ON MAXIMUMS AND MINIMUMS.*

124. IF, while a variable quantity constantly increases, a function of it first increase and then diminish, the value of the function at the end of the increase, and before it has begun to diminish, is called a *maximum;* but, if the function first decrease and then increase, its value, at the end of the decrease, is said to be a *minimum.* The height of the tide affords a familiar instance; as, while the time advances continually, the height of the tide alternately increases and diminishes, being a maximum at the end of each increase, and a minimum at the end of each decrease. In like manner, while a circular arc continually increases, its sine increases till the arc becomes $\frac{1}{2}\pi$; it then diminishes till the arc takes the value $\frac{3}{2}\pi$, when it again begins to increase, and continues to do so till the arc becomes $\frac{5}{2}\pi$: and thus it alternately increases and decreases, as the arc goes on increasing. The *essential characteristic* of the maximum is, that it is *greater*, and that of the minimum that it is *less*, than the values which *immediately* precede and follow it; as a function may have several successive increases and diminutions, and may consequently have several maximums and minimums, which may be either equal or unequal among themselves. This will be exemplified by the ordinates of a waving curve, by the sine of a variable arc, or by the elevation of the tides referred to above, which all admit of several maximums and minimums. Some functions, on the other hand, continually increase or diminish,

* To avoid unnecessary irregularity in words adopted into the English language, *maximums* and *minimums* are here employed, instead of the more usual Latin plurals *maxima* and *minima*, in the same manner in which we form the plurals of *premium, interregnum, encomium*, &c. The same is done in French, for a similar reason, by Lacroix. On the same principle, *formulas* is used instead of *formulæ*.

as the variable on which they depend increases; and, therefore, they have neither maximum nor minimum values, in the sense explained above. Such are the logarithm of a number, and the quotient of a constant quantity, divided by a variable one.

125. The following lemma will enable us to establish in a very easy way the method of determining the maximum and minimum values of functions:

If, while a variable quantity increases, a function of it likewise increase, the differential coefficient of the function is positive, but if the function decrease, its differential coefficient is negative. For (No. 6) the differential coefficient of fx is the limit to which the value of the fraction $\frac{f(x+h)-fx}{h}$, continually approaches, as h is diminished towards evanescence; and this ratio, and consequently the limit to which it tends, will be positive or negative, according as $f(x+h)$ is greater or less than fx; that is, according as the function is increased or diminished by the addition of h to the variable x; which proves the proposition.

It is plain also that *the same relation will exist between any differential coefficient and the one immediately following it;* the latter being merely the differential coefficient of the former.

This lemma may be exemplified by means of particular instances. Thus, we have $f^1 \sin x = \cos x$. Now, as x increases from zero upwards, $\sin x$ constantly increases, till x reaches the value $\frac{1}{2}\pi$, and at the same time $\cos x$ is positive. When, however, the value of x passes $\frac{1}{2}\pi$, the sine begins to diminish, and the cosine at the same time becomes negative. The functions $x^{-1}, \cos x$, and ε^x will also afford simple illustrations.

126. From what was established in the last No., and from the definitions of maximum and minimum values, it follows, that *the differential coefficient of fx changes its sign when that function becomes a maximum or a minimum;* fx first increasing and then diminishing in the one case, and first diminishing and then increasing in the other. Now, a varying quantity can change its sign only in becoming nothing, or infinite;* and hence fx can be a maximum or minimum only when $f^1x=0$, or $f^1x=\infty$.

* Thus, the sine of x is positive, while x increases from zero up to π; but when x passes π, its sine becomes negative, and it is nothing when $x=\pi$, the value of x

Let us proceed to consider the case in which $f^1x=0$, which is generally the one that is to be employed. Now, in this case, with an exception to be considered afterwards, *if fx be a maximum, f^2x will be negative; but, in case of a minimum it will be positive.* This follows from the last No.; since after passing a maximum value, the function begins to diminish, and will therefore have the next differential coefficient negative; while, in case of fx being a minimum, the contrary will take place.

The exception referred to is, that f^2x, instead of being positive or negative, may be zero, as well as f^1x. It would appear by similar reasoning, that, when this occurs, f^3x must be nothing, if fx be either a maximum or minimum: and f^4x, unless it vanish, will be negative in case of a maximum, and positive in case of a minimum. Should f^4x vanish, like reasoning would still be admissible; and it would be seen, that, universally, fx will have a maximum or minimum value, if an *odd* number of differential coefficients, commencing with the first, vanish for one or more values of x; and that fx will be a maximum or minimum, according as the first of the remaining coefficients is negative or positive.

127. Hence we have the following rule for determining the maximum and minimum values of a function fx:—Find f^1x, and determine the real value or values of x in the equation, $f^1x=0$: then any such value which, when substituted for x in f^2x gives a negative result, will make fx a maximum; but if it give a positive result it will make that function a minimum.* If, however, f^2x should vanish for that value of x, find additional differential coefficients, and in them substitute the same value: then if the first of those coefficients that does not vanish, be of an even order, such as f^4x, f^6x, &c. the value of x that produces it will render

which marks where the positive values end and the negative ones commence. The tangent of x affords an instance of the second kind, changing its sign "in passing through infinity," when $x=\frac{1}{2}\pi$, being positive immediately before x attains that value, and negative immediately after. In like manner, $a-x$ is positive, when $x<a$; negative, when $x>a$, and zero, when $x=a$; and its reciprocal is respectively positive, negative, and infinite in the same circumstances. We must not, however, fall into the error of supposing the converse proposition to be always true; as a varying quantity may pass through zero or infinity without changing its sign. As an example, the function $(a-x)^2$ and its reciprocal become respectively nothing and infinite, when $x=a$, and are positive in all other cases.

* In many cases it is readily seen from the nature of the function, whether it will be a maximum or a minimum value that it can have; and in such cases, it is generally unnecessary to work for any of the differential coefficients after the first.

fx a maximum, if that coefficient, f^4x, &c. be negative; otherwise, fx will be a minimum.

128. As an example, let it be required to determine x, so that $fx = 2x^3 - 15x^2 + 24x + 25$ may be a maximum or minimum. Here we get

$$f^1x = 6x^2 - 30x + 24, \qquad \text{and } f^2x = 12x - 30:$$

and, by putting the former equal to zero, and resolving the equation so obtained, we get $x = 1$, and $x = 4$. Then, by the successive substitutions of these in f^2x, we get -18 and 18; and therefore fx is a maximum when $x = 1$, and a minimum when $x = 4$. The first value of x gives 9 as the value of fx, while the second gives 36.*

129. As another example, let $fx = x^4 + 4x^3 + 10$. Here

$$f^1x = 4x^3 + 12x^2, \qquad \text{and } f^2x = 12x^2 + 24x.$$

Hence putting $4x^3 + 12x^2 = 0$, we get as the three roots of the equation, $x = 0$, $x = 0$, and $x = -3$. The last of these makes $f^2x = 36$; and, this being positive, fx will be a minimum when $x = -3$. Either of the other roots, however, makes $f^2x = 0$; and this being neither positive nor negative, we must differentiate again. By this means we get $f^3x = 24x + 24$; and as this, which is a coefficient of an odd order, does not vanish when $x = 0$, that value of x will not render fx either a maximum or minimum. The function, therefore, has no maximum value; and it has but one minimum value, which is given by $x = -3$, and is found to be -17.

130. In many cases, operations of the kind we are now considering may be much facilitated by means of some obvious expedients. Some of the principles on which these depend are the following:—

(1.) A constant multiplier or divisor may be omitted.

* The student will find it advantageous to illustrate some of the examples by means of curves. Thus, if, in the present instance, he trace the curve which has $y = 2x^3 - 15x^2 + 24x + 25$ as its equation, he will find that the ordinate corresponding to $x = 1$, will be greater than those immediately preceding and following it; while, on the contrary, the ordinate corresponding to $x = 4$, will be less than those on each side of it in its immediate vicinity. The same may be illustrated by means of numbers. Thus, if $x = 0, fx = 25$; and if $x = 2, fx = 29$; each of which is less than 36, the value given by $x = 1$: while, if x be taken successively equal to 3 and 5, fx is found to be 16 and 20; each of which is greater than 9, the value corresponding to $x = 4$. To gain a full and accurate knowledge of the subject in this and other instances, he ought to examine the signs of f^1x. In the present instance, using for simplicity one sixth of f^1x, a change which does not affect the signs, we have $x^2 - 5x + 4$, or $(x-1)(x-4)$; which will be positive when x is a little less than 1, and negative, when a little greater; and negative, when x is a little less than 4, and positive, when a little greater: the results agreeing accurately with what was stated in No. 126.

Thus, if fx be a maximum, its double, its half, &c. will be greater than the doubles, halves, &c. of the values of fx immediately preceding or following. A constant quantity connected by addition or subtraction may also be rejected.

(2.) If a positive quantity be a maximum or minimum, any power of it will be the same among the like powers of the other values of the same quantity: but if a negative quantity be a maximum or minimum, its odd powers will be the same, but its even ones the opposite.

(3.) The reciprocal of a maximum is a minimum, and that of a minimum is a maximum.

(4.) In case of fractions or products, logarithms may sometimes be used with advantage.

131. To exemplify the use of these principles, let it be required to make $2+\sqrt{(10x^3-15x^2-60x+40)}$ a maximum or minimum. By rejecting 2, squaring the remainder, and dividing by 5, we get $2x^3-3x^2-12x+8$; which, if we call it fx, gives $f^1x=6x^2-6x-12$, and $f^2x=12x-6$. From the first of these, when put $=0$, we get $x=2$, and $x=-1$; the first of which gives f^2x positive, and the second negative. Hence fx is a maximum, when $x=-1$; and a minimum, when $x=2$; its value for the first being 15, and for the second -12. By substituting, however, the same values of x in the given function, we get as its values $2+\sqrt{75}$ and $2+\sqrt{-60}$; the latter of which, being imaginary, is inadmissible. The proposed function, therefore, is a maximum, when $x=-1$, but it has no other maximum or minimum value.

132. As another example, let the function,

$$u=\sqrt{(x^4-32x+48)}-x^2 \dots\dots(1),$$

be proposed. As u is here an explicit function of x, the investigation might proceed according to No. 127; and the student may find it useful to get out the solution in that way. The following process, however, is perhaps preferable;—it is, at least, more instructive. By transposing x^2, squaring, contracting, &c. we obtain

$$u^2+2x^2u+32x-48=0 \dots\dots(2):$$

and hence, by differentiating, &c. we get

$$2u\frac{du}{dx}+2x^2\frac{du}{dx}+4xu+32=0 \dots\dots(3).$$

From this, by taking (No. 126) $\frac{du}{dx}=0$ in the terms in which it occurs, we obtain $4xu+32=0$, and consequently $xu=-8$; one condition which must be satisfied for determining a maximum or minimum value of u, if there be such. By finding from this an expression for u, and substituting it in (2) we get, after some easy modifications, $x^3-3x^2+4=0$; an equation, the resolution of which gives $x=-1$, $x=2$, and $x=2$; and, corresponding to $x=-1$, we have $u=8$; while $x=2$ gives $u=-4$. Now, by differentiating (3), omitting everywhere $\frac{du}{dx}$, as it is equal to zero, we get

$$2u\frac{d^2u}{dx^2}+2x^2\frac{d^2u}{dx^2}+4u=0;\text{ whence }\frac{d^2u}{dx^2}=-\frac{2u}{u+x^2}.$$

By taking in this $x=-1$, and $u=8$, we get $-\frac{16}{9}$; a result which shows that when $x=-1$, the value of u (8) is a maximum. If, however, we take $x=2$, and $u=-4$, f^2x becomes infinite, and so likewise would all the succeeding differential coefficients. Hence, the ordinary rule fails in showing whether u is a maximum, or minimum, or neither, when $x=2$. When, as in the present instance, any special cases, not manageable by the ordinary means, present themselves, we must have recourse to other expedients for enabling us to arrive at what is required. Thus, in the present case, we may examine whether the first differential coefficient changes its sign (see No. 126) when x passes through 2, the value in question. Now, from (3) and (1) we get

$$\frac{du}{dx}=-\frac{2xu+16}{u+x^2}=-\frac{2x\sqrt{(x^4-32x+48)}-2x^3+16}{\sqrt{(x^4-32x+48)}}.$$

This, however, becomes $\frac{0}{0}$, where $x=2$; so that, in its present form we cannot determine its sign. Modifying it, however, and, for simplicity, taking its half, we get

$$-x+\frac{x^3-8}{\sqrt{(x^4-32x+48)}}.$$

Then, (No. 120) substituting $2-h$ and $2+h$ for x, we obtain, after some easy modifications,

$$-2+h+\frac{-12+6h-h^2}{\sqrt{(24-8h+h^2)}}\text{ and }-2-h+\frac{12+6h+h^2}{\sqrt{(24+8h+h^2)}}.$$

If in these the radical be taken positive, and h be diminished down towards zero, it is easy to see, that the first will tend to become negative, and the second positive; but that if the radical be taken negative, the first will become positive and the second negative. Hence (No. 126) when the radical is positive, u is a minimum; but when negative, it is a maximum.*

133. As another example, let it be required to divide a given number a into a number of equal parts, such that their continual product may be a maximum. Here, if x be the number of parts, each of them will be $\frac{a}{x}$, and we shall have $u=\left(\frac{a}{x}\right)^x$. Hence, $\log u = x\log a - x\log x$: and, therefore, by differentiating, and by multiplying by u, we shall have

$$f^1x = u(\log a - \log x - 1) = u(\log a - \log x - \log \varepsilon).$$

Putting this equal to zero, we get

$$\log x = \log a - \log \varepsilon = \log\frac{a}{\varepsilon}; \text{ and, therefore, } x=\frac{a}{\varepsilon};$$

so that each of the parts will be ε. Hence it is plain that the parts† cannot all be "equal," except when a is a multiple of ε; and thus we see the impropriety of the foregoing enunciation, though it is the same in substance that has been adopted by several writers. The following enunciation would be correct:—To find a number by which, if a given number be divided, that power of the quotient which has the required number as its index, shall be a maximum.

134. We may now consider the functions, $fx=b+c(x-a)^2$, $fx=b+c(x-a)^3$, and $fx=b+c(x-a)^4$. The first of these gives $f^1x=2c(x-a)$, and $f^2x=1.2c$. Hence, therefore, (No. 126) fx will be a maximum or minimum when $x=a$, according as c is negative or positive.

In the second function, we have $f^1x=3c(x-a)^2$, $f^2x=2.3c(x-a)$, and $f^3x=1.2.3c$; and, therefore, (No. 126) fx will have no maximum or minimum value.

* It is worth remarking, that for -1, the other value of x, there is not the variety which we have here; -10, one of the values of u, when $x=-1$, being inadmissible in reference to a maximum or minimum, as it does not satisfy the condition, $xu=-8$.

This rather interesting function will be well illustrated by means of a curve having x as abscissa, and u as ordinate.

† When a does not exceed ϵ, such as when it is ϵ, 1, $\frac{1}{2}$, 2, &c. there can be *no* "parts" in the strict sense of the term.

In the next function, we have $f^1x = 4c(x-a)^3$, $f^2x = 3.4c(x-a)^2$, $f^3x = 2.3.4c(x-a)$, and $f^4x = 1.2.3.4c$: and it follows from No. 126, that when $x=a$, fx will, as in the first function, be a maximum, when c is negative, and a minimum when it is positive.

The student may readily generalise these principles, by considering the functions,

$$fx = b + c(x-a)^{2n}, \text{ and } fx = b + c(x-a)^{2n+1},$$

n being a positive integer.*

135. The functions, $u=b+c(x-a)^{-2}$, $u=b+c(x-a)^{-3}$, &c. give results very different from those last obtained, and yet still bearing to them a close analogy. From the first of them we have $f^1x = -2c(x-a)^{-3}$. This can never become nothing, but it is infinite when $x=a$; and therefore (No. 126) if there can be a maximum or minimum value, it will be that which is obtained by taking $x=a$. Now, this value of x will make f^2x, f^3x, &c. all infinite. Let us, therefore, substitute $a-h$ and $a+h$ successively for x in f^1x, to find whether the function changes its sign in passing through infinity. The first of these substitutions gives $2ch^{-3}$, and the second $-2ch^{-3}$, which have opposite signs; and, therefore, (No. 126) the function will be a maximum ($+\infty$), when c is positive, and a minimum ($-\infty$), where it is negative. We should arrive at the same conclusions from considering the proposed function itself: since, when c is positive, and $x=a$, it becomes $+\infty$; while its values, when x is either a little less or a little greater than a, will be positive, but each of course less than $+\infty$: and when c is negative the value of u is $-\infty$; while those immediately preceding and following will be also negative, but greater than $-\infty$.

The second of the functions mentioned above gives $f^1x = -3c(x-a)^{-4}$; a quantity which does not change its sign in passing through infinity; and therefore the function itself cannot become either a maximum or a minimum. The student will find it easy, indeed, to show, that, uni-

* In illustrating these functions by means of lines, the student will find, that the equation, $y = b + c(x-a)^2$, will give a common parabola with its axis vertical, and above or below the axis of x according as c is positive or negative: and the equations, $y = b + c(x-a)^4$, $y = b + c(x-a)^6$, &c. will give curves similarly situated, and of nearly the same form. He will find on the other hand, that $y = b + c(x-a)$ will give a straight line, extending, indefinitely and continuously, above and below the axis of x; that $y = b + c(x-a)^3$ will give a cubical parabola, extending similarly; $y = b + c(x-a)^5$ a curve extending similarly, and of nearly the same form; and so on. From both the theory and the graphical delineation, it will be seen there can be a maximum or minimum only when the index is $2n$.

versally, there will be a maximum or minimum, when the index of $x-a$ is of the form $-2n$, but not when it is of the form $-2n+1$; n being a positive integer.*

136. Let $u=\frac{x^2-3x+2}{x^2+3x+2}$, or $u=\frac{(x-1)(x-2)}{(x+1)(x+2)}$. Then,

if $x=\sqrt{2}$, $u=12\sqrt{2}-17=-{\cdot}029437$, a minimum;

if $x=-\sqrt{2}$, $u=-12\sqrt{2}-17=-33{\cdot}970563$, a maximum.

By dividing the numerator of the given fraction by the denominator, we get $1-\frac{6x}{x^2+3x+2}$. Now, if the given fraction be a maximum, it is evident that $\frac{6x}{x^2+3x+2}$, and consequently $\frac{x}{x^2+3x+2}$, will be a minimum; and the reciprocal of this, or $x+3+\frac{2}{x}$, will be a maximum; and a like principle is applicable when u is a minimum. It appears, therefore, that the given fraction will be a maximum or minimum when $x+\frac{2}{x}$ is a maximum or minimum, which affords an easy method of solution;—much easier than that of Euler (CALC. DIFF. Pars Posterior, art. 265). We have here an instance in which a maximum value is less than a minimum one. This arises from the circum-

* When $u=b+c(x-a)^{-1}$, the corresponding curve is an hyperbola, having its two opposite parts in two vertical or opposite right angles; and when the index is -3, or any other odd negative integer, the curve will have two separate portions similarly situated, and they will resemble those of the hyperbola; but when the index is an even negative integer, the parts of the curve are in adjacent right angles,—either the pair above the axis of x, or those below it.

The learner may perhaps think that a quantity is necessarily a maximum when it is infinite. It should be recollected, however, that by the definition of a maximum the function must have increased continuously up to the maximum value, and must then begin to diminish continuously; and similar remarks are applicable regarding the minimum. The following instance will illustrate this. The tangent of 90°, though infinite, is not a maximum; for while it has increased continuously up to the infinite value, yet it does not then begin to diminish, but passes at once, *per saltum*, as it were, to a negative infinite value, and begins to *increase* continuously, becoming less and less as a *negative* quantity, and therefore greater and greater as compared with *positive* quantities.

The student will find it useful, and not difficult, to examine, in a similar way, the function $u=b+c(x-a)^n$, when n is a positive or negative fraction in its lowest terms, and he will find that it will have a maximum or minimum value, when the numerator is even and the denominator odd, but not otherwise.

stance, that, between $-\sqrt{2}$ and $\sqrt{2}$, the values of x which produce the maximum and minimum, there is the value -1, which renders the function infinite. This might be illustrated by means of a curve, having x for its abscissa, and u for its ordinate. This example is a particular case of the general one, $u=\frac{(x-a)(x-b)}{(x+a)(x+b)}$, in which the required values of x are $\pm\sqrt{ab}$.

The solution may be easily obtained by means of logarithms. Thus we have

$$\log u=\log(x-1)-\log(x+1)+\log(x-2)-\log(x+2);$$

whence, by differentiating and multiplying by u, we get

$$\frac{du}{dx}=u\left(\frac{1}{x-1}-\frac{1}{x+1}+\frac{1}{x-2}-\frac{1}{x+2}\right)=u\left(\frac{2}{x^2-1}+\frac{4}{x^2-4}\right);$$

and the answer is readily obtained by putting this equal to zero, transposing, taking the reciprocals of the two members, &c.

137. Let $u=\frac{x^2-x+1}{x^2+x-1}$. Then,

if $x=0$, $u=-1$, a maximum;

if $x=2$, $u=\frac{3}{5}$, a minimum.

By adding 1 to each member, we get $1+u=\frac{2x^2}{x^2+x-1}$, which is evidently a maximum or minimum; and, therefore, $\frac{x^2+x-1}{x^2}$, and consequently $\frac{1}{x}-\frac{1}{x^2}$, must be a minimum or maximum. In this, as in the last example, the minimum value is greater than the maximum, which arises from the circumstance that, between $x=0$ and $x=2$, there is the value, $x=\frac{1}{2}\sqrt{5}-\frac{1}{2}$, which renders u infinite.

138. Required x, such that $x\sin x$ may be a maximum. Here, $x=-\tan x$; and the values of x may be found, by the assistance of a table of natural tangents, by trial and error. Euler, from whom this example is taken (Calc. Diff. Pars Posterior, art. 272), employs a series; but he finds only one value of x, while it has an infinite number.

Some of these values are 0, $116^\circ\ 14'\frac{2}{5}$, $281^\circ\ 30'$, $457^\circ\ 9'$. $635^\circ\ 9'\frac{1}{2}$, $814^\circ\ 1'\frac{1}{3}$, $993^\circ\ 18'$, $1172^\circ\ 47'\frac{3}{4}$, $135^\circ\ 252'\frac{1}{2}$, $1532^\circ\ 8'\frac{1}{2}$ ...... $9090^\circ\ 21'\frac{2}{3}$, &c.: the first of which makes u a minimum; the second a maximum; the third a minimum, &c. As the values of x become greater, their successive differences become more and more nearly equal to 180°; and each value is an odd number of quadrants with an excess which becomes continually smaller, the greater the values become.

139. Find the number which bears to its logarithm the least ratio possible.

Here, $u = \frac{x}{lx}$; and by the usual process, we find $lx = M = l\varepsilon$ (by No. 91): whence $x = \varepsilon$, and $u = \frac{\varepsilon}{M}$.

140. Given AB,* the height of a column, and BC that of a statue placed on it; to determine the point P in a horizontal line AP, at which BC will appear greatest. Here we may assume $AC = a$, $AB = b$, and $AP = x$: now, by an obvious principle in optics, the apparent magnitude of a line is proportional to the angle formed by straight lines drawn from the extremities of the line to the eye; and hence, in the present case, P must be the point at which the angle BPC is a maximum. We have also, by trigonometry (TRIG. No. 6) the angle $APC = \tan^{-1}\frac{a}{x}$, and $APB = \tan^{-1}\frac{b}{x}$; and, therefore, $BPC = \tan^{-1}\frac{a}{x} - \tan^{-1}\frac{b}{x}$. Denoting this by fx, and differentiating according to No. 40, we get, after some obvious modifications, $f^1x = -\frac{a}{a^2+x^2} + \frac{b}{b^2+x^2}$; and by putting this equal to zero, transposing, taking the reciprocals, &c. we readily get $x^2 = ab$.†

* In the present instance no diagram is given, as the student will feel no difficulty in making one for himself. Similar omissions will be made on other occasions, when the diagrams are of a simple kind.

† This question and others have sometimes been solved on the principle, that "if an angle be a maximum, its tangent is also a maximum." This principle depends on the circumstance, that the tangent of an angle always increases with the angle;

141. Given the sides, $AB = a$, $BC = b$, $CD = c$, and $DA = e$, of a quadrilateral; to determine it so that its area may be a maximum. Let the diagonal BD be drawn. Then, putting u to denote double the area, we have, by mensuration (EUC. APPENDIX, III. 28)

$$u = ae \sin A + bc \sin C;$$

whence, by differentiating, regarding A as the independent variable, we get

$$\frac{du}{dA} = ae \cos A + bc \cos C \frac{dC}{dA} \ldots\ldots (a).$$

Again (TRIG. No. 56) we have

$$a^2 + e^2 - 2ae \cos A = b^2 + c^2 - 2bc \cos C;$$

each being equal to BD^2.

From this, we obtain $ae \sin A = bc \sin C \frac{dC}{dA}$, by differentiating, &c. Then, from (a), by putting it equal to zero, and transposing the resulting equation, and by dividing the members of the last equation by the equals so obtained, we get $\tan A = -\tan C$; so that (TRIG. No. 18) A and C are supplements of each other; and (EUC. APP. I. 3.) a circle may be described about the quadrilateral. Lastly, by putting $\pi - A$ for C in the equations $a^2 + c^2 - 2ae \cos A = b^2 + c^2 - 2bc \cos C$, the angle A would be readily found.

142. Required the values of x which render y a maximum or minimum in $y^3 - 3axy + x^3 = 0$, the equation of the curve known by the name of the *folium* of Des Cartes.

Here we have $3y^2 f^1 x - 3axf^1 x - 3ay + 3x^2 = 0$; and hence, by taking $f^1 x = 0$, we get $ay = x^2$. Then, by eliminating y between this result and the given equation, and resolving the equation so found, we obtain $x = 0$ and $x = a\sqrt[3]{2}$; and, by substituting these separately in $ay = x^2$, we find the respective values of y to be 0 and $a\sqrt[3]{4}$. To find whether

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from which we infer, that if an angle which varies continuously as a function of an independent variable, is to be a maximum or minimum, the tangent of that angle, increasing or diminishing, as the angle itself increases or diminishes, must be a maximum or minimum when the angle is such, and only then. It may be remarked, that when the angle is $\frac{1}{2}\pi$, $\frac{3}{2}\pi$, or in general, $\frac{1}{2}(2n + 1)\pi$, the tangent loses its continuity, passing from an infinite positive value to an infinite negative one. (See the note in page 83.) The student will readily see, that if an angle be the *independent variable*, its tangent cannot have maximum or minimum values, but that its secant can.

each of these values renders y a maximum, or minimum, or neither, let the differential equation found above be differentiated, all terms being omitted that contain $f^1 x$, which is equal to zero: then $3y^2 f^2 x - 3axf^2 x + 6x = 0$; whence

$$f^2 x = \frac{2x}{ax - y^2} = \frac{2a^2 x}{a^3 x - a^2 y^2} = \frac{2a^2 x}{a^3 x - x^4},$$

because $ay = x^2$. From this, by dividing the terms of the fraction by x, and by substituting successively in the result 0 and $a\sqrt[3]{2}$ for x, we find that the former, rendering $f^2 x$ positive, makes y a minimum; while the latter makes $f^2 x$ negative, and therefore renders y a maximum.*

Since the given equation is symmetrical with regard to x and y, it is plain that when $y = 0$, x is a minimum, and that when $y = a\sqrt[3]{2}$, x is a maximum.

EXERCISES REGARDING MAXIMUMS AND MINIMUMS.

1. When will $u = 4x^3 - x^2 - 2x + 1$ be a maximum or minimum? *Answ.* When $x = \frac{1}{2}$, $u = \frac{1}{4}$, a minimum; and when $x = -\frac{1}{3}$, $u = \frac{38}{27}$, a maximum.

2. What values of x will render $u = 3x^4 - 16x^3 + 6x^2 + 72x - 1$ a maximum or minimum?

Answ. When $x = -1$, $u = -48$, a minimum;
when $x = 2$, $u = 87$, a maximum;
and when $x = 3$, $u = 80$, a minimum.

3. Find the maximum and minimum values of

$$u = 72x^5 - 195x^4 - 20x^3 + 60x^2 + 5.$$

Answ. If $x = -\frac{1}{2}$, $u = 8\frac{2}{9}$, a maximum;
if $x = 0$, $u = 5$, a minimum;
if $x = \frac{1}{2}$, $u = 12\frac{9}{16}$, a maximum;
and if $x = 2$, $u = -411$, a minimum.

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* The solution here given affords another instance in addition to the one given in No. 132 of the easy mode of finding the maximum and minimum values of implicit functions. In the present instance we might render y an explicit function by resolving the equation by Cardan's method, and we might then proceed in the usual way; but the process would be laborious and inelegant. Should the student wish to trace this curve, he may substitute vx for y in the given equation. By this means he would get $x = \frac{3av}{v^3 + 1}$. Then, by assuming values for v, he would find values for x, and thence for $y = vx$. The facility afforded by the substitution here suggested, arises from the symmetry of the given equation in reference to x and y. Without such an expedient, it would be necessary by one means or other, to resolve a cubic equation for finding each value of y corresponding to an assumed value of x.

4. Required an arc x, such that $u = \sin 2x - \sin x$ may be a maximum or minimum.

Answ. When $x = \cos^{-1} \frac{1+\sqrt{33}}{8}$, u is a maximum; and when $x = \cos^{-1} \frac{1-\sqrt{33}}{8}$, it is a minimum.

5. Required x, so that $u = x \cos x$ may be a maximum or minimum. *Answ.* $x = \cot x$; and the values of x are $49^0\ 18'$, $\pi + 16^0\ 16'$, $2\pi + 8^0\ 11\frac{1}{4}'$, $3\pi + 5^0\ 59'$, $4\pi + 4^0\ 31\frac{1}{2}'$, $5\pi + 3^0\ 37\frac{3}{4}'$, $6\pi + 3^0\ 1\frac{2}{3}'$......$30\pi + 0^0\ 36\frac{1}{4}'$, &c.

6. Required x, such that $x \operatorname{versin} x$ may be a maximum or minimum. *Answ.* $x = -\tan \frac{1}{2}x = 210^0\ 27\frac{1}{2}'$, &c.

7. Show that $u = x a^{\frac{1}{x}}$ is a minimum when $x = \log a$; and that then $u = \varepsilon$, if $a = \varepsilon$.

8. Let $u = x^x$. Then, if $x = \frac{1}{\varepsilon}$, u is a minimum.

9. Let $u = x^{\frac{1}{x}}$. Then, if $x = \varepsilon$, $u = \varepsilon^{\frac{1}{\varepsilon}}$ is a maximum.

10. Prove that the difference of an arc x and its secant is a maximum or minimum, when $\sin x = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$.*

11. "To divide a given number a into two parts, such that the product of the mth power of one of them and the nth power of the other may be a maximum or minimum."

Answ. The product will be a maximum, if the parts be $\frac{ma}{m+n}$ and $\frac{na}{m+n}$; and if m and n be one of them even and the other odd, one of the *parts* will be the whole line and the other nothing; which will give $u=0$, a minimum.†

12. On a given hypotenuse to describe a plane triangle, such that the product of the mth power of one leg, and the nth power of the other, may be a maximum. Here, the segments into which the hypotenuse is divided by the perpendicular from the right angle, must be in the ratio of m to n.

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* This is the greater of the parts into which the radius is divided, when cut in extreme and mean ratio.

† The result here obtained shows that the enunciation is incorrect. It might be thus expressed:—To find a number such that if it be taken from a given number a, the product of the mth power of the result, and the nth power of the required number, may be a maximum or minimum. The student should consider the case in which one of the indices or both of them are fractional, and that in which one or both are negative.

13. Through a given point, between the lines containing a given angle, to draw a straight line, forming with those lines the least triangle possible. Here the required line will be bisected in the given point.*

14. On a given straight line as diagonal, to describe a rectangle, such that the part of it intercepted between the two straight lines intersecting the diagonal perpendicularly, and passing through the angles which that diagonal subtends, may be a maximum. *Answ.* If a be the diagonal, the line drawn through the figure perpendicular to the diagonal will be $a(\sqrt{5}-2)$.

VIII.—ON TANGENTS TO CURVES, ASYMPTOTES, DIRECTION OF CURVATURE, AND POINTS OF INFLEXION.

143. Through P, a point in a curve AB† (*fig.* 1) referred to rectangular axes OX, OY, let a straight line RPS be drawn cutting the curve in P and R, and the axis of x in S; and let the ordinates PM, RQ, be drawn. Let also PK be drawn parallel to OX; and put $\text{OM}=x$, $\text{MP}=y$, $\text{MQ}=h$, and $\text{QR}=y'$. Then, by trigonometry, $\tan \text{RPK}$ or $\tan \text{PSM}=\frac{y'-y}{h}$. Now, if h be continually diminished towards zero, the ordinate RQ will tend to coincide with PM, and the point R with P; and by the approach of R to P, the line RPS will change its position; and, ultimately, when R coincides with P, RPS will no longer cut the curve in P and R, but will become the tangent PT at P, making with OX the angle PTM, instead of PSM. Since, also, y and y' are like functions of x and $x+h$, depending on the equation of whatever particular curve may be under consideration, we shall have, by No. 6,

$$\tan \text{PSM}=\frac{y'-y}{h} \text{ changed into } \tan \text{PTM}=\frac{dy}{dx}, \text{ or } f^1 x;$$

* In solving this problem, the lines drawn from the given point, parallel to the legs of the given angle, may be put equal to a and b. Then, if x be put for the distance from the vertex along one leg to the required line, an expression for the corresponding distance along the other will be found from similar triangles; and the product of those distances will be a minimum, since the area of a triangle is equal to half the continued product of two of its sides, and the sine of their contained angle, which last is constant, as the angle is given.

† In the present and in similar investigations, the diagrams may have four varieties: as the curve may be concave or convex towards the axis of x, and in each of these cases the ordinates may form either an increasing series or a decreasing one. With due attention, however, to the signs, one diagram is sufficient. At the same time, it will be well for the student to make and consider the other diagrams.

so that the trigonometrical tangent of the angle which the tangent to a curve at any point of it makes with the axis of the abscissas is the first differential coefficient of the ordinate at that point.*

144. If QR produced if necessary, meet the tangent in V, and if we put $OQ=\xi$ and $QV=\eta$, we shall have

$$\frac{KV}{PK}=\tan PTM, \text{ that is}$$

$$\frac{\eta-y}{\xi-x}=f^1 x, \text{ whence } \eta-y=(\xi-x)f^1 x,$$

which is the equation of the tangent to the curve at the point whose coordinates are x and y, ξ and η being the coordinates of any point in the tangent.

145. The straight line PN (produced indefinitely) which is drawn through P perpendicular to the tangent, is called the *normal*† to the curve at that point: and MT, MN, the parts of the axis of x between the ordinate and the tangent on the one hand, and between the ordinate and the normal on the other, are called respectively the *subtangent* and the *subnormal* for the point P. Now, by a principle in analytic geometry (see Trig. No. 248), we get from the equation of the tangent found in the last No. $\eta-y=\frac{x-\xi}{f^1 x}$, the equation of the normal; ξ, η being any point in that line. This may be put under the form, $(\eta-y)f^1 x+\xi-x=0$.

146. If in the equation of the tangent we take $\eta=0$, we get, after a slight modification, $x-\xi=\frac{y}{f^1 x}$, which is

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* Thus, in case of the common parabola (fig. 2), taking $y=p^{\frac{1}{2}}x^{\frac{1}{2}}$ as its equation we have $\frac{dy}{dx}=\frac{\frac{1}{2}p^{\frac{1}{2}}}{x^{\frac{1}{2}}}=\tan PTM$. Hence, if $p=100$, and if we take x successively equal to 25, 49, and 100, the respective angles made with the axis of x by the tangents to the curve at the corresponding points will be found to be 45°, 35° 32′, and 26° 33′; and if $x=0$, the angle will be 90°, so that the tangent at O is perpendicular to OX.

It may be remarked that, universally, if at any point of a curve, $f^1 x=\infty$, the tangent at that point is perpendicular to the axis of x; but if $f^1 x=0$, the tangent is either parallel to that axis, or coincides with it.

† Sometimes the term *normal* is restricted to the part PN of the perpendicular. In like manner the term *tangent* is sometimes confined to the limited portion PT. It is perhaps preferable to regard them both as lines merely having position, but as unlimited in length.

OM—OT or TM, the subtangent. In the same way we obtain from the equation of the normal, $\xi - x = yf^1x$; that is ON—OM, or the subnormal, $MN = yf^1x$.

147. The principles established above enable us to draw tangents to curves in various ways; such, for instance, as by determining the angle PTM, and making MPT equal to its complement; by determining the subtangent, MT, and joining PT; or by finding the subnormal MN, joining PN, and making NPT a right angle. The tangent and the normal may also be determined by means of their equations. The mode, however, that is generally employed is to find the subtangent.

148. If there be a curve which has an infinite branch, and which is of such a nature that there is a straight line which that branch can never meet, but to which, if it be sufficiently extended, it can approach so nearly that the distance between them may become less than anything that can be assigned, the straight line is said to be an *asymptote* to the curve. Such a straight line is plainly to be regarded as that with which a tangent to the curve would continually tend to coincide, were one of the coordinates to increase without limit. To ascertain, therefore, whether a curve can have an asymptote, and if so to determine it, we must find whether the tangent will take a determinate position when x or y is infinite. To effect this, let $\eta = 0$ in the equation of the tangent (No. 144), and we get ξ or $OT = x - \frac{y}{f^1x}$; and by taking $\xi = 0$, we get η or $OT' = y - xf^1x$.* Now, since each of these values has the fixed point O as one of its extremities, it is plain that the point T or T′ will have a determinate position, if the first or second of the foregoing expressions have a finite value when x or y is $= \pm\infty$; and thus, both the existence of the asymptote, if there be one, and its position, will be determined.

The simplest way to find whether either of the axes is an asymptote, is to try, by means of the equation of the curve, whether when one of the coordinates is taken equal to zero, the other becomes infinite.

149. To exemplify the application of these principles, let us differentiate $y^2 = mx + nx^2$, the general equation (Ap-

* It is evident that $OT' = OT \times f^1x$.

PENDIX, No. 14) of lines of the second order; then, $2y\,dy = m\,dx + 2nx\,dx$, whence $\frac{dy}{dx} = \frac{m+2nx}{2y}$. Multiplying the reciprocal of this by y, we find the subtangent (No. 146) to be

$$\frac{2y^2}{m+2nx} = \frac{2(mx+nx^2)}{m+2nx} = \frac{mx+nx^2}{\frac{1}{2}m+nx}.$$

We get, also, from 146, the subnormal equal to $\frac{1}{2}m+nx$: and we obtain (No. 144) for the equation of the tangent, $\eta - y = \frac{\frac{1}{2}m+nx}{y}(\xi - x)$, where x and y are the coordinates of the point of contact, and ξ, η, those of any point in the tangent.

150. Now, in the parabola, in which (APP. No. 14) $n=0$, the foregoing expressions for the subtangent and subnormal become simply $2x$ and $\frac{1}{2}m$: whence it appears, that, *in the parabola, the subtangent is double of the abscissa;* and that *the subnormal is equal to half the parameter, and is therefore a constant quantity.*

151. In the ellipse (APP. No. 14), $n = -\frac{b^2}{a^2}$, and in the hyperbola, $n = \frac{b^2}{a^2}$; while, in both, $m = \frac{2b^2}{a}$. By substituting these in the foregoing expressions, we get for the ellipse, the subtangent $= \frac{2ax-x^2}{a-x}$, and the subnormal $= \frac{b^2}{a^2}(a-x)$; while, for the hyperbola, we get in a similar manner, the subtangent $= \frac{2ax+x^2}{a+x}$*, and the subnormal $= \frac{b^2}{a^2}(a+x)$.

* Since, in both the ellipse and hyperbola, the expression for the subtangent is independent of the axis b, it follows that, if, with the same transverse axis and vertex, any number of ellipses or hyperbolas whatever be described, and if a perpendicular to that axis be drawn cutting them all, tangents drawn through the points of intersection will all cut that axis in the same point; and the same will hold respecting parabolas having a common axis and vertex. Hence, we may draw a tangent at a given point, P (*fig.* 13) of an ellipse, by describing on either axis AB as diameter a circle, and then drawing the ordinate OP, and producing it, if necessary, to cut the circle in Q; for, if the tangent QT be drawn, and TP be joined, it will touch the ellipse.

By adding OC $= a-x$ to the expression for the subtangent of the ellipse, and converting the result into an analogy, we obtain $a-x : a :: a :$ CT, or CO : CA ::

152. By means of No. 148, we find for lines of the second order,

$$\mathrm{OT}=\frac{mx+nx^2}{\frac{1}{2}m+nx}-x=\frac{\frac{1}{2}mx}{\frac{1}{2}m+nx}, \text{ and}$$

$$\mathrm{OD}=y-\frac{\frac{1}{2}mx+nx^2}{y}=\frac{y^2-\frac{1}{2}mx-nx^2}{y}=\frac{\frac{1}{2}mx}{y}=\frac{\frac{1}{2}mx}{\sqrt{(mx+nx^2)}}.$$

In the hyperbola these are changed into

$$\mathrm{OT}=\frac{ax}{a+x}, \qquad \text{and } \mathrm{OD}=\frac{b\sqrt{x}}{\sqrt{(2a+x)}}.$$

If the numerator and denominator of the first of these be divided by x, and x be taken infinite, the first term of the denominator vanishes, and we get simply, $\mathrm{OT}=a$; and, in a similar manner, we find $\mathrm{OD}=b$. Hence, in the hyperbola (*fig.* 8), since $\mathrm{AC}=a$, if, through A, a perpendicular be drawn on both sides, and each part of it be made equal to CD, lines drawn through its extremities and through C will be asymptotes to AP and AG; and, if continued in the opposite direction, they would be asymptotes to the opposite hyperbola. By considering the conjugate hyperbolas, it would appear that the same lines are asymptotes to them also.

The ellipse can have no asymptote, since neither x nor y can become infinite. Neither can the parabola; since, for it, the expressions for OT and OD become, respectively, x and $\frac{1}{2}\sqrt{mx}$, each of which is infinite when x is infinite.

153. Given $y^2(2a-x)=x^3$, the equation of the cissoid; to find the subtangent.

Here, by dividing by $2a-x$, differentiating, and dividing

CA : CT; so that CT is a third proportional to CO and CA; and the same property belongs to the hyperbola.

It is easy to show, that, in the ellipse, the tangent makes equal angles with the lines drawn from the foci to the point of contact; or, which is the same, that P T bisects the exterior angle, FPR. For, from the analogy last obtained, putting $\mathrm{CO}=x$, we have $\mathrm{CT}=\frac{a^2}{x}$, and consequently $\mathrm{VT}=\frac{a^2}{x}+c$, and $\mathrm{FT}=\frac{a^2}{x}-c$, the ratio of which is that of a^2+cx to a^2-cx. But (App. No. 3), $\mathrm{PV}=a+\frac{cx}{a}$, and $\mathrm{PF}=a-\frac{cx}{a}$, the ratio of which is also that of a^2+cx to a^2-cx; whence VT : FT : : VP : FP; and therefore (Euc. VI. A.) PT bisects the angle FPR. It would be shown, in a similar manner, that, in the hyperbola, the tangent bisects the angle contained by lines drawn from the foci to the point of contact; while, in the parabola (*fig.* 9), the tangent bisects the angle DPF.

by $2dx$, we get $\frac{y\,dy}{dx}=\frac{3ax^2-x^3}{(2a-x)^2}$; by multiplying the reciprocal of which by $y^2=\frac{x^3}{2a-x}$, we find the subtangent,

$$\frac{y\,dx}{dy}=\frac{2ax-x^2}{3a-x}.$$

154. Given $x^2y^2=(b^2-y^2)(a\pm y)^2$, the equation of the conchoid; to find its subtangent.

Here we have $x=\left(\frac{a}{y}\pm 1\right)\sqrt{(b^2-y^2)}$; and, by differentiating, &c. we get the subtangent $=-\frac{ab^2\pm y^3}{y\sqrt{(b^2-y^2)}}$, in which the upper sign is to be used for the superior conchoid, and the lower for the inferior. If $y=\pm b$, this becomes infinite; and therefore, in that case, the tangent is parallel to the directrix. To this there is an exception, when $b=a$, and $y=-b$; as the expression for the subtangent then becomes $\frac{0}{0}$, the value of which is easily shown (No. 118) to be nothing, and therefore (*fig.* 16) the tangent coincides with AB. In this case the conchoid has a cusp (to be explained hereafter), and the cusp is the point of contact.

155. Given $y=a^x$, the equation of the logarithmic curve; to find its subtangent.

Here we find (Nos. 29 and 91) $dy=\frac{y\,la\,dx}{\mathrm{M}}$; whence we get the subtangent $=\frac{\mathrm{M}}{la}$, a constant quantity. If the system be that whose base is a, we have (No. 26) $la=1$; and consequently, the subtangent $=\mathrm{M}$. In this case, therefore, the subtangent, at every point of the curve, is equal to the modulus.

156. Let it be required to investigate the method of drawing a tangent to the common cycloid, the equation of which is $\frac{y}{a}=\text{versin}\,\frac{x+\sqrt{(2ay-y^2)}}{a}$.

By differentiating this, by transposing the second term of the result, and by dividing the terms of the second member so obtained by $\sqrt{y}$, we get $dx = dy\sqrt{\frac{y}{2a-y}}$. Hence, we find the subtangent

$$\left(=\frac{y\,dx}{dy}\right)=\frac{y^{\frac{3}{2}}}{\sqrt{(2a-y)}}=\frac{y^2}{\sqrt{(2ay-y^2)}}*=\frac{PR^2}{RD}.$$

It is evident from this that the tangent is perpendicular to the chord PD; and that PD, therefore, is the normal. Hence, if a circle be described on the axis CV as diameter, and if PE be produced to cut it in H, the tangent is parallel to the chord VH. (See *fig.* 18.)

157. Let it be required to draw a tangent to a line of the second order from a point without it, whose coordinates are ξ and η.

Here, since $y^2 = mx + nx^2$, and (No. 149) $\frac{dy}{dx} = \frac{m+2nx}{2y}$, we have (No. 144) $\eta - y = \frac{m+2nx}{2y}(\xi - x)$, or

$$\eta - \sqrt{(mx+nx^2)} = \frac{m+2nx}{2\sqrt{(mx+nx^2)}}(\xi - x);$$

the resolution of which equation will give the two values of x belonging to the two points of contact.

158. The position of a tangent to a curve may also be readily determined from its polar equation. Thus, O (*fig.* 4) being the pole, draw OG perpendicular to the radius vector OP, and let O also be taken as the origin of the rectangular coordinates, OM, MP. Then, putting OP $= r$, and the angle XOP $= \theta$, we have $x = r\cos\theta$, and $y = r\sin\theta$; by differentiating which we obtain

$$dx = \cos\theta\,dr - r\sin\theta\,d\theta, \qquad \text{and } dy = \sin\theta\,dr + r\cos\theta\,d\theta.$$

Now, since (Euc. I. 32) the angle TPO $=$ POM $-$ PTM, we have (Trig. No. 30) $\tan \mathrm{TPO} = \frac{\tan \mathrm{POM} - \tan \mathrm{PTM}}{1 + \tan \mathrm{POM} \tan \mathrm{PTM}}$.

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* For the prolate, or curtate cycloid, the subtangent is found to be

$$\frac{y^2}{\sqrt{\left\{a^2(r^2-1)+2ay-y^2\right\}}}.$$

By substituting in this the values of $\tan \text{POM} (=\tan\theta)$, and $\tan \text{PTM}$ (given in No. 143) we get, after multiplying the numerator and denominator by dx, $\tan \text{TPO} = \frac{\tan\theta\, dx - dy}{dx + \tan\theta\, dy}$. Then, by substituting in this the values found above for dx and dy, by contracting, and by dividing the numerator and the denominator of the result by $\sin\theta \tan\theta + \cos\theta$, we get $\tan \text{TPO} = -\frac{r\,d\theta}{dr}$; and thence $\text{OG} = -\frac{r^2\,d\theta}{dr}$, which, in reference to curves thus expressed, is called the subtangent. The subnormal ON, found by dividing r^2 by the subtangent, is $-\frac{dr}{d\theta}$.

159. If OU be drawn perpendicular to the tangent, and be denoted by v, we have, in the similar triangles, NPO, OPU, NP^2, or $\text{OP}^2 + \text{ON}^2 : \text{OP}^2 :: \text{OP}^2 : \text{OU}^2$, that is (by the last No.) $r^2 + \frac{dr^2}{d\theta^2} : r^2 :: r^2 : v^2$; and hence, we readily get $v^2 = \frac{r^4 d\theta^2}{r^2 d\theta^2 + dr^2}$; a formula which is of importance in several investigations.

160. To give an instance of the use of the preceding formula, let us investigate the distance v from the focus of a parabola to the tangent at a point r, θ. Now, (App. 13, note) we have $r(1 - \cos\theta) = \frac{1}{2}p$; from which we get, by differentiating and by multiplying by r,

$$r(1-\cos\theta)\,dr + r^2 \sin\theta\, d\theta = 0, \quad \text{or} \quad \tfrac{1}{2}p\,dr + r^2\sin\theta\, d\theta = 0;$$

and hence we have $r^4 \sin^2\theta\, d\theta^2 = \frac{1}{4}p^2 dr^2$. Then, by dividing both members of this, first, by $\sin^2\theta$, and, again, by $r^2\sin^2\theta$, and by substituting the results in the formula found in the last No. we get, after easy simplifications,

$$v^2 = \frac{\frac{1}{4}p^2 r^2}{\frac{1}{4}p^2 + r^2 \sin^2\theta}.$$

But $\frac{1}{4}p^2 = r^2(1-\cos\theta)^2$; and therefore, by easy and obvious changes, we obtain,

$$v^2 = \frac{\frac{1}{4}p^2 r^2}{r^2(1-\cos\theta)^2 + r^2\sin^2\theta} = \frac{\frac{1}{4}p^2 r^2}{2r^2 - 2r^2\cos\theta} = \frac{\frac{1}{4}p^2 r}{2r(1-\cos\theta)} =$$

$$\frac{\frac{1}{4}p^2 r}{p} = \tfrac{1}{4}p\,r.$$

Hence in the parabola, the perpendicular from the focus to the tangent is a mean proportional between the distances from the focus to the vertex and the point of contact.

161. To take another instance we have (APPENDIX No. 13, note) in the ellipse $ar - cr\cos\theta = b^2$; and, therefore, $(a - c\cos\theta)dr + cr\sin\theta\, d\theta = 0$; or, by multiplying by r, &c.

$$b^2 dr + cr^2 \sin\theta\, d\theta = 0\,; \qquad \text{whence } c^2 r^4 \sin^2\theta\, d\theta^2 = b^4 dr^2.$$

Dividing the members of this successively by $c^2\sin^2\theta$ and $c^2r^2\sin^2\theta$, and substituting the results in the formula in No. 159, we get, after some modifications,

$$v^2 = \frac{b^4 r^2}{b^4 + c^2r^2\sin^2\theta} = \frac{b^4}{a^2 + c^2 - 2ac\cos\theta},$$

the latter value being obtained by substituting for b^4 in the denominator of the former its value $(ar - cr\cos\theta)^2$, and dividing the terms of the fraction so obtained by r^2. Hence, by putting $a^2 - b^2$ for c^2, and by multiplying the terms of the resulting fraction by r, dividing the results by b^2, &c. we get $v^2 = \frac{b^2r}{2a - r}$. In a similar way we should obtain for the perpendicular in question, in the hyperbola, $\frac{b^2r}{2a + r}$.

162. Since the two foci are alike related to the ellipse, it follows evidently, that if v' be put to denote the perpendicular from the other focus, we should have $v'^2 = \frac{b^2(2a - r)}{r}$; the radius vector for that focus being $2a - r$. In like manner, for the hyperbola, we should have $v'^2 = \frac{b^2(2a + r)}{r}$. Hence in both curves we have $vv' = b^2$; that is, the product of the perpendiculars from the foci to the tangent at any point of an ellipse or hyperbola, is equal to the square of half the conjugate axis.

163. As an example regarding the drawing of tangents to curves referred to polar coordinates, let it be required to find the subtangent of the hyperbolic spiral, the equation of which is $r\theta = a$. Here, by differentiating, and by means of No. 158, we readily find that the subtangent is equal to the constant quantity a.

164. From the theory of tangents to curves, we are able to derive the means of determining the direction of their

curvature. Thus, (*fig.* 1) if the curve be concave towards the axis of x, it is plain that if x be increased, the angle T (PTX) will be diminished, and therefore (No. 125) will have its differential coefficient negative; while the reverse will take place if it be convex to that axis. Now, (No. 143) we have $\tan T = \frac{dy}{dx}$; and, therefore, by differentiation (No. 35), $(1 + \tan^2 T)\frac{dT}{dx} = \frac{d^2x}{dx^2}$. Now, since $1 + \tan^2 T$ is essentially positive, it follows that the sign of $\frac{dT}{dx}$ will be the same as that of $\frac{d^2y}{dx^2}$. Hence, therefore, the curve will be convex towards the axis of x, if the second differential coefficient of y be positive, and concave if it be negative. In this reasoning it has been tacitly assumed, that y is positive; but it is easy to see, that if y be negative, the conclusion will be the reverse in reference to the signs. The following enunciation of the principle is plainly applicable in all cases:—*A curve at any point* (x, y) *is convex to the axis of* x, *if* y *and its second differential coefficient have the same sign; but concave, if the signs be different.*

165. If a curve change the direction of its curvature, the point at which the change takes place is called a *point of inflexion*, or a *point of contrary flexure.* It follows from the last No. that at such a point $\frac{d^2y}{dx^2}$ must change its sign, and must therefore (No. 126) be either nothing or infinite. If it be nothing, there will be an inflexion, unless f^3x vanish; in which case there will be no inflexion, unless f^4x also vanish; and so on. Should f^2x be infinite, it is necessary to ascertain, as in No. 132, whether it changes its sign.*

EXERCISES REGARDING TANGENTS, ETC.

Find the subtangents for the following curves:

1. The witch. *Ans.* $-\frac{2(ax - x^2)}{a}$.

* The theory given above amounts, exactly, as it ought, to the finding of the circumstances in which the angle T is a maximum or a minimum.

2. The spiral of Archimedes. *Ans.* $-\frac{a\theta^2}{2\pi}$, or $-r\theta$.

3. The curve, $y^4 = a^4 + x^4$. *Ans.* $\frac{y^3}{x^3}(a^4+x^4)^{\frac{1}{4}}$.

4. The curve, $3ay^2 + a^3 = 2x^3$. *Ans.* $\frac{x^2}{a}$.

5. The curve, $y^2 = 2a^2 \log x$. *Ans.* $\frac{a^2}{x}$.

6. Prove that the subtangent of the cissoid is a maximum, when $x = 3a - a\sqrt{3}$; and that of the witch a minimum, when $x = \frac{1}{2}a$.

7. Prove that the curve whose equation is $x^2y = x^3 - a^3$, has the axis of y continued in the negative direction as one asymptote; and as another a straight line drawn through the origin in both directions, and making, with the axis of x an angle of 45°. Show, also, that the curve is concave to the axis of x, when x is greater than a, but everywhere else convex.

8. If the equation of a curve be $y^3 - x^3 = 3ax^2$, prove that it has asymptotes passing through the points $x = -a, y = 0$, and $x = 0, y = a$.

IX.—ON THE QUADRATURE AND RECTIFICATION OF PLANE CURVES, ETC.

166. LET AB (*fig.* 5) be a curve, and let it be required to determine the area of the space ACMP bounded by the curve, one of the axes, and two ordinates. Draw the ordinate QR so that the successive ordinates between it and MP may form either an increasing series or a decreasing one, which may always be done by taking MQ sufficiently small. Then, if we put $CM = x$, $MQ = h$, $MP = y$, and $QR = y'$, the area ACMP is a function of x, and ACQR a like function of $x + h$, the nature of the function depending on the equation of the curve. Let the former area be denoted by A, and the latter by A′; and (No. 6) the differential coefficient of A will be found by dividing A′—A, that is, the space PMQR by h, and by determining the limit to which the quotient tends, when h is continually diminished towards zero. Now, drawing PD and RE parallel to MQ, we have the areas of the rectangles MD and QE equal respec-

tively to yh and $y'h$; and the area PMQR is always of a magnitude intermediate between MD and QE. But these rectangles tend continually to become equal, as h is diminished, since y' thus tends to coincide with y, and to become equal to it. Hence, therefore, the area of QE, and, *a fortiori*, that of PMQR, tend each to become yh. Ultimately, therefore, (No. 6),

$$\frac{A'-A}{h}=\frac{PMQR}{h}, \text{ becomes } \frac{dA}{dx}=y;$$

and consequently we have $dA=ydx$; and by integration, $A=\int ydx$. *To find*, therefore, *the area of a curve referred to rectangular coordinates, multiply one of the coordinates by the differential of the other, and integrate the result.**

167. If the nature of a curve be expressed by means of its polar equation, the method of finding its area may be investigated in the following manner:—Let the areas of the sectors COP and COR (*fig.* 6) be respectively A and A′: let also $OP=r$, the angle $XOP=\theta$, and the angle $XOR=\theta+h$, so that $POR=h$. Then A and A′ will be respectively the same functions of θ and $\theta+h$; and (No. 6) the differential coefficient of A will be found by dividing $A'-A$ by h, and taking the limit of the quotient, when h tends to become evanescent. Now, if OR be taken so near OP, that the intermediate radii may form either an increasing series or a diminishing one, and if the circular arcs PD, RE be described, the area of OPR will be of a magnitude intermediate between the areas of the circular sectors OPD, OER. But if h be diminished towards zero, OR will tend to coincide with OP, and to become equal to it; and therefore the sector OER, and, *a fortiori*, the area OPR, will each tend to become equal to the sector OPD. Now, (No. 30) the area of this sector is $\frac{1}{2}r\times PD=\frac{1}{2}r^2h$.† Ultimately, therefore,

$$\frac{A'-A}{h}=\frac{OPR}{h}, \text{ becomes } \frac{dA}{d\theta}=\tfrac{1}{2}r^2; \text{ and, consequently,}$$

$$dA=\tfrac{1}{2}r^2d\theta, \text{ and } A=\tfrac{1}{2}\int r^2d\theta.$$

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* Since the area of an oblique parallelogram is found by multiplying two of its adjacent sides by the sine of the contained angle, it follows, that if the axis of the coordinates be oblique, the result found as above must be multiplied by the sine of their angle of inclination.

† For (TRIG. No. 2) the angle POD is measured by the quotient obtained by dividing the arc PD by the radius OP; and therefore, conversely, $PD=OP\times POD=rh$.

168. To exemplify the use of the principles that have now been established regarding areas, let it be required to find the area of the common parabola, taking $y = p^{\frac{1}{2}} x^{\frac{1}{2}}$ as its equation. From this we have ydx or (No. 166) $d\text{A} = p^{\frac{1}{2}} x^{\frac{1}{2}} dx$; and therefore (by A_2, p. 41) $\text{A} = \frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}} + \text{C}$, the area required. If we wish to have the area between the vertex O (*fig.* 2) and any ordinate MP, we must find what C will become for this purpose. We effect this from the consideration, that if M coincide with O, we shall have, simultaneously, $x = 0$ and $\text{A} = 0$; and hence the foregoing equation becomes $0 = 0 + \text{C}$, and therefore $\text{C} = 0$; and we have simply $\text{A} = \frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}}$. This may be put under the form $\text{A} = \frac{2}{3} x p^{\frac{1}{2}} x^{\frac{1}{2}}$, or, by the equation of the curve $\text{A} = \frac{2}{3} xy$, and hence we arrive at the interesting conclusion, that the area of the parabolic space OMP is two thirds of MB, the rectangle under the coordinates.*

169. If we wish to have the area P′M′MP, intercepted between the two ordinates M′P′ and MP, let $\text{OM} = x$ and $\text{OM}' = x_0$: then the areas up to MP and M′P′, will be respectively $\frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}} + \text{C}$ and $\frac{2}{3} p^{\frac{1}{2}} x_0^{\frac{3}{2}} + \text{C}$; and by taking the difference of these we shall have $\text{P}'\text{M}'\text{MP} = \frac{2}{3} p^{\frac{1}{2}} (x^{\frac{3}{2}} - x_0^{\frac{3}{2}})$. This definite integral may be thus expressed in Fourier's notation, $\int_{x_0}^{x} ydx$, or $\int_{x_0}^{x} d\text{A} = \frac{2}{3} p^{\frac{1}{2}} (x^{\frac{3}{2}} - x_0^{\frac{3}{2}})$. If $x_0 = 0$, this becomes the same that we obtained in the last No. It becomes also, by an easy modification, $\frac{2}{3}(xy - x_0 y_0)$, where $\text{M}'\text{P}' = y_0$.†

170. As another example, let it be required to find the

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* Hence, by drawing a parallel to OM at a distance from it equal to two thirds of PM, we should have a rectangle equal to the parabolic area; and a square equal to this would be found by Euc. II. 14. The parabola therefore admits of exact *quadrature* or *squaring*. The quadrature of this curve was discovered by Archimedes, and being the first complete solution of a problem of the kind that was ever given, it was regarded as one of his principal discoveries. The simplicity and brevity of the method in which the problem can now be solved, when compared with the method discovered by that great geometer, show in a striking manner the power and excellence of the modern analysis, even in one of its simplest applications.

† Thus, if $\text{OM} = 90$ feet, $\text{MP} = 60$, and $\text{OM}' = 10$, we find $p = 40$ (by dividing 60^2 by 90), and thence $\text{M}'\text{P}' = \sqrt{(40 \times 10)} = 20$; and, by what has been shown in the text, we find the area P′M′ MP to be $3466\frac{2}{3}$ square feet. We should have separately also $\text{OMP} = 3600$, and $\text{OM}'\text{P}' = 133\frac{1}{3}$.

area of any of the curves defined by the general equation, $x^m y^n = a$.* Here we have

$$y^n = ax^{-m}, \quad y = a^{\frac{1}{n}} x^{-\frac{m}{n}}, \quad \text{and } dA = ydx = a^{\frac{1}{n}} x^{-\frac{m}{n}} dx;$$

and hence, by formula A_2, p. 41, we get

$$A = \frac{n}{n-m} a^{\frac{1}{n}} x^{\frac{n-m}{n}} + C; \text{ and, therefore,}$$

$$\int_{x_0}^{x} dA = \frac{n}{n-m} a^{\frac{1}{n}} \left(x^{\frac{n-m}{n}} - x_0^{\frac{n-m}{n}} \right) = \frac{n}{n-m} (xy - x_0 y_0).$$

In the hyperbolic curves this area will plainly be infinite, when x is infinite, if $m < n$; but finite, if $m > n$: and hence it appears, that in the first case, the area commencing at any ordinate, and comprehended between the curve and the axis of x, both extended infinitely, is of infinite magnitude; while in the other there is a limit, $\frac{n}{m-n} x_0^{\frac{n-m}{n}}$, to which the area may be made to approach as nearly as we please, but which it can never reach. This arises from the circumstance, that the curve approaches the asymptote more rapidly in the latter case than in the former. It is easy to show, by taking $x = 0$, that exactly the reverse takes place with regard to the space between the curve and the other asymptote.

171. The result found in the last No. fails in reference to the common equilateral hyperbola. For it, however, using b^2 for a, we have $dA = b^2 x^{-1} dx$; and therefore (p. 41, B) $A = b^2 \log x + C$, the required area. If we take the area as commencing from $x = b$, we get, when $x = b$, $0 = b^2 \log b + C$, and therefore $C = -b^2 \log b$. Hence, in that case we have

$$\int_{b}^{x} dA = b^2 \log \frac{x}{b},$$

where b is the value of either x or y at the vertex of the curve; and if this be taken equal to unity, we have the area simply equal to $\log x$. It thus appears, that in this

* These curves are hyperbolas when m and n have the same signs; otherwise, they are parabolas. Each curve of the hyperbolic kind will plainly have the axes of coordinates as asymptotes. When m and n are each equal to unity, the curve is the common equilateral hyperbola; but when one of them is ± 1, and the other ∓ 2, it is the common parabola. If $x = \pm 1$ and $y = \mp 1$, the locus is a straight line.

particular equilateral hyperbola, the area comprehended between the curve and the asymptote in one direction, and between the fixed ordinate through the vertex and any other ordinate in the other direction, is the Neperian logarithm of the abscissa belonging to the latter ordinate; a circumstance which, for a long time, caused these logarithms to be called *hyperbolic* ones. If, however, we had taken the area as commencing from the ordinate corresponding to $x=1$, we should have had $C=0$, and consequently $\int_1^x dA = b^2 \log x$; and, therefore, (No. 91) we should have the logarithms in any system whatever, merely by assuming b^2 equal to the modulus of the system; and hence this application of the term *hyperbolic* is improper.

172. Let it now be required to find the area of the circle, the equation of which, the centre being the origin, and a the radius, is $y^2=a^2-x^2$. Here we have ydx or $dA = \sqrt{(a^2-x^2)}\,dx$. To integrate this in an easy manner, we may assume $\sqrt{(a^2-x^2)} = zx$, from which, by squaring, &c. we get $x^2 = \dfrac{a^2}{1+z^2}$. We thus have $dA = zxdx$; whence, by integration by parts (p. 41, K), we get

$$A = \tfrac{1}{2}zx^2 - \tfrac{1}{2}\int x^2\,dz = \tfrac{1}{2}xy - \tfrac{1}{2}a^2\int\frac{dz}{1+z^2};$$

the latter form being obtained by substituting for zx^2 its value $x\sqrt{(a^2-x^2)}$ or xy, and for x^2 in the second term its value found above. From this, again, by G, p. 41, and by restoring the value of z, we get

$$A = \tfrac{1}{2}xy - \tfrac{1}{2}a^2\tan^{-1}\frac{\sqrt{(a^2-x^2)}}{x} + C = \tfrac{1}{2}xy - \tfrac{1}{2}a^2\tan^{-1}\frac{y}{x} + C.$$

Hence, if we take $x_0=0$, we shall have

$\displaystyle\int_0^x dA = \tfrac{1}{2}xy - \tfrac{1}{2}a^2\tan^{-1}\frac{y}{x} + \tfrac{1}{4}\pi a^2$; the value of y corresponding to $x=0$ being 1: and from this, by taking $x=a$, and consequently, $y=0$, we find the area of the quadrant to be $\frac{1}{4}\pi a^2$; and therefore that of the entire circle is πa^2, a result to which we should have been led by means of No. 30.

173. Since, in the ellipse we have $y=\dfrac{b}{a}\sqrt{(a^2-x^2)}$, and

consequently $A = \int y dx = \frac{b}{a}\int \sqrt{(a^2 - x^2)}\,dx$, we see, from comparing this with the investigation in the last No. that if, by what is there established, we find the area of a portion (or the whole) of the circle whose radius is a, the area of the corresponding portion (or of the whole) of the ellipse will be obtained by multiplying it by b and dividing the product by a.

174. If we wish to find the area of the hyperbola referred to its principal axes, we have $ay = b\sqrt{(x^2 - a^2)}$; and therefore

$$A = \int y dx = \frac{b}{a}\int \sqrt{(x^2 - a^2)}\,dx$$

$$= \tfrac{1}{2}xy - \tfrac{1}{2}ab\log\{x + \sqrt{(x^2 - a^2)}\} + C.$$

The integration here is easily affected in the manner followed in No. 172, except that G_2 is to be employed instead of G: and, by obvious modifications, we should get in other forms, $A = \frac{1}{2}xy - \frac{1}{2}ab\log(bx + ay) + C$,

$$\text{and } A = \tfrac{1}{2}xy - \tfrac{1}{2}ab\log\left(\frac{x}{a} + \frac{y}{b}\right) + C.$$

175. To find the area of the cycloid, we get from the two values of x and y (App. No. 19) $dx = a(1 - \cos\omega)\,d\omega$, and thence $dA = ydx = a^2(1 - \cos\omega)^2 d\omega$. This, by the actual squaring of $1 - \cos\omega$, and by putting (Trig. No. 29) $\frac{1}{2} + \frac{1}{2}\cos 2\omega$ for $\cos^2\omega$, becomes $dA = a^2(\frac{3}{2} - 2\cos\omega + \frac{1}{2}\cos 2\omega)\,d\omega$; whence ($D_2$, p. 41) $A = a^2(\frac{3}{2}\omega - 2\sin\omega + \frac{1}{4}\sin 2\omega) + C$. By taking this integral between the limits $\omega = 0$ and $\omega = \pi$, and doubling the result, or by taking it at once between $\omega = 0$ and $\omega = 2\pi$, we arrive at the interesting conclusion, that the entire area bounded by any branch of the cycloid and the axis of x is $3\pi a^2$, or (No. 171) triple the area of the generating circle. From the value found above for A, it would be easy to eliminate ω, and thus to get the area expressed in terms of x or y or both. It would be of little use, however.

176. As an exemplification of the method of finding the areas of curves by means of their polar equations, let us take the general equation (App. No. 24) $r = a\theta^n$, and we

get $\frac{1}{2}r^2d\theta$ or (No. 165) $dA=\frac{1}{2}a^2\theta^{2n}\,d\theta$: the integral of which (A_2, p. 41) is

$$A=\frac{a^2\theta^{2n+1}}{2(2n+1)}+C,\text{ or }A=\frac{r^2\theta}{2(2n+1)}+C.$$

Taking this between the limits θ_0 and θ, we get

$$\int_{\theta_0}^{\theta}dA=\frac{a^2(\theta^{2n+1}-\theta_0^{\,2n+1})}{2(2n+1)}=\frac{r^2\theta-r_0^{\,2}\theta_0}{2(2n+1)}.$$

177. By taking, in this, n successively equal to 1, —1, and $\frac{1}{2}$ (App. 24), we obtain the following areas:—

For the spiral of Archimedes, $A=\frac{1}{6}a^2(\theta^3-\theta_0^{\,3})$;

For the hyperbolic spiral, $A=\frac{1}{2}\left(\frac{a^2}{\theta^0}-\frac{a^2}{\theta}\right)$;

For the parabolic spiral, $A=\frac{1}{4}a^2(\theta^2-\theta_0^{\,2})$.

