From the PHILOSOPHICAL MAGAZINE for July 1852.

REMARKS ON LORD BROUGHAM'S "EXPERIMENTS AND OBSERVATIONS ON THE PROPERTIES OF LIGHT, &c."

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BY

THE REV. BADEN POWELL, M.A., F.R.S. &c.,
SAVILIAN PROFESSOR OF GEOMETRY, OXFORD.

THE publication of Lord Brougham's optical researches, in which a number of experimental facts connected with the phænomena now usually called "diffraction," are viewed according to a peculiar theory of certain new properties of light, and in some respects held to be irreconcileable with the principle of interference, seems to render desirable some examination into the actual bearing of the results on the theory of undulations, by which not only all the phænomena of diffraction, hitherto known, have been so perfectly explained, but which has also been applied so extensively to other large classes of facts, as to render it unphilosophical to resort to theories assumed on independent grounds to meet apparent exceptions in limited classes of phænomena.

These researches having been briefly alluded to by the Astronomer Royal in his opening address to the British Association at Ipswich*, and having also myself made a few observations on the subject at the same meeting†, my object in the present communication is to follow up the question in somewhat more detail; and without pretending to enter on any controversy as to the author's *theory*, to examine merely the *experimental evidence* adduced, and inquire how far it seems accordant or not with the undulatory theory.

During the summer of last year I took the opportunity of

* See Athenæum, No. 1236. † See Ibid. No. 1237.

repeating the experiments with the utmost care, for all the most material cases considered; since which time various causes have delayed the publication of my results.

The whole of the author's investigations are expressed with reference to his peculiar hypothesis of certain forces of "deflexion" and "inflexion" supposed to be exerted upon the rays of light by the action of the edge of an opake body near which they pass: nor is it always an easy matter to disentangle the actual facts from the language of this theory, so as to see to what the experimental evidence really amounts.

Of those of the author's propositions which refer solely to the exposition of his theoretical views, I do not propose to enter on any discussion. There are also other portions of the investigations, which, though of a more experimental character, will not call for much observation, as they either tend to establish phænomena in exact conformity with well-known results, or are of a nature not having much bearing on theory either way.

Of this class are the preliminary experiments (Prop. I. Exp. 1, 2, 3); though with respect to the last it ought to be remarked, that Newton by no means limits the number of fringes to *three*, and in one modification of the experiment expressly mentions that *four or five* were rendered visible*. When (as in Exp. 4) the origin of light is not the *single point absolutely requisite in all accurate investigations*, but an extended object, such as a flame, the moon, &c., it may be questioned how far the fringes may be properly termed images of it. In Prop. II. Exp. 2, that the nature or form of the edge makes no difference in the result, accords exactly with the long-known experiments of Biot, Haldat, and others. Indeed, as is equally well understood, the fringes may be produced without any opake edge at all, as at the junction of two faces cut on a glass, slightly inclined to each other. Again, the hyperbolic fringes of an acute angle (in Prop. V. Exp. 3), as well as the measures of the fringes at successive distances from the edge determining the locus of any given fringe (in Prop. X., and additional remarks, (2) p. 252), appear to agree with previous observations; though, according to the author's theory, each fringe seems to be regarded as an individual ray, while in the interference theory it is the locus of the intersections of a series of rays.

At another part of his discussion the author assumes (Prop. XI.) an aggressive position, and endeavours to *refute* the application of the interference theory. In reply I think it will suffice to remark, under the several heads,—(1) the theory of interference explains perfectly *both* the internal and external fringes of a shadow; (2) the *breadth* of the fringes has no dependence on

* See Opticks, book 3. part 2. obs. 2.

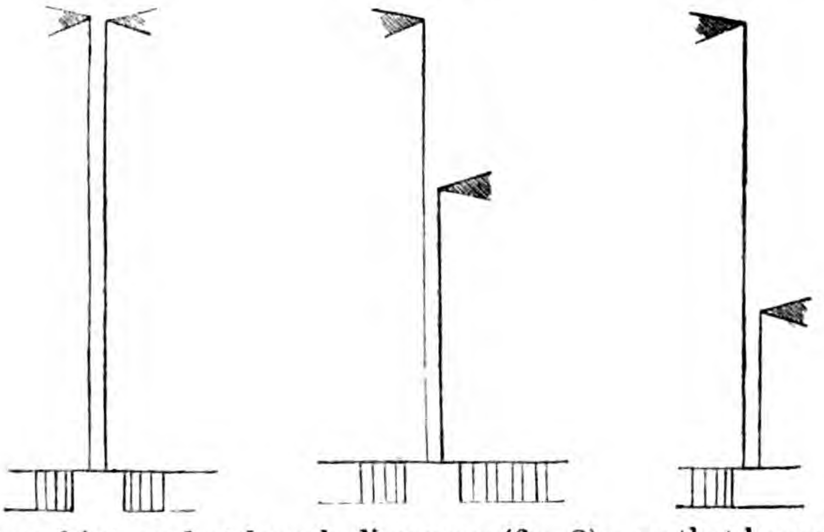
the *length of route* of the rays, but it has on the *angle* at which they intersect; so that (3) in the case represented in the author's fig. (20), supposing abstractedly two pairs of interfering rays (such as BC, AC, and BD, AD), it is evident that the fringes at D *ought to be* broader than those at C, not owing to any difference in the *routes*, but because the *angle* BDA is less than BCA; while (4) interference perfectly explains fringes, even when the action is wholly on *one side* of the ray or edge.

But passing from these points of confessedly less importance, we will proceed to the most material and fundamental experiment (Prop. II. Exp. 1), in which, when fringes are formed by the edge of an opake body, if a second edge be placed at a greater distance along the ray from the origin on the same side as the first edge, it produces no change in the fringes, but on the opposite side it does, the fringes being shifted in position towards the first side; or in other words, in the one case it has no power of producing further diffraction, in the other it has: and this is viewed by the author as supporting his theory of a peculiar action exerted by the edge upon the ray passing near it, by which it is disposed or indisposed for further flexure according to the conditions above expressed.

The experimental fact in general is easily verified. There is, however, one material condition necessary to be attended to for reproducing the result exactly as described by the author.

When two edges are at the same distance from the origin and from a narrow aperture, they give, as is well known, fringes on each side extending into the shadow, with a white centre (fig. 1). As one edge is removed successively further from the origin and nearer to the screen, the fringes on that side dilate (fig. 2),

Fig. 1. Fig. 2. Fig. 3.



become faint, and at length disappear (fig. 3); so that beyond a certain distance there remain only the fringes on the other side, or on that of the edge nearest the origin, which diverge further into the shadow on that side as the breadth of the effective aperture is diminished.

In this way, then, the second edge, if beyond the limits of distance mentioned, will cause an appearance of fringes on the side towards the first edge diverging into the shadow.

With regard to the bearing of this experiment on theory, it is in the first instance necessary to bear in mind, that, *according to the undulatory theory, neither the formation of fringes, nor any shifting of those fringes, implies a* FLEXURE *in the rays; in this theory no such idea is introduced or needed.*

In the particular case in question, when the two edges are at the same distance from the origin forming a narrow aperture, the nature of the fringes is perfectly explained and reduced to quantitative results by Fresnel's theory.

When the second edge is placed as in Lord Brougham's experiments, at a greater distance along the ray, this would be equivalent to a wide aperture placed obliquely to the direction of the ray, so as to be effectively as narrow as before. Now this case is one *which has not yet been reduced to calculation.*

The formulas of Fresnel, even in the simplest cases, are considerably complicated, and involve integrations which cannot be generally exhibited in a finite form. In the cases of a single edge, or that of an aperture when it is a long narrow parallelogram, an equilateral triangle, or a circle, the integration has been performed in a way sufficient for calculation*.

In the case of the *oblique* aperture, at my request, a friend eminently versed in the analysis of the subject, undertook to work out the formulas; and he pursued the inquiry far enough to be able to say that they became immensely complicated; still it could not be certain that they might not be made to yield to proper treatment, should anyone think it worth while to follow up the attempt.

