WATSON'S

EASY AND COMPREHENSIVE INTRODUCTION

TO

ALGEBRA,

DESIGNED FOR THE USE OF SCHOOLS:

WITH PLAIN AND FAMILIAR EXAMPLES,

AND NUMEROUS NOTES AND OBSERVATIONS,

INTENDED AS AIDS TO

PRIVATE STUDENTS.

BY W. WATSON,

AUTHOR OF "TUTOR'S ASSISTANT, OR COMPLETE SCHOOL ARITHMETIC,"
"KEY," ETC.

SECOND EDITION,
MUCH IMPROVED AND ENLARGED.

LONDON:
SIMPKIN, MARSHALL, AND COMPANY;
W. B. JOHNSON, BEVERLEY; MOZLEY & SONS, DERBY;
AND ALL BOOKSELLERS.

Price Three Shillings, neatly bound.

W. B. Johnson, Printer, Market-Place, Beverley.

PREFACE

TO THE FIRST EDITION.

It has been said, that nothing strengthens the powers of the mind more than an acquaintance with the Sciences. If this assertion be correct, Algebra seems to have a considerable claim upon our attention, as it not only in common with the other Sciences contributes to this end, but is a kind of key to all the rest.

The want of a cheap, easy, and comprehensive Introduction to pure Algebra, adapted to academical education, has been a matter of just and almost general complaint among Schoolmasters; to supply this deficiency to the Author's own pupils, was the original motive for the compilation of the following Treatise.

It is now submitted to the public at the instance of several eminent Teachers of Mathematics, in the hope not only of alleviating the labours of Teachers generally, but likewise of affording assistance to those who may wish to study Algebra without the help of a master. To such it will be of peculiar advantage; experience having taught the Author to feel for the young analyst, who sets sail on the sea of Science, unaided *by the guidance of a pilot; the impediments he generally meets with have been carefully kept in view, and where anything like difficulty has presented itself, it has been obviated by explanatory notes: so that it is hoped this manual will be found to possess that regular gradation and facility which will lead the student almost imperceptibly from its lowest elements to its highest principles.*

ADVERTISEMENT

TO THE SECOND EDITION.

The slightly altered arrangement, together with the great extension of elementary Exercises, in the present edition, have both been suggested by increased experience in Algebraical teaching.

The various cases of Equations have also been re-modelled and much enlarged, besides the addition of an entire new case of Pure Equations.

The Diophantine department too has likewise been very much extended, and several new forms added; so that now it is not only the most complete, but the most extensive popular treatise extant.

Throughout the whole Work it has been the Author's aim to make each department copious, without being redundant; easy, without being simple; and smart, without being recondite. How far he has succeeded, is left for the judicious and candid public to determine.

W. WATSON.

Beverley, June, 1844.

CONTENTS.

ALGEBRAIC DEFINITIONS.

1. ALGEBRA is that branch of the Mathematics which computes by means of signs and symbols. Its leading rules are the same as those of Arithmetic; and operations are denoted by the following signs or characters:

2. $+$ *plus* or *more*, is the sign of Addition: $a + b$ shows that the number represented by b, is to be added to that represented by a, and is read *a plus b*.

3. $-$ *minus* or *less*, is the sign of Subtraction: thus $a - b$ shews that the number represented by b is to be taken from that represented by a, and is read *a minus b*.

Also, $a \backsim b$ represents the difference of the quantities a and b when it is not known which is the greater.

4. $\times$ *into*, is the sign of Multiplication: thus $a \times b$ shews that the number represented by a is to be multiplied by that represented by b, and is read *a into b;* or, the product may be denoted thus $a\,b$; also, $5 \times x \times y$ is the same as $5\,x\,y$.

5. $\div$ *divided by*, is the sign of Division: thus $a \div b$ shews that the number represented by a is to be divided by that represented by b; the same may also be expressed by placing b under a with a line between them like a vulgar fraction, thus $\frac{a}{b}$

6. $=$ *equal to*, is the sign of equality, thus $x = a + b$, shews that the number represented by x is equal to those represented by a and b, and is read x equal a plus b.

7. *Like quantities*, are those which consist of the same letters or symbols, as $2\,a$ and $5\,a$, or $3\,a\,y$ and $5\,a\,y$, &c.

8. *Unlike quantities*, are those that consist of different letters, or different powers of the same letter, as, a, a^2, $x\,y$, $a\,y^2$, &c.

9. *Given quantities*, are those whose values are known; they are generally represented by the leading letters of the alphabet, as, a, b, c, &c.

10. *Unknown quantities*, are those whose values are saught; they are generally reqresented by the final letters of the alphabet, as, x, y, z, &c.

11. *Indefinite quantities*, are those whose values may be any number taken at pleasure; they are generally represented by the middle letters of the alphabet, as, m, n, p, &c.

12. *Simple quantities*, are those which consist of one term only, as, a, $b\,c$, $b\,c\,d\,y$, &c.

13. *Compound quantities*, are those which consist of several terms, as, $a + b$, $3\,a\,y + 7\,x^2 y$, $x + y + z$, &c.

14. *Positive quantities*, are those which are to be *added*, and have the sign plus $(+)$ before them.

15. *Negative quantities*, are those which are to be subtracted, and have the sign $(-)$ before them.

16. *Co-efficient*, is a number prefixed to a quantity, as, 3, 4, and 5, are the co-efficients of $3\,x$, $4\,x\,y^2$, and $5\,(x-y)$.

17. *A binomial,* consisteth of two terms, as, $a + b$; a *residual,* is the same with one term negative, as, $a—b$.

18. The *reciprocal* of a number is unity divided by it; as, $\frac{1}{a}$ is the reciprocal of a; and $\frac{a}{b}$ is the reciprocal of $\frac{b}{a}$.

19. *Powers of quantities,* are denoted by a small figure or letter above the quantity to the right, called the index or exponent, thus, a^2, a^3, a^4, a^n are the *square, cube, biquadrat,* and the n^{th} power of a.

20. *Roots,* are generally represented by the reciprocals of the indices which denote the powers: thus, $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, $a^{\frac{1}{n}}$ denote the *square root, cube root,* 4^{th} *root,* and n^{th} *root* of a. Also, $a^{\frac{2}{3}}$, $a^{\frac{3}{2}}$, $a^{\frac{m}{n}}$ denotes the *square* of the cube root, the *cube* of the square root, and the m^{th} power of the n^{th} root of a; the upper number representing the power, and the lower the root.

Roots also are expressed by the radical sign $\surd$: thus, $\sqrt{ab}$ = the square root of a into b. $\sqrt[3]{ab}$ = cube root of the same. Also,

$$\sqrt{a+b} \times 2y = \surd \left(a+b\right)2y = 2y\left(a+b\right)^{\frac{1}{2}}$$

are expressions for the square root of $a + b$, multiplied by $2y$. The bar (—) or parenthesis ( ) which is used to collect several quantities in one, is called the *vinculum.*

21. The *double sign* $\pm$, is either $+$ or $-$, that is, the quantity that follows it may be either *plus* or *minus*, and when that quantity is transposed, it is expressed thus $\mp$.

22. The word *therefore*, is represented by three dots, thus $\therefore$

23. A *prime number*, is a number that has no divisor between itself and unity, as, 3, 5, 7, 11, 13, 17, &c.

24. A *composite number*, is a number that has divisors, as $15 = 3 \times 5$, and $7\,a\,b = 7 \times a\,6$, which divisors are sometimes called factors.

25. *Rectangle*, in Arithmetic is the same as product; when applied to space, it is the same as parallelogram.

26. A *digit*, is any one figure from *one* to *nine* inclusive.

ALGEBRA.

ADDITION.

CASE 1.

To add quantities which are like, and have like signs.

RULE.

Add the co-efficients together, to the sum of which join the letter, or letters common to each term, and prefix the common sign + or —.

Thus $2\,x$ added to $6\,x$ make $8\,x$
$-\,2\,x$ added to $-\,6\,x$ make $-\,8\,x$

EXAMPLES.

x	$-\ x$	$2\,a\,x$	$-\ a\,z\,y^{\frac{1}{2}}$
$2\,x$	$-\,2\,x$	$3\,a\,x$	$-\,3\,a\,z\,y^{\frac{1}{2}}$
$3\,x$	$-\,3\,x$	$4\,a\,x$	$-\,6\,a\,z\,y^{\frac{1}{2}}$
$6\,x$ *	$-\,6\,x$	$9\,a\,x$	$-\,10\,a\,z\,y^{\frac{1}{2}}$ †

* When a leading term has no sign prefixed, it is understood to be positive.

† Quantities affected with indices, as $y^{\frac{1}{2}}$, y^2, $\sqrt{a\,y}$, &c. are to be considered as if they were expressed by a single letter.

$2x^2 + 3zy$	$3 - 3\sqrt{x} + xy$	$x^3 - y^{\frac{1}{3}}$
$x^2 + zy$	$4 - 4\sqrt{x} + 7xy$	$3x^3 - 5y^{\frac{1}{3}}$
$3x^2 + 2zy$	$5 - 2\sqrt{x} + 9xy$	$7x^3 - y^{\frac{1}{3}}$
$5x^2 + 4zy$	$6 - \sqrt{x} + xy$	$2x^3 - 7y^{\frac{1}{3}}$
$11x^2 + 10zy$	$18 - 10\sqrt{x} + 18xy$	$13x^3 - 14y^{\frac{1}{3}}$

CASE 2.

To add quantities which are like, but have unlike signs.

RULE.

Add all the affirmative co-efficients into one sum, and all the negative ones into another; then subtract the less sum from the greater, and to the difference prefix the sign of the greater with the common quantity or letters.

Thus $3a$ added to $-a$ makes $2a$
$-3a$ added to a makes $-2a$

EXAMPLES.

$-3x$	$7xy^2$	$-6x^3 - 8y^{\frac{1}{2}}$
$7x$	$-3xy^2$	$-9x^3 + 7y^{\frac{1}{2}}$
$5x$	$4xy^2$	$17x^3 + y^{\frac{1}{2}}$
$-8x$	$-8xy^2$	$-4x^3 + 2y^{\frac{1}{2}}$
x	0	$-2x^3 + 2y^{\frac{1}{2}}$

$x^4 - 12y^2$	$37y - 14y^{\frac{1}{2}} - y^{\frac{1}{3}} + 7xy$
$-x^4 + 7y^2$	$y + 7y^{\frac{1}{2}} + 7y^{\frac{1}{3}} - 3xy$
$-7x^4 + y^2$	$-12y + 12y^{\frac{1}{2}} - 6y^{\frac{1}{3}} - 7xy$
$9x^4 - 21y^2$	$-9y + y^{\frac{1}{2}} - 4y^{\frac{1}{3}} + 3xy$
$-x^4 + 5y^2$	$-16y - 6y^{\frac{1}{2}} + 4y^{\frac{1}{3}} - xy$
$x^4 - 20y^2$	y * * $- xy$

CASE 3.

When the quantities are unlike.

RULE.

Collect together all the like quantities as in the last case, and set down those that are unlike one after another, with their proper signs.

Thus x added to y makes $x + y$
z added to $x + y$ makes $x + y + z$
$-z$ added to $x - y$ makes $x - y - z$

EXAMPLES.

$-3xy$	$xy - 3x^2$	$5ay - 2x^{\frac{1}{2}}$
$5xy$	$3x^2 - 2xy$	$3xy^2 - 3ay$
$-3az$	$12x^2 - xy$	$2\sqrt{x} - 4xy^2$
$2xy - 3az$	$12x^2 - 2xy$	$2ay - xy^2$

$xy - y$	$x + 3y + z$
$my + x^2$	$x + 5z - 3y$
$nx^2 - 3y$	$3x - 3z + 6y$
$-x^2 + 4y$	$2z - y$
$xy + my + nx^2$	$5x + 5y + 5z$

EXERCISES IN ADDITION.

1. Add a, b, c, and d together. Ans. $a + b + c + d$.

2. Add $x^2 - 3\,y^{\frac{1}{2}}$, $2\,x^2 - y^{\frac{1}{2}}$, and $5\,x^2 - 4\,y^{\frac{1}{2}}$ together. Ans. $8\,x^2 - 8\,y^{\frac{1}{2}}$.

3. Add $17\,x\,y^{\frac{1}{3}}$, $x^3 - 10$, and $y^3 - 17\,x\,y^{\frac{1}{3}} + 10$ together. Ans. $x^3 + y^3$.

4. Add $5 - x\sqrt{y} + 2\,y^{\frac{1}{2}}$, $2 - 3\,x\sqrt{y} - 4\,y^{\frac{1}{2}}$, and $3 + 4\,x\sqrt{y} + 4\,y^{\frac{1}{2}}$ together. Ans. $10 + 2\sqrt{y}$.

5. Add $5\,x^2 - 3\,y^2 - x^{\frac{1}{2}}$, $3\,x^{\frac{1}{2}} + 3\,y^2 - 4\,x^2$, and $-2\,x^{\frac{1}{2}}$ together. Ans. x^2.

6. Add $y - a\,b + c$, $7\,y - 12\,a\,b - 9\,c$, and $12\,y + 7\,a\,b + 8\,c$ together. Ans. $20\,y - 6\,a\,b$.

7. Add $x^2 - 7\,x\,y - z$, $6\,x\,y - x^2 + z$, and $z + x\,y - x^2$ together. Ans. $z - x^2$.

8. Add $a + b - 10$, $c - d - a$, and $4\,c + 2\,a - 3\,b + d$ together. Ans. $2\,a - 2\,b + 5\,c - 10$.

9. Add $9\,a - 7\,b + 5\,x - 4\,d - 7\,c + 50$ to $2\,x - 3\,a + 5\,c + 4\,d - 40$.

Ans. $6\,a + 7\,x - 7\,b - 2\,c + 10$.

10. Add $3\,x^{\frac{1}{3}} - y^{\frac{1}{2}}$, $7\,x^{\frac{1}{2}} - y^{\frac{1}{2}} + y$, $4\,x^2 + 4\,y^{\frac{1}{2}}$, and $7\,x^{\frac{1}{3}} - 7\,x^{\frac{1}{2}} - 2\,y^{\frac{1}{2}}$ together. Ans. $10\,x^{\frac{1}{3}} + 4\,x^2 + y$.

11. Add $3\,x^2\,y^2 - 8\,y^4$, $5\,y^4 - x^2\,y^2$, $8\,x^2\,y^2 - 6\,y^4$, $4\,x^2\,y^2 + y^4$, and $z + 8\,y^4$ together.

Ans. $14\,x^2\,y^2 + z$.

12. Add $a + b + c + d$, $2\,a + 2\,b + 2\,c - 2\,d$, $2\,a + 2\,b + 2\,d - 2\,c$, $2\,a + 2\,d + 2\,c - 2\,b$, and $2\,b + 2\,c + 2\,d - 2\,a$ together.

Ans. $5\,a + 5\,b + 5\,c + 5\,d$.

13. Add $5\sqrt{x-y}+7$, $x^{\frac{1}{4}}-y^{\frac{1}{2}}$, $x^3-5\sqrt{x-y}-3$, and $3x^2-4+y^{\frac{1}{2}}$ together.

Ans. $3x^2+x^3+x^{\frac{1}{4}}$.

14. Add x^2y^3+xy, $5ab-z$, $x^{\frac{1}{2}}y^{\frac{1}{2}}-5xy$, $9xy^2+z$, $7ab-xy$, and $xy^2-7ab-x^{\frac{1}{2}}y^{\frac{1}{2}}$ together.

Ans. $x^2y^3+10xy^2+5ab-5xy$.

15. Add $3x^{\frac{2}{3}}+5y^{\frac{3}{2}}-7$, $12-4y^{\frac{3}{2}}-7x^{\frac{2}{3}}$, $x^{\frac{2}{3}}-y^{\frac{3}{2}}$, and $3x^{\frac{2}{3}}-5$ together. Ans. 0.

SUBTRACTION.

RULE.

Change all the signs of the *lower* line, or *conceive* them to be changed, then collect them together as in Addition.

EXAMPLES.

From	a	$a-b$	$a+b$
Take	b	$-b$	$a-b$
Rem.	$a-b$*	$a\ \cdot$	$\cdot+2b$

From	$a-5x$	$5xy-6$	$7a^2-3b$
Take	$-a+5x$	$-2xy+1$	$2a^2-7b$
Rem.	$2a-10x$	$7xy-7$	$5a^2+4b$

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* It is obvious that $a-b$ is the difference between a and b. And also, if this difference be subtracted from the greater number a, the less number b must remain. Whence subtraction is effected by changing the signs of the lower terms and annexing them to the upper. In order to facilitate a clear conception of this, the learner should prove the results, as in Arithmetic.

From	$11\,x^2\,(y+z)$	$20 - 6\,x \quad - 5\,x\,y$
Take	$10\,x^2\,(y+z)$	$-\ 8 + 3\,x\,y - 6\,x \quad - a\,y$
Rem.	$x^2\,(y+z)$	$28 \quad - \quad 8\,x\,y + a\,y$

EXERCISES IN SUBTRACTION.

1. From $a + b$, take $a - 2\,b$. Ans. $3\,b$.

2. From $\frac{1}{2}\,a + \frac{1}{2}\,b$, take $\frac{1}{2}\,a - \frac{1}{2}\,b$. Ans. b.

3. From $7\,x^2\,(x + y + z)$, take $5\,x^2\,(x + y + z)$.
Ans. $2\,x^2\,(x + y + z)$.

4. From $4\,x^2 + x^{\frac{1}{2}} - 4\,y$, take $3\,x^2 - x^{\frac{1}{2}} - 8x$.
Ans. $x^2 + 2\,x^{\frac{1}{2}} + 8\,x - 4\,y$.

5. From $\frac{1}{4}\,x + \frac{1}{4}\,y + \frac{1}{4}\,z$, take $\frac{1}{4}\,x - \frac{3}{4}\,y - \frac{3}{4}\,z$.
Ans. $y + z$.

6. From $x - x\,y^2 + 9$, take $y^2 - x$.
Ans. $2\,x - x\,y^2 - y^2 + 9$.

7. From $x + 2\,y - z - a$, take $x - y + x^2$.
Ans. $3\,y - x^2 - z - a$.

8. From $x^2 - (x - y)^{\frac{1}{2}} + y^3$, take $x^2 + (x - y)^{\frac{1}{2}}$
$- x\,y$. Ans. $y^3 - 2\,(x - y)^{\frac{1}{2}} + x\,y$.

9. From $x^2 + a - y^{\frac{1}{2}}$, take $x^2 + y^2 - z^2 - x\,y$
$- y^{\frac{1}{2}}$. Ans. $z^2 - y^2 + a + x\,y$.

10. From $17\,x^{\frac{1}{3}}\,y^{\frac{1}{2}} - x\,y\,\sqrt{x\,y}$, take $12\,x^{\frac{1}{3}}\,y^{\frac{1}{2}} + 4$
$x\,y\,\sqrt{x\,y}$. Ans. $5\,x^{\frac{1}{3}}\,y^{\frac{1}{2}} - 5\,x\,y\,\sqrt{x\,y}$.

11. From $6\,x^3\,y - 10\,x^2\,y^2 + 13\,x\,y^3 - 19\,y^4$, take
$2\,x^2\,y^2 + 2\,y^4 - 5\,y\,x^3 - 3\,x\,y^3$.
Ans. $11\,y\,x^3 + 16\,x\,y^3 - 12\,x^2\,y^2 - 21\,y^4$.

12. From $9\,a\,x^2 + 15\,a^3 - 17\,x^3 - 7\,a^2\,x$, take $9\,a\,x^2 + 17\,a^3 - 19\,x^3 - 9\,a^2\,x$.

Ans. $2\,x^3 + 2\,a^2\,x - 2\,a^3$.

13. From $a\,x^4 + 5\,y\,(a+b)$, take $a\,z^2 - y\,(a+b)$.

Ans. $a\,(x^4 - z^2) + 6\,y\,(a+b)$.

14. From $2\,x^2 - y^2$ added to $x - y\,(a-b)^{\frac{1}{2}}$, take $x^2 + y^2 + y\,(a-b)^{\frac{1}{2}}$.

Ans. $x^2 + x - 2\,y^2 - 2\,y\,(a-b)^{\frac{1}{2}}$.

15. From $\frac{1}{2}\,x^{\frac{1}{2}} - \frac{1}{3}\,y^{\frac{1}{3}}$ added to $\frac{1}{2}\,x^{\frac{1}{2}} - \frac{2}{3}\,y^{\frac{1}{3}}$, take the sum of $\frac{1}{2}\,y^{\frac{1}{3}} - \frac{1}{4}\,y^{\frac{1}{4}}$ and $\frac{1}{2}\,y^{\frac{1}{3}} - \frac{3}{4}\,y^{\frac{1}{4}}$.

Ans. $x^{\frac{1}{2}} - y^{\frac{1}{3}} + y^{\frac{1}{4}}$.

MULTIPLICATION.

CASE 1.

When the factors are both simple quantities.

RULE.

Multiply the co-efficients of the two terms together, and to the product affix all the letters in those terms, and the result will be the required product.*

EXAMPLES.

a	$4\,a$	$-\,5\,a\,y$	$-\,3\,x\,y$
b	$-\,3\,b$	$4\,c$	$-\,3\,x\,y$
$a\,b$	$-\,12\,a\,b$	$-\,20\,a\,c\,y$	$9\,x^2\,y^2$

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* In Multiplication, like signs produce *plus*, and unlike signs *minus:* that is, if the sign of the multiplicand and multiplier be both $+$, the product will be $+$; or, if they be both $-$, the product will also be

$7\,y^2$	$-\ 6\,a^2\,y$	$a\,b$	$-\,7\,x^3\,y$
$3\,a\,y$	$-\ 2\,a^3\,y$	$c\,d$	$x\,y$
$21\,a\,y^3$	$12\,a^5\,y^2$	$a\,b\,c\,d$	$-\,7\,x^4\,y^2$

$-\,3\,y^{\frac{1}{2}}$	$3\,\sqrt{y}$	$-\ y^{\frac{1}{3}}$	$14\,y^m$
$y^{\frac{1}{2}}$	$-\ \sqrt{y}$	$-\,5\,y^{\frac{1}{2}}$	$2\,y^n$
$-\,3\,y$	$-\,3\,y$	$5\,y^{\frac{5}{6}}$	$28\,y^{m+n}$

CASE 2.

When one of the factors is a compound quantity.

RULE.

Multiply every term of the compound factor separately by the multiplier, and their products placed one after another, with their proper signs, will be the whole product.

+; but if one be +, and the other —, then the product will be —. Like quantities are also multiplied together by adding their indices. Thus

$$a \times a^2 = a^{1+2} = a^3$$
$$a^2 \times a^3 = a^{2+3} = a^5$$
$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$$
$$a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{1}{2}+\frac{1}{3}} = a^{\frac{3}{6}+\frac{2}{6}} = a^{\frac{5}{6}}$$
$$a^{\frac{2}{3}} \times a^{\frac{1}{4}} = a^{\frac{2}{3}+\frac{1}{4}} = a^{\frac{8}{12}+\frac{3}{12}} = a^{\frac{11}{12}}$$

See the rule and note for finding a common denominator, in the Author's Arithmetic.

EXAMPLES.

$$\begin{array}{l} 3a^2 - y \\ 2a \\ \hline 6a^3 - 2ay \end{array} \qquad \begin{array}{l} 5y - 3a \\ 3x \\ \hline 15xy - 9ax \end{array}$$

$$\begin{array}{l} \quad 5ay + 4x - 7 \\ -\ 2b \\ \hline -10aby - 8bx + 14b \end{array}$$

$$\begin{array}{l} 3x(y - z) + y \\ \ \ x \\ \hline 3x^2(y - z) + xy \end{array} \qquad \begin{array}{l} 35x^2 - 4 \\ \ 2y^2 \\ \hline 70x^2y^2 - 8y^2 \end{array}$$

$$\begin{array}{l} x^{\frac{1}{2}} - y^2 - z^2 \\ x^{\frac{1}{2}} \\ \hline x \ - y^2x^{\frac{1}{2}} - z^2x^{\frac{1}{2}} \end{array}$$

EXERCISES.

1. Multiply $a y^2 + y^2 + 7$, by 7.
Ans. $7 a y^2 + 7 y^2 + 49$.
2. Multiply $7 a - 3 a b - 2 d$, by $2 x y$.
Ans. $14 a x y - 6 a b x y - 4 d x y$.
3. Multiply $2 a - 5 x - 2 d$, by $- a^2$.
Ans. $- 2 a^3 + 5 a^2 x + 2 d a^2$.
4. Multiply $3 x^{\frac{1}{2}} - 4 x y$, by $2 x^{\frac{1}{2}}$.
Ans. $6 x - 8 y x^{\frac{3}{2}}$.

5. Multiply $7\,a\,x^{\frac{2}{3}}$, by $2\,x^{\frac{1}{3}}$. Ans. $14\,a\,x$.

6. Multiply $2\,y^{\frac{3}{4}}$, by $y^{\frac{1}{2}}$. Ans. $2\,y^{\frac{5}{4}}$.

7. Multiply $6\,x^{\frac{4}{5}}$, by $12\,x^{\frac{6}{5}}$. Ans. $72\,x^2$.

8. Multiply $1\frac{1}{2}\,y^{\frac{1}{2}}$, by $2\,y^{\frac{1}{4}}$. Ans. $3\,y^{\frac{3}{4}}$.

9. Multiply $2\frac{1}{4}\,x^2$, by $2\,x^{\frac{1}{2}}$. Ans. $4\frac{1}{2}\,x^{\frac{5}{2}}$.

10. Multiply $3\,x^{\frac{1}{n}}$, by $2\,x^{\frac{1}{m}}$. Ans. $6\,x^{\frac{m+n}{m\,n}}$.

11. Multiply $y\,x^2 - 3\,x\,y^3 - 5\,x$, by $3\,x^3\,y^4$.
Ans. $3\,x^5\,y^5 - 9\,x^4\,y^7 - 15\,x^4\,y^4$.

12. Multiply $-1\frac{1}{2}\,x^2 - \frac{3}{2}\,y^{\frac{1}{2}} - 2$, by $-2\,y^{\frac{1}{2}}$.
Ans. $3\,x^2\,y^{\frac{1}{2}} + 3\,y + 4\,y^{\frac{1}{2}}$.

13. Multiply x^{m+1}, by x^{m-1}. Ans. x^{2m}.

14. Multiply $x^{\frac{2}{3}}\,y^{\frac{1}{3}} - 2\,a\,x^{\frac{5}{3}}\,y^{\frac{4}{3}}$, by $x^{\frac{1}{3}}\,y^{\frac{2}{3}}$.
Ans. $x\,y - 2\,a\,x^2\,y^2$.

15. Multiply $x^m\,y^n - a\,y^2 + a^2$, by $a\,y$.
Ans. $a\,x^m\,y^{n+1} - a^2\,y^3 + a^3\,y$.

CASE 3.

When both factors are compound quantities.

RULE.

Multiply every term of the multiplicand by each term of the multiplier, setting down the products one after another with their proper signs; then add the several lines of products together, and their sum will be the whole product.

EXAMPLES.

$$\begin{array}{l}
x + y \\
x + y \\
\hline
x^2 + x\,y \\
\qquad\;\; x\,y + y^2 \\
\hline
x^2 + 2\,x\,y + y^2 \\
\hline\hline
\end{array}
\qquad\qquad
\begin{array}{l}
x^2 - y \\
x^2 - y \\
\hline
x^4 - x^2\,y \\
\qquad - x^2\,y + y^2 \\
\hline
x^4 - 2\,x^2\,y + y^2 \\
\hline\hline
\end{array}$$

$$\begin{array}{l}
3\,x^2 - 2\,y \\
2\,x^2 + 2\,y \\
\hline
6\,x^4 - 4\,x^2\,y \\
\qquad + 6\,x^2\,y - 4\,y^2 \\
\hline
6\,x^4 + 2\,x^2\,y - 4\,y^2 \\
\hline\hline
\end{array}
\qquad\qquad
\begin{array}{l}
x^2 + x\,y - y^2 \\
x - y \\
\hline
x^3 + x^2\,y - x\,y^2 \\
\qquad - x^2\,y - x\,y^2 + y^3 \\
\hline
x^3 \quad \cdot \quad - 2\,y^2 + y^3 \\
\hline\hline
\end{array}$$

$$\begin{array}{l}
x^2 + x\,y - y^2 \\
x^2 + y^2 \\
\hline
x^4 + x^3\,y - x^2\,y^2 \\
\qquad\qquad + x^2\,y^2 + x\,y^3 - y^4 \\
\hline
x^4 + x^3\,y \quad \cdot \quad + x\,y^3 - y^4 \\
\hline\hline
\end{array}$$

EXERCISES.

1. Multiply $x - y$, by $x - y$.
   Ans. $x^2 - 2\,x\,y + y^2$.
2. Multiply $x + y$, by $x - y$. Ans. $x^2 - y^2$.
3. Multiply $x^2 + x\,y + y^2$, by $x - y$.
   Ans. $x^3 - y^3$.
4. Multiply $x^2 - x\,y + y^2$, by $x + y$.
   Ans. $x^3 + y^3$.

5. Multiply $x^2 + xy - y^2$, by $x - y$.
Ans. $x^3 - 2xy^2 + y^3$.

6. Multiply $x^4 - x^2 + x$, by $x^2 - x$.
Ans. $x^6 - x^5 - x^4 + 2x^3 - x^2$.

7. Multiply $x^3 + x^2y + xy^2 + y^3$, by $x - y$.
Ans. $x^4 - y^4$.

8. Multiply $x^3 - x^2y + xy^2 - y^3$, by $x + y$.
Ans. $x^4 - y^4$.

9. Multiply $x^4 - x^3y + x^2y^2 - xy^3 + y^4$, by $x + y$.
Ans. $x^5 + y^5$.

10. Multiply $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$, by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.
Ans. $x + y$.

11. Multiply $x^n + y^n$, by $x^n - y^n$. Ans. $x^{2n} - y^{2n}$.

12. Multiply $x^2 - y^2 - 2$, by $x^2 + y^2 + 2$.
Ans. $x^4 - y^4 - 4y^2 - 4$.

13. Multiply $x^2 + xy + y^2$, by $x^2 - xy + y^2$.
Ans. $x^4 + x^2y^2 + y^4$.

14. Multiply $3x^2 - 2xy + 5$, by $x^2 + 2xy - 3$.
Ans. $3x^4 + 4x^3y - 4x^2y^2 + 16xy - 4x^2 - 15$.

15. Multiply $a^2 + b^2 + c^2 - ab - ac - bc$, by $a + b + c$.
Ans. $a^3 + b^3 + c^3 - 3abc$.

16. Multiply $a^3 + 2a^2b + 2ab^2 + b^3$, by $a^3 - 2a^2b + 2ab^2 - b^3$.
Ans. $a^6 - b^6$.

17. Multiply $\frac{3}{2}x^3 - 5x^2 + \frac{x}{4} + 9$, by $\frac{1}{2}x^2 - x + 3$.
Ans. $\frac{3}{4}x^5 - 4x^4 + \frac{77}{8}x^3 - \frac{43}{4}x^2 - \frac{33}{4}x + 27$.

18. Multiply $x + \frac{y}{2} - \frac{y^2}{3}$, by $x - \frac{y}{2} + \frac{y^2}{3}$.
Ans. $x^2 - \frac{y^2}{4} + \frac{y^3}{3} - \frac{y^4}{9}$.

19. Multiply $x + \frac{y}{2x} - \frac{y^2}{2x}$, by $x - \frac{y}{2x} + \frac{y^2}{2x}$.

Ans. $x^2 - \frac{y^2}{4x^2} + \frac{y^3}{2x^2} - \frac{y^4}{4x^2}$.

20. Multiply $x^m - x^{m-1} - y^n$ by $x - y^2$.

Ans. $x^{m+1} - x^m(y^2+1) - xy^n + y^2 x^{m-1} + y^{n+2}$.

DIVISION.

CASE 1.

When the divisor and dividend are both simple quantities.

RULE.

Set the dividend over the divisor, in the manner of a vulgar fraction, then reduce it to its simplest form by cancelling the letters or figures that are common to both.*

EXAMPLES.

1. Divide $6\,a^3$, by $3\,a^2$.

$$\text{Here } \frac{6\,a^3}{3\,a^2} = 2\,a \text{ Ans.}$$

---

* 1. Like signs produce $+$, and unlike signs $-$, in the quotient.

2. Powers and Roots of the *same quantities* are divided by subtracting their indices. Thus

$$a \div a = a^{1-1} = a^0 = 1$$
$$a^3 \div a^2 = a^{3-2} = a^1 = a$$
$$a^{\frac{1}{3}} \div a^{\frac{1}{4}} = a^{\frac{1}{3}-\frac{1}{4}} = a^{\frac{4}{12}-\frac{3}{12}} = a^{\frac{1}{12}}$$
$$a^{\frac{3}{4}} \div a^{\frac{1}{2}} = a^{\frac{3}{4}-\frac{1}{2}} = a^{\frac{3}{4}-\frac{2}{4}} = a^{\frac{1}{4}}$$
$$a^2 \div a^{\frac{1}{2}} = a^{2-\frac{1}{2}} = a^{\frac{4}{2}-\frac{1}{2}} = a^{\frac{3}{2}}$$
$$a^2 \div a^{-2} = a^{2+2} = a^4$$

2. Divide a by a. Ans. 1.

3. Divide $17\ x^3\ y$, by $17\ x\ y$. Ans. x^2.

4. Divide $9\ a\ x^2\ y^2$, by $-9\ x^2\ y^2$. Ans. $-a$.

5. Divide $-16\ x\ y\ z^2$, by $-4\ x\ z$. Ans. $4\ y\ z$.

6. Divide $5\ y^2\ z$, by $-10\ y\ x$. Ans. $-\frac{y\ z}{2\ x}$.

7. Divide $12\ y^2\ x^3\ z$, by $8\ y\ x^2\ z$. Ans. $\frac{3}{2}\ y\ x$.

8. Divide $14\ x$, by $2\ x^{\frac{1}{3}}$. Ans. $7\ x^{\frac{2}{3}}$.

9. Divide $2\ y^{\frac{5}{4}}$, by $y^{\frac{1}{2}}$ Ans. $2\ y^{\frac{3}{4}}$.

10. Divide $12\ x^2$, by $2\ x^{\frac{6}{5}}$. Ans. $6\ x^{\frac{4}{5}}$.

11. Divide $3\ x^{\frac{3}{4}}$, by $2\ x^{\frac{1}{4}}$. Ans. $1\frac{1}{2}\ x^{\frac{1}{2}}$.

12. Divide $6\ x^{\frac{1}{m}}$, by $2\ x^{\frac{1}{n}}$. Ans. $3\ x^{\frac{n-m}{mn}}$.

CASE 2.

When the divisor is a simple, and dividend a compound quantity.

RULE.

Divide each term of the dividend, as before, and set down such as will not divide in the simplest form they will admit of.

EXAMPLES.

1. Divide $4\ a\ b + 2\ b^2$, by $2\ b$.

Here $(4\ a\ b + 2\ b^2) \div 2\ b = \frac{4\ a\ b + 2\ b^2}{2\ b} = 2\ a + b$ Ans.

2. Divide $3\ a\ b\ c + 12\ a\ b\ x - 9\ b\ a^2$, by $3\ a\ b$.
Ans. $c + 4\ x - 3\ a$.

3. Divide $20\ a^3\ b^3 - 30\ a^2\ b^2 + 10\ a\ b$, by $-10\ a\ b$.
Ans. $-2\ a^2\ b^2 + 3\ a\ b - 1$.

4. Divide $6x^4 + 3x^3y - z$, by $3x^2$.

Ans. $2x^2 + xy - \frac{z}{3x^2}$.

5. Divide $7x^2y - 28y^2x - 7$, by $14x$.

Ans. $\frac{1}{2}xy - 2y^2 - \frac{1}{2x}$.

6. Divide $-20d^2b^2 + 60ab^2$, by $-12ab$.

Ans. $\frac{5d^2b}{3a} - 5b$.

7. Divide $8axy^{\frac{1}{2}} - 12a^2x^2y$, by $12axy^{\frac{1}{2}}$.

Ans. $\frac{2}{3} - axy^{\frac{1}{2}}$.

8. Divide $8x^{\frac{3}{2}}y^{\frac{4}{3}} - 6x^{\frac{2}{3}}y^{\frac{3}{4}}$, by $4x^{\frac{1}{2}}y^{\frac{1}{2}}$.

Ans. $2xy^{\frac{5}{6}} - \frac{3}{2}x^{\frac{1}{6}}y^{\frac{1}{4}}$.

9. Divide $-10x^{\frac{1}{2}}v^3q^2 + 5xv^{\frac{5}{3}}q^{\frac{3}{2}}$, by $-5x^{\frac{1}{2}}v^{\frac{4}{3}}q^{\frac{3}{2}}$.

Ans. $2v^{\frac{5}{3}}q^{\frac{1}{2}} - x^{\frac{1}{2}}v^{\frac{1}{3}}$.