For the lituus, in which $n=-\frac{1}{2}$, the foregoing solution fails. In reference to it, however, we have

$$\tfrac{1}{2}r^2d\theta=dA=\tfrac{1}{2}a^2\frac{d\theta}{\theta},\text{ and }\int_{\theta_0}^{\theta}dA=\tfrac{1}{2}a^2(\log\theta-\log\theta_0)=\frac{1}{2}a^2\log\frac{\theta}{\theta_0}.$$

178. We may now proceed to the *rectification* of curves, that is the finding of their lengths. To effect this we must first find the differential of AP (*fig.* 19), a portion of any curve AB in terms of the coordinates, x and y (OM and MP), of the extremity P at which it is regarded as varying. For determining this, suppose OM to receive the increment $MQ=h$, and let the arc $AP=s$, the arc $AR=s'$, the chord $PR=c$, and $QR=y'$: then (Euc. I. 47),

$$c^2=h^2+(y'-y)^2,\text{ and therefore }\frac{c^2}{h^2}=1+\frac{(y'-y)^2}{h^2}.$$

Now, when h is continually diminished towards zero, $y'-y$, c, and $s'-s$ will also be continually diminished, and will vanish simultaneously with h; and the chord c, and the arc $s'-s$ will evidently tend to coincide as R approaches P; and therefore unity will be the limit to which their ratio will tend, as they become smaller and smaller; so that ultimately either of them may be used for the other. Hence, by No. 6, the foregoing equation will become

$$\frac{ds^2}{dx^2}=1+\frac{dy^2}{dx^2};\quad\text{whence }ds^2=dx^2+dy^2.$$

It thus appears, that *to find the square of the differential of an arc, we are to add together the squares of the differentials of its coordinates.*

179. What was obtained in the last No. is easily expressed in terms of polar coordinates. For, since (No. 158) $x=r\cos\theta$ and $y=r\sin\theta$, we have $dx=\cos\theta dr-r\sin\theta d\theta$, and $dy=\sin\theta dr+r\cos\theta d\theta$: and taking the sum of the squares of these, and in the result using 1 instead of $\sin^2\theta+\cos^2\theta$, we get dx^2+dy^2 or $ds^2=dr^2+r^2d\theta^2$.

180. As a first example of the application of the principles now established, let us find the length of the common parabola, taking $y^2=4ax$ as its equation. From this, by differentiating, and dividing by 2, we get $ydy=2adx$: from which, by dividing by y, squaring, putting $4ax$ for y^2, and adding dx^2, we get, after easy modifications, and, by taking the square root,

$$\sqrt{(dx^2+dy^2)}, \text{or } ds=\left(\frac{x+a}{x}\right)^{\frac{1}{2}}dx.$$

For integrating this in an easy manner we may put $\left(\frac{x+a}{x}\right)^{\frac{1}{2}}=z$, which, by squaring, &c. gives $x=\frac{a}{z^2-1}$, and $ds=zdx$. From the latter we get by integration by parts, &c. (p. 41, K and G_2)

$$s=zx-\int xdz=zx-a\int\frac{dz}{z^2-1}=$$

$$zx-\tfrac{1}{2}a\log(z-1)+\tfrac{1}{2}a\log(z+1)+C:$$

and this, by restoring the value of z, and by some easy modifications, becomes

$$s=(x^2+ax)^{\frac{1}{2}}+\tfrac{1}{2}a\log\frac{a+2x+2(x^2+ax)^{\frac{1}{2}}}{a}+C.$$

For giving the length of the parabola commencing from the vertex, we must take $C=0$, as x and s must then be each equal to zero.*

* As an instance of the method of calculating integrals, let it be required to find the length of the line described by a cannon ball, projected on a horizontal plane, so as to descend at the distance of 4,800 feet from the cannon, after having obtained the height of 1,600 feet at its greatest elevation; its path being supposed to be a parabola, as it would be very nearly, were its motion not resisted by the air. Here

181. As another example, let it be required to find the length of the cycloid. From No. 156, we have $dx^2 = \left(\frac{y}{2a-y}\right) dy^2$, and by adding dy^2, and taking the square roots of the equals so obtained, we get

$$\sqrt{(dx^2 + dy^2)} = ds = (2a)^{\frac{1}{2}} (2a-y)^{-\frac{1}{2}} dy.$$

Then, by integrating this according to A_2, p. 41, we obtain $s = C - 2(2a)^{\frac{1}{2}}(2a-y)^{\frac{1}{2}} = C - 2(4a^2 - 2ay)^{\frac{1}{2}}$. Taking this between the limits $y_0 (= RP)$ and $y = 2a$, we get the arc VP (*fig.* 18) $= 2(4a^2 - 2ay_0)^{\frac{1}{2}} = 2(CV^2 - CV.CK)^{\frac{1}{2}} = 2(CV.VK)^{\frac{1}{2}} = 2VH$: and hence we arrive at the interesting conclusion, that the arc VP, measured from the vertex, is double of the corresponding chord VH in the generating circle. Hence, also, the length of the entire branch AVB is four times the diameter of the generating circle. This curve, therefore, is rectifiable, since we can find a straight line (the double of the chord VH) to which it is exactly equal.

182. The *cubature* of bodies, formed by the revolutions of plane surfaces about fixed axes, that is, the finding of their volumes or contents, is an important problem in mensuration. To find the method of doing this, let the line AB (*fig.* 20) by revolving about the axis OX, describe the body AFGP; and let v be put to denote the volume of this corresponding to the coordinates $OM = x$, and $MP = y$. Let x receive the increment $MQ = h$: then the bodies AFGP, $AFHR (= v')$, will be like functions of x and $x + h$. Let PD and RE be drawn parallel to OX, and during the revolution the rectangles MD, QE will describe the cylinders PGKD, ELHR, one of which is greater and the other less than the body PGHR, which is the increment of AFGP. Now, the volume of a cylinder being found by multiplying the area of its base by its height or thickness, we have the

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we have $x = 1600$, and $y = 2400$; and since $y^2 = 4ax$, we find $a = 900$. Substituting these in the value found above for the arc, we get

$$s = \sqrt{(1600^2 + 900 \times 1600)} + 450 \log \frac{900 + 3200 + 2\sqrt{(1600^2 + 900 \times 1600)}}{900}.$$
$$= 2000 + 450 \log 9 = 2000 + 450 \times 2{\cdot}1972246 = 2988{\cdot}75107 \text{ feet};$$

the double of which, 5977·50214, is the entire space described by the ball.

The parabola, it may be remarked, admits only of approximate, and not of exact, rectification, since the Neperian logarithms of all numbers except those of ϵ, ϵ^2, ϵ^3, $\sqrt{\epsilon}$, &c. are incommensurable, and can, therefore, be assigned only approximately.

volume of ELHR$=\pi y'^2 h$, and that of PGKD$=\pi y^2 h$. These cylinders tend to become equal as h is diminished towards zero; and therefore the volume of ELHR, and *a fortiori*, that of PGHR, tend each to become $\pi y^2 h$. Ultimately, therefore, (No. 6)

$$\frac{v'-v}{h}=\frac{\text{PGHR}}{h}, \text{ becomes } \frac{dv}{dx}=\pi y^2,$$

and consequently we have $v=\pi\int y^2 dx$. Hence, it appears, that *to find the volume of a body of revolution, we are to multiply the area of a circle whose radius is the revolving coordinate, by the differential of the others, and to integrate the result.*

183. The area, S, of the surface of a body of revolution will be found by integrating the equation, $dS=2\pi y ds$. To prove this, let the surfaces of the bodies (*fig.* 20) AFGP and AFHR be represented by S and S′ Then, since the arc PR and its chord c, tend continually to coincide as h is diminished towards zero, it is plain that the surface described by the arc will, as h is thus diminished, tend to coincide with the conical surface described by its chord, and therefore to become equal to it. Now, by mensuration, the area of this conical surface is $\pi(y'+y)c$:* and therefore, ultimately, $\frac{S'-S}{h}$ will become $\frac{dS}{dx}=\frac{2\pi y ds}{dx}$, y' becoming ultimately equal to y, and $\frac{c}{h}$ to $\frac{ds}{dx}$. Hence, by multipling by dx, and integrating the result, we get the theorem above enunciated.

184. To exemplify the principles established in the last two Nos. let us investigate the volume and the surface of the sphere. Here, taking the extremity of the diameter of

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* If the curve surface of a cone be cut along its slant side by a straight line passing from its vertex to its base, and be then developed or spread out into a plane, it will become a sector of a circle, having the slant height as its radius. To find, then, the area of the curve surface of a frustum of the curve is nothing more than to find the difference of the areas of two sectors, having as radii the slant heights of the whole cone and of the part cut off. Let these slant heights be denoted by σ' and σ; and then from the similarity of the sectors, if the arc of one of them be $2qs'$, that of the other will be $2qs$; the two quantities, $2qs'$ and $2qs$, being plainly the circumferences of the two bases of the conical frustum. Now, (No. 30) the areas of the two sectors are qs'^2 and qs^2, the difference of which is qs'^2-qs^2, or $(s'+s)(qs'-qs)$, which, with the proper adaptations, gives the formula employed in the text.

the revolving semicircle, as the origin of the coordinates, we have, $y^2=2ax-x^2$; whence, $\pi y^2 dx=dv=\pi(2axdx-x^2dx)$, the integral of which is $v=\pi(ax^2-\frac{1}{3}x^3)+C$. This, without any constant quantity, gives the volume of a segment of the sphere having its height or axis equal to x. Taking the integral also between $x=0$ and $x=2a$, we find the content of the entire sphere to be $\frac{4}{3}\pi a^3$. Hence it appears, that the sphere is two thirds of its circumscribed cylinder; since the content of the latter, found by multiplying πa^2, the area of its base, by $2a$, its length, is $2\pi a^3$, of which $\frac{4}{3}\pi a^3$ is two thirds.*

185. To find S, the surface of the sphere, we have, taking the centre as origin, $y^2=a^2-x^2$. Differentiating this, halving, and squaring, we get $y^2dy^2=x^2dx^2$; by dividing the members of which by those of $y^2=a^2-x^2$, adding dx^2 to both members, contracting, and extracting the square root, we obtain $\sqrt{(dx^2+dy^2)}=\frac{a\,dx}{\sqrt{(a^2-x^2)}}$. Hence, (No. 182) we get $dS=2\pi a\,dx$, and consequently, $S=2\pi ax$; which for giving the surface (*fig.* 43) corresponding to the abscissa CO, requires no constant quantity. When $x=a$, this becomes $2\pi a^2$; the double of which, $4\pi a^2$, is the surface of the entire sphere. It appears, from this and from No. 277, that the surface of a sphere is equal to four times the area of one of its great circles. It also follows, that it is equal to the curve surface of its circumscribed cylinder; this latter surface being found by multiplying $2\pi a$, the circumference of its base, by $2a$, its length.†

186. It would appear, on the same principles as those employed in No. 181, that the result there obtained is not confined to solids of revolution, but is equally applicable to any body whatever, if its section, by a plane ABGQDC

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* Since the equation of the ellipse is $a^2y^2=b^2(2ax-x^2)$, we should find, in a manner exactly similar, that, in the spheroid, $v=\frac{b^2}{a^2}\cdot\pi(ax^2-\frac{1}{3}x^3)$; and that the content of the entire spheroid is $\frac{4}{3}\pi ab^2$, where a is the fixed semi-axis, and b the revolving one, the result (App. No. 5) being equally applicable in respect to the prolate and the oblate spheroid.

† If to this we add $2\pi a^2$, the sum of the areas of its ends, we get $6\pi a^2$ for the entire surface of the cylinder; of which the surface of the globe found above is two thirds. It appears from this, and from No. 184, that the volume and surface of the sphere are respectively two thirds of those of the cylinder described about it. This singular relation was discovered by Archimedes, who was so much pleased with it, that he wished to have a sphere inscribed in a cylinder engraved on his tomb.

(*fig.* 24), have its boundaries ABG and CDQ straight lines, or portions of a curve of which the equation is given; and if its sections, by parallel planes passing through any points, A, B, G, in either of these boundaries, be similar figures, whether rectilineal or curvilineal. In this case, s will be the area of the section made by the parallel plane passing through B, E, and D; and everything else is the same as in No. 181.

187. Hence we have the means of finding the content of a pyramid. Thus, putting x to denote the perpendicular distance (*fig.* 45) from the vertex V to the plane ABCDE parallel to the base, it is evident, from similar triangles, that any side of this plane is proportional to x; and, since (EUC. VI. 19, cor. 3) the area of the same plane is proportional to the square of any of its sides, it follows, that s, the area of ABCDE, is also proportional to the square of x, and may be represented by ax^2, a being constant. Hence, we have $dv = ax^2dx$; and, by integration, $v = \frac{1}{3}ax^3 + C$. This, by integrating between x' and x, will give, for the content of the frustum, $\frac{1}{3}a(x^3 - x'^3)$. If $x' = 0$, we have, for the content of the whole pyramid, $\frac{1}{3}ax^3$, or $\frac{1}{3}x \times ax^2$; and, this being one third of the product of the base and altitude, it follows, that the content of the pyramid is one third of the content of a prism of the same base and altitude.

EXERCISES.

1. Prove that the content of a cone is one third of the cylinder of the same base and altitude; and that if x be the altitude, and y and y' the radii of the bases of a frustum of a cone, the content of the frustum is equal to $\{yy' + \frac{1}{3}(y-y')^2\} \times \pi \times x$.

2. Prove that the content of the paraboloid (formed by the revolution of a parabola about its axis) is half of the circumscribed cylinder.

3. Prove that the content of the surface of the paraboloid, between x and x', is $\frac{1}{6}\pi a^{\frac{1}{2}}\{(a+4x)^{\frac{3}{2}} - (a+4x')^{\frac{3}{2}}\}$.

4. Given $ay^2 = x^3$, the equation of the semicubical parabola; prove that the length of the curve from $x=0$, is $\frac{8}{27}a\left(\frac{9}{4}\cdot\frac{x}{a}+1\right)^{\frac{3}{2}} - \frac{8}{27}a$.

5. Prove that if an oblate spheroid and a prolate one be generated from the same ellipse, the whole volume of the first is to that of the second, as the greater axis of the ellipse to the less.

6. The area of the lemniscata (APPENDIX, No. 25) is $\frac{1}{4}\sqrt{(a^4-v^4)}$; and the sum of the areas of the two loops is a^2. Required the proof.

7. Prove that, in the curve whose equation (APP. No. 27) is

$$v=a\sin n\theta, \quad s=\tfrac{1}{4}a^2(\theta-\theta')-\frac{1}{8n}a^2(\sin 2n\theta-\sin 2n\theta'),$$

s denoting the area comprehended between the curve, and the two lines making, with the fixed radius, the angles θ' and θ. Prove also, that the area of one of the leaves of the curve is equal to an nth part of a quadrant of the circle whose radius is a.

8. Prove that the content of the body formed by the revolution of the curve of sines (APP. No. 32) P′E′EP(*fig.* 40) about E′E, is $\frac{1}{2}\pi a^2(x-x')-\frac{1}{4}\pi a^3\left(\sin\frac{2x}{a}-\sin\frac{2x'}{a}\right)$, where $x'=\mathrm{OE}'$, and $x=\mathrm{OE}$. Prove, also, that the content of the body described by OCO′ is $\frac{1}{2}\pi^2a^3$, being to the inscribed sphere in the ratio of $\frac{3}{8}\pi$ to 1.

9. Prove that the content of the body described by the revolution of the cissoid about the axis AB, and commencing at A (*fig.* 14), is

$$\pi\left(8a^3\log\frac{2a}{2a-x}-4a^2x-ax^2-\tfrac{1}{3}x^3\right).$$

10. Prove that in the involute of the circle, the area AOP (*fig.* 49) is $\frac{(v^2-a^2)^{\frac{3}{2}}}{6a}$; and that the length of the arc AP is $\frac{v^2}{2a}$. See APPENDIX, No. 34.

11. If the equation of a curve be $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, prove that its length is $\frac{3}{2}a^{\frac{1}{3}}x^{\frac{2}{3}}+\mathrm{C}$.

12. If the equation of a curve be $6a^2xy=x^4+3a^4$, prove that the surface of the body described by its revolution about the axis of x is

$$\frac{\pi}{36}\left(\frac{x^6}{a^4}+12x^2-\frac{9a^4}{x^2}\right)+\mathrm{C}.$$

X.—ON CURVATURE.

188. If ϱ be the radius CP (*fig.* 19) of a circle which touches a straight line DE in any point P, it is plain that, in the immediate vicinity of P the circle will tend more and more nearly to coincide with DE, the greater the radius becomes; and that the less the radius is, the more rapidly will the circle turn away from the tangent on each side of the point of contact. Hence, the less the radius is, the greater or the more rapid is the curvature of the circle, and the greater the radius, the less is the curvature; and on this account ϱ^{-1}, the reciprocal of the radius, is naturally taken as the measure of curvature in different circles, as compared with one another.

189. Let us now consider a part PR (*fig.* 4) of any curve AB, that part having no point of inflexion, and we shall see plainly that the direction of the part extending from P to R will undergo a continuous change from point to point, and that the entire amount of this change will be measured by the magnitude of the angle TCT′ made by the tangents at P and R, the curve having at any point the direction of its tangent at that point. Now, it is plain, that the greater this change of direction is for a given distance PR, the greater must be the curvature of the line PR; and the less this change is, the less does PR deviate from the tangent TC, or, in other words, the less is its curvature. The mean curvature of the arc, therefore, will depend on the relative magnitudes of the angle TCT′ and the arc PR, and will, therefore, be represented by the ratio of the first to the second; that is, by $\frac{T'-T}{s'-s}$,* if we denote AP by s, AR by s', and the angles PTX and RT′X by T and T′ respectively. Now, if PR be continually diminished towards zero, by the approach of R to P, RT′ will tend continually to coincide with PT; and (No. 6) we obtain, ultimately, as the limit of the preceding ratio, $\frac{dT}{ds}$, which will be the *measure* or *index of curvature* at the point P.

190. Let now the normals PS and RS be drawn; and

* In reference to the diagram here referred to, T′—T is negative; but if the curve were convex to the axis, it would be positive. For simplicity, the sign — is here omitted, as it may without error, when the foregoing remark is recollected.

since a circle may be described about the quadrilateral SPCR, the angle S is equal to TCT′, that is T′—T. Now, supposing the chord of PR drawn, and denoting it by σ, and putting SP$=\varrho$, we have, in the triangle SPR, $\frac{\sin(\mathrm{T}'-\mathrm{T})}{\sigma}=\frac{\sin \mathrm{SRP}}{\varrho}$. The limit to which this (No. 6) will continually tend, as R approaches P, is $\frac{d\mathrm{T}}{ds}=\frac{1}{\varrho}$, since the chord PR and the arc PR tend to coincide and to become equal, and since the angle SPR tends to become the same as SRT′, and therefore to be a right angle. Comparing the result now obtained with that which was found in the last No. we see that the measure of curvature is $\frac{1}{\varrho}$; and this (No. 188) is the curvature of the circle whose radius is ϱ. That circle, therefore, having the same curvature as the curve at the point P, and therefore coinciding with it more nearly than any other circle touching it in the same point, is called for the first reason the *circle of curvature*, and for the second, the *osculating circle*, at the point P. Its radius ϱ, is often called the *radius of curvature*, and its centre S, the *centre of curvature*. This centre, it will be seen, is in the normal PS.

191. The equation found above enables us to determine the radius of curvature at any assigned point of a given curve. Other formulas, however, may be investigated which are in general preferable in practice. To find one of these, we have (No. 143) $\mathrm{T}=\tan^{-1}\frac{dy}{dx}$; and hence, by differentiating this according to No. 40, and multiplying the terms of the resulting fraction by dx^2, we get $d\mathrm{T}=\frac{d^2y\,dx}{dx^2+dy^2}=\frac{d^2y\,dx}{ds^2}$, the latter form being obtained by means of No. 177. Substituting this in the equation found in the last No. and resolving the result, we get $\varrho=\frac{ds^3}{d^2ydx}=\frac{(dx^2+dy^2)^{\frac{3}{2}}}{d^2ydx}$, the latter form being also obtained by means of No. 177.

In this x has been taken as the independent variable, and consequently dx as constant. Had this not been done, we should have got, by the same mode of investigation, $\varrho=\dfrac{ds^3}{dx\,d^2y-dy\,d^2x}$.

192. The radius of curvature may readily be expressed in terms of the polar coordinates of the curve. To effect this, we have simply (No. 158) to change x into $r\cos\theta$, and y into $r\sin\theta$ in the value of ϱ for rectangular coordinates. The doing of this will be facilitated by substituting for $\cos\theta$ and $\sin\theta$ their values according to No. 100. By this means we get

$$x=\tfrac{1}{2}r(\varepsilon^{\theta\sqrt{-1}}+\varepsilon^{-\theta\sqrt{-1}}),\text{ and } y\sqrt{-1}=\tfrac{1}{2}r(\varepsilon^{\theta\sqrt{-1}}-\varepsilon^{-\theta\sqrt{-1}});$$

whence $x+y\sqrt{-1}=r\varepsilon^{\theta\sqrt{-1}}$, and $x-y\sqrt{-1}=r\varepsilon^{-\theta\sqrt{-1}}$.

Then, by differentiating these, supposing θ to be the independent variable, and putting r' and r'' to denote the first and second differential coefficients of r, we obtain

$$dx+dy\sqrt{-1}=(r'+r\sqrt{-1})\varepsilon^{\theta\sqrt{-1}}d\theta\ldots\ldots\ldots\ldots\ldots(1),$$
$$d^2x+d^2y\sqrt{-1}=(r''-r+2r'\sqrt{-1})\varepsilon^{\theta\sqrt{-1}}d\theta^2\ldots\ldots\ldots(2),$$
$$dx-dy\sqrt{-1}=(r'-r\sqrt{-1})\varepsilon^{-\theta\sqrt{-1}}d\theta\ldots\ldots\ldots\ldots\ldots(3),$$
$$d^2x-d^2y\sqrt{-1}=(r''-r-2r'\sqrt{-1})\varepsilon^{-\theta\sqrt{-1}}d\theta^2\ldots\ldots(4),$$

Now, the product of the first members of (2) and (3)* is $dx\,d^2x+dy\,d^2y+(dx\,d^2y-dy\,d^2x)\sqrt{-1}$. In this the coefficient of $\sqrt{-1}$ is the denominator of the second value of ϱ in the last No.; and it is easy to see by inspection, that the corresponding coefficient in the product of their second members is $2r'^2-r(r''-r)d\theta^3$, or $(2r'^2-rr''+r^2)d\theta^3$. Using this, therefore, instead of the denominator of the value

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* We might use with equal advantage the product of (1) and (4), merely changing in that case the sign of the coefficient of $\sqrt{-1}$. The student will see that a chief principle employed in the investigation in the text is, that if an equation contain real and imaginary terms, the real terms in one member must be equal to those in the other, and the imaginary in the one to the imaginary in the other.

It would be easy to show, that if θ be not taken as the independent variable, the denominator of the value of ρ would be

$$r(dr\,d^2\theta-d\theta\,d^2\rho)+(2dr^2+r^2d\theta^2)d\theta.$$

It may be farther remarked, that the result found in No. 177 would be obtained at once by taking the products of the members of (1) and (3).

of ϱ last found, and (No. 179) $(r^2+r'^2)^{\frac{3}{2}}d\theta^3$ instead of its numerator, we get

$$\varrho=\frac{(r^2+r'^2)^{\frac{3}{2}}}{r^2-rr''+2r'^2},\quad \text{or,}\ \varrho=\frac{(r^2d\theta^2+dr^2)^{\frac{3}{2}}}{(r^2d\theta^2-rd^2r+2dr^2)d\theta},$$

by multiplying the numerator and denominator by $d\theta^3$.

193. The position of the centre of curvature will be known if its coordinates be known. These we shall find by means of the two equations $(\eta-y)dy+(\xi-x)dx=0$, and $(\eta-y)^2+(\xi-x)^2=\varrho^2$; the first of which (No. 145) is the equation of any point whatever in the normal, and the second (Euc. I. 47) fixes the position of the centre on the normal by means of its distance from the point of contact (x, y). From the first of these let us find the value of $\xi-x$, and substitute it in the second; then, by taking instead of ϱ, its first value found in No. 191, multiplying by dx^2, dividing by dx^2+dy^2, and extracting the square root, we get $\eta-y=\frac{dx^2+dy^2}{d^2y}$, and by substituting this in the first, we readily obtain $\xi-x=-\frac{dx^2+dy^2}{d^2y}\frac{dy}{dx}$. Had we taken the second value of ϱ we should have got

$$\eta-y=\frac{(dx^2+dy^2)dx}{dxd^2y-dyd^2x},\quad \text{and}\ \xi-x=-\frac{(dx^2+dy^2)dy}{dxd^2y-dyd^2x}.$$

194. In adapting the last formulas in the preceding No. to polar coordinates, we may, for the sake of simplicity, take $\theta=\frac{1}{2}\pi$, so as to make the fixed axis perpendicular to the radius vector, at whatever point of a curve we may be considering. Now, since (page 62, first note) $\varepsilon^{\frac{1}{2}\pi\sqrt{-1}}=\sqrt{-1}$, and, consequently, $\varepsilon^{-\frac{1}{2}\pi\sqrt{-1}}=-\sqrt{-1}$, the formulas, (1), (2), (3), (4), in No. 192, will become

$$dx+dy\sqrt{-1}=(r'\sqrt{-1}-r)d\theta,\ \text{or}\ dr\sqrt{-1}-rd\theta;$$

$$d^2x+d^2y\sqrt{-1}=(r''\sqrt{-1}-r\sqrt{-1}-2r')d\theta^2$$
$$=d^2r\sqrt{-1}-rd\theta^2\sqrt{-1}-2drd\theta;$$

$$dx-dy\sqrt{-1}=(-r'\sqrt{-1}-r)d\theta=-dr\sqrt{-1}-rd\theta;$$

$$d^2x-d^2y\sqrt{-1}=(-r''\sqrt{-1}+r\sqrt{-1}-2r')d\theta^2$$
$$=-d^2r\sqrt{-1}+rd\theta^2\sqrt{-1}-2drd\theta.$$

From these, by taking half the sum and half the difference of the first and third, and also of the second and fourth, and by dividing the half differences by $\sqrt{-1}$, we get

$$dx=-rd\theta,\quad dy=dr,\quad d^2x=-2drd\theta,\ \text{and}\ d^2y=d^2r-rd\theta^2;$$

and we have also $x=r\cos\frac{1}{2}\pi=0$, and $y=r\sin\frac{1}{2}\pi=r$. The substitution of these values of x, y, dx, &c. in the expressions at the end of the last No. will give

$$\eta-r=-\frac{r(dr^2+r^2d\theta^2)}{r^2d\theta^2-rd^2r+2dr^2},$$

$$\text{and}\ \xi=-\frac{dr^2+r^2d\theta^2}{r^2d\theta^2-rd^2r+2dr^2}\cdot\frac{dr}{d\theta}.$$

195. The following examples will illustrate the principles that have been established in the seven preceding Nos.

Exam. 1. Let it be required to find the radius of curvature of the common parabola, and the coordinates of its centre of curvature. Here, taking $y^2=ax$, as the equation of the parabola, we have

$$y=a^{\frac{1}{2}}x^{\frac{1}{2}};\quad dy=\frac{a^{\frac{1}{2}}dx}{2x^{\frac{1}{2}}};\quad \text{and}\ d^2y=-\frac{a^{\frac{1}{2}}dx^2}{4x^{\frac{3}{2}}}:\ \text{whence,}$$

$$dx^2+dy^2=\frac{a+4x}{4x}.\ dx^2,$$

$$(dx^2+dy^2)^{\frac{3}{2}}=\frac{(a+4x)^{\frac{3}{2}}}{8x^{\frac{3}{2}}}.\ dx^3,\quad \text{and}\ d^2ydx=-\frac{a^{\frac{1}{2}}dx^3}{4x^{\frac{3}{2}}}.$$

Hence, by No. 191 we get $\varrho=-\dfrac{(a+4x)^{\frac{3}{2}}}{2a^{\frac{1}{2}}}$. At the vertex of the curve, $x=0$, and ϱ becomes simply $-\frac{1}{2}a$. By No. 193, we readily find $\xi=3x+\frac{1}{2}a$, and $\eta=-\dfrac{4x^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, which are the coordinates of the centre.

Exam. 2. Let us now find the general value for the radius of curvature of a line of the second order. The general equation of a line of that order is $y^2=mx+nx^2$, from which we obtain

$$dy=\frac{(m+2nx)dx}{2y};\ \text{and}\ d^2y=\frac{2nydx^2-(m+2nx)dx\,dy}{2y^2}$$

or, by substituting for dy its value just found, and, by an easy reduction,

$$d^2y=\frac{\{4ny^2-(m+2nx)^2\}dx^2}{4y^3}.$$

Hence we get $dx^2+dy^2=\frac{\{4y^2+(m+2nx)^2\}dx^2}{4y^2}$, and

$$d^2ydx=\frac{\{4ny^2-(m+2nx)^2\}dx^3}{4y^3}.$$

From this, by No. 191, we find

$$\varrho=\frac{\{4y^2+(m+2nx)^2\}^{\frac{3}{2}}}{8ny^2-2(m+2nx)^2};$$

which, by substituting its value for y^2, is changed into

$$\varrho=-\frac{\{4(mx+nx^2)+(m+2nx)^2\}^{\frac{3}{2}}}{2m^2}.$$

For the parabola, in which $n=0$, this will become the same as in the foregoing example. For the ellipse, the value of ϱ will be the general one found above, with the signs of the terms containing n negative; while, for the hyperbola, no change is to be made.

Exam. 3. To find the radius of curvature of the logarithmic curve, we have (No. 155)

$$dy=\frac{yladx}{M}; \text{ whence, } d^2y=\frac{dyladx}{M}=\frac{y(la)^2dx^2}{M^2}:$$

and, by No. 191, $\varrho=\frac{\{M^2+y^2(la)^2\}^{\frac{3}{2}}}{My(la)^2};$

or, if a be the base of the system of logarithms, $\varrho=\frac{(M^2+y^2)^{\frac{3}{2}}}{My}$.

The coordinates of the centre are

$$\xi=x-\frac{M^2+y^2(la)^2}{Mla}, \text{ and } \eta=y+\frac{M^2+y^2(la)^2}{y(la)^2},$$

or, if a be the base,

$$\xi=x-M-\frac{y^2}{M}, \text{ and } \eta=2y+\frac{M^2}{y}.$$

Exam. 4. Let the curve be the cycloid. Then (No. 156), $\frac{dy}{dx}=\left(\frac{2a-y}{y}\right)^{\frac{1}{2}}$; and, by differentiating again, we get

$\frac{d^2y}{dx}=-\frac{ady}{y\sqrt{(2ay-y^2)}}$. If we multiply this by the foregoing, and divide by dy, we shall obtain $\frac{d^2y}{dx^2}=-\frac{a}{y^2}$ whence $d^2ydx=-\frac{adx^3}{y^2}$. By squaring the members of $\frac{dy}{dx}=\left(\frac{2a-y}{y}\right)^{\frac{1}{2}}$, adding 1 to the result, and multiplying by dx^2, we get $dx^2+dy^2=\frac{2adx^2}{y}$; and, consequently, $(dx^2+dy^2)^{\frac{3}{2}}=\left(\frac{2a}{y}\right)^{\frac{3}{2}}dx^3$. Hence, No. 191,

$$\varrho=-\left(\frac{2a}{y}\right)^{\frac{3}{2}}\times\frac{y^2}{a}=-2\sqrt{2ay}.$$

Now, since VC (*fig.* 18) $=2a$, and CK$=y$, the chord HC or PD$=\sqrt{2ay}$; and, consequently, the radius of curvature is double of PD: and, since (No. 156) PD is the normal, the centre of the circle will be at P′ in the continuation of PD, P′D being equal to PD.

196. It is evident that, for any curve except the circle, the position of the centre of the osculating circle will be different for different points of the curve. The locus of this centre, or the line in which it is always found, is called the *evolute* of the curve; and the proposed curve, in relation to the evolute, is called its *involute*. The equation of the evolute of a curve will be found by eliminating x and y and their differentials by means of the two equations at the end of No. 193, or those at the end of No. 194, and of the equation of the given curve.

197. As an example, let the curve be the common parabola. Then (Exam. 1, p. 116), $\xi=3x+\frac{1}{2}a$, and $\eta=-\frac{4x^{\frac{3}{2}}}{a^{\frac{1}{2}}}$. The first of these gives $x=\frac{2\xi-a}{6}$; and, by substituting this value for x in the second, squaring the result, multiplying by a, and reducing, we get $a\eta^2=\frac{2}{27}(2\xi-a)^3$, the equation of the evolute, which is the curve called the semicubical parabola.*

* The equation of this curve, in its simplest form, is $ay^2=x^3$; and that found above will be reduced to this form by multiplying it by 27, dividing the product by 2, and then putting a to denote the coefficient of η^2, and $2\xi-a=x$.

198. As a second example, let it be required to find the equation of the evolute of the hyperbola, the equation of which is $a^2y^2=b^2(x^2-a^2)$. From this, by successive differentiations, we get

$$a^2ydy=b^2xdx, \text{ and } a^2dy^2+a^2yd^2y=b^2dx^2;$$

whence, by easy reductions,

$$\frac{dy}{dx}=\frac{b^2x}{a^2y}; \quad dx^2+dy^2=\frac{a^4y^2+b^4x^2}{a^4y^2}.\,dx^2; \quad \text{and } d^2y=-\frac{b^4dx^2}{a^2y^3}.$$

By substituting these values in those of ξ and η in (No. 193), we find $\xi=\frac{(a^2+b^2)x^3}{a^4}$, and

$$\eta=-\frac{(a^2+b^2)y^3}{b^4}=-\frac{a^2+b^2}{a^3b}.(x^2-a^2)^{\frac{3}{2}}=-\frac{c^2}{a^3b}.(x^2-a^2)^{\frac{3}{2}};$$

where $c^2=a^2+b^2$. The first of these gives $x=\left(\frac{a^4\xi}{a^2+b^2}\right)^{\frac{1}{3}}=\frac{a^{\frac{4}{3}}\xi^{\frac{1}{3}}}{c^{\frac{2}{3}}}$; and the second, $x=\frac{a}{c^{\frac{2}{3}}}(b^{\frac{2}{3}}\eta^{\frac{2}{3}}+c^{\frac{4}{3}})^{\frac{1}{2}}$. Putting these values of x equal to one another, squaring, &c. we get $a^{\frac{2}{3}}\xi^{\frac{2}{3}}=b^{\frac{2}{3}}\eta^{\frac{2}{3}}+c^{\frac{4}{3}}$, which is the equation of the evolute of the hyperbola.

For the ellipse, b^2, and consequently $b^{\frac{2}{3}}$, have the contrary sign, and the equation becomes $a^{\frac{2}{3}}\xi^{\frac{2}{3}}=c^{\frac{4}{3}}-b^{\frac{2}{3}}\eta^{\frac{2}{3}}$. In this case, $c^2=a^2-b^2$. The value of η will be imaginary, when $c^{\frac{4}{3}}$ is less than $a^{\frac{2}{3}}\xi^{\frac{2}{3}}$, or, which is the same, when c^2 is less than $a\xi$: so, also, will the value of ξ, when c^2 is less than $b\eta$.

In *fig.* 9, QK and QK′ are the two branches of the evolute of the parabola; in *fig.* 8, QK, QK′, qk, qk', are the branches of the evolute of the hyperbola; and in *fig.* 25, QK, QK′, qK, and qK′, are those of the evolute of the ellipse. It is evident that, in each curve, the several branches are equal and similar.

199. To find the evolute of the cycloid, we have (No. 156) $\frac{dy}{dx}=\left(\frac{2a-y}{y}\right)^{\frac{1}{2}}$; and thence we readily get

$$\frac{d^2y}{dx^2}=-\frac{a}{y^2}, \quad \frac{dx^2+dy^2}{dx^2}=\frac{2a}{y}, \quad \frac{dx^2+dy^2}{dx^2}\times\frac{dx^2}{d^2y},$$

$$\text{or } \frac{dx^2+dy^2}{d^2y}=-2y.$$

Hence, from No. 193, we obtain

$$\eta=-y,\quad \text{and } \xi=x+2\sqrt{(2ay-y^2)}.$$

By substituting these in the equation of the cycloid, we get $-\frac{\eta}{a}=\text{versin}\frac{\xi-\sqrt{(-2a\eta-\eta^2)}}{a}$, which is possible only when η is negative. Let us now transfer the origin to A′ (*fig.* 18) in the continuation of VC, CA′ being equal to VC; and let A′B′, parallel to AB, be taken as the axis of the abscissas. This will be effected by substituting $\eta'-2a$ for η, and $\pi a-\xi'$ for ξ, πa being equal to AC. By this means, the equation found above becomes $2-\frac{\eta'}{a}=\text{versin}\left\{\pi-\frac{\xi+\sqrt{(2a\eta'-\eta'^2)}}{a}\right\}$; and as $\text{versin}(\pi-A)=2-\text{versin } A$, this equation is at once reduced to $\frac{\eta'}{a}=\text{versin}\frac{\xi'+\sqrt{(2a\eta'-\eta'^2)}}{a}$; which, since $\xi'=$ A′R′, and $\eta'=$ R′P′, is evidently the equation of a cycloid having A′ as its origin, and described by a generating circle equal to the original one, and moving in the direction A′B′.*

200. To advance another step in the theory of evolutes, we get, by differentiating $(\eta-y)dy+(\xi-x)dx=0$, the equation (No. 145) of the normal, regarding dx as constant, $(\eta-y)d^2y+dy\,d\eta-dy^2+dx\,d\xi-dx^2=0$; which becomes simply $dy\,d\eta+dx\,d\xi=0$, since, by No. 193 $(\eta-y)d^2y=dx^2+dy^2$. By equalling the values of $\frac{dy}{dx}$ found from this and from the equation of the normal, and by an easy reduction, we get

$$\eta-y=(\xi-x)\frac{d\eta}{d\xi}.$$

Now, (No. 144) this equation is that of a straight line touching the evolute at the point ξ, η, and passing through the point x, y in the original curve: and since the first of these points is the centre of the osculating circle, and the second the point in which that circle meets the curve, it follows

* From this it appears, that, if there be two parallels, and if a series of cycloids be described on each of them, on the same side having the diameters of their generating circles equal to the distance between the parallels; and if the cycloids be so situated that the extremities of the bases of the one series may fall on the vertices of the other series, those between the parallels will be the evolutes of the others. The conclusion above obtained affords the means, in connexion with No. 201, of making a pendulum move in an arc of a cycloid.

that the radius of the osculating circle touches the evolute, and that the centre of that circle is the point of contact.

201. By differentiating the second of the equations at the commencement of No. 193, we get

$$(\xi-x)d\xi-(\xi-x)dx+(\eta-y)d\eta-(\eta-y)dy=\varrho d\varrho;$$

which, by means of the first of the same two equations, is reduced to

$$(\xi-x)d\xi+(\eta-y)d\eta=\varrho d\varrho.$$

By substituting in this equation, and in $(\xi-x)^2+(\eta-y)^2=\varrho^2$, the value found for $\eta-y$ in the preceding No., and putting σ to denote the arc of the evolute corresponding to the co-ordinates ξ and η, we get, after easy modifications,

$$(\xi-x)^2\frac{d\sigma^2}{d\xi^2}=\varrho^2, \qquad \text{and } (\xi-x)\frac{d\sigma^2}{d\xi}=\varrho d\varrho;$$

and by dividing the members of the latter of these equations by the square roots of those of the former we get

$d\sigma=\pm d\varrho$, and thence, by integration, $\sigma=C\pm\varrho$.

From this we have $\sigma\mp\varrho=C$; and it thus appears that if σ and ϱ increase simultaneously, they increase equally, having the constant difference C; while, if one of them diminish as the other increases, the increment and decrement must be equal, the sum of σ and ϱ being then equal.

From this property, and from the one established in the last No., we arrive at the very interesting conclusion, that, *if an inextensible thread be applied to the evolute, and, being kept tight, be gradually unwound, a fixed point in it will describe the original curve, or involute.* Thus, if O (*fig.* 18) be taken as the commencement of the arc of the evolute, and if $OP'=\sigma$ and $PP'=\varrho$; then, if $PP'=AP'$, and if we put $OA=C$, we shall have $\sigma=C+\varrho$, wherever P is taken; which agrees with the expression found above, with the upper sign. If, again, O′ were taken as the origin of the arc, and consequently $\sigma=O'P'$, and $O'A=C$, we should have $\sigma=C-\varrho$, the expression found above, taken with the lower sign; the difference of sign resulting solely from the direction in which the arc is measured, ϱ and σ having the same sign when they both increase or both diminish; but opposite signs, when one of them diminishes as the other increases. It appears, therefore, that wherever the corresponding points P and P′ are taken, AP′ and PP′ will be always equal, so that if a thread, applied to AA′, be unwound by moving the point P from A towards V, P will describe the involute AV: and hence we have the origin of the names, *involute* and *evolute*.*

The expression found in No. 191 for the radius of curvature, consisting obviously of a finite number of terms in all algebraic curves, it follows from this No. and the last, that the evolute of every such curve is *rectifiable*, or is such that a straight line may be found exactly equal to it; and hence, of algebraic curves, there are *more* which are rectifiable than the contrary. The evolutes of many transcendental curves, also, are rectifiable.

202. As an example in reference to polar coordinates, let us find the evolute of the logarithmic spiral. In this case we have

$$\log r = \theta \log a, \qquad dr = r\,d\theta \log a,$$

$$\text{and } d^2r = dr\,d\theta \log a = r\,d\theta^2 (\log a)^2;$$

and, by substituting these in the formulas found in Nos. 194 and 192, we get

$$\eta = 0, \qquad \xi = -r \log a, \qquad \text{and } \rho = r\{1+(\log a)^2\}^{\frac{1}{2}}.$$

Hence (*fig.* 26), if P′ be the centre of the osculating circle at P, and if OG be drawn perpendicular to the radius vector PO, the point P′ will be in this perpendicular, since $\eta = 0$. We have also OP′, on the negative side of O, equal to $r \log a$; and dividing OP by OP′, we find $-\dfrac{1}{\log a}$ for the tangent of OP′P, which (App. No. 23) is the same as that of the angle OPG. Hence, since OP′ is the radius vector of the evolute, and (No. 199) PP′ a tangent to it at P′, it follows (App. No. 23), that the evolute is also a logarithmic spiral, similar to the proposed one,—a remarkable property of this curve, analogous to that of the cycloid pointed out in No. 199.

203. If two curves have a common point (x, y), they are said to have with one another at that point a *contact of the first order*, if the first differential coefficients of y derived from their equations, and only these be equal, so that (No. 143) they have a common tangent; a *contact of the*

* The French writers, for a similar reason, call the evolute the *développée*, and the involute the *développante*.

second order, if the first and second differential coefficients, and no others be respectively equal; and, in general, they have a *contact of the nth order*, if the first n differential coefficients, and none after them, in the one be respectively equal to the first n in the other. It may naturally be supposed, that the higher the order of contact is, it is likewise the closer; and this is found to be the case. To prove it, let the curves,

$$y=fx, \qquad y=f_2x, \qquad \text{and } y=f_3x,$$

have a common tangent at a common point (x, y); and suppose that the last of these curves has with the first a contact of the third order, and with the second only one of the second order, so that

$$f_3^1x=f^1x=f_2^1x, \quad f_3^2x=f^2x=f_2^2x, \quad \text{and } f_3^3x=f^3x;$$
$$\text{but } f_3^3x > \text{ or } < f_2^3x.$$

$$\text{Then will } \frac{f^1x-f_3^1x}{f_2^1x-f_3^1x} \text{ and } \frac{f^2x-f_3^2x}{f_2^2x-f_3^2x} \text{ be each } = \frac{0}{0}.$$

By changing x into $x+h$ in the latter, we get

$$\frac{f^2(x+h)-f_3^2(x+h)}{f_2^2(x+h)-f_3^2(x+h)},$$

the numerator of which, as h is continually diminished, tends to become simply $f^3x-f_3^3x$, and therefore to vanish; while the denominator tends to become $f_2^3x-f_3^3x$; and therefore (hypothesis) it is finite. Hence, therefore, the numerator tends to become infinitely small in comparison of the denominator; that is, the amount of separation in the first and third curves produced by an infinitely small change in the value of x, is infinitely small compared with the separation produced by the same means in the second and third. It would be easy to generalise this proof by supposing the contacts to be of the mth and nth orders, instead of being of the third and second.

204. From development by means of Taylor's theorem, it would be seen that if the first differential coefficients that are unequal, be of an odd order, the same odd power of h will be a multiplier of that coefficient; and, therefore, the *difference* of the ordinate will have opposite signs for $x-h$ and $x+h$. Hence, in this case, the curves, however intimately they may coincide, intersect one another at the

point of contact; so that instead of simple contact, as the term naturally implies, there are both contact and intersection. It would appear on similar grounds, that when the contact is of an odd order, there is not intersection; one of the lines lying on the same side of the other on both sides of the point of contact in its immediate vicinity. Such is the case with regard to the rectilineal tangent, except at a point of inflexion.

205. As a useful example in reference to the contact of curves, let $y=fx$ be the equation of a curve, and let it be required to find a circle having contact of the second order with that curve at any point of it (x, y). If, for this purpose, we assume, as the equation of the required circle, $(\xi-x_1)^2+(\eta-y_1)^2=\varrho^2$, where ξ, η, and ϱ are constant quantities, or as they are often called in such cases, *parameters*, determining the position and magnitude of the circle, all that is necessary is to find the values of these parameters, so that x_1, y_1, f^1x_1, and f^2x_1 may be equal respectively to x, y, f^1x, and f^2x. To effect this, let us differentiate the equation of the circle twice: then, after changing some signs, &c. we get

$$\xi-x_1+(\eta-y_1)\frac{dy_1}{dx}=0, \qquad \text{and } 1+\frac{dy_1^{\,2}}{dx_1^{\,2}}-(\eta-y_1)\frac{d^2y_1}{dx_1^{\,2}}=0.$$

Hence, by making the change above indicated, we have the three equations,

$$(\xi-x)^2+(\eta-y)^2=\varrho^2, \qquad \xi-x+(\eta-y)\frac{dy}{dx}=0,$$

$$\text{and } 1+\frac{dy^2}{dx^2}-(\eta-y)\frac{d^2y}{dx^2}=0\,;$$

which, by elimination, will give the values of ξ, η, and ϱ. Now, the first and second of these are the same as the second and first equations at the beginning of No. 193 and therefore the values of $\xi-x$ and $\eta-y$ will be the same as those found in that No.; and by substituting, for the sake of simplicity, ds^2 for dx^2+dy^2, in these values, taking the sum of their squares, &c. we readily find $\varrho=\dfrac{ds^3}{dxd^2y}$, the same as the radius of curvature obtained in No. 191. Hence, since for the circle which we have now been investigating, ξ, η, and ϱ are the same as for the osculating circle, the two circles are identical; and the osculating circle is, therefore, the circle which has contact of the second order with the given curve.

206. It may be shown, in conclusion, that at a point of maximum or minimum curvature (such as at an extremity of an axis of an ellipse, and in numberless other instances), the osculating circle has curvature of the third order with the given curve. To prove this, we have from one of the formulas in the last No., after multiplying by dx_1^2,

$$dx_1^2+dy_1^2-(\eta-y_1)d^2y_1=0\,;$$

whence, by differentiating, &c. we get

$$3dy_1d^2y_1-(\eta-y_1)d^3y_1=0.$$

From the first of these expressions we find $(\eta-y_1)d^2y_1$ to be equal to $dx_1^2+dy_1^2$; and, therefore, by substituting this in the second expression, we get

$$d^3y_1=\frac{3dy_1d^2y_1^2}{dx_1^2+dy_1^2}.$$

Again, since No. 191, $\varrho dx\,d^2y=(dx^2+dy^2)^{\frac{3}{2}}$, we have, by taking the logarithms and doubling them

$$2\log\varrho+2\log dx+2\log d^2y=3\log(dx^2+dy^2).$$

By differentiating this, and, as ϱ is a maximum or minimum, taking $d\varrho=0$, we readily get $d^3y=\dfrac{3\,dy\,d^2y^2}{dx^2+dy^2}$. Now, this is the same as the value found above for d^3y_1, when x_1 and y_1 take the same values as x and y; and therefore (No. 203) the contact is of the third order.

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207. We have recently had instances in which, contrary to what was assumed in No. 80, dx was not to be taken as a constant quantity. As instances of a similar kind occur occasionally, it may not be improper to advert briefly to the subject here. For full information, the student must have recourse to the larger works on the differential calculus.

If, viewing the subject in its full generality, we differentiate $\dfrac{du}{dx}$ without regarding either dx or du as constant, we find, after dividing by dx,

$$\frac{d^2u}{dx}-\frac{du}{dx}\frac{d^2x}{dx^2}\,\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

which is to be used instead of $\frac{d^2u}{dx^2}$ found in the usual way. By a similar process we should get from this, after easy reductions,

$$\frac{d^3u}{dx^3} - \frac{du}{dx}\frac{d^3x}{dx^3} - 3\left\{\frac{d^2u}{dx^2}\frac{d^2x}{dx^2} - \frac{du}{dx}\left(\frac{d^2x}{dx^2}\right)^2\right\}\text{........}(2);$$

which, in like manner, is to be employed instead of $\frac{d^3u}{dx^3}$, found on the supposition that dx is constant. By farther differentiations on similar principles, expressions of a higher order might be found; but they would be complicated, and they are seldom needed in practice.

If it be required simply to render u the independent variable instead of x, the differential coefficients of u above the first will vanish, and the expressions just found will become

$$-\frac{du}{dx}\frac{d^2x}{dx}\text{..................................}(3),$$

$$\text{and} - \frac{du}{dx}\frac{d^3x}{dx^3} + 3\frac{du}{dx}\left(\frac{d^2x}{dx^2}\right)^2\text{..................}(4).$$

If x is to be eliminated, as being an assigned function of some third variable, this will be effected by employing formula (1), or formulas (1) and (2), as the case may require, and substituting for x and its differentials their proper values in terms of the third variable.

208. To exemplify these principles, let us "change the independent variable" in the equation,

$$\frac{du}{dx} + \left(\frac{du}{dx}\right)^3 + u\frac{d^2u}{dx^2} = 0;$$

that is, to change the equation into another having du constant and dx variable. By means of formula (3), this will become at once

$$\frac{du}{dx} + \left(\frac{du}{dx}\right)^3 - u\frac{du}{dx}\frac{d^2x}{dx^2} = 0;$$

and by multiplying this result by dx^3, and dividing the product by du^3, we get $\left(\frac{dx}{du}\right)^2 + 1 - u\frac{d^2x}{du^2} = 0$; which is the required equation.

209. As another example, let us change the equation,

$$x^3 \frac{d^3u}{dx^3} + 3x^2 \frac{d^2u}{dx^2} + x\frac{du}{dx} + au = 0,$$

into another having y as the independent variable, y being equal to $\log x$. Here, by differentiating $\log x = y$, and multiplying by x, we obtain $dx = xdy$; and by two other differentiations, regarding dy as constant, and replacing in each instance dx by xdy, we get $d^2x = xdy^2$, and $d^3x = xdy^3$. Now, by substituting these in formula (2), No. 207, and multiplying the result by x^3, we get,

$$\frac{d^3u}{dy^3} - \frac{du}{dy} - 3\left(\frac{d^2u}{dy^2} - \frac{du}{dy}\right), \text{ or } \frac{d^3u}{dy^3} - 3\frac{d^2u}{dy^2} + 2\frac{du}{dy},$$

the new value of the first term of the proposed equation. In a similar manner we should find from formula (1), after multiplying by $3x^2$, that

$$3\left(\frac{d^2u}{dy^2} - \frac{du}{dy}\right)$$

is the new value of the second term; and the third term becomes simply $\frac{du}{dy}$. Then, by substituting these results in the given equation, we get $\frac{d^3u}{dy^3} + au = 0$, the equation required.

XI.—DIFFERENTIATION OF FUNCTIONS OF TWO OR MORE VARIABLES; AND MAXIMUMS AND MINIMUMS OF SUCH FUNCTIONS.

210. We have thus far confined our attention to functions of a single variable. In many instances, however, particularly in inquiries in physical science, functions depend on two or more variables which are independent of one another. Thus, the distance to which water is thrown by a fire-engine depends on the force applied, the direction given to the jet, and in some degree on the resistance of the air; and the path described by a planet depends not only on its distance from the sun and its projectile motion, but it is also modified by the attractions of the other planets.

211. It will appear on a slight consideration of the subject, that in the investigations regarding differentials which we have thus far had, we have virtually taken as the diffe-

rentials of a variable and its function their infinitely small simultaneous increments; since, in changing $\frac{u'-u}{h}$ into $\frac{du}{dx}$, we have written du instead of $u'-u$, and dx instead of h, when h and consequently $u'-u$, which depends on it, tend simultaneously to evanescence. By viewing the subject of differentiation in this light, and by employing due care and caution, we can shorten and simplify many investigations regarding functions, either of one variable or of more. In physical investigations, in particular, much facility results from employing this principle; and the same view of the subject enables us to investigate with much ease the differentials of functions of two or more variables.

Thus, for differentiating a function of two variables, x and y, let us assume u to denote the function, so that $u=f(x,y)$; and, supposing x to be increased by h, and y by k, let us put $u'=f(x+h, y+k)$. Then

$$u'-u=f(x+h,y+k)-f(x,y), \quad \text{or } u'-u$$

$$=\frac{f(x+h,y)-f(x,y)}{h}h+\frac{f(x+h,y+k)-f(x+h,y)}{k}k;$$

the latter form being obtained by adding $f(x+h,y)$ and subtracting it, and by multiplying and dividing in one part by h, and in the other by k. Now (No. 6), if h be diminished towards evanescence, the first of the fractions in the second member will tend to $\frac{df(x,y)}{dx}dx$, or $\frac{du}{dx}dx$, as its limit: and if both h and k be diminished towards zero, the second fraction will have $\frac{df(x,y)}{dy}dy$, or $\frac{du}{dy}dy$, as the limit to which it will tend. In the same circumstances, also, according to the preceding observations, we may replace $u'-u$ by du; and we shall thus have as the differential required,

$$du=\frac{du}{dx}dx+\frac{du}{dy}dy.$$

It is necessary to consider carefully the notation here employed. The first member, du, is called the *total differential* of u or $f(x,y)$; and it is made up of the two parts, $\frac{du}{dx}dx$, and $\frac{du}{dy}dy$, which are therefore called the *partial differentials* of u; and the coefficient of dx in the first, and that of dy in the second, are called the *partial differential coefficients* of u. The denominators dx and dy are not to be regarded as mere divisors of du, but as showing that $\frac{du}{dx}$ is the differential coefficient obtained by regarding x alone as variable, and $\frac{du}{dy}$ the one obtained by supposing y alone to vary.* It thus appears, that to get the total differential, we are to find the partial differential coefficients, first supposing x alone variable, and then y alone; to multiply the first by dx, and the second by dy; and to add the results together: and it might be shown in a manner exactly similar, that this rule may be extended to a function of any number of variables whatever; so that, if $u = f(x, y, z, \ldots)$, we should have

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz + \ldots\ldots\ldots$$

It may be remarked, in conclusion, in reference to this subject, that this reasoning would hold good, though the quantities x, y, z,......were all or some of them functions of one another, or of some other variable, such as t or v.

212. To exemplify the principles established in the last No., let it be required to differentiate $u = x^y$. By differentiating this (Nos. 9 and 29), first on the supposition that y is constant and x variable, and again on the contrary supposition, we obtain

$$\frac{du}{dx} = y\,x^{y-1} \quad \text{and} \quad \frac{du}{dy} = x^y \log x.$$

Then, multiplying the second members by dx and dy respectively, and adding the results, we get

$$du \text{ or } d\,.\,x^y = yx^{y-1}\,dx + x^y \log x dy.$$

* All ambiguity regarding the total and the partial differentials of u would be removed, if the total one were denoted by du, and if the partial ones in reference to x and y were written respectively $d_x u$ and $d_y u$. The notation in the text, however, is generally preferred on account of its simplicity; and with proper care, it will lead to no ambiguity or error. It may be remarked also that Euler, Laplace, and some others, have enclosed the partial differential coefficients in parentheses; thus

$$du = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy.$$

This mode of notation, however, is now rarely employed.

Let, again, $u = x^n \tan y$; and it will be found in a similar way, that

$$du = nx^{n-1}\tan y dx + x^n(1+\tan^2 y)dy.$$

So likewise if $u = xy$, we should have $du = ydx + xdy$; and if $u = \frac{v}{z}$ we should get $du = \frac{dv}{z} - \frac{vdz}{z^2}$, as in Nos. 11 and 12.

213. If u be a *function of a function*, such as $u = \log y$, and $y = \sin x$, so that $u = \log \sin x$, we can express its differential coefficient very simply by means of the notation explained in No. 211. For this purpose, let x be changed into $x + h$, and let $y + k$ be what y becomes in consequence, and u' what u becomes; and we shall have $\frac{u'-u}{h} = \frac{u'-u}{k}\frac{k}{h}$. Now (No. 6), when h and k are taken as evanescent, this becomes $\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}$.

Thus, in the instance just mentioned we have

$\frac{dy}{dx} = \cos x$, and $\frac{du}{dy} = \frac{1}{y}$; whence $\frac{du}{dx} = \frac{\cos x}{y} = \frac{\cos x}{\sin x} = \cot x$.

214. Let $u = f(x, y)$, and let Δ_x, prefixed to any quantity, denote the finite difference (No. 5) of that quantity, when x is changed into $x + h$, and Δ_y, the like difference, when y is changed into $y + k$. Then

$$\Delta_x u = f(x+h, y) - f(x, y), \text{ and}$$

$$\Delta_y \Delta_x u = \{f(x+h, y+k) - f(x+h, y)\} - \{f(x, y+k) - f(x, y)\}.$$

By a similar process, only commencing with $\Delta_y u$, and then getting $\Delta_x \Delta_y u$, we should find the equivalent of the latter to be identical with the value just found for $\Delta_y \Delta_x u$; and therefore $\Delta_y \Delta_x u = \Delta_x \Delta_y u$. Now (No. 204), when h and k are diminished towards evanescence, Δ_x and Δ_y will become simply d_x and d_y; and we shall have $d_y d_x u = d_x d_y u$: that is, *if* u, *a function of two variables, be subjected to successive differentiation, first with regard to one of those variables, and the result in reference to the other, the final result will be the same in whichever order we proceed.*

Thus, for example, if we differentiate $x^2 \sin y$ in reference to x, we get $2x \sin y dx$; and by differentiating this in reference to y, we obtain $2x \cos y dx dy$: while, if we first differentiate $x^2 \sin y$ in reference to y, we obtain $x^2 \cos y dy$; and if this be differentiated in reference to x, we get as before, $2x \cos y dx\, dy$.

It would be shown in a similar way, that this important principle belongs to functions of three or more variables; the final result of successive differentiations being always the same, in whatever order the several processes succeed one another.

215. We are now prepared to determine the successive differentials of functions of two or more variables. Thus, since (No. 211),

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy,$$

if we differentiate each of the terms in the second member, making du to vary in reference to both x and y; introducing also dx in the denominator of the result, when we differentiate in reference to x, and multiplying at the same time by dx; and doing the same in reference to dy, when we differentiate in reference to y; we get by adding together two terms which, by the last No., are equal,

$$d^2 u = \frac{d^2 u}{dx^2} dx^2 + 2\frac{d^2 u}{dxdy} dxdy + \frac{d^2 u}{dy^2} dy^2.*$$

In this $\frac{d^2u}{dx^2}$, $\frac{d^2u}{dxdy}$, and $\frac{d^2u}{dy^2}$ are the partial second differ-

* By removing the denominators, this might be written

$$d^2u = d^2{}_x u + 2\, d_x d_y u + d^2{}_y u;$$

and other modes of notation have been employed.

By farther successive differentiation conducted on the same principles, it would be found, that

$$d^3 u = d^3{}_x u + 3\, d^2{}_x d_y u + 3\, d_x d^2{}_y u + d^3{}_y u;$$

and, in general, $d^n u =$

$$d^n{}_x u + \frac{n}{1} d^{n-1}{}_x d_y u + \frac{n(n-1)}{1.2} d^{n-2}{}_x d^2{}_y u \ldots\ldots + \frac{n}{1} d_x d^{n-1}{}_y u + d^n{}_y u;$$

the striking resemblance of which to the binomial theorem it is at once curious and important to observe. On this account, on the principle of notation which has been called the "separation of symbols," this differential has been expressed in the following manner;

$$d^n u = (d_x + d_y)^n u;$$

where n in the second member, as well as in the first, must be taken, not as an exponent of a power, but as an index of differentiation. It may be remarked, that as in the differentiation of functions of a single variable x, its differential is generally assumed as constant, so also for similar reasons, in investigations regarding functions of several variables, x, y, z,...., their differentials, dx, dy, dz,...., are regarded as constant, unless in particular cases it is necessary to adopt the contrary supposition.

ential coefficients of u; the first denoting the second differential coefficient found by taking x alone variable; the third that which arises from taking y alone variable, and the second that which results from differentiating u, first, on the supposition that one of the quantities x and y is variable, and the result on the supposition that the other varies; and the first member d^2u is the total second differential of u.

As an example, let it be required to find the second differential of $u=\varepsilon^x y^3$. Here we have

$$\frac{d^2u}{dx^2}=\varepsilon^x y^3, \quad \frac{d^2u}{dxdy}=3\varepsilon^x y^2, \quad \text{and } \frac{d^2u}{dy^2}=6\varepsilon^x y;$$

and multiplying these by their respective denominators, and substituting the results in the expression found above, we get as the required differential,

$$d^2u=\varepsilon^x y^3dx^2+6\,\varepsilon^x y^2dxdy+6\,\varepsilon^x ydy^2.$$

216. The method of finding the maximum and minimum values of functions of single variables, has been given in Section VII.; and we may now proceed to investigate the method of determining such values of functions of two independent variables, when functions of that kind have such values. To do this, let us assume $u=f(x, y)$; and, to facilitate the investigation, we may first suppose y to be a function of x, taking care afterwards to rectify the effect of this supposition. Differentiating, then, (No. 213) on this assumption, we get

$$\frac{tdu}{dx}=\frac{du}{dx}+\frac{du}{dy}\frac{dy}{dx},$$

$$\text{and } \frac{td^2u}{dx^2}=\frac{d^2u}{dx^2}+2\frac{d^2u}{dxdy}\frac{dy}{dx}+\frac{d^2u}{dy^2}\frac{dy^2}{dx^2}+\frac{du}{dy}\frac{d^2y}{dx^2};*$$

in which tdu and td^2u are assumed to denote the total differentials of u. Now, by the same reasoning that was employed in Section VII. it would appear that $\frac{tdu}{dx}$ must

* This term is obtained by annexing $\frac{du}{dy}\frac{d^2y}{dx^2}$ to the value for the total second differential of z found in No. 215 and dividing the entire result by dx^2. The reason of this is plain from the circumstance, that in No. 215, dy was regarded as constant, y, as well as x, being supposed to vary uniformly, a supposition which cannot be assumed generally, when y is a function of x.

be equal to zero in case of either a maximum or minimum value of u; and $\frac{td^2u}{dx^2}$, with the exception mentioned in No. 126, must be negative in case of a maximum, and positive in case of a minimum. The making of the first of these coefficients equal to zero, when y is independent of x, and when, consequently, $\frac{dy}{dx}$ is indeterminate, will take place, if the two conditions,

$$\frac{du}{dx}=0, \qquad \text{and } \frac{du}{dy}=0,$$

be fulfilled; and the latter of these conditions makes the second expression found above become

$$\frac{td^2u}{dx^2}=\frac{d^2u}{dx^2}+2\frac{d^2u}{dxdy}\frac{dy}{dx}+\frac{d^2u}{dy^2}\frac{dy^2}{dx^2}.$$

This latter quantity, as we have seen, must, with the exception referred to above, be negative in case of a maximum, and positive in case of a minimum, not changing its sign whatever values may be attributed to x and y in the immediate vicinity of those which render $f(x, y)$ a maximum or minimum. Now, this quantity will retain its sign unchanged when $\frac{d^2u}{dx^2}\cdot\frac{d^2y}{dx^2} > \left(\frac{d^2u}{dxdy}\right)^2$;* a condition which, the latter member being a square, and therefore positive, evidently requires, that $\frac{d^2u}{dx^2}$ and $\frac{d^2u}{dy^2}$ shall have the same sign.

It thus appears that we are to find the partial differential coefficients of the first order in reference to x and y; and putting them separately equal to zero, to find the

* To prove this, we may assume the expression, $az^2+2bz+c$, which may be put under the form

$$\frac{(az+b)^2-b^2+ac}{a}.$$

In this, whatever may be the value of z, the numerator will be always positive, if a and c have the same sign, and if their product be greater than b^2: and therefore, if these relations exist, the trinomial will not change its sign, but will have that of the denominator a. Now this expression becomes the same as the total second differential coefficient in the text, if we take

$$a=\frac{d^2u}{dy^2}, \quad b=\frac{d^2y}{dxdy}, \quad c=\frac{d^2u}{dx^2}, \quad \text{and } z=\frac{dy}{dx};$$

and therefore the truth of what is stated is manifest.

values of x and y from the equations so obtained. • We are then to find the partial differential coefficients of the second order, $\frac{d^2u}{dx^2}$, $\frac{d^2u}{dxdy}$, and $\frac{d^2u}{dy^2}$, and to substitute in them the values of x and y previously found. If the product of the values thus obtained for the first and third of these coefficients be positive, and be greater than the square of that of the second, the values of x and y will render the function a maximum or minimum, according as the first and third coefficients are both negative or both positive. The function can be neither a maximum nor a minimum if the product of the first and third of these be less than the square of the second.*

The student will feel no difficulty in showing that the maximum and minimum values of three or more variables are determined on exactly the same principles; each partial differential coefficient of the first order being equal to zero, and the total differential coefficient of the second order being negative for a maximum, and positive for a minimum. In many instances of this kind, the processes become complicated and difficult, especially for the differential coefficients of the second and higher orders. Often, however, (see the note in page 77) it is unnecessary to proceed to any differentials above those of the first order.

217. To exemplify these principles, let the perimeter of a triangle be given equal to $2s$; and let it be required to find the sides so that the area may be a maximum.

Here, putting the sides equal to x, y, and $2s-x-y$, we have, by a well known theorem (TRIG. No. 59, note), the area $u=\sqrt{s(s-x)(s-y)(x+y-s)}$. Squaring this, and taking the logarithms, we get

$$2\log u=\log s+\log(s-x)+\log(s-y)+\log(x+y-s).$$

Then, by differentiating, first in relation to x, and then to y, and by easy reductions, we get

$$\frac{du}{dx}=\frac{u}{2}\frac{2s-2x-y}{(s-x)(x+y-s)}, \qquad \frac{du}{dy}=\frac{u}{2}\frac{2s-2y-x}{(s-y)(x+y-s)}.$$

Hence, by putting the numerators of these each equal to zero, we obtain two equations containing x and y, and

* Should the total second differential vanish, we should proceed to the higher ones, as in No. 127. In such cases, however, the operations generally become laborious, and the results obtained are seldom of much interest.

these equations give $x=\frac{2}{3}s$, $y=\frac{2}{3}s$, and consequently, $2s-x-y=\frac{2}{3}s$; whence it appears, that the triangle is equilateral. By the test pointed out in No. 216, these values would be found to give the area a maximum.

218. As another example, let it be required to find the dimensions of a rectangular reservoir, open at top, and having the least internal surface possible, its content in cubic measure being given equal to a^3.

Here, putting x and y for the length and breadth of the base, we have the depth $=\frac{a^3}{xy}$; and, adding together the areas of the base and the four perpendicular sides, we find,

$$u=xy+\frac{2a^3}{x}+\frac{2a^3}{y}.$$

By differentiating this with respect to x, and then to y, we get

$$\frac{du}{dx}=y-\frac{2a^3}{x^2}, \quad \text{and } \frac{du}{dy}=x-\frac{2a^3}{y^2}; \text{ also,}$$

$$\frac{d^2u}{dx^2}=\frac{4a^3}{x^3}, \quad \frac{d^2u}{dy^2}=\frac{4a^3}{y^3}, \quad \text{and } \frac{d^2u}{dxdy}=1.$$

Hence, by putting the first differential coefficients equal to nothing, we get x and y each equal to a multiplied into the cube root of 2, and the depth equal to half that product; whence it appears that the cistern is to be half a cube. By substituting this value of x and y in the other coefficients, we get

$$\frac{d^2u}{dx^2}=2, \quad \text{and } \frac{d^2u}{dy^2}=2;$$

which being both positive, and their product being greater than the square of $\frac{d^2u}{dxdy}$, it follows that the function is a minimum, as is evident also from the nature of the question.

219. As a third example, let it be required to inscribe in a given circle a polygon having n sides, and having its area a maximum. To solve this interesting question, let us assume x_1, x_2, x_3, &c. to represent the successive angles at the centre formed by radii drawn to the extremities of the successive sides of the polygon. These radii will divide the polygon into isosceles triangles, the area of the first of

which, by a well known principle (Euc. App. III. 28), will be $\frac{1}{2}r^2 \sin x_1$, that of the second $\frac{1}{2}r^2 \sin x_2$, &c.; and therefore the area of the entire polygon will be

$$\tfrac{1}{2}r^2(\sin x_1 + \sin x_2 + \sin x_3 + \ldots\ldots + \sin x_n).$$

Suppressing (No. 130) the constant factor $\frac{1}{2}r^2$, and substituting for $\sin x_n$ what is plainly equal to it,

$$\sin\{2\pi - (x_1 + x_2 + \ldots\ldots\ldots + x_{n-1})\}, \text{ or } \sin(2\pi - s_1),$$

if, for brevity, we assume

$$s_1 = x_1 + x_2 + x_3 + \ldots\ldots + x_{n-1};$$

we shall have

$$u = \sin x_1 + \sin x_2 + \sin x_3 + \ldots\ldots + \sin(2\pi - s_1).$$

$$\text{Hence } \frac{du}{dx} = \cos x_1 - \cos(2\pi - s_1) = 0;$$

and therefore $x_1 = 2\pi - s_1$; so that the first angle and the last are equal. It would be shown in a similar manner, that any other angle, taken as the first, is equal to the last in reference to it, that is, to the one immediately preceding it; and thus it appears, that all the angles, x_1, x_2, &c. are equal, so that the polygon must have all its sides and angles equal; and it is therefore a regular polygon.