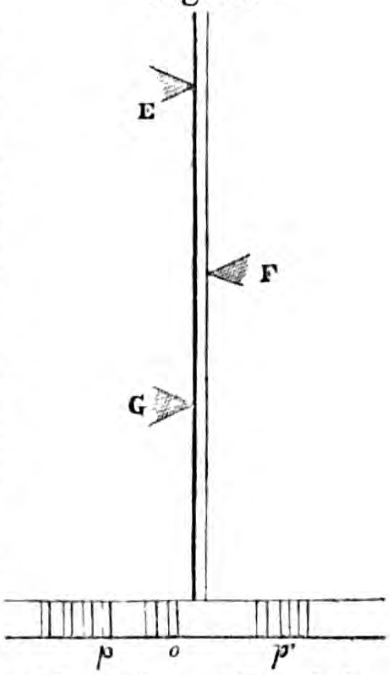
But further, this particular case *has been considered,* though only in a general way, by Fresnel†. Upon the obvious geometrical construction he points out the general conditions for determining the position of a fringe, and shows that the fringes will in this case undergo a modification, and *will not be symmetrical,* but more expanded on one side than the other, which exactly agrees with observation.

* See Airy's Tracts, Undulatory Theory, art. 73 *et seq.* Journal of Science and Phil. Mag. vol. xv. Dec. 1839; and vol. xviii. Jan. 1841.

† *Mém. sur la Diffraction. Mém. de l'Institut,* vol. v. note, p. 452, for 1821, published in 1826.

The simple facts affirmed in Prop. III. Exp. 1 and 2, when divested of all theoretical language, appear to be, that if three edges, E, F, G, be placed at successive distances from the origin in the order of the letters, E and G being on one side of the ray and F on the other; then if E and F alone give fringes as at *o* (fig. 4), and G be then made to act upon them, or if F and G alone give fringes and E be made to act upon them, in either case the fringes will be shifted to *p* towards the side on which E and G lie, and become broader; and the conclusion which the author chiefly insists upon is that all three edges act in producing the ultimate result: the same thing being further confirmed by exp. 3, in which a curved form given to the edge E, is still exhibited in the form of the fringes after the action of F and G.

Fig. 4.



That all three edges should be in some degree effective in producing the ultimate character of the fringes, would, on a general view, be obviously consistent with the wave theory; since, on that theory, a new set of waves originates at each edge, all of which conspire to produce the ultimate result; though antecedent to exact calculation, it would be impossible to say what would be the precise action of each.

On repeating the experiment, however, in regard to the *particular appearances* described by the author, I have found considerable difficulty: consistently with the conditions before remarked, if the edges E and F form a narrow aperture so as to give a white centre, and within such limits of distance along the ray as to produce fringes on *each* side (as in fig. 2), then if G be also within the same limits, and be advanced so near to the ray laterally as to make a still narrower aperture, the fringes on *each side* will expand further into the shadow. If the edges be *beyond* those limits (which seems to be implied in the author's description, since he speaks of only *one* set of fringes), then E and F will give a white centre with fringes *on the side towards* E, as at *o*; and when G is introduced it will narrow the aperture and give new fringes *on the side towards* F, at *p'*, that is, just *the opposite way* to that which the author describes. In repeating the experiment a great number of times at very different distances, and under varied conditions, I have never been able to obtain any other result: indeed it would clearly be inconsistent with the former experiments that it should be otherwise.

Prop. VI. appears precisely to express Fresnel's conclusion (above referred to), that with two edges at unequal distances from the origin, the fringes will be broader on the side towards the edge most remote from the origin, which is again more precisely exhibited in Prop. VIII., when the aperture is sufficiently narrow to give a white centred image; the same regard being had to the limits in distance as before.

In Prop. VII. the meaning is by no means obvious; but it seems to amount experimentally to this,—that with one edge only, the fringe nearest that edge is the broadest; and that when a second edge acts opposite to it at some distance along the ray, but so as to give the fringes of an aperture, then among the fringes of each set, those towards the middle of the aperture are the broadest: the first being obviously the case of the external fringes; the second easily verified, and agreeing with the ordinary case of an aperture with edges at the same distance; while as to the application of the undulatory theory, we can only make the same remarks as before.

In the Additional Observations, (3), p. 254, the truth of the general assertion, that when fringes are formed by two edges, a third can affect them only when parallel and not when at right angles to them, is indeed obvious; but the precise conditions of the experiment mentioned are difficult to understand. It would seem to consist in first forming the fringes of a narrow aperture with a broad white centre in the ordinary way; and then in that white centre producing new fringes by a third edge nearer to the screen: these, however, the author affirms, will be formed only when the third edge is parallel to the aperture and not when at right angles to it; they are also described as brighter and narrower than the ordinary fringes. The author cautions us against confounding them with the ordinary external fringes, and proceeds to argue that they are of a different nature, for several reasons, but chiefly because (Exp. 1) they do not increase in breadth when the aperture is narrowed, and (Exp. 2) because their breadth increases as the distance of the third edge from the aperture is diminished, the third edge remaining at the same distance from the screen.

The last results (which I have fully verified) do not appear to me to evince any *peculiarity*: relatively to the third edge, the aperture may be regarded as a new origin of light, in which light the third edge gives its external fringes.

But with respect to the first part of the proposition, viz. that these fringes are only formed parallel to the aperture, on repeatedly trying the experiment, I have uniformly found them

formed equally, whether the edge be parallel or perpen
the aperture; though in the latter case they may fo
reasons be less distinct and conspicuous.

It might indeed be fully admitted that the rays for
white centre may be in some respects under different c
from the ordinary rays, and that thus the fringes formec
might possibly be different: I can only say that I ha
been able to detect any such difference.

If, indeed, the author's meaning be that these fringes
in any degree into the lateral fringes, it is obvious t
would be mutually affected in a way conformable to pre
periments.

One other remark of the author deserves especial att
that, but for what he considers the incapacity for furthe
in the same direction, induced in a ray after one inflex
ray might be continually bent round an opake body; an
luminous object might be seen, though the whole of t
intervened, or in other words, that we might *see round a*

Now if such inflexion took place it would clearly be
accompanied by a considerable diffusion of the light,
after a few successive inflexions it might be so much w
as to become imperceptible.

It is however a remarkable fact, that such an appa
flexion *does take place* to a very great extent, as I have
out in a paper " On Luminous Rings round Shadows
moirs of the Royal Astronomical Society, vol. xvi. p. 30
which (as I have there mentioned) I believe to be a m
tion of the same phænomenon, described rather obscu
Newton† and more distinctly by Hooke‡, and appare
cordant with the theory of undulations (Ibid. p. 310).

* Additional Observations, 4.

† Opticks, book 3. part 1. obs. 5.

‡ As this curious point seems to have been much overlooke
perhaps be excused in annexing a brief notice of Dr. Hooke's exp
from a fragment on Light, appended to the Essay on Comets
vity, in his posthumous works: London, 1705, p. 186.

Light being admitted into a dark room through a very small
received on a screen at some distance, on holding an opake body in
of light, besides a " zone or fascia of light much brighter than th
the surface," along and outside the edge of the shadow (which
bably the first diffraction fringe), he observed *a faint light extend*
the edge into the shadow; and when the opake body was held so as
nearly the whole of the luminous circle, " *rays were seen darting do*
perpendicular to the edge of the shadow, like the tail of a comet,
downwards more than 10 times, probably 100 times their breadth.

I have thus, I trust with perfect impartiality, gone through all the main experimental points of the author's investigations, and upon the whole I can perceive *nothing substantiated* which is positively *irreconcileable with the principle of interference,* while the new modifications of the phænomena here presented, so far as *general* considerations can be relied on, seem sufficiently conformable to the undulatory theory : but as to their more *exact,* or *quantitative* explanation, no definitive opinion can be pronounced, until certain analytical investigations of almost impracticable length and complexity, shall have been gone through, by which alone that theory can be brought into exact and satisfactory comparison with experiment.

pendicular to the edge; if circular, tending to the centre; if angular, bisecting it; if concave, spreading out, &c. A representation of the appearance is given in Plate ii. fig. 8 (p. 155). At p. 190, the Editor adds a memorandum found among Dr. Hooke's papers, stating, that on March 18, 167$\frac{4}{5}$, he "read a discourse" on several new properties of light; which he sums up as follows:—

"That there is a deflexion of light differing both from reflexion and refraction, and seeming to depend on the unequal density of the constituent parts of the ray, whereby light is dispersed from the place of condensation and rarefied or gradually diverged into a quadrant;" 2ndly, that this takes place "perpendicularly to the edge;" and 3rdly, that "the parts deflected by the greatest angle are the faintest."