10. Divide $8x^{\frac{3}{2}}y^{\frac{4}{3}} - 6ax^{\frac{2}{3}}y^{\frac{3}{4}} - 4a^{\frac{1}{2}}xy$, by $6a^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{1}{2}}$.

Ans. $\frac{4xy^{\frac{5}{6}}}{3a^{\frac{1}{2}}} - a^{\frac{1}{2}}x^{\frac{1}{6}}y^{\frac{1}{4}} - \frac{2}{3}x^{\frac{1}{2}}y^{\frac{1}{2}}$.

CASE 3.

When both factors are compound quantities.

RULE.

1. Range the terms of both the quantities according to the dimensions of some letter in them, so that the *first term* may contain the highest power of that letter; the second term, the next highest power, and so on.

2. Divide the first term of the dividend by the first term of the divisor, and put the result in the quotient.

3. Multiply the whole divisor by the term just found, and subtract the result from the dividend.

4. To this remainder, bring down as many terms of the dividend as are requisite for the next operation, then divide as before, andso on as in long division in Arithmetic.

EXAMPLES.

Divide $x^2 + 2xy + y^2$, by $x + y$

$x + y$) $x^2 + 2xy + y^2$ ($x + y$ quotient

$$\begin{array}{l} x^2 + xy \\ \hline \quad xy + y^2 \\ \quad xy + y^2 \\ \hline \quad \cdot \quad \cdot \\ \hline \hline \end{array}$$

Divide $a^3 - 4a^2c + 4ac^2 - c^3$, by $a - c$

$a - c$) $a^3 - 4a^2c + 4ac^2 - c^3$ ($a^2 - 3ac + c^2$

$$\begin{array}{l} a^3 - a^2c \\ \hline \quad -3a^2c + 4ac^2 \\ \quad -3a^2c + 3ac^2 \\ \hline \qquad\qquad ac^2 - c^3 \\ \qquad\qquad ac^2 - c^3 \\ \hline \qquad\qquad \cdot \quad \cdot \\ \hline \hline \end{array}$$

Divide $x^3 - y^3$, by $x - y$ *

$$
\begin{array}{l}
x - y\ )\ x^3 - y^3\ (\ x^2 + xy + y^2 \\
\qquad\quad \underline{x^3 - x^2 y} \\
\qquad\qquad\quad x^2 y - y^3 \\
\qquad\qquad\quad \underline{x^2 y - x y^2} \\
\qquad\qquad\qquad\quad x y^2 - y^3 \\
\qquad\qquad\qquad\quad \underline{x y^2 - y^3} \\
\qquad\qquad\qquad\qquad \cdot \quad \cdot
\end{array}
$$

Divide x, by $x + y$

$$
\begin{array}{l}
x + y\ )\ x\ \ (\ 1 - \frac{y}{x} + \frac{y^2}{x^2} - \frac{y^3}{x^3} + \frac{y^4}{x^4} - \&c. \\
\qquad\quad \underline{x + y} \\
\qquad\quad \cdot - y \\
\qquad\quad \underline{- y - \frac{y^2}{x}} \\
\qquad\qquad \cdot + \frac{y^2}{x} \\
\qquad\qquad \underline{+ \frac{y^2}{x} + \frac{y^3}{x^2}} \\
\qquad\qquad\quad \cdot - \frac{y^3}{x^2} \\
\qquad\qquad\quad \underline{- \frac{y^3}{x^2} - \frac{y^4}{x^3}} \\
\qquad\qquad\qquad + \frac{y^4}{x^3} \text{ remainder.}
\end{array}
$$

* The difference of any two *equal* powers of different numbers is divisible by the *difference* of the numbers: that is $x^n - y^n$ is divisible by $x - y$. If n be an *even* number, then $x^n - y^n$ is divisible both by $x - y$ and $x + y$. Also, if n be an *odd* number, then $x^n + y^n$ is divisible by $x + y$

EXERCISES.

1. Divide $x^2 - y^2$, by $x - y$. Ans. $x + y$.*

2. Divide $x^2 - 2xy + y^2$, by $x - y$.
Ans. $x - y$.

3. Divide $x^3 - a x^2 + a^2 x - a^3$, by $x - a$.
Ans. $x^2 + a^2$.

4. Divide $x^3 + y^3$, by $x + y$. Ans. $x^2 - xy + y^2$.

5. Divide $a^3 - 3a^2 z + 3 a z^2 - z^3$, by $a - z$.
Ans. $a^2 - 2az + z^2$.

6. Divide $a^3 + 5a^2 b + 5 a b^2 + b^3$, by $a + b$.
Ans. $a^2 + 4ab + b^2$.

7. Divide $x^4 - y^4$, by $x - y$.
Ans. $x^3 + x^2 y + x y^2 + y^3$.

8. Divide $x^4 - y^4$, by $x + y$.
Ans. $x^3 - x^2 y + x y^2 - y^3$.

9. Divide $x^5 - y^5$, by $x - y$.
Ans. $x^4 + x^3 y + x^2 y^2 + x y^3 + y^4$.

10. Divide $x^5 + y^5$, by $x + y$.
Ans. $x^4 - x^3 y + x^2 y^2 - x y^3 + x^4$.

11. Divide $x^6 - x^4 - 4x^3 - x^2 - 3x - 6$, by $x^2 - x - 2$. Ans. $x^4 + x^3 + 2x^2 + 3$.

12. Divide $a^5 - 5a^4 b + 10 a^3 b^2 - 10 a^2 b^3 + 5 a b^4 - b^5$, by $a^2 - 2ab + b^2$.
Ans. $a^3 - 3a^2 b + 3 a b^2 - b^3$.

13. Divide $b^6 - 3 b^4 x^2 + 3 b^2 x^4 - x^6$, by $b^3 - 3b^2 x + 3 b x^2 - x^3$.
Ans. $b^3 + 3 b^2 x + 3 b x^2 + x^3$.

---

* The truth of the results should frequently be proved by multiplying the quotients by the divisors.

14. Divide $x^{\frac{3}{2}} + 2\,x\,y^{\frac{1}{2}} + 2\,x^{\frac{1}{2}}\,y + y^{\frac{3}{2}}$ by $x^{\frac{1}{2}} + y^{\frac{1}{2}}$.

Ans. $x + x^{\frac{1}{2}}\,y^{\frac{1}{2}} + y$.

15. Divide $\frac{3}{4}\,x^5 - 4\,x^4 + \frac{77}{8}\,x^3 - \frac{43}{4}\,x^2 - \frac{33}{4}\,x + 27$, by $\frac{1}{2}\,x^2 - x + 3$.

Ans. $\frac{3}{2}\,x^3 - 5\,x^2 + \frac{1}{4}\,x + 9$.

16. Divide $x^{2n} - y^{2n}$, by $x^n - y^n$. Ans. $x^n + y^n$.

17. Divide $a^4 + 3\,x^4$, by $a + x$.

Ans. $a^3 - a^2\,x + a\,x^2 - x^3 + \dfrac{4\,x^4}{a + x}$.

18. Divide $x^5 + y^5$, by $x - y$.

Ans. $x^4 + x^3\,y + x^2\,y^2 + x\,y^3 + y^4 + \dfrac{2\,y^5}{x - y}$.

19. Divide 1, by $1 + x$.

Ans. $1 - x + x^2 - x^3 + x^4 - x^5 \pm$ &c.

20. Divide x, by $x + y$.

Ans. $1 - \dfrac{y}{x} + \dfrac{y^2}{x^2} - \dfrac{y^3}{x^3} + \dfrac{y^4}{x^4} \pm$, &c.

INVOLUTION.

INVOLUTION is the raising of powers from any proposed root, or the method of finding the square, cube, biquadrate, &c. of any quantity.

RULE.

Multiply the quantity into itself as many times as there are units in the index *less one*, and the last product will be the power required: that is, for the square, multiply the quantity *once* into itself; for the cube, multiply it *twice* into itself; for the biquadrate, *three* times, &c.

EXAMPLES.

1. Required the square, cube, and biquadrate of $-3x$.

Here $-3x \times -3x = 9x^2 =$ square

$-3x \times -3x \times -3x = -27x^3 =$ cube

and $-3x \times -3x \times -3x \times -3x = 81x^4 =$ biquad.

2. Required the square, cube, and biquadrate of $\frac{x}{y}$.

Here $\frac{x}{y} \times \frac{x}{y} = \frac{x^2}{y^2} =$ square

$\frac{x}{y} \times \frac{x}{y} \times \frac{x}{y} = \frac{x^3}{y^3} =$ cube

And $\frac{x}{y} \times \frac{x}{y} \times \frac{x}{y} \times \frac{x}{y} = \frac{x^4}{y^4} =$ biquadrate

---

NOTE. The n^{th} power of $-3x = \pm 3^n x^n$ and the m^{th} power of $x^{\frac{1}{n}} = x^{\frac{1}{n} \times m} = x^{\frac{m}{n}}$. The sign $\pm$, is $+$, when x is an *even* number, and $-$ when an *odd* number. Whence it is evident that to raise any quantity, to any power, it is only necessary to multiply the index of the quantity, by that of the power; and *vice versa* for roots.

3. Required the square, cube, and biquadrate of $x + y$, and the 4^{th} power of $x - y$.

$$
\begin{array}{l}
x + y \\
x + y \\
\hline
x^2 + x y \\
\quad + x y + y^2 \\
\hline
x^2 + 2 x y + y^2 = \text{square} \\
x + y \\
\hline
x^3 + 2 x^2 y + x y^2 \\
\qquad x^2 y + 2 x y^2 + y^3 \\
\hline
x^3 + 3 x^2 y + 3 x y^2 + y^3 = \text{cube.} \\
x + y \\
\hline
x^4 + 3 x^3 y + 3 x^2 y^2 + x y^3 \\
\qquad x^3 y + 3 x^2 y^2 + 3 x y^3 + y^4 \\
\hline
x^4 + 4 x^3 y + 6 x^2 y^2 + 4 x y^3 + y^4 = 4^{th} \text{ power.}
\end{array}
$$

$$
\begin{array}{l}
x - y \\
x - y \\
\hline
x^2 - x y \\
- x y + y^2 \\
\hline
x^2 - 2 x y + y^2 = \text{square} \\
x^2 - 2 x y + y^2 \\
\hline
x^4 - 2 x^3 y + x^2 y^2 \\
- 2 x^3 y + 4 x^2 y^2 - 2 x y^3 \\
\qquad x^2 y^2 - 2 x y^3 + y^4 \\
\hline
x^4 - 4 x^3 y + 6 x^2 y^2 - 4 x y^3 + y^4 = 4^{th} \text{ power.}
\end{array}
$$

A *binomial* or *residual* may be raised at once to any power whatever by the following rule, invented by Sir Isaac Newton, and denominated the binomial theorem.*

1. *To find the terms without the co-efficients.*—The index of the first term is the same as the power to which it is to be raised, and *decreases* by *one* in every succeeding term; the second letter enters into the second term, and *increases* by *one* in every succeeding term, till it stands by itself with the same index as the first term.

2. *To find the co-efficients.*—The co-efficient of the first term is 1, that of the second is the index of the first. To find the third co-efficient, multiply that of the second by the index of the leading letter in the same term, and divide the product by 2, and so on: that is, multiply the co-efficient of the term last found by the index of the leading letter in the same term, and divide the product by the number of terms to that place, and it will give the co-efficient of the term next following.

The rule in general terms is

$$(x + y)^n = x^n + n\, y\, x^{n-1} + n\,\frac{(n-1)}{2}\, y^2\, x^{n-2} + n\,\frac{(n-1)\,.\,(n-2)}{2\,.\,3}\, y^3\, x^{n-3} + \&c.$$

$$(x - y)^n = x^n - n\, y\, x^{n-1} + n\,\frac{(n-1)}{2}\, y^2\, x^{n-2} - n\,\frac{(n-1)\,.\,(n-2)}{2\,.\,3}\, y^3\, x^{n-3} + \&c.$$

---

* 1 All the terms of any power of a binomial will be *plus*, (+) and those of a residual *plus* and *minus* (+ & —) alternately.

2 The co-efficients, as well as the indices, are the same at equal distances from the middle; and the mumber of terms is always one more than the index of the given power.

EXAMPLE.

Required the 6^{th} power of $x + y$.

The terms without the co-efficients are

$$x^6,\ x^5 y,\ x^4 y^2,\ x^3 y^3,\ x^2 y^4,\ x y^5,\ y^6.$$

and the co-efficients are

$$1,\ 6,\ \frac{5 \times 6}{2},\ \frac{15 \times 4}{3},\ \frac{20 \times 3}{4},\ \frac{15 \times 2}{5},\ \frac{6 \times 1}{6}$$

Or, 1, 6, 15, 20, 15, 6, 1.

But it is best to put down the co-efficients and the powers at once, in one line, thus,

$$x^6 + 6 x^5 y + 15 x^4 y^2 + 20 x^3 y^3 + 15 x^2 y^4 + 6 x y^5 + y^6.$$

EXERCISES.

1. Required the square of $3 x y^3$. Ans. $9 x^2 y^6$.
2. Required the square of $x y^{\frac{1}{2}}$. Ans. $x^2 y$.
3. Required the cube of $-5 x y^{\frac{1}{2}}$. Ans. $-125 x^3 y^{\frac{3}{2}}$.
4. Required the biquadrate of $-2 x y^{\frac{1}{2}}$. Ans. $16 x^4 y^2$.
5. Required the 4^{th} power of $x^{\frac{1}{2}} y^{\frac{1}{4}}$. Ans. $x^2 y$.
6. Required the 5^{th} power of $-2 x y^{\frac{1}{5}}$. Ans. $-32 x^5 y$.
7. Required the m^{th} power of $x^{\frac{1}{m}}$. Ans. x.
8. Required the m^{th} power of $2 a^{\frac{1}{2}}$. Ans. $2^m a^{\frac{m}{2}}$.

9. Required the m^{th} power of $a^{\frac{2}{3n}}$. Ans. $a^{\frac{2m}{3n}}$.

10. Required the square of $x - y^{\frac{1}{2}}$.

Ans. $x^2 - x y^{\frac{1}{2}} + y$.

11. Required the cube of $x + y^{\frac{1}{2}}$.

Ans. $x^3 + 3 x^2 y^{\frac{1}{2}} + 3 y + y^{\frac{3}{2}}$.

12. Required the 4^{th} power of $x - y$.

Ans. $x^4 - 4 x^3 y + 6 x^2 y^2 - 4 x y^3 + y^4$.

13. Required the 5^{th} power of $x + y$.

Ans. $x^5 + 5 x^4 y + 10 x^3 y^2 + 10 x^2 y^3 + 5 x y^4 + y^5$.

14. Required the 5^{th} power of $x - y$.

Ans. $x^5 - 5 x^4 y + 10 x^3 y^2 - 10 x^2 y^3 + 5 x y^4 - y^5$.

15. Required the 6^{th} power of $x - y$.

Ans. $x^6 - 6 x^5 y + 15 x^4 y^2 - 20 x^3 y^3 + 15 x^2 y^4 - 6 x y^5 + y^6$.

16. Required the 7^{th} power of $x + y$.

Ans. $x^7 + 7 x^6 y + 21 x^5 y^2 + 35 x^4 y^3 + 35 x^3 y^4 + 21 x^2 y^5 + 7 x y^6 + y^7$.

EVOLUTION.

EVOLUTION is the reverse of Involution, or it is the finding of the square root, cube root, &c.

CASE. 1.—RULE.

Take the root of the co-efficient and the root of the quantity, which roots joined together will be the root required.

EXAMPLES.

1. Required the square root of $16x^2$.

Here $\sqrt{16x^2} = \sqrt{16} \times \sqrt{x^2} = 4 \times x = 4x$ Ans.

2. Required the cube root of $\dfrac{8x^3y}{64z^6}$

Here $\sqrt[3]{\dfrac{8x^3y}{64z^6}} = \dfrac{\sqrt[3]{8} \times \sqrt[3]{x^3} \times \sqrt[3]{y}}{\sqrt[3]{64} \times \sqrt[3]{z^6}} =$

$$\frac{2 \times x \times y^{\frac{1}{3}}}{4 \times z^2} = \frac{xy^{\frac{1}{3}}}{2z^2} \text{ Ans.*}$$

---

* In a similar way may the square root, &c. of numbers be simplified, and their *sums*, *differences*, *products*, and *quotients*, readily ascertained; thus

$$\sqrt{48} = \sqrt{16 \times 3} = \sqrt{16} \times \sqrt{3} = 4 \times \sqrt{3} = 4\sqrt{3}.$$

and

$$\sqrt{48} + \sqrt{27} = \sqrt{16 \times 3} + \sqrt{9 \times 3} = 4\sqrt{3} + 3\sqrt{3} = 7\sqrt{3}.$$

$$\sqrt{48} - \sqrt{27} = \sqrt{16 \times 3} - \sqrt{9 \times 3} = 4\sqrt{3} - 3\sqrt{3} = \sqrt{3}.$$

$$\sqrt{48} \times \sqrt{27} = 4\sqrt{3} \times 3\sqrt{3} = 12\sqrt{9} = 12 \times 3 = 36.$$

$$8\sqrt{108} \div 2\sqrt{6} = \frac{8\sqrt{108}}{2\sqrt{6}} = 4\sqrt{18} = 4\sqrt{9 \times 2} = 12\sqrt{2}.$$

also,

$$(5 + \sqrt{3}) \times (5 - \sqrt{3}) = 25 - 3 = 22.$$

$$(\sqrt{5} + \sqrt{3}) \times (\sqrt{5} - \sqrt{3}) = 5 - 3 = 2.$$

$$(5^{\frac{1}{3}} + 3^{\frac{1}{3}}) \times (5^{\frac{2}{3}} - 3^{\frac{2}{3}}) = 5 - 3 = 2.$$

$$(5^{\frac{1}{4}} + 3^{\frac{1}{4}}) \times (5^{\frac{1}{4}} - 3^{\frac{1}{4}}) \times (5^{\frac{1}{2}} + 3^{\frac{1}{2}}) = (5^{\frac{1}{2}} - 3^{\frac{1}{2}}) \times (5^{\frac{1}{2}} + 3^{\frac{1}{2}}) = 5 - 3 = 2.$$

EXERCISES.

1. Find the square root of $7x^2$. Ans. $x\sqrt{7}$.

2. Find the square root of $\frac{9x^3y^2}{4}$ Ans. $\frac{3}{2}yx^{\frac{3}{2}}$.

3. Find the cube root of $\frac{1}{27}x^2y^3$. Ans. $\frac{1}{3}yx^{\frac{2}{3}}$.

4. Find the cube root of $\frac{1}{8}x^6y$. Ans. $\frac{1}{2}x^2y^{\frac{1}{3}}$.

5. Find the 4*th* root of $\frac{16x^3y^2}{z^4}$. Ans. $\frac{2x^{\frac{3}{4}}y^{\frac{1}{2}}}{z}$.

When surd quantities, under different radical signs, are to be multiplied or divided one by the other, their indices must be reduced to the lowest common denominator: thus

$$a^{\frac{1}{2}} \times b^{\frac{1}{3}} = a^{\frac{3}{6}} \times b^{\frac{2}{6}} = (a^3 b^2)^{\frac{1}{6}}$$

$$a^{\frac{1}{2}} \div b^{\frac{1}{3}} = \frac{a^{\frac{1}{2}}}{b^{\frac{1}{3}}} = \frac{a^{\frac{3}{6}}}{b^{\frac{2}{6}}} = \left(\frac{a^3}{b^2}\right)^{\frac{1}{6}}$$

Or, if $a = 2$ and $b = 3$, then

$$2^{\frac{1}{2}} \times 3^{\frac{1}{3}} = 2^{\frac{3}{6}} \times 3^{\frac{2}{6}} = 8^{\frac{1}{6}} \times 9^{\frac{1}{6}} = 72^{\frac{1}{6}}$$

$$2^{\frac{1}{2}} \div 3^{\frac{1}{3}} = \frac{2^{\frac{1}{2}}}{3^{\frac{1}{3}}} = \frac{2^{\frac{3}{6}}}{3^{\frac{2}{6}}} = \frac{8^{\frac{1}{6}}}{9^{\frac{1}{6}}} = \left(\frac{8}{9}\right)^{\frac{1}{6}}$$

$$\text{Also, } x^{\frac{1}{n}} \times y^{\frac{1}{m}} = x^{\frac{m}{mn}} \times y^{\frac{n}{mn}} = \left(x^m y^n\right)^{\frac{1}{mn}}$$

$$x^{\frac{1}{n}} \div y^{\frac{1}{m}} = \frac{x^{\frac{1}{n}}}{y^{\frac{1}{m}}} = \frac{x^{\frac{m}{mn}}}{y^{\frac{n}{mn}}} = \left(\frac{x^m}{y^n}\right)^{\frac{1}{mn}}$$

$$(ab)^{\frac{1}{2}} \div (a^2b)^{\frac{1}{3}} = \frac{(ab)^{\frac{1}{2}}}{(a^2b)^{\frac{1}{3}}} = \frac{(ab)^{\frac{3}{6}}}{(a^2b)^{\frac{2}{6}}} = \left(\frac{a^3b^2}{a^4b^3}\right)^{\frac{1}{6}} = \left(\frac{b}{a}\right)^{\frac{1}{6}}$$

6. Find the 4^{th} root of $3\,x^2$. Ans. $3^{\frac{1}{4}}\,x^{\frac{1}{2}}$.

7. Find the 5^{th} root of $32\,x^5\,y^{10}$. Ans. $2\,x\,y^2$.

8. Find the m^{th} root of $a\,y^{\frac{m}{n}}$. Ans. $a^{\frac{1}{m}}\,y^{\frac{1}{n}}$.

9. Find the m^{th} root of $x^n\,y^{m+n}$.

Ans. $x^{\frac{n}{m}}\,y^{\frac{m+n}{m}}$.

10. Find the m^{th} root of $x^{2m}\,y^{mn}$. Ans. $x^2\,y^n$.

11. Find the cube root of $\frac{a}{b}\sqrt{\frac{a}{b}}$. Ans. $\left(\frac{a}{b}\right)^{\frac{1}{2}}$.

12. Find the cube root of $\frac{\sqrt{a+x}}{\sqrt{b^3}}$. Ans. $\frac{(a+x)^{\frac{1}{6}}}{b^{\frac{1}{2}}}$.

CASE 2.

To find the square root of a compound quantity.

RULE.

1. Range the terms according to the powers of some letter, and put the root of the first term in the quotient.

2. Subtract the square of the root thus found from the first term, and bring down the two next terms to the remainder for a dividend.

3. Divide the dividend by *double* the root, and put the result both in the quotient and divisor.

4. Multiply the divisor thus increased, by the term last put in the quotient, and subtract the product from the dividend, and so on as in common arithmetic.*

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* 1. The pupil, with a view of acquiring expertness in operation, should occasionally prove the results by Involution.

2. The biquadrate or 4^{th} root, is the *square root* of the *square root*.

EXAMPLES.

Required the square root of $x^2 + 2xy + y^2$.

$$x^2 + 2xy + y^2 \; (\, x + y \text{ the root.}$$
$$x^2$$
$$2x + y) \; . \quad 2xy + y^2$$
$$2xy + y^2$$
$$. \quad .$$

Required the square root of $x^4 - 4x^3 + 6x^2 - 4x + 1$.

$$x^4 - 4x^3 + 6x^2 - 4x + 1 \; (\, x^2 - 2x + 1 = \text{root}$$
$$x^4$$
$$2x^2 - 2x) \; . - 4x^3 + 6x^2$$
$$- 4x^3 + 4x^2$$
$$2x^2 - 4x + 1) \; . \quad 2x^2 - 4x + 1$$
$$2x^2 - 4x + 1$$
$$. \quad . \quad .$$

Required the square root of $a^2 - x^2$

$$a^2 - x^2 \left(a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \text{\&c.}\right.$$
$$a^2$$
$$\left. 2a - \frac{x^2}{2a} \right) . - x^2$$
$$- x^2 + \frac{x^4}{4a^2}$$
$$\left. 2a - \frac{x^2}{a} - \frac{x^4}{8a^3} \right) . \quad - \frac{x^4}{4a^2}$$
$$- \frac{x^4}{4a^2} + \frac{x^6}{8a^4} + \frac{x^8}{64a^6}$$
$$\left. 2a - \frac{x^2}{a} - \frac{x^4}{4a^3} - \frac{x^6}{16a^5} \right) . - \frac{x^6}{8a^4} - \frac{x^8}{64a^6} - \text{\&c.}$$

EXERCISES.

1. Find the square root of $x^2 - 2xy + y^2$.

Ans. $x - y$.

2. Find the square root of $x^2 + 4xy + 4y^2$.

Ans. $x + 2y$.

3. Find the square root of $9x^2 - 12xy + 4y^2$.

Ans. $3x - 2y$.

4. Find the square root of $196x^2 + 14xy + \frac{1}{4}y^2$.

Ans. $14x + \frac{1}{2}y$.

5. Find the square root of $a^2 - 2abx^{\frac{1}{2}} + b^2x$.

Ans. $a - bx^{\frac{1}{2}}$.

6. Find the square root of $x^2 + 2xy + y^2 - 2zy - 2xz + z^2$.

Ans. $x + y - z$.

7. Find the square root of $4x^2 + 4xy + y^2 - 12x - 6y + 9$.

Ans. $2x + y - 3$.

8. Find the square root of $x^2 - 10xy + 25y^2 + 6xyz - 30y^2z + 9y^2z^2$.

Ans. $x - 5y + 3yz$.

9. Find the square root of $x^4 - 8x^2y^3 + 16y^6 - 2x^2y^2 + 8y^5 + y^4$.

Ans. $x^2 - 4y^3 - y^2$.

10. Find the square root of $x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{16}$.

Ans. $x^2 - x + \frac{1}{4}$.

11. Find the square root of $x^6 - 2x^5 + 3x^4 - 4x^3 + 3x^2 - 2x + 1$.

Ans. $x^3 - x^2 + x - 1$.

12. Find the square root of $\dfrac{x^2}{y^2} - 2 + \dfrac{y^2}{x^2}$.

Ans. $\dfrac{x}{y} - \dfrac{y}{x}$.

13. Find the square root of $x^{2m} - 6x^m y^{n+1} + 9y^{2n+2}$.

Ans. $x^m - 3y^{n+1}$.

14. Find the biquadrate root of $x^8 - 8x^6y + 24x^4y^2 - 32x^2y^3 + 16y^4$. Ans. $x^2 - 2y$.

15. Find the square root of $x^2 - xy$.

$$\text{Ans. } x - \frac{y}{2} - \frac{y^2}{8x} - \frac{y^3}{16x^2} - \text{\&c.}$$

CASE 3.

To find any root of a compound quantity.

RULE.

1. Find the root of the first term, and place it in the quotient.

2. Subtract its power from that term, and bring down the second term for a dividend.

3. Involve the root last found to the next lower power, and multiply it by the index of the given power for a divisor.

4. Divide the dividend by the divisor, and the quotient will be the next term of the root.

5. Involve this new root to the given power, and subtract the result from the proposed quantity, always dividing the first term of the remainder by the divisor first found, for a new term, and thus proceed till the root is completed.

EXAMPLES.

To find the cube root of $x^6 + 6x^5 - 40x^3 + 96x - 64$.

$x^6 + 6x^5 - 40x^3 + 96x - 64\ (x^2 + 2x - 4 =$ root
x^6

$3x^4)\ \ .\ 6x^5$

$\therefore (x^2 + 2x)^3 = x^6 + 6x^5 + 12x^4 + 8x^3$

$3x^4)\qquad -12x^4$

$\therefore (x^2 + 2x - 4)^3 = x^6 + 6x^5 - 40x^3 + 96x - 64.$

To find the 5^{th} root of $x^5 - 10\,x^4\,y + 40\,x^3\,y^2 - 80\,x^2\,y^3 + 80\,x\,y^4 - 32\,y^5$.

$$x^5 - 10\,x^4\,y + 40\,x^3\,y^2 - 80\,x^2\,y^3 + 80\,x\,y^4 - 32\,y^5 \;(\,x - 2\,y = \text{root}$$
$$x^5$$
$$5\,x^4\,)\;\; - 10\,x^4\,y$$

$$\therefore (x - 2\,y)^5 = x^5 - 10\,x^4\,y + 40\,x^3\,y^2 - 80\,x^2\,y^3 + 80\,x\,y^4 - 32\,y^5.$$

EXERCISES.

1. Find the cube root of $x^3 + 3\,x^2\,y + 3\,x\,y^2 + y^3$. Ans. $x + y$.
2. Find the cube root of $x^3 - 3\,x^2\,y + 3\,x\,y^2 - y^3$. Ans. $x - y$.
3. Find the cube root of $x^3 \pm 9\,x^2 + 27\,x \pm 27$. Ans. $x \pm 3$.
4. Find the 4^{th} root of $x^4 + 4\,x^3\,y + 6\,x^2\,y^2 + 4\,x\,y^3 + y^4$. Ans. $x + y$.
5. Find the 4^{th} root of $x^4 - 8\,x^3\,y + 24\,x^2\,y^2 - 32\,x\,y^3 + 16\,y^4$. Ans. $x - 2\,y$.
6. Find the 5^{th} root of $x^5 + 5\,x^4\,y + 10\,x^3\,y^2 + 10\,x^2\,y^3 + 5\,x\,y^4 + y^5$. Ans. $x + y$.
7. Find the 5^{th} root of $x^5 - 5\,x^4\,y^{\frac{1}{2}} + 10\,x^3\,y - 10\,x^2\,y^{\frac{3}{2}} + 5\,x\,y^2 - y^{\frac{5}{2}}$. Ans. $x - y^{\frac{1}{2}}$.
8. Find the 6^{th} root of $x^6 + 6\,x^5\,y + 15\,x^4\,y^2 + 20\,x^3\,y^3 + 15\,x^2\,y^4 + 6\,x\,y^5 + y^6$. Ans. $x + y$.
9. Find the 6^{th} root of $x^6 - 12\,x^5 + 60\,x^4 - 160\,x^3 + 240\,x^2 - 192\,x + 64$. Ans. $x - 2$.
10. Find the cube root of $x^6 - 6\,x^5 + 40\,x^3 - 96\,x - 64$. Ans. $x^2 - 2\,x - 4$.

ARITHMETICAL PROPORTION AND PROGRESSION.

ARITHMETICAL PROPORTION is the relation between two numbers, with respect to their difference.

If three numbers be in arithmetical proportion, the difference between the first and second is the same as the difference between the second and third: thus 1, 4, 7, are in arithmetical proportion, the common difference of which is 3. Also, a, $a + d$, $a + 2\,d$, the common difference of which is d.

ARITHMETICAL PROGRESSION is when a series of numbers or quantities increase or decrease by the same common difference: thus 1, 3, 5, 7, &c. is an *increasing* series, the common difference of which is 2. Also, a, $a + d$, $a + 2\,d$, $a + 3\,d +$, *&c.* the common difference of which is d.

25, 20, 15, 10, &c. is a *decreasing* series, the common difference of which is 5. Also, a, $a - d$, $a - 2\,d$, $a - 3\,d$, &c. the common difference of which is d. Whence arithmetical progression is formed by *addition* or *subtraction*.

THEOREM 1.

If *three* numbers or quantities are in arithmetical proportion, the sum of the extremes is *double* the *mean*.

As 1, 4, 7 and x, y, z.

Then $1 + 7 = 4 \times 2$ and $x + z = 2\,y$.

THEOREM 2.

If *four* numbers or quantities be in arithmetical proportion, the *sum* of the two extremes is equal to the *sum* of the two means.

As 1, 3, 5, 7 and x, y, z, w.

Then $1 + 7 = 3 + 5$ and $x + w = y + z$.

THEOREM 3.

The *last term* (l) of an *increasing* series is equal to the *first* term, *plus* the product of the common difference into the *number* of terms *less one*; but *minus* that product when decreasing; thus representing the *number* of terms of

$$a + (a + d) + (a + 2\,d) + (a + 3\,d) \text{ \&c. by } n$$
$$l = a + (n - 1)\,d$$

of $a + (a - d) + (a - 2\,d) + (a - 3\,d) + \text{\&c. by } n$
$$l = a - (n - 1)\,d.$$

THEOREM 4.

The sum S, of n terms, of any arithmetical series, is equal to the sum of the two extremes, multiplied by *half* the number of terms.

Thus in $a + (a + d) + (a + 2\,d) + (a + 3\,d)$ to l
$$S = (a + l)\frac{n}{2}.$$
In which if the values of l, found in the last theorem, be substituted, we obtain $S = \left\{2\,a + (n - 1)\,d\right\} \times \frac{n}{2}$ which when d is negative, becomes $S = \left\{2\,a - (n - 1)\,d\right\} \times \frac{n}{2}$.

From the foregoing theorems is deduced the following :*

FORMULA.

1. $a = \frac{2\,S}{n} - l.$
2. $a = l - (n - 1)\,d.$

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* a = the first term, d = the common difference, l = the last term, n = the number of terms, and S = the sum of all the terms.

3. $a = \frac{S}{n} - \frac{(n-1)d}{2}$.

4. $l = a + (n-1)d$.

5. $l = \frac{2S}{n} - a$.

6. $l = \frac{S}{n} + \frac{(n-1)d}{2}$.

7. $n = \frac{2S}{a+l}$.

8. $n = \frac{l-a}{d} + 1$.

9. $d = \frac{l-a}{n-1}$

10. $d = \frac{l^2 - a^2}{2S - (a+l)}$.

11. $S = (a+l) \times \frac{n}{2}$.

12. $S = \left\{ 2a + (n-1)d \right\} \times \frac{n}{2}$.*

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* If P and Q represent the p^{th} and q^{th} terms, then (theo. 3.)

$$P = a + (p-1)d, \text{ and } Q = a + (q-1)d.$$

$$\therefore P - Q = (p-q)d, \text{ and } d = \frac{P-Q}{p-q}.$$

$$\text{Whence } a = \frac{Q(p-1) - P(q-1)}{p-q}.$$

$$l = \frac{Q(p-n) - P(q-n)}{p-q}.$$

$$\text{And } S = \left\{ \frac{Q(2p-n-1) - P(2q-n-1)}{p-q} \right\} \times \frac{n}{2}.$$

Examples of the use of these formulæ.

1. In an increasing arithmetical series, the first term is 3, the common difference 2, and the number of terms 20; required the last term.

Here (form. 4.) $a = 3$, $d = 2$ and $n = 20$.

$\therefore 3 + (20 - 1) \times 2 = 3 + 19 \times 2 = 3 + 38 = 41$. Ans.

2. The first term of a decreasing arithmetical series is 50, the common difference 3, and the number of terms 15; required the last term.

Here (form. 4.) $a = 50$, $d = -3$, and $n = 15$.

$\therefore 50 + (15 - 1) \times -3 = 50 + 14 \times -3 = 50 - 42 = 8$. Ans.

3. The first term is 1, the last term is 100, and the number of terms 20; required the sum of the series.

Here (form. 11.) $a = 1$, $l = 100$, and $n = 20$.

$\therefore (1 + 100) \times \frac{20}{2} = 101 \times 10 = 1010$. Ans.

4. The first term is 100, the common difference -3, and number of terms 34; required the sum of the series.

Here (form. 12.) $a = 100$, $d = -3$, and $n = 34$.

$\therefore \left\{ 200 + (34 - 1) \times -3 \right\} \times \frac{34}{2} = (200 - 99) \times 17 = 101 \times 17 = 1717$. Ans.

5. Find the sums of ten different arithmetical series to n terms, each beginning with unity, the common difference of the first being 1, of the second 2, of the third 3, &c.

Here (form. 12.) $a = 1$ and by making $d = 1$, $d = 2$, $d = 3$, &c. the sums are as follow:

$d = 1$ the sum is $\frac{n^2 + n}{2}$.

$d = 2$ .... n^2.

$d = 3$ .... $\frac{3n^2 - n}{2}$.

$d = 4$ .... $2n^2 - n$.

$d = 5$ .... $\frac{5n^2 - 3n}{2}$.

$d = 6$ .... $3n^2 - 2n$.

$d = 7$ .... $\frac{7n^2 - 5n}{2}$.

$d = 8$ .... $4n^2 - 3n$.

$d = 9$ .... $\frac{9n^2 - 7n}{2}$.

$d = 10$ .... $5n^2 - 4n$.

Now these sums are general expressions for the first ten *polygonal numbers*, and by taking $n = 1, 2, 3$, &c. we obtain their respective series, as under.

Trigonal $\frac{n^2 + n}{2} = 1, 3, 6, 10,$ &c.

Tetragonal $n^2 = 1, 4, 9, 16,$ &c.

Pentagonal $\frac{3n^2 - n}{2} = 1, 5, 12, 22,$ &c.

Hexagonal $2n^2 - n = 1, 6, 15, 28,$ &c.

Heptagonal $\frac{5n^2 - 3n}{2} = 1, 7, 18, 34,$ &c.

Octagonal $3n^2 - 2n = 1, 8, 21, 40,$ &c.

Nonagonal $\frac{7n^2 - 5n}{2} = 1, 9, 24, 46,$ &c.

Decagonal $4n^2 - 3n = 1, 10, 27, 52,$ &c.