EXERCISES.

Required proofs of the following statements.

1. If $u = x^m y^n (a - x - y)^p$, u will be a maximum, when

$$x = \frac{ma}{m+n+p}, \quad y = \frac{na}{m+n+p}, \quad \text{and } a - x - y = \frac{pa}{m+n+p}.$$

2. To divide π into three parts, x, y, and $\pi - (x+y)$, such that the sum of their sines may be a maximum or minimum, we must have $\cos x = \cos y = -\cos 2x = -\cos 2y$. Hence, we find for one answer each of the parts equal to $\frac{1}{3}\pi$, and $u = 3 \sin 60^\circ = \frac{3}{2}\sqrt{3}$, which is a maximum.

3. To divide π into three parts, such that the product of their sines may be a maximum or minimum, we must have for a literal solution, all the parts equal; and the product of the sines will be a maximum.

4. The quotient obtained by dividing ε^{x+y} by xy is a minimum, when $x = y = 1$.

5. If $u=x-2\log x+2y-\log y$, u is a minimum, when $x=2$ and $y=\frac{1}{2}$; and it is then equal to $3-\log 2$.

6. The least polygon of a given number of sides that can be described about a given circle is a regular one.

XII.—SINGULAR POINTS OF CURVES—ANALYSES OF CURVES.

220. A *singular point* of a curve is a point at which the curve has some property inherently different from what it has at the points in the immediate vicinity of that point. Thus, a curve at a point of inflexion (No. 165) retains the same character, a change in the direction of curvature, whatever may be the system of coordinates to which it referred. On the contrary, the points in which a curve may meet its axes, and the point at which one of its coordinates may be a maximum or minimum, though it is of importance to consider them in the analyses of curves, are not, when viewed merely in this light, *singular* points, as they change, if the axes be changed. Of other singular points in addition to those of inflexion, *multiple points* are perhaps the most remarkable. Such points are those through which two or more branches or portions of a curve pass, and these are of two kinds; the first, those in which two or more branches *intersect* each other, and the second those in which branches *touch* one another. Points of the latter kind have been called points of *osculation*. In reference to points of the first kind, it is plain from No. 143, that, while the particular value of x corresponding to one of them, gives a single value for y, the differential coefficient f^1x should have as many *different* values as there are branches passing through the point; while in case of contact, there will be as many *equal* values of the same coefficient as there are branches which touch one another; and on these principles the theory for determining such points is founded.

221. To carry out analytically the theory now mentioned, let $u=f(x,y)=0$ be the equation of a curve, free of radicals and variable denominators. Then y being a function of x, we have (No. 211) $\frac{du}{dx}+\frac{du}{dy}y'=0$, where $y'=\frac{dy}{dx}$. The resolution of this equation, since u and its dif-

ferentials contain no radicals, will give one value and only one for y' (which would be 0, ∞, or a finite quantity), unless the two partial differential coefficients were each equal to zero; in which case we should have $y'=\frac{0}{0}$. Now, if in this case, by proceeding according to the method established in Section VI. we should find that, for some point x, y of the curve, y' has two or more real values, there will be as many branches of the curve passing through that point; and (No. 143), these values of y' will be the trigonometrical tangents of the angles which those branches make with the axis of x, the branches intersecting one another when the values of y' are unequal; but those of them touching another which correspond to equal values of y'.

222. Multiple points may also be determined by means of the differential coefficients of the second or higher orders. Thus double points may be discovered by finding the second differential of $u=f(x, y)$, rejecting the term of it which contains $\frac{du}{dy}$ as a factor, and dividing by dx^2. In this way we get (No. 215),

$$\frac{d^2u}{dx^2}+2\frac{d^2u}{dxdy}y'+\frac{d^2u}{dy^2}y'^2=0;$$

the resolution of which equation w give the two values of y' unequal or equal, corresponding to a double point. The expression obtained from this equation for y' will contain the radical,

$$\left\{\left(\frac{d^2u}{dxdy}\right)^2-\frac{d^2u}{dx^2}\frac{d^2u}{dy^2}\right\}^{\frac{1}{2}}$$

and it is plain from what we have seen, that if the quantity within the vinculum be positive, there will be a double point by intersection; but should the two terms in the vinculum be equal, there will be a double point by osculation.

223. As an example, let us take the equation, $a^2y^2=a^2x^2-x^4$, which is the equation of one of the lemniscates. By differentiating, and dividing by $2a^2ydx$, we get

$$\frac{dy}{dx}=\frac{a^2x-2x^3}{a^2y}=\frac{a^2x-2x^3}{a\sqrt{(a^2x^2-x^4)}}.$$

This becomes $\frac{0}{0}$, when $x=0$; but its value is found at once to be ± 1, by dividing the numerator and denominator by x, and in the result taking $x=0$. There is a double point, therefore, at the origin, and the two branches cut one another perpendicularly, 1 being the tangent of 45^o, and -1 that of 135^o, or -45^o.

In working by the second method, we have $u=a^2x^2-x^4-a^2y^2=0$, and thence

$$\frac{du}{dx}=2a^2x-4x^3, \qquad \frac{du}{dy}=-2a^2y,$$

$$\frac{d^2u}{dx^2}=2a^2-12x^2, \qquad \frac{d^2u}{dxdy}=0, \qquad \text{and } \frac{d^2u}{dy^2}=-2a^2;$$

whence $-2a^2y'^2+2a^2-12x^2=0$, and therefore,

$$y'=\pm\frac{\sqrt{(a^2-6x^2)}}{a};$$

which, when $x=0$, becomes ± 1, as before.*

224. If as a second example, we assume $a^3y^2=x^5+bx^4$ as the equation of a curve, we get, by a single differentiation, and by easy modifications, when x, and therefore $y=0$, $y'^2=\frac{(5x^4+4bx^3)^2}{4a^3(x^5+bx^4)}=\frac{0}{0}=\frac{(5x^2+4bx)^2}{4a^3(x+b)}=0$. Hence y' has two values each equal to zero, when $x=0$; and as y is then $=0$, two branches of the curve have the axis of the abscissas touching them at the origin of the coordinates; and the equation of the curve shows that on each side of the origin each branch of the curve recedes from that axis, and therefore the two branches osculate at the origin. The curve is represented in *fig.* 62.

Had we proceeded to the second differentials, we should have had,

$$\frac{du}{dx}=5x^4+4bx^3, \qquad \frac{du}{dy}=-2a^3y,$$

$$\frac{d^2u}{dx^2}=20x^3+12bx^2, \qquad \frac{d^2u}{dxdy}=0, \qquad \text{and } \frac{d^2u}{dy^2}=-2a^3;$$

whence we should get $y'^2=\dfrac{10x^3+6bx^2}{a^3}=0$, and hence,

* It would be found by means of No. 165, that the two branches of the curve have inflexions when $x=0$; so that the origin is a *doubly* singular point.

when $x=0$, we shall have two values of y' each equal to zero, as before.

225. If at a double point the two values of y' be equal, and the two branches of the curve have existence on one side of the point but not on the other, the point is a *cusp*; the two branches having a common tangent, and therefore not intersecting each other, and the figure thus presenting as it were a sharp point. If the two branches lie on opposite sides of their common tangent, the cusp is said to be of the *first species*; but if they be on the same side, it is said to be of the *second species*.* It is plain from No. 164, that the cusp will be of the first kind, when $\frac{d^2y}{dx^2}$ has opposite signs for the two branches; but of the second, when the signs are the same.

226. If $u=f(x, y)=0$ give a real value for y corresponding to a particular value of x, while that value of x renders y' imaginary, the point corresponding to this value of x is detached from all the other points of the curve, as x cannot there increase or diminish continuously, the change of x into $x \pm h$ being inadmissible, when h is taken infinitely small, the differential coefficients being thus rendered imaginary. Such a point is called an *isolated point*, and often, though with perhaps less propriety, a *conjugate point*.

227. The curves represented in figures 35, 36, 37, 38, and 39, in all of which O is the origin, and OX the axis of x, will exemplify most of the principles above established regarding the singular points of algebraic curves. The equation of the first of these is $a(y-b)^2=x^3-cx^2$, where $OG=CD=b$. Then G is an isolated point, and there are two points of inflexion corresponding to $x=\frac{4}{3}c$. The tangent can be no where parallel to the axis of x, as, though y' is equal to zero when $x=\frac{2}{3}c$, yet that value of x renders y imaginary.

The equation of the second is $a(y-b)^2=x^3+cx^2$. Here x cannot be less than $-c$; and therefore if $OC=FD=c$, the curve will lie wholly to the right of D. At this point, and at the origin, the values of y are each equal to

* Cusps of the first kind, from their spreading out like the horns of animals, have been called (from the Greek κερας, a horn) *ceratoid* points; while the others have (from ῥαμφος, the beak of a bird) been called *ramphoid* points.

zero, while everywhere between them y has two values different from one another; and, therefore, between these points there is a *loop* or *node*. At the origin we have $y=b$, and $y'=\pm\sqrt{\frac{c}{a}}$; and therefore there is a double point at F.

The equation for *fig.* 37 is $a(y-b)^2=x^3-c^2x$. In this case we find that the tangents are perpendicular to OX at D, F, and G; that both the values of y are real between C and O, and imaginary between O and E. Hence, between D and F there is a detached oval; and two infinite branches proceed from G, which, as it is easy to show, have points of inflexion corresponding to $x=c\sqrt{(1+\frac{2}{3}\sqrt{3})}$.

For *fig.* 38, the equation of the curve is $a(y-b)^2=x^3$, the curve being the semicubical parabola. From this equation it is easy to show, that the curve has no points of inflexion, no node, and no detached oval; but that it has a cusp of the first species at F, each branch having OX as a tangent at that point.

Fig. 39, the equation for which is

$$a(ay-x^2)^2=x^5, \text{ or } y=\frac{x^2}{a}\pm\frac{x^{\frac{5}{2}}}{a^{\frac{3}{2}}},$$

exhibits an instance of a cusp of the second species, both branches at the origin lying on the same side of OX, which there touches both of them.

228. Some transcendental equations give as their loci a succession of isolated points. In reference to any particular equation of this kind, such points may be regarded as being situated on a branch, all other points of which correspond to imaginary values of y, and which, therefore, have no existence when viewed in connexion with the given equation. Such a locus may be called a *dotted* or *pointed branch* (the French writers call it a *branche pointillée*). The equation $y=x^2+\sqrt{(\sin^2 x-1)}$, or, which is the same, $y=x^2+\cos x\sqrt{-1}$, affords an example. If the equation were simply $y=x^2$, the locus would be a continuous parabola, with its axis vertical. The term $\cos x\sqrt{-1}$, however, is always imaginary, except when it vanishes, which it does where $\cos x=0$, and when, consequently, $x=\pm(n+\frac{1}{2})\pi$, n being zero, or a whole number.

Hence, if we take points on the axis of x at distances from the origin equal to $\pm\frac{1}{2}\pi$, $\pm\frac{3}{2}\pi$, &c., ordinates drawn through these points will cut the parabola in the only points belonging to the locus of the given equation.

This subject would be well illustrated by means of the equation, $y=x^2\pm\sqrt{(1-a\sec^2x)}$, which, if a be between 0 and 1, will give for the locus of the equation an infinite series of detached ovals intersected by the parabola having $y=x^2$ as its equation. If $a>1$, all the values of y will be imaginary, and there will be no locus whatever: while, if $a=1$, the locus will have existence, the ovals above referred to becoming detached points on the parabola.

A more general equation than the preceding, would be $y=fx\pm\sqrt{(1-a\sec^2x)}$; and one still more general would be $y=fx\pm\varphi x\sqrt{(1-a\varphi x)}$. The consideration of these, and of others of a very general character, may be instructive to the student.

229. In curves which are the loci of transcendental equations, it sometimes happens that a branch stops abruptly at a certain point, in consequence of the immediately adjoining ordinates on one side becoming imaginary. Such a point may be called a *point of stoppage*, (the French writers call it a *point d'arret*, or a *point de rupture*).

The equation, $y=\dfrac{x^2\log x}{x^2-1}$, affords an example; since $y=0$, when $x=0$ (found by dividing the terms of the second member by x^2, and by means of No. 118); and the curve has a single continuous branch, corresponding to the *positive* values of x. No value of x, however, can be negative, as (No. 105) the value of y corresponding to such a value of x would be imaginary. Hence the curve terminates abruptly at the origin, which is therefore a point of stoppage.

230. There are still other singular points belonging to the loci of certain transcendental equations, which possess some interest. These are *salient* or *angular points* (*points saillants ou anguleux*, as the French writers call them). In such curves, two branches, not intersecting each other, meet in a point without having, as in the case of an ordinary cusp, a common tangent, the two branches having, in fact, at that point tangents inclined at a finite angle depending on the equation of the curve.

The equation, $y = \dfrac{x}{1 + \varepsilon^{\frac{1}{x}}}$, affords an example, in which the origin of the coordinates is a point of this kind: since it gives $\dfrac{dy}{dx} = \dfrac{1}{1 + \varepsilon^{\frac{1}{x}}} + \dfrac{1}{x\varepsilon^{-\frac{1}{x}}(1 + \varepsilon^{\frac{1}{x}})^2}$, which, if x be supposed to be diminished down to zero, though positive values, becomes 0; while, if x increase up to zero on the negative side of the origin, and be taken negative in vanishing, the same coefficient will become 1; and these (No. 143) are the trigonometrical tangents of the angles which the tangents to the two branches make with the axis of the abscissas at the origin. Hence it appears, that on the positive side of the origin in reference to x, the curve coincides with the axis of the abscissas at the origin, while the branch corresponding to the negative values of x, starts from the origin at an angle whose tangent is 1, and which, therefore, is 45°, or rather 225°.*

231. The principles established above are of essential importance in the analysis of curves; and they afford much assistance in the tracing or delineating of curves by means of their equations. In tracing a curve, it is generally proper to resolve its equation, so as to find one of the coordinates (whichever can be more easily determined) in terms of the others, when that can be done by means of an equation of the first or second degree. Suppose y to be the coordinate, the value of which is thus obtained. Then, by attributing to x various values between $-\infty$ and $+\infty$, we shall find among others, those that make $y=0$, $y=\pm\infty$, or y imaginary, as the case may be. When $y=0$, the curve cuts the axis of x; when $y=\pm\infty$, there is an infinite branch; and when y is imaginary, the curve has no existence, on a perpendicular to the axis of the abscissas determined by the value of x. By this means, also, we may determine as many points in the curve as we please; and a line traced through these in the natural manner that will obviously suggest itself, will approximate to the true curve as nearly as we please, the degree of approximation

* Moigno, Cournot, and other writers, give the curve whose equation is $y = x\tan^{-1} ax^{-1}$, as affording an instance of a point of this kind at the origin of its coordinates. This, however, is wrong; as an infinite number of branches pass through the origin, not giving any angular point, but intersecting each other, and thus producing a multiple point.

depending mainly on the smallness of the intervals between the points so found. In the second place, for obtaining greater accuracy, we may find $\frac{dy}{dx}$, and by this means we may determine (No. 143) the angles at which the curve cuts the axis, if it do cut it, and (No. 127) the maximum and minimum values of y, if there be such. In the third place, we may (No. 148) find the asymptotes, if there be any: and, fourthly, to ensure still more precision, we may determine (by No. 165, and by the preceding part of this Section) the singular points, properly so called, if there be such.

232. To exemplify these principles, let us consider the curve of which the equation is $y = x^4 - 5x^2 + 4$.* By assigning to x the values in the first line of the following table, we shall obtain those of y standing below them in the second line:

x......0,	$\pm\infty$,	± 1,	± 2,	± 3,	± 4,	&c.
y......4,	∞	0,	0,	40,	180,	&c.

Hence we see, that the curve meets the axis of x, when x has any of the values, 1, 2, -1, and -2,† and that it has positive infinite branches on both sides of the origin, when the abscissa is indefinitely extended in each direction. By assuming x equal to $\pm\frac{1}{4}$, $\pm\frac{1}{2}$, $\pm\frac{3}{4}$, $\pm\frac{5}{4}$, &c., we should find additional points in the curve, which would enable us to approximate still more nearly to its true form; and by taking intermediate values of x at smaller intervals asunder, we might obtain as many additional points as we should consider desirable.

To help in the analysis, however, we get, by differentiation,

$$y' = 4x^3 - 10x, \qquad \text{and } y'' = 12x^2 - 10.$$

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* If it be wished to substitute for this or any similar equation, one having all its terms of the same dimensions, we have merely to change x into $a^{-1}x$, in the equation, and to clear the result of negative indices by multiplication. When calculations are to be made, however, it is generally convenient to take the constant quantity equal to unity, when there is but one, or one of them equal to unity when there are more than one, and, if necessary, to modify the other terms proportionally.

† These values, which are here found *accidentally*, as it were, would be obtained directly by putting the value of y equal to zero, and resolving the equation so found. It may be remarked, that since the given equation contains only even powers of x, the values of y for equal positive and negative values of x are the same; and hence the two parts of the curve on the opposite sides of the axis of y are symmetrical and equal, so that one of them, if inverted, would coincide with the other by superposition.

By taking in the first of these, x successively equal to 1, -1, 2, and -2, we find, that at the points corresponding to these values of x, the curve crosses the axis of x at angles whose tangents are -6, 6, 12, and -12;* and thus its direction at each of these points is determined. By putting, also, y' equal to zero, we get $x=0$, $x=\sqrt{\frac{5}{2}}$, and $x=-\sqrt{\frac{5}{2}}$; and thus we find points of the curve at which its tangent is parallel to the axis of x, and at which, consequently, the ordinate y is a maximum or minimum. These values of y are 4, $-\frac{9}{4}$, and $-\frac{9}{4}$. It is plain also, that y' cannot be infinite; so that the tangent to the curve cannot anywhere be perpendicular to the axis of x; and that x cannot have a maximum or minimum value, but may be augmented indefinitely on both sides of the origin.

By putting the value of y'' equal to zero, we get $x=\sqrt{\frac{5}{6}}$, and $x=-\sqrt{\frac{5}{6}}$; each of which values corresponds to a point of inflexion, since neither of them makes the third differential coefficient vanish. The value of y corresponding to each of these is $\frac{19}{36}$.

Farther examination would show that this curve has no other singular points, except those of maximum curvature; and these are of little consequence in the tracing of curves.

By attending to the preceding results, and by finding a sufficient number of values of y, as referred to above, the curve may be delineated with any degree of accuracy we please. It is represented in *fig.* 63.

233. As another example, let it be required to analyse and trace the curve whose equation is $x^3-3axy+y^3=0$. Denoting this by u, and differentiating, we get

$$\frac{du}{dx}=3x^2-3ay, \qquad \text{and } \frac{du}{dy}=-3ax+3y^2;$$

which both vanish at the origin, y, as well as x, being there equal to zero, by the given equation. Differentiating again, we get

$$\frac{d^2u}{dx^2}=6x, \quad \frac{d^2u}{dy^2}=6y, \qquad \text{and } \frac{d^2u}{dx\,dy}=-3a;$$

* These angles are respectively $-80°\ 32'$, $80°\ 32'$, $85°\ 14'$, and $-85°\ 14'$. Except as a matter of curiosity, however, it is of no use to find them in degrees and minutes, as they are most easily determined by means of their tangents.

and, therefore (No. 222), $6yy'^2-6ay'+6x=0$; whence

$$y'=\frac{a\pm\sqrt{(a^2-4xy)}}{2y}.$$

This, taken with the upper sign, when $x=y=0$, gives $y'=\infty$; but with the lower it gives $y'=\frac{0}{0}$, the value of which is found in the usual way to be zero. Hence the origin is a double point, one branch of the curve touching the axis of x, and the other that of y, at that point.

To facilitate the rest of the analysis of this curve, we may assume $y=vx$;* by substituting which in the given equation, we readily get $x=\frac{3av}{1+v^3}$, and $y=\frac{3av^2}{1+v^3}$: and, by assigning various values to v, we may find as many points of the curve as we please.

To find whether the curve has asymptotes, let us differentiate severally the expressions above obtained for x and y, and, by dividing the second result by the first, we get

$$\frac{dy}{dx}\ (\text{or } f^1x)=\frac{2v-v^4}{1-2v^3}.$$

Hence (No. 148) we obtain

$$\text{OT}'=\frac{3av^2}{1+v^3}-\frac{3av}{1+v^3}\,\frac{2v-v^4}{1-2v^3}=\frac{-3av^2-3av^5}{(1+v^3)(1-2v^3)}=-\frac{3av^2}{1-2v^3};$$

the second form of this value being found by performing the actual subtraction indicated, and the third by dividing the terms of the result by $1+v^3$. Now, x and y are each infinite, when $v=-1$; and by substituting this value of v in what we have just found, we get $\text{OT}'=-a$. Also, by dividing this by -1, the value of f^1x, we get (No. 148, note) $\text{OT}=a$. The curve, therefore, has an asymptote passing through T and T′, OT, and OT′ being taken each equal to a in the negative directions in reference to x and y.

To find where the tangent is parallel to the axis of x, we put $f^1x=0$, and we thus get $2v-v^4=0$; and, therefore, $v=0$ and $v^3=2$. The first of these corresponds to the origin, agreeing with what we have already seen; and

* This substitution is suggested by the circumstance, that the equation of the curve is symmetrical in reference to x and y.

the second gives $x = a\sqrt[3]{2}$, and $y = a\sqrt[3]{4}$. In like manner, by taking $f^1x = \infty$, we should find that the tangent would be parallel to the axis of y at the point at which $x = a\sqrt[3]{4}$, and $y = a\sqrt[3]{2}$.* Hence, since x and y have no infinite values except those pointed out above, it is plain that there must be a loop or *node* bounded by the axes and the last mentioned tangents. This curve is the folium of Descartes, already referred to in No. 142, and its delineation is given in *fig.* 64.

234. If a curve be referred to polar coordinates by means of an equation, $r = f\theta$, it is proper, as a general rule in analysing and tracing it, (1.) to determine θ when $f\theta = 0$: (2.) to find r when $\theta = \pm n\pi$: and (3.) to find r when $\theta = \pm (n + \frac{1}{2})\pi$, n being 0, 1, 2, 3, &c. When real values are thus obtained for θ and r, they will show the angles which the radius vector makes with the fixed axis, when the curve passes through the pole, and the points in which it cuts the fixed axis, and those in which it cuts a perpendicular to that axis passing through the pole. In this way we shall commonly be able to discover the general course and character of the curve; and as many additional points as we please may be determined by assigning other values to θ. It is often advantageous, also, to find maximum and minimum values of r and θ when there are such.†

235. To exemplify the tracing of curves so expressed, let $r = a\sin^2\theta$. Here, since it is an even power of $\sin\theta$ that is used, it is unnecessary to consider the negative values of θ, as they would give the same curve as the positive. We have also $r = 0$, when $\theta = 0$, or $\theta = n\pi$; and $r = a$, when $\theta = (n + \frac{1}{2})\pi$. This latter value is evidently a maximum one. We should find, also, $r = \frac{1}{2}a$ when $\theta = (n \pm \frac{1}{4})\pi$; $r = \frac{1}{4}a$, when $\theta = (n \pm \frac{1}{6})\pi$; $r = \frac{3}{4}a$ when $\theta = (n \pm \frac{1}{3})\pi$: and other values may be readily found by means of trigonometry, or graphically. The equation may be put under the form $r = \frac{1}{2}a\,\text{versin}\,2\theta$, which affords, perhaps, the easiest means

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* These results would be readily found from the equation, $\frac{dy}{dx} = \frac{x^2 - ay}{ax - y^2}$, which would be obtained at once by differentiating the equation of the curve.

† Additional precision and a more complete analysis would be obtained by determining the asymptotes and singular points. The investigation of the theory of these in reference to curves defined by polar equations, presents no difficulty. The limits of the present work, however, prevents its insertion. It may be remarked, that in some cases, it is easier to find θ by means of r, than r by means of θ. The equation, $r^3 + ar = b^2\theta$, affords an instance.

of determining the values of r. The curve is exhibited in *fig.* 65; and it possesses the curious property of having the pole as a quadruple point, formed by the union of two cusps. Its equation for rectangular coordinates is easily shown to be $(x^2+y^2)^3=a^2y^4$.

236. As another example, let us take the equation, $r=a+\frac{a}{\theta}$. Here, if $\theta=+0$, we have $r=+\infty$; while if $\theta=-0$, we have $r=-\infty$: so that for positive values of θ, the values of r commence by being infinite and positive; and those of r for negative values of θ, by being infinite and negative. The curve, therefore, has infinite branches lying in opposite directions. By multiplying by $\sin\theta$ we get $r\sin\theta=a\sin\theta+a\frac{\sin\theta}{\theta}$, which is the perpendicular from the point θ, r to the fixed axis. Now, when $\theta=0$, this becomes simply a, since (No. 31) when θ, whether positive or negative, is diminished towards zero, it and its sine tend to have the ratio of equality. When, however, θ is a small positive quantity, that perpendicular will be greater than a; but if θ be a small negative quantity, the same perpendicular will be less than a. Hence, if a circle be described from the pole as centre, and with a as radius, a tangent to it on the upper side, and parallel to the fixed axis, will be an asymptote to both the infinite branches, lying below the one and above the other. It is easy to see, also, that if θ be increased indefinitely by successive revolutions of the radius vector, the second term of the value of r may be made as small as we please, and that r will thus tend ultimately to take the value a; so that the circle is an asymptote to this spiral portion of the curve, lying always within it. The other branch will approach the centre, and will reach it when $\theta=-57°\,17'\,45''$; the length of the circular arc to the radius 1, corresponding to that angle being (Exer. 3, p. 70) -1. After that, the radius vector becomes positive, and tends, by its continued revolution in the negative direction, to have a as its ultimate value. Accordingly, this branch, after passing through the centre, becomes, likewise, a spiral, to which the circle, always outside of it, is an asymptote. This curious curve is represented in *fig.* 66.

GENERAL EXERCISES REGARDING CURVES.

1. Trace the curves of which the equations are $y = x^3 - a^2x$, and $y = x^3 - ax^2$. Show that each of them has a point of inflexion, and that one of them meets the axis of x three times, and the other twice. Show how the curves become modified, when by a becoming evanescent, the equations become $y = x^3$, and the curve the cubical parabola.

2. The curve, $ay^2 = (x-a)^2(x-b)$, has a singular point when $x = a$; — a conjugate point, if $a < b$, and a double point if $a > b$.

3. In the witch, the equation of which is $xy^2 = 4a^2(2a - x)$, there are points of inflexion, where $x = \frac{3}{2}a$.

4. In the curve of sines, in which $y = \sin x$, there are points of inflexion, when $y = 0$.

5. Given $2ay^3 + 3a^2y^2 + 2a^2x^2 - x^4 - a^4 = 0$, the equation of a curve; to prove that the points $x = a$ and $y = 0$; $x = -a$ and $y = 0$; and $x = 0$ and $y = -a$, are double points: and that the tangents of the inclinations of the curve to the axis of x at the first and second of these points, are $\pm \frac{2}{3}\sqrt{3}$; and at the third $\pm \frac{1}{3}\sqrt{6}$.

6. The curve, $x^2y + aby - a^2x = 0$, has inflexions, when $x = 0$, and $x = \pm\sqrt{3ab}$: and the tangent is parallel to the axis of x, when $x^2 = ab$.

7. The curve whose equation is $y^4 - axy^2 + x^4 = 0$, has a triple point and a cusp at the origin.*

8. The curve whose equation is $y^3 - 2xy^2 + x^2y - a^3 = 0$,

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* This curve is readily traced by describing with the radius a, a semicircle passing through the origin O, and having its centre in the axis of y; and then, by drawing through a point M in the axis of x, a perpendicular to that axis extending on both sides of it, and cutting the circle in A and A_2; and lastly, by taking on that perpendicular, on opposite sides of M, MP, and MP_2, each equal to the mean proportional between OM and MA, and MP_3 and MP_4 each equal to the mean between OM and MA_2; as P, P_2, &c. are points in the curve. The curve consists of two equal and similar leaves, lying on opposite sides of the axis of the abscissas. From the method of construction here pointed out, the student will have no difficulty in deriving the equation in the text. This curve is given as an example regarding multiple points by Cramer, in his valuable old work (p. 413) on the *Analyse des Lignes Courbes Algébriques;* a work which has afforded to subsequent writers many of their best examples and illustrations regarding curves. The student will find, also, much interesting information regarding curves in Newton's *Enumeratio Linearum Tertii Ordinis*, published in 1706; and still more in the work of Stirling, published in 1717, and entitled, *Lineæ Tertii Ordinis Newtonianæ, sive Illustratio Tractatus D. Newtoni de Enumeratione Linearum Tertii Ordinis.* Stirling and Maclaurin were two highly distinguished alumni of the University of Glasgow, nearly contemporaneous. The latter gained deserved distinction in all quarters, but the merits of the former were overlooked in a great degree by the mathematicians of Britain, though they were recognised and duly appreciated by those of the continent.

has the axis of x as one asymptote; and, as another, a straight line passing through the origin, and making with the same axis an angle of 45°.

9. The curve whose equation is $y^2(a+x)=x^2(a-x)$, has a double point at the origin; a node between $x=0$ and $x=a$; and an asymptote perpendicular to the axis of x, and passing through the point of that axis at which $x=-a$.

10. Trace the curve, $r=a\cos^2 n\theta+b$; and point out its general properties and peculiarities. Determine, in particular, the curves corresponding to $n=3$, and first $b=\frac{1}{2}a$; secondly, $b=-\frac{1}{2}a$; and thirdly, $b=0$.

11. Show that lines of the second order have no inflexions.

12. Determine the loci of the following equations:

(1.) $y=x^n+\sqrt{(\sin x-1)}$;
(2.) $y^2=ax\sin x$;
(3.) $xy=a^2\sqrt{(1-\sec^2 x)}$;
(4.) $y=x^2+\log(a\sin x-1)$.

In the first of these, let n be taken successively equal to 0, ±1, ±2, &c.; and in the fourth let successively, $a=1$, $a>1$, and $a<1$.

13. Trace the curve whose equation for polar coordinates is $r^2-2a^2\cos\frac{1}{2}\theta+a^2_2-b^2=0$; and point out its peculiarities, when $b=a$, $b>a$, and $b<a$.

14. What is the locus of $y=x^3\pm\sqrt{(\sin\frac{1}{2}\pi x-1)}$?

15. Trace the curve whose equation is $y^2+\varepsilon^x=1$.

16. Prove that when n is infinite, the locus of $x^{2n}+y^{2n}=a^{2n}$ is the perimeter of a square having $2a$ as its side.

XIII.—INTEGRATION OF RATIONAL FUNCTIONS.

237. If a differential have its differential coefficient composed of terms of the form x^m, such as if it were

$$(x^2+ax^{-1}+bx^{-\frac{1}{2}})\,dx,$$

its integral will be found by means of one or both of the formulas A and B, p. 41. If, however, the coefficient be of the form,

$$\frac{ax^m+bx^{m-1}+\dots\dots\dots}{x^n+b^1x^{n-1}+\dots\dots\dots},$$

the principles that have been thus far established will in general fail to enable us to find the integral. In such a case, should m equal or exceed n, the numerator is to be divided by the denominator, and the operation to be carried out till a remainder is obtained in which the highest index of x is less than n. By this means the differential will be reduced to the form

$$\left(ax^{m-n} + b''x^{m-n-1} + \dots + \frac{\alpha x^{n-1} + \beta x^{n-2} + \dots\dots}{x^n + bx^{n-1} + \dots\dots}\right)dx;$$

the integral of the first part of which will be obtained by means of one or both of the formulas above referred to. The second part, however, cannot be integrated by means of any of the principles given in p. 41, unless it be of such a form as to admit the application of formula B, G, G_2, or occasionally of K_2. For completing, therefore, the means of integrating rational fractions of every kind, we have only to find the method of integrating such fractions, when the highest power of x in the numerator of the differential coefficient is less by one or more units than that of the highest power in its denominator. The general solution of this problem will be rendered easy to the learner by a particular example, such as the following.

238. Given $du = \dfrac{5x+1}{x^2+x-2}\,dx$, to find its integral. By putting the denominator equal to zero, we get $x=1$, and $x=-2$; and therefore (ALG. 172) the denominator is equivalent to $(x-1)(x+2)$. It is plain, also (ALG. 73 and 85), that the differential coefficient in the proposed quantity is the sum of two fractions having $x-1$ and $x+2$ as denominators; and what remains to be done, is to determine the numerators of these fractions. To effect this, let us assume

$$\frac{5x+1}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2}.$$

Then, by the actual addition of the two latter fractions, the denominator of the sum is the same as that of the first member; while the numerator is $(A+B)\,x+2\,A-B$. Putting this equal to $5x+1$, we have (ALG. 204),

$$A+B=5, \text{ and } 2\,A-B=1;$$

and from these equations we get $A=2$ and $B=3$. Hence we have,

$$du=\frac{2\,dx}{x-1}+\frac{3\,dx}{x+2};$$

and, therefore, by integrating by means of B, p. 41, we get

$$u=2\log(x-1)+3\log(x+2)=\log(x-1)^2+\log(x+2)^3.$$

By a process similar to that which has just been employed, we should succeed in integrating all functions of a similar kind. In thus employing, however, the method of indeterminate coefficients, the process generally becomes complicated and laborious, and the following method is preferable.

239. Let, as before, $\frac{5x+1}{x^2+x-2}\,dx=\frac{A\,dx}{x-1}+\frac{B\,dx}{x+2}$. Then, dividing by dx, and multiplying by x^2+x-2, we get

$$5x+1=A\frac{x^2+x-2}{x-1}+B\frac{x^2+x-2}{x+2}.$$

Now, since $x-1$ and $x+2$ are factors of x^2+x-2, the first of the fractions in the second member will become $\frac{0}{0}$, if $x=1$, and the second will vanish. The value of the first may be found by dividing x^2+x-2 by $x-1$, and taking $x=1$ in the quotient; or more easily (No. 118) by taking $x=1$ in the differential coefficients of the numerator and denominator. In this way we find its value to be 3; and by dividing $5x+1$, or 6, by this, we get $A=2$, as before. So likewise, the first fraction vanishes, when $x=-2$, and the second, becoming $\frac{0}{0}$, is found in a similar way to be -3; and by dividing $5x+1$, or -9, by this, we get $B=3$, the same that was found in the other way.

240. To generalise the method employed in the last No., let fx be the numerator, and Fx the denominator of the differential coefficient; and (ALG. 172) having resolved Fx into its simple factors $x-a$, $x-b$, ..., $x-l$, let us assume

$$\frac{fx}{Fx}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}+\ldots\ldots+\frac{L}{x-l}.$$

Then, by multiplying by Fx, we obtain

$$fx = A\,\frac{Fx}{x-a} + B\,\frac{Fx}{x-b} + C\,\frac{Fx}{x-c} + \ldots\ldots + L\,\frac{Fx}{x-l}.$$

Now, if we take $x=a$, the first term of the second member will become $A\frac{0}{0}$, and all the others will vanish. The value of the first term (No. 118) is AF^1a; and accordingly $A=\frac{fa}{F^1a}$; where fa and F^1a are put to denote what fx and F^1x become respectively, when x is changed into a. In a manner exactly similar, it would be shown, that if fb and F^1b be put to denote respectively what fx and F^1x become where x is changed into b, we should have $B=\frac{fb}{F^1b}$; and we should arrive at a similar conclusion in reference to the values of the remaining coefficients, C, D, ..., L.

It appears, therefore, that *to resolve the fraction which has fx as its numerator and Fx as its denominator, into the simple fractions of which it is the sum, we are to find* (ALG. 172) *the simple factors, $x-a$, $x-b$, ..., $x-l$, of Fx. These will be the denominators of the required fractions; and their numerators will be found by dividing fx by the first differential coefficient of Fx, and taking in the quotient x successively equal to a, b, c, ..., l.*

241. As an example of the principle established in the last No., let it be required to integrate $du=\frac{(x^2-3)dx}{x^3-7x+6}$. By the resolution of the equation, $x^3-7x+6=0$, we get $x=1$, $x=2$, and $x=-3$; whence the factors of the denominator are $x-1$, $x-2$, and $x+3$. Then, by dividing x^2-3 by $3x^2-7$ (the differential coefficient of the denominator) we get $\frac{fx}{F^1x}=\frac{x^2-3}{3x^2-7}$. By taking in this x equal successively to 1, 2, and -3, we get $\frac{1}{2}$, $\frac{1}{5}$, and $\frac{3}{10}$; and consequently we have

$$\frac{x^2-3}{x^3-7x+6}=\frac{1}{2}\,\frac{1}{x-1}+\frac{1}{5}\,\frac{1}{x-2}+\frac{3}{10}\,\frac{1}{x+3}.$$

Hence, by multiplying by dx, and integrating (by means of B, p. 41) we get

$$\int \frac{(x^2-3)dx}{x^3-7x+6} = \frac{1}{2}\log(x-1) + \frac{1}{5}\log(x-2) + \frac{3}{10}\log(x+3).$$

242. As another example, let it be required to find the integral of

$$du = \frac{(8x^2-6x+8)dx}{x^3-x^2+4x-4}.$$

Here, by resolving the equation obtained by putting the denominator equal to zero, we get $x=1$, $x=2\sqrt{-1}$, and $x=-2\sqrt{-1}$. We have also

$$\frac{fx}{F'x} = \frac{8x^2-6x+8}{3x^2-2x+4}:$$

and by taking x in this successively equal to the foregoing values, we obtain 2, 3, and 3; and therefore

$$u = \int \frac{(8x^2-6x+8)dx}{x^3-x^2+4x-4} = \int \frac{2dx}{x-1} + \int \frac{3dx}{x+2\sqrt{-1}} + \int \frac{3dx}{x-2\sqrt{-1}}$$

$$= 2\log(x-1) + 3\log(x+2\sqrt{-1}) + 3\log(x-2\sqrt{-1})$$

$$= 2\log(x-1) + 3\log(x^2+4);$$

the last expression being obtained from the one immediately preceding it, by taking the product of $x+2\sqrt{-1}$ and $x-2\sqrt{-1}$. The integral might also have been obtained by taking the sum of the second and third of the partial fractions; as we should thus have got $\frac{6x\,dx}{x^2+4}$, the integral of which is $3\log(x^2+4)$.

243. If the denominator of the differential coefficient have two or more equal factors, the foregoing method fails. Thus, in the expression $\frac{4x^3+6x^2-8x+4}{x^4+x^3-2x^2}dx$, the factors of the denominator are x, x, and x^2+x-2 (the last of which is the product of the simpler factors $x-1$ and $x+2$); and we should fail in obtaining a solution, were

we to assume A, B, C, and D as numerators to the several denominators x, x, $x-1$, and $x+2$. To get a solution let us assume

$$\frac{4x^3+6x^2-8x+4}{x^4+x^3-2x^2}=\frac{A_2}{x^2}+\frac{A}{x}+\frac{Q}{x^2+x-2}\ \ldots\ldots(1).$$

Then, by multiplying by the denominator of the first member, we get

$$4x^3+6x^2-8x+4=A_2(x^2+x-2)+$$
$$Ax(x^2+x-2)+Qx^2\ldots\ldots(2):$$

and if in this we take $x=0$ (one of its equal values), we get $4=-2A_2$, and therefore $A_2=-2$. Again, by differentiating equation (2), and taking in the result $x=0$ (its other equal value), we get $-8=A_2-2A$; whence we readily find $A=3$. In the next place, to find the partial fractions equivalent to the last term of equation (1), we have merely to divide the numerator of the first member by the differential coefficient of its denominator, and in the quotient to take x successively equal to 1 and -2. In this way we obtain 2 and -1. Hence equation (1) becomes

$$\frac{4x^3+6x^2-8x+4}{x^4+x^3-2x^2}=-\frac{2}{x^2}+\frac{3}{x}+\frac{2}{x-1}-\frac{1}{x+2}:$$

and from this, by multiplying by dx and integrating, we get

$$\int\frac{(4x^3+6x^2-8x+4)\,dx}{x^4+x^3-2x^2}=$$
$$\frac{2}{x}+3\log x+2\log(x-1)-\log(x+2).$$

244. To generalise the principle employed in the foregoing example, let us assume

$$\frac{fx}{Fx}=\frac{A_n}{(x-a)^n}+\frac{A_{n-1}}{(x-a)^{n-1}}+\ldots\ldots+\frac{A}{x-a}+\frac{Q}{\varphi x},$$

where Fx is evidently equal to $(x-a)^n\varphi x$, and φx does not contain $x-a$ as a factor. Hence, by multiplying by Fx, we plainly get

$$fx=A_n\varphi x+A_{n-1}(x-a)\varphi x+A_{n-2}(x-a)^2\varphi x$$
$$+\ldots\ldots+Q(x-a)^n;$$

and the value of A_n will be found, by taking $x=a$, to be $\frac{fa}{\phi a}$. By differentiating the last equation, and in the result taking $x=a$, we get $f^1a=A_n\phi^1a+A_{n-1}\phi a$, an equation which will give the value of A_{n-1}: and, by repeated differentiations, and by taking in each result $x=a$, we should get the means of finding the values of A_{n-2}, A_{n-3}, &c., successively.

245. For completing the means of integrating rational fractions of every kind, it is necessary that we investigate the method of integrating the expression, $\frac{dx}{(x^2+a^2)^n}$,* n being a whole number. To effect this, let us assume

$$\int\frac{dx}{(x^2+a^2)^n}=\frac{Ax}{(x^2+a^2)^{n-1}}+B\int\frac{dx}{(x^2+a^2)^{n-1}},$$ A and B being constant. By differentiating this equation, we obtain

$$\frac{dx}{(x^2+a^2)^n}=\frac{Adx}{(x^2+a^2)^{n-1}}-\frac{2(n-1)Ax^2dx}{(x^2+a^2)^n}+\frac{Bdx}{(x^2+a^2)^{n-1}}.$$

Multiplying this by $(x^2+a^2)^n$, and dividing by dx, we get

$$1=A(x^2+a^2)-2(n-1)Ax^2+B(x^2+a^2).$$

Hence, by equalling the coefficients of the like quantities, we obtain

$$Aa^2+Ba^2=1, \qquad \text{and } A-2(n-1)A+B=0.$$

These equations give $A=\frac{1}{2(n-1)a^2}$, and $B=\frac{2n-3}{2(n-1)a^2}$; by substituting which in the integral assumed above, we obtain

$$\int\frac{dx}{(x^2+a^2)^n}=\frac{1}{2(n-1)a^2}\cdot\frac{x}{(x^2+a^2)^{n-1}}+\frac{2n-3}{2(n-1)a^2}\int\frac{dx}{(x^2+a^2)^{n-1}}.$$

By a similar process, the integral of the latter term of this would be obtained in two terms, the second of which

* In this it is plain, that for x^2 we may put $(x-c)^2$, c being a constant quantity.

would contain $\int\frac{dx}{(x^2+a^2)^{n-2}}$. By a like operation, continued till the exponent of x^2+a^2 in the denominator would be unity, we should at length arrive at an expression, the only variable part in which would be $\frac{dx}{x^2+a^2}$; and the integral of this would be obtained by formula G, p. 41. The number of such operations would evidently be $n-1$.

246. In all that we have had thus far, respecting rational fractions, it has been taken for granted that we know the factors of the denominator. Now, when these are not given, the finding of them would require the resolution of the equation obtained by putting the denominator equal to zero;—a problem which, beyond certain limits, the present state of analysis does not enable us to solve in general terms. One interesting case of it, however, which we can resolve, is that in which the denominator is $x^n \pm a^n$, n being a whole number; and this we may now consider.

By assuming $x=ay$, the expression just mentioned becomes $a^n(y^n \pm 1)$, and it will be merely necessary to find the factors of $y^n \pm 1$, or, which is the same, to find the roots of the equation, $y^n = \pm 1$. Now (No. 102), if

$$y=\cos\varphi \pm \sin\varphi\sqrt{-1},\ y^n=\cos n\varphi \pm \sin n\varphi\sqrt{-1};$$

and, therefore, to suit the present case, we must have $\cos n\varphi \pm \sin n\varphi\sqrt{-1} = \pm 1$. This will be so, if φ be taken of such a value that $n\varphi$ may be an even or odd multiple of π, accordingly as the second member is $+1$ or -1; since the sine of any multiple of π is nothing, while the cosine of an even multiple of it is 1, and that of an odd multiple -1. We may assume, therefore, $n\varphi=2m\pi$ when $y=1$, and $n\varphi=(2m+1)\pi$ when $y=-1$; so that, in the former case, we shall have

$$\varphi=\frac{2m\pi}{n},\quad \text{and } y=\cos\frac{2m\pi}{n}+\sin\frac{2m\pi}{n}\sqrt{-1}\,\ldots\ldots\ldots(a),$$

and in the latter,

$$\varphi=\frac{(2m+1)\pi}{n},$$

$$\text{and } y=\cos\frac{(2m+1)\pi}{n}+\sin\frac{(2m+1)\pi}{n}\sqrt{-1}\,\ldots\ldots(b).$$

These values of y will evidently (No. 102) give respectively $y^n = 1$, and $y^n = -1$; and the different values of y will be found by taking m successively equal to 0, 1, 2, 3, &c. up to $n-1$, after which the same series of values would recur perpetually. Hence, the simple factors of $y^n - 1$ will be found by taking m successively equal to 0, 1, 2, 3, &c. in the expression,

$$y - \cos\frac{2m\pi}{n} - \sin\frac{2m\pi}{n}\sqrt{-1} \ldots\ldots\ldots\ldots\ldots\ldots (c).$$

In like manner, the factors of $y^n + 1$ will be found, by assigning to m the same values in the formula,

$$y - \cos\frac{(2m+1)\pi}{n} - \sin\frac{(2m+1)\pi}{n}\sqrt{-1} \ldots\ldots (d).$$

Of the factors which will thus be obtained, those which arise from taking two arcs such that their sum may be 2π, or such that in (c) two values of m may be together equal to n, and in (d) to $n-1$, will differ only in the signs of their last terms; since, by trigonometry, the cosines of such arcs are the same, while their sines are equal, but are one positive and the other negative. Thus, in (c), if the arc were taken first equal to $\frac{2\pi}{n}$, and then to $\frac{2(n-1)\pi}{n}$, so that in the one case, $m = 1$, and in the other $m = n - 1$, we should have, for the corresponding factors,

$$y - \cos\frac{2\pi}{n} - \sin\frac{2\pi}{n}\sqrt{-1}, \quad \text{and } y - \cos\frac{2\pi}{n} + \sin\frac{2\pi}{n}\sqrt{-1}.$$

By taking the product of the imaginary factors so related, we should have real factors of the second degree, which would be of the following forms:

$$\text{For } y^n - 1, \quad y^2 - 2y\cos\frac{2m\pi}{n} + 1 \ \ldots\ldots\ldots\ldots (e):$$

$$\text{for } y^n + 1, \quad y^2 - 2y\cos\frac{(2m+1)\pi}{n} + 1 \ldots\ldots (f).$$

In the use of these formulas, it is to be observed that, when m is equal to 0, n, $2n$, &c. formula (e) gives $(y-1)^2$ instead of $y-1$, which is the factor to be employed. This arises from the circumstance, that the factor thus obtained is produced by the multiplication of two factors which in that case are equal, namely, those arising from taking $m=0$, and $m=n$, $m=2n$, &c.*

By an extension of the same principles, the factors of $x^{2n}+px^n+q$ might be obtained; since, by the theory of equations, the second term might be taken away by substituting $x'^n-\frac{1}{2}p$ for x^n.

What has been established in this No. is the same in

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* By means of the principles above explained, the following results may be found, the investigation of which will afford useful exercise to the student.

SIMPLE FACTORS OF y^5-1.

$y-1$,
$y-\cos\frac{2}{5}\pi-\sin\frac{2}{5}\pi\sqrt{-1}$,
$y-\cos\frac{4}{5}\pi-\sin\frac{4}{5}\pi\sqrt{-1}$,
$y-\cos\frac{6}{5}\pi-\sin\frac{6}{5}\pi\sqrt{-1}$,
or $y-\cos\frac{4}{5}\pi+\sin\frac{4}{5}\pi\sqrt{-1}$,
$y-\cos\frac{8}{5}\pi-\sin\frac{8}{5}\pi\sqrt{-1}$
or $y-\cos\frac{2}{5}\pi+\sin\frac{2}{5}\pi\sqrt{-1}$,

REAL FACTORS OF y^5-1.

$y-1$,
$y^2-2y\cos\frac{2}{5}\pi+1$,
$y^2-2y\cos\frac{4}{5}\pi+1$.

SIMPLE FACTORS OF y^6-1.

$y-1$,
$y-\cos\frac{1}{3}\pi-\sin\frac{1}{3}\pi\sqrt{-1}$,
$y-\cos\frac{2}{3}\pi-\sin\frac{2}{3}\pi\sqrt{-1}$,
$y+1$,
$y-\cos\frac{4}{3}\pi-\sin\frac{4}{3}\pi\sqrt{-1}$
or $y-\cos\frac{2}{3}\pi+\sin\frac{2}{3}\pi\sqrt{-1}$
$y-\cos\frac{5}{3}\pi-\sin\frac{5}{3}\pi\sqrt{-1}$
or $y-\cos\frac{1}{3}\pi+\sin\frac{1}{3}\pi\sqrt{-1}$.

REAL FACTORS OF y^6-1.

$y-1$, $y+1$,
$y^2-2y\cos\frac{1}{3}\pi+1$,
$y^2-2y\cos\frac{2}{3}\pi+1$.

SIMPLE FACTORS OF y^5+1.

$y-\cos\frac{1}{5}\pi-\sin\frac{1}{5}\pi\sqrt{-1}$,
$y-\cos\frac{3}{5}\pi-\sin\frac{3}{5}\pi\sqrt{-1}$,
$y+1$,
$y-\cos\frac{7}{5}\pi-\sin\frac{7}{5}\pi\sqrt{-1}$
or $y-\cos\frac{3}{5}\pi+\sin\frac{3}{5}\pi\sqrt{-1}$,
$y-\cos\frac{9}{5}\pi-\sin\frac{9}{5}\pi\sqrt{-1}$
or $y-\cos\frac{1}{5}\pi+\sin\frac{1}{5}\pi\sqrt{-1}$.

REAL FACTORS OF y^5+1.

$y^2-2y\cos\frac{1}{5}\pi+1$,
$y^2-2y\cos\frac{3}{5}\pi+1$,
$y+1$.

SIMPLE FACTORS OF y^6+1.

$y-\cos\frac{1}{6}\pi-\sin\frac{1}{6}\pi\sqrt{-1}$,
$y-\cos\frac{1}{2}\pi-\sin\frac{1}{2}\pi\sqrt{-1}$,
or $y-\sqrt{-1}$,
$y-\cos\frac{5}{6}\pi-\sin\frac{5}{6}\pi\sqrt{-1}$,
$y-\cos\frac{7}{6}\pi-\sin\frac{7}{6}\pi\sqrt{-1}$
or $y-\cos\frac{5}{6}\pi+\sin\frac{5}{6}\pi\sqrt{-1}$,
$y-\cos\frac{9}{6}\pi-\sin\frac{9}{6}\pi\sqrt{-1}$
or $y+\sqrt{-1}$,
$y-\cos\frac{11}{6}\pi-\sin\frac{11}{6}\pi\sqrt{-1}$
or $y-\mathrm{cs}\frac{1}{6}\pi+\sin\frac{1}{6}\pi\sqrt{-1}$.

REAL FACTORS OF y^6+1.

$y^2-2y\cos\frac{1}{6}\pi+1$,
y^2+1,
$y^2-2y\cos\frac{5}{6}\pi+1$.

effect as *Cotes's theorem*, so called from its discoverer, Mr. Cotes of Cambridge.*

ADDITIONAL EXAMPLES OF THE INTEGRATION OF RATIONAL FRACTIONS.

247. Required the integral of $\frac{bdx}{x^2-a^2}$. Here, the factors of the denominator are $x-a$ and $x+a$, and its differential coefficient is $2x$; which becomes successively $2a$ and $-2a$ for $x=a$, and $x=-a$. Hence (No. 240)

$$\frac{bdx}{x^2-a^2}=\frac{b}{2a}\,\frac{dx}{x-a}-\frac{b}{2a}\,\frac{dx}{x+a};$$

and, therefore,

$$\int\frac{bdx}{x^2-a^2}=\frac{b}{2a}\left\{\log(x-a)-\log(x+a)\right\}=\frac{b}{2a}\log\frac{x-a}{x+a}.$$

If $b=1$, this becomes the same as G_2, p. 41; and thus we have a direct analytical investigation of that formula.

Since the given differential might be put under the form $\frac{b}{a\sqrt{-1}}\,\frac{a\sqrt{-1},dx}{x^2+(a\sqrt{-1})^2}$, the integral might (p. 41, G) be expressed under the imaginary form $\frac{b}{a\sqrt{-1}}\tan^{-1}\frac{x}{a\sqrt{-1}}$. In this, as in other cases in which a real quantity is expressed by means of imaginary symbols, one of these symbols does

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* This theorem is as follows: Let the circumference of a circle (*fig.* 41) whose radius is a, and centre C, be divided into $2n$ equal parts in the points 1, 2, 3, &c. and let O be taken within the circle or not, as the case may be, so that $OC=y$, and join O1, O2, O3, &c.; then, O1, O3, O5, O7, &c. are the factors of y^n+a^n; and O2, O4, O6, &c. are those of y^n-a^n, or a^n-y^n. To show the truth of this, with regard to the figure here referred to, which answers to $n=5$, we have, taking $a=1$ (Trig. No. 56), $O1^2$ or $O1\times O9=y^2-2y\cos\frac{1}{5}\pi+1$, $O3^2$ or $O3\times O7=y^2-2y\cos\frac{3}{5}\pi+1$, and $O5=y+1$; which, by the last note, are the factors of y^5+1. We have also, in like manner, $O2^2=O2\times O8=y^2-2y\cos\frac{2}{5}y+1$, $O4^2=O4\times O6=y^2-2y\cos\frac{4}{5}y+1$, and $O10=1-y$ or $y-1$; which, by the same note, are the factors of y^5-1 or $1-y^5$; and a like illustration might be given in other cases.

Mr. Cotes died in 1716, at the early age of thirty-four, to the great regret of Newton, and all who knew his high talents as a mathematician.

away with the imaginary quantities arising from the other. This would appear in the present instance, as well as in others, by expanding both forms of the integral into series, since the series would be reducible to one another.

By using $x-a\sqrt{-1}$ and $x+a\sqrt{-1}$ as the factors of x^2+a^2, we should find, in like manner,

$$\int \frac{adx}{x^2+a^2} = \tfrac{1}{2}\sqrt{-1}.\log\frac{x-a\sqrt{-1}}{x+a\sqrt{-1}};$$

which has already (No. 66) been shown to be equal to $\tan^{-1}\frac{x}{a}$.

248. Required the integral of $\frac{(6x^2+13x-43)\,dx}{x^3-13x-12}$. By resolving (ALG. § 189) the equation $x^3-13x-12=0$, we find $x=-1$, $x=-3$, and $x=4$, whence the factors of the denominator are $x+1$, $x+3$, and $x-4$. Then, assuming

$$\frac{6x^2+13x-43}{x^3-13x-12} = \frac{A}{x+1} + \frac{B}{x+3} + \frac{C}{x-4},$$

and proceeding by No. 240, we get $F^1x=3x^2-13$, and $\frac{fx}{F^1x}=\frac{6x^2+13x-43}{3x^2-13}$; in which, by taking x successively equal to -1, -3, and 4, we get $A=5$, $B=-2$, and $C=3$; so that

$$\frac{6x^2+13x-43}{x^3-13x-12} = \frac{5}{x+1} - \frac{2}{x+3} + \frac{3}{x-4}.$$

Hence, by multiplying by dx, and integrating (p. 41, B), we find

$$\int\frac{(6x^2+13x-43)\,dx}{x^3-13x-12} = 5\log(x+1) - 2\log(x+3) + 3\log(x-4),$$

the integral required.

The values of A, B, C, might have been found, though not so easily, by the addition of the partial fractions. By this means, after the rejection of the common denominator, there results, $6x^2+13x-43$

$$= Ax^2 - Ax - 12A + Bx^2 - 3Bx - 4B + Cx^2 + 4Cx + 3C;$$

whence, by equalling the coefficients of the like quantities, we get, $A+B+C=6$,

$$-A-3B+4C=13, \text{ and } -12A-4B+3C=-43;$$

and these three equations will give the same values for A, B, and C, as those found above.

249. Required the integral of $\frac{(x^3-2)\,dx}{x^7-4x^3}$. Here the denominator may be resolved successively into $x^3(x^4-4)$, $x^3(x^2-2)(x^2+2)$, and $x^3(x-\sqrt{2})(x+\sqrt{2})(x-\sqrt{-2})(x+\sqrt{-2})$. Assuming, therefore, $\frac{x-2}{x^7-4x^3}=$

$$\frac{A}{x-\sqrt{2}}+\frac{B}{x+\sqrt{2}}+\frac{C}{x-\sqrt{-2}}+\frac{D}{x+\sqrt{-2}}+\frac{A_3}{x^3}+\frac{A_2}{x^2}+\frac{A_1}{x},$$

we have (No. 240) $\frac{fx}{F^1x}=\frac{x^3-2}{7x^6-12x^2}$; and, substituting for x successively in this, $\sqrt{2}$, $-\sqrt{2}$, $\sqrt{-2}$, and $-\sqrt{-2}$, we find

$$A=\frac{\sqrt{2}-1}{16},\quad B=-\frac{\sqrt{2}+1}{16},\quad C=\frac{1+\sqrt{-2}}{16},\quad D=\frac{1-\sqrt{-2}}{16}.$$

Again (No. 244), we have $\frac{fx}{\varphi x}=\frac{x^3-2}{x^4-4}$; in which, taking $x=0$, we get $A_3=\frac{1}{2}$. The differential coefficients obtained by two successive differentiations of $fx=x^2-2$, are such as to vanish when $x=0$; and hence A_2 and A_1 are each equal to zero. Hence we have $\frac{x^3-2}{x^7-4x^3}=$

$$\frac{1}{16}\left(\frac{\sqrt{2}-1}{x-\sqrt{2}}-\frac{\sqrt{2}+1}{x+\sqrt{2}}+\frac{1+\sqrt{-2}}{x-\sqrt{-2}}+\frac{1-\sqrt{-2}}{x+\sqrt{-2}}\right)+\frac{1}{2x^3};$$

or, by the actual addition of the terms containing the imaginary quantities,

$$\frac{x^3-2}{x^7-4x^3}=\frac{\sqrt{2}-1}{16(x-\sqrt{2})}-\frac{\sqrt{2}+1}{16(x+\sqrt{2})}+\frac{x-2}{8(x^2+2)}+\frac{1}{2x^3}.$$

Multiplying now by dx, and integrating by B, G, and A, page 41, we obtain $\int\frac{(x^3-2)dx}{x^7-4x^3}=\frac{\sqrt{2}-1}{16}\log(x-\sqrt{2})$

$$-\frac{\sqrt{2}+1}{16}\log(x+\sqrt{2})+\tfrac{1}{16}\log(x^2+2)^{*}-\frac{\sqrt{2}}{8}\tan^{-1}\frac{x^{*}}{\sqrt{2}}-\frac{1}{4x^2},$$

the integral required. This, by some easy reductions, will become $\int\frac{(x^3-2)dx}{x^7-4x^3}=-\frac{\sqrt{2}}{16}\log\frac{x+\sqrt{2}}{x-\sqrt{2}}+\frac{1}{16}\log\frac{x^2+2}{x^2-2}-$

$$\frac{\sqrt{2}}{8}\tan^{-1}\frac{x}{\sqrt{2}}-\frac{1}{4x^2};$$

and it might be expressed in various other forms.

250. Required the integral of $\dfrac{(x^3-2x+1)dx}{x^4-6x^3+10x^2-6x+9}$.

Here the factors of the denominator being $(x-3)^2$ and x^2+1, or $(x-3)^2$, $x-\sqrt{-1}$, and $x+\sqrt{-1}$, we assume

$$\frac{x^3-2x+1}{x^4-6x^3+10x^2-6x+9}=\frac{A_2}{(x-3)^2}+\frac{A_1}{x-3}+\frac{A}{x-\sqrt{-1}}+\frac{B}{x+\sqrt{-1}}.$$

Then (No. 244), we have

$$fx=x^3-2x+1,\quad\text{and}\ \frac{dfx}{dx}=3x^2-2;$$

and hence, if $x=3$, we get $A_2=\frac{11}{5}$, and $A_1=\frac{59}{50}$. Again (No. 240), we have $F^1x=4x^3-18x^2+20x-6$; and fx being x^3-2x+1, by taking $x=\sqrt{-1}$, and $x=-\sqrt{-1}$, we find $A=\dfrac{1-3\sqrt{-1}}{12+16\sqrt{-1}}$, and $B=\dfrac{1+3\sqrt{-1}}{12-16\sqrt{-1}}$; or

$$A=-\frac{9+13\sqrt{-1}}{100},\ \text{and}\ B=-\frac{9-13\sqrt{-1}}{100},$$

as would appear by multiplying the numerator and denominator of the first by $\frac{1}{4}(12-16\sqrt{-1})$, and those of the second by $\frac{1}{4}(12+16\sqrt{-1})$. Hence, the last two fractions are

$$-\frac{9+13\sqrt{-1}}{100(x-\sqrt{-1})},\quad\text{and}\ -\frac{9-13\sqrt{-1}}{100(x+\sqrt{-1})},$$

* These two terms arise from taking xdx and $-2dx$ separately, as numerators to the denominator, $8(x^2+2)$.

the sum of which is $\frac{13-9x}{50(x^2+1)}$. We have, therefore,

$$\frac{(x^3-2x+1)dx}{x^4-6x^3+10x^2-6x+9}=\frac{11dx}{5(x-3)^2}+\frac{59dx}{50(x-3)}+\frac{(13-9x)dx}{50(x^2+1)}.$$

Now (A_2, p. 41), the integral of the first term is $-\frac{11}{5(x-3)}$; and (B, p. 41) that of the second is $\frac{59\log(x-3)}{50}$. The remaining term may be divided into two, $-\frac{9x\,dx}{50(x^2+1)}$ and $\frac{13\,dx}{50(x^2+1)}$. The integral of the first of these (B, p. 41) is $-\frac{9\log(x^2+1)}{100}$; and (G, p. 41) that of the second is $\frac{13\tan^{-1}x}{50}$. Collecting these, we have

$$\int\frac{(x^3-2x+1)\,dx}{x^4-6x^3+10x^2-6x+9}=$$
$$-\frac{11}{5(x-3)}+\frac{59\log(x-3)}{50}-\frac{9\log(x^2+1)}{100}+\frac{13\tan^{-1}x}{50}.$$

251. Required the integral of $\frac{dx}{ax^2+bx+c}$. Here, by substituting $x'-\frac{b}{2a}$ for x, so as to take away the second term of the denominator, and then putting $\frac{4ac-b^2}{4a^2}=a'^2$, we get

$$\frac{dx}{ax^2+bx+c}=\frac{1}{a}\cdot\frac{dx'}{x'^2+a'^2}=\frac{1}{aa'}\cdot\frac{a'dx'}{x'^2+a'^2}.$$

By integrating this (p. 41, G), and restoring the values of a' and x', we obtain,

$$\int\frac{dx}{ax^2+bx+c}=\frac{2}{\sqrt{(4ac-b^2)}}\tan^{-1}\frac{2ax+b}{\sqrt{(4ac-b^2)}}.$$

Should b^2 be greater than $4ac$, the integral in this form would contain imaginary quantities. In that case, let the numerator and the denominator be multiplied by $4a$, and the denominator will then be the product of the factors, $2ax+b-\sqrt{(b^2-4ac)}$ and $2ax+b+\sqrt{(b^2-4ac)}$. Assuming, then,

$$\frac{4a}{4a^2x^2+4abx+4ac}=\frac{\mathrm{A}}{2ax+b-\sqrt{(b^2-4ac)}}+\frac{\mathrm{B}}{2ax+b+\sqrt{(b^2-4ac)}},$$

and proceeding as in the foregoing examples, we get

$$\mathrm{A}=\frac{2a}{\sqrt{(b^2-4ac)}},\quad\text{and }\mathrm{B}=-\frac{2a}{\sqrt{(b^2-4ac)}};$$

and the required integral is found to be

$$\frac{1}{\sqrt{(b^2-4ac)}}\log\frac{2ax+b-\sqrt{(b^2-4ac)}}{2ax+b+\sqrt{(b^2-4ac)}}.$$

If $b^2=4ac$, both these methods fail. In this case, however, the differential becomes simply $\frac{dx}{(a^{\frac{1}{2}}x+c^{\frac{1}{2}})^2}$; and ($\mathrm{A}_2$, p. 41) the integral is $-\frac{1}{ax+a^{\frac{1}{2}}c^{\frac{1}{2}}}$ or $-\frac{1}{ax+\frac{1}{2}b}$.*

252. Required the integral of $\frac{x^2dx}{2x^5+3}$. Here, by putting the denominator under the form $2(x^5+\frac{3}{2})$, and by putting also $\frac{3}{2}=a^5$ and $x=ay$, the differential becomes $\frac{y^2dy}{2a^2(y^5+1)}$. Now (note, p. 159), the real factors of y^5+1 are $y+1$, $y^2-2y\cos\frac{1}{5}\pi+1$, and $y^2-2y\cos\frac{3}{5}\pi+1$; and (No. 240) we find $\frac{y^2dy}{2a^2(y^5+1)}=$

$$\frac{1}{10a^2}\left\{\frac{dy}{y+1}+\frac{2\cos\frac{2}{5}\pi(y+1)dy}{y^2-2y\cos\frac{1}{5}\pi+1}-\frac{2\cos\frac{1}{5}\pi(y+1)dy}{y^2-2y\cos\frac{3}{5}\pi+1}\right\}.$$

In this expression, the integral of the first term in the vin-

* This example, in connexion with No. 245, will enable us to integrate the more general equation, $\frac{dx}{(ax^2+bx+c)^n}$.

culum is $\log(y+1)$. The second, by putting $y-\cos\frac{1}{5}\pi=y'$, becomes

$$\frac{2\cos\frac{2}{5}\pi(y'+1+\cos\frac{1}{5}\pi)\,dy'}{y'^2+\sin^2\frac{1}{5}\pi};$$

the integral of which is easily found; and, by the restoring of the value of y', it becomes

$$\cos\tfrac{2}{5}\pi\log(y^2-2y\cos\tfrac{1}{5}\pi+1)+2\cos\tfrac{2}{5}\pi\cot\tfrac{1}{10}\pi\tan^{-1}\frac{y-\cos\frac{1}{5}\pi}{\sin\frac{1}{5}\pi}.$$

By a like process, the integral of the remaining term would be found to be

$$-\cos\tfrac{1}{5}\pi\log(y^2-2y\cos\tfrac{3}{5}\pi+1)-2\cos\tfrac{1}{5}\pi\cot\tfrac{1}{20}\pi\tan^{-1}\frac{y-\cos\frac{3}{5}\pi}{\sin\frac{3}{5}\pi}.$$

Collecting these, and multiplying by $\frac{1}{10a^2}$, we should obtain the required integral; and we might express it in terms of the original quantities, by substituting for y and a their respective values. It might be farther modified, also, by introducing instead of $\cos\frac{2}{5}\pi$, $\cos\frac{1}{5}\pi$, &c. their values (TRIG. No. 42, &c.), $\frac{1}{4}(\sqrt{5}-1)$, $\frac{1}{4}(\sqrt{5}+1)$, &c.

EXERCISES IN THE INTEGRATION OF RATIONAL FUNCTIONS.

1. $$\int\frac{3x\,dx}{x^2-5x-50}=\log(x+5)+2\log(x-10)=\log\{(x+5)(x-10)^2\}=\log(x^3-15x^2+500).$$

2. $$\int\frac{3x\,dx}{x^2+5x-50}=\log(x-5)+2\log(x+10)=\log(x^3+15x^2-500)$$

3. $$\int\frac{(x^3-21x+22)\,dx}{x^3+x^2-10x+8}{}^*=x-\tfrac{2}{5}\log(x-1)-2\log(x-2)+\tfrac{7}{5}\log(x+4).$$

4. $$\int\frac{(2x^3+2x^2+4x+1)\,dx}{x^2+x+1}=x^2+\log(x^2+x+1).$$

* In this, as well as in every instance in which the highest power of x in the numerator is not less than its highest power in the denominator, we are first (No. 237) to divide the numerator by the denominator. By this means we get, in the present instance, $dx-\frac{(x^2+11x-14)dx}{x^3+x^2-10x+8}$, the integration of which is effected by methods already pointed out.

5. $\int \frac{(2x^4-2x^3-14x^2+35x-25)dx}{x^3-7x+6}$

$$=x^2-2x+\log\frac{(x-1)(x-2)}{(x+3)^2}.$$

6. $\int \frac{4xdx}{x^4-1}=\log(x-1)+\log(x+1)-\log(x^2+1)=$

$$\log\frac{x^2-1}{x^2+1}.$$

7. $\int \frac{(x^2-6x+11)dx}{(x-2)^3}=$

$$-\frac{3}{2(x-2)^2}+\frac{2}{x-2}+\log(x-2)=\frac{4x-11}{2(x-2)^2}+\log(x-2).$$

8. $\int \frac{(2x^2-16x+23)dx}{(4-x)^4}=$

$$-\frac{3}{(4-x)^3}+\frac{2}{4-x}=\frac{2x^2-16x+29}{(4-x)^3}.$$

9. $\int \frac{(x+12)dx}{x^3-2x^2+3x-6}=-\log\frac{x^2+3}{(x-2)^2}-\sqrt{3}\,.\tan^{-1}\frac{x}{\sqrt{3}}.$

10. $\int \frac{3a^2dx}{x(a+bx^3)^2}=\frac{a}{a+bx^3}-\log\frac{a+bx^3}{x^3}.$

11. $\int \frac{abdx}{(x^2+a^2)(x^2+b^2)}=\frac{1}{b^2-a^2}\left(b\tan^{-1}\frac{x}{a}-a\tan^{-1}\frac{x}{b}\right).$

12. $\int \frac{xdx}{x^3+x^2+x+1}=\frac{1}{2}\log\frac{\sqrt{(x^2+1)}}{x+1}+\frac{1}{2}\tan^{-1}x.$

13. $\int \frac{dx}{1-x^4}=\frac{1}{4}\log\frac{1+x}{1-x}+\frac{1}{2}\tan^{-1}x.$

14. $\int \frac{dx}{1+x^4}=\frac{1}{4\sqrt{2}}\log\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}+\frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}.$

XIV.—INTEGRATION OF IRRATIONAL DIFFERENTIALS.