I have fully referred to and commented upon Newton's description of the same phænomenon, conveyed in terms so singularly coincident, in my paper before referred to.

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ON THE

DEMONSTRATION OF FRESNEL'S FORMULAS

FOR

REFLECTED AND REFRACTED LIGHT;

AND THEIR APPLICATIONS.

BY

THE REV. BADEN POWELL, M.A., F.R.S. ETC.,
SAVILIAN PROFESSOR OF GEOMETRY IN THE UNIVERSITY
OF OXFORD.

1. A QUESTION between two fundamentally different views of the theory of polarization, which has been long agitated among inquirers into the undulatory theory, viz. as to the direction of the plane of *vibrations* in relation to that of *polarization*, has of late excited more peculiar interest, partly from the announcement, a few years ago, of a remarkable *crucial* experiment by Professor Stokes, and partly from several subsequent investigations, especially the recent elaborate discussion of the general bearing of the experimental evidence by M. Haidinger.

The revival of this question recalls the attention of the student to the very unsatisfactory condition in which the elementary demonstration of those parts of the theory on which it depends has long been left, and from which recent speculations have done little to deliver it.

2. The well-known and remarkable formulas originally given by Fresnel to express the amplitudes of the vibrations, and thence the intensities, of reflected and refracted rays of polarized light (for singly-refracting media), which are found to represent so beautifully all the observed changes,—in fact including the whole doctrine of plane polarization, and thus invaluable as inductive laws,—yet long remained confessedly defective as to their systematic deduction from theory.

3. Fresnel, indeed, with that marvellous sagacity for which he was so conspicuous, satisfied himself of their truth by reason-

Life of Whately note p 386.

(21)

ing on assumptions in some instances confessedly empirical, in others dependent on analogies, or hypotheses not free from doubt, and at any rate little connected into a system.

4. This investigation, whose questionable points are so fairly stated, and ably grappled with, by Mr. Airy in his tract on the Undulatory Theory (1831, art. 128 *et seq.*), has been since pursued on different principles by M. Cauchy, and especially by the late Prof. Maccullagh in his memoir "On the Laws of Crystalline Reflexion and Refraction" (Mem. Roy. Irish Acad. vol. xviii. 1838), whose views have been ably but briefly expounded by Dr. Lloyd in his Lectures on the Wave Theory (part 2. p. 30, 1841). More recently, Mr. Power has investigated the subject by a systematic analysis, directed to other objects, but including an important element in these deductions ("On Absorption of Rays," &c., Phil. Trans. 1854, part 1).

5. But among these distinguished philosophers there exists considerable diversity, and even contradiction of views. Nor, so far as I am aware, has the subject been so discussed as to enable us to trace the source of these discrepancies, or fairly to estimate the claims of the opposing theories, or the force of the experimental results which bear upon them. Thus it seems highly desirable, that questions affecting so fundamental a part of the undulatory theory should be cleared up and placed on an unassailable basis.

Having long ago thrown aside some investigations on the subject, in which I was then engaged, I have of late had my attention recalled to the question, and have thus been induced to revise and extend those investigations, in the hope of contributing towards the settlement of the points involved, or at any rate of putting the whole discussion before the student in a connected point of view; with which object I have been led to commence *ab initio*, so that those who have only an elementary acquaintance with the theory may be enabled to follow the deductions without difficulty, and may here be furnished with that systematic elucidation which is not, as far as I am aware, to be found in any existing publication.

Theoretical Views.

6. The formulas for the amplitudes of the incident, reflected, and refracted rays, as given by Prof. Maccullagh and later writers, though closely corresponding with those of Fresnel, and fulfilling generally the same conditions, yet differ from them in certain cases as to the *sign*, and in others as to the *values* of the expressions.

7. But the main point of difference and difficulty consists in this: Fresnel investigates two sets of formulas; one set (H) for

the respective rays deduced on the supposition that the vibrations are *perpendicular* to the plane of incidence, another set (K) on the supposition that they are *parallel* to that plane. Now those of Maccullagh, which correspond closely to Fresnel's first set (H), are deduced on the contrary supposition of vibrations *parallel* to incidence, while those corresponding to Fresnel's second set (K) are for vibrations *perpendicular* to that plane.

8. In either investigation the formulas (K) are those which represent evanescence of the light at the polarizing angle, while the formulas (H) represent brightness at that incidence.

But when a ray vanishes at the polarizing angle, we know that its plane, of this second incidence, must be *perpendicular* to that of its first incidence or original polarization. Hence, according as the vibrations (K) may be *parallel* or *perpendicular* to this second plane of incidence, they must be respectively *perpendicular* or *parallel* to the first or plane of polarization. The question thus reduces itself to whether, in polarized light in general, the *vibrations* are *parallel* or *perpendicular* to the plane of *polarization.*

9. M. Cauchy, in an earlier paper (*Mem. Instit.* vol. x. p. 304), had inferred with Maccullagh, from dynamical views, that the vibrations are *parallel* to the plane of polarization. But in a later memoir (*Bull. Math.* July 1830) he deduces formulas corresponding to Fresnel's on the hypothesis of vibrations *perpendicular* to the plane of polarization, and even more formally renounces his earlier opinion and returns to that of Fresnel. He also connects similar equations with higher dynamical principles in the *Nouv. Exercices Math.* (liv. 7).

Synopsis of Formulas referred to.

10. Fresnel's formulas for vibrations *perpendicular* to the plane of incidence (h being the amplitude of the incident, h' of the reflected, and $h_{,}$ of the refracted rays, and dividing by h, i being the angle of incidence, r of refraction), are—

$$h'=\frac{-\sin(i-r)}{\sin(i+r)},\quad h_{,}=\frac{2\sin r\cos i}{\sin(i+r)};\quad . \quad . \quad . \quad \text{(H)}$$

and for vibrations *parallel* to the plane of incidence (similarly designated by k, k', and $k_{,}$),

$$k'=\frac{\tan(i-r)}{\tan(i+r)},\quad k_{,}=\left(1-\frac{\tan(i-r)}{\tan(i+r)}\right)\frac{\cos i}{\cos r}.\quad . \quad \text{(K)}$$

These last may be otherwise expressed thus:

$$k'=\frac{\sin 2i-\sin 2r}{\sin 2i+\sin 2r},$$

$$k_{,}=\left(\frac{2\sin 2r}{\sin 2i+\sin 2r}\right)\frac{\cos i}{\cos r}=\frac{4\sin r\cos i}{\sin 2i+\sin 2r}.$$

11. Here we may observe, in the numerator of $h_{,}$,

$$2 \sin r \cos i = \frac{1}{\mu} \sin 2i;$$

and of $k_{,}$,

$$2 \sin 2r \frac{\cos i}{\cos r} = \frac{1}{\mu} 2 \sin 2i.$$

12. It is also desirable to notice, that these expressions are the same as those given in Mr. Airy's Tract, § 129, under the slightly different form in which they directly result from the peculiar process there pursued, viz. writing $\sin (r - i)$ and $-\tan (r-i)$.

Also the numerator of k' is *positive* for all values of $(i-r)$, which is necessarily less than 90°, while the denominator becomes ∞ at $(i+r)=90°$, which, according to Brewster's law, is the polarizing angle, and for greater values continues *negative*.

13. Prof. Maccullagh's formulas are,—
for vibrations *parallel* to the plane of incidence,

$$h' = \frac{\sin (i-r)}{\sin (i+r)}, \quad h_{,} = \frac{\sin 2i}{\sin (i+r)}; \quad \ldots \ldots \quad (\mathrm{H}')$$

for vibrations *perpendicular* to the plane of incidence,

$$k' = \frac{\sin 2i - \sin 2r}{\sin 2i + \sin 2r} = \frac{\tan (i-r)}{\tan (i+r)},$$

$$k_{,} = \frac{2 \sin 2i}{\sin 2i + \sin 2r} = 1 + \frac{\tan (i-r)}{\tan (i+r)}. \quad \ldots \quad (\mathrm{K}')$$

These last are sometimes expressed under the forms

$$k' = \frac{\sin (i-r) \cos (i+r)}{\cos (i-r) \sin (i+r)}, \quad k_{,} = \frac{\sin 2i}{\cos (i-r) \sin (i+r)}.$$

14. Comparing these formulas with Fresnel's (distinguished by using roman letters), we may observe from (11),

$$h' = -\mathrm{h}', \quad h_{,} = \mathrm{h}_{,}\mu,$$
$$k' = \mathrm{k}', \quad k_{,} = \mathrm{k}_{,}\mu.$$

Here also $k_{,}$ undergoes the same change of sign at the incidence of complete polarization.