Undecagonal $\frac{9n^2 - 7n}{2} = 1, 11, 30, 58,$ &c.

Duodecagonal $5n^2 - 4n = 1, 12, 33, 60,$ &c.

From which it is obvious how any order of m-gonal numbers may be found, for if $d = m - 2$, then the m-gonal number is $\frac{(m-2)n^2 - (m-4)n}{2}$, which includes every order of *polygonal* numbers, m being the denomination of the order; and by putting $m = 3, 4, 5,$ &c. we obtain the preceding forms.

EXERCISES IN ARITHMETICAL PROGRESSION.

1. Find the sum of 100 terms of $2 + 4 + 6 + 8 +$, &c. Ans. 10100.

2. Find the sum of 50 terms of $5 + 8 + 11 + 14 +$, &c. Ans. 3925.

3. Find the sum of 16 terms of $16 + 14 + 12 + 10 -$, &c. Ans. 16.

4. Find the sum of 12 terms of $8 + 15 + 22 + 29 +$, &c. Ans. 558.

5. Find the sum of 20 terms of $1 + 3 + 5 + 7 +$, &c. Ans. 400.

6. Find the sum of 40 terms of $1 + 2 + 3 + 4 +$, &c. Ans. 820.

7. The first term is 5, the common difference 5, and the last term 505; required the number of terms. Ans. 101.

8. The first term is 1, the common difference 2, and number of terms 81; required the last term. Ans. 161.

9. The sum of the series is 2046, common difference 4, and number of terms 31; what is the first term?

Ans. 6.

10. The first term is 5, the last term 60, and number of terms, 12; required the common difference. Ans. 5.

11. The first term is 44, the last term 116, and sum of all the terms 800; required the common difference.

Ans. 8.

12. The first term is 116, the common difference — 8, and the number of terms 10; required the last term.

Ans. 44.

13. Find the sum of 12 terms of $1 + 1\frac{1}{2} + 2 + 2\frac{1}{2}$ +, &c. Ans. 45.

14. Find the sum of 15 terms of $1 + \frac{6}{7} + \frac{5}{7} + \frac{4}{7}$ +, &c.

Ans. 0.

15. Find the sum of 16 terms of $15 + 14\frac{2}{3} + 14\frac{1}{3} + 14$ + &c. Ans. 200.

GEOMETRICAL PROPORTION AND PROGRESSION.

1. GEOMETRICAL PROPORTION is the equality between two ratios, or between the quotients of the terms.

2. GEOMETRICAL PROGRESSION is when a series of quantities have the same constant ratio; that is, when they *increase* by a common multiplier, or *decrease* by a common divisor. Whence geometrical progression is formed by multiplication or division.

Thus 1, 2, 4, 8, 16, 32, &c.

And a, $a\,r$, $a\,r^2$, $a\,r^3$, $a\,r^4$, $a\,r^5$, &c.

are *increasing* series, the ratio or common *multiplier* of the former being 2, and that of the latter r.

Also, 32, 16, 8, 4, 2, 1, &c.

And $a\,r^5$, $a\,r^4$, $a\,r^3$, $a\,r^2$, $a\,r$, a, &c.

are *decreasing* series, the ratio or common divisor of the former being 2, and that of the latter r.

Some of the most useful properties of numbers in geometrical progression, are contained in the following theorems:

THEOREM 1.

When *three* numbers or quantities are in geometrical proportion, the *product* of the extremes is equal to the *square* of the mean. Thus 2, 6, 18, are in geometrical proportion, because the first divided by the second is the same as the second divided by the third.

And $2 \times 18 = 6^2$.

Also, if x, y, and z are in geometrical proportion, then $x\,z = y^2$, for by that proportion $\frac{x}{y} = \frac{y}{z}$ whence $x\,z = y^2$.

THEOREM 2.

When *four* numbers or quantities are in geometrical proportion, the product of the extremes is equal to the product of the means. Thus a, b, c, d, are in geometrical proportion, when* $\frac{a}{b} = \frac{c}{d}$, or $a\,d = b\,c$.

That is, as $a : b :: c : d$.

* Whence it is obvious that if a, b, c, and d, be multiplied or divided by any constant quantity m, they will still be proportional.

As an illustration, put $a = 2$, $b = 4$, $c = 3$, and $d = 6$, and they will be proportionable in the following forms:

1. As $a : b :: c : d$ or as $2 : 4 :: 3 : 6$
2. As $b : a :: d : c$ or as $4 : 2 :: 6 : 3$
3. As $a : c :: b : d$ or as $2 : 3 :: 4 : 6$
4. As $a : a + b :: c : c + d$ or as $2 : 6 :: 3 : 9$
5. As $a : a \backsim b :: c : c \backsim d$ or as $2 : 2 :: 3 : 3$
6. As $a + b : a \backsim b :: c + d : c \backsim d$ or as $6 : 2 :: 9 : 3$
7. As $a \backsim b : b :: c \backsim d : d$ or as $2 : 4 :: 3 : 6$

Also, the like *powers* and *roots* of these quantities will be proportionals.

8. As $a^n : b^n :: c^n : d^n$ or $2^n : 4^n :: 3^n : 6^n$
9. As $a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}$ or $2^{\frac{1}{n}} : 4^{\frac{1}{n}} :: 3^{\frac{1}{n}} : 6^{\frac{1}{n}}$

In all of which, the product of the extremes is equal to that of the means.

THEOREM 3.

The last term of a geometrical series is found by multiplying the first term by such a power of the ratio as is denoted by the number of terms less one. Thus, representing the first term by a, the ratio by r, and the number of terms by n, the series is

$$a + a\,r + a\,r^2 + a\,r^3 +, \&c.$$

and, as r is not in the first term, and its index increases by 1 from the second, it is obvious that the last power of r is $n - 1$. Whence $a \times r^{n-1} = a\,r^{n-1} =$ last term.

THEOREM 4.

The sum of any series in geometrical progression is found by multiplying the last term by the ratio, and dividing the difference between this product and the first term by the difference between one and the ratio; thus in the series $a + a\,r + a\,r^2 + a\,r^3 +$, &c. the last term is $a\,r^{n-1}$, and the sum of n terms is $\frac{a\,r^{n-1} \times r - a}{r - 1} = \frac{a\,r^n - a}{r - 1} = a\left(\frac{r^n - 1}{r - 1}\right)$; which, when r is a proper fraction, and n *infinite*, becomes $\frac{a}{1 - r}$: that is, $a + a\,r + a\,r^2 + a\,r^3 +$, &c. *ad infinitum* $= \frac{a}{1 - r}$.

Representing the first term by a, ratio by r, last term by l, number of terms by n, and sum of n terms by S, from the two last theorems, we deduce the following formula:

1. $S =$ *ad infinitum* $= \frac{a}{1 - r}$
2. $S = a\left(\frac{r^n - 1}{r - 1}\right)$
3. $S = \frac{r\,l - a}{r - 1}$
4. $S = \frac{l\,(r^n - 1)}{r^{n-1}(r - 1)}$
5. $a = \frac{l}{r^{n-1}}$
6. $a = r\,l - (r - 1)\,S$
7. $a = \frac{(r - 1)\,S}{r^n - 1}$
8. $l = a\,r^{n-1}$
9. $l = S - \frac{S - a}{r}$
10. $l = \frac{S\,(r - 1)\,r^{n-1}}{r^n - 1}$

11. $r = \frac{S - a}{S - l}$

12. $r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}$

13. $n = \frac{Ll - La}{Lr} + 1$*

Examples of the use of these formulæ.

1. To find the sum of $1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} +$, &c. *ad infinitum.*

Here (form. 1.) $a = 1$, $r = \frac{3}{4}$.

$$\therefore \frac{a}{1 - r} = \frac{1}{1 - \frac{3}{4}} = \frac{1}{\frac{1}{4}} = 4. \text{ Ans.}$$

2. To find the sum of 20 terms of $2 + 6 + 18 +$, &c.

Here (form. 2.) $a = 2$, $r = 3$, and $n = 20$.

$\therefore a\left(\frac{r^n - 1}{r - 1}\right) = 2\left(\frac{3^{20} - 1}{2}\right) = 3^{20} - 1 = 9^{10} - 1 = 81^5 - 1 = (81^4 \times 81) - 1 = (81^2 \times 81^2 \times 81) - 1 = (6561 \times 6561 \times 81) - 1 = (43046721 \times 81) - 1 = 3486784400$. Ans.†

---

* L denotes the logarithm.

† If P and Q represent the p^{th} and q^{th} terms, then $P = a\,r^{p-1}$ and $q = a\,r^{q-1}$, whence $\frac{P}{Q} = r^{p-q}$.

$\therefore r = \left(\frac{P}{Q}\right)^{\frac{1}{p-q}}$, and $a = \frac{P}{r^{p-1}} = P\left(\frac{Q}{P}\right)^{\frac{p-1}{p-q}} = \left(\frac{Q^{p-1}}{P^{q-1}}\right)^{\frac{1}{p-q}}$

also, $l = \left(\frac{Q^{p-n}}{P^{q-n}}\right)^{\frac{1}{p-q}}$, and $S = \left(\frac{Q^{p-n}}{P^{q-1}}\right)^{\frac{1}{p-q}} \left\{\frac{P^{\frac{n}{p-q}} - Q^{\frac{n}{p-q}}}{P^{\frac{1}{p-q}} - Q^{\frac{1}{p-q}}}\right\}$

3. The first term is 2, the ratio 3, and number of terms 9; required the last term.

Here (form. 8.) $a = 2$, $r = 3$, and $n = 9$.

$$\therefore a\, r^{n-1} = 2 \times 3^{9-1} = 2 \times 3^8 = 2 \times 9^4 = 2 \times 81^2 = 2 \times 6561 = 13122. \text{ Ans.}$$

4. The last term is 13122, the number of terms 9, and ratio 3; required the first term.

Here (form. 5.) $l = 13122$, $n = 9$, and $r = 3$.

$$\therefore \frac{l}{r^{n-1}} = \frac{13122}{3^8} = \frac{13122}{9^2 \times 9^2} = \frac{13122}{81 \times 81} = \frac{13122}{6561} = 2. \text{ Ans.}$$

5. The first term is 2, the last term 13122, and number of terms 9; required the ratio.

Here (form. 12.) $a = 2$, $l = 13122$, and $n = 9$.

$$\therefore \left(\frac{l}{a}\right)^{\frac{1}{n-1}} = \left(\frac{13122}{2}\right)^{\frac{1}{8}} = 6561^{\frac{1}{8}} = 81^{\frac{1}{4}} = 9^{\frac{1}{2}} = 3. \text{ Ans}$$

6. The first term is 2, last term 13122, and ratio 3; required the last term.

Here (form. 13.) $a = 2$, $l = 13122$, and $r = 3$.

$$\therefore \frac{L\,l - L\,a}{L\,.\,r} + 1 = \frac{L\,.\,13122 - L\,.\,2}{L\,.\,3} + 1 =$$

$$\frac{4\,.\,117998 - 0\,.\,301030}{0\,.\,477121} + 1 = \frac{3\,.\,816968}{.\,477121} + 1 = 8 + 1 = 9. \text{ Ans.}$$

EXERCISES.

1. Find the sum of $1 + \frac{1}{2} + \frac{1}{4} +$, &c. *ad infinitum.*
Ans. 2.

2. Find the sum of $1 + \frac{2}{3} + \frac{4}{9} +$, &c. *ad infinitum.*
Ans. 3.

3. Find the sum of $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} +$, &c. *ad infinitum.*
Ans. 1.

4. Find the sum of $\frac{1}{5} + \frac{1}{15} + \frac{1}{45} +$, &c. *ad infinitum.*
Ans. $\frac{3}{10}$.

5. Find the sum of $4 + 3 + \frac{9}{4} +$, &c. *ad infinitum.*
Ans. 16.

6. Find the value of $.6 = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} +$, &c. *ad infinitum.*
Ans. $\frac{2}{3}$.

7. Find the sum of 7 terms of $1 + 4 + 16 +$, &c.
Ans. 5461.

8. Find the sum of 8 terms of $5 + 20 + 80 +$, &c.
Ans. 109225.

9. Find the sum of 12 terms of $1 + 3 + 9 +$, &c.
Ans. 265720.

10. Find the sum of 16 terms of $1 + 2 + 4 +$, &c.
Ans. 65535.

11. In $1 + 2 + 4 +$, &c. find the sum of n terms.
Ans. $2^n - 1$.

12. In the last series, find the n^{th} or last term.
Ans. 2^{n-1}.

13. The first term is 5, last term 38880, and sum 46655; required the ratio.
Ans. 6.

14. The ratio is 6, last term 38880, and sum 46655; required the first term.
Ans. 5.

15. The third term is 52, and the seventh term 832; required the ratio and first term.
Ans. Ratio 2; first term, 13.

HARMONICAL PROPORTION.

Three quantities are said to be in *harmonical proportion*, when the first is to the third as the difference of the first and second is to the difference of the second and third. Suppose a, b, and c in harmonical proportion, then

$$a : c :: a \backsim b : b \backsim c.$$

Four quantities are in harmonical proportion, when the first is to the fourth as the difference of the first and second is to the difference of the third and fourth.

If the four quantities be a, b, c, and d, then

$$a : d :: a \backsim b : c \backsim d;$$

or, the reciprocals of numbers in arithmetical proportion are in harmonical proportion.

If a, y, and b are in *arithmetical*, *geometrical*, and *harmonical* proportion, then the

$$\text{Arithmetical mean} = \tfrac{1}{2}(a + b) = y,$$

$$\text{Geometrical mean} = (a\,b)^{\frac{1}{2}} = y,$$

$$\text{and Harmonical mean} = \frac{2\,a\,b}{a + b} = y.$$

SIMPLE EQUATIONS.

CASE 1.

AN EQUATION is when two equal quantities, differently expressed, are compared together by means of the sign $=$ placed between them.

$$\text{As } 10 - 4 = 6^*$$

A SIMPLE EQUATION is that which contains the unknown quantity in a simple form only, as $x + a = b$; $a x + b x = c$, &c.

REDUCTION OF EQUATIONS is the method of finding the value of the unknown quantity, which is shown in the following rules:

RULE 1.

Any quantity may be transposed from one side of the equation $(=)$ to the other by *changing its sign.*

Thus if $x - 5 = 5$,

Then by transposing the left hand 5,

$$x = 5 + 5 = 10,$$

which is the same as adding 5 to each side of the equation.

Also, if $x + 5 = 10$,

Then by transposing the 5,

$$x = 10 - 5 = 5,$$

which is the same as subtracting 5 from each side of the equation.

---

* Equations are founded upon this obvious principle, that if equals are added to equals, their *sums* will be equal: if equals are subtracted from equals, their *differences* will be equal: if equals are multiplied by equals, their *products* will be equal: and if equals are divided by equals, their *quotients* will be equal.

Again, if $x - a = b$,

Then $x = b + a$, by transposing a.

Or, if $x - a - b = - c$,

Then $x = a + b - c$, by transposing a and b.

RULE 2.

If the *unknown* quantity be multiplied by any other quantity, that quantity may be taken away by dividing all the terms of the equation by it.

Thus if $a x = a b - a$,

Then $x = b - 1$, by dividing by a.

Also, if $3 y + 15 = 27$,

Then $3 y = 27 - 15 = 12$;

Or, $3 y = 12$.

$\therefore y = 4$, by $\div$ by 3.

Or, if $a x - a^2 = a b$,

Then $x - a = b$, by $\div$ by a.

$\therefore x = b + a$, by transposing a.

RULE 3.

If the unknown quantity be divided by any other quantity, that quantity may be taken away by multiplying all the terms of the equation by it.

Thus if $\frac{x}{2} = 4$,

Then by multiplying both sides by 2,

$x = 8$.

Or, if $\frac{3x}{4} - 2 = 7$,

Then $\frac{3x}{4} = 9$, by transposing the 2;

And $3 x = 36$, by multiplying by 4.

$\therefore x = 12$, by $\div$ 3.

RULE 4.

If the unknown quantity be in the form of a surd, transpose the rest of the terms so that it may stand on one side of the equation (=) alone; then raise both sides to such a power as is denoted by the index of the surd.

Thus if $x^{\frac{1}{2}} = 3$, then by squaring both sides.

$$x = 9.$$

Also, if $\sqrt{3\,x + 3} = 3$,* then by squaring

$$3\,x + 3 = 9.$$

And $3\,x = 6$ by transposing the 3.

$\therefore\ x = 2$ by $\div$ by 3.

If $x^{\frac{1}{3}} = 3$, then $x = 27$ by cubeing.

Or, if $(x - a)^{\frac{1}{3}} = b$, then by cubeing.

$$x - a = b^3.$$

$\therefore\ x = b^3 + a$ by transposition.

RULE 5.

If the side of the equation containing the unknown quantity be a complete power, then extract such a root on both sides of the equation as is denoted by the index of the power.

Thus, if $x^2 = 9$,

Then $x = 3$ by extracting the square root.

Or, if $(x - a)^2 = b^2$,

Then $x - a = b$ by taking the square root.

$\therefore\ x = a + b$ by transposition.

* When a quantity is affected by the radical sign, that quantity is squared by suppressing the sign.

Also, if $(x^2 + a)^3 = b^6$, then by taking the cube root

$$x^2 + a = b^2.$$

$\therefore\ x^2 = b^2 - a$ * by transposition.

$\therefore\ x = \sqrt{b^2 - a}$ by taking the square root.

MISCELLANEOUS EXAMPLES.

1. Given $3x - 12 = 3$, to find the value of x.

First, $3x = 3 + 12$ by transposing 12.

Or, $3x = 15$.

$\therefore\ x = 5$ by $\div$ by 3.

2. Given $7x - 14 = 4x + 7$, to find x.

First, $7x - 4x = 7 + 14$ by trans. $4x$ and 14.

Or, $3x = 21$.

$\therefore\ x = 7$ by $\div$ by 3.

3. Given $8x^2 - 2x = 4x^2 + 6x$, to find x.

First, $8x^2 - 4x^2 = 6x + 2x$ by transpos.

Or, $4x^2 = 8x$.

$\therefore\ x = 2$ by $\div$ by $4x$.

4. Given $ax - 3b = dx - 3c$, to find x.

First, $ax - dx = 3b - 3c$ by transposition.

Or, $(a - d)x = 3(b - c)$

$$\therefore\ x = \frac{3(b - c)}{a - d} \text{ by } \div \text{ by } a - d.$$

---

* Equations involving the square or any power of the unknown term, are denominated simple quadratics, or pure equations.

5. Given $18\,a\,y^3 - 24\,a\,b\,y^2 = 6\,a\,y^3 + 12\,a^2\,y^2$, to find y.

First, by dividing by $6\,a\,y^2$, it becomes

$$3\,y - 4\,b = y + 2\,a$$

$\therefore\ 3\,y - y = 2\,a + 4\,b$ by transposition.

Or, $2\,y = 2\,a + 4\,b$

$\therefore\ y = a + 2\,b$ by $\div$ by 2.

6. Given $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} = 3$, to find x.

$$\left.\begin{array}{ll} \text{First, } x + \frac{2x}{3} - \frac{x}{2} = 6 & \text{by} \times \text{by } 2 \\ \text{And } 3\,x + 2\,x - \frac{3\,x}{2} = 18 & \text{by} \times \text{by } 3 \\ \text{Also, } 6\,x + 4\,x - 3\,x = 36 & \text{by} \times \text{by } 4 \end{array}\right\} *$$

Or, $7\,x = 36$.

$\therefore\ x = \frac{36}{7} = 5\frac{1}{7}$.

7. Given $x + \frac{1}{2}\,x + \frac{1}{3}\,x + \frac{1}{4}\,x + \frac{1}{6}\,x = 54$, to find x.

First, $12\,x + 6\,x + 4\,x + 3\,x + 2\,x = 648$, by $\times$ by 12.

Or, $27\,x = 648$.

$\therefore\ x = 24$, by $\div$ by 27.

8. Given $\sqrt{x} + 1 = (x + 9)^{\frac{1}{2}}$, to find x.

First, $x + 2\sqrt{x} + 1 = x + 9$, by squaring (rule 4.)

Or, $2\sqrt{x} = 8$.

$\therefore\ \sqrt{x} = 4$, by $\div$ by 2.

$\therefore\ x = 16$, by squaring.

---

* This equation might have been more readily cleared of fractions, by multiplying it by 12, the least common multiple of its denominators.

9. Given $\frac{\sqrt{x}+4}{\sqrt{x}-3}=\frac{\sqrt{x}+12}{\sqrt{x}-2}$, to find x.

First, by clearing the equation of fractions,

$(\sqrt{x}-3)\cdot(\sqrt{x}+12)=(\sqrt{x}+4)\cdot(\sqrt{x}-2)$.*

Or, $x+9\sqrt{x}-36=x+2\sqrt{x}-8$.

Or, $9\sqrt{x}-2\sqrt{x}=36-8$.

Or. $7\sqrt{x}=28$.

$\therefore \sqrt{x}=4$, by $\div$ by 7.

$\therefore x=16$, by squaring.

10. Given $(x^2+3ax+ab)^{\frac{1}{2m}}=(x+a)^{\frac{1}{m}}$, to find x.

First, by raising each side to the m^{th} power.†

$(x^2+3ax+ab)^{\frac{1}{2}}=x+a$.

And by squaring both sides.

$x^2+3ax+ab=x^2+2ax+a^2$.

Or, $ax=a^2-ab$, by transp.

$\therefore x=a-b$, by $\div$ by a.

If the student fully understands the foregoing examples, he will encounter no difficulty in the following

EXERCISES.

1. Given $2x-5=1$, to find the value of x.

Ans. $x=3$.

2. Given $20-4x=30-6x$, to find x.

Ans. $x=5$.

3. Given $3x-5+7=19+4$, to find x.

Ans. $x=7$.

---

* Multiplying each numerator by the other's denominator, is the same as multiplying both sides by the product of the denominators.

† See note to Involution.

4. Given $20x - x^2 = 3x^2 + 4x$, to find x.
Ans. $x = 4$.

5. Given $x + \frac{x}{3} = 8$, to find x. Ans. $x = 6$.

6. Given $5x - 19 = 3x - \frac{1}{2}x + 1$, to find x.
Ans. $x = 8$.

7. Given $y + \frac{y}{3} + \frac{y}{6} = 9$, to find y. Ans. $y = 6$.

8. Given $x + \frac{1}{2}x + \frac{1}{3}x + \frac{1}{6}x = 24$, to find x.
Ans. $x = 12$.

9. Given $\frac{x+7}{2} + \frac{2x}{3} + \frac{x-8}{4} = 10$, to find x.
Ans. $x = 6$.

10. Given $\frac{x+1}{2} + \frac{x+2}{3} = 16 - \frac{x+3}{4}$,* to find x.
Ans. $x = 13$.

11. Given $\frac{10-x}{2} + 6 = \frac{x-4}{4} + \frac{x+10}{3}$, to find x.
Ans. $x = 8$.

12. Given $\frac{x}{12} - \frac{x}{6} = 2 - \frac{x}{4}$, to find x.
Ans. $x = 12$.

13. Given $x + \frac{27-2x}{4} - \frac{7x+5}{6} - \frac{2x-3}{12} = 2$, to find x.
Ans. $x = 5$.

14. Given $x + \frac{11-x}{3} = \frac{19-x}{2}$, to find x.
Ans. $x = 5$.

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* When the leading sign of a compound term is minus, the other signs of it are changed, when it is freed from fractions.

15. Given $x - \frac{x}{2} + \frac{x}{3} - \frac{x}{4} = 7$, to find x.

Ans. $x = 12$.

16. Given $\frac{3x+4}{5} - \frac{7x-3}{2} = \frac{x-16}{4}$, to find x.

Ans. $x = 2$.

17. Given $\frac{x-2}{5} + \frac{x+2}{4} = 1$, to find x.

Ans. $x = 2$.

18. Given $\frac{4x+7}{5} + \frac{x-2}{3} = 3$, to find x.

Ans. $x = 2$.

19. Given $\frac{7x-5}{3} + \frac{87-6x}{7} - \frac{8x-46}{21} = 25$, to find x.

Ans. $x = 11$.

20. Given $\frac{x-1}{7} + \frac{23-x}{5} = 7 - \frac{x+4}{4}$, to find x.

Ans. $x = 8$.

21. Given $x - \frac{x-7}{3} + \frac{3x-1}{5} - \frac{2x}{7} = 9$, to find x

Ans. $x = 7$

22. Given $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{5} + \frac{x}{6} = 87$, to find x.

Ans. $x = 60$.

23. Given $\frac{3x-7}{5} - \frac{85-4x}{7} + \frac{45-x}{9} = 1$, to find x.

Ans. $x = 9$.

24. Given $\frac{x-5\frac{1}{2}}{2} + \frac{5x-16\frac{1}{2}}{8} - \frac{15\frac{1}{2}-x}{18} = 2$, to find x.

Ans. $x = 6\frac{1}{2}$.

25. Given $a - \frac{b^2}{x} = c$, to find x. Ans. $x = \frac{b^2}{a - c}$.

26. Given $a\,x - b = c\,x + d$, to find x.

Ans. $x = \frac{b + d}{a - c}$.

27. Given $a\,x + b^2 = a^2 + b\,x$, to find x.

Ans. $x = a + b$.

28. Given $b\,x + 3\,x - 2\,a = 4\,x + 2\,c$, to find x.

Ans. $x = \frac{2\,(a + c)}{b - 1}$.

29. Given $x - a : x + a :: b : c$, to find x.*

Ans. $x = a\left(\frac{c + b}{c - b}\right)$.

30. Given $\frac{\sqrt{3x}}{5} + 1 = 4$, to find x. Ans. $x = 75$.

31. Given $\sqrt{3\,x + 1} + 1 = x$, to find x.

Ans. $x = 5$.

32. Given $a - x = \frac{4\,x^2}{a - x}$, to find x. Ans. $x = \frac{1}{3}\,a$.

33. Given $\sqrt{12 + x} = 2 + \sqrt{x}$, to find x.

Ans. $x = 4$.

34. Given $\sqrt{x} + \sqrt{a + x} = \frac{3\,a}{\sqrt{a + x}}$, to find x.

Ans. $x = \frac{4}{5}\,a$.

35. Given $\sqrt{x - a} + \sqrt{2\,a} = \sqrt{x}$, to find x.

Ans. $x = \frac{9}{8}\,a$.

36. Given $a + x = \sqrt{a^2 + x\sqrt{4\,a\,b + x^2}}$, to find x.

Ans. $x = b - a$.

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* This is thrown into an equation by theorum 2, page 49.

37. Given $\sqrt{a+x}+\sqrt{a-x}=2$, to find x.

Ans. $x=2\sqrt{a-1}$.

38. Given $\left(a+x^{\frac{1}{2}}\right)^{\frac{1}{2}}+\left(a-x^{\frac{1}{2}}\right)^{\frac{1}{2}}=x^{\frac{1}{2}}$, to find x.

Ans. $x=4(a-1)$.

39. Given $\dfrac{x^{\frac{1}{3}}+3^{\frac{1}{3}}}{x^{\frac{1}{3}}-3^{\frac{1}{3}}}=\dfrac{4}{3}$, to find x.

Ans. $x=1029$.

40. Given $\dfrac{x^{\frac{1}{4}}+b^{\frac{1}{4}}}{x^{\frac{1}{4}}-b^{\frac{1}{4}}}=\dfrac{a}{b}$, to find x.

Ans. $x=b\left(\dfrac{a+b}{a-b}\right)^4$

41. Given $\dfrac{2x-1}{\sqrt{2x}+1}=2+\dfrac{\sqrt{2x}-1}{3}$, to find x.

Ans. $x=8$.

42. Given $\dfrac{\sqrt{3x}-2}{\sqrt{3x}+2}=\dfrac{\sqrt{3x}-1}{\sqrt{3x}+7}$, to find x.

Ans. $x=3$.

43. Given $\dfrac{5x-9}{\sqrt{5x}+3}-\dfrac{\sqrt{5x}-3}{2}=1$, to find x.

Ans. $x=5$.

44. Given $\dfrac{\sqrt{a-\sqrt{a-x^2}}}{\sqrt{a+\sqrt{a-x^2}}}=a$, to find x.

Ans. $x=\dfrac{2a}{a+1}$.

45. Given $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=b$, to find x.

Ans. $x=\dfrac{2ab}{b^2+1}$.

46. Given $\left(\frac{a^2}{x} + b\right)^{\frac{1}{2}} - \left(\frac{a^2}{x} - b\right)^{\frac{1}{2}} = c$, to find x.

Ans. $x = \frac{4\,a^2\,c^2}{4\,b^2 + c^4}$.

47 Given $(a - x)^{\frac{1}{n}} = (x^2 + 4\,a\,x + b^2)^{\frac{1}{2\,n}}$, to find x.

Ans. $x = \frac{a^2 - b^2}{6\,a}$.

48. Given $a + b\,(x + d)^{\frac{1}{m}} = c$, to find x.

Ans. $x = \left(\frac{c - a}{b}\right)^m - d$.

49. Given $\frac{4\,x^{m+1}}{a + x} = \frac{2\,x^m}{b - x} + 4\,x^m$, to find x.

Ans. $x = \frac{2\,a\,b + a}{2\,a - 1}$.

50. Given $\frac{a\,x^2 - b^2}{x\,\sqrt{a} + b} - c = \frac{x\,\sqrt{a} - b}{d}$, to find x.

Ans. $x = \frac{c\,d + b\,d - b}{\sqrt{a}\,(d - 1)}$.

PROBLEMS PRODUCING SIMPLE EQUATIONS.

Solved by using only one unknown letter.

1. What number is that, the *fourth* part of which is greater than its *fifth* part by 10?

Let $x =$ the required number,

Then its 4^{th} part is $\frac{x}{4}$, and its 5^{th} part $\frac{x}{5}$,

$\therefore \frac{x}{4} - \frac{x}{5} = 10$ by the question,

And $5x - 4x = 200$, by $\times$ by 20;

Or, $x = 200$ the required number.

2. Divide £100. between A, B, and C, so that A shall have £4. less than B, and C £2. more than B.

Let $x =$ A's share.

Then $x + 4 =$ B's share.

And $x + 4 + 2 =$ C's share.

$\therefore x + (x + 4) + (x + 4 + 2) = 100$ by the question.

Or, $3x + 10 = 100$.

Or, $3x = 90$.

$\therefore x = 30 =$ A's share,

$30 + 4 = 34 =$ B's share,

And, $30 + 4 + 2 = 36 =$ C's share.

100 proof.

3. A person lent a certain sum of money at 5 per cent., simple interest; which interest, in 10 years, was just £10. less than the principal, which is required.

Let $x =$ the principal, or money lent, in £.'s

Then $\frac{x \times 5 \times 10}{100} = \frac{50\,x}{100} = \frac{x}{2}$, the interest.

$\therefore\ x - 10 = \frac{x}{2}$ by the question.

Or, $2\,x - 20 = x$ by $\times$ by 2.

$\therefore\ x =$ £20.

4. Two travellers set out at the same time, from two towns, the distance between which is 30 miles, one going $3\frac{1}{2}$ miles an hour, and the other 4. In what time after starting will they meet?

Let $x =$ the required time in hours.

Then $3\frac{1}{2}\,x =$ distance travelled by one.

And $4\,x =$ ditto by the other.

$\therefore\ 3\frac{1}{2}\,x + 4\,x = 30$ by the question.

Or, $7\frac{1}{2}\,x = 30$.

Or, $15\,x = 60$ by $\times$ by 2.

$\therefore\ x = 4$ hours by $\div$ by 15.

5. A gentleman distributed 20*s.* among 30 boys and girls, giving the former 9*d.* and the latter 6*d.* each; how many were there of each?

Let $x =$ the number of boys.

Then $30 - x =$ the number of girls.

And $9\,x + (30 - x)\,6 = 240d. = 20s.$ by the question.

Or, $9\,x + 180 - 6\,x = 240$.

Or, $9\,x - 6\,x = 240 - 180$.

Or, $3\,x = 60$.

$\therefore\ x = 20$ boys, by $\div$ by 3.

$\therefore\ 30 - 20 = 10$ girls.

6. Two persons, A and B, set out together from the same place, and travel the same way, A going 10 miles the first day, 12 the second, 14 the third, and so on; but B travels every day 20 miles; how far will B travel before he is overtaken by A.

Let $x =$ the days they travel.

Then B goes $20\,x$ miles.

And by formula 12, page 44, A goes

$$\left\{20 + (x - 1)\,2\right\} \times \frac{x}{2} \text{ miles.}$$

$$\therefore \left\{20 + (x - 1)\,2\right\} \times \frac{x}{2} = 20\,x \text{ by the question,}$$

multiplying by 2, and dividing by x.

$$20 + 2\,x - 2 = 40.$$

$$\text{Or, } 2\,x = 22.$$

$$\therefore x = 11 \text{ days.}$$

$$\text{Whence } 20 \times 11 = 220 \text{ miles.}$$

7. What is the time, between five and six, when the hour and minute hands are together?

Let $x =$ the time past 5,

then since the minute-hand goes 12 times round, whilst the hour-hand goes once, we have

$$12 : 1 :: x + 5 : x$$

$$\text{Or, } 11 : 1 :: 5 : x \text{ (form 7, page 50.)}$$

$$\therefore 11\,x = 5 \quad \text{(theorem 2, page 49.)}$$

$$\text{Whence } x = \tfrac{5}{11} = 27'\ 16''\tfrac{4}{11}.$$

The result would have been precisely the same had the product of the extremes been made equal to product of the means in the first proportion.

8. A person bought a piece of cloth for £5. and another piece at the same price per yard for £6. 10*s.*; now if 10 yards be added to each piece, they will be in proportion of 5 to 6; required the price and length of each.

Let $x =$ yards of the shorter.

Then $\frac{5}{x} =$ price per yard.

And £6. 10*s.* $\div \frac{5}{x} = \frac{6\frac{1}{2}\ x}{5} =$ yards of the longer.

$\therefore x + 10 : \frac{6\frac{1}{2}\ x}{5} + 10 : 5 : 6$ by the question.

And by multiplying means and extremes.

$6\ x + 60 = 6\frac{1}{2}\ x + 50.$

Or, $10 = \frac{1}{2}\ x.$

$\therefore x = 20$ by $\times$ by 2.

And $\frac{6\frac{1}{2} \times 20}{5} = 6\frac{1}{2} \times 4 = 26$ the longer.

Whence £5. $\div$ 20 = 5*s.* the price.

9. Two persons, A and B, have both the same income; A saves $\frac{1}{5}^{th}$ of his; but B, by spending £80. a year more than A, at the end of 4 years finds himself £220. in debt; what is their income?

Let $x =$ their annual income in £'s.

Then A saves $\frac{1}{5}\ x$ }
And spends $\frac{4}{5}\ x$ } annually.

$\therefore \frac{4}{5}\ x + 80 =$ B's yearly expenditure.

$\therefore (\frac{4}{5}\ x + 80) \times 4 = 4\ x + 220$ by the question.

Or, $\frac{4}{5}\ x + 80 = x + 55$ by $\div$ by 4.

Or, $\frac{4}{5}\ x + 25 = x.$

Or, $4\ x + 125 = 5\ x$ by $\times$ by 5.

$\therefore x = 125.$

10. A farmer kept a servant for every 36 acres of land he rented; but after giving up 104 acres, he dismissed 5 servants, and had then a servant for every 40 acres only; what number of acres and servants had he at first?

$$\text{Let } x = \text{the number of acres at first.}$$

$$\text{Then } \frac{x}{36} = \text{the number of servants.}$$

$$\text{Also, } x - 104 = \text{the acres afterwards.}$$

$$\text{And } \frac{x - 104}{40} = \text{number of servants.}$$

$$\therefore \frac{x}{36} - \frac{x - 104}{40} = 5 \text{ by the question.}$$

$$\text{Or, } 10x - 9x + 936 = 1800 \text{ by } \times \text{ by } 360.$$

$$\therefore x = 864 \text{ acres.}$$

$$\text{And } 864 \div 36 = 24 \text{ servants.}$$

PROBLEMS PRODUCING SIMPLE EQUATIONS.

To be solved by the use of one unknown letter only.

1. A horse and an ass cost £12.; the horse cost twice as much as the ass; what was the price of each?

Ans. Horse, £8. Ass, £4.

2. A cow and a calf were bought for £16.; the cow cost three times as much as the calf; what was the price of each? Ans. Cow, £12. Calf, £4.

3. A lady bought cambric, calico, and silk, in all 200 yards. She had three times more cambric than silk, and four times more calico than silk; required the quantity of each.

Ans. Silk, 25 yds.; Cambric, 75 yds.; Calico, 100 yds.

4. What number is that from which if 10 be subtracted $\frac{1}{3}$ of the remainder is 30? Ans. 100.

5. What number is that, a *third* and a *fourth* of which is 7? Ans. 12.

6. Find a number such that if its *half* be taken from the *sum* of its *third* and *fourth* parts, the remainder will be *d*. Ans. 12 *d*.

7. A person distributed 10s. among 20 boys and girls, giving the former 8d. and the latter 3d. a piece; required the number of each. Ans. Boys, 12; Girls, 8.