253. The integrals of several differentials having irrational coefficients, have been exhibited in page 41; and others may sometimes be reduced, by inspection or by other obvious means, to such forms as to admit of integra-

tion by means of these. In many instances, however, the best mode is to transform them, by means of the Diophantine analysis (see ALG. Chap. XIV.), into others having rational coefficients; and then to apply to the results the principles that have already been established for the integration of such differentials. The method of effecting this, in reference to some of the principal differentials of this kind that admit of integration without series, will form the subject of this Section.

254. When the radicals are monomials, the differential is rendered rational simply by reducing the indices of the radicals to a common denominator, and substituting, for the variable, another with that denominator as index. Thus, if $\frac{(x^{\frac{1}{2}}-2x^{\frac{1}{3}})dx}{1+x^{\frac{1}{3}}}$ be proposed, since the common denominator of the indices is 6, let $x=z^6$; then $x^{\frac{1}{2}}=z^3$, $x^{\frac{1}{3}}=z^2$, and $dx=6z^5dz$. By the substitution of these, the given differential becomes $\frac{6(z^3-2z^2)z^5dz}{1+z^2}$; or, by division,

$$6\left(z^6-2z^5-z^4+2z^3+z^2-2z-1+\frac{2z+1}{z^2+1}\right)dz.$$

Hence, by integrating the terms separately, we get

$$6\left\{\frac{z^7}{7}-\frac{z^6}{3}-\frac{z^5}{5}+\frac{z^4}{2}+\frac{z^3}{3}-z^2-z+\log(z^2+1)+\tan^{-1}z\right\};$$

or, by substituting for z its equal $x^{\frac{1}{6}}$,

$$6\left\{\frac{x^{\frac{7}{6}}}{7}-\frac{x}{3}-\frac{x^{\frac{5}{6}}}{5}+\frac{x^{\frac{2}{3}}}{2}+\frac{x^{\frac{1}{2}}}{3}-x^{\frac{1}{3}}-x^{\frac{1}{6}}+\log(x^{\frac{1}{3}}+1)+\tan^{-1}x^{\frac{1}{6}}\right\}.$$

255. If the radical part be of the form, $\sqrt{(a+bx)}$, the differential will be rendered rational by putting $a+bx=z^2$. Thus, $\frac{dx}{\sqrt{(a+bx)}}$ will, by this means, become $\frac{2dz}{b}$, the integral of which is $\frac{2z}{b}$ or $\frac{2\sqrt{(a+bx)}}{b}$. This integral might likewise be obtained by means of formula A, page 41. That formula, and the principle here pointed out, would also give the integral for any index as well as $\frac{1}{2}$.

256. When a differential contains a single radical of the form, $\sqrt{(a+bx+cx^2)}$, it may be rendered rational by one or other of the three following methods.

I. Let $(a+bx+cx^2)^{\frac{1}{2}}=c^{\frac{1}{2}}(z+x)$....................$(a)$.
Then, by resolving this equation for x, and differentiating the result, we obtain successively

$$x=-\frac{a-cz^2}{b-2cz}\ldots(b), \quad \text{and } dx=-\frac{2c(a-bz+cz^2)dz}{(b-2cz)^2}\ldots\ldots(c).$$

Hence, by adding z to the value of x, and multiplying the sum by $c^{\frac{1}{2}}$, we get

$$(a+bx+cx^2)^{\frac{1}{2}}=-c^{\frac{1}{2}}\frac{a-bz+cz^2}{b-2cz}\ldots\ldots(d).$$

From (a) also, we have $z=c^{-\frac{1}{2}}(a+bx+cx^2)^{\frac{1}{2}}-x$......$(e)$.

II. Let us assume $(a+bx+cx^2)^{\frac{1}{2}}=a^{\frac{1}{2}}+xz$............$(a)$.
Thus, by easy operations, we obtain successively,

$$x=-\frac{2a^{\frac{1}{2}}z-b}{z^2-c}\ldots(b)\,;\quad dx=2\frac{a^{\frac{1}{2}}z^2-bz+a^{\frac{1}{2}}c}{(z^2-c)^2}dz\ldots\ldots(c);$$

$$(a+bx+cx^2)^{\frac{1}{2}}=\frac{a^{\frac{1}{2}}z^2-bz+a^{\frac{1}{2}}c}{z^2-c}\ldots\ldots(d);$$

$$\text{and } z=\frac{(a+bx+cx^2)^{\frac{1}{2}}-a^{\frac{1}{2}}}{x}\ldots\ldots(e).$$

III. If α and β be the values of x in the equation

$$\frac{a}{c}+\frac{b}{c}x+x^2=0, \text{ so that (ALG. 161)}$$

$$a+bx+cx^2=c(x-\alpha)(x-\beta), \text{ we may assume}$$

$$a+bx+cx^2, \text{ or } c(x-\alpha)(x-\beta)=c^2(x-\alpha)^2z^2\ldots\ldots(a)\,;$$

and we shall readily obtain, successively,

$$x=\frac{\alpha cz^2-\beta}{cz^2-1}\ldots\ldots(b);\quad dx=-\frac{2cz(\alpha-\beta)}{(cz^2-1)^2}dz\ldots\ldots(c)\,;$$

$$(a+bx+cx^2)^{\frac{1}{2}}=c(x-\alpha)z=\frac{c(\alpha-\beta)z}{cz^2-1}\ldots\ldots(d)\,;$$

$$\text{and } z=\sqrt{\frac{x-\beta}{c(x-\alpha)}}\ldots\ldots(e).$$

257. Of the three methods investigated in the last No., that one must be employed which will give real results. Hence the first method is inadmissible when c, and the second when a, is negative; and the third cannot be employed unless α and β are real.

In some particular cases the integration may be effected more easily than by means of the foregoing general principles.

EXAMPLES OF THE INTEGRATION OF IRRATIONAL DIFFERENTIALS.

258. LET $du=\dfrac{dx}{\sqrt{(a+bx+cx^2)}}$. Here (No. 256, I.) $du=\dfrac{2c^{\frac{1}{2}}dz}{b-2cz}$; whence (B. p. 41) $u=-\dfrac{1}{c^{\frac{1}{2}}}\log(2cz-b)$; or, by the restoration (No. 256, I.) of the value of z,

$$u=-\frac{1}{c^{\frac{1}{2}}}\log\{2c^{\frac{1}{2}}\sqrt{(a+bx+cx^2)}-2cx-b\}.$$

This may also be expressed in either of the following forms:*

$$u=\frac{1}{c^{\frac{1}{2}}}\log\{2cx+b+2c^{\frac{1}{2}}\sqrt{(a+bx+cx^2)}\}-\frac{1}{c^{\frac{1}{2}}}\log(4ac-b^2),\text{ or}$$

$$u=\frac{1}{c^{\frac{1}{2}}}\log\{2cx+b+2c^{\frac{1}{2}}\sqrt{(a+bx+cx^2)}\}+C.$$

259. If, in the result last obtained, we take $a=\pm a^2, b=0$, and $c=1$, we shall have, after rejecting the constant quantity, $\log 2$,

$$\int\frac{dx}{\sqrt{(x^2\pm a^2)}}=\log\{x+\sqrt{(x^2\pm a^2)}\}.$$

We have thus a direct analytical investigation of formula E_2, p. 41.

260. Let, again, $a=0$, $b=2$, and $c=1$; and, after rejecting $\log 2$, we shall have

$$\int\frac{dx}{\sqrt{(2x+x^2)}}=\log\{x+1+\sqrt{(2x+x^2)}\}.$$

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* The first of these forms is obtained by adding

$$c^{-\frac{1}{2}}\log\left\{2cx+b+2c^{\frac{1}{2}}\sqrt{(a+bx+cx^2)}\right\}$$

to the value of u found above, and by subtracting the same from the sum and modifying the result. The second form is the same as the first, except that $-c^{-\frac{1}{2}}\log(4ac-b^2)$ is made a part of C.

This gives, analytically, F_2, p. 41, if x be changed into $a^{-1}x$, and $-\log a$ be rejected.

261. As another example, let $du = \dfrac{dx}{\sqrt{(a+bx-cx^2)}}$. Here, by making the same assumption as in No. 256, III. we get, $du = \dfrac{2dz}{cz^2+1}$. By dividing the numerator and denominator of this by c, and denoting the second term of the denominator by q^2, we shall have, $du = \dfrac{2}{c^{\frac{1}{2}}} \cdot \dfrac{qdz}{z^2+q^2}$; the integral of which (G. p. 41) is

$$u = \frac{2}{c^{\frac{1}{2}}} . \tan^{-1}\frac{z}{q} = \frac{2}{c^{\frac{1}{2}}} \tan^{-1} c^{\frac{1}{2}} z.$$

But (No. 256, III.) $z = \sqrt{\dfrac{x-\beta}{c(x-\alpha)}}$; and, therefore,

$$u = \int \frac{dx}{\sqrt{(a+bx-cx^2)}} = \frac{2}{c^{\frac{1}{2}}} \tan^{-1} \sqrt{\frac{x-\beta}{x-\alpha}},$$

α and β being the roots of the equation, $x^2 - \dfrac{bx}{c} - \dfrac{a}{c} = 0$;*

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* If $a = a'^2$, $b=0$, and $c=1$, α and β will be $-a'$ and a'; and we shall have

$$\int \frac{dx}{\sqrt{(a'^2-x^2)}} = 2 \tan^{-1} \sqrt{\frac{a'-x}{a'+x}}.$$

We might also take $\alpha = a'$, and $\beta = -a'$, and we should have

$$\int \frac{dx}{\sqrt{(a'^2-x^2)}} = 2 \tan^{-1} \sqrt{\frac{a'+x}{a'-x}}.$$

Either of these forms of the integral may be employed, the apparent difference between them being done away in any particular case, by means of the constant quantity to be annexed. It is evident, indeed, that the arcs in the two forms are complements of one another.

The integral of $\dfrac{dx}{\sqrt{(a^2-x^2)}}$ has been already exhibited in page 41, E; and the integrals obtained here, and in that place, may be shown to be equivalent. To do this, put $\tan^{-1}\sqrt{\dfrac{a-x}{a+x}} = y$; or, which is the same, $\tan y = \sqrt{\dfrac{a-x}{a+x}}$. To the square of this, add 1, and the reciprocal of the result will be $\cos^2 y = \dfrac{a+x}{2a}$; by doubling which, we get (Trig. No. 27) $1+\cos 2y = \dfrac{a+x}{a} = 1 + \dfrac{x}{a}$, and, consequently, $\cos 2y$ or $\sin(\frac{1}{2}\pi - 2y) = \dfrac{x}{a}$; whence $\sin^{-1}\dfrac{x}{a} = \frac{1}{2}\pi - 2y$. Substituting in this the value of y, and, by introducing the values of these, the integral would be expressed in terms of the quantities contained in the proposed differential.

The integral of the quantity which we have been considering, might be obtained rather more directly in the following manner. For x substitute (ALG. § 179) $z+\frac{1}{2}\cdot\frac{b}{c}$, and the numerator will become dz, and the denominator $c^{\frac{1}{2}}\sqrt{\left(\frac{b^2+4ac}{4c^2}-z^2\right)}$. Comparing this with formula E, page 41, we have $\frac{b^2+4ac}{4c^2}$ instead of a^2; and, dividing z or its equal $x-\frac{1}{2}\cdot\frac{b}{c}$ by the square root of this, we obtain $\frac{2cx-b}{\sqrt{(b^2+4ac)}}$; and, consequently,

$$\int\frac{dx}{\sqrt{(a+bx-cx^2)}}=\frac{1}{c^{\frac{1}{2}}}\sin^{-1}\frac{2cx-b}{\sqrt{(b^2+4ac)}}.$$

Since $\cos v=\sqrt{(1-\sin^2 v)}$, and $\tan v=\frac{\sin v}{\cos v}$, this integral may also be exhibited under the forms, $\int\frac{dx}{\sqrt{(a+bx-cx^2)}}$

$$=\frac{1}{c^{\frac{1}{2}}}.\cos^{-1}\frac{2c^{\frac{1}{2}}\sqrt{(a+bx-cx^2)}}{\sqrt{(b^2+4ac)}}=\frac{1}{c^{\frac{1}{2}}}\tan^{-1}\frac{2cx-b}{2c^{\frac{1}{2}}\sqrt{(a+bx-cx^2)}}:$$

and it is plain that it might be expressed by means of the cotangent, secant, &c. It might also be shown, by means of easy reductions, that these results, and those obtained in the first part of this No. are equivalent.

By applying the result here obtained to the differential in No. 258, we should have

$$\int\frac{dx}{\sqrt{(a+bx+cx^2)}}=\frac{1}{\sqrt{-c}}.\sin^{-1}\frac{2cx+b}{\sqrt{(b^2-4ac)}};$$

an expression which, if c be positive, and consequently $-c$

---

we get $\sin^{-1}\frac{x}{a}=\frac{1}{2}\pi-2\tan^{-1}\sqrt{\frac{a-x}{a+x}}$; which, except the constant quantity $\frac{1}{2}\pi$, is the same as the integral found above. In a similar manner, the differentials F and H, in page 41, might be investigated.

negative, contains, virtually, two imaginary quantities, $\sqrt{-c}$, and a sine greater than the radius. In a similar manner, by means of No. 258, we should have the integral investigated in this No. expressed by means of imaginary quantities, thus, $\int \frac{dx}{\sqrt{(a+bx-cx^2)}}$

$$= \frac{1}{\sqrt{-c}} . \log\{-2cx+b+2\sqrt{-c}.\sqrt{(a+bx-cx^2)}\}.$$

Similar instances will be found in the exercises at the end of this Section.

262. As a third example, let us take $du = \frac{dx}{x\sqrt{(a+bx+cx^2)}}$. By substituting in this for dx, x, and $\sqrt{(a+bx+cx^2)}$, their values in I. No. 256, we get $du = -\frac{2c^{\frac{1}{2}}dz}{a-cz^2}$, or $du = \frac{2}{c^{\frac{1}{2}}} . \frac{dz}{z^2-a'^2}$; $\frac{a}{c}$ being put equal to a'^2. The integral of this (G_2, page 41) is $u = -\frac{1}{a'c^{\frac{1}{2}}} . \log \frac{z+a'}{z-a'}$; and from this, by putting instead of a' its value assumed above, and instead of z its value (I. No. 256), and by multiplying the numerator and denominator by $c^{\frac{1}{2}}$, we get u or

$$\int \frac{dx}{x\sqrt{(a+bx+cx^2)}} = -\frac{1}{a^{\frac{1}{2}}} \log \frac{\sqrt{(a+bx+cx^2)} - c^{\frac{1}{2}}x + a^{\frac{1}{2}}}{\sqrt{(a+bx+cx^2)} - c^{\frac{1}{2}}x - a^{\frac{1}{2}}}.$$

By multiplying, also, the numerator and denominator by $(a+bx+cx^2)^{\frac{1}{2}} + c^{\frac{1}{2}}x + a^{\frac{1}{2}}$, and rejecting the constant quantity $\log b$, we obtain

$$\int \frac{dx}{x\sqrt{(a+bx+cx^2)}} = -\frac{1}{a^{\frac{1}{2}}} \log \frac{2a+bx+2a^{\frac{1}{2}}(a+bx+cx^2)^{\frac{1}{2}}}{x},$$

which is a preferable form of the integral.

263. If $a=a^2$, $b=0$, and $c=1$, this will become $\int \frac{dx}{x\sqrt{(x^2+a^2)}}$

$$= -\frac{1}{a} \log \frac{\sqrt{(x^2+a^2)} - x + a}{\sqrt{(x^2+a^2)} - x - a} = -\frac{1}{a} \log \frac{a+\sqrt{(x^2+a^2)}}{x}.$$

This agrees with H_2, p. 41.

264. Required the integral of $du = \frac{xdx}{\sqrt{(a+bx+cx^2)}}$. Here, by substituting, as in I. No. 256, and putting $\frac{b}{2c} = a'$, and $\frac{a}{c} = b'$, we get

$$du = \frac{1}{2c^{\frac{1}{2}}} \cdot \frac{(z^2 - b')dz}{(z-a')^2} = \frac{1}{2c^{\frac{1}{2}}} \left\{ dz + \frac{(2a'z - a'^2 - b')dz}{(z-a')^2} \right\},$$

the latter form being obtained by dividing the numerator by the denominator in the former. This, again, by No. 244, becomes

$$du = \frac{1}{2c^{\frac{1}{2}}} \left\{ dz + \frac{(a'^2 - b')dz}{(z-a')^2} + \frac{2a'dz}{z-a'} \right\};$$

the integral of which is u, or

$$\int \frac{xdx}{\sqrt{(a+bx+cx^2)}} = \frac{1}{2c^{\frac{1}{2}}} \left\{ z - \frac{a'^2 - b'}{z-a'} + 2a' \log(z-a') \right\}.$$

If, in this, the values of a', b', and z, were introduced, the integral would be obtained in terms of the quantities contained in the proposed differential.

265. As another example, let it be required to find the integral of $\frac{dx}{x\sqrt{(a-cx^2)}}$. This might be integrated by means of III. No. 256; but the integral will be obtained with more ease by putting $\sqrt{(a-cx^2)} = z$, which gives

$$x = \sqrt{\frac{a-z^2}{c}}, \quad \text{and } dx = -\frac{zdz}{c^{\frac{1}{2}}\sqrt{(a-z^2)}}.$$

Hence, by substitution, the given differential becomes $-\frac{dz}{a-z^2}$, or $-\frac{1}{2a^{\frac{1}{2}}} \cdot \frac{2a^{\frac{1}{2}}dz}{a-z^2}$; the integral of which (No. 247) is

$$\frac{1}{2a^{\frac{1}{2}}} \log \frac{a^{\frac{1}{2}}+z}{a^{\frac{1}{2}}-z}, \quad \text{or } \frac{1}{2a^{\frac{1}{2}}} \log \frac{a^{\frac{1}{2}}+\sqrt{(a-cx^2)}}{a^{\frac{1}{2}}-\sqrt{(a-cx^2)}}.$$

By multiplying the numerator and denominator of the latter fraction in this by $a^{\frac{1}{2}}+\sqrt{(a-cx^2)}$, we find, in a more convenient form $\int \frac{dx}{x\sqrt{(a-cx^2)}}$

$$= \frac{1}{2a^{\frac{1}{2}}} \log \frac{\{a^{\frac{1}{2}}+\sqrt{(a-cx^2)}\}^2}{cx^2} = \frac{1}{a^{\frac{1}{2}}} \log \frac{a^{\frac{1}{2}}+\sqrt{(a-cx^2)}}{c^{\frac{1}{2}}x};$$

which, by the addition of the constant quantity, $\frac{1}{2a^{\frac{1}{2}}}\log c$, takes the simpler form,

$$\int\frac{dx}{x\sqrt{(a-cx^2)}}=\frac{1}{a^{\frac{1}{2}}}\log\frac{a^{\frac{1}{2}}+\sqrt{(a-cx^2)}}{x}.$$

266. If a be taken equal to a^2, and c to ± 1, this becomes

$$\int\frac{dx}{x\sqrt{(a^2\pm x^2)}}=\frac{1}{a}\log\frac{a+\sqrt{(a^2\pm x^2)}}{x},$$

which is virtually the same as H_2, p. 41.

EXERCISES IN THE INTEGRATION OF IRRATIONAL DIFFERENTIALS.

1. $$\int\frac{xdx}{\sqrt{(a+bx)}}=\frac{\frac{2}{3}(a+bx)^{\frac{3}{2}}-2a(a+bx)^{\frac{1}{2}}}{b^2}.$$

2. $$\int\frac{dx}{x\sqrt{(a+bx)}}=\frac{2}{\sqrt{a}}\log\frac{\sqrt{(a+bx)}-\sqrt{a}}{\sqrt{x}}$$
$$=\frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bx}{-a}}*.$$

3. $$\int\frac{dx}{\sqrt{(a+bx^2)}}=\frac{1}{\sqrt{b}}\log\{x\sqrt{b}+\sqrt{(a+bx^2)}\}$$
$$=\frac{1}{\sqrt{-b}}\sin^{-1}x\sqrt{\frac{-b}{a}}.$$

4. $$\int\frac{dx}{\sqrt{(ax+bx^2)}}=\frac{1}{\sqrt{b}}.\log\{\tfrac{1}{2}a+bx+\sqrt{b}.\sqrt{(ax+bx^2)}\}$$
$$=\frac{2}{\sqrt{-b}}\tan^{-1}\sqrt{\frac{-bx}{a+bx}}.$$

5. $$\int\frac{dx}{(a+bx)\sqrt{x}}=\frac{2}{\sqrt{(ab)}}\tan^{-1}\sqrt{\frac{bx}{a}}$$
$$=\frac{1}{\sqrt{(-ab)}}\log\frac{a-bx+2\sqrt{x}.\sqrt{(-ab)}}{a+bx}.$$

* In this exercise and the three following, and in all similar cases, that form of the integral is to be used in practice which is expressed by real quantities. The student will find it useful to investigate integrals of both forms in other similar cases. On this subject, see No. 261.

6. $\int \frac{dx}{\sqrt{(x-x^2)}} = 2\tan^{-1}\sqrt{\frac{x}{1-x}} = 2\sin^{-1}\sqrt{x}.$

7. $\int \frac{dx}{(1+x^2)\sqrt{(1-x^2)}} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{\sqrt{(1-x^2)}}.$

8. $\int \frac{dx}{(1+x^2)\sqrt{(x^2-1)}} = \frac{1}{\sqrt{2}}\log\frac{\sqrt{(x^2-1)}+x\sqrt{2}}{\sqrt{(x^2+1)}}.$ *

9. $\int \frac{dx}{(1+x)\sqrt{(1+x+x^2)}} = \log\frac{1-x-2\sqrt{(1+x+x^2)}}{1+x}.$

10. $\int \frac{dx}{(1+x)\sqrt{(1+x-x^2)}} = \tan^{-1}\frac{1+3x}{2\sqrt{(1+x-x^2)}}.$

XV.—INTEGRATION OF BINOMIAL DIFFERENTIALS.

267. The integration of binomial differentials might have been comprehended in the last Section. Such quantities form, however, a class of irrational differentials of so distinct a nature, that they may, with propriety, be considered separately.

Differentials of this kind may all be expressed under the form, $x^m(a+bx^n)^{\frac{p}{q}}dx$, where m, n, p, and q, are whole numbers, and all either positive or negative, except n, which is only positive.†

268. Now, if $q=1$ and p be positive, the integral is found by expanding $(a+bx^n)^p$, multiplying the result by $x^m dx$, and integrating the terms separately.

269. If q have any other value, let $a+bx^n=z^q$, an assumption which gives $(a+bx^n)^{\frac{1}{q}}=z$, $(a+bx^n)^{\frac{p}{q}}=z^p$, $x=\left(\frac{z^q-a}{b}\right)^{\frac{1}{n}}$, $x^m=\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}}$, and $dx=\frac{1}{n}\left(\frac{z^q-a}{b}\right)^{\frac{1}{n}-1}\times \frac{qz^{q-1}dz}{b}$. By substituting these values of x^m, $(a+bx^n)^{\frac{p}{q}}$,

* This is easily solved by assuming $z=\frac{x}{\sqrt{(x^2-1)}}$.

† When the differentials are not originally of this form, they may be reduced to it. Thus, if m and n be not integers, let m' be their common denominator. Then, by substituting $x'^{m'}$ for x, the resulting exponents will be integers. Also, should n be negative, let x'^{-1} be substituted for x, and the exponent of x' in the radical will be positive.

and dx, in the proposed differential, we get, after some modifications,

$$\frac{q}{nb} z^{p+q-1} \left(\frac{z^q - a}{b}\right)^{\frac{m+1}{n}-1} dz \ldots\ldots\ldots\ldots\ldots\ldots (a).$$

This formula will evidently be rational, if $\frac{m+1}{n}$ be a whole number; and will, according to its form, be integrated by some of the methods already explained. Hence, *such differentials are always integrable, when $m+1$ is divisible by n.*

270. The proposed differential may also be put, if necessary, under the form,

$x^m \{x^n(ax^{-n}+b)\}^{\frac{p}{q}} dx$, or $x^m . x^{\frac{np}{q}} (ax^{-n}+b)^{\frac{p}{q}} dx$; or, finally, $x^{m+\frac{np}{q}} (ax^{-n}+b)^{\frac{p}{q}} dx$. Comparing this with the given differential, we have $m+\frac{np}{q}$ instead of m; and, as the integral may be found in the original form when $m+1$ is a multiple of n, so here, by putting $ax^{-n}+b=z^q$, it may be found, if $m+\frac{np}{q}+1$ be divisible by n, or, which is the same, if $\frac{m+1}{n}+\frac{p}{q}$ be a whole number. Hence, *a binomial differential* expressed as above, *is always integrable, not only when $\frac{m+1}{n}$ is a whole number, but also when the sum of that quantity and the exponent, $\frac{p}{q}$, is an integer;* and in no other case, at present known, can such a differential be rendered rational.

To find the form of the reduced differential in this latter case, comparing

$$x^{m+\frac{np}{q}} (ax^{-n}+b)^{\frac{p}{q}} dx \text{ with } x^m(a+bx^n)^{\frac{p}{q}} dx,$$

we have, as has been mentioned already, $m+\frac{np}{q}$ instead of m, b instead of a, a instead of b, and $-n$ instead of n. Making these changes, therefore, in (a), No 269, we get

$$-\frac{q}{na} z^{p+q-1} \left(\frac{z^q - b}{a}\right)^{-\frac{m+1}{n}-\frac{p}{q}-1} dz \ldots\ldots\ldots\ldots\ldots\ldots (b).$$

In this, it is plain that $z=(b+ax^{-n})^{\frac{1}{q}}$.

EXAMPLES OF THE INTEGRATION OF BINOMIAL DIFFERENTIALS.

271. As an example, let the function, $x^{-7}(ax^3+x^7)^{\frac{2}{3}}\,dx$, be proposed. By dividing the part of this within the vinculum by x^2, or $(x^3)^{\frac{2}{3}}$, and multiplying the part without it by the same, the differential assumes the form, $x^{-5}(a+x^4)^{\frac{2}{3}}dx$. Comparing this with the general form (No. 267), we have $a=a$, $b=1$, $m=-5$, $n=4$, $p=2$, and $q=3$; and, since $\frac{m+1}{n}=-1$, a whole number, this, according to No. 269, is integrable. Substituting these values in the transformed differential (a), No. 269, we get

$$\tfrac{3}{4}z^4(z^3-a)^{-2}dz, \text{ or } \frac{3}{4}\cdot\frac{z^4dz}{(z^3-a)^2}, \text{ where } z=(a+x^4)^{\frac{1}{3}}.$$

Now, this being rational, and the denominator being equivalent to $(z-a^{\frac{1}{3}})^2.(z^2+a^{\frac{1}{3}}z+a^{\frac{2}{3}})^2$, the integral may be found by the means already pointed out for integrating rational fractions; and we might simply refer to what has already been given on that subject. It may not be improper, however, to give an outline of the operation; especially as, in one part of it, a different mode of investigation may be employed with advantage.

Instead, therefore, of the quantity above found, we obtain, by the methods already explained (Nos. 240 and 244),

$$\frac{1}{12a^{\frac{1}{3}}}\left\{\frac{a^{\frac{1}{3}}dz}{(z-a^{\frac{1}{3}})^2}+\frac{2dz}{z-a^{\frac{1}{3}}}-\frac{3adz}{(z^2+a^{\frac{1}{3}}z+a^{\frac{2}{3}})^2}-\frac{(2z-4a^{\frac{1}{3}})dz}{z^2+a^{\frac{1}{3}}z+a^{\frac{2}{3}}}\right\}.$$

The integrals of the first and second terms within the vinculum are found (A_2 and B, page 41) to be

$$-a^{\frac{1}{3}}(z-a^{\frac{1}{3}})^{-1}, \text{ and } 2\log(z-a^{\frac{1}{3}}).$$

To integrate the third term, we may employ the method of indeterminate coefficients, by assuming

$$\int-\frac{3adz}{(z^2+a^{\frac{1}{3}}z+a^{\frac{2}{3}})^2}=\frac{Mz+N}{z^2+a^{\frac{1}{3}}z+a^{\frac{2}{3}}}+\int\frac{Pdz}{z^2+a^{\frac{1}{3}}z+a^{\frac{2}{3}}},$$

where M and N are constant quantities. To determine these and P, differentiate the equation, multiply the result by $(z^2+a^{\frac{1}{3}}z+a^{\frac{2}{3}})^2$, and reject the common factor dz; then,

$$-3a=(-M+P)z^2+(Pa^{\frac{1}{3}}-2N)z+(M+P)a^{\frac{2}{3}}-Na^{\frac{1}{3}}.$$

Hence, by equalling the corresponding coefficients, we readily find $M=P=-2a^{\frac{1}{3}}$, and $N=-a^{\frac{2}{3}}$. We have, therefore,

$$\int -\frac{3a dz}{(z^2+a^{\frac{1}{3}}z+a^{\frac{2}{3}})^2} = -\frac{2a^{\frac{1}{3}}z+a^{\frac{2}{3}}}{z^2+a^{\frac{1}{3}}z+a^{\frac{2}{3}}} + \int \frac{-2a^{\frac{1}{3}}dz}{z^2+a^{\frac{1}{3}}z+a^{\frac{2}{3}}}.$$

Connecting the differential of the last term of this with the last term within the vinculum, we obtain

$$-\frac{(2z-2a^{\frac{1}{3}})dz}{z^2+a^{\frac{1}{3}}z+a^{\frac{2}{3}}}, \text{ or } -\frac{(2z+a^{\frac{1}{3}})dz}{z^2+a^{\frac{1}{3}}z+a^{\frac{2}{3}}} + \frac{3a^{\frac{1}{3}}dz}{z^2+a^{\frac{1}{3}}z+a^{\frac{2}{3}}};$$

the integral of the first term of which is $-\log(z^2+a^{\frac{1}{3}}z+a^{\frac{2}{3}})$. The second term is integrated in the manner pointed out in No. 251, and gives $2\sqrt{3}.\tan^{-1}\frac{2z+a^{\frac{1}{3}}}{a^{\frac{1}{3}}\sqrt{3}}$. Hence, by collecting the several parts of the integral, connecting the logarithmic parts, and adding the algebraic parts, we obtain, after some other slight modifications,

$$\int\frac{(a+x^4)^{\frac{2}{3}}dx}{x^5}$$

$$=-\frac{1}{4}\cdot\frac{z^2}{z^3-a}+\frac{1}{12a^{\frac{1}{3}}}\left\{\log\frac{(z-a^{\frac{1}{3}})^2}{z^2+a^{\frac{1}{3}}z+a^{\frac{2}{3}}}+2\sqrt{3}.\tan^{-1}\frac{2z+a^{\frac{1}{3}}}{a^{\frac{1}{3}}\sqrt{3}}\right\};$$

and this would be farther modified by substituting for z its equal $(a+x^4)^{\frac{1}{3}}$.

272. As a second example, let it be required to integrate $\frac{x^9dx}{(1-2x^3)^{\frac{1}{3}}}$, or $x^9(1-2x^3)^{-\frac{1}{3}}dx$. Here we have $a=1$, $b=-2$, $m=9$, $n=3$, $p=-1$, and $q=3$; and, though $\frac{m+1}{n}=3\frac{1}{3}$ is not an integer, yet $\frac{m+1}{n}+\frac{p}{q}=3$, being one, the integral may be found by No. 270. Putting, therefore, $x^{-3}-2=z^3$, and using in formula (b), No. 270, the foregoing values of a, b, &c. we get, instead of the given differential, $-\frac{zdz}{(z^3+2)^4}$; and this being a rational fraction, its integral will be found in the manner already explained.

The integral might also be found by taking $p=1$, and $q=-3$, instead of $p=-1$, and $q=3$, as was done above. In this way we should have $z=(x^{-3}-2)^{-\frac{1}{3}}$, and the differential would become $z^{-3}(z^{-3}+2)^{-4}dz=\frac{z^9 dz}{(1+2z^3)^4}$, which may likewise be integrated on the principles already explained. This form of the differential might also be derived from the other, by substituting $\frac{1}{z}$ for z.

273. Though binomial differentials cannot be rendered rational by any known method, except in the cases we have considered, yet they may frequently be modified by means of integration by parts, so as to assume a simpler form, and to be integrated by other means; and the same method may also be applied with advantage in some cases in which the integrals might be found by the methods that have now been explained.

Thus, if the proposed differential be $x^m(a+bx^n)^p dx$, we may write it,

$$x^{m-n+1}(a+bx^n)^p x^{n-1}dx.$$

Now, since (page 41, K) $\int udv=uv-\int vdu$, if we take $u=x^{m-n+1}$, and $dv=(a+bx^n)^p x^{n-1}dx$, we get, by differentiating the former and integrating (A, p. 41) the latter, $du=(m-n+1)x^{m-n}dx$, and $v=\frac{(a+bx^n)^{p+1}}{nb(p+1)}$. Substituting these in $\int udv=uv-\int vdu$, we get $\int x^m(a+bx^n)^p dx$

$$=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)}-\frac{m-n+1}{nb(p+1)}\int x^{m-n}(a+bx^n)^{p+1}dx.$$

Now, $(a+bx^n)^{p+1}=(a+bx^n)^p(a+bx^n)=a(a+bx^n)^p+bx^n(a+bx^n)^p$.

If we multiply this by $x^{m-n}dx$, there will result

$$x^{m-n}(a+bx^n)^{p+1}dx=ax^{m-n}(a+bx^n)^p dx+bx^m(a+bx^n)^p dx.$$

By substituting this in the last term of the foregoing equation, multiplying the result by $nb(p+1)$, transposing the last term of the product, and dividing the equation thus obtained by $b(np+m+1)$, we obtain $\int x^m(a+bx^n)^p dx$

$$=\frac{x^{m-n+1}(a+bx^n)^{p+1}-a(m-n+1)\int x^{m-n}(a+bx^n)^p dx}{b(np+m+1)} \ldots(a)$$

This formula presents a new differential, in which the index of x without the vinculum is less than the given index by n, while the index of the binomial remains the same; and, by a succession of similar processes, or rather by the same formula, taking $m-n$ successively instead of m, in the differentials in the second members, we might reduce the exponent still farther.

274. We may derive another formula by which the index of the binomial will be diminished. To effect this, it would be found, as in the last article, that

$$(a+bx^n)^p=a(a+bx^n)^{p-1}+bx^n(a+bx^n)^{p-1}.$$

Multiply by $x^m dx$, and integrate; then,

$$\int x^m(a+bx^n)^p dx=a\int x^m(a+bx^n)^{p-1}dx+b\int x^{m+n}(a+bx^n)^{p-1}dx.$$

The integral of the last term of this will be found, from No. 273 (a), by using $m+n$ for m, and $p-1$ for p, to be

$$\frac{x^{m+1}(a+bx^n)^p-a(m+1)\int x^m(a+bx^n)^{p-1}dx}{np+m+1}.$$

Let this be substituted instead of that term; then, by connecting the first and last terms of the second member in the result, we obtain

$$\int x^m(a+bx^n)^p dx=\frac{x^{m+1}(a+bx^n)^p+anp\int x^m(a+bx^n)^{p-1}dx}{np+m+1} \ldots(b)$$

275. Formula (a), No. 273, serves the intended purpose, that is, to reduce the exponent without the vinculum as nearly as possible to 0, only when m is positive. If it be negative, a formula which will succeed may be derived thus: From (a) we obtain, by freeing it of fractions, and by transposition and division, $\int x^{m-n}(a+bx^n)^p dx$

$$=\frac{x^{m-n+1}(a+bx^n)^{p+1}-b(np+m+1)\int x^m(a+bx^n)^p dx}{a(m-n+1)}.$$

In this, write $-m$ instead of $m-n$, and consequently $-m+n$ instead of m; and there will result, $\int x^{-m}(a+bx^n)^p dx$

$$=\frac{x^{-m+1}(a+bx^n)^{p+1}-b(np-m+n+1)\int x^{-m+n}(a+bx^n)^p dx}{a(1-m)}\ldots(c)$$

276. In like manner, formula (b), No. 274, fails when p is negative. In this case we get, from that formula,

$\int x^m(a+bx^n)^{p-1}dx$

$$=\frac{-x^{m+1}(a+bx^n)^p+(np+m+1)\int x^m(a+bx^n)^p dx}{anp}.$$

In this, substitute $-p$ for $p-1$, and consequently $-p+1$ for p; and there will be obtained, $\int x^m(a+bx^n)^{-p}dx$

$$=\frac{-x^{m+1}(a+bx^n)^{-p+1}+(-np+m+n+1)\int x^m(a+bx^n)^{-p+1}dx}{an(1-p)}\ldots(d)$$

277. As an example of the use of these formulas, let it be required to integrate $x^3(a+bx^2)^{\frac{5}{2}}dx$.* Here we have $m=3$, $n=2$, and $p=\frac{5}{2}$; and we obtain, by formula (a),

$$\int x^3(a+bx^2)^{\frac{5}{2}}dx=\frac{x^2(a+bx^2)^{\frac{7}{2}}-2a\int x(a+bx^2)^{\frac{5}{2}}dx}{9b}.\dagger$$

Again, applying formula (b) to the last term of this, we have $m=1$, $n=2$, and $p=\frac{5}{2}$; and we find $\int x^3(a+bx^2)^{\frac{5}{2}}dx$

$$=\frac{x^2(a+bx^2)^{\frac{7}{2}}}{9b}-\frac{2a}{9b}\cdot\frac{x^2(a+bx^2)^{\frac{5}{2}}+5a\int x(a+bx^2)^{\frac{3}{2}}dx}{7}.$$

By applying again the same formula to the last term now obtained, p being $=\frac{3}{2}$, we find

$$\int x^3(a+bx^2)^{\frac{5}{2}}dx=\frac{x^2(a+bx^2)^{\frac{7}{2}}}{9b}$$

$$-\frac{2a}{7.9b}x^2(a+bx^2)^{\frac{5}{2}}-\frac{2.5a^2}{7.9b}\cdot\frac{x^2(a+bx^2)^{\frac{3}{2}}+3a\int x(a+bx^2)^{\frac{1}{2}}dx}{5};$$

* This is also integrable by No. 269.

† Were it not for exemplifying the principles just established, the integral might at once be found by A_2, p. 41, from what is here obtained.

and, by a farther application of it to the last term of this, p being $=\frac{1}{2}$, we get $\int x^3(a+bx^2)^{\frac{5}{2}}dx$

$$=\frac{x^2(a+bx^2)^{\frac{7}{2}}}{9b}-\frac{2a}{7.9b}x^2(a+bx^2)^{\frac{5}{2}}-\frac{2.5a^2}{5.7.9b}x^2(a+bx^2)^{\frac{3}{2}}$$

$$-\frac{2.3.5a^3}{5.7.9b}\cdot\frac{x^2(a+bx^2)^{\frac{1}{2}}+a\int x(a+bx^2)^{-\frac{1}{2}}dx}{3}=$$

$$\frac{x^2(a+bx^2)^{\frac{1}{2}}}{9b}\left\{(a+bx^2)^3-\frac{2a(a+bx^2)^2+2a^2(a+bx^2)+2a^3}{7}\right\}$$

$$-\frac{2a^4(a+bx^2)^{\frac{3}{2}}}{7.9b^2};$$

the integral of the last term being found by A_2, p. 41.

278. As another example, let it be required to integrate $x^2(a+bx^2)^{\frac{5}{2}}dx$, which differs from the foregoing only in an index. Here, by formula (a), we get

$$\int x^2(a+bx^2)^{\frac{5}{2}}dx=\frac{x(a+bx^2)^{\frac{7}{2}}-a\int(a+bx^2)^{\frac{5}{2}}dx}{8b};$$

and, by successive applications of formula (b), we obtain

$$\int x^2(a+bx^2)^{\frac{5}{2}}dx=\frac{x(a+bx^2)^{\frac{7}{2}}}{8b}-\frac{ax(a+bx^2)^{\frac{5}{2}}}{6.8b}$$

$$-\frac{5a^2x(a+bx^2)^{\frac{3}{2}}}{4.6.8b}-\frac{3.5a^3x(a+bx^2)^{\frac{1}{2}}}{2.4.6.8b}-\frac{3.5a^4}{2.4.6.8b}\int\frac{dx}{(a+bx^2)^{\frac{1}{2}}};$$

the integral of the last term of which will be found by means of No. 256.

279. The following particular cases of the formulas marked (a) and (c) in Nos. 273 and 275, in which m is supposed to be a whole positive number, are often useful.

1. $$\int\frac{x^m dx}{\sqrt{(a+bx^2)}}=\frac{x^{m-1}\sqrt{(a+bx^2)}}{mb}-\frac{m-1}{m}\cdot\frac{a}{b}\int\frac{x^{m-2}dx}{\sqrt{(a+bx^2)}}.$$

2. $$\int\frac{x^m dx}{\sqrt{(a-bx^2)}}=-\frac{x^{m-1}\sqrt{(a-bx^2)}}{mb}+\frac{m-1}{m}\cdot\frac{a}{b}\int\frac{x^{m-2}dx}{\sqrt{(a-bx^2)}}.$$

3. $$\int\frac{dx}{x^m\sqrt{(a+bx^2)}}=-\frac{\sqrt{(a+bx^2)}}{(m-1)ax^{m-1}}-\frac{m-2}{m-1}\cdot\frac{b}{a}\int\frac{dx}{x^{m-2}\sqrt{(a+bx^2)}}.$$

4. $$\int\frac{dx}{x^m(a-bx^2)}=-\frac{\sqrt{(a-bx^2)}}{(m-1ax^{m-1}}+\frac{m-2}{m-1}\cdot\frac{b}{a}\int\frac{dx}{x^{m-2}\sqrt{(a-bx^2)}}.$$

By the application of these as often as may be necessary, the index of x in the rational part will be reduced to nothing or unity, and the integration will be finally effected by means of the following elementary formulas:

$$\int \frac{dx}{\sqrt{(a+bx^2)}} = \frac{1}{\sqrt{b}} \log \{x\sqrt{b} + \sqrt{(a+bx^2)}\}. \quad \text{No. 258.}$$

$$\int \frac{xdx}{\sqrt{(a \pm bx^2)}} = \pm \frac{\sqrt{(a \pm bx^2)}}{b}. \quad \text{A, page 41.}$$

$$\int \frac{dx}{\sqrt{(a-bx^2)}} = \frac{1}{\sqrt{b}} \sin^{-1} x \sqrt{\frac{b}{a}}. \quad \text{E, p. 41.}$$

$$\int \frac{dx}{x\sqrt{(a \pm bx^2)}} = \frac{1}{\sqrt{a}} \log \frac{\sqrt{(a \pm bx^2)} - \sqrt{a}}{x}. \quad \text{H}_2\text{, page 41.}$$

$$\int \frac{dx}{x\sqrt{(-a+bx^2)}} = \frac{1}{\sqrt{a}} \sec^{-1} x \sqrt{\frac{b}{a}} = \frac{1}{\sqrt{a}} \cos^{-1} \frac{\sqrt{a}}{x\sqrt{b}}. \quad \text{H. p. 41.}$$

ADDITIONAL EXERCISES IN THE INTEGRATION OF IRRATIONAL FUNCTIONS.

1. $\displaystyle\int \frac{x^3dx}{\sqrt{(1-x^2)}} = -(\tfrac{1}{3}x^2 + \tfrac{2}{3})\sqrt{(1-x^2)}.$

2. $\displaystyle\int \frac{x^5dx}{\sqrt{(1-x^2)}} = -(\tfrac{1}{5}x^4 + \tfrac{4}{15}x^2 + \tfrac{8}{15})\sqrt{(1-x^2)}.$

3. $\displaystyle\int \frac{x^2dx}{\sqrt{(1-x^2)}} = -\tfrac{1}{2}x\sqrt{(1-x^2)} + \tfrac{1}{2}\sin^{-1}x.$

4. $\displaystyle\int \frac{x^4dx}{\sqrt{(1-x^2)}} = -(\tfrac{1}{4}x^3 + \tfrac{3}{8}x)\sqrt{(1-x^2)} + \tfrac{3}{8}\sin^{-1}x.$

5. $\displaystyle\int \frac{dx}{x^3\sqrt{(1-x^2)}} = -\frac{\sqrt{(1-x^2)}}{2x^2} - \tfrac{1}{2}\log\frac{\sqrt{(1-x^2)}+1}{x}.$

6. $\displaystyle\int \frac{dx}{x^5\sqrt{(1-x^2)}} = -\frac{\sqrt{(1-x^2)}}{4x^4} - \frac{3\sqrt{(1-x^2)}}{8x^2} - \tfrac{3}{8}\log\frac{\sqrt{(1-x^2)}+1}{x}.$

7. $\int \frac{dx}{x^2\sqrt{(1-x^2)}} = -\frac{1}{x}\sqrt{(1-x^2)}$.

8. $\int \frac{dx}{x^4\sqrt{(1-x^2)}} = -\left(\frac{1}{3x^3}+\frac{2}{3x}\right)\sqrt{(1-x^2)}$.

9. $\int \frac{dx}{x^6\sqrt{(1-x^2)}} = -\left(\frac{1}{5x^5}+\frac{4}{15x^3}+\frac{8}{15x}\right)\sqrt{(1-x^2)}$.

10. $\int \frac{dx}{\sqrt{(1-x-x^2)}} = \tan^{-1}\frac{2x+1}{2\sqrt{(1-x-x^2)}} = \sin^{-1}\frac{2x+1}{\sqrt{5}}$.

11. $\int \frac{xdx}{(a+bx)^{\frac{9}{2}}} = -\left(\frac{a+bx}{5}-\frac{a}{7}\right)\times\frac{2}{b^2(a+bx)^{\frac{7}{2}}}$

12. $\int \frac{dx}{(1+x^2)^{\frac{7}{2}}} = \left(\frac{8}{15}x^5+\frac{4}{3}x^3+x\right)\frac{1}{(1+x^2)^{\frac{5}{2}}}$.

13. $\int \frac{dx\sqrt{x}}{1-x} = -2\sqrt{x}+\log\frac{(1+\sqrt{x})^2}{1-x}$.

XV.—INTEGRATION OF TRANSCENDENTAL FUNCTIONS.

280. In Section IV. for the purpose of illustrating the method of integration by parts, a few examples and exercises were given, in which the differential coefficients were logarithmic, exponential, or circular functions. We may now consider differentials with such coefficients somewhat more particularly than it would have been proper to do at that part of the work; and we may commence with those which have logarithmic and exponential coefficients.

281. Let the integral of $x^m \log x dx$ be required. To obtain this, we may assume $u=\log x$, and $dv=x^m dx$, which give

$du=\frac{dx}{x}$, and $v=\frac{1}{m+1}x^{m+1}$; and hence (page 41, K) we obtain $\int x^m \log x dx$

$$=\frac{1}{m+1}x^{m+1}\log x-\frac{1}{(m+1)^2}x^{m+1}=\frac{x^{m+1}}{m+1}\left(\log x-\frac{1}{m+1}\right).$$

282. If it be required to integrate $x^m(\log x)^2dx$, we may put $(\log x)^2=u$, and $x^m dx=dv$, and we get (page 41, K)

$$\int x^m(\log x)^2dx=\frac{x^{m+1}(\log x)^2}{m+1}-\frac{2}{m+1}\int x^m\log x dx,$$

the last term of which will be integrated as in the preceding No.

283. In like manner, to integrate the general expression, $x^m(\log x)^n dx$, by putting $(\log x)^n=u$, and $x^m dx=dv$, we should find

$$\int x^m(\log x)^n dx=\frac{x^{m+1}(\log x)^n}{m+1}-\frac{n}{m+1}\int x^m(\log x)^{n-1}dx.$$

This formula diminishes the index of $\log x$ by unity; and, by its successive application to the last term here found, to the last term of the result so obtained, &c. the exact integral will at length be found, by means of No. 281, if n be a whole positive number. The same principles would also be applicable in integrating $X(\log x)^n dx$, X denoting any rational function of x.

The formula found above fails when $m=-1$. The integration, however, is easily effected, and the integral is exhibited in Ex. 12, page 44.

284. The formula found in the last No. fails, also, when n is negative; as, by its application, the index of $\log x$ becomes continually a greater negative number, instead of approaching zero. In this case, we may put the proposed differential under the form,

$$x^{m+1}(\log x)^{-n}\frac{dx}{x}, \text{ or } x^{m+1}(\log x)^{-n}d\log x.$$

Then, assuming $u=x^{m+1}$, and $dv=(\log x)^{-n}d\log x$, we get (p. 41, K) $\int x^m(\log x)^{-n}dx$, or

$$\int\frac{x^m dx}{(\log x)^n}=-\frac{x^{m+1}}{(n-1)(\log x)^{n-1}}+\frac{m+1}{n-1}\int\frac{x^m dx}{(\log x)^{n-1}}.$$

If n be a whole number, the successive application of this formula will at length give, for the term to be integrated, $\frac{x^m dx}{\log x}$. The form of this may be simplified by putting

$x^{m+1}=z$, which gives $x^m dx=\frac{dz}{m+1}$, $\log x=\frac{\log z}{m+1}$, and, consequently, $\frac{x^m dx}{\log x}=\frac{dz}{\log z}$; a formula which has never been integrated except by series. The mode of integrating it will be given in No. 288.

285. By using, in No. 76, a^x instead of ε^x, we should find, by a like process,

$$\int x a^x dx=\frac{a^x}{\log a}\left(x-\frac{1}{\log a}\right).$$

We may now integrate the general expression, $x^m a^x dx$. To effect this, put $u=x^m$, and $dv=a^x dx$; and (page 41, K) there will be obtained

$$\int x^m a^x\, dx=\frac{x^m a^x}{\log a}-\frac{m}{\log a}\int x^{m-1} a^x\, dx.$$

By the successive application of this formula, the index of x, if positive, will be continually diminished; and, if it be also a whole positive number, the integral will be exactly determined. If it be a fraction, the integral cannot be found without series.

286. If m be negative, the formula last found fails. In this case, put $u=a^x$, and $dv=x^{-m}dx$, and there will result, by page 41, K,

$$\int\frac{a^x dx}{x^m}=-\frac{a^x}{(m-1)x^{m-1}}+\frac{\log a}{m-1}\int\frac{a^x dx}{x^{m-1}}.$$

287. This formula also fails when $m=1$, as the denominator, becomes nothing. The integral in this case has never been determined, except by substituting for a^x its equal according to No. 93, and integrating the terms separately. Multiplying the expression there obtained by dx, dividing the product by x, and integrating, we find

$$\int\frac{a^x dx}{x}=\log x+\log a\,.\,x+\tfrac{1}{2}(\log a)^2.\frac{x^2}{1.2}+\tfrac{1}{3}(\log a)^3.\frac{x^3}{1.2.3}+\&c.$$

288. The expression $\frac{dx}{\log x}$, mentioned in No. 284, may be reduced to this form by putting $\log x=z$, or (No. 26)

$x = \varepsilon^z$; the differential of which (No. 29) is $dx = \varepsilon^z dz$. These values give $\frac{dx}{\log x} = \frac{\varepsilon^z dz}{z}$; the integral of which, by the last No. and by restoring x, we find to be

$$\int \frac{dx}{\log x} = \log^2 x + \log x + \frac{1}{2} \cdot \frac{(\log x)^2}{1.2} + \frac{1}{3} \cdot \frac{(\log x)^3}{1.2.3} + \text{\&c.};$$

where $\log^2 x$ denotes the logarithm of $\log x$.

EXERCISES IN THE INTEGRATION OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS.

1. $\int x^5 (\log x)^3 dx = \frac{x^6}{6} \left\{ (\log x)^3 - \tfrac{1}{2} (\log x)^2 + \tfrac{1}{6} \log x - \tfrac{1}{36} \right\}.$

2. $\int \frac{\log x dx}{(1-x)^2} = \frac{x \log x}{1-x} + \log(1-x).$

3. $\int \frac{x^4 dx}{(\log x)^3} = -\frac{x^5}{2(\log x)^2} - \frac{5x^5}{2 \log x} + \frac{25}{2} \int \frac{x^4 dx}{\log x}.$*

4. $\int x^3 a^x dx = a^x \left\{ \frac{x^3}{\log a} - \frac{3x^2}{(\log a)^2} + \frac{6x}{(\log a)^3} - \frac{6}{(\log a)^4} \right\}.$

5. $\int \frac{a^x dx}{x^3} = -\frac{a^x}{2x^2} - \frac{a^x \log a}{2x} + \frac{(\log a)^2}{2} \times \int \frac{a^x dx}{x}.$†

6. $\int \varepsilon^x (1+x) dx = x\varepsilon^x$, or

$$= \varepsilon^x \left(1 + \frac{x^2}{1.2} - \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} - \text{\&c.}\right).‡$$

289. When the differential contains a variable arc, it may generally be changed by integration by parts, so that the arc shall not be contained in the part remaining to be

* The integral of this term may be found by Nos. 284 and 288.

† For the integral of this, see No. 287.

‡ The first of these results is obtained by connecting dx with ϵ^x in the operation; while the latter is derived by taking dx along with x, x^2, &c. and integrating by parts. The series within the vinculum is equivalent to $\epsilon^{-x} + x$, as appears from No. 93, by taking x negative. Hence the latter form of the integral is the same as $\epsilon^x (\epsilon^{-x} + x)$, or $1 + x\epsilon^x$, differing from the other by the constant quantity 1.

integrated. Thus, if the differential were $x^2\sin^{-1}xdx$, by putting $\sin^{-1}x=u$, and $dv=x^2dx$, we should obtain

$$\int x^2\sin^{-1}xdx=\tfrac{1}{3}x^3\sin^{-1}x-\frac{1}{3}\int\frac{x^3dx}{\sqrt{(1-x^2)}};$$

the last term of which is integrable by methods already established. Had there been a power of $\sin^{-1}x$, the integration by parts must have been repeated.

290. We may confine our attention, therefore, to those differentials which contain only trigonometrical functions of the variable in their coefficients; and of these it is sufficient to consider those which contain $\sin x$ and $\cos x$, as all the rest may be reduced to these; $\tan x$ being equal to $\frac{\sin x}{\cos x}$, $\sec x$ to $\frac{1}{\cos x}$, &c.

291. If the proposed differential be of the form, $\cos^m xdx$ or $\sin^m xdx$, m being a whole positive number, $\cos^m x$ or $\sin^m x$ may be reduced (TRIG. formula 234, 237, or 238) to a series of terms which, without the coefficients, will be of the forms, $\cos mx$, $\cos(m-2)x$, $\cos(m-4)x$, &c. or $\sin mx$, $\sin(m-2)x$, &c. Then, multiplying by dx, we obtain the integral by formula D or D_2, page 41.

292. To exemplify this, let $\int\sin^4 xdx$ be required. By one of the trigonometrical formulas referred to, we get $\sin^4 x=\frac{1}{8}\cos 4x-\frac{1}{2}\cos 2x+\frac{3}{8}$; then, multiplying by dx and integrating (by D_2, page 41), we find

$$\int\sin^4 xdx=\tfrac{1}{32}\sin 4x-\tfrac{1}{4}\sin 2x+\tfrac{3}{8}x.$$

293. The general formula, $\sin^m x\cos^n xdx$, may now be considered. This, when m and n are whole positive numbers, might be integrated by expanding $\sin^m x$ and $\cos^n x$ in the manner already explained, taking the product of the results, and reducing the terms of that product by TRIGONOMETRY, No. 24; and, lastly, by multiplying the terms of the result by dx, and integrating them separately.

294. The same may also be effected, in the following manner, by integration by parts. The proposed differential may be put under the form, $\sin^{m-1}x\cos^n x\sin xdx$; and, if we assume

$$\sin^{m-1}x=u,\quad\text{and }\cos^n x\sin xdx=dv,$$

which give, by differentiation and integration,

$$(m-1)\sin^{m-2}x\cos x dx = du, \quad \text{and} -\frac{\cos^{n+1}x}{n+1} = v,$$

we shall have (K, page 41), $\int \sin^m x \cos^n x dx$

$$= -\frac{\sin^{m-1}x\cos^{n+1}x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2}x\cos^{n+2}x dx.$$

Here, though in the part still to be integrated the index of one quantity is diminished, that of the other is as much increased. To obviate this, substitute for $\cos^{n+2}x$, its equal, $\cos^n x(1-\sin^2 x)$; then,

$$\int \sin^m x \cos^n x dx = -\frac{\sin^{m-1}x\cos^{n+1}x}{n+1}$$

$$+\frac{m-1}{n+1}\int \sin^{m-2}x\cos^n x dx - \frac{m-1}{n+1}\int \sin^m x \cos^n x dx.$$

By transposing the last term, and incorporating it with the left-hand member by addition, the coefficient of the integral will be found to be $\frac{m+n}{n+1}$. Divide, then, both members by this, and there will be obtained, $\int \sin^m x \cos^n x dx$

$$= -\frac{\sin^{m-1}x\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}x\cos^n x dx.$$

295. Putting $\sin^m x\cos^n x dx$ under the form, $\sin^m x\cos^{n-1}x \cos x dx$, assuming $\cos^{n-1}x = u$ and $\sin^m x \cos x dx = dv$, and proceeding as before, we should find $\int \sin^m x \cos^n x dx$

$$= \frac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x dx.$$

By means of the former expression, the index of $\sin x$ is diminished; by this, that of $\cos x$ is lessened; and by their successive or alternate application, the indices of both these terms may be reduced to 0 or 1; m and n being thus far considered to be whole positive numbers.

296. To investigate formulas to serve when m or n is negative, in the formula obtained at the end of No. 294, transpose the left-hand member to the right; transpose, also, the last term of the second member to the left, and

divide by $-\frac{m-1}{m+n}$; then, instead of m, take $-m+2$, and, after the quantities with negative indices are taken to the denominator, there results

$$\int\frac{\cos^n x dx}{\sin^m x}=-\frac{\cos^{n+1}x}{(m-1)\sin^{m-1}x}+\frac{m-n-2}{m-1}\int\frac{\cos^n x dx}{\sin^{m-2}x}.$$

By a similar process, and by putting $n-2=-n$, we should obtain (from No. 295)

$$\int\frac{\sin^m x dx}{\cos^n x}=\frac{\sin^{m+1}x}{(n-1)\cos^{n-1}x}-\frac{m-n+2}{n-1}\int\frac{\sin^m x dx}{\cos^{n-2}x}.*$$

297. The limits of the present work will not permit the full consideration of these four formulas. Into a few particulars, however, it may be proper to examine. If $n=0$ in No. 294, and $m=0$ in No. 295, we get

$$\int\sin^m x dx=-\frac{\sin^{m-1}x\cos x}{m}+\frac{m-1}{m}\int\sin^{m-2}x dx,$$

and

$$\int\cos^n x dx=\frac{\cos^{n-1}x\sin x}{n}+\frac{n-1}{n}\int\cos^{n-2}x dx.$$

These formulas afford another method of effecting what is pointed out in No. 291.

298. In a similar manner, from the formulas in No. 296, we get

$$\int\frac{dx}{\sin^m x}=-\frac{\cos x}{(m-1)\sin^{m-1}x}+\frac{m-2}{m-1}\int\frac{dx}{\sin^{m-2}x},$$

---

* The expression, $\sin^m x\cos^n x dx$, may at once be reduced to a binomial differential, and may be integrated by the methods delivered in Section XV. by putting $\sin x=z$; as this gives $\cos x=(1-z^2)^{\frac{1}{2}}$, and $dx=\frac{dz}{(1-z^2)^{\frac{1}{2}}}$, and the required integral becomes

$$\int z^m(1-z^2)^{\frac{n}{2}}\cdot\frac{dz}{(1-z^2)^{\frac{1}{2}}}=\int z^m(1-z^2)^{\frac{n-1}{2}}dx.$$

It might also be made to assume an exponential form, by substituting $\frac{1}{2}(\epsilon^{x\sqrt{-1}}+\epsilon^{-x\sqrt{-1}})$ for $\cos x$, and $\frac{1}{2\sqrt{-1}}(\epsilon^{x\sqrt{-1}}-\epsilon^{-x\sqrt{-1}})$ for $\sin x$, according to No. 100.

and

$$\int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}.$$

These formulas reduce the integration of the proposed formula finally to that of dx, if m and n be even; or to that of $\frac{dx}{\sin x}$ or $\frac{dx}{\cos x}$, if they be odd.

299. To integrate the two formulas at the end of the last No. we have

$$\int \frac{dx}{\sin x} = \int \frac{\sin x dx}{\sin^2 x} = -\int \frac{d\cos x}{\sin^2 x} = -\int \frac{d\cos x}{1-\cos^2 x}.$$

Put $\cos x = z$, and this will be transformed into

$$\int \frac{dx}{\sin x} = \int \frac{-dz}{1-z^2}.$$

Integrating this by No. 247, and restoring the value of z, we get

$$\int \frac{dx}{\sin x} = \log \sqrt{\frac{1-\cos x}{1+\cos x}};$$

which (TRIG. formulas 31 and 32) is the same as

$$\int \frac{dx}{\sin x} = \log \tan \tfrac{1}{2} x.$$

By a similar process, we should find

$$\int \frac{dx}{\cos x} = \log \cot \tfrac{1}{2} (\tfrac{1}{2}\pi - x) = \log \tan \tfrac{1}{2} (\tfrac{1}{2}\pi + x).*$$

---

* Otherwise $\frac{dx}{\sin x} = \frac{d(\frac{1}{2}x)}{\sin \frac{1}{2}x \cos \frac{1}{2}x}$. By dividing the numerator and denominator of the second member by $\cos^2 \frac{1}{2}x$, we get for numerator (No. 35) $d \tan \frac{1}{2}x$ and for denominator $\tan \frac{1}{2}x$; and therefore (page 41, B) the integral is $\log \tan \frac{1}{2}x$. In like manner we should have

$$\frac{dx}{\cos x} = \frac{dx}{\sin (\frac{1}{2}\pi - x)} = \frac{d(\frac{1}{2}x)}{\sin (\frac{1}{4}\pi - \frac{1}{2}x) \cos (\frac{1}{4}\pi - \frac{1}{2}x)}.$$

which may be integrated in a similar way.

EXERCISES IN THE INTEGRATION OF TRIGONOMETRICAL FUNCTIONS.

1. $\int \tan x dx = \int \frac{\sin x dx}{\cos x} = \int - \frac{d \cos x}{\cos x} = -\log \cos$

2. $\int \cot x dx = \log \sin x.$

3. $\int \frac{dx}{\sin x \cos x} = \int \frac{2dx}{\sin 2x} = \log \tan x.$

4. $\int \frac{dx}{\sin^2 x} = -\cot x.$

5. $\int \frac{dx}{\cos^2 x} = \tan x.$

6. $\int \sin^m x \cos x dx = \int \sin^m x d \sin x = \frac{\sin^{m+1} x}{m+1}.$*

7. $\int \sin^3 x dx = -\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x = -\frac{1}{3} \cos x (\sin^2 x + 2).$

8. $\int \frac{dx}{\sin^3 x} = -\frac{\cos x}{2 \sin^2 x} + \frac{1}{2} \log \tan \frac{1}{2} x.$

9. $\int \sin^3 x \cos^2 x dx = (\frac{1}{5} \sin^4 x - \frac{1}{15} \sin^2 x - \frac{2}{15}) \cos x.$

10. $\int \frac{\sin^3 x dx}{\cos^2 x} = \cos x + \sec x.$

11. $\int \frac{dx}{\sin^3 x \cos^2 x} = \frac{1}{\sin^2 x \cos x} - \frac{3 \cos x}{2 \sin^2 x} + \frac{3}{2} \log \tan \frac{1}{2} x.$

12. $\int x \sin x dx = -x \cos x + \sin x.$

XVII.—INTEGRATION OF FUNCTIONS OF TWO OR MORE VARIABLES.

300. When the terms of differentials are of the forms, $fxdx$, $fydy$, &c. the integrals are found in the modes already explained. If, however, independent variables and their differentials be mingled together, so that there may be

* This fails when $m = -1$; but it then becomes the same as Ex. 2.

terms of the forms, $fydx, fxdy, f(x,y)\,dx$, &c. those methods fail. We may now, therefore, proceed to investigate the more useful of the principles on which such quantities are integrated: and, first, we may consider those which have exact integrals. The method of ascertaining when they are of this kind may be investigated in the following manner.

Let $du = \text{M}dx + \text{N}dy$, where M is a function of y, or of x and y; and N a function of x, or of x and y. Then, if this be the differential of some actual function of the two variables x and y, it follows, from No. 211, that $\text{M} = \frac{du}{dx}$, and $\text{N} = \frac{du}{dy}$; the first of these being the differential coefficient obtained by considering y constant and x variable, and the second that which results from considering y variable and x constant. Now, since (No. 214) $\frac{d^2u}{dxdy} = \frac{d^2u}{dydx}$, it follows, that $\frac{d\text{M}}{dy} = \frac{d\text{N}}{dx}$; which is the *criterion of integrability*, and which may be thus expressed:

If there be a differential function, $\text{M}dx + \text{N}dy$, *in which* M *is a function either of* y, *or of* x *and* y, *and* N *a function either of* x, *or of* x *and* y; *and, if* M *and* N *be differentiated, the first, on the supposition that* y *is variable and* x *constant, and the second, on the supposition that* x *is variable and* y *constant: then, if* $\frac{d\text{M}}{dy}$ *be equal to* $\frac{d\text{N}}{dx}$, $\text{M}dx + \text{N}dy$ *has an exact integral.*

301. Now, it is evident, from what precedes, that

$$u = \int \text{M}dx + \text{Y}, \qquad \text{and } u = \int \text{N}dy + \text{X};$$

the first of these being integrated on the supposition that y, and the second, that x is constant; and, since the constant quantity annexed in the first integral may contain y, it is denoted by Y, while X is used for a similar reason in the second.

302. To find the value of Y, it is plain, that, by differentiating $u = \int \text{M}dx + \text{Y}$ with respect to y, we get $\frac{d\int \text{M}dx}{dy} + \frac{d\text{Y}}{dy} = \text{N}$; whence,

$$dY = Ndy - \frac{d\int Mdx}{dy}.dy, \quad \text{and } Y = \int\left(N - \frac{d\int Mdx}{dy}\right)dy.$$

Hence, by substituting this in the first form of the integral found above, we obtain

$$u = \int Mdx + \int\left(N - \frac{d\int Mdx}{dy}\right)dy.$$

which is the form of the integral to be employed in practice.*

303. To exemplify the use of these principles, let it be required to integrate the equation, $du = \frac{ydx - xdy}{x^2 + y^2}$. Here we have $M = \frac{y}{x^2 + y^2}$ and $N = -\frac{x}{x^2 + y^2}$, by differentiating which, we get $\frac{dM}{dy}$ and $\frac{dN}{dx}$ each equal to $\frac{x^2 - y^2}{(x^2 + y^2)^2}$; and hence (No. 300) the proposed equation is integrable. Multiplying, therefore, $M = \frac{y}{x^2 + y^2}$ by dx, and integrating on the supposition that y is constant, we get $\int Mdx = \tan^{-1}\frac{x}{y}$. By differentiating this on the supposition that y is variable and x constant, we obtain $\frac{d\int Mdx}{dy} = -\frac{x}{x^2 + y^2}$; and hence, from the formula in No. 302, we get, finally, $u = \tan^{-1}\frac{x}{y} + C$, the integral required.†

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* Another form, which would be found in a similar manner, and which is equally adapted for use, is

$$u = \int N\,dy + \int\left(M - \frac{d\int N\,dy}{dx}\right)dx.$$

† The integral of this equation, which occurs at the beginning of the "Mécanique Céleste" of Laplace, may be easily found by dividing the numerator and denominator of the proposed differential by y^2. By this means, the numerator becomes $\frac{ydx - xdy}{y^2}$, or $d\left(\frac{x}{y}\right)$, and the denominator becomes $1 + \frac{x^2}{y^2}$. The integral, therefore, is (page 41, G) $\tan^{-1}\frac{x}{y} + C$.

304. Let us now integrate $du=\frac{xdx+2ydx+ydy}{(x+y)^2}$. In this example, we have

$$M=\frac{x+2y}{(x+y)^2}=\frac{1}{x+y}+\frac{y}{(x+y)^2}, \quad \text{and } N=\frac{y}{(x+y)^2};$$

and hence we find $\frac{dM}{dy}$ and $\frac{dN}{dx}$ each equal to $-\frac{2y}{(x+y)^3}$, so that the criterion of integrability, or *equation of condition* established in No. 300, is satisfied. We next find, taking y constant,

$$\int Mdx=\log(x+y)-\frac{y}{x+y};$$

and, differentiating this with respect to y, we get

$$\frac{d\int Mdx}{dy}=\frac{1}{x+y}-\frac{x}{(x+y)^2}.$$

Hence,

$$N-\frac{d\int Mdx}{dy}=\frac{x+y}{(x+y)^2}-\frac{1}{x+y}=\frac{1}{x+y}-\frac{1}{x+y}=0.$$

We have simply, therefore (by No. 302), $u=\int Mdx=\log(x+y)-\frac{y}{x+y}+C$, the integral required. This might also be expressed under the form, $u=\log(x+y)+\frac{x}{x+y}+C$, which exceeds the foregoing result by unity.