Densities and Vibrating Masses.

15. In deducing these formulas, it is in all cases necessary to express the ratio of the *masses* of æther simultaneously vibrating without and within the medium; and the differences in the respective formulas are mainly dependent on the very opposite suppositions made by the several philosophers as to the *density* of the æther in different media,—Fresnel supposing it *more* dense

within the denser medium; Cauchy, *less* dense; and Maccullagh, *equally* dense in all media. The last-named writer has argued that refraction cannot be dependent on the *density* of the æther as such. He especially observes, that "in doubly-refracting crystals, the density, being independent of the direction, could not be conceived to vary with the refractive index" (p. 39). And Prof. Stokes has observed, that in the vibrations of æther, "diminution of velocity seems capable of being accounted for on several distinct hypotheses."

16. The expressions for the masses of æther vibrating in the same time without and within the denser medium, are obtained on these different suppositions as to the density of the æther, as follows:—

If v be the velocity of the incident ray, $v_{\prime}$ that of the refracted, and the index $\mu = \frac{v}{v_{\prime}}$, then at a *perpendicular incidence*, the simultaneously vibrating masses will be simply

$$\frac{(m)}{(m_{\prime})} = \frac{v}{v_{\prime}} = \mu = \frac{\sin i}{\sin r}.$$

If the densities be δ, $\delta_{\prime}$, then, according to the view of Fresnel, $\delta_{\prime} > \delta$, and

$$\frac{\delta}{\delta_{\prime}} = \frac{\sin^2 r}{\sin^2 i} = \frac{1}{\mu^2},$$

and we must multiply in this ratio, which gives

$$\frac{(m)}{(m_{\prime})} = \frac{1}{\mu}.$$

If $\delta = \delta_{\prime}$, according to the view of Maccullagh,

$$\frac{(m)}{(m_{\prime})} = \mu.$$

17. In either case, for *oblique incidences* we must multiply by the rectangular breadth of the rays on the same base or section of the surface, which will be as $\cos i : \cos r$, or

$$\frac{m}{m_{\prime}} = \frac{(m)\cos i}{(m_{\prime})\cos r}.$$

Thus, according to Fresnel,

$$\frac{m}{m_{\prime}} = \frac{1}{\mu}\frac{\cos i}{\cos r} = \frac{\sin r \cos i}{\sin i \cos r}.$$

According to Maccullagh,

$$\frac{m}{m_{\prime}} = \mu\frac{\cos i}{\cos r} = \frac{\sin 2i}{\sin 2r}.$$

18. It may here be observed, that if we admit equal densities, we must nevertheless suppose some retarding power in the æther within the denser medium. It is still conceivable that this may follow the *same law* as that of increased density, and that thus Fresnel's formula might still apply.

Or again, this reduces itself to the condition, that for perpendicular incidence we should have

$$\frac{(m)}{(m_{\prime})}=\frac{1}{\mu},$$

which might be simply the original condition, without involving the division by μ^2, as above, and might be dependent directly on some hypothesis assumed as to the constitution of the æther.

19. Expressing Fresnel's values by roman letters, and comparing with Maccullagh's, we have

$$\frac{\mathrm{m}}{\mathrm{m}_{\prime}}=\frac{m}{m_{\prime}}\frac{1}{\mu^2}.$$

20. According to the view of Cauchy, the density is *diminished* in the denser medium. If we suppose it diminished, according to the same law,

$$\frac{\delta}{\delta_{\prime}}=\mu^2 \text{ and } \frac{m}{m_{\prime}}=\mu^3\frac{\cos i}{\cos r}.$$

21. Mr. Power, taking α for the distances of the molecules without, and $\alpha_{\prime}$ within, the medium, obtains what is equivalent to

$$\frac{m}{m_{\prime}}=\frac{\alpha_{\prime}^{3}v\cos i}{\alpha^3 v_{\prime}\cos r}=\frac{\delta^3 v\cos i}{\delta_{\prime}^{3}v_{\prime}\cos r};$$

but having avoided any assumption of the law of refraction at the outset, he deduces (§§ 18, 28) the value

$$\mu=\frac{\alpha_{\prime}^{3}v}{\alpha^3 v_{\prime}}=\frac{\delta^3 v}{\delta_{\prime}^{3}v_{\prime}},$$

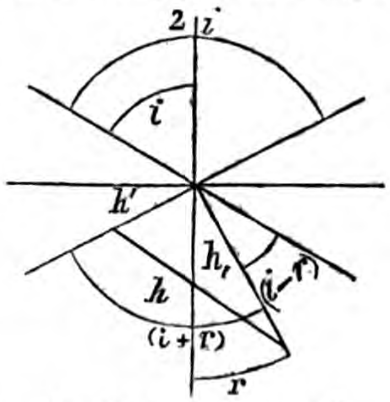
which seems irreconcileable with the admitted principle $\mu=\frac{v}{v_{\prime}}$, unless by supposing $\delta=\delta_{\prime}$, which would agree with Maccullagh's view. Or if we could have $\frac{\delta^3}{\delta_{\prime}^{3}}=\frac{1}{\mu^2}$, the expression would agree with Fresnel's view; or if $\frac{\delta^3}{\delta_{\prime}^{3}}=\mu^2$, with that of diminished density. But as neither of these suppositions seem reconcileable with admitted principles, it will not be material to discuss them further.

Equivalent Vibrations.

22. As to the general nature of the vibratory forces concerned, it will be on all hands admitted that the vibratory force of the

external æther or incident ray is the sole exciting cause of that communicated, partly to the reflected, partly to the refracted ray; so that the vibratory force of the incident waves must be distributed between the reflected and refracted.

23. An obvious geometrical relation is derived from the known *directions* of the incident, reflected, and refracted rays, which, with the parallel to one of them, form a triangle, whose angles being known, the sides are in the ratios of their *sines*: and the same relation subsists between the portions of the amplitudes at right angles to the rays, and supposed to lie in the same plane.



24. The triangle formed by the directions of the incident, reflected, and refracted rays, will have—

The angle formed by the incident and reflected rays $=2i$
... ... incident and refracted rays $=(i-r)$
... ... reflected and refracted rays $=(i+r)$.

Then the sides, or parts of the rays or amplitudes intercepted will be

$$\frac{h'}{h}=\frac{\sin(i-r)}{\sin(i+r)},\quad \frac{h_{\prime}}{h}=\frac{\sin 2i}{\sin(i+r)}. \quad . \quad . \quad . \quad (\mathrm{H}')$$

25. Again, these sides have mechanically the relation of the resultant and components of the vibratory motions in their respective directions. Hence this simple relation is adopted by Prof. Maccullagh to express the relative amplitudes or velocities: and this is the more remarkable, since (as he observes) they so nearly resemble the expressions adopted by Fresnel, on the opposite hypothesis of vibrations *perpendicular* to the plane of incidence, with which this construction can have no relation.

26. For vibrations *perpendicular* to the plane of incidence, or when the vibrations of all the three rays are *parallel to the surface* of the medium, it is also inferred that the amplitudes must be mechanically equivalent, or, as more distinctly argued by Mr. Power, that "a particle at the surface of separation will be at one and the same moment performing its phase to the incident, to the reflected, and to the refracted rays, with transverse velocities proportional to the amplitudes of those rays respectively;" and that "since this particle cannot move in more than one way at once, it is clear that the two latter must be equivalent to the former, according to the law of the composition of velocities."