8. A person rented 8 acres of land, for which he paid 10 guineas a year; it was in two fields, for one of which he paid 30s., and for the other 24s. an acre; required the acres in each. Ans. 5 Acres and 3 Acres.

9. A and B between them borrow £50., A to pay 6, and B 5 per cent. simple interest; now at the end of 25 years they had paid £67. 10s. interest; how much did each borrow? Ans. A, £20.; B, £30.

10. A gambler being asked how many sovereigns he had left, said, three times as many as he had lost; and being asked how many he had lost, said, as many as being multiplied into $\frac{1}{3}$ of the number left, would give as many as he had at first; query that number. Ans. 16.

11. Two persons, A and B, have both the same income; A saves $\frac{1}{5}$ of his yearly; but B, by spending £30. a year more than A, at the end of 5 years finds himself £50. in debt; what is their income? Ans. £100.

12. Two persons, A and B, travelling together, (A with £200. and B with £90.) met a company of robbers, who took from A twice as much as they took from B, leaving A three times as much as they left B; what did they take from each? Ans. from A, £140.; from B, £70.

13. A lady bought $3\frac{1}{2}$ dozen of oranges and lemons; for the oranges she gave $1\frac{1}{2}$d. a piece, and for the lemons 20d. a dozen; required the number of each, she having laid out 5s. 6d. Ans. Oranges, 24; Lemons, 18.

14. A smuggler had a quantity of brandy which he expected to sell for £9. 18s.; but after selling 10 gallons, a revenue officer seized $\frac{1}{3}$ of the remainder, in consequence of which he only made £8. 2s.; required the number of gallons, and price per gallon.

Ans. Gals., 22; 9s. per gal.

15. If 60 quarters of wheat cost £250., for what must it be sold a quarter, to gain by the whole as much as 6 quarters are sold for? Ans. £4. 12s. $7\frac{1}{9}$d.

16. Sold 20 quarters of oats for 20 guineas, and cleared as much by the whole as 12 bushels cost; what was the gain per quarter? Ans. 1s. $5\frac{25}{43}$d.

17. A market woman bought in a certain number of eggs, at the rate of 5 for 2d., one half of which she sold at 2 for a penny, and the other half at 3 for a penny, and cleared 4d. by so doing; what number had she?

Ans. 240.

18. What is the wage of that mechanic who pays $\frac{3}{7}$ of it for board and lodgings, $\frac{3}{8}$ of the remainder for books and clothes, besides laying by 10s. a week? Ans. 28s.

19. Divide £100. among A, B, and C, so that A's share may be to B's as 3 to 4, and B's to C's as 6 to 5.

Ans. A's, = £29. 0s. $7\frac{23}{31}$d. B's, = £38. 14s. $2\frac{10}{31}$d. and C's, = £32. 5s. $1\frac{29}{31}$d.

20. A General ranging his army in the form of a solid square, finds he has 100 men to spare; but increasing the side by 1 man, he wants 25 to fill up the square; how many soldiers had he? Ans 3944.

21. Two travellers, A and B, set out together from the same place, and travel the same way, A going 28 miles the first day, 26 the second, 24 the third, and so on; but B travels uniformly 20 miles a day; how far will A travel before he is overtaken by B? Ans. 180 miles.

22. A farmer kept a servant for every 40 acres he rented, and after taking 70 acres more he hired 4 additional servants, and then had a servant for every 30 acres; required the number of servants and acres at first.

Ans. 5 servants, and 200 acres.

23. A corn-factor bought a quantity of oats, on condition that they weighed 23 stones a quarter; but when they were delivered there were 6 quarters more than there ought to have been for the weight, each quarter weighing too light by three stones; how many quarters were delivered?

Ans. 46.

24. A man being asked how much money he had, said, I have 4s. in silver for every sovereign; but if I had 3 sovereigns more I should then have only 2s. 6d. for every sovereign; how much money had he? Ans. £6.

25. A person being asked the time of the day, said, it is between one and two, and the minute and hour hands are directly opposite to each other; query the time.

Ans. 38′ 10″$\frac{10}{11}$ past 1.

26. A gentleman who had forgot to wind up his watch at night, found that it had stopped early in the morning; and in order to set it right, put it back 5 hours and 40′, then observed that the time at which it had stopped, which is required, was to the true time as 29 to 105. Ans. 25′ past 2.

27. A huckster bought 160 eggs at 2 for a penny, and a certain number at three for a penny, and sold the whole at 5 for 2d., gaining thereby $\frac{1}{10}$ of what they cost; how many did he buy? Ans. 880.

28. A person started from York at 6 o'clock in the morning, to travel towards Beverley, at the rate of $3\frac{1}{2}$ miles an hour; at 20 minutes to 7, another set out from Beverley to meet him, travelling $4\frac{1}{2}$ miles an hour, and he went just half a mile beyond the middle of the distance before they met; required the distance from York to Beverley. Ans. 29 miles.

CASE 2.

Equations involving two unknown quantities may be reduced to one by the following rules:

RULE 1.

Find the value of one of the unknown quantities in terms of the other, in each equation; then these values being made equal to each other, there will arise a new equation, with only one unknown quantity in it, the value of which may be found as before.

EXAMPLE.

$$\left.\begin{array}{l}\text{Given } 2x + 3y = 21 \\ \text{And } \quad 3x - 2y = 12\end{array}\right\} \text{to find } x \text{ and } y.$$

From the first $2x = 21 - 3y$ by transp. $3y$.

$$\therefore x = \frac{21 - 3y}{2} \text{ by } \div \text{ by } 2.$$

And from the second $3x = 12 + 2y$ by transp. $2y$.

$$\therefore x = \frac{12 + 2y}{3} \text{ by } \div \text{ by } 3.$$

Whence by the rule

$$\frac{21 - 3y}{2} = \frac{12 + 2y}{3}.$$

And multiplying both by 6.

$$63 - 9y = 24 + 4y.$$

Or, $13y = 39$.

$$\therefore y = 3.$$

$$\text{Whence } x = \frac{12 + 6}{3} = 6.$$

EXERCISES.

1. Given $\left.\begin{array}{r} x + y = 20 \\ \text{And } x - y = 10 \end{array}\right\}$ to find x and y. *

   Ans. $x = 15$, $y = 5$.

2. Given $\left.\begin{array}{r} x + 3\,y = 30 \\ \text{And } 3\,x - y = 30 \end{array}\right\}$ to find x and y.

   Ans. $x = 12$, $y = 6$.

3. Given $\left.\begin{array}{r} 2\,x + 3\,y = 23 \\ \text{And } 5\,x - y = 15 \end{array}\right\}$ to find x and y.

   Ans. $x = 4$, $y = 5$.

4. Given $\left.\begin{array}{r} 4\,x + y = 34 \\ \text{And } x + 4\,y = 16 \end{array}\right\}$ to find x and y.

   Ans. $x = 8$, $y = 2$.

5. Given $\left.\begin{array}{r} 3\,x + 2\,y = 42 \\ \text{And } 2\,x + 3\,y = 48 \end{array}\right\}$ to find x and y.

   Ans. $x = 6$, $y = 12$.

6. Given $\left.\begin{array}{r} 5\,x + 4\,y = 58 \\ \text{And } 3\,x + 7\,y = 67 \end{array}\right\}$ to find x and y.

   Ans. $x = 6$, $y = 7$.

7. Given $\left.\begin{array}{r} 11\,x + 3\,y = 100 \\ \text{And } 10\,x - 8\,y = 48 \end{array}\right\}$ to find x and y.

   Ans. $x = 8$, $y = 4$.

8. Given $\left.\begin{array}{r} \frac{1}{2}\,x + \frac{1}{3}\,y = 7 \\ \text{And } x + y = 16 \end{array}\right\}$ to find x and y.

   Ans. $x = 10$, $y = 6$.

9. Given $\left.\begin{array}{r} \frac{1}{2}\,x + \frac{1}{3}\,y = 4 \\ \text{And } \frac{1}{3}\,x + \frac{1}{2}\,y = 3\frac{1}{2} \end{array}\right\}$ to find x and y.

   Ans. $x = 6$, $y = 3$.

---

* When the sum and difference of two numbers are given, half the sum added to half the difference, gives the greater; and half the difference taken from half the sum, gives the less.

10. Given $\frac{1}{4}x + \frac{1}{5}y = 23$ And $\frac{1}{6}x - \frac{1}{8}y = 5$ } to find x and y.

Ans. $x = 60, y = 40$.

11. Given $\frac{x}{7} + 7y = 99$ And $\frac{y}{7} + 7x = 51$ } to find x and y.

Ans. $x = 7, y = 14$.

12. Given $x : y :: 3 : 7$ And $5x + 2y = 87$ } to find x and y.

Ans $x = 9, y = 21$.

RULE 2.

Find the value of one of the unknown quantities in terms of the other, in one of the equations, by the last rule; then substitute this value for the letter it is equal to in the other equation, and there will result an equation involving only one unknown term, which may be reduced as before.

EXAMPLES.

Given $2x + 3y = 17$ And $5x - 2y = 14$ } to find x and y.

From the first $x = \frac{17 - 3y}{2}$ by the last rule; which *value* of x, substituted for x in the second

$$\text{Gives } \frac{17 - 3y}{2} \cdot 5 - 2y = 14.$$

$$\text{Or, } \frac{85 - 15y}{2} - 2y = 14.$$

$$\therefore 85 - 15y - 4y = 28 \text{ by } \times \text{ by } 2.$$

$$\text{Or, } 15y + 4y = 85 - 28.$$

$$\text{Or, } 19y = 57.$$

$$\therefore y = 3 \text{ by } \div \text{ by } 19.$$

$$\text{Whence } x = \frac{17 - 9}{2} = \frac{8}{2} = 4.$$

EXERCISES.

1. Given $\left.\begin{array}{l} 2x + 3y = 29 \\ \text{And } 3x - 2y = 11 \end{array}\right\}$ to find x and y.

   Ans. $x = 7$, $y = 5$.

2. Given $\left.\begin{array}{l} 5x - 3y = 13 \\ \text{And } 7x - 2y = 27 \end{array}\right\}$ to find x and y.

   Ans. $x = 5$, $y = 4$.

3. Given $\left.\begin{array}{l} x + y = 14 \\ \text{And } x - y = 2 \end{array}\right\}$ to find x and y.

   Ans. $x = 8$, $y = 6$.

4. Given $\left.\begin{array}{l} 12x - 7y = 67 \\ \text{And } 7x + 12y = 216 \end{array}\right\}$ to find x and y.

   Ans. $x = 12$, $y = 11$.

5. Given $\left.\begin{array}{l} \frac{x}{3} + 3y = 21 \\ \text{And } \frac{y}{3} + 3x = 29 \end{array}\right\}$ to find x and y.

   Ans. $x = 9$, $y = 6$.

6. Given $\left.\begin{array}{l} \frac{1}{2}x + \frac{1}{3}y = 8 \\ \text{And } \frac{1}{3}x + \frac{1}{2}y = 7 \end{array}\right\}$ to find x and y.

   Ans. $x = 12$, $y = 6$.

7. Given $\left.\begin{array}{l} \frac{1}{2}x + \frac{3}{4}y = 19 \\ \text{And } \frac{3}{5}x + \frac{2}{3}y = 20 \end{array}\right\}$ to find x and y.

   Ans. $x = 20$, $y = 12$.

8. Given $\left.\begin{array}{l} x : y :: 3 : 2 \\ \text{And } x^2 - y^2 = 20 \end{array}\right\}$ to find x and y.

   Ans. $x = 6$, $y = 4$.

9. Given $\left.\begin{array}{l} 30 - \frac{x}{2} = \frac{y}{3} + 4y \\ \text{And } \frac{x - y}{2} + \frac{x}{4} - 1 = \frac{3y - x}{5} \end{array}\right\}$ to find x and y.

   Ans. $x = 8$, $y = 6$.

10. Given $\left.\begin{aligned} x + 2y &= a \\ \text{And } x^2 - 4y^2 &= b^2 \end{aligned}\right\}$ to find x and y.

Ans. $x = \dfrac{a^2 + b^2}{2a}$, $y = \dfrac{a^2 - b^2}{4a}$.

RULE 3.

Multiply or divide one or both equations by such numbers, or quantities, as will make the term that contains one of the unknown quantities the same in both equations; then by adding or subtracting, as the case may require, there will arise a new equation, with only one unknown term in it, which may be resolved as before.

EXAMPLE.

Given $\left.\begin{aligned} 2x + 5y &= 16 \\ \text{And } 5x - 3y &= 9 \end{aligned}\right\}$ to find x and y.

Here $(2x + 5y = 16) \times 5 = 10x + 25y = 80$
And * $(5x - 3y = 9) \times 2 = 10x - 6y = 18$

$\therefore$ by subtraction $31y = 62$
$\therefore y = 2$.

Which value of y, substituted for y in the second equation, gives

$$5x - 6 = 9; \text{ or, } 5x = 15 \therefore x = 3.$$

Had the first been multiplied by 3, and the second by 5, the co-efficients of y,

Then $(2x + 5y = 16) \cdot 3 = 6x + 15y = 48$
And $(5x - 3y = 9) \cdot 5 = 25x - 15y = 45$

$\therefore$ by addition $31x = 93$
Whence $x = 3$

And by substituting 3 for x, in either of the equations, y will be found $= 2$, as before.

* Two unequal terms may be made equal, by multiplying each by the co-efficient of the other.

EXERCISES.

1. Given $3x + 4y = 38$ And $4x + 3y = 39$ } to find x and y.

Ans. $x = 6$, $y = 5$.

2. Given $2x + y = 11$ And $2x - y = 5$ } to find x and y.

Ans. $x = 4$, $y = 3$.

3. Given $2x + 7y = 20$ And $5x + 3y = 21$ } to find x and y.

Ans. $x = 3$, $y = 2$.

4. Given $\frac{1}{2}x + \frac{3}{4}y = 19$ And $\frac{3}{5}x + \frac{2}{3}y = 20$ } to find x and y.

Ans. $x = 20$, $y = 12$.

5. Given $\frac{x}{6} + \frac{y}{5} = 15$ And $\frac{3x}{7} - \frac{3y}{8} = 3$ } to find x and y.

Ans. $x = 42$, $y = 40$.

6. Given $\frac{5x + 3y}{3} - \frac{x + y}{4} = 9\frac{1}{3}$ And $\frac{7x - 5y}{2} - \frac{2y}{9} = 9\frac{1}{3}$ } to find x and y.

Ans. $x = 5$, $y = 3$.

7. Given $\frac{y}{9} - \frac{4x - 1}{18} + \frac{y + 4}{3} - \frac{x - y}{6} = 1$ And $x - \frac{y - 3}{5} - \frac{17 - y}{10} - \frac{7x}{20} = 6$ } to find x and y.

Ans. $x = 12$, $y = 7$.

8. Given $\frac{x - 2}{5} - \frac{10 - x}{3} = y - 10$ And $\frac{2y + 4}{3} - \frac{2x + y}{8} = \frac{x + 13}{4}$ } to find x and y.

Ans. $x = 7$, $y = 10$.

9. Given $2x + 3 : y :: 11 : 3$
And $\frac{2x}{3} - \frac{5 - y}{2} = \frac{41}{12} - \frac{2x - 1}{4}$ } to find x and y.

Ans. $x = 4$, $y = 3$.

10. Given $\frac{4x + 5y}{20} + y = x + \frac{1}{20}$
And $\frac{2x - y}{3} + 2y = \frac{1}{2}$ } to find x and y.

Ans. $x = \frac{1}{4}$, $y = \frac{1}{5}$.

EQUATIONS WITH THREE UNKNOWN QUANTITIES.

When there are three unknown quantities, and three independent simple equations containing them, they may be reduced to two unknown quantities and two equations, by the following*

RULE.

Find the value of one of the unknown quantities in each of the equations; then make the first value equal to the second, and the second equal to the third, and there will result two equations with only two unknown quantities, which may be solved by any of the three preceding rules.

EXAMPLE.

Given $x + y + z = 9$
$x + 2y + 3z = 16$
And $x + 3y + 4z = 21$ } to find x, y, and z.

---

* If there are more unknown quantities than independent equations, the problem is not limited.

From the first, $x = 9 - y - z.$

From the second, $x = 16 - 2y - 3z.$

From the third, $x = 21 - 3y - 4z.$

Making the first value of x equal to the second, and the second equal to the third, we have

$$9 - y - z = 16 - 2y - 3z.$$

And $16 - 2y - 3z = 21 - 3y - 4z.$

Or, $2z + y = 7$

And $z + y = 5$

* $\therefore z = 2$ by subtraction.

Whence $y = 5 - 2 = 3$, and $x = 9 - 3 - 2 = 4.$

EXERCISES.

1. Given $\left.\begin{array}{r} x + y + z = 18 \\ x + y - z = 14 \\ \text{And } x - y - z = 2 \end{array}\right\}$ to find x, y, and z.

Ans. $x = 10, y = 6, z = 2.$

2. Given $\left.\begin{array}{r} x - 2y + 3z = 6 \\ x + y - 2z = 3 \\ \text{And } x + 4y - 4z = 9 \end{array}\right\}$ to find x, y, and z.

Ans. $x = 5, y = 4, z = 3.$

3. Given $\left.\begin{array}{r} 3x - 3y - 2z = 1 \\ 4x - 4y - 3z = 1 \\ \text{And } x + y + z = 6 \end{array}\right\}$ to find x, y, and z.

Ans. $x = 3, y = 2, z = 1.$

* If the learner has made himself master of the foregoing rules, practice will teach him to exterminate three or more unknown quantities far more expeditiously than can be described by any rule.

4. Given $\left.\begin{aligned} x + y + z &= 29 \\ x + 2y + 3z &= 62 \\ \frac{x}{2} + \frac{y}{3} + \frac{z}{4} &= 10 \end{aligned}\right\}$ to find x, y, and z.

Ans. $x = 8$, $y = 9$, $z = 12$.

5. Given $\left.\begin{aligned} x + y &= 12 \\ x - z &= 3 \\ \text{And } y - z &= 1 \end{aligned}\right\}$ to find x, y, and z.

Ans. $x = 7$, $y = 5$, $z = 4$.

6. Given $\left.\begin{aligned} 2x + 4y - 3z &= 22 \\ 4x - 2y + 5z &= 18 \\ \text{And } 6x + 7y - 8z &= 35 \end{aligned}\right\}$ to find x, y, and z.

Ans. $x = 3$, $y = 7$, $z = 4$.

7. Given $\left.\begin{aligned} \tfrac{1}{2}x + \tfrac{3}{4}y + \tfrac{4}{5}z &= 100 \\ \tfrac{1}{3}x + \tfrac{4}{5}y + \tfrac{3}{5}z &= 82 \\ \text{And } \tfrac{5}{6}x + \tfrac{5}{8}y + \tfrac{2}{5}z &= 95 \end{aligned}\right\}$ to find x, y, and z.

Ans. $x = 60$, $y = 40$, $z = 50$.

8. Given $\left.\begin{aligned} x + y &= a \\ x + z &= b \\ \text{And } y + z &= c \end{aligned}\right\}$ to find x, y, and z.

Ans. $x = \dfrac{a + b - c}{2}$, $y = \dfrac{a + c - b}{2}$, $z = \dfrac{b + c - a}{2}$.

PROBLEMS PRODUCING SIMPLE EQUATIONS.

Solved by using more than one unknown letter.

1. The sum of two numbers is 10, and the difference of their squares 20; required their numbers.

Let x = the greater number,

And y = the less.

$$\left.\begin{array}{l}\text{Then } x + y = 10 \\ \text{And } x^2 - y^2 = 20\end{array}\right\}\text{by the question.}$$

From the first, $x = 10 - y$,

Which value of x, substituted for x in the second, gives

$$(10 - y^2) - y^2 = 20,$$

$$\text{Or, } 100 - 20\,y + y^2 - y^2 = 20.$$

$$\therefore 20\,y = 80.$$

$$\therefore y = 4.$$

$$\text{Whence } x = 10 - 4 = 6.$$

2. Divide 100 into two parts, such that those parts shall be as 5 to 3.

Let $x =$ the greater part, and $y =$ the less.

$$\left.\begin{array}{l}\text{Then } x + y = 100 \\ \text{And } x : y :: 5 : 3\end{array}\right\}\text{by the question.}$$

$\therefore 3\,x = 5\,y$ from the proportion.

$$\text{Or, } x = \tfrac{5}{3}\,y.$$

Which substituted in the first, gives

$$\tfrac{5}{3}\,y + y = 100.$$

$$\text{Or, } 8\,y = 300.$$

$$\therefore y = 37\tfrac{1}{2}.$$

$$\text{Whence } x = 62\tfrac{1}{2}.$$

3. What fraction is that, to the denominator of which if 1 be added, its value will be $\frac{1}{3}$; but if 1 be added to the numerator, its value will be $\frac{1}{2}$.

Let $\frac{x}{y}$ represent the required fraction.

$$\left.\begin{array}{l}\text{Then } \dfrac{x}{y+1} = \frac{1}{3} \\ \text{And } \dfrac{x+1}{y} = \frac{1}{2}\end{array}\right\}\text{by the question.}$$

$$\begin{aligned}\text{Or, } 3x &= y + 1 \text{ from the first,}\\ \text{And } 2x &= y - 2 \text{ from the second.}\end{aligned}$$

$$\therefore \quad x = 3 \qquad \text{by subtraction.}$$

$$\text{Whence } y = 8$$

$$\therefore \tfrac{3}{8} \text{ is the required fraction.}$$

4. There is a certain number, consisting of two digits or figures, which is equal to 4 times the sum of its digits; and if 18 be added to it, the digits will be inverted, what is the number?

Let $x =$ first digit and $y =$ second.

Then $10x + y =$ the required number.

And $10y + x =$ the number inverted.

$$\left.\begin{aligned}\therefore 10x + y &= 4x + 4y\\ \text{And } 10x + y + 18 &= 10y + x\end{aligned}\right\} \text{by the question.}$$

$$\begin{aligned}\text{Or, } 2x &= y \text{ from the first.}\\ \text{And } x &= y - 2 \text{ from the second.}\end{aligned}$$

$$\therefore x = 2 \qquad \text{by subtraction.}$$

$$\text{Whence } y = 4.$$

$$\therefore 24, \text{ is the number.}$$

5. There is a number, consisting of three figures, the sum of which is 9, and the difference of the two last 1; now if the sum of the digits be subtracted from the number, the remainder will be equal to nine times the sum of the two first digits, plus 360; required the number.

Let $z =$ digit in the units place.

$y =$ ditto tens place.

And $x =$ ditto hundreds place.

$$\text{Then } 100x + 10y + z = \text{the required number.}$$
$$\text{* And } x + y + z = \text{sum of digits} = 9.$$

$$\therefore \text{By subt. } 99x + 9y \;.\; = 9x + 9y + 360 \text{ by the [question.}$$
$$\text{Or, } 90x = 360$$
$$\therefore x = 4$$
$$\text{Whence } y + z = 5$$
$$\text{But } y - z = 1$$

$$\therefore y = 3 \text{ by addition, \&c.}$$
$$\text{And } z = 2 \text{ by subtraction, \&c.}$$
$$\therefore 432 \text{ is the required number.}$$

6. A person in a party at cards bets 3*s*. to 2*s*. upon every deal, and after 20 deals finds he has gained 5*s*.; how many deals does he win?

$$\text{Let } x = \text{the deals he wins,}$$
$$\text{And } y = \text{the deals he loses.}$$
$$\left.\begin{array}{r}\text{Then } x + y = 20\\ \text{And } 2x - 3y = 5\end{array}\right\} \text{by the question.}$$
$$\text{But } 3x + 3y = 60 \text{ thrice the first.}$$

$$\therefore 5x = 65 \text{ by addition.}$$
$$\text{Whence } x = 13 \text{ the Answer.}$$

7. A waterman can row from A to B (two towns 11 miles apart, on the side of a river) and back again, in 5 hours and 52 minutes; now, with the stream he rows 5

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* From this the learner will readily perceive how to express any number algebraically, and also to prove that from *any* number, if the sum of its digits be subtracted, the remainder is divisible by 9. Also, that the difference between any number of two digits, and the same with the digits inverted is divisible by 9, and the difference of their squares by 99.

miles, in the time he rows 3 miles against it; required the time of going and returning, and rate of the stream.

Let $x =$ miles he rows per hour,

And $y =$ rate of the stream.

Then $x + y =$ miles he rows with the stream.

And $x - y =$ miles per hour against it.

$$\left.\begin{array}{l} \therefore \dfrac{11}{x+y} + \dfrac{11}{x-y} = 5\frac{13}{15} \\ \text{And } x+y : x-y :: 5 : 3 \end{array}\right\} \text{by the question.}$$

Clearing the first of fractions, it becomes

$$165\,x = 44\,(x^2 - y^2)$$

And from the proportion,

$$x = 4\,y.$$

Which, substituted for x in the last equation, it becomes

$$660\,y = 660\,y^2.$$

$$\therefore y = 1.$$

Whence $x = 4$.

$$\left.\begin{array}{l} \therefore \frac{11}{5} = 2 \text{ hours and 12 minutes} \\ \text{And } \frac{11}{3} = 3 \text{ hours and 40 minutes} \end{array}\right\} \text{time.}$$

8. A mercer bought two pieces of silk for £50. the price of 2 yards of the shorter was 6*s*. 8*d*. more than the price of 3 yards of the longer; and each piece cost the same sum; he cut off 2 yards from each, and sold the rest for £53. 12*s*.; now if he had sold the whole at that rate he would have gained £5. by each piece; how many yards did each piece contain?

Let $x =$ yards in the longer piece,

And $y =$ yards in the shorter,

Then $\dfrac{25}{x} =$ cost of a yard of the longer, And $\frac{25}{y}$ = cost of a yard of the shorter.

Also, $\frac{30}{x}$ = what a yard of the longer sold for.

And $\frac{30}{y}$ = what a yard of the shorter sold for.

$$\left.\begin{array}{r} \text{Whence } (y-2)\times\frac{30}{y}+(x-2)\times\frac{30}{x}=53\tfrac{3}{5} \\ \text{And } \frac{25}{y}\times 2-\frac{25}{x}\times 3=\tfrac{1}{3} \end{array}\right\}\text{by the question}$$

Which reduces to

$$\frac{30}{y}+\frac{30}{x}=3\tfrac{1}{5}.$$

$$\text{And } \frac{30}{y}-\frac{45}{x}=\tfrac{1}{5}.$$

$$\therefore \quad \frac{75}{x}=3 \text{ by subtraction.}$$

$$\therefore x=25.$$

$$\text{Whence } y=15.$$

PROBLEMS PRODUCING SIMPLE EQUATIONS.

To be solved by using two or more unknown letters.

1. A man has two silver cups of unequal weight, having one cover for both, of 5 ounces. Now if the cover be put on the greater cup, it is thrice as heavy as the less cup; but on the less cup, it is twice as heavy as the greater; what is the weight of each?

Ans. 4oz. and 3oz.

2. Bought at one time 30 yards of cambric and 20 of calico, for £5. and at another time, 15 yards of cambric and 30 of calico, for £3. required the price of each, per yard. Ans. cambric, 3*s.* calico, 6*d.*

3. Find three numbers in arithmetical progression, such that the first may be to the second as 5 to 7, and the sum of all three 63. Ans. 15, 21, and 27.

4. There is a certain number, to the product of whose digits, if twice the right hand digit be added, the result will be seven times that digit; and if to six times the sum of the digits, 5 be added, it will be equal to the number; which is required. Ans. 53.

5. A certain number, consisting of two places of figures, is equal to four times the sum of its digits; and three times the number is equal to the *square* of the sum of its digits; required the number. Ans. 48.

6. What fraction is that, to the numerator of which if 5 be added, its value is 2; but if 5 be added to the denominator, its value is $\frac{1}{3}$? Ans. $\frac{3}{4}$.

7. What fraction is that, the numerator of which being multiplied by 3, and 1 added to the denominator, its value is $\frac{3}{4}$, but the denominator being multiplied by 3, and 1 added to the numerator, its value is $\frac{1}{7}$?
Ans. $\frac{2}{7}$.

8. Find two numbers, such that the greater is to the less as their sum is to 20, and as their difference is to 4.
Ans. 18 and 12.

9. What two numbers are they, to each of which if 4 be added, the sums will be as 5 to 4; but if from each the same number be taken, the remainders will be to each other as 4 to 3? Ans. 36 and 28.

10. When a company at a tavern came to pay their reckoning, they found that if there had been 4 persons more, they would have had a shilling a piece less to pay; but if there had been 3 less, they would have had a shilling a piece more to pay; required the number and quota of each. Ans. 24 persons, 7*s*. each.

11. A composition of tin and copper, containing 50 cubic inches, weighs 245oz.; how many ounces of each did it contain, supposing the weight of a cubic inch of copper to be $5\frac{1}{2}$oz. and that of a cubic inch of tin $4\frac{1}{2}$oz.?

Ans. copper, 110oz. tin, 135oz.

12. A principal of £100. in an unknown time, and at an unknown rate, amounted to £160. and in three years more, at the same rate, to £178.; required the rate and time. Ans. rate, 6 per cent., time, 10 years.

13. Required the *principal* and *time* when the amount at 5 per cent. = £625. and at 6 per cent. = £650.

Ans. £500., time, 5 years.

14. A draper bought two pieces of cloth for £12. 13*s*. one being 8*s*. and the other 9*s*. a yard, and sold them at an advance of 2*s*. a yard, by which he gained £3.; what was the length of each? Ans. 13 and 17 yards.

15. A and B speculate with different sums, A gains £150. but B loses £50.; and now A's stock is to B's as 3 to 2; but had A lost £50, and B gained £100. then A's stock would have been to B's as 5 to 9; what was the stock of each? Ans. A's = £300, B's = £350.

16. Some smugglers discovered a cave, which would just hold the cargo of their boat, namely, 200lbs. of tobacco and 50 kegs of gin; whilst they were unloading, a revenue cutter coming in sight, they sailed away with

30 kegs and 50lbs. leaving the cave just half full; how many kegs or lbs. would it hold?

Ans. 700 lbs. or 70 kegs.

17. A and B can perform a piece of work in 12 days; they work together for 4 days, when A being called off, B finishes it in 20 days more; in what time would each do it separately? Ans. A in 20, and B in 30 days.

18. A corn-factor mixed as much wheat at 5*s.* a bushel with barley at 3*s.* a bushel, as cost him £30. 16*s.* and sold the mixture at 5*s.* 6*d.* a bushel, by which he gained $12\frac{1}{2}$ per cent.; required the number of bushels of each.

Ans. wheat 119, barley 7.

19. A person travelled a journey at a certain rate; had he travelled half a mile an hour faster, he would have performed the journey in $\frac{4}{5}$ of the time; but had he travelled half a mile an hour slower, he would have been $2\frac{1}{2}$ hours longer on the road; find the distance and rate of travelling.

Ans. distance, 15 miles, rate, 2 miles an hour.

20. A, B, and C, set up in trade together, but after sustaining several losses, agree to dissolve partnership, when it was found that A and C's shares exceeded B's by £250.; A and B's exceeded C's by £200; and A's was equal to the sum of the other two, *minus* £10.; required the share of each.

Ans. A's = £225. B's = £105. C's = £130.

21. A man and his wife could drink a barrel of beer in 15 days; after drinking 6 days, the wife alone drank the remainder in 30 days; in what time would either alone drink a barrel?

Ans. woman 50 days, man $21\frac{3}{7}$ days.

22. A footman laid out a certain sum in oranges, to treat his fellow servants with, and observed that if there had been 14 more, they would have cost a half-penny each less, but had there been 7 less, they would have cost a half-penny each more: what was the sum laid out?

Ans. 3s. 6d.

23. A corn-factor bought wheat and beans, for each of which he paid £105.; 2 quarters of wheat costing more than 3 quarters of beans by 14s.; now after taking 3 quarters of each for his own use, he sold the remainders for £220. 16s. and had he sold the *whole* at the same rate, he would have cleared £15. by each: how many quarters were there of each?

Ans. wheat 30, beans 50.

24. A and B are two towns on the side of a river which runs at the rate of 4 miles an hour, and a waterman finds that he is 39 minutes longer in rowing from A to B and back again, than he would be were there no stream; but with the assistance of his son he can row half as fast again, and then he is only 8 minutes longer in performing the voyage: required the rate at which he rows, and the distance of A and B?

Ans. Rate 6 miles an hour; distance 2 miles 3 fur. 20 p.

PURE EQUATIONS.

Pure Equations are such as involve some power or root of the unknown quantity, and are solvible without completing the square.

EXAMPLES.

1. Given $(x+y)^2 : (x-y)^2 :: 64 : 4$ And $x\,y = 15$ } to find x and y.

From the first $x+y : x-y :: 8 : 2$ (9 theo. 2, page 50.)

Or, $2\,x : 2\,y :: 10 : 6$ (6 theo. 2, page 50.)

Or, $x : y :: 5 : 3$ (note, theo. 2.)

$\therefore 3\,x = 5\,y.$

Or, $x = \frac{5}{3}\,y.$

Which substituted in the second,

Gives $\frac{5}{3}\,y^2 = 15.$

Or, $\frac{y^2}{3} = 3$, by $\div$ by 5.

$\therefore\ y^2 = 9.$

$\therefore\ y = 3$ (rule 5, page 58.)

Whence $x = 5.$

2. Given $x^{\frac{1}{3}} + y^{\frac{1}{3}} = 6$ And $x^{\frac{1}{3}} - y^{\frac{1}{3}} = 2$ } to find x and y.

By addition $2\,x^{\frac{1}{3}} = 8.$

And by sub. $2\,y^{\frac{1}{3}} = 4.$

$\therefore\ x^{\frac{1}{3}} = 4$ And $y^{\frac{1}{3}} = 2$ } by $\div$ by 2.

Whence $x = 64$ And $y = 8$ } by cubing.

3. Given $x^2 + xy = 40$ } to find x and y.
And $y^2 + xy = 24$ }

By addition $x^2 + 2xy + y^2 = 64$.

$\therefore x + y = 8$, by evolution,

Which substituted in the first and second, they

Become $8x = 40$,

And $8y = 24$.

$\therefore x = 5$, and $y = 3$.

4. Given $x^4 - y^4 = 609$ } to find x and y.
And $x^2 + y^2 = 29$ }

Dividing the first by the second, gives

$x^2 - y^2 = 21$,

To which add $x^2 + y^2 = 29$, the second.

Then $2x^2 = 50$,

And $2y^2 = 8$, by subtraction.

$\therefore x^2 = 25$, } by $\div$ by 2.
And $y^2 = 4$, }

$\therefore x = 5$, } by evolution.
And $y = 2$, }

5. Given $x^{\frac{1}{3}} + y^{\frac{1}{3}} = 7$ } to find x and y.
And $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 25$ }

By squaring the first equation,

$x^{\frac{2}{3}} + 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = 49$,

Subtract $x^{\frac{2}{3}} \quad + \quad y^{\frac{2}{3}} = 25$, the second.

And $\quad 2x^{\frac{1}{3}}y^{\frac{1}{3}} \quad = 24$.

Which subtracted from the second,

Gives $x^{\frac{2}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = 1$

$\therefore x^{\frac{1}{3}} - y^{\frac{1}{3}} = 1$ by evolution.

To which add $x^{\frac{1}{3}} + y^{\frac{1}{3}} = 7$ the first.

Then $2x^{\frac{1}{3}} = 8$

And $2y^{\frac{1}{3}} = 6$ by subtract.

Or, $x^{\frac{1}{3}} = 4$ \
And $y^{\frac{1}{3}} = 3$ } by $\div$ by 2.

$\therefore x = 64$ \
And $y = 27$ } by cubing.

EXERCISES.

1. Given $x - y = 2$ \
And $x^2 - y^2 = 24$ } to find x and y.

Ans. $x = 7$, $y = 5$.

2. Given $x + y = 9$ \
And $x^2 + y^2 = 45$ } to find x and y.

Ans. $x = 6$, $y = 3$.

3. Given $x - y = 2$ \
And $x^3 - y^3 = 56$ } to find x and y.

Ans. $x = 4$, $y = 2$.

4. Given $x + y = 5$ \
And $x^3 + y^3 = 35$ } to find x and y.

Ans. $x = 3$, $y = 2$.

5. Given $x^4 y^2 + x^2 y^4 = 3600$ \
And $x^2 + y^2 = 25$ } to find x and y.

Ans. $x = 4$, $y = 3$.

6. Given $x^3 y^2 + x^2 y^3 = 180$ \
And $x^4 y^3 + x^3 y^4 = 1080$ } to find x and y.

Ans. $x = 3$, $y = 2$.

7. Given $x^2 + y^2 = 20$ \
And $x^4 + y^4 = 272$ } to find x and y.

Ans. $x = 4$, $y = 2$.

8. Given $x + y = 6$
And $x^5 + y^5 = 1056$ } to find x and y.

Ans. $x = 4$, $y = 2$.

9. Given $x^3 + y^3 = 468$
And $(x + y) \cdot x y = 420$ } to find x and y.

Ans. $x = 7$, $y = 5$.