305. As another example, let us take $du=\frac{ydx-xdy}{y\sqrt{(x^2-y^2)}}$. Here

$$M=\frac{1}{\sqrt{(x^2-y^2)}}, \quad \text{and } N=\frac{-x}{y\sqrt{(x^2-y^2)}}.$$

By differentiating these, we find $\frac{dM}{dy}$ and $\frac{dN}{dx}$ each equal to $\frac{y}{(x^2-y^2)^{\frac{3}{2}}}$. We have, also,

$$Mdx=\frac{dx}{\sqrt{(x^2-y^2)}}, \text{and (p. 41, } E_2) \int Mdx=\log\{x+\sqrt{(x^2-y^2)}\};$$

$$\frac{d\int \mathrm{M}dx}{dy}=-\frac{y}{\{x+\sqrt{(x^2-y^2)}\}\sqrt{(x^2-y^2)}}=-\frac{x-\sqrt{(x^2-y^2)}}{y\sqrt{(x^2-y^2)}};$$

$$\mathrm{N}-\frac{d\int \mathrm{M}dx}{dy}=-\frac{1}{y};$$

and $u=\log\{x+\sqrt{(x^2-y^2)}\}-\log y=\log\frac{x+\sqrt{(x^2-y^2)}}{y}+\mathrm{C}.$

306. If $ydx+xdy=0$ were proposed, we should have $\mathrm{M}=y$, $\mathrm{N}=x$,

$$\frac{d\mathrm{M}}{dy}=1=\frac{d\mathrm{N}}{dx},\ \int \mathrm{M}dx=xy,\ \ \frac{d\int \mathrm{M}dx}{dy}=x,\ \mathrm{N}-\frac{d\int \mathrm{M}dx}{dy}=0;$$

and, therefore, we have simply, for the required integral, $xy=\mathrm{C}$, agreeing with Nos. 11 and 8.

307. The equation in the last No. may be very easily integrated in the following manner. Divide it by xy, and there is obtained $\frac{dx}{x}+\frac{dy}{y}=0$; the integral of which is $\log+x\log y=\log \mathrm{C}$, the constant quantity being put under the form, $\log \mathrm{C}$. Now, this is the same as $\log xy=\log \mathrm{C}$; and, therefore, $xy=\mathrm{C}$, as before.

308. The method employed in the last No.—that of *separating the variables*,—may often be used with advantage, both in cases in which the principles established in Nos. 300, 301, and 302, are applicable, and in others. This method may always be employed when the coefficients M and N are homogeneous, that is, are of the same dimensions with respect to the variables.* In this case, we simply substitute xz for y; and, consequently, $zdx+xdz$ for dy.

309. To exemplify this, let us take the equation, $xdx+ydx=ydy$, which does not possess the criterion of integrability, pointed out in No. 300. By substituting in this xz for y, and dividing by x, we obtain, after transposition, $(1+z-z^2)dx-xzdz=0$; which, by division by x and $1+z-z^2$, gives $\frac{dx}{x}-\frac{zdz}{1+z-z^2}=0$. The integral of

* Equations that are not homogeneous, may sometimes be transformed in such a manner as to become so. This may, in several instances, be effected by substituting $x'+a$ for x, and $y'+b$ for y; and then assigning such values to a and b as may cause the terms to vanish which are not homogeneous with the rest.

the first term of this is $\log x$. That of the second term is found by means of Section XIII. and the final result is

$$\log x + \tfrac{1}{2}\log(1+z-z^2) - \tfrac{1}{10}\sqrt{5}.\log\frac{2z-1+\sqrt{5}}{2z-1-\sqrt{5}} = C;$$

which, by putting $\frac{y}{x}$ for z, and by some reductions, becomes

$$\tfrac{1}{2}\log(x^2+xy-y^2) - \tfrac{1}{10}\sqrt{5}.\log\frac{2y+x(\sqrt{5}-1)}{2y-x(\sqrt{5}+1)} = C.$$

311. If the result last obtained be differentiated, there is found $\frac{xdx+ydx-ydy}{x^2+xy-y^2}=0$, which, by multiplication by the denominator, becomes the same as the proposed differential; and such a multiplication is necessary in every case in which the criterion of integrability does not hold.* There are, however, innumerable quantities, by any of which if such a differential be multiplied, the product will possess the criterion, and will accordingly be an exact differential. If we suppose z, a function of x and y, or of either, to be such a multiplier in reference to the differential equation, $\mathrm{M}dx + \mathrm{N}dy = du$, we shall have (No. 300)

$$\frac{d(z\mathrm{M})}{dy} - \frac{d(z\mathrm{N})}{dx} = 0;$$

$$\text{that is, } \frac{\mathrm{M}dz + zd\mathrm{M}}{dy} - \frac{\mathrm{N}dz + zd\mathrm{N}}{dx} = 0;$$

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* What is here pointed out arises from the circumstance, that some constant quantity contained in the primitive function has been eliminated by the introduction of a variable one instead of it. The same effect may also be produced without elimination, by the multiplication of the differential by a variable quantity, such as for the purpose of clearing it of fractions. The following example illustrates both these principles. Let $y\log x + \mathrm{C}x = 0$; then, by differentiation, we get $\frac{ydx}{x} + \log xdy + \mathrm{C}dx = 0$, an expression which possesses the criterion of integrability; while, if we either multiply by x, or substitute for C its value, $-\frac{y\log x}{x}$, found from the primitive equation, the criterion will belong to neither result. The same might be farther illustrated in the present instance, by dividing the primitive equation by x, and differentiating the quotient; as C thus disappears, the result being $\frac{x\log xdy + y(1-\log x)dx}{x^2} = 0$. This equation possesses the criterion, though C is virtually eliminated. If, however, it be multiplied by x^2, the product will not possess the criterion, as it will not be the exact differential of any function. We see, therefore, two causes which prevent a differential function, of the kind here considered, from having the criterion of integrability: first, the elimination of a constant quantity; and, secondly, the multiplication of an exact differential by any variable quantity, except u or ϕu, as explained in No. 312. Only the former cause is mentioned in any work that the author has examined.

an equation, from which, if we could determine the value of z, we should be able to integrate every differential equation of the first order, containing two variables, by the method pointed out in No. 302. The finding of z, however, except in some particular cases, is, for the most part, more difficult than the integrating of the proposed differential.

312. When one multiplier, z, is known, innumerable others, as was remarked in the last No. can be found. For, let $z\mathrm{M}dx+z\mathrm{N}dy=du$, du being a function of x and y; then, multiplying by φu, any function whatever of u, we obtain, $\varphi uz\mathrm{M}dx + \varphi uz\mathrm{N}dy = \varphi udu$, the second member of which being integrable, the first is also integrable.

313. One case in which a multiplier, z, can be readily found, is when it is a function of only one of the variables, suppose x. On this supposition, the equation in No. 311 becomes

$$\frac{zd\mathrm{M}}{dy} - \frac{\mathrm{N}dz+zd\mathrm{N}}{dx} = 0, \quad \frac{dz}{dy} \text{ vanishing.}$$

Hence, we get, by transposition and division,

$$\frac{dz}{z} = \frac{1}{\mathrm{N}}\left(\frac{d\mathrm{M}}{dy} - \frac{d\mathrm{N}}{dx}\right)dx; \text{ and, thence,}$$

$$\log z = \int \frac{1}{\mathrm{N}}\left(\frac{d\mathrm{M}}{dy} - \frac{d\mathrm{N}}{dx}\right)dx;$$

and, putting X to denote the integral in the second member, we have by the nature of logarithms, $z = \varepsilon^{\mathrm{X}}$.

314. The method pointed out in the last No. is applicable when N does not contain y, and when M contains only its first power; as in the equation, $dy + \mathrm{P}ydx = \mathrm{Q}dx$, in which P and Q are any functions of x.* In this, we have

$$\mathrm{M} = \mathrm{P}y - \mathrm{Q}, \qquad \mathrm{N} = 1, \qquad \frac{1}{\mathrm{N}}\left(\frac{d\mathrm{M}}{dy} - \frac{d\mathrm{N}}{dx}\right) = \mathrm{P},$$

$$\mathrm{X} = \int \mathrm{P}dx, \qquad \text{and } z = \varepsilon^{\int \mathrm{P}dx}.$$

* This equation is generally called a *lineal equation of the first order*, a denomination to which Lacroix objects, on account of its containing the word *lineal*, which, instead of being a general term, is a geometrical one. The denomination which he prefers is, *an equation of the first degree and of the first order*.

The equation, $dy + (ay^2 + bx^m)\,dx = 0$, called the *equation of Riccati*, after the name of an Italian mathematician, by whom it was first considered, admits the separation of the variables when $m = 0$. Also, if we substitute $\frac{z}{x^2} + \frac{1}{ax}$ for y, a

Multiplying the proposed equation by this, we obtain

$$\varepsilon^{\int Pdx}dy+(Py-Q)\varepsilon^{\int Pdx}dx=0,$$

an equation which is integrable by the methods already explained.

315. If there be an equation, $Mdx+Ndy+M'dx+N'dy=0$, which can be separated into parts, $Mdx+Ndy$, and $M'dx+N'dy$, such that we can find multipliers z and z', which will render them integrable, or such that

$$\int(zMdx+zNdy)=u, \qquad \text{and } \int(z'M'dx+z'N'dy)=u';$$

then, by putting $z\varphi u=z'\varphi'u'$, we shall be able to determine the functions expressed by φ and φ', so that the multipliers $z\varphi u$ and $z'\varphi'u'$ may be identical; and then, if the proposed equation be multiplied by the quantity so found, the result will be integrable.

316. The integration of differential equations of the first order, which are functions of more variables than two, is effected on principles similar to those established above. Thus, if $du=Mdx+Ndy+Pdz$, where M is the partial differential coefficient of u, found on the supposition that x alone is variable, and N and P those which arise respectively from supposing y and z variable, it would be shown, as in No. 292, that, if $Mdx+Ndy+Pdz$ be an exact differential,

$$\frac{dM}{dy}=\frac{dN}{dx}, \qquad \frac{dM}{dz}=\frac{dP}{dx}, \qquad \frac{dN}{dz}=\frac{dP}{dy}.$$

The *criterion of integrability*, therefore, in this case is, that these three equations shall all hold true.

317. To integrate the equation, $du=Mdx+Ndy+Pdz$, when, by the criterion pointed out in the last No. it is found to admit of an exact integral, we may find the integral of its first term on the supposition that x alone is variable; and we shall thus obtain $u=\int Mdx+f(y,z)$: and, by differentiating this with respect to y, we get

$$\frac{du}{dy}=\frac{d\int Mdx}{dy}+\frac{df(y,z)}{dy}.$$

Now, by the proposed equation, $\frac{du}{dy}=N$; by equalling

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result is obtained, which is homogeneous when $m=-2$; and which, when $m=-4$, may have its variables separated by dividing the equation by $x^2(az^2+b)$. The same equation may be integrated in various other cases, which it would be too tedious here to particularize.

which with the foregoing value of $\frac{du}{dy}$, we find

$$\frac{d\int \mathrm{M}dx}{dy}+\frac{df(y,\ z)}{dy}=\mathrm{N}.$$

From this we get

$$\frac{df(y,\ z)}{dy}=\mathrm{N}-\frac{d\int \mathrm{M}dx}{dy};$$

whence, by multiplying by dy, and integrating with respect to y, we obtain

$$f(y,\ z)=\int \mathrm{N}dy-\int\frac{d\int \mathrm{M}dx}{dy}.dy+\varphi z.$$

We then substitute this value of $f(y,\ z)$ in the value of u already found, which gives

$$u=\int \mathrm{M}dx+\int \mathrm{N}dy-\int\frac{d\int \mathrm{M}dx}{dy}.dy+\varphi z.$$

By differentiating this with respect to z, we get the value of $\frac{du}{dz}$; and, putting the result so obtained equal to P, we find the value of φz; the substitution of which, with the constant quantity C, in the last value of u, gives the complete integral.

318. As an example, let it be required to integrate

$$du=\frac{ydx}{z}+\frac{(x-2y)dy}{z}+\frac{(y^2-xy)dz}{z^2}.$$

Here $\mathrm{M}=\frac{y}{z}$, $\mathrm{N}=\frac{x-2y}{z}$, $\mathrm{P}=\frac{y^2-xy}{z^2}$; also,

$$\frac{d\mathrm{M}}{dy}=\frac{1}{z}=\frac{d\mathrm{N}}{dx},\quad \frac{d\mathrm{M}}{dz}=-\frac{y}{z^2}=\frac{d\mathrm{P}}{dx},\quad \frac{d\mathrm{N}}{dz}=-\frac{x-2y}{z^2}=\frac{d\mathrm{P}}{dy},$$

so that (No. 316) the proposed expression is an exact differential. Hence, by integrating the first term in relation to x, we get $u=\frac{xy}{z}+f(y,\ z)$.

Finding the differential coefficient of this in respect to y, putting it equal to N, and rejecting $\frac{x}{z}$, we obtain

$$\frac{df(x,\ y)}{dy}=-\frac{2y}{z}.$$

Multiply this by dy, and integrate the product in relation to y; then

$$f(y,z) = -\frac{y^2}{z} + \varphi z; \text{ and, therefore, } u = \frac{xy}{z} - \frac{y^2}{z} + \varphi z.$$

Differentiating this with respect to z, dividing by dz, and putting the quotient equal to P, we get, by contraction, $\frac{d\varphi z}{dz} = 0$; and, consequently, $\varphi z = C$. Hence the required integral is $u = \frac{xy - y^2}{z} + C$.

EXERCISES IN THE INTEGRATION OF FUNCTIONS OF TWO OR MORE VARIABLES.

1. $du = \frac{ydx - xdy}{y\sqrt{(y^2 - x^2)}}$; $u = \sin^{-1}\frac{x}{y} + C$.

2. $du = \left(\frac{1}{y} - \frac{y^{\frac{1}{2}}}{2x^{\frac{3}{2}}}\right)dx - \left(\frac{x}{y^2} + \frac{1}{2x^{\frac{1}{2}}y^{\frac{1}{2}}}\right)dy$;

$$u = \frac{x}{y} + \sqrt{\frac{y}{x}}.$$

3. Let $\sin y\,dx + x\cos y\,dy + \sin x\,dy + y\cos x\,dx = 0$; then, $x\sin y + y\sin x = C$.

4. $du = \frac{(x+y)dx - (x-y)dy}{x^2 + y^2}$;

$$u = \tfrac{1}{2}\sin^{-1}\frac{x^2 - y^2}{x^2 + y^2} + \log\sqrt{(x^2 + y^2)} + C.$$

5. $du = (dx - dy)\sqrt{\frac{y+x}{y-x}} - dy\sin^{-1}\frac{x}{y}$;

$$u = \sqrt{(y^2 - x^2)} - y\sin^{-1}\frac{x}{y} + C.$$

6. $du = \frac{xdx - 2ydx + ydy}{x - y} \times \epsilon^{\frac{x}{x-y}}$; $u = (x - y)\epsilon^{\frac{x}{x-y}} + C$.

7. Let $3ydx = 2xdy$; then, $x^3 = Cy^2$.

8. Let $(x^2y - 3y^3)dx + (3xy^2 - x^3)dy = 0$; then, $y^3 - x^2y - Cx^3 = 0$.

9. Let $x^2dy + 3y^2dx = 2xydx$; then, $3xy - x^2 = Cy$.

10. Let $xydx = (x^2 + xy + y^2)dy$; then, $\log(x+y) - \frac{x}{y} = C$.

11. Let $(a-xy)dx=(1-x^2)dy$; then, $y=ax+C\sqrt{(1-x^2)}$.

12. $du=\left(\frac{1}{y}-\frac{y}{x}-\frac{2z^2}{x^3}\right)dx-\left(\frac{x}{y^2}+\log x\right)dy+\frac{2z}{x^2}dz$;

$$u=\frac{x}{y}-y\log x+\frac{z^2}{x^2}+C.$$

XVIII.—INTEGRATION OF DIFFERENTIAL EQUATIONS OF THE HIGHER ORDERS.

319. If a primitive equation between two variables, x and y, be differentiated, the result is said to be a differential equation of the *first order*; and, after division by dx, it will contain x, y, and $\frac{dy}{dx}$. This, after differentiation, and division by dx, will give an equation involving x, y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$. Such an equation is said to be of the *second order*; and, by continuing the process, we should obtain other differential equations of a similar kind; the general expression for the differential coefficient of the highest order in each, after n operations, being $\frac{d^ny}{dx^n}$, and the equation being said to be of the nth *order*.

Differential equations are also of different *degrees*, according to the highest power of the differential coefficient that marks their order. Thus, if the differential coefficient be $\frac{d^ny}{dx^n}$, and if the highest power of it that occurs in an equation be the mth power, the equation is said to be of the mth degree and nth order.

The integration of differential equations of the higher degrees and orders, is of such extent and difficulty, that, even in the present advanced state of science, the solutions of only some of the simplest cases have been effected; and, even of these, the nature of the present publication will admit the consideration of but a few of the easiest.

320. With a view to illustrate in a simple manner the nature of the equations which we are now considering, let us differentiate the primitive equation, $y=x^2+Cx+C_2$, and we shall find the successive differential coefficients,

$$\frac{dy}{dx}=2x+C, \qquad \text{and } \frac{d^2y}{dx^2}=2;$$

one of the constants, C_2, disappearing in the first operation, and the other, C, in the second; so that, after two differentiations, two constants disappear: and a similar disappearance of constants takes place in other cases, either directly or by elimination. Thus, if $Cy = x^2 + C_2x$, we obtain successively, $C\frac{dy}{dx} = 2x + C_2$, and $C\frac{d^2y}{dx^2} = 2$; and from these three equations we derive, by elimination, $2y - 2x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = 0$, a differential equation of the first degree and second order. The same result might also be obtained by changing the primitive equation into $C\frac{y}{x} - x = C_2$, so that C_2 may stand by itself; as, by differentiating this, we obtain $C\left(\frac{dy}{x} - \frac{ydx}{x^2}\right) = dx$, an equation which does not contain C_2. From this we derive $\frac{1}{x}.\frac{dy}{dx} - \frac{y}{x^2} = \frac{1}{C}$, in which C stands by itself; and the differentiation of this gives $2y - 2x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = 0$, the same as before. We might also arrive at the same final result by finding the value of C from the primitive equation, differentiating it, and, from the result, deriving the value of C_2, as the differential of this would be the same as has been already obtained. In this way, we should have, successively,

$$\frac{x^2 + C_2x}{y} = C, \qquad \frac{2xydx - x^2dy}{xdy - ydx} = C_2,$$

$$\text{and } 2y - 2x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = 0.$$

321. In a similar manner, in general terms, if we have $F(x, y, C, C_2) = 0$, we get, by successive differentiations,

$$F^1\left(x, y, \frac{dy}{dx}, C, C_2\right) = 0, \text{and } F^2\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, C, C_2\right) = 0;$$

and, by eliminating C and C_2 from these three equations, we should obtain a result of the form,

$$f^2\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0.$$

We might also arrive at the same conclusion, by putting the primitive equation under either of the forms,

$$f(x, y, C) = C_2, \quad \text{and } \varphi(x, y, C_2) = C;$$

as, by differentiating these, we should find

$$f^1\left(x, y, \frac{dy}{dx}, C\right) = 0, \quad \text{and } \varphi^1\left(x, y, \frac{dy}{dx}, C_2\right) = 0;$$

either of which, by differentiation, would give

$$f^2\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0,$$

as before.

322. A like illustration would be applicable, if the primitive equation contained a greater number of distinct constants; and it would be seen that, by each differentiation, one constant, and only one, would disappear, either directly or by elimination. The only apparent exception is when an equation is not general, such as $y^2 = ax^3$. This, however, may be regarded as what $y^2 + by = ax^3 + cx^2 +$ &c. becomes when b, c, &c. become nothing.

From this it appears, conversely, that in the successive integration of differential equations of the higher orders, one constant quantity is to be annexed after each operation.*

323. We may now investigate the methods of integrating differential equations of the second order, in some of the easiest cases; taking x, as usual, as the *independent variable*, that is, the variable whose differential is constant. If we put, then, $\frac{dy}{dx} = y'$, and $\frac{d^2y}{dx^2}$ or $\frac{dy'}{dx} = y''$, it is plain that the most general form of a differential equation of the second order is $f(x, y, y', y'') = 0$, the first member of which may contain constant quantities. The equation in this form has never been integrated, except in some particular cases. When, however, the equation wants one or more of the three quantities, x, y, y', the difficulty is much

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* It is also plain that there may be as many differential equations of the first order as there are constants in the primitive equation, since any one of these constants may be eliminated. On the same principle, these, by differentiation and elimination, will give differential equations of the second and higher orders, the number of which will depend on that of the constants in the primitive equation.

If n be the number of those constants, it may be shown that the numbers of the differential equations of the first, second, and mth orders will be, respectively,

$$n, \quad \frac{n(n-1)}{1.2}, \quad \text{and } \frac{n(n-1)\,(n-2)\ldots\ldots[n-(m-1)]}{1.2.3\ldots\ldots m}.$$

The proof of this is easy; but it is rather tedious to be given here.

lessened, and the solution can be effected in the five cases in which, in addition to y'', the equation contains

$$x; \quad y; \quad y'; \quad x, y'; \quad \text{or } y, y'.$$

324. In the first of these cases, in which $f(x, y'') = 0$, we substitute $\frac{dy'}{dx}$ for y'', and the result gives a value for dy' of the form $\mathrm{X}dx$; the integral of which may be found by the methods already explained, and is of the form, y' or $\frac{dy}{dx} = \mathrm{X}' + \mathrm{C}$; X and X′ being functions of x, and C a constant quantity. Then, by multiplying by dx, and integrating, y is obtained in terms of x.

325. In the second case, in which $f(y, y'') = 0$, we obtain by resolving the equation, y'' or $\frac{d^2y}{dx^2} = \mathrm{Y}$; where Y is a function of y. Multiply this by $2dy$, and integrate; then $\frac{dy^2}{dx^2} = 2\int \mathrm{Y}\,dy + \mathrm{C}$. Extract the square root of this, multiply by dx, and divide by $\sqrt{(2\int \mathrm{Y}\,dy + \mathrm{C})}$, and the integral of the result will be the primitive equation.

326. In the third case, we have $f(y'', y') = 0$, or $f\left(\frac{dy'}{dx}, y'\right) = 0$; whence we derive $dx = \varphi y' dy'$, the product of which by y' gives $y'dx$ or $dy = y'\varphi y' dy'$. By integrating these two equations, we obtain $x = \int \varphi y' dy' + \mathrm{C}$, and $y = \int y' \varphi y' dy' + \mathrm{C}'$; and the equation obtained from these by the elimination of y', is the primitive equation.

327. In the fourth case, we have $f(x, y', y'') = 0$. This, by substituting $\frac{dy'}{dx}$ for y'', is reduced to an equation of the first order with respect to x and y'; and may therefore be integrated by some of the methods already explained. The integral thus found, containing x and y' or $\frac{dy}{dx}$, is also of the first degree; and its integral will be the primitive equation.

328. In the fifth case, in which $f(y, y', y'') = 0$, by resolving the equation we find $y'' = \varphi(y, y')$; whence, $y''dy = \varphi(y, y')dy$. Now, by dividing the value of y'' by that of y' (No. 323), and clearing the result of fractions, we get $y''dy = y'dy'$; which changes the last equation into $y'dy' = \varphi(y, y')dy$, an equation of the first order with respect

to y and y'. Its integral may be found, therefore, and will be of the form, $\varphi'(y, y', \text{C})=0$, C being a constant quantity.

The principles here pointed out, as well as some others that need not be separately explained, will be illustrated in the following examples.

EXAMPLES OF INTEGRATING DIFFERENTIAL EQUATIONS OF THE HIGHER ORDERS.

329. Let $d^3y=axdx^3$. Here, by dividing by dx^2, and integrating, we obtain $\frac{d^2y}{dx^2}=\frac{1}{2}ax^2+\text{C}$. Multiplying this by dx, and integrating the result, we find

$$\frac{dy}{dx}=\frac{1}{2.3}\,ax^3+\text{C}x+\text{C}_2;$$

whence, by a similar operation,

$$y=\frac{1}{2.3.4}ax^4+\tfrac{1}{2}\text{C}x^2+\text{C}_2x+\text{C}_3,$$

which is the integral required.

330. Let it be required to integrate $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{d^2ydx}=r$, r being constant. This gives, by reduction,

$$\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}=r\frac{d^2y}{dx^2};$$

which, by putting $\frac{dy}{dx}=y'$, and, consequently, $\frac{d^2y}{dx}=dy'$, will become

$$(1+y'^2)^{\frac{3}{2}}=r\,\frac{dy'}{dx}.$$

Multiplying this by dx, and dividing by $(1+y'^2)^{\frac{3}{2}}$, we obtain $dx=r.\frac{dy'}{(1+y'^2)^{\frac{3}{2}}}$; whence, $y'dx$ or $dy=\frac{ry'dy'}{(1+y'^2)^{\frac{3}{2}}}$.

Hence, by integration, we find

$$x=\frac{ry'}{\sqrt{(1+y'^2)}}+\text{C},\quad \text{and } y=\frac{r}{\sqrt{(1+y'^2)}}+\text{C}_2;$$

from which, by eliminating y', we get $(x-\text{C})^2+(y-\text{C}_2)^2=r^2$, a constant quantity.

By comparing this example with No. 191, we find that it exhibits the nature of the curve in which the radius of curvature is a constant quantity; the result here obtained showing it to be a circle, as we know otherwise from the nature of the circle. This question exemplifies No. 326.

331. As another example, let it be required to find the equation of the curve which has the radius of its osculating circle a third proportional to a constant quantity, a, and the abscissa.

Here, by No. 191, we have $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{d^2ydx}=\frac{x^2}{a}$; which, by putting $dy=y'dx$, becomes

$$\frac{(1+y'^2)^{\frac{3}{2}}dx}{dy'}=\frac{x^2}{a}, \text{ whence } \frac{adx}{x^2}=\frac{dy'}{(1+y'^2)^{\frac{3}{2}}};$$

and, by integration (page 41, A_2, and No. 276)

$$-\frac{a}{x}=\frac{y'}{\sqrt{(1+y'^2)}}+\frac{b}{a},$$

where $\frac{b}{a}$ is put for the constant quantity. This gives

$$y'=\frac{bx+a^2}{\sqrt{\{a^2x^2-(bx+a^2)^2\}}}=\frac{bx+a^2}{\sqrt{\{(a^2-b^2)x^2-2a^2bx-a^4\}}};$$

whence, $y'dx$ or $dy=\frac{(bx+a^2)dx}{\sqrt{\{a^2-b^2)x^2-2a^2bx-a^4\}}}$.

Hence, by integration, $y=$

$$\frac{b}{c^2}\sqrt{(c^2x^2-2a^2bx-a^4)}+\frac{a^4}{c^2}\int\frac{dx}{\sqrt{(c^2x^2-2a^2bx-a^4}};^*$$

where c^2 is put for brevity, instead of a^2-b^2.

The integral of the last term of this (Nos. 258 and 261) is

$$\frac{a^4}{c^3}\log\{c^2x-a^2b+c\sqrt{(c^2x^2-2a^2bx-a^4)}\};$$

$$\text{or } \frac{a^4}{c^3}\sin^{-1}\frac{c^2x+a^2b}{a^3};$$

---

* The integral in this form may be easily found, by assuming

$$\int\frac{(bx+a^2)dx}{\sqrt{\{(a^2-b^2)x^2-2a^2bx-a^4\}}}=A\sqrt{\{(a^2-b^2)x^2-2a^2bx-a^4\}}$$
$$+B\int\frac{dx}{\sqrt{\{(a^2-b^2)x^2-2a^2bx-a^4\}}};$$

as, after differentiating this equation, the values of A and B will be readily found by the method of indeterminate coefficients.

Hence, putting the denominator, $\sqrt{(c^2x^2-2a^2bx-a^4)}=$ D, we shall have for the required integral,

$$y=\frac{b\mathrm{D}}{c^2}+\frac{a^4}{c^3}\log(c^2x-a^2b+c\,\mathrm{D})+\mathrm{C}\text{; or,}$$

$$y=\frac{b\,\mathrm{D}}{a^2-b^2}+\frac{a^4}{(b^2-a^2)^{\frac{3}{2}}}\sin^{-1}\frac{(b^2-a^2)x+a^2b}{a^3}+\mathrm{C}.$$

Both these integrals fail when $b^2=a^2$. In this case, however, the differential becomes

$$dy=\frac{(a^2-ax)\,dx}{\sqrt{(2a^3x-a^4)}}=\frac{1}{\sqrt{a}}.\frac{(a-x)dx}{\sqrt{(2x-a)}};$$

the negative value, $-a$, being alone admissible for b, as the positive one would render the formula imaginary. The integral of this (No. 255) is

$$y=\frac{1}{3\sqrt{a}}(2a-x)\sqrt{(2x-a)}+\mathrm{C};$$

which is the equation of an algebraic line of the third order. This equation, or either of the two found above, will give a curve answering to the conditions of the question.

This problem affords an exemplification of No. 327.

332. Let a body, T (*fig.* 46), move uniformly along the straight line, O B, while another body, P, moves uniformly in pursuit of it; and let it be required to determine the nature of the line described by P, the velocity of P being to that of T as 1 to a.

Let A and O be the two contemporaneous positions of the bodies at which A O is perpendicular to O B; and let P and T be any other contemporaneous positions. Then it is plain, that P T is a tangent to the required curve; and, by the question, O $\mathrm{T}=a\times\mathrm{A\,P}$. But, if O $\mathrm{R}=x$, and R $\mathrm{P}=y$, we have (No. 178) A $\mathrm{P}=\int\sqrt{(dx^2+dy^2)}$; and, therefore, O $\mathrm{T}=a\int\sqrt{(dx^2+dy^2)}$. Now (No. 146), R $\mathrm{T}=-\frac{ydx}{dy}$, and, consequently, O T or $a\int\sqrt{(dx^2+dy^2)}=x-\frac{ydx}{dy}$. To integrate this equation, let us first differentiate it, taking y as the independent variable, and, consequently, dy constant: then, $a\sqrt{(dx^2+dy^2)}=\frac{yd^2x}{dy}$.* This,

* The sign — is omitted before this quantity, because the other member may have either + or — before it.

by putting $dx = x'dy$, will become

$$a\sqrt{(1+x'^2)}dy = ydx'; \text{ whence, } \frac{ady}{y} = \frac{dx'}{\sqrt{(1+x'^2)}}.$$

Hence, by integration (page 41, B and E_2),

$$a\log y + a\log b = \log\{x' + \sqrt{(1+x'^2)}\},$$

where the constant quantity is assumed of the form $a\log b$; and from this we obtain, by the nature of logarithms, $b^a y^a = x' + \sqrt{(1+x'^2)}$. Hence, by transposing x', squaring, contracting, restoring the value of x', &c. we get

$$x = \frac{b^a y^{a+1}}{2(a+1)} + \frac{1}{2(a-1)b^a y^{a-1}} + c,$$

the equation required. This is the simplest case of the *curve of pursuit.* Other cases would arise from supposing one of the motions, or both, not to be uniform, but to be regulated according to a given law; or from supposing the given line to be a circle or other curve.

If $a = 1$ or -1, the foregoing equation fails. In these cases, by taking a successively equal to these values in the equation, $b^a y^a = x' + \sqrt{(1+x'^2)}$, we readily find

$$x = \tfrac{1}{4}by^2 - \frac{1}{2b}\log y + c, \quad \text{and } x = \frac{1}{2b}b\log y - \frac{by^2}{4} + c.$$

If $b = 1$, these will be the same, except their signs. It is plain indeed that, in the problem as here proposed, the values of x obtained for any particular value of y, will be the same, but with contrary signs, for equal values of a, one positive and the other negative. From this condition, in connexion with the property that $x = 0$, when y has a given value, y', we may determine the values of the constants, b and c.

The process here employed exemplifies No. 328.

EXERCISES IN INTEGRATING EQUATIONS OF THE HIGHER ORDERS.

1. LET $\frac{d^4y}{dx^4} = x^n$; then,

$$y = \frac{x^{n+4}}{(n+1)(n+2)(n+3)(n+4)} + \frac{Cx^3}{2.3} + \frac{C_2x^2}{2} + C_3x + C_4.$$

2. Let $\frac{d^3y}{dx^3} = \frac{a}{x^3}$; then, $y = \frac{1}{2}a\log x + \frac{1}{2}Cx^2 + C_2x + C_3$.

3. Let $\frac{d^2y}{dx^2}=\log x+\frac{x}{b}$; then, $y=\frac{1}{2}x^2\log x-\frac{3}{4}x^2+\frac{x^3}{6b}+$
$\mathrm{C}x+\mathrm{C}_2$.

4. Let $\frac{d^2y}{dx^2}+y=0$; then, $y=\mathrm{C}\sin x+\mathrm{C}_2\cos x$.

5. Let $\frac{d^2y}{dx^2}-y=0$; then, $y=\mathrm{C}\,\varepsilon^{x}+\mathrm{C}_2\varepsilon^{-x}$; or
$y=a\,\varepsilon^{\mathrm{C}+x}\pm b\varepsilon^{\mathrm{C}_2-x}$; or $x=\log\{y+\sqrt{(y^2+\mathrm{C})}\}+\mathrm{C}_2$.

6. Let $ad^2y-dydx=0$; then, $x=a\log\frac{y-\mathrm{C}_2}{a}+\mathrm{C}$.

7. Let $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxd^2y}=\frac{a^2}{2x}$; then, $y=\int\frac{(x^2+\mathrm{C})dx}{\sqrt{\{a^4-(x^2+\mathrm{C})^2\}}}$.*

8. Let $yd^2y+dy^2-bdx^2=0$; then, $(bx-b\mathrm{C}_2)^2=by^2+\mathrm{C}$.†

XIX.—APPROXIMATION OF INTEGRALS.

333. In many instances in which, in the present state of science, integrals cannot be exactly assigned, their values may be approximated. We have already had some instances of the application of series in finding integrals; and we may now consider how the same method may be extended still farther. In effecting this, the method of indeterminate coefficients (Alg. chap. XI.) may often be applied advantageously, as will appear in some of the subjoined examples.

334. As an instance of the use of this method, let it be required to integrate $d^2y+px^mydx^2=0$. Assume

$$y=\mathrm{A}x^a+\mathrm{B}x^{a+b}+\mathrm{C}x^{a+c}+\text{\&c.}$$

Then, by adding together the second differential of this

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* In the answers of the foregoing Exercises, it is plain that C, C_2, may each have any constant value whatever, or may be zero.

† To obtain the integral here indicated, put $x^2=z$. By this means we obtain

$$y=\int\frac{\frac{1}{2}(z+\mathrm{C}(z^{-\frac{1}{2}}dz}{\sqrt{(a^4-\mathrm{C}^2-2\mathrm{C}z-z^2)}},\text{ or}$$

$$=\frac{1}{2}\int\frac{z^{\frac{1}{2}}dz}{\sqrt{(a^4-\mathrm{C}^2-2\mathrm{C}z-z^2)}}+\frac{1}{2}\int\frac{\mathrm{C}dz}{z^{\frac{1}{2}}\sqrt{(a^4-\mathrm{C}^2-2\mathrm{C}z-z^2)}};$$

and the integral may be found in the manner pointed out in Section XV. This integral is the equation of the line called the *elastic curve;* and hence it appears, that this curve has its radius of curvature inversely proportional to its abscissa.

series, and the product of the series by $px^m dx^2$, we obtain, after dividing by dx^2,

$$\left\{\begin{array}{l} a(a-1)Ax^{a-2}+(a+b)(a+b-1)Bx^{a+b-2} \\ \qquad +(a+c)(a+c-1)Cx^{a+c-2}+\&c. \\ +p\{Ax^{a+m}+Bx^{a+b+m}+Cx^{a+c+m}+\&c.\} \end{array}\right\}=0.$$

Now, the exponents of the several terms will evidently be the same if $m=-2$; which, however, would answer to only a particular case of the proposed equation. To obviate this, we may assume a of such a value that the term $a(a-1)Ax^{a-2}$ may vanish; which will be effected by making $a=0$, or $a=1$. Taking the former value, and equalling the indices of the corresponding terms, we get

$$b-2=m, \quad c-2=b+m, \quad d-2=c+m, \&c.;$$

whence, we find $b=m+2$, $c=2m+4$, $d=3m+6$, &c. Again, by equalling the coefficients of the like terms, we get $b(b-1)B=pA$, $c(c-1)C=pB$, &c.; from which equations we find the values of B, C, &c.; and, substituting these in the assumed value of y, we obtain $y=$

$$A\left\{1-\frac{px^{m+2}}{(m+1)(m+2)}+\frac{p^2x^{2m+4}}{(m+1)(m+2)(2m+3)(2m+4)}\right.$$
$$\left.-\frac{p^3x^{3m+6}}{(m+1)(m+2)(2m+3)(2m+4)(3m+5)(3m+6)}+\&c.\right\};$$

where A is an arbitrary constant. This integral, however, is incomplete, as (No. 322) it should involve two such constants, the proposed equation having been of the second order. We must therefore derive another series, on the supposition that $a=1$; and this series will be another incomplete or particular integral, containing a constant, which, for the sake of distinction, may be denoted by A_2. The integral so produced will be found, by a process in every respect resembling the foregoing, to be $y=$

$$A_2x\left\{1-\frac{px^{m+2}}{(m+2)(m+3)}+\frac{p^2x^{2m+4}}{(m+2)(m+3)(2m+4)(2m+5)}\right.$$
$$\left.-\frac{p^3x^{3m+6}}{(m+2)(m+3)(2m+4)(2m+5)(3m+6)(3m+7)}+\&c.\right\}:$$

and the sum of the two series will be the complete integral required, of which each series is a particular case.*

335. As another example, let us integrate the equation, $dy+ydx+px^m dx=0$; and, let it be required to assign the integral so that we may have $y=q$ when $x=r$. To effect this, let us assume

$$y=q+At^a+Bt^{a+b}+Ct^{a+c}+\&c.$$

where t is put for $x-r$; and, therefore, $x=r+t$, and $dx=dt$. Hence,

$$dy=at^{a-1}dt+(a+b)Bt^{a+b-1}dt+(a+c)Ct^{a+c-1}dt+\&c.$$

By substituting these values of y, and dy, x, and dx in the given equation, we get, after dividing by dt,

$$\left.\begin{array}{l} aAt^{a-1}+(a+b)Bt^{a+b-1}+(a+c)Ct^{a+c-1}+\&c. \\ +q+At^a+Bt^{a+b}+Ct^{a+c}+\&c. \\ +pr^m+pmr^{m-1}t+pm.\frac{m-1}{2}r^{m-2}t^2+\&c. \end{array}\right\}=0.$$

Hence, by taking $a=1$, which renders the first terms in the three lines similar, we find that, to render the next terms similar, we must take $b=1$, $c=2$, &c. Then, by equalling the coefficients of the corresponding terms, we get

$$A=-(pr^m+q), \qquad B=\frac{pr^m-pmr^{m-1}+q}{2},$$

$$C=-\frac{pr^m-pmr^{m-1}+pm(m-1)r^{m-2}+q}{2.3}, \&c.;$$

and hence the required integral is determined by substituting these values for A, B, &c. in the series assumed for y.

336. We may now consider the method of approximating the values of integrals by means of continued fractions; and, perhaps, in the limited space which can here be given to the subject, the purpose will be best effected by means of examples.

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* For other particulars regarding this integral, see Lacroix, *Calcul*, tome ii. No. 661, &c.

Let, then, the differential $(1+x^n)\,dy - dx = 0$, be proposed,

and assume $y = \cfrac{Ax^a}{1+\cfrac{Bx^b}{1+\cfrac{Cx^c}{1+\cfrac{Ex^e}{1+\text{ \&c.}}}}}$*

where A, a, B, b, &c. are constant coefficients and exponents, to be determined. To find the values of A and a, take y simply equal to Ax^a; and, consequently, $dy = a\,Ax^{a-1}dx$. The substitution of this in the proposed differential gives $a\,Ax^{a-1}dx + a\,Ax^{a+n-1}\,dx - dx = 0$; from which, by rejecting the term containing the highest power of x, we get $a\,Ax^{a-1} = 1$, an equation which will be satisfied by taking a and A each $=1$. Using these values, therefore, and employing another term in the assumed value of y, we have

$$y = \frac{x}{1+Bx^b}, \quad \text{and } dy = \frac{dx}{1+Bx^b} - \frac{bBx^b dx}{(1+Bx^b)^2}.$$

By substituting this in the proposed differential, by clearing the result of fractions, by rejecting the terms in which x has the index $n+b$ or $2b$, and by dividing by $x^b dx$ and transposing, we get $x^{n-b} = bB + B$; an equation which will be satisfied by taking $b = n$ and $B = \frac{1}{n+1}$. Substituting these in the assumed value of y, and employing an additional term, we get

$$y = \cfrac{x}{1+\cfrac{\frac{1}{n+1}x^n}{1+Cx^c}}$$

or, by reduction,

$$y = \frac{(n+1)x(1+Cx^c)}{(n+1)(1+Cx^c)+x^n}.$$

By a process exactly similar to the two foregoing, we should find $c = n$, and $C = \frac{n^2}{(n+1)(2n+1)}$; and the values of e,

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* Some recent writers express continued fractions in a way which will be exemplified by writing the one in the text in the following manner:—

$$y = \frac{Ax^a}{1+}\;\frac{Bx^b}{1+}\;\frac{Cx^c}{1+}\;\frac{Ex^e}{1+}, \text{ \&c.}$$

E, &c. would be derived in the same way. We should finally obtain, therefore, y or $\int \frac{dx}{1+x^n} =$

$$\cfrac{x}{1+\cfrac{\frac{1}{n+1}x^n}{1+\cfrac{\frac{n^2}{(n+1)(2n+1)}x^n}{1+\cfrac{\frac{(n+1)^2}{(2n+1)(3n+1)}x^n}{1+\cfrac{(2n)^2}{(3n+1)(4n+1)}x^n}}}}$$
&c.

337. As particular cases of this integral, we may take n successively equal to 1 and 2, and (page 41, B and G) we get

$$\log(1+x) = \cfrac{x}{1+\cfrac{\frac{1}{2}x}{1+\cfrac{\frac{x}{2.3}}{1+\cfrac{\frac{4x}{3.4}}{1+\cfrac{4x}{4.5}}}}} \quad \text{\&c.} \qquad \tan^{-1}x = \cfrac{x}{1+\cfrac{\frac{1}{3}x^2}{1+\cfrac{\frac{4x^2}{3.5}}{1+\cfrac{\frac{9x^2}{5.7}}{1+\cfrac{16x^2}{7.9}}}}} \quad \text{\&c.}$$

338. It might be shown, in a similar manner, that, if $mydx+(1+x)dy=0$, we should have y, or $(1+x)^m=$

$$\cfrac{1}{1+\cfrac{mx}{1-\cfrac{(m-1).\frac{1}{2}x}{1+\cfrac{\frac{1}{3}(m+1).\frac{1}{2}x}{1-\cfrac{\frac{1}{3}(m-2).\frac{1}{2}x}{1+\cfrac{\frac{1}{5}(m+2).\frac{1}{2}x}{1-\text{\&c.}}}}}}}$$

The investigation is left for exercise to the student. For farther information on this subject, recourse may be had to Lacroix's large work on the Differential and Integral Calculus, vol. ii. chap. 6, where the examples here given will be found along with others.

XX.—DIRECT METHOD OF FINITE DIFFERENCES.*

339. If, in a function of a variable quantity, that variable be assumed successively equal to 0, 1, 2, 3, ..., x, or to -1, -2, ..., $-x$, the corresponding values of the function may be denoted by u_0, u_1, u_2, u_3, ..., u_x; or by u_{-1}, u_{-2}, ..., u_{-x}. If the variable be taken successively equal to x and $x+1$, so that the corresponding values of the function may be u_x and u_{x+1}, their difference, $u_{x+1}-u_x$, is called (No. 5) the *difference*, *finite difference*, or *increment* of u_x, and it is denoted by Δu_x. In like manner, u_1-u_0 is denoted by Δu_0; u_2-u_1, by Δu_1; $u_{-1}-u_{-2}$, by Δu_{-2}, &c. The term u_x is called the *general term* of the series.

340. Since the difference Δu_x will evidently have different values according to those assigned to the variable, the difference of those obtained by taking in it x and $x+1$ for the variable, that is, $\Delta u_{x+1}-\Delta u_x$, is called the *second difference* of the function, and is denoted by $\Delta^2 u_x$. In like manner, $\Delta^2 u_{x+1}-\Delta^2 u_x$ is called the *third difference*, and is expressed by $\Delta^3 u_x$; and so on to the nth *difference*, which is denoted by $\Delta^n u_x$.

341. Hence it is plain that, *to find the first difference of a function whose form is known, we are to substitute $x+1$ for x in the function; and, from the result, to subtract the function itself. By a like process, the second difference will be derived from the first; and, in general, any difference from the one immediately preceding it.* Thus, if

$$u_x = a + bx^2, \text{ and, consequently, } u_{x+1} = a + b(x+1)^2,$$

we obtain, by taking the former from the latter, $\Delta u_x = b(2x+1)$. In like manner, by subtracting this from what it becomes when $x+1$ is written instead of x, we get $\Delta^2 u_x = 2b$; and thence $\Delta^3 u_x = 0$. We see from this, that

* A valuable treatise on Finite Differences, by Sir John Herschel, is annexed as an Appendix to the English translation of Lacroix's smaller work on the Differential and Integral Calculus. The limits of the present publication admit the insertion of only a short abstract of the theory, with a few of its most interesting applications. The subject divides itself into two great branches:—the first, in which *we derive, from a proposed function, another called its difference;* and the second, in which, conversely, *the difference is given to determine the primitive function.* The former is termed the *Direct Method of Differences;* and the latter, the *Inverse Method.* These two branches are analogous—the first to the Differential, and the latter to the Integral Calculus. The Differential and Integral Calculus may be deduced, in fact, from the Calculus of Differences; and may be regarded, when considered in a particular light, as a case of the latter.

a constant quantity (a), connected by addition or subtraction with the variable part of a function, disappears in taking the difference.

342. The rule in the last No. enables us to find the differences of sums, products, &c. of functions of the same variable. Thus, let $u_x = fx \pm \varphi x$, and $u_{x+1} = f(x+1) \pm \varphi(x+1)$. Then, by subtracting, and by considering that $f(x+1) - fx = \Delta fx$, and $\varphi(x+1) - \varphi x = \Delta\varphi x$, we get

$$\Delta u_x \text{ or } \Delta(fx \pm \varphi x) = \Delta fx \pm \Delta\varphi x.$$

343. Let it now be required to find the difference of the product, $u_x u'_x$. Here we have

$$\begin{aligned}\Delta(u_x u'_x) &= u_{x+1} u'_{x+1} - u_x u'_x = (u_x + \Delta u_x)(u'_x + \Delta u'_x) - u_x u'_x \\ &= u'_x \Delta u_x + u_x \Delta u'_x + \Delta u_x \Delta u'_x \\ &= u'_x \Delta u_x + u_{x+1} \Delta u'_x, \quad \text{or} = u_x \Delta u'_x + u'_{x+1} \Delta u_x.\end{aligned}$$

If, instead of u'_x, we had the constant quantity a, its difference would be 0, and we should have simply $\Delta a u_x = a\Delta u_x$.

344. If the fraction $\frac{u_x}{u'_x}$ be proposed, we have

$$\Delta\frac{u_x}{u'_x} = \frac{u_{x+1}}{u'_{x+1}} - \frac{u_x}{u'_x};$$

which, by actual subtraction, and by substituting $u_x + \Delta u_x$ and $u'_x + \Delta u'_x$ for their equals u_{x+1} and u'_{x+1}, becomes

$$\Delta\frac{u_x}{u'_x} = \frac{u'_x \Delta u_x - u_x \Delta u'_x}{u'_x u'_{x+1}}.$$

345. In like manner, we have $\Delta x^n = (x+1)^n - x^n$, or (No. 14)

$$\Delta x^n = nx^{n-1} + n\frac{n-1}{2}x^{n-2} + n\frac{n-1}{2}\frac{n-2}{3}x^{n-3} + \text{ \&c.}$$

346. The difference of the continued product of the factors, $u_x, u_{x+1}, u_{x+2}, \ldots\ldots\ldots, u_{x+n-1}$, would be shown, on similar principles, to be

$$u_{x+1}\, u_{x+2} \,\ldots\ldots\ldots\, u_{x+n-1}(u_{x+n} - u_x).$$

If, in this, $u_x = a + bx$, so that the successive factors increase by equal differences, the factor, $u_{x+n} - u_x$, becomes nb; and, this being constant, it appears that the difference of a function of this kind is of the same form with the function itself, but has one variable factor (u_x) fewer; that is,

$$\Delta.(a+bx)\{a+b(x+1)\}\{a+b(x+2)\}\dots\{a+b(x+n-1)\}$$
$$=\{a+b(x+1)\}\{a+b(x+2)\}\ \dots\ \{a+b(x+n-1)\}nb.$$

If x be written for $a+bx$, this assumes the simpler form,

$$\Delta.x(x+b)(x+2b)\ \dots\dots\ \{x+(n-1)b\}$$
$$=(x+b)(x+2b)(x+3b)\ \dots\dots\ \{x+(n-1)b\}nb.$$

In this form, the increment of x, instead of being necessarily 1, is b.

347. It would be shown, in a similar manner, that

$$\Delta\frac{1}{u_x\ u_{x+1}\ \dots\dots\ u_{x+n-1}}=-\frac{u_{x+n}-u_x}{u_x\ u_{x+1}\ \dots\dots\ u_{x+n}}.$$

In this difference there is one factor more in the denominator than there are in the given denominator. By making, also, the substitutions indicated in the last No. we obtain

$$\Delta\frac{1}{(a+bx)\{a+b(x+1)\}\ \dots\dots\ \{a+b(x+n-1)\}}$$
$$=-\frac{nb}{(a+bx)\{a+b(x+1)\}\ \dots\dots\ \{a+b(x+n)\}};\text{ and}$$

$$\Delta\frac{1}{x(x+b)(x+2b)\ \dots\dots\ \{x+(n-1)b\}}$$
$$=-\frac{nb}{x(x+b)(x+2b)\ \dots\dots\ (x+nb)}.$$

348. If $u_x=a^x$, we get, in the usual manner,

$$\Delta u_x,\text{ or }\Delta a^x=a^{x+1}-a^x=a^x(a-1).$$

From this again, by taking the successive differences in a similar manner, we should obtain

$$\Delta^2a^x=a^x(a-1)^2,\ \dots\dots\ \Delta^na^x=a^x(a-1)^n.$$

It might be proved, in a similar way (from No. 87), that

$$\Delta\log x=\log\frac{x+1}{x}=\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{3x^3}-\&c.$$

To find the difference of $\sin x\varphi$, we have (TRIG. No. 25)

$$\Delta\sin x\varphi=\sin(x+1)\varphi-\sin x\varphi=2\cos(x\varphi+\tfrac{1}{2}\varphi)\sin\tfrac{1}{2}\varphi.$$

In a similar manner, we should find

$$\Delta\cos x\varphi=-2\sin(x\varphi+\tfrac{1}{2}\varphi)\sin\tfrac{1}{2}\varphi.$$

If the differences of powers of the cosine or sine be required, those powers may be modified by the formulas, TRIG. Section VIII. and the differences of the results may be found in the manner now shown.

The differences of other trigonometrical functions are found on similar principles.

349. *If the successive differences of any rational integral function of the nth order be taken, the nth difference is constant, and all the differences of a higher order vanish.* Thus, the first, second, and third differences of x^3+Ax^2+Bx+C are, respectively,

$$3x^2+(3+2A)x+1+A+B, \quad 6x+6+2A, \quad \text{and } 6.$$

In like manner, if the general formula,

$$Ax^n+Bx^{n-1}+Cx^{n-2}+ \ldots\ldots +Hx^2+Kx+L,$$

be assumed, it would be readily shown that the first, second, &c. differences would be of the forms,

$$nAx^{n-1}+B_2x^{n-2}+C_2x^{n-3}+\ldots\ldots\ldots\ldots\ldots +H_2x+K_2,$$
$$n(n-1)Ax^{n-2}+B_3x^{n-3}+C_3x^{n-4}+\ldots\ldots +H_3,$$
$$\&c. \qquad \&c. \qquad \&c.;$$

the detached constant quantities, L, K_2, H_3, &c. vanishing in the successive operations: and it would appear, finally, that the nth difference would be

$$n(n-1)(n-2) \ldots\ldots\ldots 3.2.1\,A;$$

which, being constant, will have no difference.

350. To illustrate this by an example in numbers, let $u_x=3x^3+x^2+2$. Then, by taking x successively equal to 0, 1, 2, 3, &c. we get $u_0=2$, $u_1=6$, $u_2=30$, &c. Subtracting the first of these values from the second, the second from the third, &c. we find $\Delta u_0=4$, $\Delta u_1=24$, &c.; and again, by subtracting the first of these differences from the second, the second from the third, &c. there result $\Delta^2 u_0=20$, $\Delta^2 u_1=38$, &c.; the operation standing as follows, the third differences being all equal, and the fourth being each 0:—

Terms,	2	6	30	92	210	402, &c.
1st differences,	4	24	62	118	192, &c.	
2d differences,	20	38	56	74, &c.		
3d differences,	18	18	18, &c.			
4th differences,	0	0, &c.				

351. Since (No. 339) $u_1 = u_0 + \Delta u_0$, we have (No. 342) $\Delta u_1 = \Delta u_0 + \Delta^2 u_0$; whence, by addition, $u_1 + \Delta u_1$, or (No. 339) $u_2 = u_0 + 2\Delta u_0 + \Delta^2 u_0$. From this we get, in like manner, $\Delta u_2 = \Delta u_0 + 2\Delta^2 u_0 + \Delta^3 u_0$; and, by addition,

$$u_2 + \Delta u_2, \text{ or } u_3 = u_0 + 3\Delta u_0 + 3\Delta^2 u_0 + \Delta^3 u_0.$$

The process might thus be continued as far as we please; and it would readily appear, from an examination of the mode in which the coefficients of the several terms are formed, that they are the same as would arise from the continued multiplication of a binomial (such as $a+b$) into itself. Hence we shall have (No. 14) the general expression, $u_x = u_0 + \frac{x}{1} . \Delta u_0 +$

$$\frac{x(x-1)}{1.2} . \Delta^2 u_0 + \frac{x(x-1)(x-2)}{1.2.3} . \Delta^3 u_0 + \text{\&c.} \dots \text{A.}$$

If we put $\frac{n'}{n}$ instead of x, this will become $u_{\frac{n'}{n}} = u_0 +$

$$\frac{n'}{n} . \Delta u_0 + \frac{n'(n'-n)}{2n^2} . \Delta^2 u_0 + \frac{n'(n'-n)(n'-2n)}{2.3n^3} . \Delta^3 u_0 + \text{\&c.. B.}$$

352. A general formula, exhibiting any difference of the function u_x, in an interesting form, may be thus investigated, by means of Nos. 339 and 342:—

$\Delta u_x = u_{x+1} - u_x$; whence

$\Delta^2 u_x = u_{x+2} - u_{x+1} - (u_{x+1} - u_x) = u_{x+2} - 2u_{x+1} + u_x$. Hence,

$$\Delta^3 u_x = u_{x+3} - u_{x+2} - 2(u_{x+2} - u_{x+1}) + u_{x+1} - u_x$$
$$= u_{x+3} - 3u_{x+2} + 3u_{x+1} - u_x.$$

In like manner, we should find

$$\Delta^4 u_x = u_{x+4} - 4u_{x+3} + 6u_{x+2} - 4u_{x+1} + u_x:$$

and, by considering, as in the last No. the manner in which the coefficients are formed, we should find that they are those of the powers of a residual (such as $a-b$); and, therefore, we have (No. 14) the general formula, $\Delta^n u_x =$

$$u_{x+n} - \frac{n}{1} u_{x+n-1} + \frac{n(n-1)}{1.2} u_{x+n-2} - \frac{n(n-1)(n-2)}{1.2.3} u_{x+n-3} + \text{\&c.}$$

353. The results obtained in No. 351 are useful in enabling us to determine a proposed term of a series by means of other terms, without computing it from the law of the series.

As an example of this, let it be required to find the fiftieth term of the series, 2, 6, 30, &c. in No. 350. Here, as we have seen,

$u_0 = 2$, $\Delta u_0 = 4$, $\Delta^2 u_0 = 20$, $\Delta^3 u_0 = 18$, and $\Delta^4 u_0 = 0$. Then, taking $x = 49$, we find, by means of series A in No. 351, $u_{50} =$

$$2 + 49 \times 4 + \frac{49 \times 48}{2} \times 20 + \frac{49 \times 48 \times 47}{2 \times 3} \times 18 = 355350,$$

the term required. This is the same that would be found by taking $x = 49$ in $3x^3 + x^2 + 2$, the expression from which the series was derived.

354. The series in No. 351 always give the exact value of the required term, when, as in the foregoing example, and as (No. 349) in all cases of rational integral functions, the differences of any order become constant. In every other case, the method fails in giving accurate results. In such circumstances, however, when the series is such that the successive differences continually diminish, and when certain terms are known, the value of any intermediate term may be approximated in a manner that is often very useful. The effecting of this constitutes what is called the *Method of Interpolation.*

As an example, let it be supposed that, in computing a table of logarithms, the common logarithms of 100, 102, 104, and 106, have been found to be 2·0000000, 2·0086002, 2·0170333, and 2·0253059; and that it is required to compute from these the logarithm of 103. Here we have the following table:—

$u_0 = 2·0000000$			
	·0086002		
$u_1 = 2·0086002$		—·0001671	
	·0084331		·0000066
$u_2 = 2·0170333$		—·0001605	
	·0082726		
$u_3 = 2·0253059$			

The second column of this table is found by subtracting each number in the first from the one immediately below it; and each of the other columns is derived in the same manner from the column before it. We have, therefore, $u_0 = 2$, $\Delta u_0 = ·0086002$, $\Delta^2 u_0 = —·0001671$, and $\Delta^3 u =$ ·0000066. Then, since the intervals between the numbers whose logarithms are employed are each 2, and since the required logarithm is that of 103, let us take $n = 2$, and $n' = 103 - 100 = 3$, and we shall have, by formula B, No. 351, the common logarithm of 103 = ·0128372, which is true in all its figures. We should find, in like manner, by taking n' successively equal to 1 and 5, the logarithms of 101 and 105 also true.

355. In every application of this method, the accuracy of the result depends on the smallness of the last difference used in the computation. Had the logarithm of 108 been employed in the foregoing exercise, we should have had a fourth difference, ·0000008; but this is so small, that it would have influenced none of the results found above. If, however, the logarithms had been carried out to more places, a greater number of them must have been used in the table in the last No. so as to have given differences of a higher order.

It will be readily seen, that this principle may be advantageously employed in the calculation of tables of various kinds. Thus, the logarithms of prime numbers, such as 101 and 103, may be easily computed from those of the composite numbers between which they are placed. So, likewise, in Astronomy, when the numbers expressing the situations of the heavenly bodies, their rising, setting, the apparent distance of the moon and the sun or stars, &c. are accurately calculated for certain times, the like numbers for the intermediate times may often be found much more easily by interpolation, than by computations founded on rigorous principles. The same method may also be employed with advantage, in determining intermediate quantities from others found by observation, particularly in Astronomy.

356. In the method of interpolation, so far as we have yet considered it, the given terms have been supposed to be placed at equal intervals. Thus, in No. 354, the numbers 100, 102, 104, 106, whose logarithms were employed, had the common difference, 2; and, in deducing the formulas in No. 351, the variable was supposed to increase from term to term by unity. When this is not the case, other principles are necessary. To investigate these, let u be a function of a variable quantity z, and suppose that, when z takes certain given values, z_0, z_1, z_2, ..., z_{n-1}, u is changed into u_0, u_1, u_2, ..., u_{n-1}, which will also be known;

and let us assume, for the expression to be determined,

$$u_x = A + Bz_x + Cz_x^2 + \dots\dots\dots + Lz_x^{n-1}.$$

If in this we take, successively, $x=0$, $x=1$, $x=2$, ..., $x=n-1$, we get

$$\begin{aligned} u_0 &= A + Bz_0 + Cz_0^2 + \dots\dots\dots + Lz_0^{n-1}, \\ u_1 &= A + Bz_1 + Cz_1^2 + \dots\dots\dots + Lz_1^{n-1}, \\ u_2 &= A + Bz_2 + Cz_2^2 + \dots\dots\dots + Lz_2^{n-1}, \\ \dots & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ u_{n-1} &= A + Bz_{n-1} + Cz_{n-1}^2 + \dots + Lz_{n-1}^{n-1}. \end{aligned}$$

We have thus n equations, which will give, by elimination, the values of the n constant coefficients, A, B, C, ..., L. This elimination will be effected by subtracting each equation from the one following it; as, by this means, A will be eliminated, being excluded from the $n-1$ resulting equations. If we divide the first of these by $z_1 - z_0$, the second by $z_2 - z_1$, &c. B will be eliminated from the results, by a like subtraction of each equation from the one following it. In a similar manner, C, D, &c. will be eliminated; and we shall at length find the value of L. Then, by substituting this value in all the n assumed equations except one, the value of the coefficient K, immediately preceding L, will be determined by a series of operations similar to those employed in finding L; and thus all the coefficients may be found. By following out the process here indicated, and putting, for the sake of brevity,

$$\frac{u_1 - u_0}{z_1 - z_0} = U_0, \quad \frac{u_2 - u_1}{z_2 - z_1} = U_1, \quad \frac{u_3 - u_2}{z_3 - z_2} = U_2, \text{ \&c.}$$

$$\frac{U_1 - U_0}{z_2 - z_0} = U_0', \quad \frac{U_2 - U_1}{z_3 - z_1} = U_1', \text{ \&c.}$$

$$\frac{U_1' - U_0'}{z_3 - z_0} = U_0'', \text{ \&c.}$$

we obtain, for the general expression required,

$$u_x = u_0 + U_0(z_x - z_0) + U_0'(z_x - z_0)(z_x - z_1) + U_0''(z_x - z_0)(z_x - z_1)(z_x - z_2) + \text{\&c.}$$

357. The following method of deriving, in a different and a very elegant form, the same result as that found by

the method explained in the last No. was discovered by Lagrange. Since u_x is to become u_0, u_1, u_2, &c. when z_x becomes z_0, z_1, z_2, &c. we may assume

$$u_x = au_0 + bu_1 + cu_2 + \text{.........} + lu_{n-1},$$

provided that a, b, c, &c. are of such values as to answer these conditions. Now, when $x=0$, the second member must, by hypothesis, become u_0, which will take place if $a=1$, $b=0$, $c=0$, ..., $l=0$; and, therefore, a, b, c, ..., l, must be such functions of x, that

$$\text{when } x=0, \quad a=1, \quad b=0, \quad c=0, \ldots\ldots, l=0.$$

It would be shown in a similar manner, that,

$$\begin{array}{lllll} \text{when } x=1, & a=0, & b=1, & c=0, \ldots\ldots, & l=0; \\ \text{when } x=2, & a=0, & b=0, & c=1, \ldots\ldots, & l=0; \\ \ldots\ldots\ldots\ldots, & \ldots\ldots, & \ldots\ldots, & \ldots\ldots, & \ldots\ldots, \\ \text{when } x=n-1, & a=0, & b=0, & c=0, \ldots\ldots, & l=1. \end{array}$$

Since, therefore, $z_x = z_0$, or $z_x - z_0 = 0$, when $x=0$, it follows that $z_x - z_0$ must be a factor of all the coefficients, except a; and it would appear, in like manner, that $z_x - z_1$ must be a factor of them all, except b; $z_x - z_2$, of all, except c; $z_x - z_{n-1}$, of all, except l. Hence, we may assume,

$$a = \text{A}(z_x - z_1)(z_x - z_2)(z_x - z_3) \ldots\ldots (z_x - z_{n-1});$$

which, when $x=0$, becomes

$$1 = \text{A}(z_0 - z_1)(z_0 - z_2)(z_0 - z_3) \ldots\ldots (z_0 - z_{n-1}):$$

and, by dividing the last equation by this, we get

$$a = \frac{(z_x - z_1)(z_x - z_2)(z_x - z_3) \ldots\ldots (z_x - z_{n-1})}{(z_0 - z_1)(z_0 - z_2)(z_0 - z_3) \ldots\ldots (z_0 - z_{n-1})}.$$

In a similar manner, we should find

$$b = \frac{(z_x - z_0)(z_x - z_2)(z_x - z_3) \ldots\ldots (z_x - z_{n-1})}{(z_1 - z_0)(z_1 - z_2)(z_1 - z_3) \ldots\ldots (z_1 - z_{n-1})};$$

and the other coefficients would obviously be found in the same way. Hence we get, for the required value,

$$u_x = \frac{(z_x - z_1)(z_x - z_2)(z_x - z_3) \ldots\ldots (z_x - z_{n-1})}{(z_0 - z_1)(z_0 - z_2)(z_0 - z_3) \ldots\ldots (z_0 - z_{n-1})} u_0$$

$$+ \frac{(z_x - z_0)(z_x - z_2)(z_x - z_3) \ldots\ldots (z_x - z_{n-1})}{(z_1 - z_0)(z_1 - z_2)(z_1 - z_3) \ldots\ldots (z_1 - z_{n-1})} u_1$$

$$+ \frac{(z_x - z_0)(z_x - z_1)(z_x - z_3) \ldots\ldots (z_x - z_{n-1})}{(z_2 - z_0)(z_2 - z_1)(z_2 - z_3) \ldots\ldots (z_2 - z_{n-1})} u_2$$

$$+ \ \&c.$$

This formula is very convenient in practice, as its terms are fitted for computation by means of logarithms. It may be shown to be, in reality, the same as the one given at the end of No. 356. To do this, it would be merely necessary to find the actual values of the symbols, U_0, U_0', U_0'', &c. in terms of z_0, z_1, z_2, &c. and u_0, u_1, u_2, &c. and to substitute them in the last-mentioned formula; as the result, when arranged according to u_0, u_1, u_2, &c. would be the same as that which has been obtained in this No.

It would be easy to show, also, that formula A, No. 351, is a particular case of the formulas found in this No. and the foregoing. This would be proved by substituting, in No. 356, $p + xh$ instead of z_x. By this means, we should have U_0, or, which is the same, $\frac{u_1 - u_0}{z_1 - z_0} = \frac{\Delta u_0}{h}$, and $z_x - z_0 = xh$; the introduction of which into the second term of the value of u_x in No. 356, renders it the same as the second term of formula A, No. 351: and the remaining terms of the latter might be derived, in a similar manner, from those of the former.

358. The formulas obtained in the last two Nos. besides their use in interpolation, enable us also to find the equation of a curve of a given species, passing through any proposed number of given points. As an instance, let the coordinates of the given points be x', x'', x''', and y', y'', y'''; and let it be required to find a curve passing through them, whose equation is $y = A + Bx + Cx^2$; that is, let it be required to find the values of A, B, and C, so that a curve of the species expressed by the equation, $y = A + Bx + Cx^2$, may pass through the given points. Here, we are to take

$$u_x = y, \qquad u_0 = y', \qquad u_1 = y'', \qquad u_2 = y''',$$
$$z_x = x, \qquad z_0 = x', \qquad z_1 = x'', \qquad z_2 = x''';$$

and, by substituting these in the formula found in the last No. we obtain, for the required equation,

$$y = \frac{(x-x'')(x-x''')}{(x'-x'')(x'-x''')}y' + \frac{(x-x')(x-x''')}{(x''-x')(x''-x''')}y'' + \frac{(x-x')(x-x'')}{(x'''-x')(x'''-x'')}y'''.$$

Thus, if $x'=0$, $x''=1$, $x'''=3$, and $y'=2$, $y''=4$, $y'''=5$, we should have, by reduction, $2y=4+5x-x^2$, the equation of a common parabola passing through the three given points. The same result might also be derived from formula A, No. 351.

From the mode of investigation pursued in Nos. 356 and 357, it is plain that we can determine the values of as many constants, A, B, C, &c. as there are points given. Hence, if n denote the number of points, the equation to be assumed is $y = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \ldots + \mathrm{L}x^{n-1}$.

In the method here followed, the curve to be determined is assumed as a parabola, the general equation of a parabola of any order being $y = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \ldots\ldots + \mathrm{L}x^n$. It is evident, however, that there are innumerable other curves, as well as parabolic ones, that will pass through the given points. Parabolas are generally to be preferred, because they are more simple in their nature than any others, on account of their ordinates being rational functions of their abscissas.

359. By the method that has now been pointed out, we can find a parabolic curve nearly coinciding with a proposed one, and therefore having nearly the same area: and thus we have an easy and useful means of approximating the areas of curves. As an example, let it be required to approximate the area of O P P″ R″, a portion of the circle (*fig.* 50) whose centre is O, and radius O P; the co-ordinates of P, P′, and P″, being

$$x'=0 \qquad x''=28, \qquad x'''=60,$$
$$y=100, \qquad y''=96, \qquad y'''=80.$$

Hence, by the method pointed out in the last No. we find the equation of the common parabola passing through P, P′, P″, to be

$$y = 100 + \frac{x}{42} - \frac{x^2}{168};$$

by multiplying which by dx, and integrating, we get (No. 166)

$$s = 100x + \frac{x^2}{84} - \frac{x^3}{504}.$$

If, in this, we take $x = 60$, we find the area of the space contained by the parabola and the straight lines, PO, OR″, and R″P″, to be $5614\frac{2}{7}$; while the space bounded by the circle and the same straight lines is found, when computed on the strict principles already explained (Section IX.) to be 5617·514, differing from the foregoing by only about a 1740th part of itself: and hence, for ordinary purposes, the parabolic area, which is easily computed, might be used for the circular one.

Had another intermediate ordinate been employed, the parabola would have been a line of the third order, and the approximation would have been more nearly true; and a farther increase in the number of the ordinates would have diminished the error farther still. With the same number of ordinates, also, the degree of approximation is greater, the less the distance is between the extreme ordinates.

In the present instance, the parabola falls above the arc PP′, and below P′P″; and thus it will be seen how the errors tend to neutralize each other.

It would be easy to establish general formulas of approximation on the principles here employed; and such are given by writers on Mensuration.

EXERCISES IN THE DIRECT CALCULUS OF DIFFERENCES.*

1. $\Delta \frac{x^2}{x-1} = 1 + \frac{1}{x} - \frac{1}{x-1}.$

2. $\Delta \frac{a^x}{x} = a^x \left\{ \frac{a}{x+1} - \frac{1}{x} \right\}.$

3. $\Delta \tan x\varphi = \frac{\sin \varphi}{\cos x\varphi \cos(x+1)\varphi}.$

4. $\Delta \cot x\varphi = \frac{-\sin \varphi}{\sin x\varphi \sin(x+1)\varphi}.$

* The reader will find a valuable "Collection of Examples of the Application of the Calculus of Finite Differences" in the work with that title by Sir John Herschel. From it, several of the exercises on the subject in this work are taken.