27. But these are merely particular cases of the general theory of "equivalence of vibrations," first, I believe, systematically proposed by Prof. Maccullagh as the basis of his higher investigation of the laws of reflexion and refraction at the surfaces of crystals. Yet some cases of it appear to have been assumed by Fresnel, though under a slight, but material, difference of view. The general principle common to both is, that in "two contiguous media, the incident, reflected, and refracted vibrations are mechanically equivalent:" but a difference in conception of the *distribution* of the force among them gives rise to a corresponding difference in the form of the expressions; on either view, however, these expressions indicate conditions *distinctively* applying to vibrations respectively parallel and perpendicular to the plane of incidence.

28. Taking three co-ordinate planes, XY that of the surface, XZ that of incidence, and YZ perpendicular to incidence, any vibration h passing through the origin taken at the point of incidence, and inclined to XZ by an angle θ, and to XY by ϕ, may be resolved into

$$y=h\sin\theta,\quad x=h\cos\theta\cos\phi,\quad z=h\cos\theta\sin\phi.$$

Then for h of the incident ray we have $\phi=i$ and θ

... h' of the reflected ray we have $\phi=i$ and θ'

... $h_{,}$ of the refracted ray we have $\phi=r$ and $\theta_{,}$.

The law of equivalent vibrations, according to Prof. Maccullagh, is expressed by these relations between the resolved parts respectively:—

in z, $h\cos\theta\sin i+h'\cos\theta'\sin i=h_{,}\cos\theta_{,}\sin r$;

in y, $h\sin\theta+h'\sin\theta'=h_{,}\sin\theta_{,}$;

in x, $h\cos\theta\cos i+h'\cos\theta'\cos i=h_{,}\cos\theta_{,}\cos r$.

29. Hence if the plane of vibration coincide with YZ perpendicular to incidence, $\theta=\theta'=\theta_{,}=90°$, and the law becomes simply

$$h+h'=h_{,}.$$

If it coincide with XZ, $\theta=\theta'=\theta_{,}=0$, and it becomes

$$h\cos i+h'\cos i=h_{,}\cos r,$$

or

$$h+h'=h_{,}\frac{\cos r}{\cos i}=(h_{,}).$$

These two may be included in the formula

$$(h_{,})=h_{,}\left(\sin\theta+\frac{\cos r}{\cos i}\cos\theta\right).\quad .\quad .\quad .\quad \text{(L)}$$

$\theta=90°$ gives $(h_{,})=h_{,}$, and $\theta=0$ gives $(h_{,})=h_{,}\frac{\cos r}{\cos i}$.

Such is the principle of equivalent vibrations, according to Prof. Maccullagh; in other words, it is founded on the proposition that "*the incident and reflected vibrations are equivalent to the refracted.*"

30. It does not appear to what extent Fresnel had adopted any *generalized* view of this kind; but in the deduction of his formula, he introduces an equation which is equivalent to

$$h-h'=\frac{\cos r}{\cos i}h_{\prime}$$

for vibrations *parallel* to the plane of incidence; and if we suppose this to be the expression for the law of equivalence, in that case we must also take for that of *perpendicular* vibrations,

$$h-h'=h_{\prime}.$$

In other words, the law of equivalent vibration, according to Fresnel's view, will differ from that of Maccullagh in that it affirms the proposition that "*the refracted and reflected vibrations are equivalent to the incident.*"

These values of $h_{\prime}$, which we will call (L'), may be combined in the same expression as before (L).

Equation of vis viva.

31. That the *amplitudes* of the vibrations are the measures of the *velocities* of those vibrations whose time is constant, is obvious; and further, that the *square* of the amplitude or velocity, *multiplied* by the *vibrating mass*, is the true measure of the *intensity*; and consequently that the principle of *vis viva*, of which it is a simple application, is true, undoubtedly receives ample proof *à posteriori*, inasmuch as all the calculations founded on this principle agree to such extreme accuracy with the experimental results, whatever question may have existed as to its establishment *à priori*.

32. Mr. Power, assuming the general principle, but following an original analytical method, deduces directly expressions for the *vis viva* of the incident, reflected, and refracted ray respectively; and applies them by introducing the same expressions for the amplitudes as those of Maccullagh, and on the same supposition as to the directions of the planes of vibration, though not on the same hypothesis as to the constitution of the æther, with respect to which he is led to conclusions of a peculiar kind, and which have been considered by some as questionable, but the consideration of which is foreign to the present inquiry.

33. Assuming the ordinary formula for vibrations in *any plane*,

$$u=h\sin\frac{2\pi}{\lambda}(vt-x);$$

if n be the number of vibrations in a unit of time, $v=n\lambda$; and for two media,

$$\frac{v}{v_{/}}=\mu, \quad \frac{v}{\lambda}=\frac{v_{/}}{\lambda_{/}}=n \text{ for homogeneous light.}$$

34. If we consider, first, a single line of vibrating molecules, we may investigate the *vis viva* of that *line*; but from this proceeding to the vibrating *mass*, we must multiply the previous expression by the density (or equivalent retarding property), which we may express generally by δ and $\delta_{/}$ for the external and internal æther; and by the rectangular breadths of the oblique rays on the same base or section of the surface, which will be respectively proportional to $\cos i$ and $\cos r$: thus we shall have in general for the multipliers, $\delta \cos i$ and $\delta_{/} \cos r$; the former for the incident and reflected, the latter for the refracted ray.

35. Mr. Power's investigation is restricted to a particular hypothesis as to the density, but may be more simply and generally followed out thus:—

For a length dx, in which the molecules have a common velocity, we may take for that velocity,

$$\mathrm{v}=\frac{du}{dt}=\frac{2\pi hv}{\lambda}\cos\frac{2\pi}{\lambda}(vt-x);$$

and for a single *line* of vibrating molecules, the *vis viva*

$$\int \mathrm{v}^2 dx=\frac{4\pi^2h^2v^2}{\lambda^2}\int \cos^2\frac{2\pi}{\lambda}(vt-x)dx.$$

For a portion from x to $x+\lambda$, the integral is easily found to reduce to $\frac{\lambda}{2}$; and thus for those limits,

$$\int_{x+\lambda}^{x} \mathrm{v}^2 dx=\frac{2\pi^2h^2v^2}{\lambda}=2\pi^2nh^2v=(p);$$

and for the vibrating *mass*, as before explained (34),

For the incident ray $p=2\pi^2nh^2v\delta\cos i$

... reflected ray $p'=2\pi^2nh'^2v\delta\cos i$

... refracted ray $p_{/}=2\pi^2nh_{/}^2v_{/}\delta_{/}\cos r$.

But since the principle of *vis viva* gives $p=p'+p_{/}$, we have the equation connecting the *vires vivæ*,

$$\mu\delta\cos i(h^2-h'^2)=\delta_{/}\cos r h_{/}^2,$$

or generally

$$m(h^2-h'^2)=m_{/}h_{/}^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(M)}$$

where, on substituting the values of m and $m_{/}$ on the respective hypotheses, we can express the *vires vivæ* equally on the respective views of Fresnel, Maccullagh, and others.

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$\delta_{\prime}\cos rh_{\prime}^2$,

. . . . (M)

m and $m_{\prime}$ on the respective
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l others.

36. In some of these investigations, reference has been made to the mechanical doctrine of the impact of elastic bodies, to which the communication of vibratory motion in æther presents so striking an analogy. The well-known equations expressing the law of impact are adopted directly as the basis of Fresnel's proof in the case of vibrations perpendicular to the plane of incidence, giving rise to his formulas (H).

The very same equation is deduced by Maccullagh from his abstract dynamical principles, without reference to the abovementioned analogy; but applied on the same hypothesis as to the vibrations, and on his assumption as to the density, it produces his formulas (K).

37. This deduction, which is independent of any particular supposition as to densities ~~or direction of vibrations~~, is as follows:—

Combining equations (L) and (M), we have

$$\frac{k^2-k'^2}{k_{,}^2}=\frac{m_{,}}{m}=\frac{(k+k')(k-k')}{(k+k')^2}=\frac{k-k'}{k+k'};$$

whence,

$$\frac{m-m_{,}}{m+m_{,}}=\frac{k'}{k}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (N)$$

whence again,

$$(k+k_{,})=2m=k_{,}, \text{ or } \frac{k_{,}}{k}=\frac{2m}{m+m_{,}}. \quad . \quad . \quad . \quad . \quad (O)$$

Deduction of Expressions for the Amplitudes.