10. Given $(x^2 + y^2) \cdot (x - y) = 260$
And $(x - y) \cdot x y = 126$ } to find x and y.

Ans. $x = 9$, $y = 7$.

11. Given $x^2 y - x y^2 = 30$
And $x^3 y^2 - x^2 y^3 = 450$ } to find x and y.

Ans. $x = 5$, $y = 3$.

12. Given $x^{\frac{1}{3}} + y^{\frac{1}{3}} = 5$
And $x + y = 35$ } to find x and y.

Ans. $x = 27$, $y = 8$.

13. Given $x^{\frac{1}{2}} - y^{\frac{1}{2}} : x^{\frac{1}{2}} + y^{\frac{1}{2}} :: 1 : 3$
And $3 x^{\frac{1}{2}} - y^{\frac{1}{2}} = 15$ } to find x and y.

Ans. $x = 36$, $y = 9$.

14. Given $x - y : x^{\frac{1}{2}} - y^{\frac{1}{2}} :: 8 : 1$
And $x y = 225$ } to find x and y.

Ans. $x = 25$, $y = 9$.

15. Given $x^4 - y^4 : x^2 - y^2 :: 58 : 1$
And $x y = 21$ } to find x and y.

Ans. $x = 7$, $y = 3$.

16. Given $x^{\frac{1}{3}} + y^{\frac{1}{2}} : x^{\frac{1}{3}} - y^{\frac{1}{2}} :: 5 : 1$
And $x^{\frac{1}{3}} y^{\frac{1}{2}} = 6$ } to find x & y.

Ans. $x = 27$, $y = 4$.

17. Given $x^{\frac{1}{3}} - y^{\frac{1}{3}} = 1$
And $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 13$ } to find x and y.

Ans. $x = 27$, $y = 8$.

18. Given $\left.\begin{array}{l} x^{\frac{1}{4}} + y^{\frac{1}{4}} = 5 \\ \text{And } x^{\frac{1}{2}} + y^{\frac{1}{2}} = 13 \end{array}\right\}$ to find x and y.

Ans. $x = 81$, $y = 16$.

19. Given $\left.\begin{array}{l} \frac{1}{x} + \frac{1}{y} = \frac{7}{10} \\ \text{And } \frac{1}{x^2} + \frac{1}{y^2} = \frac{29}{100} \end{array}\right\}$ to find x and y.

Ans. $x = 5$, $y = 2$.

20. Given $\left.\begin{array}{l} x^{\frac{4}{3}} + y^{\frac{2}{5}} = 85 \\ \text{And } x^{\frac{2}{3}} + y^{\frac{1}{5}} = 11 \end{array}\right\}$ to find x and y.

Ans. $x = 27$, $y = 32$.

21. Given $\left.\begin{array}{l} x^2 + 2\,y^2 = x\,y^{\frac{1}{2}} + 2\,x\,y^{\frac{3}{2}} \\ \text{And } x^2 + x\,y^{\frac{1}{2}} - 2\,y^2 = 256 \end{array}\right\}$ to find x and y.

Ans. $x = \pm\, 16$, $y = 4$.

22. Given $\left.\begin{array}{l} (x + y) \times (x - y)^2 = 3\,x\,y \\ \text{And } (x^2 + y^2) \times (x^2 - y^2)^2 = 15\,x^2\,y^2 \end{array}\right\}$ to find x and y.

Ans. $x = 4$, and $y = 2$.

23. Given $\frac{x}{2} = \frac{x^2}{2\,(1 + \sqrt{1 + x})^2} + 2$, to find x.

Ans. $x = 8$.

24. Given $\frac{x}{2} = (\sqrt{1 + x} - 1) \cdot (\sqrt{1 - x} + 1)$, to find the value of x.

Ans. $x = \frac{24}{25}$.

25. Given $\left.\begin{array}{l} x^{\frac{1}{2}}\left(\frac{y^2 + y^{\frac{1}{2}}}{y}\right) = 36 + \left(\frac{x}{y}\right)^{\frac{1}{2}} \\ \text{And } y^{\frac{1}{2}}\left(\frac{x^2 + x^{\frac{1}{2}}}{x}\right) = 48 + \left(\frac{y}{x}\right)^{\frac{1}{2}} \end{array}\right\}$ to find x and y.

Ans. $x = 16$, $y = 9$.

ADFECTED QUADRATICS.

An adfected quadratic equation is one that involves both the first and second power of the unknown quantity, and when properly prepared for solving, falls under one of the three following forms:

$$x^2 + a\,x = b$$
$$x^2 - a\,x = b$$
$$x^2 - a\,x = -\,b$$

The roots or value of x in each of which may be found by the following

RULE.

1. Prepare the equation, if necessary, so that all the known terms may stand on the right hand side, and the unknown ones on the left.

2. If the square of the unknown quantity has a co-efficient prefixed to it, all the terms must be divided by that co-efficient.

3. To *complete the square*, add the *square* of *half* the co-efficient of the *second* term to both sides of the equation, and that side which involves the unknown quantity will then be a complete square.

4. Extract the square root on both sides of the equation, and the unknown quantity will then become known.

EXAMPLES.

1. Given $x^2 + 4\,x = 45$, to find the value of x.*

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* Here the co-efficient of the second term is 4, the square of its half is $2^2 = 4$. All quadratics have at least two roots, generally one positive and one negative, though sometimes they are both positive.

First $x^2 + 4x + 4 = 49$, by completing the square,

And $x + 2 = 7$, by evolution.

$\therefore x = 5$.

2. Given $x^2 - 12x + 24 = 4$, to find x.

First $x^2 - 12x = -20$, by transposing 24,

And $x^2 - 12x + 36 = 16$, by completing the $\square$.

$\therefore x - 6 = 4$, by evolution.

Whence $x = 10$, by transposition.

3. Given $7x^2 - 14x + 14 = 35$, to find x.

First $7x^2 - 14x = 21$, by transposing 14,

And $x^2 - 2x = 3$, by $\div$ by 7.*

Also, $x^2 - 2x + 1 = 4$, by completing the $\square$.

$\therefore x - 1 = 2$, by evolution.

Whence $x = 3$.

4. Given $3x^2 + 9x = 54$.

First $x^2 + 3x = 18$, by $\div$ by 3,

And x^2† $+ 3x + \frac{9}{4} = 18 + \frac{9}{4} = \frac{81}{4}$, by completing $\square$.

Also, $x + \frac{3}{2} = \frac{9}{2}$, by evolution.

$\therefore x = \frac{9}{2} - \frac{3}{2} = \frac{6}{2} = 3$.

5. Given $\frac{1}{2}x^2 - \frac{1}{3}x + 20\frac{1}{2} = 42\frac{2}{3}$, to find x.

First, $x^2 - \frac{2}{3}x + 41 = 85\frac{1}{3}$, by $\times$ by 2.‡

And $x^2 - \frac{2}{3}x = 44\frac{1}{3} = \frac{133}{3}$, by transp. 41.

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* See rule, part 2.

† When the co-efficient of the second term is an odd number, put 2 underneath, and square the fraction. If the co-efficient of the second term is a fraction, divide the numerator by 2, if it is an even number; if not, multiply the denominator by 2, which is the same thing.

‡ Multiplying by 2, is the same as dividing by $\frac{1}{2}$, the co-efficient of the square.

Also, $x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{133}{3} + \frac{1}{9} = \frac{400}{9}$, by comp. □

$\therefore x - \frac{1}{3} = \frac{20}{3}$, by evolution.

Whence $x = \frac{21}{3} = 7$.

6. Given $9x - x^2 = 14$, to find x.

First, $x^2 - 9x = -14$, by making x^2 positive.*

Then $x^2 - 9x + \frac{81}{4} = \frac{81}{4} - 14 = \frac{25}{4}$ by comp. □

And $x - \frac{9}{2} = \pm \frac{5}{2}$ by evolution.†

$\therefore x = \frac{9}{2} \pm \frac{5}{2}$.

When the upper sign is used,

$x = \frac{14}{2} = 7$.

When the under sign is used,

$x = \frac{4}{2} = 2$.

$\therefore$ both roots are positive.

7. Given $x^4 - 3x^2 = 550$, to find x. ‡

First $x^4 - 3x^2 + \frac{9}{4} = 550 + \frac{9}{4} = \frac{2209}{4}$, by comp. □

And $x^2 - \frac{3}{2} = \pm \frac{47}{2}$.

$$\therefore x^2 = \frac{3 \pm 47}{2} = 25, \text{ or } -22.$$

$\therefore x = \pm \sqrt{25} = 5$, or -5.

Or, $x = \pm \sqrt{-22}$.

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* When the square of the unknown quantity is *negative*, it must be made *positive*, by changing *all* the signs.

† The double sign $\pm$ is usually put before the root, because the square root of any positive quantity, as m^2 is either $+m$ or $-m$ for $+m \times +m = m^2$, and $-m \times -m = m^2$. The square root of $-m^2$ is imaginary, there being no quantity either negative or positive multiplied into itself will produce it.

‡ When the *power* of the first term is *double* the power of the second, whatever be the power, the equation is a quadratic, and it has as many roots as there are units in its highest power.

8. Given $x - x^{\frac{1}{2}} = 30$, to find x.*

First, $x - x^{\frac{1}{2}} + \frac{1}{4} = 30\frac{1}{4} = \frac{121}{4}$ by comp. □

And $x^{\frac{1}{2}} - \frac{1}{2} = \frac{11}{2}$ by evolution.

$\therefore x^{\frac{1}{2}} = \frac{1}{2} \pm \frac{11}{2} = 6$, or -5.

Whence $x = 36$ or 25.

9. Given $x^{\frac{2}{3}} - \frac{3}{2} x^{\frac{1}{3}} = 1$, to find x.

First, $x^{\frac{2}{3}} - \frac{3}{2} x^{\frac{1}{3}} + \frac{9}{16} = \frac{25}{16}$, by comp. □

And $x^{\frac{1}{3}} - \frac{3}{4} = \pm \frac{5}{4}$ by evolution.

$\therefore x^{\frac{1}{3}} = \frac{3 \pm 5}{4} = 2$ or $-\frac{1}{2}$.

Whence $x = 8$, or $-\frac{1}{8}$ by cubing.

10. Given $x^{2n} - 6ax^n = 7a^2$, to find x.

First, $x^{2n} - 6ax^n + 9a^2 = 16a^2$ by comp. □

And $x^n - 3a = \pm 4a$ by evolution.

$\therefore x^n = 3a \pm 4a = 7a$, or $-a$.

Whence $x = (7a)^{\frac{1}{n}}$, or $-a^{\frac{1}{n}}$.

11. Given $\left.\begin{array}{l} x^2y^2 + 2xy = 168 \\ \text{And } x^2 + y^2 = 25 \end{array}\right\}$ to find x and y.

Completing the square of the first.

$x^2y^2 + 2xy + 1 = 169$.

$\therefore xy + 1 = \pm 13$ by evolution.

$\therefore xy = 12$, or -14.

Or, $2xy = 24$, or -28 by $\times$ by 2.

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* Here the co-efficient of $x^{\frac{1}{2}}$ is 1, and the square of its half $= \frac{1}{4}$.

Which added to, and subtracted from the second equation,

Gives $x^2 + 2xy + y^2 = 49,$

And $x^2 - 2xy + y^2 = 1.$

Or, $x^2 + 2xy + y^2 = -3.$

And $x^2 - 2xy + y^2 = 63.$

$\therefore$ by extracting the square root,

$$\left.\begin{array}{r} x + y = \pm 7 \\ \text{And } x - y = \pm 1 \end{array}\right\}\text{ from the former,}$$

$\therefore x = \pm 4$ by addition, &c.

And $y = \pm 3$ by subtraction, &c.

$$\left.\begin{array}{r} \text{Or, } x + y = \pm \sqrt{-3} \\ \text{And } x - y = \pm \sqrt{63} \end{array}\right\}\text{ from the latter.}$$

$$\therefore x = \pm \frac{\sqrt{-3} \pm \sqrt{63}}{2}$$

$$\text{And } y = \pm \frac{\sqrt{-3} \mp \sqrt{63}}{2}$$

EXERCISES.

1. Given $x^2 + 2x = 8$, to find the value of x.
   Ans. $x = 2$.*

2. Given $x^2 + 4x = 21$, to find x.
   Ans. $x = 3$.

3. Given $x^2 + 6x = 40$, to find x.
   Ans. $x = 4$.

4. Given $x^2 + 8x = 65$, to find x.
   Ans. $x = 5$.

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* The negative, and imaginary roots, being of minor importance, are omitted in the following exercises.

5. Given $x^2 + 10x - 7 = 89$, to find x. Ans. 6.

6. Given $x^2 + 12x + 9 = 142$, to find x.
Ans. $x = 7$.

7. Given $x^2 - 14x + 27 = 3$, to find x.
Ans. $x = 12$.

8. Given $x^2 - 16x + 35 = 7$, to find x.
Ans. $x = 14$.

9. Given $3x^2 - 78x + 99 = 24$, to find x.
Ans. $x = 25$.

10. Given $7x^2 + 224x - 714 = 581$, to find x.
Ans. $x = 5$.

11. Given $x^2 - x + 1 = 43$, to find x.
Ans. $x = 7$.

12. Given $x^2 + 3x = 88$, to find x. Ans. $x = 8$.

13. Given $x^2 - 5x + 9 = 59$, to find x.
Ans. $x = 10$.

14. Given $2x^2 + 14x = 396$, to find x.
Ans. $x = 11$.

15. Given $3x^2 - 39x + 36 = 0$, to find x.
Ans. $x = 12$.

16. Given $\frac{1}{2}x^2 + \frac{17}{2}x = 217$, to find x.
Ans. $x = 14$.

17. Given $\frac{1}{6}x^2 - \frac{7}{2}x = 12$, to find x.
Ans. $x = 24$.

18. Given $3x^2 - 2x = 1045$, to find x.
Ans. $x = 19$.

19. Given $7x^2 + 4x = 3795$, to find x.
Ans. $x = 23$.

20. Given $x^2 - \frac{3}{4}x = 763$, to find x.
Ans. $x = 28$.

21. Given $5x^2 + 3x = 4292$, to find x.
Ans. $x = 29$.

22. Given $\frac{1}{2} x^2 + \frac{5}{8} x = 532$, to find x.
Ans. $x = 32$.

23. Given $\frac{1}{3} x^2 - \frac{1}{4} x - 41 \frac{3}{4} = 100$, to find x.
Ans. $x = 21$.

24. Given $13 x^2 + 28 x = 201$, to find x.
Ans. $x = 3$.

25. Given $x^3 - 2 x^{\frac{3}{2}} - 48 = 0$, to find x.
Ans. $x = 4$.

26. Given $x^{\frac{2}{3}} + 3 x^{\frac{1}{3}} = 10$, to find x. Ans. $x = 8$.

27. Given $\frac{1}{3} x^{\frac{3}{2}} + x^{\frac{3}{4}} = 29 \frac{1}{3}$, to find x.
Ans. $x = 16$.

28. Given $x^{\frac{4}{3}} + 5 x^{\frac{2}{3}} - 6 = 30$, to find x.
Ans. $x = 8$.

29. Given $4 x^{\frac{1}{3}} - 3 x^{\frac{1}{6}} = 27$, to find x.
Ans. $x = 729$.

30. Given $2 x^{\frac{1}{2}} + 3 x^{-\frac{1}{2}} = 7$, to find x.
Ans. $x = 9$.

31. Given $x^{\frac{5}{2}} - 648 x^{-\frac{1}{2}} = 3 x$, to find x.
Ans. $x = 9$.

32. Given $x^{2n} - 4 x^n = 5$, to find x.
Ans. $x = 5^{\frac{1}{n}}$.

33. Given $x^{\frac{2}{n}} + 8 x^{\frac{1}{n}} = 20$, to find x.
Ans. $x = 2^n$.

34. Given $x^n - 2 a x^{\frac{n}{2}} = b^2$, to find x.
Ans. $x = (a \pm \sqrt{a^2 + b^2})^{\frac{2}{n}}$.

35. Given $x^2 + 5x + 2(x^2 + 5x)^{\frac{1}{2}} = 48$, to find x.
Ans. $x = 4$.

36. Given $x^2 + x + 6 + 4(x^2 + x + 6)^{\frac{1}{2}} = 60$, to find x.
Ans. $x = 5$.

37. Given $(x + x^{\frac{1}{2}})^2 + 11(x + x^{\frac{1}{2}}) = 102$, to find x.
Ans. $x = 4$, or 9.

38. Given $x^{\frac{1}{2}} - x^{\frac{1}{3}} : x^{\frac{1}{2}} + x^{\frac{1}{3}} :: 5x^{\frac{1}{2}} : 30x^{\frac{1}{3}}$, to find x.
Ans. $x = 64$, or 729.

39. Given $x + x^{\frac{1}{2}} : x - x^{\frac{1}{2}} :: 3x^{\frac{1}{2}} + 6 : 2x^{\frac{1}{2}}$, to find x.
Ans. $x = 4$.

40. Given $x^2 + 3x + 2\sqrt{x^2 + 3x + 7} = 28$, to find x.
Ans. $x = 3$.

41. Given $\left.\begin{array}{l} (x+y)^2 + x + y = 42 \\ \text{And } x - y = 2 \end{array}\right\}$ to find x and y.
Ans. $x = 4, y = 2$.

42. Given $\left.\begin{array}{l} x + y + \sqrt{x+y} = 12 \\ \text{And } xy = 18 \end{array}\right\}$ to find x and y.
Ans. $x = 6$ or 3, $y = 3$ or 6.

43. Given $\left.\begin{array}{l} x^2y^4 - 5xy^2 = 84 \\ \text{And } y^4 + x^2 = 25 \end{array}\right\}$ to find x and y.
Ans. $x = 3, y = 2$.

44. Given $\left.\begin{array}{l} (x^2 + y^2)^2 + x^2 + y^2 = 182 \\ \text{And } x^2 + x^2y^2 + y^2 = 49 \end{array}\right\}$ to find x and y.
Ans. $x = 3$ or 2, $y = 2$ or 3.

45. Given $\left.\begin{array}{l} x^2 + y^2 + x + y = 32 \\ \text{And } (x^2 + y^2) \cdot (x + y) = 175 \end{array}\right\}$ to find x and y.
Ans. $x = 4, y = 3$.

46. Given $\left.\begin{array}{l} x^4 - x^2 + y^4 - y^2 = 312 \\ \text{And } x^2 + x^2y^2 + y^2 = 169 \end{array}\right\}$ to find x and y.
Ans. $x = 4, y = 3$.

47. Given $x^{\frac{1}{2}} + \sqrt{5x^{\frac{1}{2}} + 5y^{\frac{1}{2}}} = 10 - y^{\frac{1}{2}}$ } to find
And $x - y + 2(x^{\frac{1}{2}} + y^{\frac{1}{2}})\sqrt{x^{\frac{1}{2}} - y^{\frac{1}{2}}} = 15$ } x & y.

Ans. $x = 9$, $y = 4$.

48. Given $x + y - 2\sqrt{\dfrac{x+y}{x-y}} = \dfrac{3}{x-y}$ } to find
And $x^4 - y^4 = 369$ } x and y.

Ans. $x = 5$, $y = 4$.

49. Given $(x^{\frac{3}{2}} + y^{\frac{3}{2}})^{\frac{3}{2}} - 184384512 \div (x^{\frac{3}{2}} + y^{\frac{3}{2}})^{\frac{3}{2}} = 486$
And $x^3 + 4y^3 + 2x^{\frac{3}{2}} = 148224 + 4x^{\frac{3}{2}}y^{\frac{3}{2}} + 4y^{\frac{3}{2}}$,
to find x and y. Ans. $x = 64$, $y = 16$.

50. Given $25y + \sqrt{x^2 - 15y - 14} = \frac{5}{3}x^2 - 180$
And $\dfrac{x}{8y} + \dfrac{y}{2x} + \dfrac{2}{3} = \sqrt{\dfrac{x}{3y} + \dfrac{1}{4}}$, to find x and y.
Ans. $x = 12$ and $y = 2$, besides which, x admits of 15 other values, and y of 7 other values.

PROBLEMS PRODUCING QUADRATICS.

1. What two numbers are those the *sum* of which are 24, and the *product* 63?

Let $x =$ the greater number,
Then $24 - x =$ the less,
And $(24 - x) \cdot x = 63$ by the question.
Or, $24x - x^2 = 63$.
$\therefore x^2 - 24x = -63$ by changing the signs.
And $x^2 - 24x + 144 = 81$ by completing the □
$\therefore x - 12 = 9$ by evolution.

Or, $x = 21$.

$\therefore 24 - 21 = 3$ the less.

The same by two unknown letters;

Let x and y represent the required numbers.

Then $x + y = 24$ } by the question.
And $x y = 63$ }

$\therefore x = 24 - y$ from the first.

Which substituted in the second,

Gives $(24 - y) \cdot y = 63$,

The same as the first equation in the solution above.

But it may be solved without completing the square,

For $x^2 + 2 x y + y^2 = 576$ square of the first.

And $4 x y = 252$ four times the second.

$\therefore x^2 - 2 x y + y^2 = 324$ by subtraction.

And $x - y = 18$ by evolution.

But $x + y = 24$ the first.

$\therefore 2 x = 42$ by addition.

And $2 y = 6$ by subtraction.

$\therefore x = 21$ and $y = 3$.

2. Divide 458 into two square numbers, such that the root of one shall exceed the root of the other by 4.

Let $x =$ the less root.

Then $x + 4 =$ the greater.

And $x^2 + (x + 4)^2 = 458$ by the question.

Or, $2 x^2 + 8 x + 16 = 458$.

Or, $x^2 + 4 x = 221$ a quadratic.

$\therefore x^2 + 4 x + 4 = 225$ by comp. $\square$.

And $x + 2 = 15$ by evolution.

$\therefore x = 13.$

Whence $13^2 = 169$, and $(13 + 4)^2 = 289$, are the required numbers.

3. Bought a cottage for a certain sum, and sold it again for £171., by which I gained as much per cent. as it cost; required the price.

Let $x =$ what it cost in £.'s

Then $171 - x =$ the gain.

And, as $100 : x :: x : 171 - x$ by the question.

$\therefore x^2 = 17100 - 100\,x$ by $\times$ means and extremes.

Or, $x^2 + 100\,x = 17100$ a quadratic.

$\therefore x^2 + 100\,x + 2500 = 19600$ by comp. the $\square$.

And $x + 50 = 140$ by evolution.

$\therefore x = 90.$

4. A vintner sold 7 dozen of sherry and 12 dozen of claret for £50.; he sold 3 dozen more of sherry for £10. than he did claret for £6.; required the price of each per dozen.

Let $x =$ dozens of claret for £6.

Then $x + 3 =$ dozens of sherry for £10.

And $\frac{6}{x}$ and $\frac{10}{x+3}$ the price of each per dozen.

Whence $\frac{6}{x} \times 12 + \frac{10}{x+3} \times 7 = 50$ by the question,

which reduces to

$$x^2 + \tfrac{4}{25}\,x = \tfrac{108}{25}.$$

And by completing the square,

$$x^2 + \tfrac{4}{25}\,x + \tfrac{4}{625} = \tfrac{108}{25} + \tfrac{4}{625} = \tfrac{2704}{625}.$$

$\therefore x + \frac{2}{25} = \frac{52}{25}$ by evolution.

$\therefore x = 2.$

Whence $\frac{6}{2} = £3.$, and $\frac{10}{2+3} = £2.$

The price of each per dozen.

5. Find four numbers in arithmetical progression, such that the product of the means may be 40, and the product of the extremes 22.

Let $x =$ the first term.

And $y =$ common difference.

Then x, $x + y$, $x + 2y$, and $x + 3y$ are the numbers.

$\therefore (x+y) \cdot (x+2y) = x^2 + 3xy + 2y^2 = 40$ } by the

And $(x + 3y) \cdot x = x^2 + 3xy \quad = 22$ } quest.

$\therefore \quad 2y^2 = 18$ by subt.

$\therefore \quad y = 3.$

which subtracted for y in the second,

gives $x^2 + 9x = 22,$

comp. the square $x^2 + 9x + \frac{81}{4} = 22 + \frac{81}{4} = \frac{169}{4}.$

$\therefore x + \frac{9}{2} = \frac{13}{2}.$

$\therefore x = 2.$

Whence the numbers are 2, 5, 8, and 11.

6. Find three numbers in geometrical progression, the sum of which shall be 7 and the sum of their squares 21.

Let x, y, and z denote the numbers.

Then $\begin{cases} xz = y^2 & \text{by theo. 1, p. 49.} \\ x + y + z = 7 \\ x^2 + y^2 + z^2 = 21 \end{cases}$ by the question

Transposing y in the second, and squaring

$x^2 + 2xz + z^2 = 49 - 14y + y^2.$

And by putting $2\,y^2$ for $2\,x\,z$,

$$x^2 + y^2 + z^2 = 49 - 14\,y.$$

$\therefore\ 21 = 49 - 14\,y$ by equality.

$\therefore\ y = 2.$

Whence $x\,z = 4$ from the first

And $x + z = 5$ from the second.

The *sum* and *product* of x and z, which is problem 1, solved, $x = 1$, $z = 4$.

$\therefore$ 1, 2, and 4, are the numbers required.

7. Find four numbers in geometrical progression, the sum of which shall be 15, and the sum of their squares 85.

Let x and y be the two means.

Then $\frac{x^2}{y}$ and $\frac{y^2}{x}$ will be the extremes.

$$\left.\begin{aligned} \therefore\ \frac{x^2}{y} + x + y + \frac{y^2}{x} &= 15 = a \\ \text{And } \frac{x^4}{y^2} + x^2 + y^2 + \frac{y^4}{x^2} &= 85 = b \end{aligned}\right\}\text{by the question.}$$

Put $x + y = s$, and $x\,y = p$.

Then $\frac{x^2}{y} + \frac{y^2}{x} = a - s$ from the first.

And $\frac{x^4}{y^2} + 2\,x\,y + \frac{y^4}{x^2} = (a - s)^2$ by squaring.

But $2\,x\,y = 2\,p$

$\therefore\ \frac{x^4}{y^2} + \frac{y^4}{x^2} = (a - s)^2 - 2\,p$ by subt.

*But $\quad x^2 + y^2 \quad = s^2 - 2p.$

$$\therefore \quad \frac{x^4}{y^2} + x^2 + y^2 + \frac{y^4}{x^2} = a^2 + 2s^2 - 2as - 4p \text{ by add.}$$

And $a^2 + 2s^2 - 2as - 4p = b$ by equality.

From $\frac{x^2}{y} + \frac{y^2}{x} = a - s$, we get

$$x^3 + y^3 = (a - s)p.$$

But $x^3 + y^3 = s^3 - 3sp.$

$\therefore ap - sp = s^3 - 3sp$ by equality.

Whence $p = \frac{s^3}{2s + a}$,

which substituted for p in

$$a^2 + 2s^2 - 2as - 4p = b,$$

gives $a^2 + 2s^2 - 2as - \frac{4s^3}{2s + a} = b.$

Or, $s^2 + \frac{b}{a}s = \frac{a^2 - b}{2}$ by reduction.

$\therefore s^2 + \frac{b}{a}s + \frac{b^2}{4a^2} = \frac{a^2 - b}{2} + \frac{b^2}{4a^2}$ by comp. □.

And $s + \frac{b}{2a} = \sqrt{\frac{a^2 - b}{2} + \frac{b^2}{4a^2}}.$

$\therefore s \quad = \sqrt{\frac{a^2 - b}{2} + \frac{b^2}{4a^2}} - \frac{b}{2a}.$

And by restoring the values of a and b,

$s = 6$, and $p = 8$.

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* The *sum* of the squares of two numbers, is less than the *square* of their *sum* by twice their product. And the *sum* of the *cubes* of any two numbers, is less than the *cube* of their *sum* by three times their *product* into their *sum*.

$\therefore\ x + y = 6,$

and $x\,y = 8,$

the *sum* and *product*. See prob. 1.

Whence $x = 4$, and $y = 2$.

$\therefore$ the numbers are 1, 2, 4, and 8.

The same solved without substituting for the sum *and* product *of the means.*

Multiplying the first equation by $x\,y$,

and $x^3 + x\,y\,(x + y) + y^3 = 15\,x\,y.$

Or, $(x + y)^3 - 2\,x\,y\,(x + y) = 15\,x\,y.$

$$\therefore\ x\,y = \frac{(x + y)^3}{2\,(x + y) + 15}.$$

Also, by transposing the means in the first, and squaring

$$\frac{x^4}{y^2} + 2\,x\,y + \frac{y^4}{x^2} = 225 - 30\,(x + y) + (x + y)^2.$$

Or, $\frac{x^4}{y^2} + \frac{y^4}{x^2} = 225 - 30\,(x + y) + x^2 + y^2.$

And by adding $x^2 + y^2$ to each side,

$$\frac{x^4}{y^2} + x^2 + y^2 + \frac{y^4}{x^2} = 225 - 30\,(x + y) + 2\,(x^2 + y^2).$$

$\therefore$ by equality, from the second,

$2\,(x^2 + y^2) - 30\,(x + y) + 225 = 85.$

Or, $x^2 + y^2 - 15\,(x + y) = -\,70,$

which is the same as

$(x + y)^2 - 2\,x\,y - 15\,(x + y) = -\,70.$

And by substituting $\frac{(x + y)^3}{2\,(x + y) + 15}$ for $x\,y$, there

results $(x + y)^2 + \frac{17}{3}\,(x + y) = 70$, a quadratic.

And completing the square,

$$(x+y)^2 + \tfrac{17}{3}(x+y) + (\tfrac{17}{6})^2 = 70 + (\tfrac{17}{6})^2 = \tfrac{2809}{36}.$$

And $x + y + \frac{17}{6} = \frac{53}{6}$ by evolution.

$\therefore x + y = 6$, and $x\,y = 8$,

the sum and product. (See prob. 1.)

Whence $x = 4$, and $y = 2$.

$\therefore$ the numbers are 1, 2, 4, and 8.

8. Find two numbers, such that their product added to their sum may be 47, and their sum taken from the sum of their squares may leave 62.

Put $m + n$ and $m - n$ for the two numbers.

$$\left.\begin{array}{l}\text{Then } (m+n)\,.\,(m-n) + 2m = 47 \\ \text{And } (m+n)^2 + (m-n)^2 - 2m = 62\end{array}\right\}\text{ by the quest.}$$

Or, $m^2 - n^2 + 2m = 47$.

And $m^2 + n^2 - m = 31$.

$\therefore 2m^2 + m = 78$ by addition.

Or, $m^2 + \frac{1}{2}m = 39$.

$\therefore m^2 + \frac{1}{2}m + \frac{1}{16} = \frac{625}{16}$ by comp. $\square$.

And $m + \frac{1}{4} = \frac{25}{4}$ by evolution.

$\therefore m = 6$.

Whence $n^2 = 1$, $\therefore n = 1$.

$\therefore$ the numbers are 7 and 5.

Although the above artifice of substituting the *sum* and *difference* of two quantities for the required numbers simplifies the solution, it may nevertheless be solved without it; for, putting x and y for the required numbers, we have by the question

$$x + y + x\,y = 47.$$

$$\text{And } x^2 + y^2 - (x + y) = 63.$$

And by adding the second to twice the first,

$$(x + y)^2 + (x + y) = 156,$$

a quadratic in terms of their *sum*.

$\therefore\ (x + y)^2 + (x + y) + \frac{1}{4} = \frac{625}{4}$ by comp. the $\square$.

And $x + y + \frac{1}{2} = \frac{25}{2}$ by evolution.

$$\therefore\ x + y = 12.$$

And $x\,y = 35$ from the first.

The *sum* and *product*. (Prob. 1.)

Or, by adding the first and second,

$$x^2 + x\,y + y^2 = 109.$$

And by transposing $x\,y$ in the first, and squaring

$$x^2 + 2\,x\,y + y^2 = 2209 - 94\,x\,y + x^2\,y^2.$$

Subt. $x^2 + \quad x\,y + y^2 = \quad 109.$

And $\qquad x\,y \qquad = 2100 - 94\,x\,y + x^2\,y^2.$

$$\therefore\ x^2\,y^2 - 95\,x\,y = -2100,$$

a quadratic in terms of their *product*.

$\therefore\ x^2\,y^2 - 95\,x\,y + \frac{9025}{4} = \frac{625}{4}$ by completing the $\square$.

And $x\,y - \frac{95}{2} = \pm \frac{25}{2}$.*

$$\therefore\ x\,y = 35.$$

$\therefore$ from the first, $x + y = 12$, whence we have the *sum* and *product* as before.

Solved, $x = 7$, and $y = 5$.

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* Here, if the upper sign be used, $x\,y = 60$, which, though a root of the quadratic, does not solve the problem, for if it be substituted in the first, there results $x + y = -13$, an absurdity.

9. A set out from a certain town, and travelled 7 miles an hour; after he had gone 32 miles, B set out from another town to meet him, and travelled every hour $\frac{1}{19}$ of the whole journey, and when he had travelled as many hours as he went miles in an hour, he met A; required the distance of the two places.

Let x = the required distance.

Then $\frac{x}{19}$ = time A and B travelled together.

$$\therefore 32 + \frac{x}{19} \times 7 + \frac{x}{19} \times \frac{x}{19} = x \text{ by the question,}$$

which reduces to

$$x^2 - 228x = -11552.$$

$$\therefore x^2 - 228x + 12996 = 1444 \text{ by completing the } \square.$$

$$\text{And } x - 114 = \pm 38 \text{ by evolution.}$$

$$\therefore x = 114 \pm 38.$$

If the upper sign be taken, $x = 152$ miles.

But if the lower be taken, $x = 76$ miles.

$\therefore$ the question admits of two answers.

10. If A start from R to go to S at the same time B starts from S to go to R, both travelling uniformly, and t hours after they meet, A arrives at S, and t' hours after, B arrives at R; what time is each on the road?

$$\overset{\displaystyle Q}{R\text{————},\text{—————}S}$$

Suppose Q the place of meeting, and T the time before meeting: then because the space passed over by uniform motion is as the time in which it is passed over,

$$RQ : QS :: T : t.$$

$$\text{Also, } RQ : QS :: t' : T.$$

$$\therefore \quad T : t :: t' : T.$$

$$\text{Whence } T^2 = t\,t'.$$

$$\therefore \quad T = \sqrt{t\,t'}.$$

$$\therefore \quad \sqrt{t\,t'} + t = \text{A's time.}$$

$$\text{And } \sqrt{t\,t'} + t' = \text{B's time.}$$

PROBLEMS PRODUCING QUADRATICS.

1. What two numbers are those, the difference of which is 4, and the product 21? Ans. 7 and 3.

2. What number is that, which added to its square, the sum is 30? Ans. 5.

3. Find two numbers such that their product may be 24, and sum of the greater and half the less may be 8? Ans. 4 and 6.

4. Find a number such that if 7 times its square be taken from its cube, the remainder shall be 8 times the required number? Ans. 8.

5. A boy being asked the age of his father, said, three times as old as I am; and been asked how old *he* was, said if my age be multiplied by 36, the product will be equal to the difference between the *cube* of my age, and the square of my father's; required the age of each?

Ans. father 36, son 12.

6. A draper bought a piece of linen for £3. 15s.; by selling which at 3s. a yard, he gained as much as 6 yards cost him; what was the length of the piece?

Ans. 30 yards.

7. A person bought as many oxen as cost £400.; now if he had given £5. less for each, he would have had 4 more for the same money; what number did he buy?

Ans. *16*.

8. A person bought a bale of cloth for £160., and after distributing 50 yards among his friends, sold the remainder for £15. more than the whole cost, by which he gained 2s. a yard; required the quantity, and prime cost?

Ans. 400 yards, 8s. a yard.

9. What number is that, which being divided by the product of its two digits, the quotient is $5\frac{1}{4}$, but when 18 is subtracted from it there remains a number having the same digits inverted? Ans. 42.

10. A lady bought 36 oranges and 20 lemons for 15s.; she had 4 more oranges for 2s. 6d., than she had lemons for 3s.; what was the price of each?

Ans. oranges $2\frac{1}{2}$d., lemons $4\frac{1}{2}$d.

11. A lady being asked her age, height, and fortune; said, my height in inches is three times as many as my age in years, my fortune in half-crowns is the square of their sum, and the sum of all three is 6480; required her age, height, and fortune? Ans. age 20, height 5 feet, fortune £800.

12. A traveller sets out and goes 9 miles a day; 9 days after, another follows who travels the first day 5 miles, the second 6, the third 7, and so on; in what time will he overtake the first? Ans. 18 days.

13. Query the age of a man when x is the first digit, and y the second of the number which express his age; also $x^2 + y = y^2 - x$, and $2\,x\,y + 5 =$ to his age? *

Ans. 45.

* This may also be solved by a simple equation.

14. Two travellers set out at the same time from London to go to a town 240 miles distant; one travelled 2 miles an hour faster than the other, and arrived 6 hours before him; required the rates of travelling?

Ans. 8 and 10 miles an hour.