5. $\Delta \cot 2^x \varphi = -\dfrac{1}{\sin 2^{x+1}\varphi}$.

6. $\Delta \tan^{-1} x\varphi = \tan^{-1}\left(\dfrac{\varphi}{1+\varphi^2 x+\varphi^2 x^2}\right)$.

7. $\Delta \sin(x\varphi + h) = 2\sin\frac{1}{2}\varphi\cos\{(x+\frac{1}{2})\varphi + h\}$.

8. $\Delta^{2n}\sin(x\varphi+h) = (2\sin\frac{1}{2}\varphi)^{2n}\sin\{(x+n)\varphi+h\}$.

9. $\Delta\dfrac{1}{2^x}\cot\dfrac{\varphi}{2^x} = \dfrac{1}{2^{x+1}}\tan\dfrac{\varphi}{2^{x+1}}$.

10. $\Delta 2^x \sin\dfrac{\varphi}{2^x} = 2^{x+2}\sin\dfrac{\varphi}{2^{x+1}}\sin^2\dfrac{\varphi}{2^{x+2}}$.

11. $\Delta^n x^m = (x+n)^m - \dfrac{n}{1}(x+n-1)^m + \dfrac{n(n-1)}{1.2}(x+n-2)^m - \text{\&c.}$*

12. $(x+n)^n - \dfrac{n}{1}(x+n-1)^n + \dfrac{n(n-1)}{1.2}(x+n-2)^n - \text{\&c.} = 1.2.3 \ldots\ldots n.$

13. $\Delta^n 0^m = n^m - \dfrac{n}{1}(n-1)^m + \dfrac{n(n-1)}{1.2}(n-2)^m - \text{\&c.}$

14. $n^n - \dfrac{n}{1}(n-1)^n + \dfrac{n(n-1)}{1.2}(n-2)^n - \text{\&c.} = 1.2.3 \ldots n.$

15. Required the 121st term, and also the xth term of the series, 1, 7, 25, 61, 121, 211, &c.
Answ. $u_{121} = 1771441$, and $u_x = x^3 - x + 1$.

16. If it be found, by computation, that the moon rises at a given place, on the first day of a month, at $5^h\ 24^m$; on the third, at $6^h\ 46^m$; and on the fifth, at $8^h\ 32^m$: required the times of rising on the second and fourth days.
Answ. $6^h\ 2^m$ and $7^h\ 36^m$.

17. Prove, from the following data, that the common logarithm of 3·1415926536 is 0·4971498726:

l3·14 = 0·4969296481,	l3·17 = 0·5010592622,
l3·15 = 0·4983105538,	l3·18 = 0·5024271200.
l3·16 = 0·4996870826,	

* This formula is obtained from No. 352 by taking $u_x = x^m$. The next Exercise is solved by means of this one in connexion with No. 349, m being taken equal to n. The 13th and 14th Exercises show what the 11th and 12th become when $x = 0$.

18. Given the cube roots of 123, 124, and 125, equal respectively to 4·973190, 4·986631, and 5; to show, by interpolation, that the cube root of 123456789 is 497·79339.*

19. Given the logarithmic tangents of $2^0\ 9'$, $2^0\ 10'$, and $2^0\ 11'$, equal to 8·5745197, 8·5778766, and 8·5812077, respectively; required the tangent of $2^0\ 10'\ 23''$.
Answ. 8·5791566.

20. Given AB, BC, CD, and DE (*fig.* 51), each equal to 5 perches, and the perpendicular ordinates drawn through A, B, C, D, and E, to the curvilineal boundary FG, equal to 5, 6, 9, 11, and 11 perches, respectively; to find the equation of a parabolic curve, meeting FG at the extremities of those ordinates, and to compute the area of the parabola.

Answ. $y=5-\frac{3x}{10}+\frac{41x^2}{300}-\frac{x^3}{125}+\frac{x^4}{7500}$; area, $169\frac{7}{9}$.

XXI.—INVERSE METHOD OF FINITE DIFFERENCES.

360. THE sign used to denote the *integral* of a proposed difference, that is, the primitive function from which it is derived, is the Greek letter Σ prefixed to the difference. Thus, since (No. 341) $\Delta(bx^2+C)=b(2x+1)$, we have, conversely, $\Sigma b(2x+1)=bx^2+C$. The arbitrary quantity C may be either constant, or such a function of x as will be unchanged when $x+1$ is substituted for x, such as when, x being a whole number, C is any function of $\cos 2\pi x$. This will be illustrated by taking the difference of $bx^2+a\cos 2\pi x$, as it will also be found to be $b(2x+1)$, the same as the difference of bx^2+C when C is considered constant. In the applications of this method, however, which will be given in this Section, it will be unnecessary to assign any values to C, except constant ones.

361. It follows, from Nos. 342 and 343, that the integral of a difference consisting of terms connected by addition or

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* Since $123000000=123\times 1000000$, it follows, by taking the cube roots, that the cube root of 123000000 is 100 times the cube root of 123; and is therefore 497·3190. In like manner, we find the roots of 124000000 and 125000000; and thence (No. 351) the root of 123456789, by taking $x=\frac{456789}{1000000}$. It will thus be seen how, by means of tables of square and cube roots up to 100 or 1000, the roots of other numbers, whether intermediate or not, may be readily computed. Such tables are given in Hutton's Mathematics, in his Tracts, &c.

subtraction, is found by integrating the terms separately, and connecting the results by their proper signs; and that a constant multiplier in a difference remains in the integral. Thus, since $\Delta(ax^2 \pm bx + C) = a\Delta x^2 \pm b\Delta x$, we have, conversely,

$$\Sigma(a\Delta x^2 \pm b\Delta x) = a\Sigma\Delta x^2 \pm b\Sigma\Delta x = ax^2 \pm bx + C.$$

362. The integral of any power of a variable, having a whole positive index, may be found by the method of indeterminate coefficients. Thus, to find the integral of x^3, let us assume

$$\Sigma x^3 = Ax^4 + Bx^3 + Cx^2 + Dx.$$

By taking (No. 345) the difference of this, we obtain $x^3 =$

$$4Ax^3 + 6Ax^2 + 4Ax + A + 3Bx^2 + 3Bx + B + 2Cx + C + D:$$

and hence, by putting the coefficients of the like powers equal to each other, we have $4A = 1$,

$$6A + 3B = 0, \quad 4A + 3B + 2C = 0, \quad A + B + C + D = 0.$$

These equations give $A = \frac{1}{4}$, $B = -\frac{1}{2}$, $C = \frac{1}{4}$, $D = 0$; and, therefore

$$\Sigma x^3 = \tfrac{1}{4}x^4 - \tfrac{1}{2}x^3 + \tfrac{1}{4}x^2 + \text{Const.}$$

363. In the same manner, if we put

$$\Sigma x^m = Ax^{m+1} + Bx^m + Cx^{m-1} + Dx^{m-2} + \&c.$$

we should find (No. 345), by taking the difference, $x^m =$

$$\begin{aligned}\frac{m+1}{1}Ax^m + \frac{(m+1)m}{1.2}Ax^{m-1} &+ \frac{(m+1)m(m-1)}{1.2.3}Ax^{m-2} + \&c.\\ + \frac{m}{1}Bx^{m-1} &+ \frac{m(m-1)}{1.2}Bx^{m-2} + \&c.\\ &+ \frac{m-1}{1}Cx^{m-2} + \&c.\\ &+ \&c.\end{aligned}$$

Hence, by equalling the corresponding coefficients, reducing, &c. we get

$$A = \frac{1}{m+1}, \quad B = -\tfrac{1}{2}, \quad C = \frac{m}{3.4}, \quad D = 0, \quad \&c.;$$

and, therefore, we should at length find the following general formula for the integral of a power having a whole positive index:

$$\Sigma x^m = -\tfrac{1}{2}x^m + \frac{1}{m+1}x^{m+1} + \frac{1}{6}\frac{m}{2}x^{m-1}$$

$$-\frac{1}{30}\frac{m(m-1)(m-2)}{1.2.3.4}x^{m-3}$$

$$+\frac{1}{42}\frac{m(m-1)(m-2)(m-3)(m-4)}{1.2.3.4.5.6}x^{m-5}$$

$$-\frac{1}{30}\frac{m(m-1)\ \ldots\ldots\ (m-6)}{1.2.3\ \ldots\ldots\ 8}x^{m-7}$$

$$+\frac{5}{66}\frac{m(m-1)\ \ldots\ldots\ (m-8)}{1.2.3\ \ldots\ldots\ 10}x^{m-9}$$

$$-\frac{691}{210.13}\frac{m(m-1)\ \ldots\ldots\ (m-10)}{1.2.3\ \ldots\ldots\ 12}x^{m-11}$$

$$+\frac{7}{6}\frac{m(m-1)\ \ldots\ldots\ (m-12)}{1.2.3\ \ldots\ldots\ 14}x^{m-13}$$

$$-\frac{3617}{30.17}\frac{m(m-1)\ \ldots\ldots\ (m-14)}{1.2.3\ \ldots\ldots\ 16}x^{m-15}$$

$$+\frac{43867}{42.19}\frac{m(m-1)\ \ldots\ldots\ (m-16)}{1.2.3\ \ldots\ldots\ 18}x^{m-17}$$

$$-\frac{1222277}{110.21}\frac{m(m-1)\ \ldots\ldots\ (m-18)}{1.2.3\ \ldots\ldots\ 20}x^{m-19}$$

$$+\quad \&c. \qquad + \text{constant.}^*$$

* The numeral coefficients, $\frac{1}{6}, \frac{1}{30}, \frac{1}{42}$, &c. of the terms following the first and second, have been called by Euler, the *Numbers of Bernoulli*, from James Bernoulli, by whom some of their properties were first pointed out. These numbers are of much use in the theory of series. Many of their properties are developed in the Appendix to the English translation of Lacroix's Elementary Treatise on the Differential and Integral Calculus, and in Herschel's Examples on the Calculus of Finite Differences.

364. If, in the last No. m be taken successively equal to 0, 1, 2, 3, &c. the following particular integrals are obtained:—

$$\Sigma 1 = x + C \qquad \Sigma x = -\frac{x}{2} + \frac{x^2}{2} + C.$$

$$\Sigma x^2 = -\frac{x^2}{2} + \frac{x^3}{3} + \frac{x}{6} + C. \qquad \Sigma x^3 = -\frac{x^3}{2} + \frac{x^4}{4} + \frac{x^2}{4} + C.$$

$$\Sigma x^4 = -\frac{x^4}{2} + \frac{x^5}{5} + \frac{x^3}{3} - \frac{x}{30} + C$$

$$\Sigma x^5 = -\frac{x^5}{2} + \frac{x^6}{6} + \frac{5x^4}{12} - \frac{x^2}{12} + C$$

$$\Sigma x^6 = -\frac{x^6}{2} + \frac{x^7}{7} + \frac{x^5}{2} - \frac{x^3}{6} + \frac{x}{42} + C$$

$$\Sigma x^7 = -\frac{x^7}{2} + \frac{x^8}{8} + \frac{7x^6}{12} - \frac{7x^4}{24} + \frac{x^2}{12} + C$$

$$\Sigma x^8 = -\frac{x^8}{2} + \frac{x^9}{9} + \frac{2x^7}{3} - \frac{7x^5}{15} + \frac{2x^3}{9} - \frac{x}{30} + C$$

$$\Sigma x^9 = -\frac{x^9}{2} + \frac{x^{10}}{10} + \frac{3x^8}{4} - \frac{7x^6}{10} + \frac{x^4}{2} - \frac{3x^2}{20} + C$$

$$\Sigma x^{10} = -\frac{x^{10}}{2} + \frac{x^{11}}{11} + \frac{5x^9}{6} - x^7 + x^5 - \frac{x^3}{2} + \frac{5x}{66} + C$$

&c. &c.

365. If a compound quantity, such as $3x^5 - 5x^3$, be proposed for integration, we may find the integrals of its parts in the manner now explained, and connect them by their proper signs. In this way, we find

$$\Sigma(3x^5 - 5x^3) = \frac{x^6 - 3x^5 + 5x^3 - 3x^2}{2} + C.$$

366. Such quantities as we have now been considering, may often be integrated very easily by means of the principle established in No. 346. If, in the concluding formula in that No. we write $x - b$ instead of x, and, consequently, x instead of $x + b$, &c. and add a factor, or change n into $n + 1$, we shall have, by integrating, and dividing by $(n+1)b$,

$$\Sigma x(x+b)(x+2b) \ldots\ldots \{x+(n-1)b\}$$
$$= \frac{(x-b)x(x+b) \ldots\ldots \{x+(n-1)b\}}{(n+1)b}.$$

In this, the factors are equidifferent, and the integral is found by multiplying the given difference by $x-b$, and dividing the product by $(n+1)b$; n denoting the number of factors in the given difference. Hence, *to integrate a function whose successive factors have equal differences, multiply by the factor next less than the least in the proposed function, and divide the product by the common difference, and by the number of factors so increased.*

367. Quantities of the form, ax^m+bx^n+ &c. are often very easily modified so as to be expressed by means of products of equidifferent factors; and, consequently, to admit of integration in the manner pointed out in the last No. Thus, for example, let the integral of x^4 be required. Then,

$$x^4=x^3(x+1)-x^3=(x-1)x^2(x+1)+x^2(x+1)-x^3$$
$$=(x-1)x^2(x+1)+x^2$$
$$=(x-2)(x-1)x(x+1)+2(x-1)x(x+1)+x^2$$
$$=(x-2)(x-1)x(x+1)+2(x-1)x(x+1)+x(x+1)-x.$$

Hence, the proposed quantity is expressed by *factorials*, consisting of products of equidifferent factors; and, therefore, by the preceding No. we get

$$\Sigma x^4=\frac{(x-3)(x-2)(x-1)x(x+1)}{5}+\frac{(x-2)(x-1)x(x+1)}{2}$$
$$+\frac{(x-1)x(x+1)}{3}-\frac{(x-1)x}{2}+\text{C}.$$

This result is easily reduced to the form in which it appears in No. 344. The proposed quantity might also have been reduced into factorials in many other ways, each giving a different form of the integral. Thus, we may have

$$x^4=(x-1)x(x+1)(x+2)-2x(x+1)(x+2)$$
$$+7(x+1)(x+2)-15x-14, \text{ and}$$

$$\Sigma x^4=\frac{(x-2)(x-1)x(x+1)(x+2)}{5}-\frac{(x-1)x(x+1)(x+2)}{2}$$
$$+\frac{7x(x+1)(x+2)}{3}-\frac{15(x-1)x}{2}-14x+\text{C}.$$

In like manner, if it were required to integrate $2x^3-3x^2$, we should have

$$2x^3-3x^2=2x^2(x+1)-2x^2-3x^2=2x^2(x+1)-5x^2$$
$$=2(x-1)x(x+1)+2x(x+1)-5x^2$$
$$=2(x-1)x(x+1)+2x(x+1)-5x(x+1)+5x$$
$$=2(x-1)x(x+1)-3x(x+1)+5x.$$

Hence, by No. 366, we get $\Sigma(2x^3-3x^2)=$

$$\tfrac{1}{2}(x-2)(x-1)x(x+1)-(x-1)x(x+1)+\tfrac{5}{2}(x-1)x+C.$$

A little practice will enable the student to reduce any rational integral function into factorials of the kind here pointed out.

368. The factorials of a function may also be found by means of indeterminate coefficients. Thus, we might assume,

$$x^4=Ax(x+1)(x+2)(x+3)+Bx(x+1)(x+2)+Cx(x+1)+Dx;$$

and, by performing the actual multiplications, and equalling the like coefficients, we should get $A=1$,

$$B+6A=0,\quad C+3B+11A=0,\quad D+C+2B+6A=0.$$

Hence, $A=1$, $B=-6$, $C=7$, and $D=1$; so that

$$x^4=x(x+1)(x+2)(x+3)-6x(x+1)(x+2)+7x(x+1)-x, \text{ and}$$

$$\Sigma x^4=\frac{(x-1)x(x+1)(x+2)(x+3)}{5}-\frac{3(x-1)x(x+1)(x+2)}{2}+\frac{7(x-1)x(x+1)}{3}-\frac{(x-1)x}{2}+C;$$

another form of the integral, which is also equivalent to those already found.

369. It would appear, in a manner exactly similar, from No. 347, that

$$\Sigma\frac{a}{x(x+b)(x+2b)\ldots\ldots\{x+(n-1)b\}}=C-\frac{a}{(n-1)bx(x+b)(x+2b)\ldots\ldots\{x+(n-2)b\}};$$

whence we see, that, *to integrate a rational fraction having* *its numerator constant, and its denominator the product of factors having equal differences, we are to efface the greatest factor, and to divide by the common difference, and by the number of factors remaining; changing the sign of the result, and annexing a constant quantity.*

Thus, for example, we have

$$\Sigma\frac{1}{(x-1)x(x+1)}=\text{C}-\frac{1}{2}\frac{1}{(x-1)x};$$

$$\Sigma\frac{3}{x(x+1)(x+2)(x+3)}=\text{C}-\frac{1}{x(x+1)(x+2)}.$$

370. If, in such fractions, one or more factors be wanting in the denominator, the numerator and denominator may be multiplied so as to supply the deficiency; and the result can be resolved into parts that will admit of integration in the manner pointed out in the last No. To illustrate this by an example, let it be required to integrate $\frac{1}{x(x+1)(x+3)}$. By multiplying the numerator and denominator of this by $x+2$, the resulting numerator will be equivalent either to the sum of x and 2, or to $x+3-1$; and hence the given expression becomes either

$$\frac{x}{x(x+1)(x+2)(x+3)}+\frac{2}{x(x+1)(x+2)(x+3)},\text{ or}$$

$$\frac{x+3}{x(x+1)(x+2)(x+3)}-\frac{1}{x(x+1)(x+2)(x+3)}.$$

Now, in the first form we can divide the numerator and denominator of one term by x: and, in the second, we can divide those of the first term by $x+3$. Doing this therefore, and integrating by the rule in the last No. we get, for the required integral,

$$\text{C}-\frac{1}{2}\frac{1}{(x+1)(x+2)}-\frac{2}{3}\frac{1}{x(x+1)(x+2)},\text{ or}$$

$$\text{C}-\frac{1}{2}\frac{1}{x(x+1)}+\frac{1}{3}\frac{1}{x(x+1)(x+2)}.$$

371. The method of indeterminate coefficients enables us to integrate many rational fractions, which cannot be in-

tegrated by the methods that have thus far been explained. As an instance, let it be required to integrate

$$\frac{x^2-5x+1}{x(x+1)(x+2)(x+3)}.$$

To effect this, assume the numerator,

$$x^2-5x+1=Ax(x+1)+Bx+D=Ax^2+(A+B)x+D.$$

Hence, by comparing the coefficients, and reducing, we have $A=1$, $B=-6$, and $D=1$. The given expression, therefore, becomes

$$\frac{x(x+1)}{x(x+1)(x+2)(+3)}-\frac{6x}{x(x+1)(x+2)(x+3)}$$
$$+\frac{1}{x(x+1)(x+2)(x+3)}:$$

and, by dividing the numerator and denominator of the first and second terms, respectively, by $x(x+1)$ and x, we obtain, for the required integral, by No. 369,

$$C-\frac{1}{x+2}+\frac{3}{(x+1)(x+2)}-\frac{1}{3}\frac{1}{x(x+1)(x+2)}.$$

372. It was shown, in No. 348, that $\Delta a^x=a^x(a-1)$; and we have, conversely, $\Sigma a^x=\frac{a^x}{a-1}+C$.

373. From the same No. we obtain, by integrating and dividing by the constant quantity $2\sin\frac{1}{2}\varphi$,

$$\Sigma\cos(x\varphi+\tfrac{1}{2}\varphi)=\frac{\sin x\varphi}{2\sin\frac{1}{2}\varphi}+C;$$

$$\text{and } \Sigma\sin(x\varphi+\tfrac{1}{2}\varphi)=-\frac{\cos x\varphi}{2\sin\frac{1}{2}\varphi}+C;$$

which, by writing x for $x+\frac{1}{2}$, will become

$$\Sigma\cos x\varphi=\frac{\sin(x-\frac{1}{2})\varphi}{2\sin\frac{1}{2}\varphi}+C;$$

$$\text{and } \Sigma\sin x\varphi=-\frac{\cos(x-\frac{1}{2})\varphi}{2\sin\frac{1}{2}\varphi}+C.$$

374. Since (No. 343) $\Delta(u_x u'_x)=u_x\Delta u'_x+u'_{x+1}\Delta u_x$, we find, by integrating and transposing, that

$$\Sigma u_x\Delta u'_x=u_x u' -\Sigma u'_{x+1}\Delta u_x.$$

This formula is evidently analogous to $\int u dv = uv - \int v du$, and it is often of much use in the integrating of differences. To exemplify the mode of applying it, let the integral of $x a^x$ be required. This (No. 348) may be put under the form, $x \Delta \frac{a^x}{a-1}$. Taking, therefore, $u_x = x$, and $u'_x = \frac{a^x}{a-1}$, we have $\Delta u_x = 1$; and substituting these values in the foregoing formula, we obtain

$$\Sigma x a^x = x \frac{a^x}{a-1} - \Sigma \frac{a^{x+1}}{a-1} = x \frac{a^x}{a-1} - \frac{a^{x+1}}{(a-1)^2};$$

the last term being found by means of No. 372.

EXERCISES IN THE INTEGRATION OF DIFFERENCES.

1. $\Sigma(2x^4 - 4x^2 + 1) = \frac{2x^5}{5} - x^4 - \frac{2x^3}{3} + 2x^2 + \frac{4x}{15}$.

2. $\Sigma(5x^4 + 10x^3 + 4x^2 - x) = x^5 - 2x^3 + x$.

3. $\Sigma \frac{x+3}{x(x+1)(x+2)} = -\frac{1}{x} - \frac{1}{2x(x+1)}$, or

$$= -\frac{1}{x+1} - \frac{3}{2x(x+1)}$$

4. $$\Sigma \frac{1}{x(x+3)(x+4)} = -\frac{1}{2(x+1)(x+2)} + \frac{1}{3(x+1)(x+2)(x+3)} - \frac{1}{2x(x+1)(x+2)(x+3)}.$$

5. $\Sigma \frac{1}{3(3x-1)x(3x+1)} = -\frac{1}{6(3x-1)x}$.

6. $$\Sigma a^x (pa^x + q)(pa^{x+1} + q)(pa^{x+2} + q) \ldots\ldots (pa^{x+n-1} + q) = \frac{a}{p} \cdot \frac{(pa^{x-1} + q)(pa^x + q)(pa^{x+1} + q) \ldots\ldots (pa^{x+n-1} + q)}{a^{n+1} - 1}.$$

7. $$\Sigma \frac{a^x}{(pa^x + q)(pa^{x+1} + q) \ldots\ldots (pa^{x+n-1} + q)} = C - \frac{1}{p(a^{n-1} - 1)} \cdot \frac{1}{(pa^x + q)(pa^{x+1} + q) \ldots\ldots (pa^{x+n-2} + q)}.$$

8. $\Sigma(5\sin^3 x\varphi - 4\sin^5 x\varphi) = -\frac{5}{8}\frac{\cos(x-\frac{1}{2})\varphi}{\sin\frac{1}{2}\varphi}$

$$+\frac{1}{8}\frac{\cos 5(x-\frac{1}{2})\varphi}{\sin\frac{5}{2}\varphi}.$$

XXII.—APPLICATION OF THE CALCULUS OF DIFFERENCES IN THE SUMMATION OF SERIES.

375. LET the successive terms of a series be denoted by u_1, u_2, u_3, ... u_x, and their sum by S_x; that is, let

$$S_x = u_1 + u_2 + u_3 + \ldots\ldots + u_x.$$

Then, by annexing another term, we have

$$S_{x+1} = u_1 + u_2 + u_3 + \ldots\ldots + u_x + u_{x+1}.$$

Subtracting the former expression from this, and substituting ΔS_x for its equal, $S_{x+1} - S_x$, we obtain

$$\Delta S_x = u_{x+1}, \quad \text{and, consequently,} \ S_x = \Sigma u_{x+1} + C.$$

To obtain the value of C, let $x=0$; in which case, by the nature of the inquiry, $S_x = 0$, and, therefore, $0 = \Sigma u_1 + C$. Take this from the sum found above, and there results, for the required sum,

$$S_x = \Sigma u_{x+1} - \Sigma u_1.$$

Hence it appears, that, *to determine the sum of a series, we are to find the integral of the term that would follow the last of the proposed terms, and to subtract from that integral what it becomes when x (the number of terms) is nothing.*

376. To exemplify this, let the terms of the series be

$$a, \quad a+b, \quad a+2b, \ \ldots\ldots, a+(x-1)b,$$

and we shall have $\Sigma u_{x+1} = \Sigma(a+xb)$; the value of which is found (No. 364) to be $\Sigma u_{x+1} = xa + \frac{1}{2}x(x-1)b$. Taking in this $x=0$, we get $\Sigma u_1 = 0$; and, therefore (No. 375), we have, for the required sum,

$$S_x = xa + \tfrac{1}{2}x(x-1)b, \quad \text{or } S_x = \tfrac{1}{2}x\{a+a+(x-1)b\};$$

whence it appears, that the sum of a series of equidifferent numbers is found by multiplying the number of terms into the sum of the extremes, and taking half the product. Thus, for example, if $a=1$, and $b=1$, we get

$$1+2+3+\ldots\ldots+x = \tfrac{1}{2}x(x+1).$$

377. To find the sum of the series,

$$a, \quad ar, \quad ar^2, \ldots\ldots, ar^{x-1},$$

we have $\Sigma u_{x+1} = \Sigma ar^x = \frac{ar^x}{r-1}$, by No. 372; whence, by taking $x=0$, we get $\Sigma u_1 = \frac{a}{r-1}$; and, consequently, (No. 375),

$$S_x = \frac{ar^x - a}{r-1} = \frac{a(r^x - 1)}{r-1}.$$

If $r < 1$, and x infinite, this will become $S = \frac{a}{1-r}$, where S is put for S_∞, the sum of an infinite number of terms.

378. To find the sums of any assigned powers of the numbers 1, 2, 3, ..., x, we have merely to integrate $(x+1)^n$, n denoting the index of the power; and this is effected by No. 363 or 364.

Thus, if $n=2$, we should have (Nos. 375 and 364)

$$S_x = -\frac{(x+1)^2}{2} + \frac{(x+1)^3}{3} + \frac{x+1}{6}, \text{ or } S_x = \frac{2x^3+3x^2+x}{6}.$$

If, again, $n=3$, we should obtain, in a similar manner,

$$S_x = \frac{(x+1)^4 - 2(x+1)^3 + (x+1)^2}{4} = \{\tfrac{1}{2}(x+1)^2 - \tfrac{1}{2}(x+1)\}^2$$

$$= \{\tfrac{1}{2}x(x+1)\}^2.$$

Comparing this with the conclusion of No. 376, we see that

$$1^3 + 2^3 + 3^3 + \ldots\ldots + x^3 = (1+2+3+\ldots\ldots+x)^2,$$

which is a curious relation of these series.

379. Required the sum of the series,

$$\frac{1}{4} + \frac{1}{4.5} + \frac{1.2}{4.5.6} + \frac{1.2.3}{4.5.6.7} + \frac{1.2.3.4}{4.5.6.7.8} + \text{\&c.}$$

The xth, or general term of this series, and the following term, are

$$\frac{1.2.3}{x(x+1)(x+2)(x+3)}, \quad \text{and} \quad \frac{1.2.3}{(x+1)(x+2)(x+3)(x+4)}:$$

by integrating the latter of which, and subtracting from the result what it becomes when $x=0$, we get (No. 375)

$$S_x = \frac{1}{3} - \frac{2}{(x+1)(x+2)(x+3)}, \quad \text{and } S = \frac{1}{3}.$$

380. To sum the series, $1^2-2^2+3^2-4^2+\ldots\ldots\pm x^2$, we may put it under the form, $1^2+2^2.(-1)+3^2.(-1)^2+\ldots\ldots+x^2.(-1)^{x-1}$; and then, to obtain S_x, we have to integrate $(x+1)^2.(-1)^x$, which will be effected by successive applications of No. 374. Thus (No. 372), the integral of $(-1)^x$ being $-\frac{1}{2}(-1)^x$, and the difference of $(x+1)^2=2x+3$, we have

$$\Sigma(x+1)^2.(-1)^x, \text{ or } -\tfrac{1}{2}\Sigma(x+1)^2\Delta(-1)^x$$
$$=-\tfrac{1}{2}(x+1)^2(-1)^x+\tfrac{1}{2}\Sigma(-1)^{x+1}(2x+3).$$

The second term of this may be put under the form,

$$\Sigma x(-1)^{x+1}+\tfrac{3}{2}\Sigma(-1)^{x+1}, \text{ or } -\Sigma x(-1)^x-\tfrac{3}{2}\Sigma(-1)^x;$$

the actual integrals of the parts of which (Nos. 374 and 372) are

$$\tfrac{1}{2}x(-1)^x+\tfrac{1}{4}(-1)^{x+1}, \text{ and } \tfrac{3}{4}(-1)^x.$$

Hence, putting the term, $\frac{1}{4}(-1)^{x+1}$, under the form, $-\frac{1}{4}(-1)^x$, and collecting the several terms, we get

$$\Sigma(x+1)^2.(-1)^x$$
$$=-\tfrac{1}{2}(x+1)^2.(-1)^x+\tfrac{1}{2}x(-1)^x-\tfrac{1}{4}(-1)^x+\tfrac{3}{4}(-1)^x;$$

which, by expanding the first term, contracting, &c. becomes $-\frac{1}{2}x(x+1).(-1)^x$; and this is the required sum, as, by taking $x=0$, it vanishes without any quantity being annexed.

Comparing this with the conclusions of Nos. 376 and 378, we find the following remarkable relation between the sums of three series:

$$(1+2+3+\ldots\ldots+x)^2=(1^2-2^2+3^2-4^2+\ldots\ldots\pm x^2)^2$$
$$=1^3+2^3+3^3+\ldots\ldots+x^3.$$

381. The following series are called *figurate numbers* of the *first order, second order, third order*, &c.:

1,	1,	1,	1,	1,	1,	&c.
1,	2,	3,	4,	5,	6,	&c.
1,	3,	6,	10,	15,	21,	&c.
1,	4,	10,	20,	35,	56,	&c.
1,	5,	15,	35,	70,	126,	&c.
	&c.		&c.		&c.	

Their nature is such, that the xth term, in any order after the first, is equal to the sum of the first x terms of the preceding order. Thus, 35, the fifth term of the fourth order, is equal to the sum of 1, 3, 6, 10, and 15, the first five terms of the third order.*

382. Now, each term of the first order being 1, we have the sum of x terms of that order equal to x; which, by the nature of the series, is the xth or general term of the second order.

The sum of x terms in the latter order is shown, in No. 367, to be $\frac{1}{2}x(x+1)$, which, again, is the general term of the third order; and the term after this, in the third order, is $\frac{1}{2}(x+1)(x+2)$.

The sum of x terms, therefore, in this order is found, by means of the integral of $\frac{1}{2}(x+1)(x+2)$, to be

$$\frac{x(x+1)(x+2)}{1.2.3}.$$

A continuation of this mode of proceeding would show, that the sum of x terms in the nth order is

$$\frac{x(x+1)(x+2)(x+3)\ldots\ldots(x+n-1)}{1.2.3.4\ldots\ldots n}.$$

383. To sum the reciprocals of the figurate numbers, since the general term of the nth order is

$$\frac{x(x+1)(x+2)\ldots\ldots(x+n-2)}{1.2.3\ldots\ldots n-1}.$$

we are to take the reciprocal of the term following this, and to integrate it. We thus obtain (No. 369)

$$S_x = C - \frac{1.2.3\ldots\ldots(n-1)}{(n-2)(x+1)(x+2)(x+3)\ldots\ldots(x+n-2)};$$

* The several orders of figurate numbers may also be expressed thus:

1st,	$\frac{1}{1}$,	$\frac{1}{1}$,	$\frac{1}{1}$,	......	$\frac{1}{1}$;
2d,	$\frac{1}{1}$,	$\frac{2}{1}$,	$\frac{3}{1}$,	....	$\frac{x}{1}$;
3d,	$\frac{1.2}{1.2}$,	$\frac{2.3}{1.2}$,	$\frac{3.4}{1.2}$,	....	$\frac{x(x+1)}{1.2}$;
4th,	$\frac{1.2.3}{1.2.3}$,	$\frac{2.3.4}{1.2.3}$,	$\frac{3.4.5}{1.2.3}$,	....	$\frac{x(x+1)(x+2)}{1.2.3}$;
n th,	$\frac{1.2\,..\,n-1}{1.2\,..\,n-1}$,	$\frac{2.3\,..\,n}{1.2\,..\,n-1}$,	$\frac{3.4\,..\,n+1}{1.2\,..\,n-1}$,	....	$\frac{x(x+1)\,..\,(x+n-2)}{1.2\,..\,n-1}$.

whence, $S_1 = C - \frac{n-1}{n-2}$; and, therefore, for the sum of x terms and of an infinite number, we get

$$S_x = \frac{n-1}{n-2} - \frac{1.2.3\ldots\ldots(n-1)}{(n-2)(x+1)(x+2)(x+3)\ldots\ldots(x+n-2)},$$

$$\text{and } S = \frac{n-1}{n-2}.$$

As examples, let n be taken successively equal to 3, 4, and 5, and we shall have, for the sum of x terms,

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \&c. = 2 - \frac{2}{x+1} = \frac{2x}{x+1};$$

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{10} + \&c. = \frac{3}{2} - \frac{3}{(x+1)(+2)} = \frac{3}{2} \cdot \frac{x(x+3)}{(x+1)(x+2)};$$

$$\frac{1}{1} + \frac{1}{5} + \frac{1}{15} + \&c. = \frac{4}{3} - \frac{2.4}{(x+1)(x+2)(x+3)}$$

$$= \frac{4}{3} \cdot \frac{x(x^2+6x+11)}{(x+1)(x+2)(x+3)};$$

while the sums of an infinite number of terms of the same series are respectively, 2, $\frac{3}{2}$, and $\frac{4}{3}$.*

384. We may now sum, by the method here explained, the series,

$$\sin\theta + \sin 2\theta + \sin 3\theta + \ldots\ldots + \sin x\theta,$$

---

* For the first and second orders, the expression found above fails. In the first order, the sum of x terms is evidently x, and the sum of an infinite number is infinite. In the second order, also, the sum of an infinite number is infinite. To show this subtract the members of second formula in No. 88, taking $M = 1$, from those of $\log 1 = 0$, and there will remain

$$\log 1 - \log(1-h), \text{ or } \log\frac{1}{1-h} = \frac{h}{1} + \frac{h^2}{2} + \frac{h^3}{3} + \&c.$$

which, when $h = 1$, becomes

$$\log \infty = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.;$$

an infinite quantity, since the logarithm of a number infinitely great is obviously infinite.

We may prove, also, that the sum of the infinite series, $\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \&c.$ is infinite, by transposing lx in formula (6), page 53; as, taking $M = 1$, we thus obtain

$$\log(x+1) - \log x, \text{ or } \log\left(1+\frac{1}{x}\right) = 2\left\{\frac{1}{2x+1} + \frac{1}{3}\left(\frac{1}{2x+1}\right)^3 + \&c.\right\}$$

Now, when $x = 0$, the first member of this is infinite, while the second becomes $2(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \&c.)$; whence the proof is evident.

the sum of which is also found, TRIG, No. 217. In this case, we have

$$\Sigma u_{x+1} = \Sigma \sin(x+1)\theta = C - \frac{\cos(x+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta} \text{ (No. 373)};$$

$$\Sigma u_1 = C - \frac{\cos\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta};$$

$$S_x = \Sigma u_{x+1} - \Sigma u_1 = \frac{\cos\frac{1}{2}\theta - \cos(x+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta} = \frac{\sin\frac{1}{2}x\theta\sin\frac{1}{2}(x+1)\theta}{\sin\frac{1}{2}\theta}.$$

The sum of the cosines might be found in a similar manner.

385. The method that has now been explained, while it enables us to sum a great many series, yet fails in numberless instances, in consequence of the limited advances that have yet been made in the Inverse Calculus of Differences; as it succeeds only when we are able to integrate the general term of the series. By the Direct Method, however, we may find as many series as we please, which may be summed by the Inverse Method; and thus, though we may not be able to sum series that may be proposed, we may discover series which we can sum, that may be curious in their form, or that may possess interesting properties. To find series in this way, from assuming their sum, we are merely to find the difference of some assumed function of x, which becomes nothing when $x=0$, and in that difference to take x successively equal to 0, 1, 2, 3, &c. Then, the values thus obtained will form a series, the sum of which will be the assumed function.

As an instance we have

$$\Delta\, 3^x \sin\frac{\theta}{3^x} = 3^{x+1}\sin\frac{\theta}{3^{x+1}} - 3^x\sin\frac{\theta}{3^x}$$

$$= 3^x\left(3\sin\frac{\theta}{3.3^x} - \sin\frac{\theta}{3^x}\right) = 3^x . 4\sin^3\frac{\theta}{3^{x+1}};$$

the last modification being made by means of the formula, $\sin 3A = 3\sin A - 4\sin^3 A$ (TRIG. No. 196). Hence, suppressing the multiplier 4, and taking x successively equal to 0, 1, 2, 3, &c. we get

$$\sin^3\frac{\theta}{3} + 3\sin^3\frac{\theta}{3^2} + 3^2\sin^3\frac{\theta}{3^3} + \ldots\ldots + 3^{x-1}\sin^3\frac{\theta}{3^x};$$

the sum of which, by No. 375, in connexion with the difference just found, or rather with its integral, is

$$\frac{1}{4}\left(3^x \sin\frac{\theta}{3^x} - \sin\theta\right).$$

If x, and consequently the series itself, be taken infinite, the arc in the first term of the sum now found will become infinitely small; and its sine may therefore be taken equal to the arc itself. By this means, we shall have the infinite series,

$$\sin^3\frac{\theta}{3} + 3\sin^3\frac{\theta}{3^2} + 3^2\sin^3\frac{\theta}{3^3} + 3^3\sin^3\frac{\theta}{3^4} + \&c. = \tfrac{1}{4}(\theta - \sin\theta);$$

a formula from which the length of the circumference of a circle might be found by means of the sines of an arc, and of the arcs obtained from its continual trisection. The computation, however, would be laborious.

EXERCISES IN THE SUMMATION OF SERIES.

1. $2+5+9+19+40+77+$ &c. (to x terms)* $=$

$$\frac{5x(x-1)(x-2)(x-3)}{2.3.4} + \frac{x(x-1)(x-2)}{2.3} + \frac{3x(x-1)}{2} + 2x.$$

2. $S_x = 1+5+15+35+70+126+$ &c.

$$= x + x(x-1)\left\{x + \frac{(x-2)(x-3)}{2.3} + \frac{(x-2)(x-3)(x-4)}{2.3.4.5}\right\}.$$

3. $S_x = \frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} + \frac{1}{4.6} + \frac{1}{5.7} +$ &c.

$$= \frac{3}{4} - \frac{2x+3}{2(x+1)(x-2)} = \frac{x(3x+5)}{4(x+1)(x+2)}; \quad S = \tfrac{3}{4}.$$

4. $S_x = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} +$ &c. $= \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2x+1} = \frac{x}{2x+1}$†

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* By using $x-1$ for x in series A, No. 351, in the manner pointed out in No. 353, we may find the general term of this to be

$$\frac{5(x-1)(x-2)(x-3)}{1.2.3} + \frac{(x-1)(x-2)}{1.2} + 3x - 1.$$

† In this exercise we have plainly $S = \frac{1}{2}$; in the next $S = \frac{n}{n+1}$; and in Exer. 7, $S = \frac{3}{2}$.

5. $S_x = 1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} + \&c. = \frac{n}{n+1}\left\{1 - \frac{1}{n^x}.(-1)^x\right\}.$

6. $S_x = \frac{1}{2.4} + \frac{1}{4.6} + \frac{1}{6.8} + \&c. = \frac{1}{4}\cdot\frac{x}{x+1}; \quad S = \frac{1}{4}.$

7. $S_x = \frac{5}{1.2.3} + \frac{6}{2.3.4} + \frac{7}{3.4.5} + \&c. = \frac{3}{2} - \frac{x+3}{(x+1)(x+2)}.$

8. Find the series in which $S_x = 4\left(\text{cosec}^2\theta - \frac{1}{4^x}\text{cosec}^2\frac{\theta}{2^x}\right)$,

and, consequently, $S = 4\left(\text{cosec}^2\theta - \frac{1}{\theta^2}\right)$.

Ans. $\sec^2\frac{1}{2}\theta + \frac{1}{4}\sec^2\frac{1}{4}\theta + \frac{1}{16}\sec^2\frac{1}{8}\theta + \frac{1}{64}\sec^2\frac{1}{16}\theta + \&c.$

9. Given $S_x = \frac{1}{8^x}\cdot\frac{2\cos\frac{\theta}{2^x}}{\sin^3\frac{\theta}{2^x}} - \frac{2\cos\theta}{\sin^3\theta}$, and, consequently,

$S = \frac{2}{\theta^3} - \frac{2\cos\theta}{\sin^3\theta}$; to find the equivalent series.

Ans. $\frac{\text{versin}\,\theta}{2\sin\theta + \sin 2\theta} + \frac{1}{8}\cdot\frac{\text{versin}\,\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta + \sin\theta} + \frac{1}{8^2}\cdot\frac{\text{versin}\,\frac{1}{4}\theta}{2\sin\frac{1}{4}\theta + \sin\frac{1}{2}\theta} + \&c.$

10. Prove by means of Ex. 10, page 228, that

$S_x = \sin\theta\sin^2\frac{1}{2}\theta + 2\sin\frac{1}{2}\theta\sin^2\frac{1}{4}\theta + 4\sin\frac{1}{4}\theta\sin^2\frac{1}{8}\theta + \&c.$

$= \frac{1}{2}\left(2^{x-1}.\sin\frac{\theta}{2^{x-1}} - \frac{1}{2}\sin 2\theta\right)$, and $S = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta$.

11. Prove by means of Ex. 6, page 228, that

$S_x = \tan^{-1}\frac{1}{1+1+1^2} + \tan^{-1}\frac{1}{1+2+2^2} + \tan^{-1}\frac{1}{1+3+3^2} + \&c.$

$= \frac{1}{4}\pi - \tan^{-1}\frac{1}{x+1}$, and $S = \frac{1}{4}\pi$.

12. From the following enumerations of the population

of Scotland, find what it may be expected to be in 1850 and 1851.

In 1801,	1,599,000;	In 1831,	2,365,000;
1811,	1,806,000;	1841,	2,620,000.
1821,	2,093,000;		

Ans. 2,908,255 in 1850, and 2,949,000 in 1851.

XXIII.—VARIATIONS OF TRIANGLES.

386. The differentiation of the fundamental formulas in Trigonometry, forms an interesting and useful application of some of the principles established in the earlier parts of the present work. The following are the most important results thus obtained. Their investigations are given in the tenth chapter of Délambre's "Astronomie Théorique et Pratique." To give them here would occupy more space than is consistent with the design of the present publication. The notes exhibit, however, such outlines of modes of investigation—often shorter and easier than those given by Delambre—as will render it easy for the student to follow out the processes at full length.

387. Of the three sides, a, b, c, of a spherical triangle, and of the three angles, A, B, C, respectively opposite to them, if any three be given or constant, the rest are determined. If only two be constant, however, all the remaining four quantities are variable; and, as six different combinations of two quantities may be made out of four, it follows, that whatever two of the six quantities are taken as constant, there will be six formulas expressing the variations of the rest. Thus, a and b being constant, c, A, B, C, will be variable, and the combinations that can be made of these by pairs are c, A; c, B; c, C; A, B; A, C; and B, C. Now, the constant quantities may be 1. Two sides; 2. Two angles; 3. A side and the adjacent angle; and 4. A side and the opposite angle. Hence, the entire number of formulas expressing the variations is twenty-four.

VARIATIONS OF SPHERICAL TRIANGLES.*

I. *a and b constant.* (*Two sides.*)

$$1.\ \frac{dc}{dC} = \sin A \sin b = \sin B \sin a \dagger$$

$$\left.\begin{array}{l} 2.\ \dfrac{dA}{dc} = -\dfrac{\cot B}{\sin c} = -\dfrac{\cos B}{\sin C \sin b} \ddagger \\ 3.\ \dfrac{dB}{dc} = -\dfrac{\cot A}{\sin c} = -\dfrac{\cos A}{\sin C \sin a} \end{array}\right\}$$

$$4.\ \frac{dA}{dB} = \frac{\tan A}{\tan B}$$

$$\left.\begin{array}{l} 5.\ \dfrac{dA}{dC} = -\dfrac{\sin A \cos B}{\sin C} = -\dfrac{\sin a \cos B}{\sin c} \\ 6.\ \dfrac{dB}{dC} = -\dfrac{\sin B \cos A}{\sin C} = -\dfrac{\sin b \cos A}{\sin c} \end{array}\right\}$$

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* The following are the fundamental formulas of Spherical Trigonometry. Their investigations will be found in the third section of the author's work on Trigonometry. They are marked, for the sake of reference, with the numbers attached to them in that publication.

"$\cos a = \cos A \sin b \sin c + \cos b \cos c$ .... (85)
$\cos b = \cos B \sin a \sin c + \cos a \cos c$ .... (86)
$\cos c = \cos C \sin a \sin b + \cos a \cos b$ .... (87)

"$\dfrac{\sin A}{\sin a} = \dfrac{\sin B}{\sin b} = \dfrac{\sin C}{\sin c}$ .... (94)

"$\cot a \sin b = \cot A \sin C + \cos C \cos b$ .... (95)
$\cot a \sin c = \cot A \sin B + \cos B \cos c$ .... (96)
$\cot b \sin a = \cot B \sin C + \cos C \cos a$ .... (97)
$\cot b \sin c = \cot B \sin A + \cos A \cos c$ .... (98)
$\cot c \sin a = \cot C \sin B + \cos B \cos a$ .... (99)
$\cot c \sin b = \cot C \sin A + \cos A \cos b$ .... (100)

"$\cos A = \cos a \sin B \sin C - \cos B \cos C$ .... (101)
$\cos B = \cos b \sin A \sin C - \cos A \cos C$ .... (102)
$\cos C = \cos c \sin A \sin B - \cos A \cos B$ .... (103)"

† To obtain this expression, differentiate (87), last note, taking a and b constant; then, by changing the signs, $\sin c\, dc = \sin C\, dC \sin a \sin b$. Hence, the formulas in the text are obtained by dividing by $\sin c\, dC$, and modifying the result by (94).

‡ The formula, which, besides a and b, contains also c and A, is (85). Differentiating this formula, therefore, we obtain, after transposing, and dividing by $\sin A \sin b \sin c\, dc$,

$$\frac{dA}{dc} = \frac{\cos A \sin b \cos c - \cos b \sin c}{\sin A \sin b \sin c} = -\frac{\cot b \sin c - \cos A \cos c}{\sin A \sin c}.$$

Now, the numerator of the second member is equal to $\cot B \sin A$, as appears from (98) by transposition; whence $\dfrac{dA}{dc} = -\dfrac{\cot B}{\sin c}$. The other value is found by multiplying the numerator and denominator by $\sin B$, and substituting, according to (94), $\sin C \sin b$ for $\sin c \sin B$. The next formula is derived, in a similar manner, from (86). Formula 4 is found by dividing formula 2 by formula 3, and multiply-

II. A *and* B *constant.* (*Two angles.*)

7. $\frac{dC}{dc} = \sin b \sin A = \sin B \sin a$*

$$\left.\begin{array}{l} 8.\ \frac{da}{dC} = \frac{\cot b}{\sin C} = \frac{\cos b}{\sin c \sin B} \\ 9.\ \frac{db}{dC} = \frac{\cot a}{\sin C} = \frac{\cos a}{\sin c \sin A} \end{array}\right\}$$

10. $\frac{da}{db} = \frac{\tan a}{\tan b}$

$$\left.\begin{array}{l} 11.\ \frac{da}{dc} = \frac{\sin A \cos b}{\sin C} = \frac{\sin a \cos b}{\sin c} \\ 12.\ \frac{db}{dc} = \frac{\sin B \cos a}{\sin c} = \frac{\sin b \cos a}{\sin C} \end{array}\right\}$$

III. *a and* B *constant.* (*A side and the adjacent angle.*)

13. $\frac{db}{dc} = \cos A$†

14. $\frac{db}{dA} = -\frac{\tan b}{\tan A}$

15. $\frac{dA}{dC} = -\cos b$

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ing the numerator and denominator by tan A tan B; formula 5, by taking the product of 1 and 2; and formula 6, by taking the product of 1 and 3. The other modifications are made by (94). It will be readily perceived, that formulas 2 and 3 do not exhibit distinct properties, but that they contain merely the same principle applied to different parts of the triangle. The same is the case with respect to every pair of formulas in this section, connected by a circumflex.

* Formula 7 is derived from (103). To find formula 8, differentiate (101), and divide by $\sin a \sin B \sin C\, dC$; then,

$$\frac{da}{dC} = \frac{\cos a \cos C + \cot B \sin C}{\sin a \sin C};$$

for the numerator of which, substitute its equal according to (97), and the other modifications are easy. Formula 9 is derived, in a similar manner, from (102).

To find 10, divide 8 by 9; to get 11, take the product of 7 and 8; and to find 12, take the product of 7 and 9.

† To obtain formula 13, differentiate (86), taking a and B constant; then, by dividing by $-\sin b\, dc$, we get

$$\frac{db}{dc} = \frac{\cos a \sin c - \cos B \sin a \cos c}{\sin b} = \frac{\sin a (\cot a \sin c - \cos B \cos c)}{\sin b};$$

and, if this be modified by (96), the rest is easy.

Formula 14 is obtained by differentiating $\sin a \sin B = \sin b \sin A$. To find 15, differentiate (101), and apply (97) in the same manner in which (96) was applied in deriving 13.

The product of 14 and 15 gives 16, while the division of 13 by 14 gives 17; and that of 17 by 15 gives 18.

16. $\frac{db}{dC} = \frac{\sin b}{\tan A}$

17. $\frac{dA}{dc} = -\frac{\sin A}{\tan b}$

18. $\frac{dC}{dc} = \frac{\sin A}{\sin b}$

IV. *a and* A *constant.* (*A side and the opposite angle.*)

19. $\frac{dc}{db} = -\frac{\cos C}{\cos B}$*

$$\left.\begin{array}{l} 20.\ \frac{db}{dB} = \frac{\tan b}{\tan B} \\ 21.\ \frac{dc}{dC} = \frac{\tan c}{\tan C} \end{array}\right\}$$

22. $\frac{dB}{dC} = -\frac{\cos b}{\cos c}$

$$\left.\begin{array}{l} 23.\ \frac{dC}{db} = -\frac{\cos c \tan B}{\sin b} = -\frac{\cot c \sin C}{\cos B} \\ 24.\ \frac{dB}{dc} = -\frac{\cos b \tan C}{\sin c} = -\frac{\cot b \sin B}{\cos C} \end{array}\right\}$$

388. The formulas given above hold true in all spherical triangles. In rightangled triangles, they admit of several modifications, by means of the formulas, Trig. No. 84. It may be also remarked, that the first six are not admissible in rightangled triangles; since, if the right angle and two sides be given, the triangle is determined.

389. Most of the same formulas will be applicable with respect to plane triangles, if—as is pointed out, Trig. No. 162—the terms *sine* and *tangent*, before a side, be omitted,

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* To investigate 19, differentiate (85), taking a and A constant; and, after transposition and division, there will result

$$\frac{dc}{db} = -\frac{\cos A \cos b \sin c - \sin b \cos c}{\cos A \cos c \sin b - \sin c \cos b} = -\frac{\sin c}{\sin b} \cdot \frac{\cos A \cos b - \sin b \cot c}{\cos A \cos c - \sin c \cot b}.$$

Then, by applying (100) and (96), the rest of the process is easy.

Formulas 20 and 21 are both derived from (94). Formula 23 is obtained by dividing 19 by 21; and 24, by multiplying 19 by 20, and taking the reciprocal. To obtain 22, multiply 24 by 21.

and the cosine be taken equal to the radius. Thus, formula 5 is reduced to $\frac{d\,A}{d\,C}=-\frac{a\cos B}{c}$; while 15, which is the same as $\frac{d\,A}{d\,C}=-\frac{\cos b}{R}$, will become $\frac{d\,A}{d\,C}=-\frac{1}{1}$, showing that the variations of the angles are equal, the one diminishing as much as the other increases, as is indicated by the sign *minus*. When the differential of a side of a plane triangle is compared with that of an angle, the latter must first be divided by the radius. Thus, formula 1, by supplying the radius and multiplying by dC, becomes $dc=\frac{b\sin A}{R}\cdot\frac{d\,C}{R}$.*

390. As an example of the application of the principles that have now been established, let the angle B and the side a be given, and let it be required to find the change produced on the side c, by a change in the opposite angle C.

The formula applicable in solving this problem is 18, which, in a plane triangle, becomes $dc=\frac{b\,d\,C}{\sin A}$. Hence, multiplying the numerator and denominator by $2\sin C$, and substituting in the numerator $c\sin B$ for its equal $b\sin C$, we get $dc=\frac{2c\sin B\,d\,C}{2\sin A\sin C}$, or (Trig. No. 24)

$$dc=\frac{2c\sin B\,d\,C}{\cos(A-C)-\cos(A+C)}=\frac{2c\sin B\,d\,C}{\cos(A-C)+\cos B}.$$

Now, the less the difference of A and C is, the greater is $\cos(A-C)$; and hence we infer, that the denominator will

* To give an example of this in numbers, let $a=5738$, $b=4260$, and $C=56°\ 45'$, which (Trig. No. 61) give $A=76°\ 56'$, and $c=4926$; and let it be required to find the increase of c corresponding to an increase of half a degree in C. Now, (No. 108) the length of the circumference to the radius 1 is 6·283185 nearly; and, the number of seconds in the circumference being $360\times60\times60$, we find, by proportion, that the radius is equal in length to 206265 seconds very nearly. Hence, therefore, using this for the radius, and $1800''$ $(=30')$ instead of d C, we get, by the formula in the text, the logarithm of $d\,c$, by adding together $\log b$, $\log\sin A$, and log 1800, and subtracting from the sum $10+\log 206265$. By this means, $d\,c$ is found to be 36·2; and, adding this to 4926, we get 4962·2, the value of c when C becomes $57°\ 15'$.

It may be observed, that some of the formulas are not applicable to plane triangles. Such is 7, which is derived from a formula that does not belong to such triangles; and, in 8 and 9, the cotangent becomes infinite, showing that the differential of a or b is infinitely greater than that of C, which must therefore be nothing; so that C is constant, which we know also from Euc. I. 32.

be the greatest possible, and dc the least possible, when $A=C$, and consequently $a=c$. Taking $A=C$, therefore, we get $dc=\frac{2c\sin B\,dC}{1+\cos B}=2c\tan\frac{1}{2}B dC$. If $B=90^\circ$, $dc=\frac{2c\,dC}{\cos(A-C)}$; or, when dc is the least possible, $dc=2c\,dC$.*

391. As another example, let it be required to compare the variations of the sun's altitude, and the corresponding time at a given place, on a given day. Here (TRIG. No. 172), if we put a for the polar distance, b for the colatitude, C for the hour angle, and, consequently, A for the azimuth, and c for the zenith distance, we have, from formula 1, $d\,C=\frac{1}{\sin b}.\frac{1}{\sin A}.dc$. Hence it appears, that the error in the time computed from the altitude will exceed the error in the altitude, in the compound ratio of the cosine of the latitude to the radius, and of the sine of the azimuth to the radius. The less the latitude, therefore, and the nearer the body is to the prime vertical, the less is the error in the computed time, other things being alike.†

392. If, in the last No. we suppose the zenith distance $c=90^\circ$, the triangle becomes quadrantal, and we have (TRIG. No. 109) $\cos a=\sin b\cos A$. Hence, by dividing by $\sin b$, we get the value of $\cos A$; by taking the square of which from 1, extracting the square root, and multiplying by $\sin b$, we get

$$\sin b\sin A=\surd(\sin^2 b-\cos^2 a)=\surd(\cos^2 l-\sin^2 d),$$

where l is the latitude, and d the declination; and this again, TRIG. page 109, Exercise 1, (2), is equivalent to

* Hence it appears, that an error made in observing the angle C, produces the least effect on the computed length of B A, when C and A are equal; that is, when each is the complement of half the angle B. If, therefore, B be a right angle, as in measuring the height of an object on a horizontal plane, the nearer the angle C is to 45°, the less effect will an error in the measurement of that angle produce on the height as obtained by computation. If d C be one minute, we have $dc=\frac{2c\times 60''}{206265''}=\frac{120c}{206265}=\frac{c}{1719}$ nearly; so that an error of one minute in the angle would produce an error in the computed height of a 1719th part of itself.

† The student will find it easy to show, that the error in seconds of time arising from an error of s seconds in altitude will be $\frac{s}{15\cos\text{lat.}\sin azim.}$ and that, for instance, the error at the latitude of 55° 52′ will be $0{\cdot}12\times s\times\text{cosec}\,azim.$ nearly.

$\sqrt{\cos(l+d)\cos(l-d)}$. Substituting this in the formula in the last No. we get $dC = \frac{dc}{\sqrt{\cos(l+d)\cos(l-d)}}$, a formula which shows the change produced on the hour angle, by a small change of altitude or depression, about the time of rising or setting; and it is reduced to time by dividing by 15, because fifteen degrees correspond to an hour.* The variation dC is evidently least when the declination is nothing.

393. It is to be recollected, that since, when these formulas are employed in computations, *finite differences* are used instead of *differentials*, the results will not be rigorously true, but merely approximations. When the variations are small, however, in comparison of the other quantities, the errors are minute, and the results are obtained with much more ease than they would be by means of the ordinary modes of resolving triangles.

XXIV.—MISCELLANEOUS INVESTIGATIONS.

394. Given the latitude of a place, and two circles parallel to the horizon; to find the declination of a body which, in its apparent diurnal motion, will pass from one of them to the other in the shortest time possible.†

Let Z and P (*fig.* 54) be the zenith and pole, and S and

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* As an example of the use of this, let it be required to find what time is occupied, at Belfast, on the twenty-first of December, by the body of the sun, in rising or setting; that is, the time between the instants at which the upper limb and the lower are on the horizon. Here we have the latitude $= 54^\circ\ 36'$; and, by the tables in the Nautical Almanack and other books, the sun's declination is $23^\circ\ 28'$, and his diameter $32'\ 35''$, or $1955''$. Hence, we have $dc = 1955''$, $l + d = 78^\circ\ 4'$, and $l - d = 31^\circ\ 8'$. Then, by taking half the sum of the logarithmic cosines of $l + d$ and $l - d$, and adding to the result the logarithm of 15, we get 0·80006; and, by taking this from the logarithm of 1955, we obtain 2·49109, the number answering to which is 310 nearly, the number of seconds required. Hence the time is $5^m\ 10^s$.

† "A simple case of this problem, viz. to find the day of shortest twilight in a given latitude, employed, for several years, the two brothers, James and John Bernoulli, without success. By treating it algebraically, they were led to an equation of the fourth order, in which they were embarrassed to separate the useful roots from those which ought to be rejected; but, afterwards, by employing the synthetic method, they separately obtained answers very convenient for astronomical computation. In the year 1780, Fontaine attempted a solution by algebraic analysis. In this manner, he obtained an equation of the fourth order, which he required twenty quarto pages to reduce and explain."—The foregoing extract is taken from a paper in the Mathematical Companion for 1805. Much information on the subject, with several solutions of the problem, will be found collected in the fourth volume of Leybourn's edition of the Mathematical Questions propose din the Ladies' Diary. The easy method here given of solving this and similar problems, was first pointed out by the Author of this work, in the Belfast Diary for 1828 and 1829.

S′ the required points on the given parallels, having equal polar distances, P S and P S′. Now, since the time of describing the arc S S′ is a minimum, the angle S P S′ must also be a minimum. Hence (Sect. VII.),

$$\frac{d\,\mathrm{SPS}'}{dx}, \text{ or } \frac{d(\mathrm{P}'-\mathrm{P})}{dx}=0; \text{ whence } \frac{d\mathrm{P}'}{dx}=\frac{d\mathrm{P}}{dx}:$$

where P and P′ denote the angles Z P S and Z P S′, and x the polar distance P S or P S′. Now, putting the latitude $=l$, $\mathrm{ZS}=a$, and $\mathrm{ZS}'=a'$, we have (formula 2, page 247)

$$\frac{d\mathrm{P}}{dx}=-\frac{\cot \mathrm{S}}{\sin x}, \qquad \text{and } \frac{d\mathrm{P}'}{dx}=-\frac{\cot \mathrm{S}'}{\sin x};$$

and, therefore, by what we have just seen,

$$\frac{\cot \mathrm{S}}{\sin x}=\frac{\cot \mathrm{S}'}{\sin x}; \quad \text{whence, } \mathrm{S}=\mathrm{S}'.$$

Now, in the triangles P Z S and P Z S′, we have (TRIG. No. 71)

$$\cos \mathrm{S}=\frac{\sin l-\cos a \cos x}{\sin a \sin x}, \quad \text{and } \cos \mathrm{S}'=\frac{\sin l-\cos a' \cos x}{\sin a' \sin x}.$$

Putting the second members of these equal to one another, and multiplying by $\sin x$, $\sin a$, and $\sin a'$, we obtain, after transposition,

$$(\sin a' \cos a-\cos a' \sin a) \cos x=(\sin a'-\sin a)\sin l, \text{ or}$$
$$\sin (a'-a) \cos x=(\sin a'-\sin a)\sin l;$$

whence, by dividing by $\sin (a'-a)$, and by TRIGONOMETRY, No. 28.

$$\cos x=\frac{\cos\frac{1}{2}(a'+a)}{\cos\frac{1}{2}(a'-a)}.\sin l,$$

where $\cos x$ being the cosine of the polar distance, is the sine of the declination.

If $a=\frac{1}{2}\pi$, and $a'=\frac{1}{2}\pi+2b$, this becomes $\sin \text{dec.}=-\tan b \sin l$. This formula solves the well-known problem in which it is required to determine the time at which the twilight is shortest in a given latitude, $2b$ denoting the sun's depression below the horizon, at the beginning of the morning, or the end of the evening, twilight. If $2b$, as is generally supposed, be 18^0, the result which we have obtained may be expressed by the following analogy, in which the negative sign shows that the latitude and the required declination are of contrary kinds:

Radius : sin lat. : : tan 9° : —sin dec.

395. "Given the latitude of the place, and the positions of two hour circles with respect to the meridian; to determine the declination of that star whose change in altitude shall be the greatest possible in passing over the interval between those hour circles."

Here, in addition to the notation adopted in the last No. let Z S and Z S′ (*fig.* 54) be represented by z and z'; and the angles P Z S and P Z S′, by Z and Z′. Then, since Z S′— Z S is a maximum, it might be shown, as in the last problem, and by formula 13, page 248, that

$$\frac{dz'}{dx}=\frac{dz}{dx}, \qquad \frac{dz'}{dx}=\cos S', \quad \text{and} \frac{dz}{dx}=\cos S;$$

whence $\cos S'=\cos S$, and $S'=S$. Now (Trig. No. 77), we have

$$\cot S=\frac{\tan l \sin x-\cos P \cos x}{\sin P},$$

$$\text{and } \cot S'=\frac{\tan l \sin x-\cos P' \cos x}{\sin P'}.$$

By putting these equal to each other, dividing by $\sin x$, multiplying by the denominators, and transposing, we get

$$(\sin P' \cos P-\cos P' \sin P) \cot x=(\sin P'-\sin P)\tan l, \text{ or}$$

$$\sin (P'-P) \cot x=(\sin P'-\sin P) \tan l.$$

Hence we obtain, by dividing by $\sin (P'-P)$, and by Trigonometry, No. 28,

$$\cot x, \quad \text{or } \tan \text{dec.}=\frac{\cos \frac{1}{2}(P'+P)}{\cos \frac{1}{2}(P'-P)}.\tan l.$$

This question is taken from Gregory's Trigonometry, page 243, where an erroneous answer,

$$\tan \text{dec.}=\frac{\sin \frac{1}{2}(P'-P)}{\sin \frac{1}{2}(P'+P)}.\tan l, \text{ is given.}$$

Since $S=S'$, it would appear, by means of the formulas of the four sines (Trig. No. 76), that $Z=\pi-Z'$; whence it appears, that the azimuths of the body at the required positions are supplements of each other. It is also plain, that, if the angle S P S′ be bisected by the arc P Q, we shall have $\tan \text{dec.} = \frac{\cos \text{Z P Q}}{\cos \text{S P Q}} . \tan l$; that, at the equator, where $l = 0$, the declination must be nothing for every value of P and P′; and that, if P = P′, the formula will become $\tan \text{dec.} = \cos \text{P} \tan l$, an expression which will determine the declination of a star that, in crossing a given hour circle, will be increasing or diminishing its altitude more rapidly than any other star would in crossing the same hour circle. In this last case, it is evident, from the equation, $\text{Z} = \pi - \text{Z}$, that the star will cross the given hour circle and the prime vertical at the same time. Similar results for particular cases, besides that of the shortest twilight, might be derived, in a similar manner, from the solution of the problem in the last No. Other questions of this kind will be found in the next Section.

396. Let the points D and C (*fig.* 47) move uniformly, and with equal velocities, on the perpendicular lines A O and O B, in the directions O B and A F; required the equation of the curve to which the straight line joining C and D is always a tangent.

Let P be the point of contact, and put $\text{O R} = x$, $\text{R P} = y$, and $\text{C O} + \text{O D} = a$. Then (No. 146),

$$\text{D R} = -\frac{y\,dx}{dy}, \quad \text{and C G} = -\frac{x\,dy}{dx};$$

and, therefore, by the nature of the question,

$$x + y - \frac{x\,dy}{dx} - \frac{y\,dx}{dy} = a.$$

The integration of this is easily effected by first differentiating it, as, by taking dx constant, it thus becomes

$$dx + dy - dy - \frac{x\,d^2 y}{dx} - dx + \frac{y\,dx\,d^2 y}{dy^2} = 0.$$

By contracting this, &c. we get

$$\frac{dy}{y^{\frac{1}{2}}} = \pm \frac{dx}{x^{\frac{1}{2}}}; \text{ and thence, } y^{\frac{1}{2}} = \pm x^{\frac{1}{2}} + a^{\frac{1}{2}};$$

the constant quantity, $a^{\frac{1}{2}}$, being assumed so that y may be equal to a when $x = 0$. By squaring this, we obtain $y = x + a \pm 2 a^{\frac{1}{2}} x^{\frac{1}{2}}$, which is the equation of a parabola having, for its axis, the line bisecting the angle A O B, and such that its chords, parallel to O B, are bisected by a diameter drawn through B.

397. Required the nature of the curve (*fig.* 48) to which the straight line C D, given in length, is always a tangent, C moving along the straight line B F, and D on A E, which is perpendicular to B F.

Here, putting $OR = x$, $RP = y$, and $CD = a$, we have

(No. 146) $RD = -\frac{y\,dx}{dy}$; whence

$$PD = -\sqrt{\left(\frac{y^2 dx^2}{dy^2} + y^2\right)} = -\frac{y\sqrt{(dx^2+dy^2)}}{dy}, \text{ and}$$

$$OD = x - \frac{y\,dx}{dy} = \frac{x\,dy - y\,dx}{dy}.$$

But RD : PD : : OD : CD; that is,

$$-\frac{y\,dx}{dy} : -\frac{y\sqrt{(dx^2+dy^2)}}{dy} :: \frac{x\,dy - y\,dx}{dy} : a.$$

Hence we easily find $\frac{a\,dx}{\sqrt{(dx^2+dy^2)}} = x - \frac{y\,dx}{dy}$; whence, by differentiating and contracting, there is obtained

$$-\frac{a\,dy}{(dx^2+dy^2)^{\frac{3}{2}}} = \frac{y}{dy^2}:$$

and from this we get

$$-\frac{(dx^2+dy^2)^{\frac{3}{2}}}{a} = \frac{dy^3}{y}, \quad \text{and} -\frac{(dx^2+dy^2)^{\frac{1}{2}}}{a^{\frac{1}{3}}}, \text{ or} -\frac{ds}{a^{\frac{1}{3}}} = \frac{dy}{y^{\frac{1}{3}}};$$

where s (No. 178) denotes the arc of the curve corresponding to the co-ordinates x and y. Hence, by integration, and by assuming the constant so as to give $s = 0$, when $x = 0$ and $y = a$, we get, for the equation of the required curve, $\frac{3}{2} a^{\frac{1}{3}} y^{\frac{2}{3}} = \frac{3}{2} a - s$. We may also obtain the equation in another form, by dividing the members of the equation, $\frac{a\,dx}{\sqrt{(dx^2+dy^2)}} = x - \frac{y\,dx}{dy}$, by dx, and multiplying by dy; as, by performing on the result an operation in every respect similar to the foregoing, we should find $\frac{3}{2}a^{1/3}x^{2/3}=s$. Then, by eliminating s from these two equations, we get $x^{2/3}+y^{2/3}=a^{2/3}$, the equation for rectangular co-ordinates; an equation which indicates a line of the sixth order. This curve, both in its form and equation, resembles the evolute of the ellipse (No. 198).

From what we have seen, it appears, that the curve is quadrable; and that the length of APB, one of its four equal and similar parts, is $\frac{3}{2}a$.

The student will find it easy to prove, that the radius of the osculating circle is equal to $3(axy)^{1/3}$, which is a very elegant property of this curve.

398. Suppose the interest of one pound, for the xth part of a year, to be the xth part of a given number r; it is required to find its interest for a year at compound interest, when x becomes infinitely great; or, which amounts to the same, when the interest begins to bear interest each instant as it becomes due.

Putting r' to denote the amount of one pound at the end of a year, we have, by the theory of compound interest, (ALG. p. 272),

$$r'=\left(1+\frac{r}{x}\right)^x; \text{ whence } \log r'=x\log\left(1+\frac{r}{x}\right).$$

By expanding the second member of this by No. 88, and taking x infinite in the result, so that the terms which contain x in the denominator may vanish, we get $\log r'=r$; and, consequently, $r'=\varepsilon^r$, or, by No. 93,

$$r'=1+r+\frac{r^2}{2}+\frac{r^3}{2.3}+\frac{r^4}{2.3.4}+\text{ \&c.}$$

which is the required value.