38. (I.) On the hypothesis of *equal* densities, we have directly, on substituting in the equation (N) the values of the masses and dividing by k, supposed $=1$,

$$m=\sin 2i, \quad m_{,}=\sin 2r,$$

$$k'=\frac{\sin 2i-\sin 2r}{\sin 2i+\sin 2r}, \quad k_{,}=\frac{2\sin 2i}{\sin 2i+\sin 2r}. \quad . \quad . \quad (K')$$

These are Maccullagh's formulas deduced in the same way, and here (as before) k' changes sign at $i+r=90°$.

39. It is also evident that these values fulfil the equations of *vis viva* (M), viz.

$$(k^2-k'^2)=k_{,}^2\frac{\cos r}{\mu\cos i}=\sin 2i\sin 2r,$$

as well as that of equivalence of vibrations (L),

$$k+k'=k_{,}=2\sin 2i;$$

which last shows that they belong necessarily to vibrations *perpendicular* to the plane of incidence.

40. For vibrations *parallel* to the plane of incidence:—since we

are here concerned only with the *ratios* of the amplitudes, we may suppose one of them the same as in the last case, and thence find the others, which will be different in order to fulfil the different conditions. Thus assuming $k_{,}=h_{,}$, the condition of equivalence (L) gives

$$(h+h')=h_{,}\frac{\cos r}{\cos i}=2\sin 2i\frac{\cos r}{\cos i}=4\sin i\cos r\,;$$

and from this, with that of *vis viva* (M), viz.

$$(h^2-h'^2)=h_{,}^2\frac{\cos r}{\mu\cos i}=16\sin i\cos i\sin r\cos r=(h+h')(h-h')$$

$$(h-h')=4\cos i\sin r$$

$$2h=4\sin i\cos r+4\cos i\sin r$$

$$2h'=4\sin i\cos r-4\cos i\sin r\,;$$

and since always $\sin i > \sin r$ and $\cos r > \cos i$, the last value is always positive. Hence dividing by h, considered $=1$,

$$h'=\frac{\sin(i-r)}{\sin(i+r)},\quad h_{,}=\frac{\sin 2i}{\sin(i+r)}\,;\quad .\ .\ .\ \text{(H)}$$

the same values as those derived from the geometrical construction before mentioned (24).

41. (II.) On the hypothesis of *increased* density (or of increased retardation supposed to be represented by the same law), we have for the masses $m=\sin r\cos i$, $m_{,}=\sin i\cos r$, in which, since $\sin r<\sin i$ and $\cos i<\cos r$, we have always $m<m_{,}$, and therefore when we substitute in equation (N), we have

$$m-m_{,}=-(\sin i\cos r-\sin r\cos i)=-\sin(i-r).$$

Thus from that equation we have directly, dividing by h, as before,

$$h'=\frac{-\sin(i-r)}{\sin(i+r)},\quad h_{,}=\frac{2\sin r\cos i}{\sin(i+r)}.\quad .\ .\ \text{(H)}$$

These are Fresnel's formulas as derived originally from assuming the same equation (N) on the analogy of impact.

42. But it is also evident from the process of elimination by which that equation is here obtained, that these values fulfil the equation of *vis viva*,

$$(h^2-h'^2)=h_{,}^2\mu\frac{\cos r}{\cos i}=\sin 2i\sin 2r,$$

as well as that of equivalence of vibrations (L),

$$h+h'=h_{,},\ \text{or}\ \sin(i+r)-\sin(i-r)=2\sin r\cos i\,;$$

which last proves that they *necessarily* belong to vibrations *perpendicular* to the plane of incidence.

43. For vibrations *parallel* to the plane of incidence:—as

before, if we assume $k_{\prime}=2h_{\prime}$ to find k and k', the condition of equivalent vibrations (L) gives

$$k+k'=k_{\prime}\frac{\cos r}{\cos i}=4\sin r\cos i\frac{\cos r}{\cos i}=2\sin 2r.$$

Also the equation of *vis viva* (M) becomes

$$(k^2-k'^2)=k_{\prime}^2\mu\frac{\cos r}{\cos i}=4\sin 2i\sin 2r=(k+k')(k-k'),$$

whence

$$k-k'=2\sin 2i$$
$$2k=\sin 2r+\sin 2i$$
$$2k'=\sin 2r-\sin 2i.$$

Hence, as before,

$$k'=\frac{\sin 2r-\sin 2i}{\sin 2r+\sin 2i},\quad k_{\prime}=\frac{4\sin r\cos i}{\sin 2r+\sin 2i}.\ .\ .\quad \text{(K)}$$

And since for $i=\varpi$ or $(i+r)=90°$, $\sin 2r=\sin 2i$, we have $k'=0$; for incidences, $i<\varpi$, we have $\sin 2r<\sin 2i$, and therefore $-k'$; and for $i>\varpi$ $\sin 2r>\sin 2i$ or $+k'$.

44. Or again, *without assuming the value of* $k_{\prime}$, we may proceed thus: from the equation of *vis viva*,

$$(k^2-k'^2)\frac{\sin r\cos i}{\sin i\cos r}=k_{\prime}^2;$$

from the law of equivalence,

$$(k+k')^2\frac{\cos^2 i}{\cos^2 r}=k_{\prime}^2.$$

Equating these, we have

$$(k^2-k'^2)\sin r\cos^2 r\cos i=(k+k')^2\sin i\cos^2 i\cos r$$
$$(k^2-k'^2)\sin 2r=(k+k')^2\sin 2i$$
$$k^2(\sin 2r-\sin 2i)=2kk'\sin 2i+k'^2(\sin 2i+\sin 2r);$$

and observing that

$$2kk'\sin 2i=kk'\big((\sin 2i-\sin 2r)+(\sin 2i+\sin 2r)\big)$$
$$(k+k')k(\sin 2r-\sin 2i)=(k+k')k'(\sin 2i+\sin 2r)$$
$$k(\sin 2r-\sin 2i)=k'(\sin 2r+\sin 2i)$$
$$k+k'=2\sin 2r,\ \text{and thence}\ k_{\prime}=2\sin 2r\frac{\cos i}{\cos r};$$

or as before, dividing by k,

$$k'=\frac{\sin 2r-\sin 2i}{\sin 2r+\sin 2i},\quad k_{\prime}=\frac{4\sin r\cos i}{\sin 2r+\sin 2i}.$$

45. Now it is to be observed that these formulas differ from

those of Fresnel, as before given, in that the numerator of k' is here *negative* for $i < \varpi$ and *positive* for $i > \varpi$.

This difference is traceable to the process by which the deduction of Fresnel's original form was effected; it being made by means of two equations (see Airy's Tract, § 129), one of which is the same as the equation of *vis viva*, viz.

$$\sin r \cos i(k^2 - k'^2) = \sin i \cos r k_i^2;$$

the other is

$$\cos i(k - k') = \cos r\, k_{i},$$

which involves a different assumption as to the principle of equivalent vibrations, as before observed (30).

46. It will be desirable to follow the deduction of the formulas on this supposition.

For vibrations *perpendicular* to the plane of incidence, from the *vis viva*,

$$(h^2 - h'^2)\frac{\sin r \cos i}{\sin i \cos r} = h_i^2.$$

From *this* law of equivalence (30),

$$(h - h')^2 = h_i^2.$$

Hence

$$(h^2 - h'^2) \sin r \cos i = (h - h')^2 \sin i \cos r,$$

or

$$(h + h') \sin r \cos i = (h - h') \sin i \cos r$$

$$h(\sin i \cos r - \sin r \cos i) = h'(\sin r \cos i + \sin i \cos r),$$

whence

$$h' = \frac{\sin(i - r)}{\sin(i + r)}, \quad h_i = \frac{2 \sin r \cos i}{\sin(i + r)}.$$

These differ from Fresnel's *original* formulas in the *sign* of h'.