15. A person bought 12 chickens, 10 ducks, and 6 turkeys, for £2. 13s. 6d.; he had twice as many chickens for 12s. as he had ducks for 7s., and 2 ducks more for 7s. than he had turkeys for 6s.; required the price of each.

Ans. chickens, 1s. 6d. ducks, 1s. 9d. turkeys, 3s.

16. Find three numbers having equal differences, such that the square of the least added to the product of the two greater may make 28; but the square of the greater added to the product of the two less may make 44.

Ans. 2, 4, and 6.

17. A boy counting his money, found that a third and a fourth of it in pence, together with four times the square root of a third of it, was just 4d. short of the whole; what was he worth? Ans. 4s.

18. A boy being asked his age and height, said, my age is a square number, to which if 14 times it root be added, the sum is my height in inches; and from which height, if 3 be subtracted, the remainder is 3 times the next consecutive square to my age; required his age and height. Ans. age 9 years, height 4 feet 3 inches.

19. A draper sold 54 yards of cotton and 56 of dowlas for seven guineas, selling more by 4 yards of cotton for a guinea than of dowlas; required the price per yard of each.

Ans. cotton 1s. 2d, dowlas 1s. 6d.

20. A started from York to go to Leeds at the same time B started from Leeds to go to York; they both travelled

uniformly, and $3\frac{1}{2}$ hours after they met, A arrived at Leeds; and 4 hours after, B arrived at York; how long was B on the road? Ans. 7 hours $44\frac{1}{2}$ *minutes.*

21. A coach starts from A to go to B at the same time a man starts from B to walk to A; now the coach is just an hour and $37\frac{1}{2}$ minutes on the road; and the man, (who travels 4 miles an hour,) 2 hours and 10 minutes after meeting the coach; what is the distance from A to B;
Ans. 13 miles.

22. A and B start from two towns to meet on the road, A travelling $3\frac{1}{2}$ miles an hour, and B $\frac{1}{13}$ of the whole journey per hour; now when B had travelled two hours more than he travelled miles in an hour, he met A; required the distance between the two towns, A having started two hours before B. Ans. 52 or $45\frac{1}{2}$ miles.

23. Find 4 numbers in geometrical progression, the difference of the means of which is 12, and the difference of their extremes 52. Ans. 2, 6, 18, and 54.

24. A cattle-dealer bought as many cows as cost him £102. 8s.; and he sold them at £6. 16s. a piece, by which he gained as much as one cost him; what was the number of cows? Ans. 16.

25. A and B hired a pasture, into which A put 30 sheep, and B as many as cost him 12s. a week; afterwards B put in 6 additional sheep, and found that he must then pay 13s. 6d. a week; at what rate was the pasture hired?
Ans. 27s. a week.

26. Two men, A and B, set forth together from C to go to D, at which place they also arrived together; A went 30 miles a day, B 12 miles the first day, and increased his speed every day by equal differences, and when they had

travelled half the time, A was 50 miles before B; required the distance from C to D. Ans. 300 miles.

27. Find 3 numbers in harmonical proportion such that the difference of their differences shall be 2, and the product of the first and third 72. Ans. 12, 8, and 6.

28. The sum of 3 numbers in harmonical proportion is 13, and the product of their extemes 18; required the numbers. Ans. 6, 4, and 3.

29. A gentleman being asked the age of his son and daughter, said, if from the cube of my son's age multiplied by my daughter's, you subtract the cube of my daughter's multiplied by my son's, 96 remains; but if from the product of their ages you subtract the square of my daughter's, the remainder is 4; required their ages.

Ans. son 4, daughter 2.

30. Find two numbers such that the cube of the greater multiplied by the less, *plus* the cube of the less multiplied by the greater may be 78, and the sum of their biquadrates 97. Ans. 3 and 2.

31. Find two numbers such that their sum added to their product may be 51, and their sum taken from the sum of their squares may leave 138. Ans. 3 and 12.

32. *Required the side of a square, when the diagonal is 4 yards longer than the side. Ans. $4 (1 + \sqrt{2}.)$

33. The diagonal of a rectangle is 17, and the sum of the length and breadth 23; required the sides.

Ans. 15, and 8.

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* These questions are solvable by the rules and notes in the Author's 'Tutor's Assistant;' several of the solutions exhibit a demonstration of his rule, for finding the sides of a right-angled triangle when the hypotenuse and area are given.

34. A slate, 13 inches by 8, was framed with a frame of equal width, the area of which was $\frac{9}{13}$ of that of the slate; required its width. Ans. $1\frac{1}{2}$ inches.

35. In a right-angled triangle there is given the base 6, and difference between the hypotenuse and perpendicular 2; to find those two sides.

Ans. hypotenuse 10, perpendicular 8.

36. The hypotenuse of a right-angled triangle is 20, and the sum of the base and perpendicular added to their product is 220; required the legs. Ans. 16 and 12.

37. The perimeter of a right-angled triangle is 198, and the diameter of its inscribed circle 28; required the legs.

Ans. 77 and 36.

38. The perimeter of a right-angled triangle is 60, and the perpendicular demitted from the right-angle upon the hypotenuse is 12; determine the legs.

Ans. 20 and 15.

39. The hypotenuse of a right-angled triangle is 35, and the side of its inscribed square 12; required the other two sides. Ans. 28 and 21.

40. The hypotenuse of a right-angled triangle is 25, the sum of the legs and perpendicular drawn from the right-angle upon the hypotenuse in one sum is 47; required the legs and perpendicular.

Ans. 20, 15 legs; perpendicular 12.

41. In a right-angled triangle the sine of the least angle is $\frac{3}{5}$, and its area 150; required the sides.

Ans. hypotenuse 25, legs 20 and 15.

42. In a right-angled triangle the hypotenuse is 75, and tangent of the least angle $\frac{3}{4}$; find the legs.

Ans. 60 and 45.

43. In a right-angled triangle the secant of the least angle is $\frac{5}{4}$, and diameter of the inscribed circle 10; required the legs. Ans. 20 and 15.

44. A water cistern, three feet deep, when filled to the depth of 2 feet, holds less than when completely filled by 24 cubic feet, together with the number of feet in the semiperimeter of the base; also the diagonal, from any one of the corners at the top to its opposite corner at the bottom, is equal to one-eighth of the square of the diagonal of the bottom; required its contents in gallons.

Ans. 673 gallons, nearly.

COMPOUND QUADRATICS.

(Art. 1.) A compound quadratic is a cubic, or higher equation put under a quadratic form, as

$$x^6 - 6x^4 + 3x^3 + 9x^2 - 9x = 10,$$

is the same as

$$(x^3 - 3x)^2 + 3(x^2 - 3x) = 10.$$

And by completing the square,

$$(x^3 - 3x)^2 + 3(x^2 - 3x) + \tfrac{9}{4} = \tfrac{49}{4}.$$

$\therefore$ by evolution,

$$x^3 - 3x + \tfrac{3}{2} = \tfrac{7}{2}.$$

$$\therefore x^3 - 3x = 2, \text{ a cubic equation.}$$

(Art. 2.) Now all cubic equations, falling under one of the following forms, namely,

$$x^3 + ax = b$$

$$x^3 - ax = b$$

$$\text{And } x^3 - ax = -b$$

May, when the value of x is rational, be converted into quadratics by the following

RULE. *

1. Multiply the equation by the unknown letter x.

2. Resolve the right hand, or absolute term, into two such factors, that when the square of one of them, together with the square of the unknown quantity, is added to both sides of the equation, the co-efficient of the second term will be the same as the other factor.

3. Add the square of half the co-efficient of the second term, to both sides of the equation, and each side will then be a perfect square; and by extracting the square root, &c. there results a quadratic equation.

Now to reduce $x^3 - 3x = 2$, as found above, to a quadratic, by the foregoing rule, we have, by multiplying by x

$$x^4 - 3x^2 = 2x;$$

and by resolving 2 into its factors, 2 and 1; it is evident that the square of either of them, multiplied into the square of x, will make the co-efficient of the second term equal to the other factor.

Thus $x^4 + x^2 = 4x^2 + 2x$ by adding $4x^2$ } to each
And $x^4 - 2x^2 = x^2 + 2x$ by adding x^2 } side.

And by completing the squares by the third part of the rule,

$$x^4 + x^2 + \tfrac{1}{4} = 4x^2 + 2x + \tfrac{1}{4}$$

$$\text{And } x^4 - 2x^2 + 1 = x^2 + 2x + 1.$$

$\therefore$ by evolution, &c.

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* Though this may be considered equivalent to solving the equation, it will nevertheless be found of considerable advantage when quadratic solutions are required.

$$x^2 - 2x = 0 \text{ from the former,}$$

$$\text{And } x^2 - x = 2 \text{ from the latter.}$$

Whence $x = 2$ by the rule for adfected quadratics.

(Art. 3.) Should the equation be of the form

$$x^3 \pm a x^2 \pm b x = \pm c,$$

it may be reduced to one of the foregoing forms, by taking away its second term, which may be done by taking a new unknown letter, and subjoining to it a *third* part of the co-efficient of the second term, with its sign changed; then this being substituted for the original unknown quantity, in the proposed equation, there will result an equation wanting its second term.

EXAMPLES.

1. To exterminate the second term of the equation.

$$x^3 + 6x^2 - 4x = 69.$$

Here the third part of the co-efficient of the second term is 2, assume therefore

$$x = z - 2, \text{ and}$$

$$\begin{aligned} x^3 &= (z-2)^3 = z^3 - 6z^2 + 12z - 8 \\ 6x^2 &= 6(z-2)^2 = \quad 6z^2 - 24z + 24 \\ -4x &= -4(z-2) = \quad -4z + 8 \end{aligned}$$

$$\therefore \text{ by addition, } z^3 \quad . \quad -16z + 24 = 69$$

$$\text{Or, } z^3 - 16z = 45.$$

Which equation falls under one of the foregoing forms, and may be reduced to a quadratic accordingly.

2. To exterminate the second term from

$$y^3 - 15y^2 + 81y = 243$$

$$\text{Put } y = x + 5, \text{ then}$$

$$y^3 = (x+5)^3 = x^3 + 15x^2 + 75x + 125$$
$$-15y^2 = -15(x+5)^2 = -15x^2 - 150x - 375$$
$$81\,y = 81\,(x+5) = \qquad 81\,x + 405$$

$\therefore$ by addition $x^3 \quad . \quad + 6\,x + 155 = 243$

Or, $x^3 + 6\,x = 88.$

(Art. 4.) To know when any biquadratic equation, as

$$x^4 + a\,x^3 + b\,x^2 + c\,x + s = 0,$$

is of such a form as to be solvable by quadratics; assume $x = y + h$, and the equation becomes

$$\left.\begin{array}{lllll} x^4 = y^4 + 4\,h\,y^3 & + 6\,h^2\,y^2 & + 4\,h^3\,y & + h^4 \\ a\,x^3 = & a\,y^3 + 3\,a\,h\,y^2 & + 3\,a\,h^2\,y & + a\,h^3 \\ b\,x^2 = & & b\,y^2 + 2\,b\,h\,y & + b\,h^2 \\ c\,x = & & c\,y & + c\,h \\ s = & & & + s \end{array}\right\} = 0.$$

Which will evidently be a quadratic when the second and fourth columns vanish; that is, when

$$4\,h + a = 0$$

And $4\,h^3 + 3\,a\,h^2 + 2\,b\,h + c = 0.$

From the former $h = -\dfrac{a}{4}$

Which substituted in the latter, it becomes

$$a^3 - 4\,a\,b + 8\,c = 0,$$

the necessary relation of the co-efficients, in order to a quadratic form.

If m, n, p, q, represent the roots of the foregoing biquadratic, then

$$x - m = 0,\ x - n = 0,\ x - p = 0,\ x - q = 0,$$

And $(x-m)\cdot(x-n)\cdot(x-p)\cdot(x-q) = 0.$

Or, $x^4 - (m+n+p+q)\,x^3 + (m\,n + m\,p + m\,q + n\,p + n\,q + p\,q)\,x^2 - (m\,n\,p + m\,n\,q + m\,p\,q + p\,q\,n)\,x + m\,n\,p\,q = 0.$

$\therefore\ a = -(m + n + p + q)$

$b = mn + mp + mq + np + nq + pq.$

$c = -(mnp + mnq + mpq + pqn).$

And $s = mnpq.$

Now if we take $m = -1$, $n = -2$, $p = 3$ and $q = 3$, then $a = -3$, $b = -7$, $c = 15$, and $s = 18$, and the equation becomes

$$x^4 - 3x^3 - 7x^2 + 15x + 18 = 0,$$

the co-efficients of which possess not the necessary relation for transforming it into a quadratic; but if it be divided by $x - 3 = 0$, it becomes

$$x^3 - 7x - 6 = 0 \text{ (Art. 2.)}$$

Or, $x^3 - 7x = 6$

And by adding $6x$ to each side,

$$x^3 - x = 6x + 6.$$

Or, $x(x^2 - 1) = 6(x + 1).$

$\therefore\ x(x - 1) = 6$ by $\div$ by $x + 1$.

Or, $x^2 - x = 6$ a quadratic.

Which has been obtained without the aid of the rule in Art. 2.

(Art. 5.) Given $x^3 - 4x^2 - 8x^{\frac{3}{2}} - 17x - 8x^{\frac{1}{2}} - 12 = 0$, to find x by quadratics.

Dividing by $x + 1 = 0$

$$x^2 - 5x - 8x^{\frac{1}{2}} - 12 = 0$$

Or, $x^2 - 5x = 8x^{\frac{1}{2}} + 12.$

And by adding $x + 4$ to each side,

$$x^2 - 4x + 4 = x + 8x^{\frac{1}{2}} + 16 \text{ two squares.}$$

$\therefore\ x - 2 = \sqrt{x} + 4$ by evolution.

$\therefore\ x - \sqrt{x} = 6$ a quadratic.

(Art. 6.) Given $x^2 + \frac{1}{x^2} + 5\,x + \frac{5}{x} = 51\,\frac{1}{25}$.

By adding 2 to each side it becomes

$$x^2 + 2 + \frac{1}{x^2} + 5\,x + \frac{5}{x} = 53\,\frac{1}{25}.$$

$$\text{Or, } \left(x + \frac{1}{x}\right)^2 + 5\left(x + \frac{1}{x}\right) = 53\,\frac{1}{25}.$$

And by completing the square,

$$\left(x + \frac{1}{x}\right)^2 + 5\left(x + \frac{1}{x}\right) + \frac{25}{4} = \frac{5929}{100}.$$

$$\therefore\ x + \frac{1}{x} + \frac{5}{2} = \frac{77}{10}.$$

$$\text{Or, } x + \frac{1}{x} = \frac{26}{5}.$$

$$\therefore\ x^2 - \frac{26}{5}\,x = -1, \text{ a quadratic.}$$

EXERCISES,

To be solved by Quadratics.

1. Given $x^4 + 4\,x^3 - 8\,x = 165$, to find x.
Ans. $x = 3$.

2. Given $x^4 - 2\,x^3 + x = 30$, to find x.
Ans. $x = 3$.

3. Given $x^4 + 2\,x^3 - x = 3080$, to find x.
Ans. $x = 7$.

4. Given $x^4 - 8\,x^3 + 14\,x^2 + 8\,x = 15$, to find x.
Ans. $x = 5$.

5. Given $x^4 + 2\,x^3 - 7\,x^2 - 8\,x + 12 = 0$, to find x.
Ans. $x = 2$.

6. Given $x^4 + 6\,x^3 + 29\,x^2 + 60\,x = 6300$, to find x.
Ans. $x = 7$.

7. Given $x^2 - 6x^{\frac{3}{2}} + 12x - 9x^{\frac{1}{2}} = 28$, to find x.
Ans. $x = 16$.

8. Given $x + 4x^{\frac{3}{4}} + 8x^{\frac{1}{2}} + 8x^{\frac{1}{4}} = 285$, to find x.
Ans. $x = 81$.

9. Given $x^{\frac{4}{3}} - 4x + 6x^{\frac{2}{3}} - 4x^{\frac{1}{3}} = 15$, to find x.
Ans. $x = 27$.

10. Given $\frac{4}{9}x^{\frac{4}{3}} - \frac{4}{9}x - \frac{5}{9}x^{\frac{2}{3}} + \frac{1}{3}x^{\frac{1}{3}} = 20$, to find x.
Ans. $x = 27$.

11. Given $x^6 - 4x^4 + 4x^3 + 4x^2 - 8x = 3360$, to find x.
Ans. $x = 4$.

12. Given $x^6 - 2x^4 + 8x^3 + x^2 - 8x = 15360$, to find x.
Ans. $x = 5$.

13. Given $x^6 + 6x^5 + 11x^4 + 5x^3 - 2x^2 - x = 462$, to find x.
Ans. $x = 2$.

14. Given $x^6 + 4x^5 - 8x^4 - 18x^3 + 48x^2 - 36x = 5616$, to find x.
Ans. $x = 4$.

15. Given $x^3 + 50x = 375$, to find x. Ans. $x = 5$.

16. Given $x^{\frac{3}{2}} + x^{\frac{1}{2}} = 10$, to find x. Ans. $x = 4$.

17. Given $(x + 1)x^{\frac{1}{2}} = 30$, to find x. Ans. $x = 9$.

18. Given $x - 5x^{\frac{1}{3}} = 12$, to find x. Ans. $x = 27$.

19. Given $x^2 + x - 6x^{\frac{1}{2}} = 8$, to find x.
Ans. $x = 4$.

20. Given $x^2 + x - 14x^{\frac{1}{2}} = 48$, to find x.
Ans. $x = 9$.

21. Given $x^2 - 5x - 8x^{\frac{1}{2}} = 12$, to find x.
Ans. $x = 9$.

22. Given $x^4 + x^2 - 14x = 48$, to find x.
Ans. $x = 3$.

EQUATIONS IN GENERAL.

All Equations have as many roots, real and imaginary, as there are units in the exponent of its highest power: that is, a simple equation has but one root; a quadratic has two roots; a cubic, three roots; and a biquadrate, four roots, &c. &c.

Thus $(x - 1) \times (x - 2) \times (x - 3) \times (x - 4) = x^4 - 10^3 + 35\,x^2 - 50\,x + 24 = 0$, is a biquadrate equation, produced by the continual multiplication of four simple ones, namely, $x - 1 = 0$, $x - 2 = 0$, $x - 3 = 0$, and $x - 4 = 0$, the roots of which are evidently 1, 2, 3, and 4; any of which, being substituted for x, will make the equation vanish: that is, $x = 1$, $x = 2$, $x = 3$, or, $x = 4$.

Real roots are such as are expressed by real possible quantities, either negative or positive.

Imaginary roots are those into which any imaginary or impossible quantity enters; as, $a \pm \sqrt{-b}$, which is impossible, because no negative quantity has a square root: such expressions, though they cannot be valued, are such, that when they are substituted for the unknown quantity, make the equation vanish.

Imaginary quantities always enter equations by pairs; if one of them be $a + \sqrt{-b}$, the other will be $a - \sqrt{-b}$.

In the given equation $x^3 - 64 = 0$, we find by transposition and evolution, $x = 4$, one root; and by dividing the given equation by $x - 4 = 0$, there results

$$x^2 + 4\,x + 16 = 0;$$

$$\text{Or, } x^2 + 4\,x = -16;$$

and by completing the square, and evolution, the other two roots are found to be

$$x = -2 \pm \sqrt{-12},$$

which are both imaginary, or impossible, admitting of no assignable value, yet they are such that, when substituted for x, they make the equation vanish; that is, the cube of each is $= 64$;

$$\text{or, } (-2 + \sqrt{-12})^3 = 64 = (-2 - \sqrt{-12})^3.$$

SCHOLIUMS.

1. In every equation the sum of all the roots is equal to the co-efficient of the second term, with its sign changed from $+$ to $-$, or from $-$ to $+$.

2. The sum of the product of every two of the roots is equal to the co-efficient of the third term, without any change in its sign.

3. The sum of the product of every three of the roots is equal to the co-efficient of the fourth term, with its sign changed, and so on to the last or absolute term, which is equal to the product of all the roots.

4. When all the roots are positive, the terms of the equation will be $+$ and $-$ alternately.

5. When all the roots are negative, the terms will all be $+$.

6. When the roots are partly positive and partly negative, there will be as many positive roots as there are changes in the signs, and as many negative roots as there are permanencies of signs.

In order to elucidate the above, let it be required to find the roots of the following general cubic equation, when they are in arithmetical progression:

$$x^3 + m\,x^2 - n\,x = p.$$

Assume $z - y$, z, and $z + y$, for the required roots, which are obviously in arithmetical progression. Their sum is $3z$; therefore, (Sch. 1.) $3z = -m$, whence $z = -\frac{m}{3}$, and $(z - y) \times z + (z + y) \times z + (z + y) \times (z - y) = -n$. (Sch. 2.)

Or, $\left(-\frac{m}{3} - y\right) \times -\frac{m}{3} + \left(-\frac{m}{3} + y\right) \times -\frac{m}{3} +$

$$\left(-\frac{m}{3} + y\right) \times \left(-\frac{m}{3} - y\right) = -n,$$

which reduces to $\frac{m^2}{3} - y^2 = -n$. $\therefore y = \sqrt{\frac{m^2}{3} + n}$.

Hence the roots are

$$\left(-\frac{m}{3} - \sqrt{\frac{m^2}{3} + n}\right), -\frac{m}{3}, \text{and} \left(-\frac{m}{3} + \sqrt{\frac{m^2}{3} + n}\right),$$

Where m and n may be taken at pleasure.

If $m = 6$, and $n = 4$, the roots will be -6, -2, and $+2$, and $p = -6 \times -2 \times 2 = 24$. (Sch. 3.)

CUBIC EQUATIONS.

To find the roots of cubic equations, according to Cardan's method:—

First reduce the equation to the form of $x^3 + a x = b$, and make $x = p + q$.

Then $\begin{cases} x^3 = p^3 + 3 p q (p + q) + q^3 \\ a x = \qquad\qquad a (p + q) \end{cases}$

Whence, by addition, since $p + q = x$, we have

$$x^3 + a x = p^3 + (3 p q + a) x + q^3 = b;$$

Assume $3 p q = -a$,

Then $p^3 + q^3 = b$.

Whence, from the last equation but one,

$$p = -\frac{a}{3\,q}, \text{ and } q = -\frac{a}{3\,p},$$

Which substituted separately in the last, we have

$$-\frac{a^3}{27\,q^3} + q^3 = b,$$

$$\text{And } -\frac{a^3}{27\,p^3} + p^3 = b,$$

Which reduces to the quadratic forms,

$$q^6 - b\,q^3 = \frac{a^3}{27},$$

$$p^6 - b\,p^3 = \frac{a^3}{27},$$

And by completing the square, and evolution, &c.

$$q^3 = \tfrac{1}{2}\,b \pm \sqrt{\tfrac{1}{4}\,b^2 + \frac{a^3}{27}},$$

$$p^3 = \tfrac{1}{2}\,b \pm \sqrt{\tfrac{1}{4}\,b^2 + \frac{a^3}{27}}.$$

But as their sum is $= b$, it is obvious that the double sign must be $+$ in one, and $-$ in the other; whence,

$$p = \left(\tfrac{1}{2}\,b + \sqrt{\tfrac{1}{4}\,b^2 + \frac{a^3}{27}}\right)^{\frac{1}{3}}$$

$$q = \left(\tfrac{1}{2}\,b - \sqrt{\tfrac{1}{4}\,b^2 + \frac{a^3}{27}}\right)^{\frac{1}{3}}, \text{ and}$$

$$x = \left(\tfrac{1}{2}\,b + \sqrt{\tfrac{1}{4}\,b^2 + \frac{a^3}{27}}\right)^{\frac{1}{3}} + \left(\tfrac{1}{2}\,b - \sqrt{\tfrac{1}{4}\,b^2 + \frac{a^3}{27}}\right)^{\frac{1}{3}}$$

In the above formula, $\frac{1}{4}\,b^2$ is always positive, it being the square of $\frac{1}{2}\,b$; but $\frac{a^3}{27}$ will be negative or positive, according as a is negative or positive in the proposed equation. If a be negative, and $\frac{a^3}{27}$ greater than $\frac{b^2}{4}$, the solution cannot be obtained by the above formula.

To find the value of x, by the above formula in the equation $x^3 + 18\,x = 6$. Here $a = +\,18$ and $b = +\,6$,

$$\therefore\ \tfrac{1}{2}\,b = 3,\ \tfrac{1}{4}\,b^2 = 9, \text{ and } \frac{a^3}{27} = 216; \text{ whence}$$

$$x = \sqrt[3]{3 + \sqrt{9 + 216}} + \sqrt[3]{3 - \sqrt{9 + 216}}$$

$$\text{Or, } x = \sqrt[3]{3 + \sqrt{225}} + \sqrt[3]{3 - \sqrt{225}}$$

$$\text{Or, } x = \sqrt[3]{18} + \sqrt[3]{-12} = \sqrt[3]{18} - \sqrt[3]{12} = .331313^*$$

BIQUADRATIC EQUATIONS.

BIQUADRATIC EQUATIONS are of the 4th degree, or in which the unknown quantity rises to the 4th power:—

As $x^4 + a\,x^3 + b\,x^2 + c\,x + d = 0$, where a, b, c, d, may be any numbers whatever, positive or negative, or any of them may be equal to 0.

These equations are the highest that admit of any direct solution, except in particular cases.

The first method of solution was invented by Louis Ferrari, a pupil of Cardan, and published by the latter in 1540; since which, several other methods have been

* When one of the roots has been found by any method, the other two roots may be found as follows: Bring all the terms to the left hand side of the equation; then change the sign of the root, and connect it with the unknown quantity, by which divide all the terms of the equation; the quotient will be a quadratic containing the other two roots.

invented by Descart, Waring, Euler, Simpson, &c.; all of which differ more in form than principle.

Ferrari's method, as generalized by Simpson, is as follows, supposing the general equation given above to be the same:—

As $(x^2 + \frac{1}{2} a x + p)^2 - (q x + r)^2 = 0$, where p, q, and r are unknown, the values of which are to be determined so as to make the latter equation equal to the general or proposed equation, which is done as follows:

$$\left.\begin{array}{ll}(x^2 + \frac{1}{2} a x + p)^2 = x^4 + a x^3 + & \left\{\begin{array}{l}\frac{1}{4} a^2 x^2 \\ 2 p x^2\end{array}\right. + a p x + p^2 \\ -(q x + r)^2 \quad = \quad - & \quad q x^2 - 2 q r x - r^2\end{array}\right\}$$

$$= x^4 + a x^3 + b x^2 + c x + d.$$

And by equating the homologous, or like terms, we have

$$\frac{1}{4} a^2 + 2 p - q^2 = b.$$
$$a p - 2 q r = c.$$
$$p^2 - r^2 = d.$$

And by multiplying the first and third of these equations together, their product will evidently be equal to $\frac{1}{4}$ of the square of the second, whence we have

$$8 p^3 - 4 b p^2 + (2 a c - 8 d) p - a^2 d + 4 b d - c^2 = 0,$$

a cubic equation, from which p may be found by Cardan's formula, and q and r are found from

$$q = \sqrt{\tfrac{1}{4} a^2 + 2 p - b}$$
$$r = \frac{a p - c}{2 q}.$$

Now, having found the values of p, q, and r, the four values of x are found from the assumed equation,

$$(x^2 + \tfrac{1}{2} a x + p)^2 - (q x + r)^2 = 0;$$

Or, $x^2 + \frac{1}{2} a x + p = \pm (q x + r)$ by trans. & evol.

$$\left.\begin{array}{l}\text{Whence } x^2 + (\tfrac{1}{2}a - q)\,x = r - p \\ \text{Or, } x^2 + (\tfrac{1}{2}a + q)\,x = -r - p\end{array}\right\}\text{two quadratics}$$

And, by taking the double sign $\pm$, by which $(q\,x + r)$ is affected, both $+$ and $-$,

$$\text{we have}\left\{\begin{array}{l} x = -\dfrac{\frac{1}{2}a - q}{2} + \sqrt{\left(\dfrac{\frac{1}{2}a - q}{2}\right)^2 + r - p} \\ x = -\dfrac{\frac{1}{2}a - q}{2} - \sqrt{\left(\dfrac{\frac{1}{2}a - q}{2}\right)^2 + r - p} \\ x = -\dfrac{\frac{1}{2}a + q}{2} + \sqrt{\left(\dfrac{\frac{1}{2}a + q}{2}\right)^2 - r - p} \\ x = -\dfrac{\frac{1}{2}a + q}{2} - \sqrt{\left(\dfrac{\frac{1}{2}a + q}{2}\right)^2 - r - p} \end{array}\right\}\text{the four values of } x.$$

In the equation $x^4 + 12\,x - 17 = 0$, we have, by comparing it with the general one, $a = 0$, $b = 0$, $c = 12$, and $d = -17$; therefore the resulting cubic equation is $8\,p^3 + 136\,p = 144$, or $p^3 + 17\,p = 18$.

$$\therefore \left\{\begin{array}{l} p = 1 \\ q = \sqrt{2} \\ r = -\dfrac{12}{2\sqrt{2}} = -3\sqrt{2}. \end{array}\right.$$

Which values being substituted in the above formula,

$$\text{gives}\left\{\begin{array}{l} x = \tfrac{1}{2}\sqrt{2} + \sqrt{-3\sqrt{2} - \tfrac{1}{2}} \\ x = \tfrac{1}{2}\sqrt{2} - \sqrt{-3\sqrt{2} - \tfrac{1}{2}} \\ x = -\tfrac{1}{2}\sqrt{2} + \sqrt{3\sqrt{2} - \tfrac{1}{2}} \\ x = -\tfrac{1}{2}\sqrt{2} - \sqrt{3\sqrt{2} - \tfrac{1}{2}} \end{array}\right.$$

The two first of which are imaginary, and the two last irrational.

RESOLUTION OF EQUATIONS BY APPROXIMATION.

A great number of approximating rules have been invented by different mathematicians, for the solution of equations of all denominations; the most easy and expeditious of which is the following rule of Double Position, or, as it is commonly called,

TRIAL AND ERROR.

RULE.*

1. Find, by trial, two numbers as near the truth as possible, noting how much their sums differ from the absolute term of the equation.

2. Then, as the sum of the errors is to the difference of the assumed numbers, so is either of the errors to the correction of the number belonging to the error used, which correction being added to, or subtracted from, the said number, according as the error is too big or too little, will give the true root *nearly*.

3. Take this root and the nearest of the two former, or any other that may be found nearer, and by proceeding as above, a root will be found still nearer than before, and so on to any degree of exactness required.

---

* 1. It is generally best to assume two numbers that differ from each other only by unity in the last figure.

2. In the operation, it is best always to use the least error.

3. In working with the *first* supposed numbers, find only one figure more.

4. In working with the *second* supposed numbers, take the quotient only to four places of figures; and in working with the *third* supposed numbers, take the quotient to eight places of figures, &c.

EXAMPLE.

To find the roots of the equation $x^3 - 15x^2 + 63x = 50$.

Here it is soon found by a few trials that x is but a very little above 1; therefore, suppose 1 and 1 . 1.

1st sup.		*2nd sup.*	
1	$= x =$	1 . 1	
63	$63x$	69 . 3	
— 15	$-15x^2$	— 18 . 15	
1	x^3	1 . 331	
49	sums	52 . 481	
50	but should be	50 .	
too little 1.	errors*	2 . 481	too big

2.481

3.481 : .1 :: 1 : .03

therefore $x = 1.03$ nearly.

Again, suppose the two numbers 1.03 and 1.02, then

1st sup.		*2nd sup.*
1.03	$= x =$	1.02
64.89	$63x$	64.26
— 15.9135	$-15x^2$	— 15.6060
1.092727	x^3	1.061208
50.069227	sums	49.715208

* If the errors are both alike, *i. e.* both too little or both too big, then their difference instead of their sum must be used for the first term of the proportion; the second term is the difference of the supposed numbers, and the third term is the less error.

	50.	should be	50.
too big	.069227	errors	.284792 too little
	.214792		

.354019 : .01 :: .069277 : .0019555.

$\therefore x = 1.03 - .0019555 = 1.02804$ very nearly.

And the given equation divided by $(x - 1.02804)$, there results $x^2 - 13.97196\,x = -48.63627$, a quadratic; the roots of which, by completing the square and evolution, are 6.57653 and 7.39543; therefore the three roots of the given equation $x^3 - 15\,x^2 + 63\,x = 50$

$$\text{are} \left\{\begin{array}{l} 1.02804 \\ 6.57653 \\ 7.39543 \end{array}\right. \quad \text{Sch. 1.}$$

$$\text{sum } 15.00000$$

EXPONENTIAL EQUATIONS.

An EXPONENTIAL EQUATION is one into which an exponential quantity enters; as, $a^x = b$, $a^{b^x} = c$, $x^x = d$, &c. The readiest method of solving exponential equations is by means of a table of logarithms; thus, taking the first equation, $a^x = b$,

$$x \times \log.\, a = \log.\, b; \quad \therefore\ x = \frac{\log.\, b}{\log.\, a}.$$

In the second equation, $a^{b^x} = c$, assume $b^x = y$; then

$$a^y = c, \text{ and } y \times \log.\, a = \log.\, c; \ \therefore\ y = \frac{\log.\, c}{\log.\, a} = h;$$

$$\therefore\ b^x = h, \text{ or } x = \frac{\log.\, h}{\log.\, b}.$$

In the third equation $x^x = d$, we have $x \times \log.\, x = \log.\, d$, in which the value of x may be determined to any degree of exactness by the last rule.

EXAMPLES.

1. Given $x^{x+1} = y^m$

$$y^{x+1} = x^n.$$

From the first $y = x^{\frac{x+1}{m}}$,

From the second $y = x^{\frac{n}{x+1}}$.

$$\therefore x^{\frac{x+1}{m}} = x^{\frac{n}{x+1}}.$$

Or, $\frac{x+1}{m} = \frac{n}{x+1}$.

Whence $(x+1)^2 = m\,n$.

$$\therefore x = \sqrt{m\,n} - 1,$$

$$\text{and } y = (\sqrt{m\,n} - 1)^{\left(\frac{n}{m}\right)^{\frac{1}{2}}}$$

If $m = 4$, and $n = 9$, then

$$x = 5, \text{ and } y = 5^{\frac{3}{2}}.$$

2. Given $x^x = 100$, to find an approximate value of x.

Here, $x \times \log. x = \log. 100 = 2$;

and by a few trials we find x to be nearly in the middle between 3 and 4; assume therefore

$$x = 3.5 \text{ and } x = 3.6.$$

First sup. $x = 3.5$

log. $3.5 = 0.544068$

then $3.5 \times 0.544068 = 1.904238$

but should be 2

error, too little .095762

Second sup. $x = 3.6$.

log. $3.6 = 0.556303$
then $3.6 \times 0.556303 = 2.002689$
but should be 2

.002689 error, too big.

.095762 } errors
.002689 }

.098451 : .1 :: .002689 : 0.00273, the correction.

$\therefore\ x = 3.6 - 0.00273 = 3.59727$ nearly;

On trial this is found to be a very small matter too little, therefore take

$x = 3.59727$ and $x = 3.59728$, and repeat the operation.

First $x = 3.59727$

log. $3.59727 = 0.555973$
$\therefore\ 3.59727 \times 0.555973 = 1.9999854$
but should be 2.

error, too little 0.0000146

Second $x = 3.59728$

log. $3.59728 = 0.555974$
$\therefore\ 3.59728 \times 0.555974 = 1.9999953$
but should be 2

0.0000047 error, too little.

0.0000146 } errors
0.0000047 }

0.0000099 : .00001 :: .000047 : 0.00000474747, the cor.
which added to 3.56728

gives $x = 3.59728474747$

3. To find the value of x, such that x pounds being put out at $2x$ per cent. per annum, compound interest, for $3x$ years, the whole amount may be $5x$.

By the nature of compound interest, we have $1 + 0 2 x$ = the amount of £1. for a year.

And $x(1 + .02\,x)^{3x} = 5x$, by the question;

Or, $(1 + .02\,x)^{3x} = 5$, by $\div$ by x.

Whence $3x \times \log. (1 + .02\,x) = \log. 5 = .69897$.

Where, by a few trials, x is soon found to be between 5 and 6, or even between 5.3 and 5.4. And by approximation $x = 5.312728$; or from the equation $(1 + .02\,x)^{3x} = 5$; we obtain $(1 + .02\,x)^x = 1.70976$, by extracting the cube root; whence $x \times \log. (1 + .02\,x) = \log. 1.70976$, from which by approximation x will be found as above.

EXERCISES.

1. Given $x^3 - 2x = 50$, to find x.

Ans. $x = 3.864$.

2. Given $x^3 + 10x^2 + 5x = 260$, to find x.

Ans. $x = 4.1178$.

3. Given $x^4 - 3x^2 - 75x = 10000$, to find x.

Ans. 10.2607.

4. Given $\left.\begin{array}{l} x^{x+y} = y^4 \\ \text{And } y^{x+y} = x^9 \end{array}\right\}$ to find x and y.

Ans. $x = 2.3643$.

$y = 2.3643^{\frac{3}{2}}$.