Hence, if $r=\cdot05$, we should easily find $r'=1\cdot051271$, nearly; so that, if interest be payable *momently* at 5 per cent, per annum, the interest of £100, at the end of a year, instead of being £5, would be £5 2*s*. 6½*d*. For ·06, we should find, in like manner, 1·06183654 and £6 3*s*. 8*d*.

This problem might also be solved by expanding

$$r'=\left(1+\frac{r}{x}\right)^x, \text{ or } r'=\frac{(\quad+\quad)}{x^x},$$

by the binomial theorem, and taking x infinite in the result.

399. To find the weight of a column of uniform density and thickness, rising with its axis perpendicular to the earth's surface, and of given or of infinite height.

Let r be the earth's radius, x any height above the surface, m the specific gravity, or the weight of a horizontal lamina, of the column at the surface of the earth, and w the required weight. Then, since the earth's attraction is inversely proportional to the square of the distance from the centre, we have

$$\frac{1}{r^2} : \frac{1}{(r+x)^2} :: m : \frac{r^2 m}{(r+x)^2},$$

the weight of a like lamina at the height x. Multiplying this by dx, and integrating, we obtain

$$w = rm - \frac{r^2 m}{r+x} = \frac{rx}{r+x}.m,$$

the constant quantity C being taken equal to rm, so that w may vanish when $x=0$.

If the height, x, be infinite, we have simply $w=rm$; which shows, that the actual weight, or pressure, of a column of infinite height, would be the same as that of one on the same base, of an altitude equal to the earth's radius, and of uniform weight throughout; that is, having its density increasing as the square of the distance from the earth's centre increases.

400. Let half the greater axis of an ellipse $=1$, and the eccentricity $=e$; required the length of any assigned portion of the curve.

Here (App. No. 3), since $b^2=1-e^2$, we have $y^2=(1-e^2)(1-x^2)$; whence, by differentiating, squaring, dividing one member of the result by $4y^2$, and the other by its equal, $4(1-e^2)(1-x^2)$, and by adding dx^2 to the result, we get (No. 178), by extraction, $ds = \frac{\sqrt{(1-e^2x^2)}\,dx}{\sqrt{(1-x^2)}}$. To integrate this, expand the numerator by means of the binomial theorem, and there will result

$$ds = \frac{dx}{\sqrt{(1-x^2)}}\left\{1 - \frac{1}{2}e^2x^2 - \frac{1.1}{2.4}e^4x^4 - \frac{1.1.3}{2.4.6}e^6x^6 - \&c.\right\}.$$

Now, the terms of the second member, exclusive of their coefficients, are all of the form, $\frac{x^m dx}{\sqrt{(1-x^2)}}$, the method of integrating which is shown in No. 279; and, therefore, applying the method there given, we get

$$s=\sin^{-1}x+\tfrac{1}{2}e^2\{\tfrac{1}{2}x\sqrt{(1-x^2)}-\tfrac{1}{2}\sin^{-1}x\}$$

$$+\frac{1.1}{2.4}e^4\left\{\left(\frac{1}{4}x^3+\frac{1.3}{2.4}x\right)\sqrt{(1-x^2)}-\frac{1.3}{2.4}\sin^{-1}x\right\}$$

$$+\frac{1.1.3}{2.4.6}e^6\left\{\left(\frac{1}{6}x^5+\frac{1.5}{4.6}x^3+\frac{1.3.5}{2.4.6}x\right)\sqrt{(1-x^2)}\right.$$

$$\left.-\frac{1.3.5}{2.4.6}\sin^{-1}x\right\}+\quad\&c.$$

This series, since e must be less than 1, and x cannot exceed 1, will always converge; and, if the ellipse be such that e is small, the convergence will be rapid. If we take $x=1$, we find the length of the elliptic quadrant to be

$$\tfrac{1}{2}\pi\left(1-\frac{1.1}{2.2}e^2-\frac{1.1.1.3}{2.2.4.4}e^4-\frac{1.1.1.3.3.5}{2.2.4.4.6.6}e^6-\&c.\right);$$

and hence, since the length of a quadrant of the circle is $\frac{1}{2}\pi$, we may have as near an approximation as we please to the ratio of the periphery of the circle to that of the ellipse, by computing the value of the series.

By putting $a=1$, and a^2+b^2, or $1+b^2=e^2$, we should find the differential of an arc of the hyperbola to be $\frac{\sqrt{(e^2x^2-1)}\,dx}{\sqrt{(x^2-1)}}$; the integral of which might be approximated in a similar manner.

401. Let A C B (*fig.* 55) be a vertical section of a hemispherical basin, having its diameter, A B, horizontal; and let D C be a straight rod of uniform density and thickness, and of a given length: it is required to find in what position the rod will rest, its motion not being impeded by the edge or bottom of the basin.

The centre of gravity of the rod will evidently be at G, its point of bisection; and, by an obvious mechanical principle, the rod will rest in equilibrium, when G is the lowest possible. Putting, therefore, $AB=a$, CG or $GD=b$, and $AC=x$, we readily find $EG=\frac{(x-b)\sqrt{(a^2-x^2)}}{a}$, which is to be a maximum. Squaring this, therefore, and rejecting the denominator, we get $u=(x-b)^2(a^2-x^2)$; by differentiating which, and proceeding according to No. 127, we get $x=\frac{1}{4}b+\frac{1}{4}\sqrt{(b^2+8a^2)}$, which will determine the required position. Should the value thus found be less than b, it is plain, that the point G would fall outside of the basin, and the rod could not rest in it, if allowed to move freely. It is also plain, that the locus of the point G is a curve, and that the solving of this problem is the same as finding the greatest ordinate of that curve.

Many other mechanical problems may be solved, on the principle here employed—that of finding the position of a body, when its centre of gravity is the highest or lowest possible; as it will then rest, if acted on by no disturbing force.

402. As another useful investigation, let it be required to find the positions of the parallels of latitude in Mercator's, or rather Wright's projection of the sphere. In this projection, so valuable in navigation, the meridians, instead of meeting at the poles, are parallel straight lines, placed at equal distances asunder. To compensate for the obvious error thus produced, the parallels of latitude, which are also represented by parallel straight lines, perpendicular to the meridians, are so placed, that the line on the chart representing any minute portion of the meridian, such as one minute, has to the line representing a like portion of the parallel, the same ratio as that of the lines themselves on the globe. Now, by the nature of the circle, the circumference of the equator, and that of any parallel of latitude, are in the same ratio as their radii, which is easily shown to be that of the radius to the cosine of the latitude; and the like parts of the circles are obviously in the same ratio. Hence, putting x to represent the true length of any portion of the meridian on the globe, commencing at the equator, and y to denote the corresponding portion of the enlarged meridian on the chart, we shall have $dx : dy :: \cos x : 1$; the ratio of dx to dy being that of the increment of x to the first term of the increment of y, or the ratio to which the increments themselves approach when they are taken indefinitely small. We have, therefore,

$$dy=\frac{dx}{\cos x}, \quad \text{and (No. 299) } y=\log\tan\tfrac{1}{2}(\tfrac{1}{2}\pi+x),$$

which requires no constant quantity to be annexed for assigning the length of the enlarged meridian, commencing at the equator, as it vanishes when $x = 0$.

This formula is adapted to the radius 1, and to Neperian logarithms. To adapt it to common logarithms, multiply (No. 91) by 2·302585; then

$$y = \{\text{com.}\log\tan(45^\circ + \tfrac{1}{2}x) - 10\} \times 2{\cdot}302585,$$

10 being taken from the index of the tabular logarithmic tangent, to reduce it to the radius 1 from the radius 10,000,000,000, for which (No. 115) it is computed. To render the formula applicable with respect to the earth, we are to multiply by the earth's radius in nautical miles, or, which is the same, to multiply by 10800, the minutes in 180°, and to divide by 3·141593. By actually performing the arithmetical operations here indicated, we get

$$y = \{\text{com.}\log\tan(45^\circ + \tfrac{1}{2}x) - 10\} \times 7915{\cdot}7;$$

and, instead of multiplying by 7915·7, we may divide by its reciprocal ·00012633. The numbers thus found are called *meridianal* or *meridional parts.* Hence, we have the following rule: *To find, in nautical miles, the meridianal parts, or the length of the enlarged meridian, corresponding to a given latitude, add half the latitude to* 45°, *from the index of the common logarithmic tangent of the sum subtract* 10, *and either multiply the remainder by* 7915·7, *or divide it by* ·00012633.

In the foregoing investigation, the earth has been taken as an exact sphere. The method of computing the meridianal parts, on the supposition that it is a spheroid, will be found in Simpson's and Maclaurin's Fluxions, and in several other works. As, however, the earth is very nearly spherical, the differences in the results are so trifling, that the meridianal parts found by the preceding rule, are constantly employed in the practice of navigation.

403. The object of another interesting problem is, to find a curve intersecting a series of other curves at a given angle; that is, crossing them in such a manner, that, at the point of intersection with any of the curves, the tangents of that curve, and of the curve to be found, may form an angle of a given magnitude.

To solve this, which is known by the name of the *Problem of Trajectories*, let AP (*fig.* 56) be one of the given curves, and BP the required one; and let PC and PD be their tangents at the point of intersection, and E P the ordinate of that point to the axis A E. Now if, to the same axis, we denote the coordinates of A P by x' and y', and those of the required curve, BP, by x and y, we have (No. 143) $\tan \mathrm{CPE} = \cot \mathrm{PCE} = \frac{dx'}{dy'}$, and $\tan \mathrm{DPE} = \cot \mathrm{PDE} = \frac{dx}{dy}$. But, putting t to express the tangent of the given angle C P D, since that angle is the difference of C P E and D P E, we have (TRIG. No. 30)

$$\tan \mathrm{CPD} = \frac{\tan \mathrm{CPE} - \tan \mathrm{DPE}}{1 + \tan \mathrm{CPE} \tan \mathrm{DPE}};$$

or, by substituting their values for tan C P E and tan D P E, and multiplying by the denominator,

$$t\left\{1 + \frac{dx}{dy} \cdot \frac{dx'}{dy'}\right\} = \frac{dx'}{dy'} - \frac{dx}{dy};$$

$$\text{whence, } \frac{dy'}{dx'} = \frac{dy - t\,dx}{dx + t\,dy} \;\ldots\ldots\ldots\ldots\; \text{(A)}.$$

Now, all the given curves, of which A P is one, are regulated by the same law, and must therefore be expressed by the same equation with respect to their coordinates, the curves differing only on account of having one of their constants of different values. It is also plain, that, at the point P, the given curve and the required one have the same ordinate, y, and have their abscissas either the same, or differing only by a constant quantity. If we, therefore, find $\frac{dy'}{dx'}$ from the equation of the given species of curves, and eliminate from it the constant quantity which is different in the different curves, by substituting the result in the equation found above, taking $y' = y$, we shall obtain the equation of the required curve.

When C P D is a right angle, t is infinite. Dividing, therefore, the numerator and denominator of the second member of (A) by this, we get simply

$$\frac{dy'}{dx'} = -\frac{dx}{dy} \;\ldots\ldots\ldots\ldots\ldots\ldots\; \text{(B)}.$$

404. To exemplify what has been now established, let it be required to find the equation of a curve, cutting, at right angles, an infinite number of common parabolas, having the same axis and vertex, the required curve passing through a given point in the common axis.

Here, since (App. No. 11) $y'^2 = ax'$, we have

$$\frac{dy'}{dx'} = \frac{a}{2y'}, \quad \text{or } \frac{dy'}{dx'} = \frac{y'}{2x'},$$

by substituting for a its value found from $y'^2 = ax'$. Taking, in the result thus obtained, $y' = y$ and $x' = x$, we find, from (B), last No.

$$\frac{y}{2x} = -\frac{dx}{dy}; \text{ whence, } \tfrac{1}{2}y^2 = c^2 - x^2,$$

c^2 being the constant quantity annexed to the integral. To find the value of c, let us put $y = 0$, then $c = \pm x$; whence it appears, that c is the distance from the common vertex to the point in which the required curve cuts the axis. The curve, therefore, is an ellipse, having the given vertex for its centre; and having one of its semi-axes equal to c, and the other equal to $c\sqrt{2}$.

Had the given equation been $y'^n = ax'$, the equation of the required curve would have been $y^2 + n\,x^2 = c^2$. On this supposition, the given curves would be parabolas when n is positive, and hyperbolas when it is negative. In the former case, the trajectory would be an ellipse having for centre the common vertex of the parabolas, and having its axes in the ratio of 1 to $\sqrt{n}$. In the latter case, the trajectory is a common hyperbola; and, if the given curves be equilateral hyperbolas, the required one is of the same kind.

405. The problem in which it is required to find a curve touching a series of lines which are related to one another according to a given law, is one of interest, to which we may now attend.

Let APD, AP′D′ (*fig.* 58), be two of the proposed lines referred to the same axis AY, and let PP′ be the required line. Then, it is easy to see that the required curve must pass through the intersection of two of the curves which are infinitely near each other. This being premised, let us assume $f(x, y, p) = 0$, as the equation of the curves APD, &c. the quantity p, like the radius of a circle, or the parameter of a parabola, continuing the same in each of the individual curves, but being different in different ones. On this supposition, we may take A P′D′ as the curve which is obtained by substituting $p+h$ for p, or such that its equation may be $f(x, y', p+h)=0$; or, by Taylor's theorem,

$$f(x, y', p) + \frac{df(x, y', p)}{dp}.h + \&c. = 0;$$

p alone being supposed variable, and y and y' denoting the ordinates, in the two curves, belonging to the common abscissa x. Now, at the point of intersection, we have $y'=y$, and, consequently, $f(x, y', p)=0$; by rejecting which from the preceding development, and dividing by h, we get $\frac{df(x, y, p)}{dp} + \&c. = 0$. This expression, at P, the point of contact with the required curve, will become simply $\frac{df(x, y, p)}{dp} = 0$, h vanishing. Hence, therefore, we shall have the equation of the required curve by eliminating p from $f(x, y, p) = 0$, the equation of the given species of curves, by means of $\frac{df(x, y, p)}{dp} = 0$, the differential coefficient of the same equation, found on the supposition that p alone is variable.

406. To exemplify the theory now established, let it be required to find the nature of a curve, P P′ (*fig.* 58), touching an infinite number of common parabolas, A P D, A P′D′, &c. passing through the point A, and having their axes perpendicular to A D, their areas being all equal.

Let the given area be put $=\frac{4}{3}a^2$, so that (No. 168) $AD.BC = 2a^2$, and let p denote the parameter, varying for the different parabolas. Then (App. No. 11), $AC^2 = p.BC$; whence, by multiplying by AC, putting a^2 for AC.BC, and extracting the cube root, we obtain $AC = a^{\frac{2}{3}}p^{\frac{1}{3}}$; and if we divide the equation, $AC.BC = a^2$, by this, we get $BC = a^{\frac{4}{3}}p^{-\frac{1}{3}}$. Hence, CE, or $QP = x - a^{\frac{2}{3}}p^{\frac{1}{3}}$, and $BQ = a^{\frac{4}{3}}p^{-\frac{1}{3}} - y$. But

$$QP^2 = p.BQ, \quad \text{or } (x - a^{\frac{2}{3}}p^{\frac{1}{3}})^2 = p(a^{\frac{4}{3}}p^{-\frac{1}{3}} - y.)$$

from which, by performing the actual operations, contracting, &c. we derive

$$py - 2a^{2/3}p^{1/3}x + x^2 = 0,$$

an equation which will answer for any of the parabolas, by giving p the proper value. By differentiating this, on the supposition that x and y are constant, and p variable, and by dividing by dp, we obtain $y - \frac{2}{3}xa^{2/3}p^{-2/3} = 0$. Finding from this an expression for p, and substituting it in the equation found above, we get, after due reductions, $xy = \frac{32}{27}a^2$, the equation of the required curve; which is, therefore, (App. No. 33) an equilateral hyperbola, having A Y and A Y′ as asymptotes.*

XXV.—MISCELLANEOUS EXERCISES.

1. Given the latitude of a place, and the positions of two parallels to its horizon; to find the declination of a star which, in passing from one of the parallels to the other, will change its azimuth by the least quantity possible. *Answ.* $\text{Sin dec.} = \frac{\cos\frac{1}{2}(a'+a)\cos\frac{1}{2}(a'-a)}{\sin l}$; where, as well as in the next question, the same notation is employed as in Nos. 394 and 395.

2. Given the latitude of a place, and the positions of two hour circles with respect to its meridan; to find the declination of a star, which, in passing from one of the circles to the other, in its apparent diurnal revolution, will change its azimuth by the greatest quantity possible. *Answ.* tan dec. =

$$\frac{\tan l \cos\frac{1}{2}(P'-P) \pm \sqrt{\{[\tan^2 l + \cos^2\frac{1}{2}(P'+P)]\sin P' \sin P\}}}{\cos\frac{1}{2}(P'+P)}.$$

* Since $AC \cdot CB = a^2$, it follows (App. No. 33), that the locus of the vertices of the several parabolas is also an equilateral hyperbola, having the same asymptotes; and it is plain that, in the two hyperbolas, the ordinates corresponding to the same abscissa are in the ratio of 27 to 32. We have thus an interesting relation between two curves which are produced by a slight variation in the same general hypothesis.

The principle established in No. 405, might be farther exemplified in the solutions of the questions already solved in Nos. 396 and 397.

3. Prove, that, in a given latitude, l, the sine of half the angle of duration of the shortest twilight, is equal to $\frac{\sin 9^0}{\cos l}$.

4. Investigate the following integral:

$$\int \frac{x^6 dx}{(1+x^3)^{2/3}} = \frac{x^7}{7} - \frac{2}{3}\frac{x^{10}}{10} + \frac{2.5}{3.6}\frac{x^{13}}{13} - \frac{2.5.8}{3.6.9}\frac{x^{16}}{16} + \&c.$$

5. Prove, by the Integral Calculus, from the formula,

$d\tan^{-1}x = \frac{dx}{1+x^2}$, or $d\tan^{-1}x = \frac{dx}{x^2+1}$, that $\tan^{-1}x =$

$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \&c.$;* or $\tan^{-1}x = \frac{1}{2}\pi - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \&c.$

6. Prove, also, by substituting z^2 for $1+x^2$ in the differential of $\tan^{-1}x$, that $\tan^{-1}x = \frac{1}{2}\pi -$

$$\sqrt{(1+x^2)}\left\{\frac{1}{1+x^2} + \frac{1}{3}\frac{1}{2(1+x^2)^2} + \frac{1}{5}\frac{1.3}{2.4(1+x^2)^3} + \&c.\right\}.$$

7. Prove, by the Integral Calculus, from No. 41, that

$$\sec^{-1}x = \frac{1}{2}\pi - \left\{\frac{1}{x} + \frac{1}{3}\frac{1}{2x^3} + \frac{1}{5}\frac{1.3}{2.4x^5} + \frac{1}{7}\frac{1.3.5}{2.4.6x^7} + \&c.\right\};$$

and show from this, by taking $x=2$, that

$$\tfrac{1}{3}\pi = 1 + \frac{1}{3}.\frac{1}{2}.\frac{1}{4} + \frac{1}{5}.\frac{1.3}{2.4}\left(\frac{1}{4}\right)^2 + \frac{1}{7}.\frac{1.3.5}{2.4.6}\left(\frac{1}{4}\right)^3 + \&c.$$

8. Prove that, when $x=a$, the value of the fraction,

$$\frac{a^{a-n} - x^{x-n}}{a-x}, \quad \text{is } a^{a-n-1}(a + a\log a - n).$$

9. If a given arc, a, be divided into parts, x and $a-x$, such that $\sin^m x \sin^n(a-x)$ is a maximum, prove that the sine of the difference of those parts is a fourth proportional to $m+n$, $m-n$, and $\sin a$.†

10. If a given arc be divided into any number of parts, $x, y, z, \ldots$, such that $\sin^m x \sin^n y \sin^p z \ldots\ldots$, may be a maxi-

* See the note, page 57.

† This may be proved, in perhaps the easiest manner, by taking the logarithms of $u = \sin^m x \sin^n(a-x)$, and differentiating them.

mum; prove that the tangents of the parts x, y, z, ..., are proportional to the indices m, n, p, ...

11. If there be two straight lines inclined at a given angle, and if a third straight line vary its position, so as always to form with them a triangle of the same area; what is the curve to which that line is, in every position, a tangent? *Answ.* An hyperbola.

12. When the sides of a quadrilateral described about a circle are given, the circle is a maximum, if the quadrilateral be such that a circle may be described about it.

13. Prove that when $x=0$, the values of $\frac{x+\sin x-\sin 2x}{2x+\sin x-\sin 3x}$ and $\frac{x+\tan x-\tan 2x}{2x+\tan x-\tan 3x}$, are each $\frac{7}{26}$; and that the value of $\frac{x+\sin x-\sin 2x}{2x+\tan x-\tan 3x}$ is $-\frac{1}{2}.\frac{7}{26}$.

14. Given two sides of a plane triangle equal to a and b; to find the third side, x, such that, if the triangle revolve about it, the double cone so formed may have its volume a maximum.

$$Answ.\ x=\sqrt{\{\tfrac{1}{3}(a^2+b^2)+\tfrac{2}{3}\sqrt{(a^4-a^2b^2+b^4)}\}};$$

$$\text{or, if } b=a, \quad x=\tfrac{2}{3}a\sqrt{3}.$$

15. Prove that, if S, the curve surface of a cone, be given, its volume will be a maximum, when the radius of its base is $\sqrt{\frac{S\sqrt{3}}{3\pi}}$.

16. If P denote the semidiurnal arc, d the declination, and m the minutes in a small increase of the declination; prove that the corresponding change, in seconds, of the time of rising or setting, is $8m \cot P \operatorname{cosec} 2d$, nearly.

17. Prove that the sum of x terms of the series,

$$\sin 4\varphi \sin 5\varphi+\sin 6\varphi \sin 10\varphi+\sin 8\varphi \sin 17\varphi+$$
$$\sin 10\varphi \sin 26\varphi+ \&c.$$

is $\frac{1}{2}\cos\varphi+\frac{1}{2}\cos 4\varphi-\frac{1}{2}\cos(x+1)^2\varphi-\frac{1}{2}\cos(x+2)^2\varphi$.

18. Prove that the sum of x terms, and the sum of an infinite number of terms, of the series,

$$\tan^2\varphi \tan 2\varphi + 2\tan^2\tfrac{1}{2}\varphi \tan\varphi + 4\tan^2\tfrac{1}{4}\varphi \tan\tfrac{1}{2}\varphi +$$

$$8\tan^2\tfrac{1}{8}\varphi \tan\tfrac{1}{4}\varphi + \&c.$$

are, respectively, $\tan 2\varphi - 2^x \tan\frac{\varphi}{2^{x-1}}$, and $\tan 2\varphi - 2\varphi$.

19. The greatest triangle that can be inscribed in a given circle, is an equilateral one. Required the proof.

20. The curve whose equation is $(x^2+y^2)^2 = ax^3$, is an oval, which has its entire area equal to five eights of the circle whose diameter is a; and its greatest ordinate corresponds to $x = \frac{9a}{16}$. Required the proof.

21. Prove that, in the curve of versed sines (the curve whose equation is $y = a \operatorname{versin} x$) there are points of inflexion when $\cos x = 0$, and, consequently, $x = (n+\frac{1}{2})\pi$. Prove, also, that the area of the curve is $ax - a^2 \sin\frac{x}{a}$, and that the curvature is greatest when $x = n\pi$.

22. If a trapezoid, of given area, have the sum of one of its parallel sides and the two adjacent ones a minimum, these three sides are equal, and each of them is half the remaining side; so that they are the three equal chords of a semicircle, of which the remaining side is the diameter. Required the proof.

23. Prove that the area of the curve of sines (App. No. 32) is $a\{a - \surd(a^2 - y^2)\}$.

24. Prove that, if the greatest ellipse possible be inscribed in a semicircle whose radius is a, and if a line be drawn joining the points of contact of the two curves, the distance between that line and the diameter of the semicircle is $\frac{1}{2}a\surd 2$.

25. The area of the catenary (App. No. 35) is $s = xy - a(z-y)$. Required the proof.

26. Find two curves such that, in the first, the subtangent is to the subnormal as the abscissa to the ordinate; and in the second the subtangent is to the subnormal as the

ordinate to the abscissa. *Answ.* The first is a parabola, having, for equation, $(y-x-a)^2=4ax$; and the equation of the second is $(y^3-x^3-a^3)^2=4a^3x^3$.

27. The point within a triangle, from which if straight lines be drawn to the three angles, the sum of their squares is the least possible, is the intersection of straight lines drawn from the angles to the points of bisection of the opposite sides. Required the proof.

28. Prove that, in the tractory (APP. No. 37), the length of the arc, VP (*fig.* 57), is equal to $a\log\frac{a}{y}$.

29. Prove, also, that in the same curve, the area is,

$$\tfrac{1}{4}\pi a^2-\tfrac{1}{2}a^2\sin^{-1}\frac{y}{a}-\tfrac{1}{2}y\sqrt{(a^2-y^2)};$$

that the content of the body formed by the revolution of the curve about OB is $\frac{1}{3}\pi(a^2-y^2)^{\frac{3}{2}}$; and that the surface of the same body is $2\pi a(a-y)$.*

30. Show also, that, in the tractory, straight lines drawn from P and T, respectively, perpendicular to PT and OB, meet in the centre of the osculating circle; and that the evolute is a catenary having its vertex at V, and its constant quantity (a, APP. No. 35) equal to OV.†

31. What is the curve in which the sine of the angle made by the radius vector and the tangent, is inversely proportional to the square of the radius vector? *Answ.* The equilateral hyperbola.

32. Required the equation of a curve, such that its area is equal to twice the rectangle of its co-ordinates. *Answ.* The equation is $xy^2=a^3$.

33. Prove that, if the radius of the base of a cone be bisected, and, through the point of bisection, a plane be drawn parallel to the plane touching the convex surface of the cone at the extremity of the same radius, the area of the parabolic section thus formed is a maximum.

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* Hence, the entire area contained between VO, the curve, and the asymptote, OB, is equal to a quadrant of the circle whose radius is OV; while the entire content of the body is equal to one half of the hemisphere whose radius is OV, and its surface is equal to that of the same hemisphere.

† Hence, if a thread applied to a catenary, and having its extremity at the lowest point, be wound off, the extremity will describe a tractory.

34. Prove that the least cone that can be described about a given cylinder, has its altitude treble of that of the cylinder.

35. If, in any curve, PB (*fig.* 1), the ordinate QR be given in position, and if the point P be such that, PM and PK being drawn parallel to the axes, the rectangle PMQK is the greatest possible, the subtangent TM is equal to MQ. Required the proof.

36. In the curve of a given ellipse, ADBE (*fig.* 7), it is required to find a point, P, the distance of which from E, the remote extremity of the conjugate axis, is the greatest possible. *Answ.* The position of the point is determined by the expression, $x=\frac{a^2\sqrt{(a^2-2b^2)}}{a^2-b^2}$, where $a=AC$, $b=CD$, and $x=CM$. In the same case, also, $CK=\frac{b^3}{a^2-b^2}$.

37. If one of the equal sides of an isosceles triangle be given, the area will be a maximum when the vertical angle is a right angle; but if the slant height of a right cone be given, the volume will be a maximum when the angle at the vertex, on a plane passing through the axis, is 109° 28′, being double of the angle whose cosine is $\sqrt{\frac{1}{3}}$. Required the proofs.

38. Prove that

$$\frac{1}{2^2.4^2}+\frac{1^2}{2^2.4^2.6^2}+\frac{1^2.3^2}{2^2.4^2.6^2.8^2}+\frac{1^2.3^2.5^2}{2^2.4^2.6^2.8^2.10^2}+\&c.=\frac{32}{27\pi}-\frac{13}{36}.$$

39. From the formula $\frac{dx}{a+x}$, prove by integration by parts, that

$$l(a+x)=la+M\left\{\frac{x}{a+x}+\frac{1}{2}\left(\frac{x}{a+x}\right)^2+\frac{1}{3}\left(\frac{x}{a+x}\right)^3+\&c.\right\}.$$

40. In series (4) and (2), No. 104, multiply by dx, and by integrating between the limits 0 and x, prove that

$$\frac{1-\cos x}{1^2}+\frac{1-\cos 2x}{2^2}+\frac{1-\cos 3x}{3^2}+\&c.=\frac{\pi x}{2}-\frac{x^2}{4}\ldots(1).$$

$$\frac{1-\cos x}{1^2}-\frac{1-\cos 2x}{2^2}+\frac{1-\cos 3x}{3^2}-\&c.=\frac{x^2}{4}\ldots\ldots\ldots(2).$$

From the two formulas thus obtained, derive in a similar way the two following:

$$\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\&c.\right)x-\left(\frac{\sin x}{1^3}+\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}+\&c.\right)$$

$$=\frac{\pi x^2}{4}-\frac{x^3}{12}\ \ldots\ldots\ (3)$$

$$\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\&c.\right)x-\left(\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}-\&c.\right)$$

$$=\frac{x^3}{12}\ \ldots\ldots\ldots\ldots\ (4)^*.$$

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* By taking $x=\pi$ in series (3) and (4) in the text, we get

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\&c.=\frac{\pi^2}{6}\ \ldots\ldots\ (5),$$

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\&c.=\frac{\pi^2}{12}\ \ldots\ldots\ (6):$$

and by taking half the sum and half the difference of these, we obtain

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\&c.=\frac{\pi^2}{8}\ \ldots\ldots\ (7),$$

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\&c.=\frac{\pi^2}{24}\ \ldots\ldots\ (8).$$

The former of these two might also be obtained by taking $x=\pi$ in series (1) or (2), and halving the result. The sum in (8), also, would be had at once from that in (5); the terms in the one being severally quadruple of those in the other. On the same principle, we should get the following formula from (6):

$$\frac{1}{2^2}-\frac{1}{4^2}+\frac{1}{6^2}-\frac{1}{8^2}+\&c.=\frac{\pi^2}{48}\ \ldots\ldots\ (9):$$

and innumerable others might be obtained similarly. From (3) and (5), or from (4) and (6), by taking $x=\frac{1}{2}\pi$, we readily get

$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\&c.=\frac{\pi^3}{32}\ \ldots\ldots\ (10).$$

From series (1) and (2), also, we obtain, by taking $x=\frac{1}{2}\pi$,

$$\frac{1}{1^2}+\frac{2}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{2}{6^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{2}{10^2}+\&c.=\frac{3\pi^2}{16}\ \ldots\ldots\ (11),$$

$$\frac{1}{1^2}-\frac{2}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}-\frac{2}{6^2}+\frac{1}{7^2}+\frac{1}{9^2}-\frac{2}{10^2}+\&c.=\frac{\pi^2}{16}\ \ldots\ldots\ (12);$$

and from these two, by subtraction, &c. we get

$$\frac{1}{2^2}+\frac{1}{6^2}+\frac{1}{10^2}+\&c.=\frac{\pi^2}{32}\ \ldots\ldots\ (13);$$

which might also be obtained from (7). By taking $x=\frac{1}{3}\pi$ in formulas (1) and (2) we should get after some slight modifications,

$$\frac{1}{1^2}+\frac{1}{4^2}+\frac{1}{7^2}+\&c.+3\left(\frac{1}{2^2}+\frac{1}{5^2}+\frac{1}{8^2}+\&c.\right)=\frac{5\pi^2}{18}\ \ldots\ (14),$$

$$\frac{1}{1^2}-\frac{1}{4^2}+\frac{1}{7^2}-\&c.-3\left(\frac{1}{2^2}-\frac{1}{5^2}+\frac{1}{8^2}-\&c.\right)=\frac{\pi^2}{72}\ \ldots\ldots\ (15):$$

41. Prove that

$$\log x = \frac{1}{x}\left\{\frac{x-1}{1} + \frac{(x-1)^2}{1.2} - \frac{(x-1)^3}{2.3} + \frac{(x-1)^4}{3.4} - \&c.\right\}: *$$

and hence show that

$$\frac{1}{1.2.3} + \frac{1}{3.4.5} + \frac{1}{5.6.7} + \&c. = \log 2 - \tfrac{1}{2}.$$

---

and other series would be found by taking the sum and difference of these. If in the same two series we take $x = \frac{2}{3}\pi$, we get

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{10^2} + \&c. = \frac{4\pi^2}{27} \ldots\ldots (16),$$

$$\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} - \frac{1}{10^2} + \&c. = \frac{2\pi^2}{27} \ldots\ldots (17).$$

It is plain, also, that sums of many other series might be found by taking $x = \frac{1}{4}\pi$, $x = \frac{1}{6}\pi$, &c. in the four series in the text. Many of them, however, would be of little or no interest.

By multiplying (3) and (4), however, when modified by means of (5) and (6), by dx, and integrating the products; by treating the integrals so obtained, similarly, and so on, the student would find other general series from which, by taking $x = \pi$, $x = \frac{1}{2}\pi$, &c. and by various modifications, he would obtain in terms of π the sums of many series of much curiosity and interest. The following are a few out of many of this kind.

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \&c. = \frac{\pi^4}{90} \ldots\ldots\ldots\ldots (18),$$

$$\frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \&c. = \frac{7\pi^4}{720} \ldots\ldots\ldots (19),$$

$$\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \&c. = \frac{\pi^6}{945} \ldots\ldots\ldots (20),$$

$$\frac{1}{1^6} - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \&c. = \frac{31\pi^6}{30240} \ldots\ldots (21),$$

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \&c. = \frac{\pi^4}{96} \ldots\ldots\ldots\ldots (22),$$

$$\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \&c. = \frac{\pi^6}{960} \ldots\ldots\ldots (23),$$

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \&c. = \frac{\pi^3}{33} \ldots\ldots\ldots (24),$$

$$\frac{1}{1^5} - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \&c. = \frac{5\pi^5}{1536} \ldots\ldots (25).$$

* This may be readily obtained from the identity $dx = x\frac{dx}{x}$ by integration by parts; and employing instead of $\log x\, dx$ what (No. 88) is equivalent, and integrating again.

APPENDIX.

1. At the foundation of the analysis and theory of curves there are two great problems, which are converses of one another. The object of one of these is to determine the equation of a curve from the mode in which it is generated; that is, to find an equation, such that if we substitute in it for one of the coordinates any value which it can have, the resolution of the equation so obtained, will give the value or values of the other coordinate. In the converse problem, the equation of the curve is given, and the object in view is to describe the curve, and to determine its various properties. The solution of the former problem in reference to a number of interesting curves, will form the chief object of the following Appendix; and it will be proper to commence with the consideration of the three important curves, the ellipse, the hyperbola, and the parabola.

2. If F and V (*fig.* 7) be two given points, and P another point which is always so situated that the *sum* of the straight lines F P and P V is a constant quantity, the locus of the point P is called an *ellipse;** but, if the *dif-*

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* Hence it is plain, that, if the ends of a thread, F P V, be fixed at the points F and V, at a less distance asunder than the length of the thread, and if the point of a pencil, P, be carried round in such a manner as to keep the thread always uniformly stretched, the line thus described is an ellipse. It is also evident, that, either according to this view or that given in the text, if the foci be made to approach one another, while the sum of F P and P V continues the same, the ellipse will become more and more nearly a circle, and will actually become one if the foci coalesce. Hence, a circle may be regarded as an ellipse having its axes (App. No. 4) equal; and the properties of the ellipse belong also to the circle, when they are modified on the supposition of the equality of the axes. If, on the other hand, F and V recede to A and B, so that $FV = FP + PV$, the ellipse becomes a straight line. The ellipse may also be described in other modes by means of continued motion; or it may be traced by finding points in the manner explained in page 144. The hyperbola may be described by means of a straight ruler and a thread, which differ in length by the difference of F P and P V; since, if one end of the ruler and one

ference of F P and P V (*fig.* 8) be always a constant quantity, the locus of P is called an *hyperbola.* It is evident, that there may be two equal and similar portions or branches, P A G and p B g, which will answer to the definition of the hyperbola. These are called the *opposite hyperbolas.** In both the ellipse and the hyperbola, the points F and V are called *foci.*

3. To find the equations of these two curves, let F P V (*fig.* 3) be a triangle, having F V bisected in C, and P M perpendicular to F V; and let $FC=CV=c$, $CM=x$, and $MP=y$; and, consequently, $FM=c+x$ and $MV=c-x$. Then, by a property of the triangle (Euc. II. 5, cor. 4), we have

$$(FP+PV)(FP-PV)=(FM+MV)(FM-MV)=4cx.$$

Now, in case of the ellipse, since $FP+PV$ is given, it may be represented by $2a$; while in reference to the hyperbola, $2a$ may be used to denote $FP-PV$; which for that curve is constant. Hence, from the foregoing equation we get, for the ellipse, $FP-PV$, and for the hyperbola, $FP+PV$, each equal to $\frac{2cx}{a}$; and, consequently, for each curve we have $FP=a+\frac{cx}{a}$. By subtracting the square of $c+x$ from the square of the value of F P, just found, we obtain (Euc. I. 47) after multiplying by a^2,

$$a^2y^2=a^4-a^2c^2+c^2x^2-a^2x^2.$$

Now, since (Euc. I. 20, and cor.) two sides of a triangle are together greater than the remaining side, and the difference of two sides is less than the third side, it follows, that in the ellipse a is greater than c, but in the hyperbola less. Hence, for the ellipse, the equation will naturally be arranged thus:

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of the thread be fixed together, and the others be placed at the foci, and if, while the ruler is made to revolve, a pencil be used to keep as much of the thread as possible applied to the ruler, the pencil will describe one branch of the hyperbola. The opposite hyperbola will be obtained by interchanging the positions of the unconnected ends of the ruler and thread. The hyperbola may also be described by finding points, as in page 144.

* The ellipse may be regarded as consisting of two branches commencing from A and B, but approaching each other so as to coalesce; while the opposite hyperbolas continually recede from one another.

$$a^2y^2 = a^4 - a^2c^2 - (a^2x^2 - c^2x^2), \text{ or}$$

$$a^2y^2 = (a^2 - c^2)(a^2 - x^2);*$$

while, for the hyperbola it will take the form,

$$a^2y^2 = c^2x^2 - a^2x^2 - (a^2c^2 - a^4), \text{ or}$$

$$a^2y^2 = (c^2 - a^2)(x^2 - a^2).*$$

Hence, if in the first case, we put $a^2 - c^2 = b^2$, and in the second, $c^2 - a^2 = b^2$, we get the following results;

The equation of the ellipse, $a^2y^2 = b^2(a^2 - x^2)$......(1);†

The equation of the hyperbola, $a^2y^2 = b^2(x^2 - a^2)$...(2).

4. By taking $y = 0$, we find from each of these equations $x = \pm a$; which shows, that each of the curves cuts F V or its continuations on both sides of C, at distances from that point, each equal to a; the points of intersection falling beyond F and V in the ellipse, but in the hyperbola between them; because a is greater than c in the ellipse, but less in the hyperbola. If, again, in the same equations, we take $x = 0$, we get, for the ellipse, $y = \pm b$, and, for the hyperbola, $y = \pm b\sqrt{-1}$. Hence it appears, that a perpendicular to F V, passing through C, is cut by the ellipse in two points on the opposite sides of C, and each at a distance from it equal to b; while the hyperbola does not cut the corresponding perpendicular, the values of y being imaginary. Farther, also, if we resolve the two equations for y, we get

$$y = \pm \frac{b}{a}\sqrt{(a^2 - x^2)}, \text{ and } y = \pm \frac{b}{a}\sqrt{(x^2 - a^2)}.$$

From the first of these expressions, it appears that, in the ellipse, x may have any value not exceeding $\pm a$, but none

* By these two arrangements, when the square roots of the members are taken, the coefficients, $\sqrt{(a^2 - c^2)}$ in the one case, and $\sqrt{(c^2 - a^2)}$ in the other, are real; while, otherwise, they would be imaginary.

† By dividing the members of this equation by a^2b^2, and transposing the last term of the result, we have the equation of the ellipse expressed under the curious form, $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$. If $b = a$, this, by multiplying by a^2, becomes $x^2 + y^2 = a^2$, the equation of the circle. It would appear in a similar manner from equation (2), that the equation of the hyperbola may be expressed in the form, $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$.

beyond these limits, as values of the latter kind would render the radical imaginary; while, in the hyperbola, on the contrary, x cannot be taken between those limits, but may have any value whatever beyond them. Hence, therefore, if for each curve, values, commencing at C, and each equal to a, be taken along F V in each direction, and if, through their remote extremities, perpendiculars be drawn to F V, the ellipse will lie wholly between those perpendiculars: the hyperbola, however, will be wholly excluded from the space between the same perpendiculars; but, lying outside of them, it will extend in each direction to an unlimited distance, thus having, as it is generally expressed, *infinite branches.* It may be farther remarked, that from the equality of the positive and negative values of y, the parts of each curve on the opposite sides of F V will be equal and symmetrical, so that by inversion and superposition, one of one of them would exactly coincide with the other. It is plain, also, that both curves are equally symmetrical in reference to the straight line drawn through C perpendicular to F V. Lastly, if we resolve, successively, the two equations for x, we get

$$x = \pm \frac{a}{b}\sqrt{(b^2-y^2)}, \text{ and } x = \pm \frac{a}{b}\sqrt{(b^2+y^2)}.$$

From this it appears, that in the hyperbola y may have any value whatever from ∞ to $-\infty$, as no value that can be assigned to it will render x imaginary. In the ellipse, on the contrary, y cannot exceed the limits b and $-b$; and, therefore, if parallels to F V be drawn, one on each side, at distances each equal to b, the curve will be confined between these parallels. All these conclusions will be illustrated by means of the delineations of the curves exhibited in figures 7 and 8; and from what has been established, it will be seen, that, in the ellipse, A B is the sum, and in the hyperbola, the difference of F P and P V; that, in the ellipse, C D or C E is equal to b; and that, if in the hyperbola, a circle be described from A or B as centre, with a radius equal to C F, and cutting D C E in D and E, C D or C E will be equal to b. In the ellipse, also, the straight line joining F D or F E is equal to A C or C B.

In both the ellipse and the hyperbola C is called the *centre.* In the ellipse A B is called the *greater* or the *transverse axis*, and DE the *less* or the *conjugate axis;* and in the hyperbola, AB is termed the *transverse*, and DE the *conjugate* or *second axis*.* In both the ellipse and the hyperbola also, any straight line drawn through the centre, and terminated both ways by the curve is called a *diameter*.

5. From the equation of the ellipse found in No. 3, we have $a^2 : b^2 :: a^2 - x^2$ or $(a+x)(a-x) : y^2$; and from the same equation, after having transposed a^2y^2 and $-b^2x^2$, we get, in a similar manner, $b^2 : a^2 :: (b-y)(b+y) : x^2$. By the restoration of the values of a, b, x, and y, these analogies, if the first and second terms of each be quadrupled, become obviously $AB^2 : DE^2 :: AM . MB : MP^2$, and $DE^2 : AB^2 :: DK . KE : KP^2$. Hence it appears, that *the square of either axis of an ellipse is to the square of the other, as the rectangle of any two parts into which the first is divided, is to the square of the ordinate drawn through the point of section parallel to the second.*

It would be shown in a similar manner from the equation of the hyperbola in No. 3, that $AB^2 : DE^2 :: AM . MB : MP^2$, and $DE^2 : AB^2 :: CD^2 + MP^2 : CM^2$. The first of these is analogous to the first of the proportions found above in reference to the ellipse; but the case is different with regard to the latter analogy of each pair, as $MP^2 + DE^2$ is not the product of real binomial factors.

6. Another interesting form of the equation of each of these curves is obtained by taking A as the origin of the coordinates, and putting $AM = x'$, so that $x' = x + a$ in the ellipse, and $x' = x - a$ in the hyperbola. Hence, in the former curve we have $x = x' - a$, and in the latter $x = x' + a$; and if these be substituted in the equations in No. 3, App. and the results be simplified, we get, after dropping the accents,

For the ellipse, $a^2y^2 = b^2(2ax - x^2)$ ...... (3),

For the hyperbola, $a^2y^2 = b^2(2ax + x^2)$ ...... (4).

7. If x be taken equal to $\pm c$, so that $x^2 = c^2$, or in the

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* It may be remarked, that, if with DE as transverse axis and AB as conjugate, the opposite hyperbolas, P′DG′ and p′Eg′, be described, they are called *conjugate hyperbolas*, in relation to PAG₁ and pBg; and they have analogous properties. They have, in particular, the same asymptotes.

ellipse $x^2=a^2-b^2$, and in the hyperbola $x^2=a^2+b^2$, we shall have by equations (1) and (2), $a^2y^2=b^4$ for each of the curves; whence $ay=b^2$, and $a:b::b:y$; that is, by doubling the terms, AB : DE :: DE : GH. The double ordinate GH, therefore, which passes through one of the foci, is a third proportional to the greater axis and the less; and it has been called the *latus rectum*, or the *parameter to the greater axis.* In like manner, a third proportional to the less axis and the greater is called the *parameter to the less.*

It follows from this and from Euc. III. 35, that if, in case of the ellipse (*fig.* 7), a circle be described on AB as diameter, and if GH be produced to meet it in R and L, RL is equal to DE.

8. Since, in *fig.* 7 (Euc. III. 35), $AM.MB=MN^2$, we have (App. No. 3) $AB^2:DE^2::MN^2:MP^2$, and thence (Euc. VI. 22) AB : DE :: MN : MP. Hence we see, that, if we take MP such that MN to MP as the diameter AB is to a given line DE, the locus of P is an ellipse having AB for its transverse axis, and DE for its conjugate.*

If a circle were described on DE as diameter, and an ordinate were drawn to DE, it would be shown in a similar manner, that the ordinate of that circle would be to the ordinate of the ellipse as DE to AB.†

9. An hyperbola in which a and b are equal is called an *equilateral* or *rectangular hyperbola;* and the two equations (2) and (4) become $y^2=x^2-a^2$, and $y^2=2ax+x^2$. This curve bears to other hyperbolas, having the same transverse axis, a relation analogous to that of the circle to ellipses, having one of their axes the same as the diameter of the circle.

10. If DR (*fig.* 9) be a straight line, and F a point

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* Hence it follows, that if, from every point in the circumference of a circle perpendiculars be drawn to a plane which is oblique to that of the circle, the curve marked out by their intersection with the plane is an ellipse; the greater axis of which is to the less as the radius is to the cosine of the angle of inclination. In this case, the circle is said to be *projected orthographically;* the eye, in the *orthographic projection* of the sphere, being supposed to be situated at an infinite distance, so that the lines on the object to be projected are marked out on the plane of projection by perpendiculars to it from their several points. The line separating the enlightened and dark parts of the moon, as seen from the earth, affords an instance; that line being the projection of half the circle of light and darkness, on a plane to which the line drawn from the earth to the moon is perpendicular.

† It follows from this, that an oblique section of a cylinder by a plane, is an ellipse; and that an ellipse may be projected orthographically into a circle.

without it, both given in position, and if P be another point which is always so situated that the straight line drawn from P to F is equal to the perpendicular PD, the locus of the point P is called a *parabola.** The point F is called the *focus;* and the straight line DR, the *directrix.* The straight line BFM perpendicular to DR is named the *axis,* and sometimes the *principal axis;* and the point A, in which it meets the curve, is called the *vertex.*

11. To find the equation of the parabola, put AF or (by the last No.) its equal $AB = a$, $AM = x$, and $MP = y$. Then, $FP = DP = BM = AM + AB = x + a$, and $FM = AM - AF = x - a$. But (Euc. I. 47) $PM^2 = PF^2 - FM^2 = (PF - FM)(PF + FM) = 2a \times 2x$; that is, $y^2 = 4ax$, or $y^2 = px$, where $p = 4a$.

If, in this, we take $x = a = AF$, we find $y^2 = 4a^2$; and, consequently, $y = 2a = \frac{1}{2}p$; whence, $LR' = p$. The line LR' is called the *parameter,* or the *latus rectum* to the axis.†

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* The ellipse and hyperbola may also be defined in a similar manner. Thus, if PD and PF be unequal, but in a constant ratio, the curve is an ellipse or hyperbola, accordingly as PD is greater or less than PF. To prove this, let AF : AB : : m : 1; and, putting $AF = q$, we have $AB = \frac{q}{m}$: whence, $BM = DP = +\frac{q}{m}$. But 1 : m : : DP : PF; from which, and from the foregoing value of DP, we get $PF = mx + q$. From the square of this, take the square of $x - q$, which is the value of FM, and there will remain MP^2 or

$$y^2 = 2(m+1)qx + (m^2 - 1)x^2,$$

the equation required. If $m = 1$, this becomes simply $y^2 = 4qx$, which (App. No. 11) is the equation of the parabola. If $m < 1$, the term $(m^2 - 1)x^2$ becomes negative. Assuming, therefore, $m^2 - 1 = -\frac{b^2}{a^2}$, and putting $2(m+1)q$ under the form,

$$\frac{2(m+1)(m-1)q}{m-1}, \text{ or } \frac{2(m^2-1)q}{m-1};$$

and in this, again, substituting $-2a$ for $\frac{2q}{m-1}$ the equation becomes

$$y^2 = \frac{b^2}{a^2}(2ax - x^2),$$

which (App. No. 6) is the equation of the ellipse. In a manner exactly similar, we should find that

$$y^2 = \frac{b^2}{a^2}(2ax + x^2),$$

when $m > 1$; which (App. No. 6) is the equation of the hyperbola. In case of the circle, the directrix is to be conceived to be at an infinite distance from the point A, so that m becomes infinitely small, or vanishes.

† Out of the many other interesting properties of the ellipse, hyperbola, and parabola, besides those that have been given above, it may be proper, in this place, to

12. To illustrate the principles that have been thus far established, it may be useful to take a few numerical examples. Thus, if in the ellipse (*fig.* 7), we have A B = 200, D E = 120, and C M = 60, we get by means of either equation (1) or (3), M P = 48.

In like manner, if we have in the hyperbola (*fig.* 8), C A = 39, C D = 84, and M P = 35, we may find by equation (2) or (4), that C M = $42\frac{1}{4}$.

If again in the parabola (*fig.* 9), AF = 32, and AM = 98, we get, by the last No. MP = 112.

13. It will be readily seen also, how, if we have in an ellipse or hyperbola, two cotemporaneous values of x and y, we may find the constant quantities a and b, and thence describe the curve; and how, if we have two ordinates of a parabola and their perpendicular distance asunder, we may find p, and then trace the curve: and the student will find it easy to form various other questions, and thus to render himself familiar with the elementary properties of these curves.

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exhibit the following. The equation of the ellipse for *any* diameters, GH and KL (*fig.* 60) taken as axes of coordinates, may be thus found. Retaining the same notation as in App. No. 3, let us draw the chord PM′P′ parallel to GH; draw also, M′Q parallel, and M′R perpendicular to AB; and let CM′ = x', M′P = y', BCM′ = p, and ACG or QM′P = q. Then, CR = $x' \cos p$, RM′ or OQ = $x' \sin p$, QP = $y' \sin q$, and QM′ or OR = $y' \cos q$; and, therefore,

$$\text{C M or } x = x' \cos p - y' \cos q, \quad \text{and M P or } y = x' \sin p + y' \sin q.$$

Hence, by substituting these values of x and y in the equation, $a^2 y^2 + b^2 x^2 = a^2 b^2$ (App. No. 3), we get, for the required equation,

$$\left.\begin{matrix} a^2 \sin^2 q \\ + b^2 \cos^2 q \end{matrix}\right\} y'^2 + 2 \left\{ \begin{matrix} a^2 \sin p \sin q \\ - b^2 \cos p \cos q \end{matrix} \right\} x'y' + \left\{ \begin{matrix} a^2 \sin^2 p \\ + b^2 \cos^2 p \end{matrix} \right\} x'^2 = a^2 b^2 \ldots\ldots (\text{A}).$$

This admits of an interesting modification, by taking p and q such that the second term may vanish, which is effected by putting

$$a^2 \sin p \sin q = b^2 \cos p \cos q; \quad \text{whence, } \tan q = \frac{b^2 \cot p}{a^2} \ldots\ldots\ldots (\text{B}).$$

Here, p may be assumed at pleasure, and then q will be determined by this equation. Now, after this change, let us put

$$a^2 \sin^2 q + b^2 \cos^2 q = \frac{a^2 b^2}{b'^2}, \quad \text{and } a^2 \sin^2 p + b^2 \cos^2 p = \frac{a^2 b^2}{a'^2},$$

and equation (A), after this substitution, and after dividing the result by $a^2 b^2$, and multiplying by $a'^2 b'^2$, will become

$$a'^2 y'^2 + b'^2 x'^2 = a'^2 b'^2 \ldots\ldots\ldots (\text{A}_2).$$

From this equation, by taking first $y'=0$, and then $x'=0$, we find that a' = C K, and b' = C G. Hence it appears, that the equation of the ellipse referred to *any* two diameters whose relative positions are determined by equation (B) *is of exactly the same form as the equation in reference to the conjugate axes*, A B and D E. This equation, also, gives, for each value of x, two equal values for y, one positive and one negative; and hence it follows, that K L *bisects* G H, *and all the chords*, P P′, &c. *parallel to it.* It is also plain, that, if P, and consequently M′, be taken at K, *the*

14. By examining the preceding results, it will be seen, that the equations of the ellipse, hyperbola, and parabola, are all of the second degree; and hence, these are *lines of*

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parallel through that point to G H *will not cut the curve, but will be a tangent to it.* By drawing P M″ P″, parallel to K L, it would be shown, by a process similar to that employed, App. No. 5, that $CG^2 : CK^2 :: GM''M''H : M''P^2$; that G H bisects L K, and the chords parallel to it; and that L K is parallel to a tangent at G or H. Diameters having any one of these properties, and consequently all of them, are called *conjugate diameters.*

It would be shown, in a similar manner, that analogous properties belong also to the hyperbola.

In the parabola A C P (*fig.* 61), let A′Q be parallel, and A A′ perpendicular to the axis AM, and draw any chord, P P′, cutting A′Q in K. Then, by putting $A'K = x''$, $KP = y'$, $AA' = a$, and $PKQ = q$, we have $PQ = y' \sin q$, and $KQ = y' \cos q$; whence A M or $x = y' \cos q + x''$, and MP or $y = y' \sin q + a$. Substituting these values of x and y in the equation, $y^2 = p\,x$, we get, after transposition,

$$\sin^2 q \,.\, y'^2 + (2\,a \sin q - p \cos q)\, y' = p\,x'' - a^2.$$

Now, putting the coefficient of $y' = 0$, we find $a = \frac{1}{2}\, p \cot q$; and, by substituting this value for a in the second member, and dividing by $\sin^2 q$, we obtain

$$y'^2 = \frac{p\,x'' - \frac{1}{4}\,p^2 \cot^2 q}{\sin^2 q}, \quad \text{or } y'^2 = p'x';$$

where x' is put for $x'' - \frac{1}{4}\,p \cot^2 q$, and p' for $\dfrac{p}{\sin^2 q}$.

Now, from the equation, $y'^2 = p'x'$, we see that $x' = 0$ when $y' = 0$, so that x' is the line C K. Also, since $y' = \pm \sqrt{(p'x')}$, we have $KP = KP'$; and, when $x' = 0$, P P′, instead of cutting the curve, becomes a tangent to it. Hence it appears, that, *in the parabola, any line,* Q C, *parallel to the axis,* AM, *bisects all chords parallel to the tangent passing through the intersection of* C Q *with the curve; and that, if* C Q, *and the tangent through* C, *be taken as axes of coordinates, the equation of the curve is of the same form as its equation in relation to the principal axis* A M. Any line, C Q, parallel to the axis A M, is called a *diameter.*

We saw (App. No. 3) that, in the ellipse (*fig.* 7), $FP = a + \dfrac{c\,x}{a}$. Then, putting $FP = r$, and $PFM = \theta$, we have FM or $c + x = r \cos\theta$, and, therefore, $x = r \cos\theta - c$; by substituting which in the foregoing expression, we get, after some modification, $r = \dfrac{b^2}{a - c\cos\theta}$, which is the polar equation of the ellipse; the focus being pole, and FB the fixed axis. If FA were taken as the fixed axis, $\cos\theta$ would be negative, and therefore the last term of the denominator would have the contrary sign. The polar equation of the hyperbola would be found to be exactly the same; the difference in the curves arising from the circumstance, that in one of them $b^2 = a^2 - c^2$, and in the other $b^2 = c^2 - a^2$.

To find the polar equation of the ellipse, the centre, C, being pole, and C B the fixed axis, let $CP = r$, $BCP = \theta$, and, consequently, C M or $x = r \cos\theta$: also (App. No. 3), $FP = a + \dfrac{c\,x}{a}$, and $PV = a - \dfrac{c\,x}{a}$. But, by a well-known theorem, (Euc. II. A.) $FP^2 + PV^2 = 2\,FC^2 + 2\,CP^2$; whence, by the substitutions above indicated, and by dividing by 2, transposing, and substituting b^2 for $a^2 - c^2$, we get

$$r^2 - \frac{c^2 r^2 \cos^2\theta}{a^2} = b^2; \quad \text{and, thence, } r^2 = \frac{a^2 b^2}{a^2 - c^2 \cos^2\theta},$$

the required equation. If $b^2 + c^2$ be substituted for a^2 in the denominator, and $\sin^2\theta$ for $1 - \cos^2\theta$ in the result, this will become, under another form,

$$r^2 = \frac{a^2\, b^2}{b^2 + c^2 \sin^2\theta}.$$

the second order. These curves may also be obtained by the section of a cone by a plane;* and hence they are called *conic sections*. All their equations may be combined in the one, $y^2 = mx + nx^2$; this being the equation of the parabola, when $m = p$ and $n = 0$; of the ellipse, when $m = \frac{2b^2}{a}$ and $n = -\frac{b^2}{a^2}$; and of the hyperbola, when $m = \frac{2b^2}{a}$ and $n = \frac{b^2}{a^2}$.

15. If AD (*fig.* 14), a chord of a given circle, be produced to cut BC, a tangent at the extremity of the diameter AB, in C, and if CP be taken equal to AD, the locus of the point P is the curve called the *cissoid*. This curve was invented by Diŏcles, a geometrician, who is supposed to have lived about the sixth century.

To find its equation, draw PM and DE perpendicular to AB, and put $AB = 2a$, $AM = x$, and $MP = y$. Then, BE

If, in the parabola (*fig.* 9), we put FP or $PD = r$, and $PFQ = \theta$, we have $r = x + \frac{1}{4}p$. But $x = FM + \frac{1}{4}p = r \cos\theta + \frac{1}{4}p$; and, therefore, $r = r\cos\theta + \frac{1}{2}\theta$.

Hence, $r = \dfrac{\frac{1}{2}p}{1 - \cos\theta}$, which is the polar equation of the parabola; the focus being pole, and FQ the fixed axis. If FA were taken as the fixed axis, the denominator would become $1 + \cos\theta$, the numerator continuing unchanged.

* To prove this, let DE (*fig.* 10) be the diameter of a circle parallel to the base of the cone (either right or oblique) whose vertex is V, and let APBP′ be another section of the cone by a plane oblique to the base, and cutting the plane of the circle in the straight line PP′; the latter section of the cone is an ellipse. For, draw AB perpendicular to PP′, and put it equal to $2a$; put, also, $AM = x$, and $MP = y$. Then, $MB = 2a - x$; and, by trigonometry, $\sin D : \sin A :: AM$ or $x : MD$, and $\sin E : \sin B :: MB$ or $2a - x : ME$. Finding values from these for MD and ME, and taking their product, we have (Euc. III. 35) MP^2 or,

$$y^2 = (2ax - x^2) . \frac{\sin A \sin B}{\sin D \sin E}, \text{ which, if } \frac{\sin A \sin B}{\sin D \sin E} \text{ be put} = \frac{b^2}{a^2},$$

becomes the same as equation (3) already found for the ellipse.

In *fig.* 11, in which AB cuts the slant side EV, produced through the vertex, we have $MB = 2a + x$; and, by proceeding as before, we obtain the equation of the hyperbola. If we conceive a similar cone vertically opposite to DVE, we shall see, by continuing the intersecting plane into it, how the opposite hyperbola is produced.

Lastly, in *fig.* 12, if the section be parallel to a plane touching the cone in the line VE, we find the same expression as before for DM; and, ME being now a constant quantity wherever M is taken in AM, we get $y^2 = x . ME . \frac{\sin A}{\sin D}$, or, by putting p for the multiplier of x, $y^2 = px$, the equation of the parabola.

It appears, therefore, first, that if the intersecting plane pass through both sides of the same cone, so as not to be parallel to the base, the section is an ellipse; secondly, if it cut one slant side of the cone, and the continuation of the other produced through the vertex, the section is an hyperbola; and, lastly, the section is a parabola, if the intersecting plane be parallel to a plane touching the slant side of the cone.

is evidently equal to AM; and, by the nature of the circle, $ED^2 = AE.EB = 2ax - x^2$. Also (Euc. VI. 4 and 22) $AM^2 : MP^2 :: AE^2 : ED^2$, or $x^2 : y^2 :: (2a-x)^2 : 2ax-x^2$. Hence, by dividing the third and fourth terms by $2a-x$, and equalling the products of the extremes and means, we get $y^2(2a-x) = x^3$, the equation for rectangular coordinates.

To find the polar equation, putting $AP = r$, and the angle $BAP = \theta$, we have in the triangle ABC, $AC = 2a \sec\theta$; and, in ABD, $AD = 2a \cos\theta$. The difference of these gives $r = 2a(\sec\theta - \cos\theta)$, or, by an easy reduction, $r = 2a \sin\theta \tan\theta$.

16. The straight line BE (*fig.* 15, 16, 17) and the point A, without it, being given in position, if from A any straight line, AP, or AP′, be drawn cutting BE in F, and if FP or FP′ be taken always equal to a given line, BV or BV′, the locus of P (*fig.* 15) is called a *superior conchoid;* and that of P′ (*fig.* 15, 16, 17), an *inferior conchoid.* BE is called the *directrix,* and A the *pole* of the curve. If, in the inferior conchoid, FP′ (*fig.* 16) be equal to AB, so that V′ may coincide with A, the curve has a cusp at A, which is its vertex; but, if FP′ (*fig.* 17) be greater than AB, the curve has a *nodus.* The conchoid was invented by the ancient geometrician, Nicomēdes.

17. To find the equation of the superior conchoid, put $AB = a$, $FP = b$, $BR = x$, and $RP = y$. Then, we shall have $AC = a + y$, $AP^2 = x^2 + (a + y)^2$; and, by similar triangles,

$$AC^2 : AP^2 :: PR^2 : PF^2;$$

$$\text{that is, } (a+y)^2 : x^2+(a+y)^2 :: y^2 : b^2.$$

Hence, by equalling the products of the extremes and means, and by transposition, we find $x^2y^2 = (b^2-y^2)(a+y)^2$, the equation of the curve, which is therefore a line of the fourth order. The equation of the inferior conchoid is $x^2y^2 = (b^2-y^2)(a-y)^2$, as would be shown in a similar manner, or simply by taking y negative in the equation of the superior one.

18. The *logarithmic* or *logistic curve* is a line of such a nature, that the abscissa has a constant ratio to the logarithm of the ordinate. Expressing this ratio, for the Neperian logarithms, by that of 1 to $\log a$, we have $1 : \log a :: x : \log y$; whence, $\log y = x \log a$, which is

one form of the equation of the curve, and which (No. 26) may also be transformed into $y = a^x$.

19. If a circle GPD (*fig.* 18), continuing always in the same plane, roll, like the wheel of a carriage, without sliding, along a fixed straight line AB, the line APVB described by any point P in its circumference, is called a *cycloid*.

To find equations for this curve, let $AR = x$ and $RP = y$, be the coordinates of P in any of its positions, and let a be put to denote PF, the radius of the generating circle. Let also the angle $DFP = \omega$, D being the point of contact of the generating circle with AB. Then, since all the points in the circumference are applied successively to points in AB, the arc PD is evidently equal to AD; and, drawing PE parallel to AB, we have (Trig. Nos. 2, 7, and 8)

$$PD = a\omega, \quad PE = RD = a\sin\omega, \quad \text{and } FE = a\cos\omega;$$

and therefore

$$x = a(\omega - \sin\omega) \ldots (a); \quad \text{and } y = a(1 - \cos\omega) = a\,\text{versin}\,\omega \ldots (b).$$

It is often convenient in investigations to use both the equations now found. We may eliminate ω, however, in the following manner, and thus get an equation containing only a, x, and y. We have, from equation (b), $\text{versin}^{-1}\frac{y}{a} = \omega$. We have also, by the nature of the circle, $PE = \sqrt{(DE.EG)}$; that is, as we have seen above,

$$a\sin\omega = \sqrt{\{y(2a - y)\}} = \sqrt{(2ay - y^2)} \ldots (z).$$

Then, by substituting these in equation (a), we get as the required equation,

$$x = a\,\text{versin}^{-1}\frac{y}{a} - \sqrt{(2ay - y^2)} \ldots\ldots\ldots (c).$$

From this, by transposing, by dividing by a, &c. we obtain the following as another equation of this curve,

$$\frac{y}{a} = \text{versin}\,\frac{x + \sqrt{(2ay - y^2)}}{a} \ldots\ldots\ldots (d).$$

By differentiating equations (a) and (b), and by making substitutions by means of equations (b) and (z) we get

$$dx = a(1 - \cos\omega)\,d\omega = y\,d\omega, \text{ and}$$
$$dy = a\sin\omega\,d\omega = \sqrt{(2ay - y^2)}\,d\omega:$$

and hence, by dividing the latter by the former, we obtain

$$\frac{dy}{dx}=\frac{\sqrt{(2ay-y^2)}}{y}=\sqrt{\frac{2a-y}{y}}^{*}\ \ldots\ldots\ldots(e).$$

This differential equation of the curve is often employed with advantage.

From the way in which this curve is generated, it is easily seen, that it may be continued indefinitely in both directions in relation to the axis of x; and that it consists of an infinite number of equal and similar branches or parts, each commencing and terminating at points on that axis at the distance of $2\pi a$ from one another. The consideration of the equations above established would lead, analytically, to the same conclusion; and it would readily appear, that any two adjacent branches form a cusp at the point where they meet.†

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* This (No. 143) is the trigonometrical tangent of T, the angle made by the tangent to the cycloid at P with the axis of x; and its reciprocal is cot T. Now (Trig. No. 11) $1+\tan^2 T=\sec^2 T$, and $1+\cot^2 T=\operatorname{cosec}^2 T$; and (by the same No.) the cosine of an angle is the reciprocal of its secant, and its sine that of its cosecant. Hence, from the value of tan T we readily find

$$\cos T=\sqrt{\frac{y}{2a}}=\frac{\sqrt{(2ay)}}{2a}=\frac{CH}{2a},\text{ and}$$

$$\sin T=\sqrt{\frac{2a-y}{2a}}=\frac{\sqrt{\{2a(2a-y)\}}}{2a}=\frac{VH}{2a};$$

the two concluding modifications being obtained from the rightangled triangle CHV, in which $CH^2=CV.CK$, and $VH^2=CV.VK$. Now (No. 181) $VP=2VH$; and therefore $\sin T=\frac{\frac{1}{2}VP}{2a}=\frac{VP}{4a}$. Farther, also, if VK′ be taken equal to CK and K′P′ be drawn parallel to CA, we have (No. 181) $VP'=2VH'=2CH$; and therefore $\cos T=\frac{VP'}{4a}$: and if we divide the value of sin T by this, we get $\tan T=\frac{VP}{VP'}$. These conclusions might be readily derived from the rightangled triangle VHC, in which (No. 156) the angle VHK or VCH is equal to T, and therefore $\sin T=\frac{VH}{2a}$, &c.

† The curve considered above is the *common* cycloid, and it is the line which is described by a point in the circumference of a wheel rolling along a straight line, such as the head of a nail in the wheel of a carriage. If the point P be *within* the circle, the curve which it describes is called a *prolate* or *inflected cycloid*; if *without* it, a *curtate cycloid*. The equation of either of these may be investigated in nearly the same manner as that of the common cycloid, and it will be found to be $a\text{ versin}^{-1}\frac{a(r-1)+y}{ar}=x+\sqrt{\{a^2(r^2-1)+2ay-y^2\}}$, where a denotes the radius of the generating circle, and ar the distance from the centre to the describing point. If $r=1$, the curve becomes the common cycloid. It is evident, that,

20. If the sine of an arc be produced, till the line made up of the sine and the part added is equal to a fourth proportional to the versed sine of the supplement, the diameter, and the sine, the locus of the extremity of the produced line is the curve called the *witch*. If the extremity of the diameter, remote from the origin of the arc, be taken as the origin of the coordinates, and the diameter be put equal to a, the equation of the curve is easily found to be $xy^2 = a^2(a-x)$. This curve was invented by Madame Agnesi, an Italian lady of high scientific attainments. It resembles in form the superior conchoid; and a straight line, drawn through the origin of the coordinates perpendicular to the axis of x, is an asymptote to both its branches.

21. If the straight line OA (*fig.* 21) be given in position, and if the point P, in the line OP, be so taken that OP is proportional to the angle AOP, the curve OPBA, which is the locus of P, is called the *spiral of Archimēdes*, or *Conon*.*

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if the motion of the generating circle be continued, the given point will describe other equal and similar cycloids without end.

It would occupy too much space here to investigate the properties of the *epicycloid*. It may be sufficient to state, that this curve is generated by the motion of one circle rolling, either externally or internally, on the circumference of another, which is in the same plane, and is fixed; the curve being produced by the motion of a point in, within, or without the circumference of the generating circle. The fixed circle is called the *base* of the epicycloid. As in the common cycloid, the describing point is generally taken in the circumference of the generating circle; and, in this case, if the diameter of the fixed circle become infinite, the epicycloid is changed into the common cycloid.