47. For vibrations *parallel* to the plane of incidence, on the same supposition (30) we have

$$k - k' = k_i \frac{\cos r}{\cos i};$$

and thence, as in (44),

$$(k^2 - k'^2) \sin 2r = (k - k')^2 \sin 2i;$$

whence, in like manner,

$$(k - k')k(\sin 2i - \sin 2r) = (k - k')k'(\sin 2i + \sin 2r),$$

and thence

$$k' = \frac{\sin 2i - \sin 2r}{\sin 2i + \sin 2r}, \text{ and } k_i = \frac{4 \sin r \cos i}{\sin 2i + \sin 2r},$$

which are the formulas of Fresnel *as originally deduced,* and in which (as before) we have $+k'$ for $i < \varpi$, and $-k'$ for $i > \varpi$.

48. Here, again, we might otherwise deduce the same formulas; as before (43) assuming $k_{,}=2h_{,,}$, we should have

$$k-k'=k\frac{\cos r}{\cos i}=2\sin 2r,$$

and $(k^2-k'^2)=4\sin 2r\sin 2i$ gives $k+k'=2\sin 2i$, whence

$$k'=\frac{\sin 2i-\sin 2r}{\sin 2i+\sin 2r}, \text{ and } k_{,}=\frac{4\sin r\cos i}{\sin 2i+\sin 2r}.$$

49. In another point of view we may observe the relation subsisting between all the above formulas (generalizing a remark in Mr. Power's paper), as follows:—

Trigonometrically, we have evidently,

$$\sin^2(i+r)-\sin^2(i-r)=\sin 2i\sin 2r, \quad . \quad . \quad . \quad . \quad . \quad \text{(P)}$$

$$[\sin 2i+\sin 2r]^2-[\sin 2i-\sin 2r]^2=4\sin 2i\sin 2r. \quad \text{(Q)}$$

50. Also from $\mu\sin r=\sin i$, we have the identical equation

$$4\mu\sin r\sin i\cos^2 i\cos r=4\sin^2 i\cos^2 i\cos r,$$

or

$$\mu\cos i(\sin 2r\sin 2i)=\cos r(\sin^2 2i);$$

whence, substituting in this equation the value from (P), it becomes

$$\mu\cos i[\sin^2(i+r)-\sin^2(i-r)]=\cos r(\sin^2 2i);$$

or from (Q) we have

$$\mu\cos i[(\sin 2i+\sin 2r)^2-(\sin 2i-\sin 2r)^2]=\cos r\,4\sin^2 2i;$$

which agree with the formulas of Maccullagh, and the equation of *vis viva* on the hypothesis of equal densities.

51. Again, we have the identical equation

$$4\cos^2 i\sin i\sin r\cos r=4\mu\sin r\cos^2 i\sin r\cos r,$$

or

$$\cos i(\sin 2i\sin 2r)=\mu\cos r(4\cos^2 i\sin^2 r);$$

whence from (P) we have

$$\cos i[\sin^2(i+r)-\sin^2(i-r)]=\mu\cos r(4\cos^2 i\sin^2 r);$$

or from (Q) we have

$$\cos i[(\sin 2i+\sin 2r)^2-(\sin 2i-\sin 2r)^2]=\mu\cos r(16\cos^2 i\sin^2 r);$$

which agree with Fresnel's formulas, and the equation of *vis viva* on the hypothesis of increased density.

52. But thus far nothing determines *the plane of vibration.* For this purpose we must apply the law of equivalent vibrations; and in either case, since it is the *squares* of the amplitudes which enter the formulas, we may take $\pm h'$, $\pm h_{,,}$, $\pm k'$, $\pm k_{,,}$, according

as they fulfil the conditions with the upper or lower sign: in this way we reproduce the foregoing results, which it is needless here to repeat.

Experimental Evidence.

53. That the whole series of experimental results relative to polarization are accurately represented by the foregoing formulas, has been fully substantiated by the researches of Fresnel, Arago, Brewster, and others; but these changes afford no distinctive test between the several theories here adverted to.

54. It may be desirable more particularly here to notice the interpretation of *the change of sign* in these formulas.

In general it is obvious, that in the elementary formula

$$u=h\sin\frac{2\pi}{\lambda}(vt-x);$$

if we suppose x increased to $x+\frac{\lambda}{2}$, we have

$$u'=h\sin\frac{2\pi}{\lambda}\left(vt-x-\frac{\lambda}{2}\right)=h\sin\left[\frac{2\pi}{\lambda}(vt-x)-\pi\right]=-h\sin\frac{2\pi}{\lambda}(vt-x),$$

or that a *difference of sign* indicates a *change of* 180° *in phase*, and that this is equivalent to *a change of* $\frac{\lambda}{2}$ *in route.*

55. This conclusion includes the solution of the difficulty as to the loss or gain of half a wave-length between reflexion at the first surface of a dense medium represented by $\sin(i-r)$, where we have $i>r$; and at the second, where we have $i<r$, which gives $-\sin(i-r)$,—a difficulty which so long embarrassed the early history of the theory, and which it is perhaps desirable to state thus clearly, though of so elementary a character, as it would sometimes appear to be still felt.

There are, however, other cases where this change of sign is material.

Intensities of Light.

56. The intensities of the reflected and refracted rays at different incidences, as compared with that of the incident light, are measured by the *vires vivæ*, that is, by the vibrating masses multiplied by the squares of the amplitudes or velocities.

Thus, considering the intensity of the incident ray as unity, for (H) let

$$I=mh'^2,\quad I_{,}=m_{,}h_{,}^2,$$

and for (K) let

$$J=mk'^2,\quad J_{,}=m_{,}k_{,}^2;$$

or dividing,

$$I = h'^2, \quad I_{,} = \frac{m_{,}}{m} h_{,}^2,$$

$$J = k'^2, \quad J_{,} = \frac{m_{,}}{m} k_{,}^2;$$

whence, calculating the values of h', $h_{,}$ and k', $k_{,}$ for the several incidences, we have the intensities directly.

But from the equation of *vis viva* we have

$$(1 - h'^2) = \frac{m_{,}}{m} h_{,}^2,$$

$$(1 - k'^2) = \frac{m_{,}}{m} k_{,}^2.$$

Thus, knowing only the values h' and k', we obtain at once those of the refracted intensities, instead of having to calculate $h_{,}$ and $k_{,}$, which is a troublesome process; thus we take simply

$$I_{,} = 1 - h'^2, \quad J_{,} = 1 - k'^2.$$

And this, it will be observed, is independent of any hypothesis as to *densities*.

57. If we calculate $I_{,}$ and $J_{,}$ *directly* as above, the values of the amplitudes will differ according to the hypotheses of the density: but then we have to observe, as before, that (denoting Fresnel's values by roman letters) we have (14) and (19)

$$h_{,} = \mathrm{h}_{,}\mu, \quad k_{,} = \mathrm{k}_{,}\mu, \quad \frac{m_{,}}{m} = \frac{\mathrm{m}_{,}}{\mathrm{m}} \frac{1}{\mu^2},$$

and thus

$$\frac{m_{,}}{m} h_{,}^2 = \frac{\mathrm{m}_{,}}{\mathrm{m}} \mathrm{h}_{,}^2, \quad \frac{m_{,}}{m} k_{,}^2 = \frac{\mathrm{m}_{,}}{\mathrm{m}} \mathrm{k}_{,}^2.$$

Thus, either way, the *calculated intensities will be the same on either hypothesis.*

58. To trace the intensities at different incidences:—

(1.) At the perpendicular incidence $i=0$ and $r=0$. Thus the formulas become $h' = \frac{0}{0}$, $k' = \frac{0}{0}$. But the actual values may be found, since at perpendicular incidences we have (16),

For equal densities, $\frac{m_{,}}{m} = \frac{1}{\mu}$;

For increased density, $\frac{m_{,}}{m} = \mu$.

In either case the formula $h' = \frac{m - m_{,}}{m + m_{,}}$ will give

$$h'^2 = \frac{(\mu - 1)^2}{(\mu + 1)^2}.$$

This was Dr. Young's formula, deduced in an early stage of these inquiries.