5. Given $x^x = 2000$, to find x.

Ans. $x = 4.8277$.

6. Given $x^{x\,y} = y^8$ And $y^{x\,y} = x^{18}$ } to find x and y.

Ans. $x = (144)^{\frac{1}{5}}$.

$y = (144)^{\frac{3}{10}}$.

7. Given $(6\,x)^x = 96$, to find x.

Ans. $x = 1.8826$.

8. In what time will the compound interest of *any* sum at $3\frac{1}{2}$ *per cent.* be equal to the simple interest of the same sum at 5 *per cent?* Ans. 20.53502 years.

UNLIMITED PROBLEMS.

CASE 1.

Problems are said to be unlimited, when there are more unknown quantities than there are equations expressing them,

$$\text{as, } a\,x \pm b\,y = c.$$

Where it is obvious that x and y will admit of an indefinite number of values, unless they be limited to whole numbers, in which case the number of answers are sometimes bound within certain limits.

To solve the equation,

$$a\,x + b\,y = c,$$

Find the value of one of the unknown quantities in terms of the other and absolute number c,

$$\text{as } y = \frac{c - a\,x}{b}.$$

Divide each term of the numerator by the denominator b, rejecting the whole numbers, should b be contained in either c or a, and let the resulting fraction be of the form

$$\frac{e\,x - m}{b}*$$

Then divide $b\,x$ by $e\,x - m$, the divisor by the remainder, the last divisor by the last remainder, and so on till the remainder become of the form $x \pm m\,p$; divide $m\,p$ by b, and call the remainder r, which will be the least value of x when the sign of $m\,p$ is negative; but when the sign of $m\,p$ is positive, subtract r from b, and the remainder will be the least value of x; the greatest value of y will be found from

$$y = \frac{c - a\,x}{b}$$

Or we may express as follows, the foregoing

RULE.

Transpose one of the unknown terms, and divide by the co-efficient of the other, rejecting whole numbers.

Annex the unknown letter to the denominator of the resulting fraction, which divide by the numerator, the divisor by the remainder, the last divisor by the last remainder, until the co-efficient of the unknown letter is unity; then if the *last term* of the *last remainder* be *minus*, and less than the divisor, it is the least value of the unknown letter; but if *plus*, subtract it from the divisor, the remainder is the least value. But if the *last term* be greater than the divisor, then divide it by the divisor, and the *remainder* will be the least value when *minus*; and when *plus*, it must be subtracted from the divisor.

---

* When m is a multiple of e, then dividing the numerator by e, the quotient may be considered as the *last remainder*.

EXAMPLES.

1. Given $21x + 17y = 2000$, to find all the possible values of x and y in whole numbers.

$$\text{Here } y = \frac{2000 - 21x}{17} = 117 - x - \frac{4x - 11}{17}.$$

$$\begin{array}{rl} 4x - 11)\ 17x & (4 \\ 16x - 44 & \\ \hline x + 44 & = \text{last remainder.} \end{array}$$

The last term of which is positive.

$$\begin{array}{l} \therefore\ 17)\ 44\ (2 \\ \quad\ 34 \\ \hline \quad\ 10 = r. \end{array}$$

$\therefore\ 17 - 10 = 7$ least value of x.

The greatest value of

$$y = \frac{2000 - 147}{17} = 109.$$

Which being divided by 21, the co-efficient of x, the quotient is 5, which indicates that there are five other answers besides the one already found,* which may be obtained as follows.

From the greatest value of y, subtract continually 2 , the co-efficient of x, and add continually 17, the co-efficient of y to the least value of x.

Then $\begin{cases} x = 7 \mid 24 \mid 41 \mid 58 \mid 75 \mid 92 \\ y = 109 \mid 88 \mid 67 \mid 46 \mid 25 \mid 4 \end{cases}$

* When the greatest value of one letter is divided by the co-efficient of the other, and nothing remains, the number of answers is denoted by the quotient.

To find in the same, the least value of y first, we have

$$x = \frac{2000 - 17\,y}{21} = 95 - \frac{17\,y - 5}{21}.$$

$$\begin{array}{l} 17\,y - 5\,)\,21\,y \quad (\,1 \\ \qquad\qquad\; 17\,y - 5 \\ \hline \qquad\qquad\; 4\,y + 5\,)\,17\,y - 5\,(\,4 \\ \qquad\qquad\qquad\qquad\; 16\,y + 20 \\ \hline \qquad\qquad\qquad\qquad\quad y - 25 = \text{last remainder.} \end{array}$$

$$\therefore\; 21\,)\,25\,(\,1$$
$$21$$
$$4 = \text{least value of } y.$$

And the greatest value of

$$x = \frac{2000 - 68}{21} = 92, \text{ as before.}$$

2. Given $19\,y - 14\,x = 28$, to the value of x and y in whole numbers.

$$\text{Here } y = \frac{14\,x + 28}{19}.$$

And by dividing the numerator by 14, the quotient is $x + 2$, where the co-efficient of x is unity; therefore $x + 2$ may be considered as the *last remainder*.

$\therefore\; 19 - 2 = 17$ the least value of x, and the least value of

$$y = \frac{28 + 14 \times 17}{19} = 14,$$

the number of answers being evidently indefinite, because y may be increased *ad infinitum* by continually adding 19 to the value of x.

3. Given $20x + 4y + z = 400$,
And $x + y + z = 100$,
to find all the values of x, y, and z, in whole numbers.

Here, by subtracting the second from the first,

$$19x + 3y = 300.$$

$$\therefore x = \frac{300 - 3y}{19} = 15 - \frac{3y - 15}{19}.$$

And dividing the numerator by 3, we have $y - 5$ for the last remainder.

$\therefore 5 =$ the least value of y, and the greatest value of

$$x = \frac{300 - 15}{19} = 15, \text{ whence}$$

$x =$	15	12	9	6	3
$y =$	5	24	43	62	81
$z =$	80	64	48	32	16

4. Given $5x + 11y = 200$, to find all the values of x and y in whole numbers.

Ans. $\begin{cases} x = 7 \mid 18 \mid 29 \\ y = 15 \mid 10 \mid 5 \end{cases}$

5. Given $2x + 3y = 25$, to find all the values of x and y in whole numbers.

Ans. $\begin{cases} x = 2 \mid 5 \mid 8 \mid 11 \\ y = 7 \mid 5 \mid 3 \mid 1 \end{cases}$

6. Given $11x + 5y = 254$, to find all the values of x and y in whole numbers.

Ans. $\begin{cases} x = 4 \mid 9 \mid 14 \mid 19 \\ y = 42 \mid 31 \mid 20 \mid 9 \end{cases}$

7. Given $19y - 14x = 11$, to find the least values of x and y in whole numbers.

Ans. $x = 6$, $y = 5$.

8. Given $11x + 35y = 500$, to find all the values of x and y in whole numbers.

Ans. $x = 20$, $y = 8$ only.

9. Given $87x + 256y = 15410$, to find all the values of x and y in whole numbers.

Ans. $x = 30, y = 50$ only.

10. Exhibit all the different ways in which it is possible to pay £20. with half crowns and half guineas.

If $x =$ half crowns, and $y =$ half guineas, then

$$\left\{ \begin{array}{l|l|l|l|l|r|r} x = 13 & 34 & 55 & 76 & 97 & 118 & 139 \\ y = 35 & 30 & 25 & 20 & 15 & 10 & 5 \end{array} \right\}$$

11. A person distributed £1. 13*s*. among a number of poor men and women, giving the former 3*s*. 6*d*. and the latter 1*s*. 4*d*. each; how many were there?

Ans. 6 men, and 9 women.

12. How many ways is it possible to make 500, by the addition of the numbers 3 and 7?

Ans. 24 ways.

13. How many ways is it possible to pay £100. with seven-shilling pieces, and three shilling tokens?

Ans. 95 ways.

14. At a sale of farming stock, a person bought 100 animals for £100., giving £10. a-piece for oxen, £2. a-piece for sheep, and 10*s*. each for pigs; required the number of each.

Ans. oxen 4, sheep 8, and pigs 88.

CASE 2.

To find a number x such, that being divided by given divisors, shall leave given remainders; that is, that

$$\frac{x - a}{b}, \; \frac{x - c}{d}, \; \frac{x - e}{f}, \text{ \&c.}$$

may be whole numbers.

[illegible] then
[illegible] being
[illegible] will
[illegible] made of
[illegible] action.
[illegible] be in-
[illegible] tion.
[illegible] required.

[illegible] being i-
[illegible] by

[illegible] assuredly.

[illegible] whose counsel

$$\therefore 17p - 6)\,26p\ \ (1$$
$$\underline{17p - 6}$$
$$\begin{array}{r|l}
9p + 6 & 17p - 6\ \ (5 \\
\text{Or, } 3p + 2 & \underline{15p + 10}
\end{array}$$
$$\begin{array}{r|l}
2p - 16 & 3p + 2\ \ (2 \\
\text{Or,* } p - 8 & \underline{2p - 16} \\
 & p + 18 = \text{last.}
\end{array}$$

$$\therefore 26 - 18 = 8 = \text{least value of } p.$$

Whence $x = 17p + 7 = 17 \times 8 + 7 = 143$.

2. To find the least whole number, which being divided by 11, 19, and 29, shall leave the remainders 3, 5, and 10 respectively.

Let $x =$ the required number,

then $\frac{x - 3}{11}$, $\frac{x - 5}{19}$ and $\frac{x - 10}{29}$

are to be whole numbers;

put $\frac{x - 3}{11} = p$, then $x = 11p + 3$,

which value of x substituted in the second,

gives $\frac{11p - 2}{19} =$ a whole number.

In which the least value of $p = 14$.

$\therefore 19 - 14 = 5$, and $\frac{p + 5}{19} = r$.

$\therefore p = 19r - 5$.

And $x = 11(19r - 5) + 3 = 209r - 52$.

---

* When any quantity is common to both terms of the divisor, it may be divided by that quantity; and if by such division the co-efficient of the unknown letter be unity, as in the example, this quotient may be considered as the last.

RULE.

Put the first fraction $\frac{x - a}{b} = p$.

Then $x = b\,p + a$

which substituted for x in the second fraction,

gives $\frac{b\,p + a - c}{d}$.

In which find the least value of p by the last rule; then the difference between p and the denominator d being added to p, and the sum divided by d, the quotient will evidently be an unit, which put $= r$. Find the value of x in terms of r, and substitute it for x in the third fraction. Find the least value of r as before, to which add the difference between it and the denominator f, and put their sum, divided by f equal to s, and so on for any numbers required.

EXAMPLES.

1. To find the least whole number, which being divided by 17, shall leave a remainder of 7; but, being divided by 26, the remainder shall be 13.

Let $x =$ the required number,

then $\frac{x - 7}{17}$, and $\frac{x - 13}{26}$

are to be whole numbers;

put $\frac{x - 7}{17} = p$,

then $x = 17\,p + 7$.

Which substituted for x in the second,

gives $\frac{17\,p + 7 - 13}{26} = \frac{17\,p - 6}{26} =$ a whole number.

$$
\begin{array}{l}
\therefore 17p - 6)\,26p \quad (1 \\
\qquad\qquad\; \underline{17p - 6} \\
\qquad\qquad\;\; 9p + 6 \;|\; 17p - 6 \;(5 \\
\qquad \text{Or, } 3p + 2 \;|\; \underline{15p + 10} \\
\qquad\qquad\qquad\qquad 2p - 16 \;|\; 3p + 2 \;\;(2 \\
\qquad\qquad\quad \text{Or,* } \;\; p - \;\; 8 \;|\; \underline{2p - 16} \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{\underline{p + 18}} = \text{last.}
\end{array}
$$

$$\therefore 26 - 18 = 8 = \text{least value of } p.$$

Whence $x = 17\,p + 7 = 17 \times 8 + 7 = 143$.

2. To find the least whole number, which being divided by 11, 19, and 29, shall leave the remainders 3, 5, and 10 respectively.

Let $x =$ the required number,

then $\frac{x - 3}{11}$, $\frac{x - 5}{19}$ and $\frac{x - 10}{29}$

are to be whole numbers;

put $\frac{x - 3}{11} = p$, then $x = 11\,p + 3$,

which value of x substituted in the second,

gives $\frac{11\,p - 2}{19} =$ a whole number.

In which the least value of $p = 14$.

$\therefore$ $19 - 14 = 5$, and $\frac{p + 5}{19} = r$.

$\therefore$ $p = 19\,r - 5$.

And $x = 11\,(19\,r - 5) + 3 = 209\,r - 52$.

---

* When any quantity is common to both terms of the divisor, it may be divided by that quantity; and if by such division the co-efficient of the unknown letter be unity, as in the example, this quotient may be considered as the last.

Which substituted for x in the third, gives

$$\frac{209\,r - 62}{29} = 7\,r - 2 + \frac{6\,r - 4}{29}$$

$$\therefore \frac{6\,r - 4}{29} = \text{a whole number,}$$

in which the least value of

$$r = 20.$$

$$\therefore x = 209\,r - 52 = 209 \times 20 - 52 = 4128.$$

3. Find a whole number, such that when it is divided by 7 and 9, shall leave the remainders 5 and 6.

Ans. 33.

4. Find the least whole number, that being divided by 9 and 15, the remainder shall be 2 and 5. Ans. 20.

5. Find the least whole number, which being divided by 35 and 45, shall leave the remainder 2 and 27.

Ans. 72.

6. Find the least whole number, that being divided by 9, 19, and 29, the remainders shall be 0, 9, and 19.

Ans. 2745.

7. Find the least whole number, which to the divisors 30, 40, and 50, shall leave 7, 17, and 27 for remainders.

Ans. 577.

8. Find the least whole number, which to the divisors 3, 5, 7, 2, shall leave 2, 4, 6, 0 respectively for remainders.

Ans. 104.

DIOPHANTINE PROBLEMS.

The diophantine analysis derives its name from Diophantus, a celebrated Mathematician, of Alexandria, who flourished in the latter part of the third century, and who is the earliest writer on the subject, or indeed on Algebra; whose labours have been handed down to the present time. It relates chiefly to the finding of square and cube numbers;* or the finding such a value or values of the unknown letter, as the terms involving it shall be a square, cube, &c. Which forms are so multifarious and intricate, that no general rule can be laid down applicable to all cases; almost every question requiring a different mode of management, though each may be handled in a great variety of ways. The difficulty involved is not unfrequently such as to require, not only the purest Algebra, but the most refined reasoning, and the subtilist artifice.

The usual method however is, when the formula to be made a square, has one of its terms a square, to connect some other letter to the root of that square, by the sign + or —, and then make the given formula equal to its

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* 1. Any square number multiplied by a square number, the product is a square. And any cube number, multiplied by a cube number, the product is a cube.

2. Any square number, divided by a square number, the quotient is a square. And any cube number, divided by a cube number, the quotient is a cube.

3. Four times the square of *half* of any number, is equal to the square of the *whole*.

4. Any square number divided into two equal parts, the sum of the cubes of those parts is a square.

5. No square number can terminate with 2, 3, 7, or 8.

6. Any triangular number being multiplied by 8, and unity added, the result is a square.

7. The area of a rational right angled triangle cannot be a square.

square; whence the square term vanishes, and the unknown term becomes known. But the various methods will be the best understood from the following

EXAMPLES.

1. To find the value of x, such that $x^2 + a\,x$ shall be a square. *

Here one of the terms is x^2, assume therefore $x - m$ for the root of the required square.

$$\text{Then } x^2 + a\,x = (x - m)^2,$$
$$\text{Or, } x^2 + a\,x = x^2 - 2\,m\,x + m^2,$$
$$\text{Or, } 2\,m\,x + a\,x = m^2,$$
$$\therefore\ x = \frac{m^2}{2\,m + a}.$$

Where a is any *given* number, and m may be taken any number at pleasure. If we take $a = 1$ and $m = 2$, then $x = \frac{4}{5}$, and the required square is

$$\left(\frac{4}{5}\right)^2 + \frac{4}{5} = \frac{16}{25} + \frac{20}{25} = \frac{36}{25} = \left(\frac{6}{5}\right)^2$$

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* 1. In any formula to be rendered a square if the unknown quantity is in a simple form only, then any arbitrary square (m^2) may be assumed equal to the formula:

$$\text{as, } a\,x + b\,x = \square = m^2 \quad \therefore\ x = \frac{m^2}{a + b},$$
$$\text{or, } a\,x + b = \square = n^2 \quad \therefore\ x = \frac{n^2 - b}{a}, \text{ \&c.}$$

Also, $(a\,x + b) \times (c\,x + d) = \square$, may be assumed $= (c\,x + d)^2\,m^2$.

2. The formula $a\,x^2 + b\,x + c$, is always reducible to a square when $a\,x^2 + b\,x + c = 0$ has rational roots; for then it is equal to two rational factors, as $(m\,x + n) \times (p\,x + q) = \square$, which is solvible by the preceding note.

*** As the given formula contains the common factor x, we may assume it equal to any square, with x^2 for a factor.

Suppose $x^2 + a\,x = m^2\,x^2$

then $x + a = m^2\,x$ by $\div$ by x,

Or, $m^2\,x - x = a$.

$$\therefore x = \frac{a}{m^2 - 1}.$$

And by taking a and m as before, the required square is

$$\left(\frac{1}{3}\right)^2 + \frac{1}{3} = \frac{1}{9} + \frac{3}{9} = \frac{4}{9} = \left(\frac{2}{3}\right)^2.$$

2. To find the value of x, such that

$$x^2 - a\,x + b = \square$$

assume its root $= x - m$,

then $x^2 - a\,x + b = (x - m)^2$,

Or, $x^2 - a\,x + b = x^2 - 2\,m\,x + m^2$.

Or, $\quad - a\,x + b = -2\,m\,x + m^2$.

$\therefore \quad 2\,m\,x - a\,x = m^2 - b$.

$$\therefore x = \frac{m^2 - b}{2m - a},$$

Where a and b are *given*, and m may be taken at pleasure.

If $a = 1$, $b = 2$, and $m = 4$, then $x = 2$, and the required square $x^2 - a\,x + b = 4$.

3. To find the value of x, such that $x^2 + 5\,x - 3$ shall be a square.

Assume as before its root $= x - m$.

Then $x^2 + 5\,x - 3 = x^2 - 2\,m\,x + m^2$,

Or, $2\,m\,x + 5\,x = m^2 + 3$.

$$\therefore \quad x = \frac{m^2 + 3}{2m + 5}.$$

Where m may be any number whatever.

If $m = 1$, then $x = \frac{4}{7}$, and the required square is

$$\frac{16}{49} + \frac{20}{7} - 3 = \frac{156 - 147}{49} = \frac{9}{49}.$$

4. To find the value of x, such that $3x^2 - 7x + 9$ shall be a square.

Here x^2 having a prime co-efficient, and the last term being a square, assume therefore $3 - mx$ for its root.

Then $3x^2 - 7x + 9 = (3 - mx)^2$.

Or, $3x^2 - 7x + 9 = 9 - 6mx + m^2x^2$.

Or, $3x^2 - 7x = - 6mx + m^2x^2$.

Or, $3x - 7 = - 6m + m^2x$ by $\div$ by x.

$\therefore m^2x - 3x = 6m - 7$.

$$\therefore x = \frac{6m - 7}{m^2 - 3}.$$

Where m may be taken at pleasure.

If $m = 1$, then $x = \frac{1}{2}$, and the required square is $\frac{3}{4} - \frac{7}{2} + 9 = \frac{25}{4}$.

If $m = 2$, then $x = 5$, and the required square is 49.

5. To find the value of x, such that $2x^2 - 4$, shall be a square.

Here the formula, which contains not a rational square, is the same as

$$x^2 + (x^2 - 4) = x^2 + (x + 2) \cdot (x - 2) = \square.$$

And by assuming its root $= x + m(x - 2)$

$$x^2 + (x + 2) \cdot (x - 2) = \left\{x + m(x - 2)\right\}^2.$$

Or, $x^2 + (x + 2) \cdot (x - 2) = x^2 + 2mx(x - 2) + m^2(x - 2)^2$.

$\therefore$ by cancelling x^2 and dividing by $x - 2$,

$$x + 2 = 2\,m\,x + m^2\,x - 2\,m^2$$

$$\text{Or, } m^2\,x + 2\,m\,x - x = 2\,m^2 + 2.$$

$$\therefore \quad x = \frac{2\,m^2 + 2}{m^2 + 2\,m - 1},$$

Where m may be taken at pleasure.

If $m = 1$, then $x = 2$, and the required square is 4.

If $m = \frac{1}{2}$, then $x = 10$, and $2\,x^2 - 4 = 196 = 14^2$.

⁂ *Had $y + 2$ been assumed equal to x, then the given formula would have been transformed to

$$2\,y^2 + 8\,y + 4 = \square.$$

And by assuming its root $= 2 - m\,y$, we have

$$2\,y^2 + 8\,y + 4 = 4 - 4\,m\,y + m^2\,y^2.$$

Cancelling the 4, and dividing by y,

$$2\,y + 8 = m^2\,y - 4\,m.$$

$$\text{Or, } m^2\,y - 2\,y = 4\,m + 8$$

$$\therefore \quad y = \frac{4\,(m + 2)}{m^2 - 2}.$$

Where m may be taken at pleasure,

If $m = 2$, then $y = 8$, and $x = 10$ as before.

6. To find such a value of x, that shall render $2\,x^2 + 8$ a square.

Here the given formula is the same as

$$x^2 + 4 + x^2 + 4 = (x + 2)^2 + (x - 2) \,.\, (x - 2) = \square.$$

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* When the given formula has not a square term, that is, not a *rational* square, which is always to be understood, and can not readily be reduced to the form of the former solution; if a value of x can be found, by trial or otherwise, that will render the formula a square, then if that value be annexed to some new letter (with the sign $+$ or $-$) and substituted for x, the original formula will be transformed to one with a square term.

Assume its root therefore $= (x + 2) + m\,(x - 2)$.

$\therefore (x+2)^2 + (x-2).(x-2) = \left\{ (x+2) + m\,(x-2) \right\}^2$

Or, $(x + 2)^2 + (x - 2) . (x - 2) = (x + 2)^2 + 2\,m\,(x + 2) . (x - 2) + m^2\,(x - 2)^2$.

Cancelling $(x + 2)^2$ and dividing by $x - 2$.

$$x - 2 = 2\,m\,x + 4\,m + m^2\,x - 2\,m^2.$$

$$\therefore\ x = \frac{2\,m^2 - 4\,m - 2}{m^2 + 2\,m - 1}.$$

Where m may be taken at pleasure,

If $m = 1$, then $x = -2$, and the required square is 16; if $m = -3$, then $x = 14$, and the square is 400.

*** If we put $x = y + 2$, as in the last problem, the given formula is transformed to

$$2\,y^2 + 8\,y + 16 = \square = \text{say to } (4 - m\,y)^2,$$

from which $y = \dfrac{8\,(m + 1)}{m^2 - 2}$.

Where m may be taken at pleasure,

If $m = 2$, then $y = 12$, and $x = 14$ as before.

7. To find such a value of x, that shall render $5\,x^2 - 20\,x + 15$ a square.

Here, from $5\,x^2 - 20\,x + 15 = 0$, we readily find $x = 1$ or 3. (note 2, page 152.)

$$\therefore\ 5\,x^2 - 20\,x + 15 = 5\,(x - 1) . (x - 3) = \square.$$

Assume its root $= (x - 3)\,m$,

then $5\,(x - 1) . (x - 3) = (x - 3)^2\,m^2$.

Or, $5\,x - 5 = m^2\,x - 3\,m^2$ by $\div$ by $(x - 3)$.

Or, $m^2\,x - 5\,x = 3\,m^2 - 5$.

$$\therefore\ x = \frac{3\,m^2 - 5}{m^2 - 5}.$$

Where m may be taken at pleasure,

If $m = 1$, then $x = \frac{1}{2}$, and the square $= \frac{25}{4}$.

If $m = 2$, then $x = -7$, and the square $= 400$.

If $m = 3$, then $x = \frac{11}{2}$, and the square $= \frac{225}{4}$.

8. To find two numbers such that their *sum* and *difference* shall be both squares.

Let x and y represent the numbers.

Then $x + y = \square = m^2$.

And $x - y = \square = n^2$.

$\therefore$ by addition, and dividing by 2,

$$x = \tfrac{1}{2}(m^2 + n^2).$$

And by subtraction, and dividing by 2,

$$y = \tfrac{1}{2}(m^2 - n^2).$$

Where m and n may be taken at pleasure, m being greater than n.

If $n = 2$ and $m = 4$,

then $x = 10$ and $y = 6$.

⁂ Had $4\,m^2$ and $4\,n^2$ been assumed for the required squares, then we should have had

$$x = 2(m^2 + n^2)$$
$$y = 2(m^2 - n^2).$$

In which if $n = 1$ and $m = 2$,

then $x = 10$ and $y = 6$, as before.

9. To find three numbers (x, y, and z) such that the sum of every two of them, minus the third, shall be a square.

$$\text{Assume } x + y - z = \square = m^2$$
$$x + z - y = \square = n^2$$
$$\text{And } y + z - x = \square = p^2.$$

Then by addition, &c. we find

$$x = \tfrac{1}{2}(m^2 + n^2)$$
$$y = \tfrac{1}{2}(m^2 + p^2)$$
$$z = \tfrac{1}{2}(n^2 + p^2)$$

Where m, n, and p, may be taken at pleasure.

10. To find the value of x and y, such that

$$x^2 + y^2 = \square \text{*}$$

put $m^2 - n^2 = x$

and $2\,m\,n = y$.

Then $x^2 + y^2$ becomes equal to

$$(m^2 - n^2)^2 + (2\,m\,n)^2 = m^4 + 2\,m^2\,n^2 + n^4,$$

which is a square, its root being $m^2 + n^2$,

where m and n may be taken any numbers at pleasure, m being the greater.

11. To find the value of x and y, such that

$$x^2 - y^2 = \square.$$

Assume $x = m^2 + n^2$,

and $y = 2\,m\,n$,

then $(m^2 + n^2)^2 - (2\,m\,n)^2 = m^4 - 2\,m^2\,n^2 + n^4$,

which is a square, its root being $m^2 - n^2$,

where m and n may be taken at pleasure.

12. To find x and y, when $2\,x^2 - 2\,y^2 = \square$.

Assume $x = 2\,p^2 + q^2$,

and $y = 2\,p^2 - q^2$,

then $2\,(2\,p^2 + q^2)^2 - 2\,(2\,p^2 - q^2)^2 = 16\,p^2\,q^2$,

which is a square, its root being $4\,p\,q$,

where p and q may be taken at pleasure, provided $2\,p^2$ is greater than q^2.

If $p = 2$ and $q = 1$, then $x = 9$ and $y = 7$.

$\therefore 81 \times 2 - 49 \times 2 = 64 = 8^2$.

---

* If x and y represent the base and perpendicular of a right angled triangle, then by substituting as above, the base $= m^2 - n^2$ perpendicular $= 2\,m\,n$, and hypotenuse $= m^2 + n^2$.

If $p = 1$ and $q = 1$, then $x = 3$ and $y = 1$.

$$\therefore 2(x^2 - y^2) = 16 = 4^2.$$

*** This problem, as well as the preceding one, together with the general one,

$$n(x^2 - y^2) = \square = v^2,$$

may be readily and generally solved by the following method.

$n(x^2 - y^2) = v^2$, is the same as

$$n \cdot (x - y) \times (x + y) = v \times v.$$

multiplying both by m,

$$m\,n \cdot (x - y \times (x + y) = v \times m\,v,$$

assume $x + y = m\,v$,

then $m\,n\,(x - y) = v$,

which multiplied by m becomes

$$m^2\,n\,x - m^2\,n\,y = m\,v.$$

$\therefore$ by equality $m^2\,n\,x - m^2\,n\,y = x + y$.

$$\therefore x = \frac{(m^2\,n + 1)\,y}{m^2\,n - 1}.$$

Where, if $y = m^2\,n - 1$,

then $x = m^2\,n + 1$, and $v = 2\,m\,n$.

If $n = 1$, it solves prob. 11, if $n = 2$, it solves prob. 12, m being arbitrary.

13. To find integral values for x, y, and z, so that

$$x^2 + y^2 + z^2 = \square,$$

assume $x = 4\,p^2 + 4\,q^2$,

$y = 6\,p\,q$,

and $z = 3\,p^2 - 3\,q^2$,

then $(4\,p^2 + 4\,q^2)^2 + 36\,p^2\,q^2 + (3\,p^2 - 3\,q^2)^2 = 25\,(p^4 + 2\,p^2\,q^2 + q^4)$,

which is a square, its root being $5\,(p^2 + q^2)$,

where p and q may be taken at pleasure, if p be greater than q.

⁂ The result would have been precisely the same, had we made

$$x = 4p^2 - 4q^2,\ y = 8pq, \text{ and } z = 3p^2 + 3q^2.$$

14. To find integral values for x, y, and z.

When $2x^2 + 2y^2 + 2z^2 = \square$,

assume $x = p^2 + q^2$,

$y = 2pq$,

and $z = p^2 - q^2$.

Then $2(p^2 + q^2)^2 + 2(2pq)^2 + 2(p^2 - q^2)^2 = 4(p^4 + 2p^2q^2 + q^4)$.

Which is a square, its root being $2(p^2 + q^2)$.

If $q = 1$ and $p = 2$, then $x = 5$, $y = 4$, and $z = 3$.

$\therefore 2(25 + 16 + 9) = 100$, the required square.

15. To find integral values for x, y, z, and w, such that $x^2 + y^2 + z^2 + w^2 = \square$.

Assume $x = 8p^2 - 4q^2$,

$y = 6p^2 - 3q^2$,

$z = 14pq$,

and $w = 2pq$.

Then $(8p^2 - 4q^2)^2 + (6p^2 - 3q^2)^2 + (14pq)^2 + (2pq)^2 = 25(4p^4 + 4p^2q^2 + q^4)$,

which is a square, its root being

$5(2p^2 + q^2)$,

where p and q may be taken at pleasure.

If p and q each $= 1$,

then $x = 4$, $y = 3$, $z = 14$, and $w = 2$,

and the required square is

$$4^2 + 3^2 + 14^2 + 2^2 = 225 = 15^2.$$

16. To find two square numbers, the *difference* of which shall be any *given number*.

Resolve the given difference into two *unequal* factors; and put x for the side of the less square, and x plus the less factor, for the side of the greater square.

Suppose it is required to find two squares, the difference of which is 21.

The factors of 21 are 3 and 7.

$$\therefore (x+3)^2 = x^2 + 6x + 9 = \text{greater } \square,$$

$$\text{and} \quad x^2 = \text{less } \square.$$

$$\therefore \quad 6x + 9 = 21 \text{ by the quest.}$$

$$\text{or,} \quad x = 2.$$

$$\therefore (3+2)^2 = 5^2 = 25 \text{ the greater } \square.$$

$$\text{and} \quad 2^2 = 4 \text{ the less } \square.$$

17. To find three squares, x^2, y^2, and z^2, when they are in arithmetical progression.

$$\text{Here } x^2 + z^2 = 2y^2 \text{ (theo. 1, page 42,)}$$

$$\text{put } m + n = x, \text{ and } m - n = z.$$

$$\text{then } (m+n)^2 + (m-n)^2 = 2y^2,$$

$$\text{or, } m^2 + n^2 = y^2.$$

Put now $m = p^2 - q^2$, and $n = 2pq$ (prob. 10.)

$$\text{then } (p^2 - q^2)^2 + (2pq)^2 = y^2.$$

$$\text{or, } y^2 = p^4 + 2p^2q^2 + q^4.$$

$$\left.\begin{array}{r} \therefore y = p^2 + q^2 \\ \text{whence } x = p^2 + 2pq - q^2 \\ \text{and } z = p^2 - 2pq - q^2 \end{array}\right\} \text{the roots of the required squares.}$$

⁂ Had we put $x = mn - y$, and $z = n - y$, then we should have found

$$y = \frac{n(m^2+1)}{2(m+1)}, \quad x = \frac{n(m^2+2m-1)}{2(m+1)}$$

$$\text{and } z = \frac{n\,(1 + 2\,m - m^2)}{2\,(m + 1)}.$$

which will be integral when $n = 2\,(m + 1)$.

Now, in the former of these, if we take

$p = 2$ and $q = 1$, then $m = 3$, and $n = 4$.

$\therefore\ x = 7,\ y = 5$, and $z = 1$.

$\therefore$ the required squares are 49, 25, and 1.

And by taking $m = 2$ in the latter, we have the same result.

*** Had we assumed x^2, $25\,x^2$, and $49\,x^2$ for the required numbers, the assumption would have solved the problem, because they are squares in arithmetical progression, their common difference being $24\,x^2$. Where x may be any number whatever; if $x = 1$, then the numbers are 1, 25, and 49 as before.

18. To find a number (x) such that $x^2 - x$, and $x^2 + x$, shall be both square numbers.

Assume $x^2 - x = (x - p)^2 = x^2 - 2\,p\,x + p^2$,

then $2\,p\,x - x = p^2$.

$$\therefore\ x = \frac{p^2}{2\,p - 1}.$$

This value substituted for x in $x^2 + n = \square$, it becomes

$$\frac{p^4}{(2\,p - 1)^2} + \frac{p^2}{2\,p - 1} = \square.$$

Or, $\dfrac{p^2}{(2p-1)^2} + \dfrac{1}{2p-1} = \square$ by $\div$ by p^2 (note 2, p. 151)

Or, $p^2 + 2p - 1 = \square$ by $\times$ by $(2p - 1)^2$ (note 1, p. 151.)

Assume it $= (p - q)^2$.

$\therefore\ p^2 + 2\,p - 1 = p^2 - 2\,p\,q + q^2$.

$$\text{Whence } p = \frac{q^2 + 1}{2\,(q + 1)}.$$

When q may be taken at pleasure.

If $q = 2$, then $p = \frac{5}{6}$, and $x = \frac{25}{24}$.

$\therefore x^2 - x = (\frac{25}{24})^2 - \frac{25}{24} = \frac{25}{576} = (\frac{5}{24})^2$.

and $x^2 + x = (\frac{25}{24})^2 + \frac{25}{24} = \frac{1225}{576} = (\frac{35}{24})^2$.

⁂ Had we put $x^2 - x = r^2 x^2$,

and $x^2 + x = s^2 x^2$,

then by dividing each by x,

from the former $x = \dfrac{1}{1 - r^2}$;

and from the latter $x = \dfrac{1}{s^2 - 1}$.

$\therefore \dfrac{1}{1 - r^2} = \dfrac{1}{s^2 - 1}$; from which

$$2 - r^2 = s^2.$$

$$\therefore 2 - r^2 = \square$$

Now this is obviously a square when $r = 1$,

put therefore $r = p - 1$ (see note prob. 5.)

then $2 - r^2 = 2 - (p - 1)^2 = 1 + 2p - p^2 = \square$.

which assume $= (1 - pq)^2$.

$$\therefore 1 + 2p - p^2 = 1 - 2pq + p^2 q^2.$$

Or, $2 - p = -2q + pq^2$.

$$\therefore p = \frac{2(q + 1)}{q^2 + 1}.$$

Where q may be taken at pleasure.

If $q = 2$, then $p = \frac{6}{5}$, $r = \frac{1}{5}$, $s = \frac{7}{5}$, and $x = \frac{25}{24}$, as before

19. To find two numbers (x and y), such that their product, added to the square of each, shall be squares.

Here $x^2 + xy = \square$.

and $y^2 + xy = \square$.

Put $m^2 = x$, and $n^2 = y$,

and the above formula becomes

$$m^4 + m^2 n^2 = m^2 (m^2 + n^2) = \square.$$

$$n^4 + m^2 n^2 = n^2 (m^2 + n^2) = \square.$$

Which will evidently be squares when

$$m^2 + n^2 = \text{a square.}$$

Now this solved after the manner of example 10,

$$m = r^2 - s^2.$$

$$\text{and } n = 2\,r\,s.$$

$$\therefore x = (r^2 - s^2)^2$$

$$\text{and } y = (2\,r\,s)^2.$$

Where r and s may be taken at pleasure.

If $r = 2$, and $s = 1$, then $x = 9$, and $y = 16$.

But in order to have x and y nonquadrate numbers, assume $x = p\,(r^2 - s^2)^2$, and $y = p\,(2\,r\,s)^2$; p, as well as r and s, being any number whatever.

If $p = 2$, and r and s as before, then $x = 18$ and $y = 32$.

20. To find x, y, and z, such that

$$(x + y)^2 + (x - y)^2 + 2\,z^2 = \square.$$

$$(x + z)^2 + (x - z)^2 + 2\,y^2 = \square.$$

$$(y + z)^2 + (y - z)^2 + 2\,x^2 = \square.$$

$$\text{and } \quad x + y + z = \square.$$

Here the three first formula are each equal to

$$2\,(x^2 + y^2 + z^2) = \text{a square,}$$

which when $x = m^2 + n^2$, $y = 2\,m\,n$ and $z = m^2 - n^2$, becomes $4\,(m^2 + n^2)^2$ (see prob. 14.)