If we now put a, a', a'', to denote the radius of the base, the radius of the generating circle, and the distance from the centre of that circle to the describing point; and if we take, as the axis of the ordinates, the straight line joining the centres of the two circles when the describing point is in that line, and as the axis of the abscissas, the diameter of the fixed circle perpendicular to the other axis, it is easy to show that

$$x = (a + a')\cos\theta + a''\cos\frac{a+a'}{a'}\theta,$$

$$y = (a + a')\sin\theta + a''\sin\frac{a+a'}{a'}\theta;$$

where θ is the angle contained by the axis of the ordinates, and the line drawn from the centre of the fixed circle to that of the generating one, when in the position corresponding to x and y. The elimination of this angle, which can always be effected in finite terms when a and a' are commensurable, will give the equation of the curve for rectangular coordinates. For the interior curve, or, as it is sometimes called, the *hypocycloid*, a'', and consequently the last terms in the foregoing equations, are to be taken negative. It is plain from what we have seen, that, like the logarithmic and numberless other curves, the common, the prolate, and the curtate cycloids are all transcendental curves; since each of them is such, that its equation cannot be expressed in a finite number of common algebraic monomials.

* The student will have a familiar idea of the nature of this curve by regarding it as the path described by a fly creeping uniformly along a spoke of a wheel which revolves uniformly.

Hence, putting the angle AOP$=\theta$, the radius vector OP$=r$, and OA, what the radius vector becomes after a complete revolution, $=a'$, we have $a' : r :: 2\pi : \theta$; whence we obtain $2\pi r = a'\theta$, the equation of the curve; or $r = a\theta$, where $a = \frac{a'}{2\pi}$. It is evident, that this spiral may be continued without end, and will extend to an unlimited distance, by the continued revolution of the radius vector. Thus, after n revolutions, or when $\theta = 2n\pi$, we have $r = na'$, where n may be any number, whole or fractional. If θ be negative, r will also be negative; and an equal and similar branch will be produced, lying in the opposite direction.

22. The *logarithmic* or *logistic spiral* is a line of such a nature, that the angle contained by the fixed axis and the radius vector has a constant ratio to the logarithm of the radius vector. Expressing this ratio, as in (App. No. 18) by that of 1 to $\log a$, we have $1 : \log a :: \theta : \log r$; whence, $\log r = \theta \log a$; and, therefore (No. 26), $r = a^{\theta}$.

23. Hence, by differentiation, and by No. 158, we get the subtangent $= -\frac{r}{\log a}$; whence it follows, that the subtangent bears a constant ratio to the radius vector. Hence, OBP (*fig.* 26) being the curve, and OG perpendicular to the radius vector OP, we have OP to OG in a constant ratio, wherever P is taken; and therefore the angle OPG is always the same, so that all the radii vectors are cut by the curve at the same angle, which is a distinguishing property of this spiral, and from which it is sometimes called the *equiangular spiral.* By dividing QG by OP, we find the tangent of the angle OPG to be $-\frac{1}{\log a}$.

24. The equation $r = a\theta^n$ comprehends an infinite number of spirals, according to the value of n. Thus, if $n = 1$, we have $r = a\theta$, the equation (App. No. 21) of the spiral of Archimedes. If $n = -1$, the equation becomes $r\theta = a$; and the curve is called the *hyperbolic* or *reciprocal spiral.* If, again, $n = -\frac{1}{2}$, and consequently $r^2\theta = a^2$, the curve is called the *lituus.* The hyperbolic spiral is represented in *fig.* 27, and the lituus in *fig.* 28; and it is easy to see,

from their equations, that if, from O as centre, circular arcs PA, P′A′, &c. be described, these arcs are equal in the hyperbolic spiral; while, in the lituus, the sectors POA, P′OA′, &c. are all equal. It is also plain, that BC is an asymptote to the one curve, and OA to the other. The curve whose equation is $v = a\theta^{1/2}$, is called the *parabolic spiral*, or the *helicoid parabola*.

25. The name *lemniscata* or *lemniscate* (derived from the Latin word *lemniscus*, *a ribbon*) is applied to several different curves. One of the most interesting is that of James Bernoulli; the equations of which are

$$(x^2+y^2)^2 = a^2(x^2-y^2), \quad \text{and } r^2 = a^2\cos 2\theta.$$

This curve is the locus of the intersection of a tangent to an equilateral hyperbola by a perpendicular from the centre. It is represented in *fig.* 29.

26. Another lemniscate, bearing a close resemblance to the foregoing, has for its equation, $a^2y^2 = a^2x^2 - x^4$. This curve arises from drawing perpendiculars to the diameter of a circle, and making each of them equal to half the sine of twice the arc which it cuts off, measured from the same extremity of the diameter.

This lemniscate is comprehended in a more general class, in which the ordinate OP (*fig.* 30) is taken equal to an nth part of the sine of BQ, which is n times BR; and a still more general class would be obtained by taking OP in any given ratio, 1 to n', to the sine of BQ. Putting $\text{CO} = x$, and $\text{OP} = y$, we shall evidently have $n'y = \sin nz$, and $x = \cos z$, z denoting BR; and from this, by eliminating (TRIG. No. 195) $\cos z$ and $\sin nz$, the equation for rectangular coordinates would be obtained.

27. The equation, $r = a\sin n\theta$, n being a whole number, comprehends lines of a lemniscate, a *foliated*, or a leaf-like form, such as the curve represented in *fig.* 31, in which $n = 5$. The number of leaves is n, when n is odd; but when n is even, the number is $2n$. If $\cos n\theta$ were used instead of $\sin n\theta$, the position of the curve but not its figure would be changed.

A curve of this kind is the locus of the point of bisection of a chord in a circle, joining two points, which move in contrary directions, with velocities in the ratio of $n-1$ to $n+1$. It may also be obtained from the motion of two points, both in the same direction, with velocities in the ratio of $1-n$ to $1+n$.* Such a curve would be obtained from the motions of the hour and minute hands of a common clock, if they were equal.

28. If a straight line, PE (*fig.* 32), move with a uniform lateral motion from the position CD, and always continue parallel to the line CD; and if, at the same time, a straight line, CP, passing through C, revolve with a uniform angular motion from the position CD, the curve GDF, HK, LM, &c. which is the locus of P, the intersection of CP and EP, is called the *quadrătrix of Dinostrătus.*

To find its equation, let $AC=a$, $AE=x$, and $EP=y$; then it is evident, that the parallel will move from A to C, while CP describes the right angle ACD: and, therefore, $AC : AE :: \frac{1}{2}\pi : ACP$; whence, $ACP=\frac{\frac{1}{2}\pi x}{a}$. We have, also, by Trigonometry, $EP=EC.\tan ACP$; that is, $y=(a-x)\tan\frac{\frac{1}{2}\pi x}{a}$, the equation required. If $x=a$, this becomes $0\times\infty$, the value of which is found (No. 122) to be $\frac{a}{\frac{1}{2}\pi}$, which is CD.

The value of y is nothing when $x=2na$, n being any number in the series, 0, ± 1, ± 2, &c.; and it is infinite when $x=(2n+1)a$, n having any of the same values except 0. Hence, the curve has an infinite number of infinite branches intersecting the axis in the points at which $y=0$; and the generating parallel is an asymptote to two branches, when it passes through the points at which y is infinite.

From the value found above for CD, we have $CD : AB :: 1 : \pi$; that is, as the diameter of a circle to its circumference. Hence, if this curve could be described geometrically, we should have the ratio of the diameter and circumference; and thence the area. It is from this property that the curve has obtained its name.

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* Every curve, whose equation is of the form,

$$r=a\sin n\theta+b\sin n'\theta+c\sin n''\theta+\&c.$$

will consist of *loops* or *leaves* of finite dimensions; but the forms and magnitudes of these will be infinitely varied, according to the values of a, b, &c. n, n, &c.

29. The *quadratrix of Tschirnhausen* possesses a similar property. This curve (*fig.* 33) is generated by the parallel motion of two lines, BK and AL, perpendicular to each other; the one at the commencement of the motion touching the circle at O, and the other coinciding with OF; their motions being so regulated, that, at the same time, one of them will become a tangent at D, and the other will coincide with DM, and that OE : OA : : OC : OD. Hence, if we put $OE = x$, $EP = y$, and $OC = a$, it is easy to show that $y = a \sin \frac{\frac{1}{2}\pi x}{a}$. If $x = 2na$, n being a whole number, this becomes 0; if $x = (4n + 1)a$, $y = a$; but if $x = (4n + 3)a$, $y = -a$: and it thus appears, that the curve consists of an infinite number of branches, equal and similar to ODF, and lying alternately on opposite sides of OFH, and in opposite directions.

30. Let OD (*fig.* 34) be any chord of a given circle, ODB, and take DP equal to the radius OA; the curve APOP′P″P‴, which is the locus of P, is called the *trisectrix*. Hence, putting $OA = a$, $OP = r$, and the angle $AOP = \theta$, we have $OD = 2a \cos\theta$; and, consequently, $r = a(2\cos\theta - 1)$, the polar equation.

If another circle be described from O as centre, with OA as radius, since OC = PD, we have CD = OP; and, since AO = AD, and the angle ADO being equal to AOD, we have also (Euc. I. 4) AC = AP; and, therefore (Euc. I. 5 and 32), APC = ACP = OAC = AOP + OAP. Hence, by taking away OAP, we have AOP = PAC; and, therefore, the arc EC is double of AC, so that the arc ACE is trisected in C. This curve, therefore, affords the means of trisecting an arc or angle; and hence it gets its name.

It is easy to show, by the usual substitution (Trig. No. 256), that the equation for rectangular coordinates is

$$y^4 + (2x^2 - 4ax - a^2)y^2 + x^2(x^2 - 4ax + 3a^2) = 0.$$

It may also be remarked, that we might take DP″ equal to OA, and that the curve thus produced would still be the same.

31. The generation of the *cardioid* differs from that of the trisectrix merely in making DP or DP″ equal to the diameter OB, instead of the radius OA. In this curve,

the interior part, APOP‴, is wanting; and, a denoting the diameter, the equations of the curve are

$$r = a(1+\cos\theta), \quad \text{and}$$

$$y^4 - 6ay^3 + (2x^2 + 12a^2)y^2 - (6ax^2 + 8a^3)y + x^4 + 3a^2x^2 = 0.$$

The equation, $r = a\cos\theta + b$, comprehends the trisectrix, the cardioid, and numberless other curves of a similar kind. It may also be remarked that the cardioid is what the epicycloid becomes, when the radii of the base and the generating circle are equal.

32. The *curve of sines*, or, as it is sometimes called, the *companion of the cycloid*, is a curve of such a nature, that, if arcs be taken in a given circle equal to its abscissas, the corresponding ordinates are the sines of those arcs. Hence, if a denote the radius, its equation is $y = a\sin\frac{x}{a}$, and it is represented in *fig.* 40; OO′, O′O″, being each equal to the semi-circumference of the circle, and the greatest and least ordinates, BC, B′C′, &c. each equal to the radius.

If the ordinates of this curve be all either increased or diminished in a constant ratio, the curve passing through the extremities of the new ordinates is called an *harmonic curve*. Hence, the equation of the harmonic curve is $y = a'\sin\frac{x}{a}$. The curve of sines, therefore, is itself a species of the harmonic curve.

33. It was shown (No. 152) that, if C (*fig.* 44) be the centre of an hyperbola, A its vertex, and consequently $CA = a$; and if, also AM and AN be each made equal to b, CM and CN produced will be asymptotes to the curve. We may now establish a curious property, which gives the equation of the curve in relation to the asymptotes as axes of the coordinates. For this purpose, through any point, P, in the curve, draw PG and PH parallel to the asymptotes. Then, the triangles PHQ and PGK are similar to CMN; and, therefore,

$$MN : MC :: KP : GP \text{ or } CH;$$

$$\text{also, } MN : MC :: PQ : PH.$$

Finding the values of CH and PH from these analogies, and taking their product, we get

$$CH.HP = \frac{MC^2}{MN^2}.KP.PQ.$$

But $KP.PQ = OQ^2 - OP^2$. Now, by the equation of the curve, $OP^2 = \frac{b^2}{a^2}(x^2 - a^2)$; and, in the triangles, CAN, COQ, we have $CA : AN :: CO : OQ$, or $a : b :: x : OQ = \frac{bx}{a}$; from the square of which, if we take the foregoing value of OP^2, we get $KP.PQ = b^2$. Now, $MN = 2b$, and $MN^2 = 4b^2$; and (Euc. I. 47) $MC^2 = a^2 + b^2$; and the expression found above for CH.HP becomes $\frac{1}{4}(a^2 + b^2)$, a constant quantity; whence it appears, that the parallelogram CHPG is always of the same magnitude, wherever P is taken, which is the property to be established.

Hence, if $CH = x$, $HP = y$, and CD, the side of the rhombus DE, $= a$, we shall have $xy = a^2$, the equation of the curve referred to the asymptotes as axes.

In the equilateral hyperbola, the equal parallelograms are rectangles, and the rhombus is a square. It is plain, that the foregoing equation does not determine the curve, unless the angle made by the asymptotes be given.

34. The involute of the circle,* that is (No. 196), the curve having the circle as its evolute, is a line which possesses several curious properties; and we may, therefore, investigate its equation. Let P (*fig.* 49) be a point in the involute AP, and PC a tangent to the generating circle; then, by the nature of the involute, CP is equal to CA. Hence, putting $OA = a$, $OP = r$, and the angle $AOP = \theta$, we have (Euc. I. 47) CP or $CA = \sqrt{(r^2 - a^2)}$; and, by Trigonometry, $\cos COP = \frac{a}{r}$, or $COP = \cos^{-1}\frac{a}{r}$; whence, $AOC = \theta + \cos^{-1}\frac{a}{r}$. Now, the semi-circumference, ACB, is πa; and we have, by proportion, $\pi : AOC :: ACB : AC$; that is, $\pi : \theta + \cos^{-1}\frac{a}{r} :: \pi a : \sqrt{(r^2 - a^2)}$, whence $a\left(\theta + \cos^{-1}\frac{a}{r}\right) = \sqrt{(r^2 - a^2)}$, the polar equation;

* If a thread, wrapped round the circumference of a fixed circle, be gradually unwound, it is plain, from the nature of involutes and evolutes (No. 201), that any point of it will describe the involute.

which might also be put under the form,

$$r\cos\left\{\frac{\sqrt{(r^2-a^2)}}{a}-\theta\right\}=a.$$

35. If a perfectly flexible and inextensible chain or cord, AVB (*fig.* 53), of uniform thickness and weight, be suspended from any two points, A and B, so as to hang freely, the curve into which it forms itself is called a *catenary*. Now, V being the lowest point, and VC a vertical line, if VR be put $=x$, $\text{RP}=y$, $\text{VP}=s$, and a be a constant quantity, it is shown by writers on mechanics, that $x^2+2ax=s^2$, which is the equation of the curve in terms of its arc and abscissa. Its equation cannot be exhibited in a common algebraic form, by means of the coordinates. It may be expressed, however, in either of the following forms:

$$adx=\sqrt{(2ax+x^2)}\,dy, \quad \text{and } \frac{y}{2a}=\log\frac{\sqrt{x}+\sqrt{(2a+x)}}{\sqrt{2a}}$$

If we take $\text{AC}=x'$, $\text{CV}=y'$, $\text{AR}'=x$, and $\text{R}'\text{P}=y$, the foregoing differential equation is easily transformed into

$$ady=\sqrt{\{2a(y'-y)+(y'-y)^2\}}\,dx.$$

36. The *elastic curve* is that into which a spring of uniform strength and elasticity is formed, when one of its ends is fixed, and the other is acted on by a weight or other force. It was so named by James Bernoulli, who proposed it for solution to the mathematicians of his day; and who, at length, at the end of three years, gave his own solution in 1694, no other having appeared. The equation of the curve, according to his principles, is

$$dy=\frac{x^2dx}{\sqrt{(a^4-x^4)}};$$

which may obviously be exhibited under various other forms.

It was shown by the same mathematician, that, if a perfectly flexible plane surface, in the form of a rectangle, be attached loosely by its extremities to two parallel horizontal lines, and if the cavity so formed be filled with a liquid which is prevented from escaping, a vertical section of that surface, in the direction of its length, will be the same as the elastic curve: and hence it follows, that this is the curve whose length being given, the centre of gravity of its area will be the lowest possible.

This curve is erroneously confounded by Dr. Hutton (*Mathematical Dictionary*), and several other writers, with the catenary.

37. If AB (*fig.* 57) be a straight line given in position, and the line VC be such, that, at every point of it, the tangent PT is equal to the given straight line VO; the line VC is called a *tractory* or *tractrix*,* and AB is called its *directrix*.

Hence, if $VO = a$, $OE = x$, and $EP = y$, we find, from No. 146,

$$y dx = -dy\sqrt{(a^2 - y^2)}; \quad \text{or, by integration,}$$

$$x = a \log \frac{a + \sqrt{(a^2 - y^2)}}{y} - \sqrt{(a^2 - y^2)}.$$

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38. A series, $t, t_2, t_3, \ldots\ldots, t_n$, the terms of which depend on one another according to an assigned law, is said to be *convergent*, when s_n, the sum of its first n terms, approaches a fixed limit, from which it may be made to differ by a quantity less than anything that can be assigned, if n be taken sufficiently great: and this limit is termed the *sum* of the series. If, however, the sum of n terms approach no fixed limit, as n is indefinitely increased, the series is said to be *divergent*, and to have no sum. The series, 1, 1, 1, ..., is evidently of the latter kind; and so also are the series, 1, 2, 3, ...; 1, —3, 9, —27, ...; and innumerable others. On the other hand, the series, $\frac{3}{10}, \frac{3}{100}, \frac{3}{1000}, \ldots\ldots$, is convergent; its sum being the interminate decimal fraction, ·3333 ......, the aggregate of any finite number of terms of which is less than $\frac{1}{3}$, but

* This curve derives its name from its having been considered to be the line described by a heavy body, P, attached to a cord, PT, of given length; the end, T, moving along OB, and the friction of the plane, VOBC, being such, that the motions of P and T will always cease together. There may be innumerable other tractories formed with peculiar properties, if OB be not taken as a straight line, but as a curve. These, in general, so far as they have yet been considered, present great difficulties in the analytical investigation of their properties, without any corresponding simplicity or elegance in the results.

which aggregate may be made to differ from $\frac{1}{3}$ by as small a quantity as we please, if a sufficient number of terms or figures be employed.* The following rules will be of use in enabling us to determine whether series are convergent or divergent;—a matter which is of importance in many analytical inquiries.

39. *If in a series,* $t, -t_2, t_3, -t_4, \ldots\ldots,$ *which has its terms positive and negative alternately, the absolute values of the terms continually diminish as n increases, so that they shall tend ultimately to become evanescent, the series is convergent, and its sum is a quantity of a value intermediate between its first term and zero.*

To prove this, let us first consider the sum of an odd number of consecutive terms, commencing with the first term of the series. This, by a change of arrangement, may be written thus;

$$t+(t_3+t_5+t_7+\ldots\ldots)-(t_2+t_4+t_6+\ldots\ldots),$$

the number of terms in each vinculum being the same as the number of those in the other. Now, since, by hypothesis, t_2 is greater than t_3, t_4 than t_5, &c. the foregoing sum will be less than t, the amount of the terms in the second vinculum being greater than that of those in the first. Again, if an additional term be taken, so as to make the number of terms even, that term, occupying a place of an

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* That a series may converge, it is plainly necessary, that its terms from its commencement, or at least those after a certain number of the leading ones, shall be each less than the one immediately preceding it. Such is the decimal fraction $\cdot 3333 \ldots$, mentioned above. On the contrary, the sum of the series, $1, 1, 1, \ldots$, tends to no limit, being 2 if we take two terms, 10 if we take ten terms; and in general the sum of n terms is n, and it may therefore be as great as we please. In like manner, by taking two, three, four, five, &c. terms of the series, $1, -3, 9, -27, \ldots$, we get, as the respective sums, $-2, 7, -20, 61$, &c. results which make no approach to a fixed limit, but which exhibit increasing divergence as the sums of more and more terms are taken. The continued diminution of the terms, however, is not the sole requisite for rendering a series convergent, as innumerable series may have this property, and yet be divergent. It may be farther remarked, that series which are divergent in the sense above explained, have a definite and determinable sum for an assigned number of their terms. Thus the sum of n terms of the series, $1, -3, 9, -27, \ldots$, is $\frac{1}{4}-\frac{1}{4}(-3)^n$. This series may be generated, in fact, by dividing 1 by 4 under the form $1+3$; and yet its sum is never $\frac{1}{4}$, but $\frac{1}{4}$ with the supplementary term, $-\frac{1}{4}(-3)^n$, a term which perpetually increases in absolute magnitude as n is augmented. On the contrary, if we divide 1 by 4 under the form $3+1$, we get the very different series, $\frac{1}{3}-\frac{1}{9}+\frac{1}{27}-\frac{1}{81}+\ldots$, the sum of which is not $\frac{1}{4}$, but $\frac{1}{4}-\frac{1}{4}(-\frac{1}{3})^n$. In this case, however, the supplementary term may be made as small as we please by the continued augmentation of n, and the series therefore converges; tending to $\frac{1}{4}$ as its limit or sum, an amount from which it may be made to differ by a quantity less than anything that can be assigned, if n be continually increased.

even order, will be negative, and will therefore still farther diminish the sum below the value of t. Even then, however, the sum will (hyp.) exceed zero, as $t+t_3+t_5+$&c. will exceed $t_2+t_4+t_6+$&c. carried to the same number of terms.

The infinite series, $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+$&c. is evidently convergent according to this rule. We know, in fact (note, p. 61), that it is the finite number 0·69314......, the Neperian logarithm of 2. It is worthy of remark, that the corresponding series, $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+$ &c. is divergent. (See ALG. p. 246.)

The series, $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+$ &c. affords another instance of convergence according to the principle above established. We know too (note, p. 61), that its value is $\frac{1}{4}\pi$, that is, 0·785398 ....

As a third example, we may consider the general series, $x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+$ &c. (No. 97), which becomes the preceding one, when $x=1$. This will plainly be convergent when x does not exceed 1, as the terms will then be perpetually diminishing in absolute magnitude. When x is greater than 1, the case is different; as though a certain number of the earlier terms of the series may go on diminishing, those that follow them will successively increase, and thus render the series divergent. To show this, let us compare the nth and the $(n+1)$th terms; that is $\pm\dfrac{x^{2n-1}}{2n-1}$ and $\mp\dfrac{x^{2n+1}}{2n+1}$. Neglecting the signs of these, which it, is merely to be recollected, are opposite, multiplying by $2n+1$, and dividing by x^{2n-1}, we get $\dfrac{2n+1}{2n-1}$ and x^2, which have evidently the same ratio as the two terms themselves: and it is easy to show that the latter of these is greater than the former whenever n is taken so great as to exceed $\dfrac{1}{2}\,\dfrac{x^2+1}{x^2-1}$, which, it is evident, may always be done, when x is greater than 1. When, therefore, x exceeds unity, such as when it is $1\frac{1}{2}$, $1\frac{1}{10}$, or the like, though at first each term may be less in absolute magnitude than the one immediately preceding it, the reverse will at length begin to take place, and accordingly the series is divergent.

This series, therefore, fails in giving the value of any arc that has its superior limit beyond $\pm \frac{1}{4}\pi$.*

40. The sum of n terms of the geometrical progression which has its first term t and its ratio r, is (ALG. 137), $t\left(\frac{1}{1-r}-\frac{r^n}{1-r}\right)$. Now, the last term of this will diminish without limit towards zero when r is less than 1; but it will increase without limit when r is greater than 1; so that in the former case the series is convergent,—in the latter divergent. It is plain, also, that when $r=1$, the series becomes $t, t, t, \ldots\ldots$, and is therefore divergent. These simple principles, as we shall see in the next No. often enables us to determine whether series of other kinds are convergent or divergent.

41. *If a series, t, t_2, t_3, &c. composed of positive terms, be such that if it be continued without limit, the quotients obtained by dividing the successive terms each by the one immediately preceding it, tend to a fixed value* r, *the series is convergent, if $r<1$; but divergent, if $r>1$.* For if $r<1$, the series tends ultimately to become a geometrical one with its successive terms diminishing, and therefore (by the preceding No.) it is convergent; while, if $r>1$, the series will tend to become one which (by the same No.) is divergent.

By means of this rule, we find that the series, $1, \frac{1}{1}, \frac{1}{1.2}, \frac{1}{1.2.3}, \ldots\ldots, \frac{1}{1.2.3\ldots(n-1)}$, is convergent; since the quotient obtained by dividing the nth term by the one preceding it is $\frac{1}{n-1}$, which tends to zero as its limit, when n is continually increased. This conclusion agrees with what we know otherwise, the limit or sum of the series being (No. 27) $2{\cdot}71828\ldots\ldots$. The general series also, $1, \frac{x}{1}, \frac{x^2}{1.2}, \frac{x^3}{1.2.3}, \ldots\ldots$ † (the developement of ε^x. No. 93) is likewise conver-

* The student will readily see that the series (No. 95) for $\sin h$ and $\cos h$ are always convergent. When h is large, however, the convergence is slow.

† In strictness the general term of this series is not, as it might at first sight appear, $\frac{x^{n-1}}{1.2.3\ldots.(n-1)}$, but the product of n factors of $1, \frac{x}{1}, \frac{x}{2}, \frac{x}{3}, \frac{x}{4}.\ldots.$; the

gent for every finite value of x: the corresponding quotient being $\frac{x}{n-1}$, which tends to evanescence as n is continually increased.

On the same principle it will appear that the series, $\frac{x}{1}, \frac{x^2}{2}, \frac{x^3}{3}, \frac{x^4}{4}, \ldots\ldots$ is convergent when x is less than 1, but divergent when it exceeds 1; since the general quotient referred to in the rule is $\frac{(n-1)x}{n}$, which, when n is infinite, becomes simply x.*

42. When r is neither greater nor less than unity, but is unity itself, the rule given in the last No. fails; and it is often difficult to decide, in that case, whether the series converges or diverges. We shall be assisted in determining this by considering the *complement* of the series, that is, the sum of the part of it which follows an assigned number (n) of its terms; as it is plain, that if this complement tend to become evanescent, as n is increased indefinitely, the series is convergent; but that otherwise it is divergent. Now, if, after the nth term, we suppose the series to become a geometrical one with decreasing terms, having the common multiplier $r=\frac{t_{n+1}}{t_n}$, we shall have its sum (ALG. 137) $\frac{t_{n+1}}{1-r}$, or $\frac{t_n t_{n+1}}{t_n - t_{n+1}}$; the latter form being obtained by restoring the value of r, and multiplying the numerator and denominator by t_n. *If, then, when n is continually increased in a series, the fraction,* $\frac{t_n t_{n+1}}{t_n - t_{n+1}}$, or its equivalent, $\frac{1}{(t_{n+1})^{-1}-(t^n)^{-1}}$, *tends to become evanescent, the series is convergent: otherwise, it is divergent.*†

This rule might be exemplified by means of the series, 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ..., already considered. We may take, how-

former expression not giving the first term (1) of the series. See De Morgan's Algebra, p. 181. This however does not affect the correctness of the result arrived at in the text in reference to the convergence of the series.

* See APP. No. 39, regarding the case in which $x=1$.

† This rule will effect all that can be done by means of the two preceding ones. When they are applicable, however, they give the required conclusions with more facility.

ever, the more general one, $\frac{1}{1^x}, \frac{1}{2^x}, \frac{1}{3^x}, \frac{1}{4^x}, \ldots\ldots$ In reference to this series, we have, after a slight reduction, and by means of the binomial theorem,*

$$\frac{1}{(t_{n+1})^{-1}-(t_n)^{-1}}=\frac{1}{(n+1)^x-n^x}=\frac{1}{xn^{x-1}+\frac{1}{2}x(x-1)n^{x-2}+\ldots\ldots}.$$

Now, if x be greater than 1, and n infinite, any term in the denominator having a positive index will be infinite, while the terms having negative indices will vanish; but if x be less than 1, all the terms in the denominator will vanish. Hence, in the former case, the series will be convergent, in the latter divergent. When $x=1$, the denominator will be 1 also, and the series divergent, as we know otherwise.†

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43. In section IX. we saw the method of determining a definite integral,‡ when the indefinite or general one has

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* In many other instances besides the present one, the character of the complement is discovered by developing one more of the quantities concerned, by means of the binomial theorem, by multiplication, or otherwise.

† The student may find it useful to prove the following propositions :

1. The series $\frac{1}{3}, \frac{1}{5}, \frac{1}{9}, \ldots, \frac{1}{2^n+1}$ is convergent. So also is $\frac{1}{1}, \frac{1}{3}, \frac{1}{7}, \ldots \frac{1}{2^n-1}$.

2. The series, $x\sin\theta$, $x^2\sin 2\theta$, $x^3\sin 3\theta, \ldots$, is convergent when x is between 1 and -1. So likewise is the series $x\cos\theta$, $x^2\cos 2\theta$, $x^3\cos 3\theta \ldots$

3. The series, $\frac{x}{x+1}, \frac{x^2}{x^2+2}, \frac{x}{x^3+3}, \ldots$, is convergent when $x < 1$.

4. The series, $\frac{1}{x+1}, \frac{1}{x^2+2}, \frac{1}{x^3+3}, \ldots$, is convergent when x exceeds the limits 1 and -1.

5. The series, $\frac{1}{1.2}, \frac{1}{2.3}, \frac{1}{3.4}, \ldots; \frac{1}{3.5}, \frac{1}{4.6}, \frac{1}{5.7}, \ldots$; and $\frac{5}{1.2.3}, \frac{6}{2.3.4}, \frac{7}{3.4.5}, \ldots$, are all convergent.

6. The series, $\frac{x}{1}, -\frac{x^2}{2}, \frac{x^3}{3}, -\frac{x^4}{4}, \ldots$, is convergent when x is either 1, or lies between 1 and -1.

7. The series, $\frac{x}{1}, -\frac{x^3}{3}, \frac{x^5}{5}, -\frac{x^7}{7}, \ldots$ converges only when x does not exceed the limits of 1 and -1.

‡ The following illustrations will throw some light on the nature of definite integrals, and on that of integration generally.

Let P_0P (*fig.* 52) be a curve, and let us consider the area of the space bounded by AB, the curve, and the ordinates AP_0, BP; the ordinates between A and B been found. In many instances, however, of an interesting kind, definite integrals may be readily obtained without the previous determination of the indefinite ones; and in cases also in which the latter are not determinable in finite

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forming either an increasing series or a decreasing one; and the abscissas OA and OB being denoted respectively by a and b. Let AB be divided into equal parts AM_1, M_1M_2, &c.; and construct the diagram by drawing parallels to the axes OX, OY. Then it is plain that the area under consideration is of a magnitude intermediate between the area of the exterior polygon $AC_0P_1C_1P_2C_2PB$, and that of the interior one $AP_0I_1P_1I_2P_2IB$. In this construction (Euc. I. 36) the rectangle C_1D is equal to IC_2, and C_0C_1 to I_2C_1; and there would be like equalities were AB divided into more equal parts, however great their number might be. Hence, therefore, the rectangle I_1D is obviously equal to the difference of the areas of the two polygons. In this rectangle, the side P_0D is the difference of the extreme ordinates AP_0, BP, and is, therefore, always of the same magnitude so long as those ordinates retain the same position. By increasing, however, the number of the equal parts AM_1, M_1M_2, &c. and thus lessening the magnitude of each, we may render the breadth AM_1 or P_0I_1 as small as we please; and therefore the polygons may be made to differ from one another, and from the intermediate curvilineal area by a quantity less than anything that can be assigned. Hence, if the number of equal parts into which AB is divided, be increased indefinitely, the curvilineal area may ultimately be considered as equal to either of the polygons, and may, therefore, be regarded as the sum of an infinite number of infinitely small rectangles. But (Sect. IX.) this area is equal to $\int_a^b fxdx$; and it therefore follows, that this integral is the *sum* of an infinite number of infinitely small elements. It was for this reason that Leibnitz gave the name of *sum* to every result thus obtained;—a term instead of which the Bernoullis and subsequent writers have used the nearly synonymous one, *integral*. For this reason also, the term *quadrature* is sometimes applied to the operation by which a definite integral, such as $\int_a^b fxdx$, is found. Hence, also, the finding of the value of $\int fxdx$ between assigned limits by any of the established rules of the integral calculus, may be regarded merely as a short and easy method of determining the sum of an infinite number of infinitely small quantities, each depending on the nature of fx.

The subject under consideration will be still farther illustrated by means of the following particular example of a simple kind. Let the curve (*fig.* 52) be a common parabola, having $y = x^2$ as its equation; and let it be required to find the area of the curvilineal quadrilateral AP_0PB; OA, as already mentioned, being denoted by a, and OB by b. Then, by dividing AB into any number n of equal parts, and, for brevity, calling each of them h, so that $b - a = nh$, we can find the area of either the exterior or the interior polygon; and, for our present purpose, it is a matter of indifference which we employ. Taking the exterior one, we have its area equal to

$$BP \times h + M_2P_2 \times h + \&c. \text{ or } \left\{b^2 + (b-h)^2 + (b-2h)^2 + \ldots + [b-(n-1)h]^2\right\} h.$$

By actual development this becomes

$$nb^2h - 2\left\{1 + 2 + 3 + \ldots + (n-1)\right\} bh^2 + \left\{1^2 + 2^2 + \ldots + (n-)1^2\right\} h^3;$$

or (Alg. Nos. 133 and 258),

$$nb^2h - n(n-1)bh^2 + \tfrac{1}{6}(2n^3 + 3n^2 + n)h^3.$$

If in this we substitute $b-a$ for nh, we get

$$(b-a)b^2 - (b-a)^2 x \frac{n-1}{n} + (b-a)^3 \frac{2n^3 + 3n^2 + n}{6n^3},$$

which is the area of the exterior polygon, whatever may be the number of its sides. If we take n infinite, this area will become simply

$$(b-a)b^2 - (b-a)^2 b + \tfrac{1}{3}(b-a)^3;$$

terms. The limits of the present work will permit the insertion of only one or two examples; and we may first take the following.

Let it be required to find the integral of $\frac{x^{2n}dx}{(1-x^2)^{\frac{1}{2}}}$ between the limits $x=0$ and $x=1$. The general integral of this, according to the second integral in No. 279, is

$$\int\frac{x^{2n}dx}{\sqrt{(1-x^2)}}=-\frac{x^{2n-1}\sqrt{(1-x^2)}}{2n}+\frac{2n-1}{2n}\int\frac{x^{2n-2}dx}{\sqrt{(1-x^2)}}.$$

Now, the first term of the second member of this equation vanishes both when $x=0$ and $x=1$; and therefore integrating between these limits, we get, simply,

$$\int_0^1\frac{x^{2n}dx}{\sqrt{(1-x^2)}}=\frac{2n-1}{2n}\int_0^1\frac{x^{2n-2}dx}{\sqrt{(1-x^2)}}.$$

By repeated integrations of the second member, and by rejecting, as above, in every instance, the first term of the integral, we should evidently get, at length,

$$\int_0^1\frac{x^{2n}dx}{\sqrt{(1-x^2)}}=\frac{1.3.5.7\ldots\ldots(2n-1)}{2.4.6.8\ldots\ldots\ldots\ldots 2n}.\ \tfrac{1}{2}\pi,$$

the integral of $\frac{dx}{\sqrt{(1-x^2)}}$ being $\sin^{-1}x$, the value of which is 0 for $x=0$, and $\frac{1}{2}\pi$ for $x=1$.

44. As a second example, let it required to integrate $\frac{x^{2n+1}dx}{\sqrt{(1-x^2)}}$ between the same limits, $x=0$ and $x=1$. By

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or, by contraction, $\frac{1}{3}(b^3-a^3)$, the area of the curvilineal quadrilateral AP_0PB;—a result which is exactly the same as that which is obtained by integrating $dA=x^2dx$ between the limits a and b; so that this definite integral is the *sum* of an infinite number of infinitely small elements.

From these views it will be seen, that a definite integral is not a function of the variable of which the proposed differential is a function; but that it depends solely on the magnitudes determining the limits between which it is taken, and on the constant quantities involved in the differential. As a particular example, if we integrate $fxdx=(x^2+px+q)\,dx$, we get, as the indefinite or general integral, $\int fxdx=\frac{1}{3}x^3+\frac{1}{2}px^2+qx+C$, which is a function of x, and which may be adapted to any particular supposition by assigning the requisite value to C. If, however, we integrate between the limits $x=a$ and $x=b$, we get the definite integral

$$\int_a^b fxdx=\tfrac{1}{3}(b^3-a^3)+\tfrac{1}{2}p\,(b^2-a^2)+q\,(b-a);$$

an expression which depends solely on the limits a and b, and on the original constant quantities p and q.

a process in every respect similar to the preceding, we should find

$$\int_0^1 \frac{x^{2n+1}dx}{\sqrt{(1-x^2)}} = \frac{2.4.6.8 \ldots\ldots\ldots 2n}{3.5.7.9 \ldots\ldots (2n+1)} \int_0^1 \frac{xdx}{\sqrt{(1-x^2)}}.$$

The integral of the part of this affected by the sign $\int$ is $-\sqrt{(1-x^2)}$; and as this becomes 0 for $x=1$, and -1 for $x=0$, we get 1 for its value between the limits $x=0$ and $x=1$; and we have, therefore, finally

$$\int_0^1 \frac{x^{2n+1}dx}{\sqrt{(1-x^2)}} = \frac{2.4.6.8 \ldots\ldots 2n}{3.5.7.9 \ldots\ldots (2n+1)}.$$

45. If $n=\infty$, the results in the two preceding Nos. become equal; since, in that case $2n$ and $2n+1$ are equal. Hence we have

$$\frac{1.3.5.7 \ldots\ldots}{2.4.6.8 \ldots\ldots} \times \tfrac{1}{2}\pi = \frac{2.4.6.8 \ldots\ldots}{3.5.7.9 \ldots\ldots};$$

and therefore, by division,

$$\tfrac{1}{2}\pi = \frac{2.2.4.4.6.6.8.8 \ldots\ldots}{1.3.3.5.5.7.7.9 \ldots\ldots};$$

a remarkable formula which was discovered by Dr. Wallis of Oxford, in the seventeenth century. This may obviously be put under several different forms. One, for instance, is

$$\tfrac{1}{2}\pi = \frac{2^2}{2^2-1} \cdot \frac{4^2}{4^2-1} \cdot \frac{6^2}{6^2-1} \cdot \frac{8^2}{8^2-1} \cdot \ldots\ldots\ldots$$

46. As another example, let it be required to find the integral of $\sin^{2m}xdx$ between the limits $x=0$ and $x=\frac{1}{2}\pi$, m being a whole positive number. For effecting this, we have (No. 297)

$$\int \sin^{2m}xdx = -\frac{\sin^{2m-1}x\cos x}{2m} + \frac{2m-1}{2m}\int \sin^{2m-2}xdx;$$

and, therefore,

$$\int_0^{\frac{1}{2}\pi} \sin^{2m}xdx = \frac{2m-1}{2m}\int_0^{\frac{1}{2}\pi} \sin^{2m-2}xdx.$$

In a similar manner we should find the integral of the second member now obtained to be

$$\frac{(2m-3)(2m-1)}{(2m-2)2m}\int_0^{\frac{1}{2}\pi}\sin^{2m-4}x dx.$$

By continuing this process sufficiently far, we should evidently get, at length,

$$\int_0^{\frac{1}{2}\pi}\sin^{2m}x dx=\frac{1.3.5.7\ldots\ldots(2m-1)}{2.4.6.8\ldots\ldots 2m}\cdot\frac{\pi}{2};$$

the part affected by the sign $\int$ becoming at last simply

$$\int_0^{\frac{1}{2}\pi} dx, \text{ or } \frac{1}{2}\pi.$$

Had we used $2m+1$ instead of $2m$, we should have got by a process exactly similar,

$$\int_0^{\frac{1}{2}\pi}\sin^{2m+1}x dx=\frac{2.4.6\ldots\ldots 2m}{3.5.7\ldots\ldots(2m+1)}.$$

From these two formulas, by taking m infinite, and equalling the results, we should obtain Wallis's formula already found in No. 45. It may be farther remarked, that we should have arrived at exactly the same final results, had we employed cosines instead of sines, and used the second formula in No. 297 instead of the first.*

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47. It may be interesting for the pupil to be made acquainted, in some degree, with the principal methods of investigation that have been employed in some of the higher parts of Mathematics, in addition to the method which has been made the foundation of the present work. Of these, it may be sufficient to mention the Method of

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* The reason of this is plain, since the entire range of the sines of arcs from 0 up to $\frac{1}{2}\pi$ is exactly the same as that of the cosines of the corresponding arcs from $\frac{1}{2}\pi$ down to 0; and 0 and $\frac{1}{2}\pi$ are the limits under consideration. From this relation, we have a neat and easy method of determining the particular integral,

$$\int_0^{\frac{1}{2}\pi}\sin^2 x dx, \text{ or } \int_0^{\frac{1}{2}\pi}\cos^2 x dx:$$

for, since these are equal, we get, by taking half their sum,

$$\int_0^{\frac{1}{2}\pi}\sin^2 x dx, \text{ or } \int_0^{\frac{1}{2}\pi}\cos^2 x dx=\tfrac{1}{2}\int_0^{\frac{1}{2}\pi}dx=\tfrac{1}{4}\pi.$$

The same would be readily found from the indefinite integral of $\sin x^2 dx$, or $\cos^2 x dx$.

Exhaustions, the Method of Indivisibles, the Infinitesimal Calculus, the Method of Fluxions, and the Method of Lagrange.

The Method of Exhaustions was employed by the ancients, and was strictly rigorous in its principles; but it was too tedious and operose in its application to be of extensive utility as an instrument of investigation. It is exemplified in the second proposition of the twelfth book of Euclid, in which it is proved, that circles are to one another as the squares of their diameters. In demonstrating this, it is first shown, by inscribing successively, in one of the circles, regular polygons of four sides, eight sides, sixteen sides, &c. and thus tending to *exhaust* the area of the circle, that a polygon may be found which will differ from the circle by a quantity less than any magnitude that can be assigned; and then, since similar polygons inscribed in the circles, are (EUC. XII. 1) proportional to the squares of the diameters, the truth of the proposition is established by means of an indirect proof.

48. The Method of Indivisibles was published by its inventor, Cavalerius of Milan, in 1635. In this method, all magnitudes are regarded as resolvable into *indivisible* elements, or elements so small as not to admit of farther division; bodies being conceived to be composed of surfaces, surfaces of lines, and lines of points. Thus, a parallelogram is supposed to be made up of straight lines parallel to one of its sides; and the number of these lines being denoted by the number of points contained in the straight line which cuts them perpendicularly, it follows, that the area of the parallelogram will be found by multiplying one of the lines by their number; that is, the length of the figure by its perpendicular breadth. In like manner, a plane triangle may be conceived to be resolved into lines parallel to its base, forming an equidifferent series, the first term of which is zero, and the last the base; and the sum of such a series being equal to half the product of the greatest term and the number of terms, it follows, that the area of the triangle is half the product of the base and perpendicular.

This method is evidently erroneous in principle; since no number of points can make a line, no number of lines a surface, and no number of surfaces a body. It gives, however, true results from a compensation of errors. It is, in fact, only the Method of Exhaustions disguised and contracted; and, though it is, in general, much easier and more simple in its application, it is less satisfactory to the mind.

49. Both these methods sink almost into insignificance, when compared with the Method of Fluxions and the Infinitesimal Calculus, which are, in reality, virtually the same as the Differential and Integral Calculus. These were both published to the world in the latter part of the seventeenth century, the first by Newton, and the second by Leibnitz.

In the Infinitesimal Calculus, magnitudes are regarded as composed of an infinite number of infinitely small magnitudes of their own kind, any one of which is called an *infinitesimal*. Thus, a parallelogram may be conceived to be made up of an infinite number of infinitely small parallelograms, formed by drawing lines parallel to one of its sides. Any of these infinitesimals, again, is supposed to consist of an infinite number of parts, infinitely small in comparison of itself. These are *infinitesimals of the second order;* and from these are derived, in a similar manner, *infinitesimals of the third order;* and so on, as far as we please. Hence it follows, that, "without sensible error," an infinitesimal of any order may be rejected in comparison of the magnitude of which it is an infinitely small part, producing no change on that magnitude by being added to it, or taken from it.

Now, in this Calculus, if any variable magnitude receive an infinitely small increase, that increase, or infinitesimal, is called its differential; and the amount of the change so produced on any function of the variable, is called the differential of the function. By means of this principle, the differentials of functions may be found with great ease; and it is shown by Carnot, in his *Réflexions sur la Métaphysique du Calcul Infinitésimal*—a work of great acuteness and ingenuity, that the results so obtained are not merely free from "sensible error," but that they are rigorously exact.

50. To illustrate this method, let us find the differential of $u = xy$. Here, by increasing x by dx, and y by dy, denoting by u' what the function then becomes, and subtracting, we obtain

$$u' - u = xdy + ydx + dxdy;$$

from which we are to reject the last term, as it is infinitely small compared with either xdy or ydx.* By this means, therefore, we find $du = xdy + ydx$, the same result as in No. 11.

In like manner, if $u = \sin x$, we have

$$u' - u = \sin(x + dx) - \sin x, \text{ or}$$

$$u' - u = \sin x \cos dx + \cos x \sin dx - \sin x.$$

Now, if dx be infinitely small, we have $\cos dx = 1$, and $\sin dx = dx$; and, therefore, $d\sin x = \cos xdx$, as in No. 32.

On the same principles, we may find the differential of the curvilineal area, ACMP (*fig.* 5). For, if MQ be taken infinitely small, so as to become dx, the increment PMQR, which will then become the differential of the area, will differ only by a quantity infinitely small from the rectangle PMQD, when diminished indefinitely by the same means; and, therefore, this differential will become simply ydx, the same as in No. 166.

If, again, the differential of the arc AP be required, we shall have this differential equal to PR, when MQ or PD is taken infinitely small, and the rightangled triangular space PDR will be infinitely nearly rectilineal, having $PD = dx$, $DR = dy$; and, therefore, PR or $ds = \sqrt{(dx^2 + dy^2)}$, as in No. 178.

The other formulas that we have already obtained in the Differential Calculus, might be derived by the same method, and, in general, by much shorter investigations, than those which are conducted on stricter principles. The same method will also afford useful abbreviations in many other cases, particularly in physical inquiries.

The foregoing examples will give the reader some idea of the way in which investigations were formerly conducted by the Infinitesimal Method. Such investigations, when properly managed, always lead to true results, and by short and easy routes. The mode of reasoning, however, has been very generally objected to; and, accordingly, several late writers who have made this Method the basis of elementary treatises, have exhibited it in a different and a

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* The second member may be written in either of the forms, $(x+dx)\, dy + ydx$, and $(y + dy)\, dx + xdy$; in each of which the second term in the vinculum may be omitted.

very improved form. Of such writers, it may be sufficient to mention Duhamel and Cournot.

50. In the Method of Fluxions, quantities are regarded as being generated by motion; a line being supposed to be produced by the motion of a point, a surface by the motion of a line, and a body by the motion of a surface. The comparative velocities or rates of such motions are called the *fluxions* of the magnitudes thus generated, and the magnitudes so produced are called *fluents*; names which are derived from the supposed motion, or *flowing*, of the generating point, line, or surface. By the assumption of this principle, all the rules for the differentiating of functions may be derived on strictly demonstrative principles: and they will be found to be the same as those obtained by the other methods. Their investigation, on this principle, will be found in the English treatises on Fluxions, one of the best and plainest of which is that of Simpson.

It has been objected to the Method of Fluxions, that, though correct in principle, it involves the ideas of motion and time, which are foreign to the nature of the magnitudes that are the subject of investigation in pure mathematics. Besides this, without any necessity in the nature of the system, Newton and his countrymen unfortunately adopted a notation far inferior to that of Leibnitz. In this notation, instead of the letter d being prefixed, one or more dots were placed over the symbol representing the variable. Some idea of the clumsiness belonging, in many cases, to the English notation, may be formed from the following instances of the modes of expressing the same quantities by the two systems; the second and fourth lines exhibiting the mode of expressing, in that notation, the corresponding quantities in the first and third:

$$dy,\ d^2y,\ d^3y,\ d^4y,\ d^ny, \qquad d(a^x) \qquad d\sin x,$$

$$\dot{y},\ \ddot{y},\ \dddot{y},\ \ddddot{y},\ \overset{n}{\dot{y}}, \qquad \dot{(a^x)},\ \text{or}\ \dot{\overline{a^x}}, \qquad \dot{(\sin x)},\ \text{or}\ \dot{\overline{\sin x}},$$

$$d^4\frac{1}{\sqrt{(1-x^2)}}, \qquad d^n\log(x^2+a^2), \qquad \&c.$$

$$\left\{\frac{1}{\sqrt{(1-x^2)}}\right\}^{::}, \quad \text{or}\ \overline{\frac{1}{\sqrt{(1-x^2)}}}^{::}, \quad \{\log(x^2+a^2)\}^{\overset{n}{\cdot}}, \quad \&c.$$

These instances show, plainly, the inconvenience of this system of notation; and, in many other cases, it is still greater. Its inferiority, indeed, has been considered, and perhaps justly, as one cause of the small progress made in later times, till recently, by the British mathematicians, in the higher departments of science.

51. In the Method of Lagrange, propounded, with all the talent and ingenuity of the author, in his two works on the Theory of Functions, he endeavours to do away with the consideration of infinitely small or evanescent elements, and to conduct the investigations in the differential and integral calculus by means of finite algebraical quantities. This theory, whether from the fame of its distinguished author, or from other causes, was for some time very generally adopted. The objections, however, of Woodhouse and others have lessened its hold on the minds of mathematicians; and the latest and best writers both on the continent and in this country, such as Cournot, Duhamel, Moigno, O'Brien, Walton, Hemming, Price, and Professors De Morgan and Young, have made the Method of Limits, or, what is virtually equivalent, the Infinitesimal Method, the basis of their treatises. Investigations founded on either of these principles, are in general more concise in a considerable degree than those which are furnished by the Method of Lagrange. It may be remarked also, that the alleged purely algebraic character of Lagrange's Method is only apparent; as when investigations are conducted according to its principles, it is necessary, directly or indirectly, as in other methods, to employ the consideration of infinitely small quantities, however ingeniously their presence may be kept out of view: and it may be farther stated, that Lagrange's proof of the possibility of developing $f(x+h)$ in a series of ascending powers of h with whole positive indices,—a development which forms the groundwork of his system,—has been objected to, as being insufficient. For these reasons, the Method of Limits has been adopted in this edition of the present work.

TABLE OF NUMBERS AND FORMULAS.*

$\varepsilon = 2{\cdot}718281828$, p. 26. Modulus of the common logarithms $= 0{\cdot}4342944819$, p. 65. $\pi = 3{\cdot}14159265$, p. 66.

Binomial Theorem, $(x+h)^n = x^n + nx^{n-1}h + \frac{n}{1}\cdot\frac{n-1}{2}x^{n-2}h^2 + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}x^{n-3}h^3 +$ &c. p. 19.

Taylor's Theorem, $f(x+h) = fx + \frac{dfx}{dx}\cdot\frac{h}{1} + \frac{d^2fx}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3fx}{dx^3}\cdot\frac{h^3}{1.2.3} +$ &c. p. 47.

$a^x = 1 + \frac{x}{1}\log a + \frac{x^2}{1.2}(\log a)^2 + \frac{x^3}{1.2.3}(\log a)^3 +$ &c. p. 55.

$$\left.\begin{array}{l}\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \text{\&c.} \\ \cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \text{\&c.}\end{array}\right\}\text{p. 56.}$$

$\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 +$ &c. p. 57.

$\sin^{-1}x = x + \frac{1}{2}\,\frac{x^3}{3} + \frac{1.3}{2.4}\,\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\,\frac{x^7}{7} +$ &c. p. 58.

$$\left.\begin{array}{l}\varepsilon^{x\sqrt{-1}} + \varepsilon^{-x\sqrt{-1}} = 2\cos x; \\ \varepsilon^{x\sqrt{-1}} - \varepsilon^{-x\sqrt{-1}} = 2\sqrt{-1}.\sin x.\end{array}\right\}\text{p. 60.}$$

$(\cos x \pm \sqrt{-1}.\sin x)^n = \cos nx \pm \sqrt{-1}.\sin nx$, p. 60.

$$\left.\begin{array}{l}\text{Equation of Tangent, } \eta - y = (\xi - x)f^1x; \\ \text{———— Normal, } (\eta - y)f^1x + \xi - x = 0,\end{array}\right\}\text{p. 90.}$$

Subtangent $= \frac{y}{f^1x}$; Subnormal $= yf^1x$, pp. 90, 91.

$d\mathrm{A} = ydx$; and $d\mathrm{A} = \frac{1}{2}r^2d\theta$, p. 100.

$ds^2 = dx^2 + dy^2$; and $ds^2 = dr^2 + r^2d\theta^2$, pp. 105, 106.

$dv = \pi y^2dx$; and $d\mathrm{S} = 2\pi yds = 2\pi y\sqrt{(dx^2+dy^2)}$, p. 108.

$\varrho = \frac{ds^3}{d^2ydx} = \frac{(dx^2+dy^2)^{\frac{3}{2}}}{d^2ydx}$; p. 113.

$\varrho = \frac{(r^2d\theta^2+dr^2)^{\frac{3}{2}}}{(r^2d\theta^2 - rd^2r + 2dr^2)d\theta}$, p. 115.

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* This table, like the one in p. 41, will be useful to the student by affording him easy means of reference to a number of important results scattered up and down in the body of the work.

EQUATIONS OF CURVES.

Ellipse, $a^2y^2 = b^2(a^2 - x^2)$; and $a^2y^2 = b^2(2ax - x^2)$. p. 275.
Hyperbola, $a^2y^2 = b^2(x^2 - a^2)$; $a^2y^2 = b^2(2ax + x^2)$; and $xy = a^2$. p. 277. p. 292.
Parabola, $y^2 = 4ax = px$, p. 279.

General equation of the Conic Sections, $y^2 = mx + nx^2$; in which, for the Parabola, $n = 0$; for the Ellipse, $m = \frac{2b^2}{a}$, and $n = -\frac{b^2}{a^2}$; and for the Hyperbola, $m = \frac{2b^2}{a}$, and $n = \frac{b^2}{a^2}$, p. 282.

Cissoid, $y^2(2a - x) = x^3$, and $r = 2a \sin\theta \tan\theta$, p. 283.
Conchoid, $x^2y^2 = (b^2 - y^2)(a \pm y)^2$, p. 283.
Logarithmic Curve, $\log y = x \log a$, and $y = a^x$, pp. 283, 284.

Cycloid, $x = a(\omega - \sin\omega)$, and $y = a \text{ versin}\,\omega$: also, $\frac{y}{a} = \text{versin}\frac{x + \sqrt{(2ay - y^2)}}{a}$, and $\frac{dy}{dx} = \sqrt{\frac{2a - y}{y}}$, pp. 284, 285.

Witch, $xy^2 = a^2(a - x)$, p. 286.
For Spirals, *see* page 286, 287, and 288.

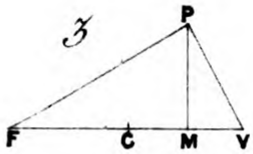
ERRATA.

In the following list of Errata, which the reader is requested to correct, the letter *b* in the second column means *from the bottom of the page.*

Page.	Line.	Mistake.	Correction.
11	4 *b.*	O	Q
12	12 *b.*	24	26
15	11	u	n
25	6	$(1+hx^{-1})$	$\log(1+hx^{-1})$
27	11 *b.*	straight lines	a straight line
34	4 *b.*	$-\frac{1}{2}x$	$-\frac{1}{2x}$
35	3, 15	u, x_2	a, x^2
,,	6	$\sqrt{\frac{dx'}{x'^2 - a^2}}$	$\frac{dx'}{\sqrt{(x'^2 - a^2)}}$
41	9	a^1	$a^2)$
58	3	2^2	3^2
59	1 *b.*	64	66
61	2, 12	88, $= x$	87, $= \frac{1}{2}x$
72	4	$\frac{1}{6}$, $\frac{1}{24}$	$\frac{1}{2}$, $\frac{1}{6}$
87	17*b.* 16*b.*	$-20x^3$, $\frac{1}{2}$	$+20x^3$, $\frac{1}{3}$
89	11	$a($	$a\sqrt{(}$
91	1 *b.*	OT	—OT
99	10 *b.*	CM	OM
108	10	others	other
111	6, 9	v, v, s	r, r, A
112	2, 14	19, 4	22, 23
130	5	v	$\frac{v}{z}$
141	17	OX	FG
157	4	variable	unintegrated
185	8	XV.	XVI.
257	3 *b.*	(+)	$(x+r)^x$
272	5 *b.*	33	32
280	24, 25	O	M
288	Heading	u	a
294	Heading	F	T

In page 33, omit ( ) in lines 4 *b.* and 3 *b.* In No. 82, page 45, insert, throughout, a before x^{-3}, x^{-4}, and x^{-5}. In page 71, line 16, for "that will be shown hereafter," put "shown in the last Section." In lines 11 and 12, page 78, interchange 9 and 36. In page 93, change D into T′; and in line 3, change the signs of the expressions for OT. In page 99, before Exer. 4, insert, "Find the subnormals for the two following curves." In page 211, the foot-notes should interchange their places.

3



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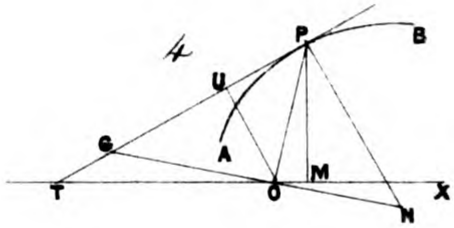

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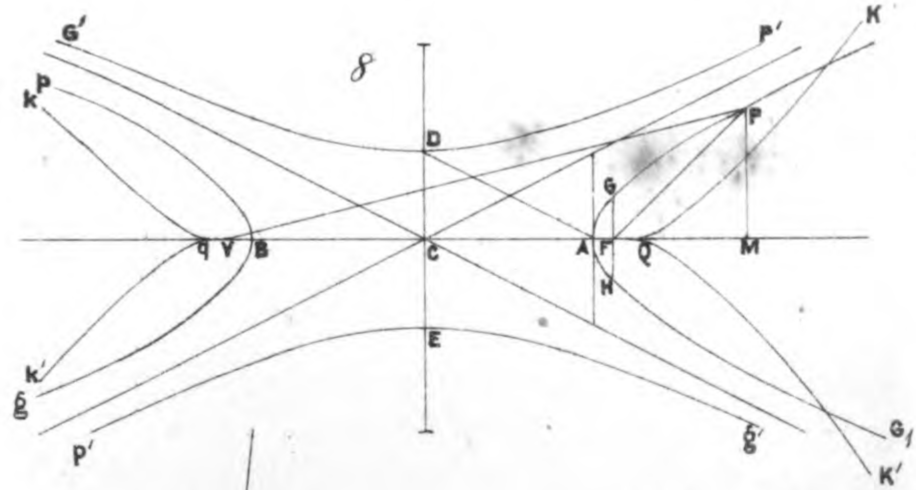

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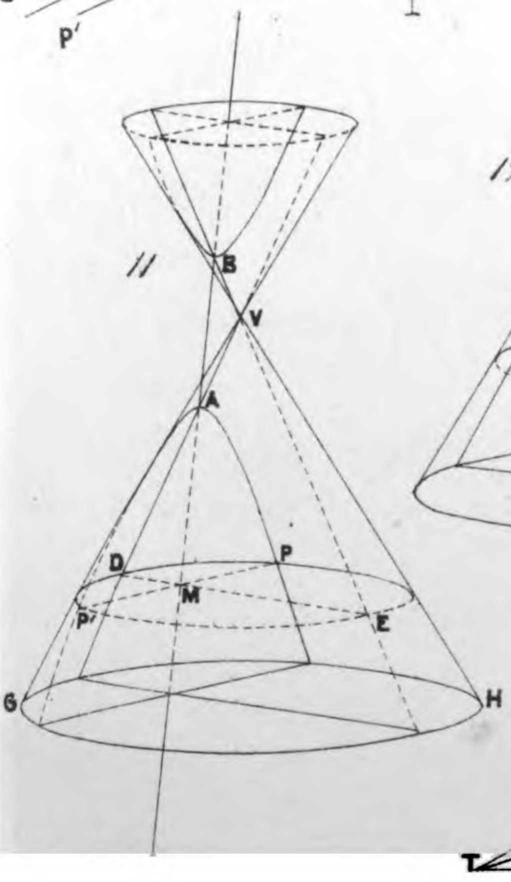

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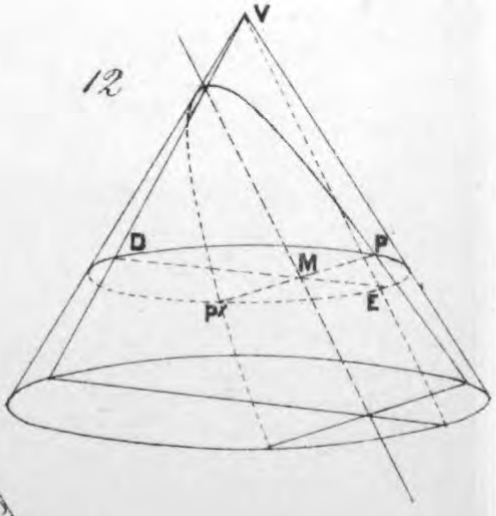

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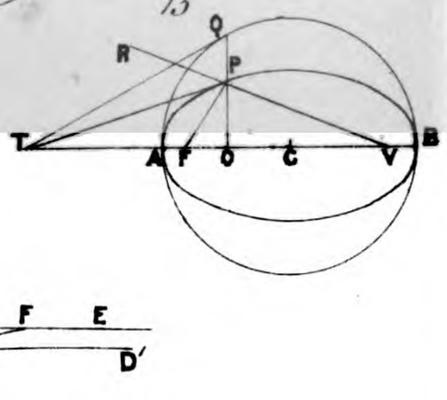

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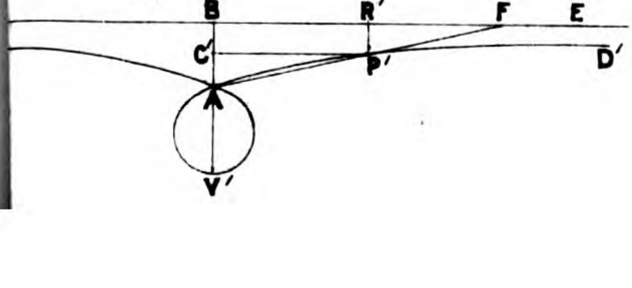

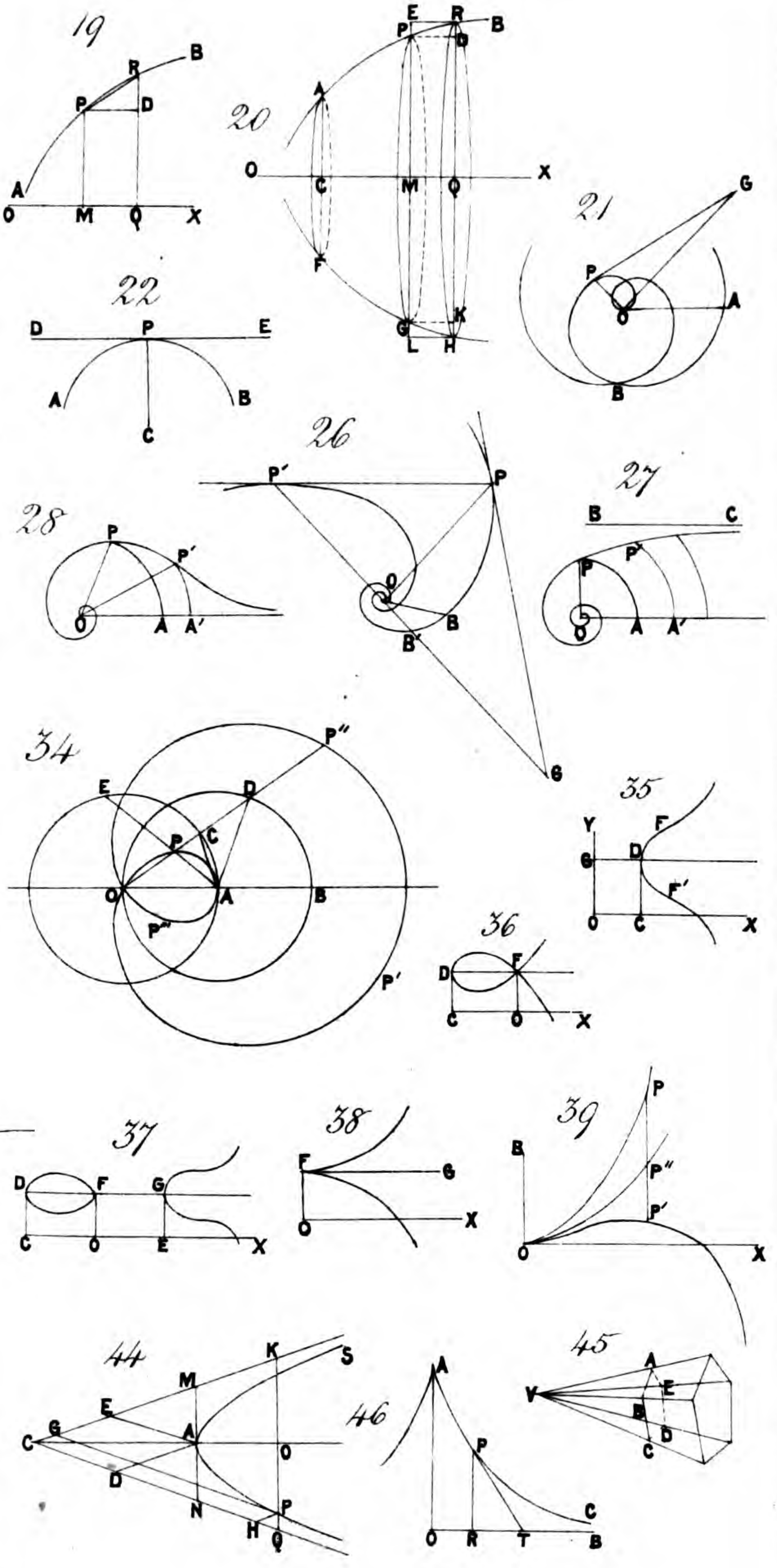
19
A B R P D O M Q X
20
E R P B D A O X C M Q F G K L H
21
G P O A B
22
D P E A B C
26
P′ P O B B′ G
27
B C P P′ O A A′
28
P P′ O A A′
34
P″ E D P C O A B P‴ P′
35
Y F G D F′ O C X
36
D F C O X
37
D F G C O E X
38
F G O X
39
P B P″ P′ O X
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A P C O R T B


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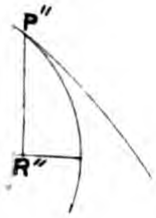

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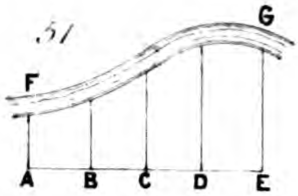

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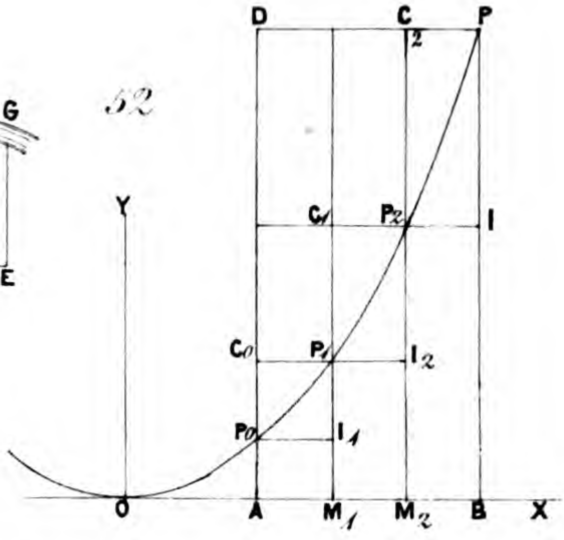

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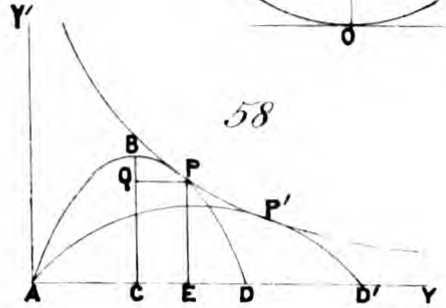

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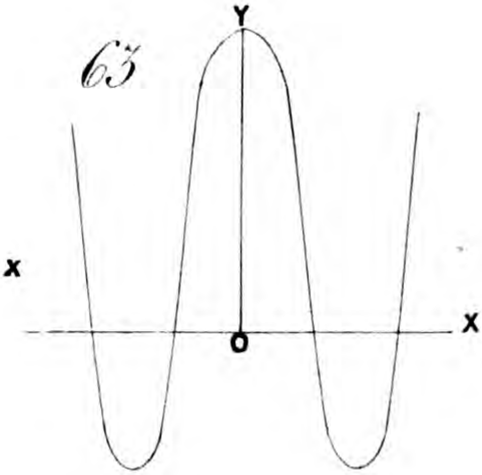

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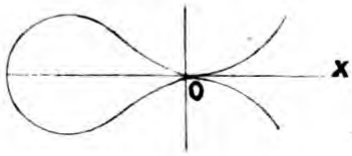

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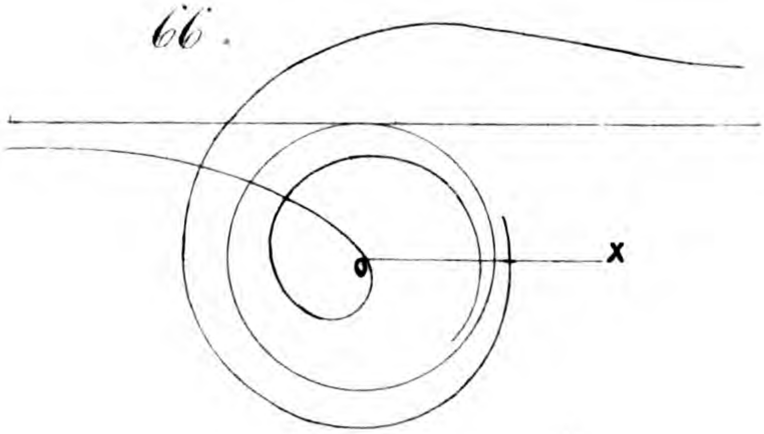

Marcus Ward & Co. Litho Belfast