Therefore we have only to make their sum

$$(m^2 + n^2) + 2\,m\,n + (m^2 - n^2) = 2\,m^2 + 2\,m\,n = \square.$$

put $m = n + p$, and it becomes

$$4\,n^2 + 6\,n\,p + 2\,p^2 = \text{a square.}$$

which assume $= (2\,n - p\,q)^2$.

$$\therefore 4\,n^2 + 6\,n\,p + 2\,p^2 = 4\,n^2 - 4\,n\,p\,q + p^2\,q^2,$$

cancelling $4\,n^2$ and $\div$ by p, &c.

$$p q^2 - 2 p = 4 q n + 6 n.$$

$$\therefore p = \frac{(4 q + 6) n}{q^2 - 2}.$$

In order to have whole numbers take $n = q^2 - 2$, then $p = 4 q + 6$, where q may be taken at pleasure.

If $q = 2$, then $n = 2, p = 14$, and $m = 16$.

Whence $x = 360$, $y = 64$ and $z = 252$.

21. To divide a *given* square (a^2) into two other squares.

Let x^2 and y^2 represent the required squares.

Then $x^2 + y^2 = a^2$;

or, $a^2 - y^2 = x^2$.

Assume $a + y = \frac{p x}{q}$,

and $a - y = \frac{q x}{p}$.

Then by addition, $2 a = \frac{p x}{q} + \frac{q x}{p}$,

from which $x = \frac{2 a p q}{p^2 + q^2}$.

And by subtraction, $2 y = \frac{p x}{q} - \frac{q x}{p}$.

$$\therefore y = \frac{(p^2 - q^2) x}{2 p q},$$

in which if we substitute the value of x,

$$y = \frac{(p^2 - q^2) a}{p^2 + q^2},$$

where p and q may be taken at pleasure.

*** If a be the sum of two squares, p and q may be so taken that $p^2 + q^2 = a$, or any factor of a, in which case x and y will be integral, and as many integral values may be found for x and y as there are different ways of resolving a into the sum of two squares, or any of its factors; in illustration of which, let us resolve 65^2 into two other square numbers.

$$\text{Then } x = \frac{2\,p\,q \times 65}{p^2 + q^2}, \text{ and } y = \frac{(p^2 - q^2)\,65}{p^2 + q^2}.$$

And since $65 = 8^2 + 1^2 = 7^2 + 4^2$, we may

$$\left.\begin{array}{l}\text{take } p = 8,\\ \text{and } q = 1;\end{array}\right\}\text{ then }\left\{\begin{array}{l}x = 16,\\ y = 63;\end{array}\right.$$

$$\left.\begin{array}{l}\text{or, } p = 7,\\ \text{and } q = 4;\end{array}\right\}\text{ then }\left\{\begin{array}{l}x = 56,\\ y = 33.\end{array}\right.$$

$$\left.\begin{array}{l}\text{Also, since } 13 = 3^2 + 2^2,\\ \text{and } 5 = 2^2 + 1^2,\end{array}\right\}\text{are factors of 65,}$$

$$\left.\begin{array}{r}\text{we may take } p = 3,\\ \text{and } q = 2;\end{array}\right\}\text{ then }\left\{\begin{array}{l}x = 60,\\ y = 25;\end{array}\right.$$

$$\left.\begin{array}{r}\text{or, } p = 2,\\ \text{and } q = 1;\end{array}\right\}\text{ then }\left\{\begin{array}{l}x = 52,\\ y = 39.\end{array}\right.$$

$$\therefore\ 65^2 = 16^2 + 63^2 = 56^2 + 33^2 = 60^2 + 25^2 = 52^2 + 39^2,$$

which are the only integral solutions the question admits of.

22. To divide a given square (a^2) into three other squares.

Let x^2, y^2, and z^2 represent the required squares.

$$\text{Then } x^2 + y^2 + z^2 = a^2;$$

$$\text{or, } y^2 + z^2 = a^2 - x^2.$$

Putting $y = (a - x)\,p$ and $z = (a - x)\,q$, it

$$\text{becomes } (a - x)^2\,p^2 + (a - x)^2\,q^2 = a^2 - x^2;$$

$$\text{or, } (a - x)\,p^2 + (a - x)\,q^2 = a + x \text{ by } \div\ a - x;$$

$$\text{or, } x\,(p^2 + q^2 + 1) = a\,(p^2 + q^2 - 1).$$

$$\therefore\ x = \frac{a\,(p^2 + q^2 - 1)}{p^2 + q^2 + 1},$$

whence $y = \dfrac{2ap}{p^2 + q^2 + 1}$ and $z = \dfrac{2aq}{p^2 + q^2 + 1}$,

the roots of the required squares, where p and q may be taken at pleasure.

Scholium 1. Having divided a given square into two squares, (see last prob.) also into three squares; and as each of these squares is divisible by a similar process into two squares, also into three squares, therefore any square number may be divided into any number of squares.

2. Any cube number also can be divided into any number of cubes, greater than two, for

$$(\tfrac{3}{6})^3 + (\tfrac{4}{6})^3 + (\tfrac{5}{6})^3 = 1^3,$$

$$\text{and } (\tfrac{4}{18})^3 + (\tfrac{7}{18})^3 + (\tfrac{8}{18})^3 + (\tfrac{17}{18})^3 = 1^3;$$

therefore, multiplying each by n^3, we have three cubes, the sum of which is n^3; also, four cubes, the sum of which is n^3; and as any one of these cubes can, in a similar way, be divided into three cubes, also into four cubes, therefore any cube number may be divided into any number of cubes greater than two.

23. To find two numbers, such that if the cube of their difference be subtracted from each, the remainder shall be squares.

Putting x = the greater number, and y = the less;

$$\text{then } x - (x - y)^3 = \square = m^2.$$

$$\text{and } y - (x - y)^3 = \square = n^2.$$

$$\therefore x - y = m^2 - n^2 \text{ by subtraction,}$$

which substituted for $x - y$, we get

$$x = (m^2 - n^2)^3 + m^2.$$

$$\text{and } y = (m^2 - n^2)^3 + n^2.$$

Where m and n may be taken at pleasure.

If $m = 2$ and $n = 1$, then $x = 31$ and $y = 28$.

24. To find two numbers, such that their sum and difference shall be both squares; also if unity be subtracted from each, the remainders shall be squares.

Representing the numbers by x and y, we have to find

$$x + y = \square = 4\,m^2.$$
$$x - y = \square = 4\,n^2.$$
$$x - 1 = \square.$$
$$y - 1 = \square.$$

From the two former, (see prob. 8.)

$$x = 2\,(m^2 + n^2).$$
$$y = 2\,(m^2 - n^2).$$

which substituted in the two latter, they become

$$2\,m^2 + 2\,n^2 - 1 = \text{a square.}$$
and $2\,m^2 - 2\,n^2 - 1 =$ a square.

putting $m = n + 1$, the former becomes

$4\,n^2 + 4\,n + 1$, which is a square,

and the latter $4\,n + 1 = \square = p^2$.

$$\therefore\ n = \frac{p^2 - 1}{4}.$$

Where p may be taken at pleasure greater than one; if it be an *odd* number, then x and y will be integral.

If $p = 3$, then $n = 2$ and $m = 3$.

$\therefore\ x = 26$ and $y = 10$.

If $p = 5$, then $n = 6$ and $m = 7$.

$\therefore\ x = 170$ and $y = 26$.

25. To find x and y in whole numbers, such that

$$x^2 + y^2 - 1 = \square = m^2.$$
and $x^2 - y^2 - 1 = \square = n^2$

by addition, &c. $x^2 = \dfrac{m^2 + n^2 + 2}{2}.$

and by subtraction, &c. $y^2 = \frac{m^2 - n^2}{2}$.

multiplying each by 4.

then $2\,m^2 + 2\,n^2 + 4 =$ a square.

and $2\,m^2 - 2\,n^2 =$ a square.

By assuming $m = 2\,p^2 + q^2$, and $n = 2\,p^2 - q^2$, (see prob. 12.) the latter becomes $16\,p^2\,q^2$, a square, and the former

$$16\,p^4 + 4\,q^4 + 4 = \text{a square.}$$

$$\text{or,}\quad 4\,p^4 + q^4 + 1 = \text{a square.}$$

Which, by putting $p = 2\,r^2$ and $q = 2\,r$, becomes

$$64\,r^8 + 16\,r^4 + 1,$$

which is a square, its root being

$$8\,r^4 + 1.$$

Whence $m = 8\,r^4 + 4\,r^2$, and $n = 8\,r^4 - 4\,r^2$.

$\therefore\ x = 8\,r^4 + 1$, and $y = 8\,r^3$.

Where r may be taken at pleasure.

If $r = 1$, then $x = 9$, and $y = 8$.

If $r = 2$, then $x = 129$, and $y = 64$.

26. To find x and y, such that

$$x + y = \text{a square.}$$

$$x^2 + y = \text{a square.}$$

$$y^2 + x = \text{a square.}$$

$$\text{and}\quad x^2 + y^2 = \text{a square.}$$

assuming $x = \frac{p}{4\,(p + q)}$ and $y = \frac{q}{4\,(p + q)}$.

answers the three first conditions, whence it only remains to make the last a square, or

$$\frac{p^2}{16\,(p + q)^2} + \frac{q^2}{16\,(p + q)^2} = \text{a square.}$$

or, multiplying by the denominator,

$p^2 + q^2 =$ a square, which is prob. 10.

⁂ Had we assumed

$$x = \frac{p(p-2)}{4n^2(p^2-2)}, \text{ and } y = \frac{p-1}{2n^2(p^2-2)},$$

then, not only the sum, and sum of the squares, would have been squares, but the square of n times the one plus the other, squares also.

27. To find x, y, and z, in whole numbers, such that

$$2(x + y + z) = \text{a square.}$$
$$2(x^2 + y^2 + z^2) = \text{a square.}$$
$$2(x^4 + y^4) - z^4 = \text{a square.}$$
$$2(x^4 + z^4) - y^4 = \text{a square.}$$
$$\text{and } 2(y^4 + z^4) - x^4 = \text{a square.}$$

by assuming $x = m^2 + n^2$.

$y = m^2 - n^2$.

and $z = 2mn$.

then $2(x^2 + y^2 + z^2) = 4(m^2 + n^2)^2$.

$2(x^4 + y^4) - z^4 = 4(m^4 + n^4)^2$.

$2(x^4 + z^4) - y^4 = (m^4 + 6m^2n^2 + n^4)^2$.

$2(y^4 + z^4) - x^4 = (m^4 - 6m^2n^2 + n^4)^2$.

Whence we have only to make

$2(x + y + z) = 4m^2 + 4mn =$ a square.

or, $m^2 + mn =$ a square.

Where, in order to have positive numbers, m must be greater than n.

Assume therefore $m = n + p$, and it becomes

$p^2 + 3np + 2n^2 = \square =$ say to $(p - nq)^2$.

from which $p = \dfrac{n(q^2-2)}{2q+3}$.

Now in order to have whole numbers,

take $n = 2q + 3$, then $p = q^2 - 2$.

Where q may be taken at pleasure, greater than unity.

If $q = 2$, then $p = 2$, $n = 7$, and $m = 9$.

$\therefore x = 130, y = 32$, and $z = 126$.

28. To find three numbers, such that their sum added to the square of each, the three results shall be squares.

Let $p\,x$, $q\,x$, and $r\,x$, represent the required numbers, and $s\,x^2$ their sum.

$$\text{Then } p\,x + q\,x + r\,x = s\,x^2.$$

$$\therefore x = \frac{p+q+r}{s}.$$

$$\left.\begin{array}{r} \text{also } p^2 x^2 + s\,x^2 = \square \\ q^2 x^2 + s\,x^2 = \square \\ \text{and } r^2 x^2 + s\,x^2 = \square \end{array}\right\} \text{ or } \left\{\begin{array}{l} p^2 + s = \text{a square.} \\ q^2 + s = \text{a square.} \\ r^2 + s = \text{a square.} \end{array}\right.$$

$$\text{Assume } p^2 + s = (p - g)^2.$$

$$q^2 + s = (q - h)^2.$$

$$\text{and } r^2 + s = (r - m)^2.$$

Whence by reduction,

$$p = \frac{g^2 - s}{2g},\ q = \frac{h^2 - s}{2h}, \text{ and } r = \frac{m^2 - s}{2m}.$$

$$\therefore x = \left(\frac{g^2 - s}{2g} + \frac{h^2 - s}{2h} + \frac{m^2 - s}{2m}\right) \times \frac{1}{s}.$$

Where g, h, m, and s, may be taken at pleasure, s being less than g^2, h^2, and m^2.

If $s = 1, g = 2, h = 3$, and $m = 4$.

then $p = \frac{3}{4}$, $q = \frac{4}{3}$, and $r = \frac{15}{8}$.

$\therefore x = \frac{95}{24}$, whence $p\,x = \frac{95}{32}$, $q\,x = \frac{95}{18}$, and $r\,x = \frac{475}{64}$.

From which it is obvious that any number of numbers may be found, having similar properties.

29. To find three numbers in arithmetical progression, such that their sum shall be a square number, and if unity be added to each number, the three sums thence arising shall be square numbers also.

Let $x^2 - 1$, $y^2 - 1$, and $z^2 - 1$, represent the required numbers, then three of the conditions are answered, as each is a square when unity is added. Whence we have only to make their sums $x^2 + y^2 + z^2 - 3 = \square$, and $x^2 + z^2 = 2y^2$.

Putting $x = mn - y$, and $z = n - y$, then from the latter we find

$$x = \frac{n(m^2 + 2m - 1)}{2(m+1)},\ y = \frac{n(m^2+1)}{2(m+1)},$$

$$\text{and } z = \frac{n(1 + 2m - m^2)}{2(m+1)}.$$

or, by taking $n = 2(m+1)$

$$x = m^2 + 2m - 1, y = m^2 + 1, \text{and } z = 1 + 2m - m^2.$$

which values substituted for x, y, and z,

in $x^2 + y^2 + z^2 - 3 =$ a square, it becomes

$$(m^2 + 2m - 1)^2 + (m^2 + 1)^2 + (1 + 2m - m^2)^2$$
$$- 3 = 3m^4 + 6m^2 = \text{a square}.$$

or, $3m^2 + 6 =$ a square.

which is obviously a square when $m = 1$.

assume therefore $m = p + 1$, and it becomes

$$3p^2 + 6p + 9 = \square = \text{say to } (3 - pq)^2.$$

$$\text{from which } p = \frac{6(q+1)}{q^2 - 3}.$$

If $q = 3$, then $p = 4$, $m = 5$, $x = 34$, $y = 26$, and $z = -14$.

∴ the numbers are 1155, 675, and 195.

Schol. In $3m^2 + 6 = \square$, m may be taken any number in the series 1, 5, 19, 71, &c. which may be continued, by multiplying the last term by four, and subtracting the preceding term from it, the remainder is the next term of the series.

30. To divide 4 into four positive parts, such that unity being added to the square of each part, the several sums shall be squares.

assume $\frac{a^2 - b^2}{2ab}$, $\frac{2b^2 - \frac{a^2}{2}}{2ab}$, $\frac{3b^2 - \frac{a^2}{3}}{2ab}$, and $\frac{6b^2 - \frac{a^2}{6}}{2ab}$

for the four parts into which 4 is to be divided.

Then $\left(\frac{a^2 - b^2}{2ab}\right)^2 + 1 = \frac{a^4 + 2a^2b^2 + b^4}{4a^2b^2}$.

$$\left(\frac{2b^2 - \frac{a^2}{2}}{2ab}\right)^2 + 1 = \frac{4b^4 + 2a^2b^2 + \frac{a^4}{4}}{4a^2b^2}.$$

$$\left(\frac{3b^2 - \frac{a^2}{3}}{2ab}\right)^2 + 1 = \frac{9b^4 + 2a^2b^2 + \frac{a^4}{9}}{4a^2b^2}.$$

and $\left(\frac{6b^2 - \frac{a^2}{6}}{2ab}\right)^2 + 1 = \frac{36b^4 + 2a^2b^2 + \frac{a^4}{36}}{4a^2b^2}$.

are all squares, whence it only remains to make the sum of the parts equal to 4, or

$$\frac{a^2 - b^2}{2ab} + \frac{2b^2 - \frac{a^2}{2}}{2ab} + \frac{3b^2 - \frac{a^2}{3}}{2ab} + \frac{6b^2 - \frac{a^2}{6}}{2ab} = 4.$$

$$\text{Or, } \frac{10b^2}{2ab} = 4. \quad \text{or, } \frac{5b}{a} = 4.$$

$$\therefore b = \frac{4a}{5}.$$

Where a may be taken at pleasure.

If $a = 5$, then $b = 4$, and the required parts are

$\frac{9}{40}$, $\frac{39}{80}$, $\frac{119}{120}$, and $\frac{551}{240}$.

31. To find a series of numbers in arithmetical progression, such that the sum of *any* number of terms shall be a square.

Representing the first term by f, the common difference by d, and the number of terms by n, then the series is $f + (f + d) + (f + 2\,d) +$ &c. the sum of which to n terms is (theo. 4, page 43)

$$(2\,f + d\,n - d) \times \frac{n}{2} = \square = m^2.$$

$$\therefore\ d = \frac{2\,(m^2 - f\,n)}{n^2 - n}.$$

Where the three indeterminates m, n, and f, may be taken at pleasure.

If $f = 1$, $n = 2$, and $m = 3$, then $d = 7$ and $1 + (1 + 7) = 9$.

If $f = 2$, $n = 3$, and $m = 4$, then $d = 3\frac{1}{3}$, and

$$2 + (2 + 3\tfrac{1}{3}) + (2 + 6\tfrac{2}{3}) = 16.$$

32. To find a series of numbers in geometrical progression, such that if any one of them be added to its adjacent number, whether preceding or following, the sum shall be a square.

Let x, $r\,x$, $r^2\,x$, &c. represent the required series, then $x + r\,x = \square$, and $r\,x + r^2\,x = (x + r\,x)\,r = \square$, which will be a square when divided by the former.

$\therefore\ r = \square$, put $r = p^2$ and assume $x + p^2\,x = m^2$.

$$\text{Then } x = \frac{m^2}{p^2 + 1}.$$

Where m and p may be taken at pleasure.

If $p = 2$, and $m = 5$, then $x = 5$,

and the required series is 5, 20, 80, 320, &c.

33. To find three numbers, such that the square of the sum of every two, minus the square of the third, shall be a cube, and that the sum of all three shall be a cube also.

Let x, y, and z represent the required numbers, then

$$(x + y)^2 - z^2 = \text{a cube.}$$
$$(x + z)^2 - y^2 = \text{a cube.}$$
$$(y + z)^2 - x^2 = \text{a cube.}$$
$$\text{and } x + y + z = \text{a cube.}$$

And by resolving the three first into their factors, we have

$$(x + y + z) \times (x + y - z) = \text{a cube.}$$
$$(x + y + z) \times (x + z - y) = \text{a cube.}$$
$$\text{and } (x + y + z) \times (y + z - x) = \text{a cube.}$$

Now these will evidently be cubes when their factors are, assume therefore

$$x + y - z = m^3.$$
$$x + z - y = n^3.$$
$$\text{and } y + z - x = p^3.$$

from which we readily find

$$x = \tfrac{1}{2}(m^3 + n^3).$$
$$y = \tfrac{1}{2}(m^3 + p^3).$$
$$\text{and } z = \tfrac{1}{2}(n^3 + p^3).$$

and their sum $x + y + z = m^3 + n^3 + p^3 = \text{a cube}$. put therefore $m = 3r$, $n = 4r$, and $p = 5r$. (schol. 2, p. 167) then $m^3 + n^3 + p^3 = 27r^3 + 64r^3 + 125r^3 = 216r^3 = (6r)^3$.

Where r may be taken any number whatever,

take $r = 1, 2, 3, 4$, &c. and

$$3^3 + 4^3 + 5^3 = 6^3.$$
$$6^3 + 8^3 + 10^3 = 12^3.$$
$$9^3 + 12^3 + 15^3 = 18^3.$$
$$\&c. \quad \&c. = \&c.$$

When $r = 2$, then $x = 364$, $y = 608$, and $z = 756$.

34. To find four numbers, such that their sum shall be a square, and the difference of every two of them a square also.

Let x, y, z, and w, represent the required numbers, then we have to find

$$x+y+z+w=\square=s^2.$$
$$x-y=\square=a^2, x-z=\square=b^2, x-w=\square=c^2.$$
$$y-z=\square=d^2, y-w=\square=e^2, z-w=\square=f^2.$$

from the last six, we readily find

$$y-z=d^2=b^2-a^2.$$
$$y-w=e^2=c^2-a^2.$$
$$z-w=f^2=c^2-b^2.$$

Now from the first equation,

$$x=s^2-y-z-w;$$

and from the second,

$$x=a^2+y.$$
$$\therefore\ a^2+y=s^2-y-z-w;$$
or, $a^2+2y=s^2-z-w.$

But $\quad y=b^2-a^2+z$, from the 7th.

$\therefore\ a^2+3y=s^2+b^2-a^2-w$ by add.

Or, $3y+w=s^2+b^2-2a^2.$

But $\quad y-w=c^2-a^2.$

$$\therefore\quad 4y=s^2+b^2+c^2-3a^2.$$

$$\text{Whence } y=\frac{s^2+b^2+c^2-3a^2}{4}.$$

$$\therefore\ z=\frac{s^2+a^2+c^2-3b^2}{4}, w=\frac{s^2+a^2+b^2-3c^2}{4},$$

$$\text{and } x=\frac{s^2+a^2+b^2+c^2}{4},$$

whence we have only to find

$$b^2 - a^2 = d^2,$$
$$c^2 - a^2 = e^2,$$
$$\text{and } c^2 - b^2 = f^2,$$

that is, to find three squares, the difference of every two of which is a square. The least numbers answering which, is $a^2 = 23409$, $b^2 = 34225$, and $c^2 = 485809$ (see Euler Algebra) and by using which

$$x = \frac{543443 + s^2}{4}, \; y = \frac{449807 + s^2}{4}.$$

$$z = \frac{406543 + s^2}{4}, \text{ and } w = \frac{s^2 - 1399793}{4}.$$

Now in order that w may be positive, s must not be taken less than 1184, but may be taken any greater number.

If $s = 1184$, then

$$x = \frac{1945299}{4}, \; y = \frac{1851663}{4}, \; z = \frac{1808399}{4},$$

$$\text{and } w = \frac{2063}{4}.$$

Or, if each be multiplied by 4, then 1945299, 1851663, 1808399, and 2063, are four integral numbers, which answer, and others, may be had at pleasure.

35. To find two triangular numbers, such that their sum and difference shall be triangular numbers.

$$\text{Let } \frac{m^2 + m}{2} \text{ and } \frac{n^2 + n}{2},$$

represent the required trigonal numbers, then by the question.

$$\frac{m^2 + m + n^2 + n}{2} \text{ and } \frac{(m^2 + m) - (n^2 + n)}{2},$$

are to be trigonal numbers.

$$\left.\begin{array}{l}\text{Or, } 4\,m\,(m+1)+4\,n\,(n+1)+1=\square \\ \text{and } 4\,m\,(m+1)-4\,n\,(n+1)+1=\square\end{array}\right\}\text{ by note 6.}$$

which are the same as

$$(2\,m+1)^2+(2\,n+1)^2-1=\text{a square,}$$

$$(2\,m+1)^2-(2\,n+1)^2+1=\text{a square,}$$

put $2\,m+1=p$, and $2\,n+1=q$.

then $p^2+q^2-1=$ a square,

and $p^2-q^2+1=$ a square.

Let $p^2+q^2-1=g^2$,

then $q^2-1=g^2-p^2$,

or, $(q+1)\cdot(q-1)\,r=(g+p)\cdot(g-p)\,r$.

take $q+1=r\,(g-p)$,

then $(q-1)\,r=g+p$.

Whence $g=(q-1)\,r-p=\dfrac{q+r\,p+1}{r}$.

from which $q=\dfrac{r^2+2\,r\,p+1}{r^2-1}$ ...... (1)

Again, put $p^2-q^2+1=h^2$,

then $(p+q)\cdot(p-q)\,s=(h+1)\cdot(h-1)\,s$.

take $p+q=(h-1)\,s$,

then $(p-q)\,s=h+1$.

from which $h=s\,p-s\,q-1=\dfrac{p+q+s}{s}$.

$\therefore\ q=\dfrac{p\,(s^2-1)-2\,s}{s^2+1}$ ...... (2)

and by equating the values of q . (1) and (2),

we find $p=\dfrac{(r^2+1)\cdot(s^2+1)+(r^2-1)\cdot 2\,s}{(r^2-1)\cdot(s^2-1)-(s^2+1)\cdot 2\,r}$.

Where the indeterminates r and s may be taken any numbers that will render p integral.

If $r = 3$, and $s = 4$, then $p = 13$, and $q = 11$, and the required numbers are 15 and 21. If $r = 3$, and $s = 3$, then $p = 37$, and $q = 29$, and the triangular numbers are 105 and 171.

36. To find a number of numbers, each of which shall be both a triangular and a pentangular number.*

Let $x =$ the root of the triangular,

and $y =$ that of the pentangular number.

then $\frac{x^2 + x}{2} = \frac{3\,y^2 - y}{2}$ by the question.

$$\text{or, } x^2 + x = 3\,y^2 - y.$$

$$\text{put } m\,p = x, \text{ and } n\,p = y.$$

$$\text{then } m^2\,p^2 + m\,p = 3\,n^2\,p^2 - n\,p.$$

$$\text{or, } 3\,n^2\,p - m^2\,p = m + n.$$

$$\therefore p = \frac{m + n}{3\,n^2 - m^2}.$$

Where m and n may be taken any number that will render p integral, which will be the case when m and n are taken any number in the following series, which may be continued by multiplying the last term by 4, and subtracting the next preceding one.

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* It is stated in Barlow's Theory of Numbers, that "no triangular number, except unity, can be equal to a pentagonal number," and great numbers of what has been deemed demonstrations, have appeared in support of the assertion. Mr. J. M. F. Wright, B. A., in 'Hints and Answers to the Cambridge Problems,' attempting to prove the non-existence of such a number, says, to effect which, that

$$\frac{2\,y^2}{x + y}$$

must be an integer, which can only be when $x = y = 1$. Now if x and y be assumed as in the text, and n taken $= m + p$, the result is $2\,p^2$ an integer.

If $m =$ 1	and $n =$ 1,	then $p = 1$,	$\therefore$ the number is	1
.. 3	.. 5	.. 4	..	210
.. 11	.. 19	.. 15	..	40755
.. 41	.. 71	.. 56	..	7906276
.. 153	.. 265	.. 209	..	1533776805
.. 571	.. 989	.. 780	..	297544793910
.. 2131	.. 3691	.. 2911		57722156241751

37. To determine a triangle, such that its sides and area shall be whole numbers, and its perimeter a square. Representing two of its sides by x and y, and the perpendicular by z, then the third side is

$$\sqrt{x^2 - z^2} + \sqrt{y^2 - z^2},$$

$$\text{and area } \left\{ \sqrt{x^2 - z^2} + \sqrt{y^2 - z^2} \right\} \times \frac{z}{2},$$

Which is to be a whole number.

Putting $m^2 + n^2 = x$, and $2\, m\, n = z$,

then $x^2 - z^2 = (m^2 - n^2)^2$, and $y^2 - z^2 = y^2 - 4\, m^2\, n^2$.

$\therefore\ y^2 - 4\, m^2\, n^2 =$ a square.

Assume $y^2 - 4\, m^2\, n^2 = (y - p)^2$, from which

$$y = \frac{p^2 + 4\, m^2\, n^2}{2\, p}.$$

and by taking $p = 2\, n$,

$$y = (m^2 + 1)\, n.$$

$\therefore$ the base or third side is

$$(m^2 - n) \cdot (n + 1).$$

Whence we have only the perimeter to make a square, or

$$(m^2 + n^2) + n\,(m^2 + 1) + (m^2 - n) \cdot (n + 1) = \square.$$

or, $2\, m^2\, n + 2\, m^2 =$ a square.

or, $2\, n + 2 = \square = 4\, q^2$.

$\therefore\ n = 2\, q^2 - 1$.

Where q may be taken at pleasure.

Therefore the sides become

$$m^2 + (2q^2 - 1)^2,\ (m^2 + 1) \cdot (2q^2 - 1) \text{ and}$$

$$\left\{ m^2 - (2q^2 - 1) \right\} \cdot 2q^2.$$

Where m may be taken at pleasure, if greater than $2q^2$.

If $q = 1$, and $m = 2$, the sides are 5, 5 and 6. If $q = 2$, and $m = 3$, they are 113, 455, and 456.

38. To determine a triangle, such that its sides shall be integral, and in arithmetical progression; also that its area, and diameters of its inscribed and circumscribing circles, shall be integral too.

Representing the sides by

$$2xy - z,\ 2xy, \text{ and } 2xy + z$$

solves one condition, as they are in arithmetical progression.

$$\text{Whence } xy\sqrt{3(x^2y^2 - z^2)} = \text{area}.$$

Now, in order to make the quantity under the radical sign rational.

$$\text{assume } 3(x^2y^2 - z^2) = \frac{m^2}{n^2}(xy + z)^2.$$

$$\text{Then } 3(xy - z) = \frac{m^2}{n^2}(xy + z), \text{ by } \div \text{ by } xy + z.$$

$$\text{Whence } z = xy\frac{(3n^2 - m^2)}{3n^2 + m^2}.$$

And by taking $y = 3n^2 + m^2$,

then $z = x(3n^2 - m^2)$.

$\therefore$ the sides are

$$2xy - z = 3x(n^2 + m^2),$$

$$2xy \quad = 2x(3n^2 + m^2),$$

$$2xy + z = x(9n^2 + m^2);$$

and area $= 6mnx^2(3n^2 + m^2)$.

And by note 2, page 127, of the Author's 'Tutor's Assistant,' the diameter of the inscribed circle is

$$4\,m\,n\,x;$$

and by note 3, the diameter of the circumscribing circle is

$$\frac{x\,(n^2 + m^2)\,.\,(9\,n^2 + m^2)}{2\,m\,n},$$

which will be integral, when x is taken any multiple of $2\,m\,n$.

If $x = 2\,m\,n$, then it becomes

$$(n^2 + m^2)\,.\,(9\,n^2 + m^2),$$

where m and n may be taken at pleasure, provided $3\,n^2$ is greater than m^2.

If $m = 1$, and $n = 1$, then $x = 2$, and the sides are 12, 16, and 20; area $= 96$, and diameters 8 and 20.

If $m = 1$, and $n = 2$, the sides are 60, 104, and 148; area $= 2496$, and diameters 32 and 185.

39. To determine a triangle, such that the perpendicular, the segments of the base made by the perpendicular, the area, and the other two sides, shall be integral. Also, that the diameters of the inscribed and circumscribing circles shall be integral.

Let x and y represent the segments of the base, and p the perpendicular.

Then $\sqrt{x^2 + p^2}$ and $\sqrt{y^2 + p^2}$ are the other two sides.

The diameter of the circumscribing circle is

$$\frac{\sqrt{x^2 + p^2} \times \sqrt{y^2 + p^2}}{p}$$

and that of the inscribed circle

$$(x + y)\,.\,p \div \left\{\frac{x + y + \sqrt{x^2 + p^2} + \sqrt{y^2 + p^2}}{2}\right\}$$

assuming $x^2 + p^2 = (x - r\,p)^2$

and $y^2 + p^2 = (y - q\,p)^2$

$$\text{Then } x = \frac{(r^2 - 1)\,p}{2\,r} \text{ and } y = \frac{(q^2 - 1)\,p}{2\,q}.$$

Or, by putting $p = 2\,q\,r$,

$x = (r^2 - 1)\,q$, and $y = (q^2 - 1)\,r$.

Whence the other two sides are

$(r^2 + 1)\,q$, and $(q^2 + 1)\,r$,

and area $\left\{(r^2 - 1)\,q + (q^2 - 1)\,r\right\} . \,r\,q$,

all of which are integral.

The diameter of the circumscribing circle is

$$\frac{(r^2 + 1) \cdot (q^2 + 1)}{2},$$

and of the inscribed circle,

$$2 \cdot \left\{\frac{(r^2 - 1)\,q + (q^2 - 1)\,r}{q + r}\right\}$$

Now, in order to render these integral, take $r = m + 1$, and $q = 2\,m - 1$, and the former becomes

$$2\,m^4 + 2\,m^3 + m^2 - 2\,m + 2,$$

and the latter, $4\,m^2 + 2\,m - 4$,

where m may be taken any number greater than unity.

If $m = 2$, then $r = 3$ and $q = 3$, whence the sides are 30, 30, and 48, area 432, diameters 50 and 16.

If $m = 3$, then $r = 4$ and $q = 5$, and sides 85, 104, and 171, area 3720, and diameters 221 and 38.

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EXERCISES.

1. Find two numbers, such that their *product* and *difference* shall both be squares. Ans. 3 and 12.

2. Find two numbers, such that both their *sum* and *product* shall be squares. Ans. 5 and 20.

3. Find two numbers, such that their *difference*, and the sum of their squares, may be squares. Ans. 3 and 4.

4. Find two numbers, such that their *sum*, and sum of their squares, may be squares. Ans. 21 and 28.

5. Find two numbers, such that their sum shall be equal the difference of their squares, and the sum of their squares a square number. Ans. 3 and 4.

6. Divide 49 into three other square numbers. Ans. 4, 9, and 36.

7. Find two numbers, such that the sum of their squares *minus* one, and the difference of their squares, *plus* one, shall be squares. Ans. 11 and 13.

8. Find two square numbers, such that the square of each, plus their product, shall be squares. Ans. 9 and 16.

9. Find two numbers, such that their difference, the difference of their squares, and the difference of their cubes, shall be squares. Ans. 6 and 10.

10. Find two numbers, such that their sum, the sum of their squares, and the sum of their cubes, may be squares. Ans. 184 and 345.

11. Find two integral squares, such that their arithmetical and harmonical means may be squares, and their geometrical mean a cube. Ans. 56^2 and 392^2.

12. Find two integral cubes, such that their arithmetical and harmonical means shall be squares, and their geometrical mean a cube. Ans. 130^3 and 520^3.

13. Find two numbers, such that the sum of their cubes divided by their sum, and the difference of their cubes divided by their difference, the sum of their quotients may be a square. Ans. 1 and 7.

14. Find three numbers, such that the sum of every two, plus 2, may be a square number, also that the sum of these squares shall be a square also.

Ans. 225, 349, and 673.

15. Find three numbers, such that their sum, the sum of every two, minus the third, and the square of the sum of every two, minus the square of the third, shall be squares. Ans. 26, 80, and 90.

16. Find three numbers in arithmetical progression, such that their sum shall be a square, and if unity be added to each number, the three sums shall also be squares. Ans. 195, 675, and 1155.

17. Find three numbers, such that the sum of every two of them may be a square, and that the sum of the three squares thus found may be a square also.

Ans. 226, 350, and 674.

18. Find three numbers, such that the difference of every two, multiplied by the third, may be a square number. Ans. 3, 5, and 8.

19. Find three square numbers, the cube roots of which shall be squares, also the difference between the cube roots of the first and second, and sum of the cube roots of second and third, shall be square numbers.

Ans. 729, 4096, and 15625.

20. Find three positive numbers, such that the sum of every two shall be a square, also the first being added to twice the second, plus three times the third, the sum shall be a square. Ans. 19, 6, and 30.

21. Find three numbers, such that the sum of every two, minus the third, shall be a cube, and the sum of all three a cube also. Ans. 364, 608, and 756.

22. Find three numbers in arithmetical progression, such that if three be added to each, the three sums shall be squares; also, if three be taken from their sum, the difference shall be a square.

Ans. 4621, 2701, and 781.

23. Find three numbers in arithmetical progression, such that the sum of every two, plus 2, shall be squares.

Ans. 481, 3361, and 6241.

24. Find four square numbers, such that eight times the first shall be equal to the sum of the other three.

Ans. 25, 100, 36, and 64.

25. Find four square numbers, such that the three first shall be in arithmetical progression, and the three last in geometrical progression.

Ans. 19600, 10000, 400, and 16.

26. Find four numbers, such that twice their sum, twice the square of the sum of every three, minus twice the square of the fourth, shall be squares.

Ans. 9, 31, 49, and 73.

27. Find four numbers, such that the sum of every three of their squares, minus the square of the fourth, shall be a square.

Ans. 2, 9, 20, and 22.

28. Find four numbers, such that the square of each added to the product of the other three, the sums in each case shall be a square.

Ans. 4, 8, 12, and 20.

29. Determine the sides of a right angled triangle, such that its peremeter shall be a square, and the diameter of its inscribed circle a cube.

Ans. 3136, 12348, and 12740.

30. Find a scalene triangle, the sides of which are in arithmetical progression, the common difference being a square, and such that the diameter of the inscribed circle shall be a square; also, both the sum and difference of the diameters of the inscribed and circumscribing circles shall be squares, the peremeter a cube, and the area integral.

Ans. 22932, 24696, 26460 } sides.
7056, 28665 } diameters.
261382464 area.

THE END.

W. B. Johnson, Printer, Market-Place, Beverley.