If we suppose $\mu=1{\cdot}5$, this will give

$$I=h'^2=\left(\frac{\cdot 5}{2{\cdot}5}\right)^2=\cdot 04,$$

$$I_{,}=1-h'^2=\cdot 96.$$

Also
$$k'=h'\frac{\cos(i-r)}{\cos(i+r)}=h',$$

whence
$$J=k'^2=h'^2=\cdot 04, \quad J_{,}=\cdot 96.$$

(2.) For the angle of complete polarization $i+r=90^\circ$,

$\sin(i-r)=\cos 2r, \quad \sin(i+r)=1, \quad \tan i=\mu=1{\cdot}5=\tan 56^\circ\ 19'$,

$$I=h'^2=\cos^2 2r, \quad J=1-h'^2=\sin^2 2r,$$

$$J=k'^2=\frac{\tan^2(i-r)}{\infty}=0, \quad J_{,}=1-k'^2=1.$$

(3.) In the limit of oblique incidence,

$i=90^\circ, \quad \sin(i-r)=\cos r, \quad \sin(i+r)=\cos r,$

$I=h'^2=1, \quad J=1-h'^2=0,$

$\tan(90-r)=\tan(90+r),$

$I_{,}=k'^2=1, \quad J_{,}=1-k'^2=0.$

(4.) For these and other incidences generally, the following table will show the *approximate* intensities.

Intensities at different Incidences.

$\mu=1{\cdot}5$.		Formulas H.		Formulas K.	
i.	r.	$I=h'^2$.	$I_{,}=(1-h'^2)$.	$J=k'^2$.	$J_{,}=(1-k'^2)$.
0° 0′	0° 0′	·040	·960	·040	·960
20	13 10	·047	·953	·034	·966
40	25 22	·077	·923	·014	·986
56 19	33 41	·152	·848	·000	1·000
60	35 16	·176	·824	·002	·998
80	41 2	·515	·485	·197	·803
90	41 49	1·000	·000	1·000	·000

59. The necessary imperfections of photometry preclude any accurate verification of these numerical results; but by throwing the two images of a round hole in an opake plate covering one end of a rhomboid of iceland spar on a surface of glass, at different incidences, the eye can readily compare the intensities in some of the more marked cases, both of the reflected and the transmitted rays, of the two beams polarized in planes at right angles to each other; which show a general agreement with theory.

Opportunities (concerning or the subject of accurate experimental determinations which is partly impracticable from the essential defects of photometry) would afford any test of the rival theories, & how they agree.

If such ~~certain~~ deviations are then applied to a series of values for diff. incidences it might serve to test the entire theory. Such a series (only approximately computed) is subjoined from which it appears that the reflected ray ~~then~~ in H. Refl. shows a ~~decrease~~ a regular increase from incidence up to 90°. more rapid near ~~the~~ 90°. in K a decrease up to 70 & then a rapid increase.

The refracted ray in H shows then a slight diminution, but rapid near. in K a slight increase to 70 & then a rapid decrease to 90°.

60. The question whether the plane of vibration is parallel or

plate of glass capable of being [illegible] inclined to the rays. The annexed construction is a very convenient one for the purpose. It is a tube capable of turning about its [illegible] & containing a long [illegible]

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rhomboid of calc spar, whose principal section can therefore be turned so as to throw the two images h k, polarised in opposite planes, on the glass g in a vertical line giving the two reflected images h' k' and the two refracted h, k,. The glass is fixed perpendicularly to the plane of a graduated circle &

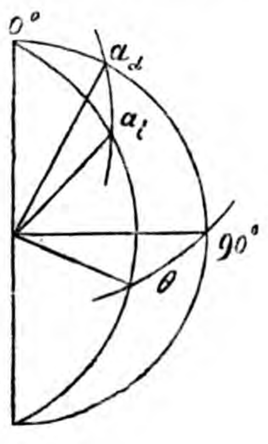
Continued [illegible]

[illegible]

60. The question whether the plane of vibration is parallel or perpendicular to the plane of polarization, has been lately discussed at large by M. Haidinger, who, after an extended comparative view of the experimental consequences involved in connexion with various optical phænomena *on either supposition*, decides in favour of the superior simplicity and consistency of *Fresnel's* view, that the plane of vibration is *perpendicular* to the plane of polarization. (See Phil. Mag. March 1856, vol. xi. p. 242, No. 71. The original is in Poggendorff's *Annalen*, Oct. 1855. See also Silliman's Journal, Jan. 1856.)

61. A very refined and ingenious suggestion for a direct "*experimentum crucis*" was made by Mr. J. A. Dale to the British Association in 1846, but practical difficulties appear to stand in the way of its application. (See 'Report,' 1846, Section A.)

62. But a more recent result of Prof. Stokes is perfectly conclusive on the question of the direction of the plane of vibration. In his paper "On the Dynamical Theory of Diffraction" (Cambridge Transactions, vol. ix. part 1, 1849), he has established theoretically the conclusion, that the *plane of vibration of a diffracted ray previously polarized, is in general different from that of the incident, except when perpendicular, and when parallel, to the plane of diffraction.* For intermediate positions their directions are connected by a simple law, which may be expressed by saying, that if the arcs of inclination of the vibrations to the perpendicular to the plane of diffraction be measured on two semicircles intersecting in a common diameter coinciding with that perpendicular, and inclined to each other at the angle of diffraction, then the arc of inclination of the *incident* vibration, measured from the point of intersection, being taken as *longitude*, that of the *diffracted* vibration will be the corresponding *right ascension.*



If the two semicircles, whose planes are respectively perpendicular to the directions of the incident and the diffracted rays, be inclined at an angle θ, equal to the angle of diffractive deviation, and if α_d and α_i be the inclinations of the diffracted and incident vibrations measured from the intersection 0°, by Mr. Stokes's theory they are connected by the equation

$$\tan \alpha_d = \cos \theta \tan \alpha_i,$$

which leads to the construction here represented by the arc of a great circle passing through the extremities of α_d and α_i perpendicular to the plane of α_d. Whence it follows, that *near the intersection* (0°), that is the *perpendicular* position, the changes in the inclination of the diffracted ray will be much less rapid, or the indications more *crowded* together, than near the *parallel* position (90°), or the plane of diffraction.

But we can *observe* the changes only in the plane of *polarization.* If, then, these changes are more slow near the *perpendicular* (0°), the planes of *polarization* are *parallel* to those of *vibration*; if near the *parallel* (90°), then they are *perpendicular* to those of vibration.

A series of experiments of the most elaborate and accurate kind unequivocally show *the latter to be the fact.*

63. *Thus experiment obliges us to adopt Fresnel's hypothesis of vibrations perpendicular to the plane of polarization,* and by consequence (as here shown) *either increased density or some other property expressed by the same law.*

In thus being compelled to relinquish the hypothesis of equal density, or at least the formulas expressing it, we do not, in fact, sacrifice anything in point of simplicity, the same amount of analysis being requisite to deduce the expressions on either supposition; and in giving up the beautiful geometry of Maccullagh, we do ample justice to his more substantial discoveries,—the general laws of equivalence of vibrations (whatever difference may arise on a subordinate point), and their connexion with the principle of conservation of *vis viva,* on which the whole theory reposes. There are, however, some other points hinted at in what precedes, which may demand further inquiry at a future opportunity.

From the PHILOSOPHICAL MAGAZINE for August 1856.

ON THE

DEMONSTRATION OF FRESNEL'S FORMULAS

FOR

REFLECTED AND REFRACTED LIGHT;

AND THEIR APPLICATIONS.

BY

THE REV. BADEN POWELL, M.A., F.R.S. ETC.,

SAVILIAN PROFESSOR OF GEOMETRY IN THE UNIVERSITY OF OXFORD.

1. IN a former paper (see Phil. Mag. &c. July 1856) I have placed in a connected point of view the several principles and deductions leading to the well-known formulas of Fresnel, as well as to certain modifications of them, for the amplitudes of the vibrations of the incident, reflected, and refracted rays, whether polarized parallel or perpendicular to the plane of incidence. I have also remarked on the question which has so long divided opinions, whether the *vibrations* are *parallel* or *perpendicular* to the plane of *polarization*, and on the decisive evidence lately obtained in favour of the latter hypothesis. Some other questions relative to the same subject still demand examination, to which I propose now to refer.

2. For this purpose it will be necessary briefly to premise a *recapitulation* of the primary principles on which the several investigations proceed, and which are fully discussed in my former paper. These are,—

I. The principle of *vis viva*; (1) that the square of the velocity multiplied by the vibrating mass is the true measure of force; (2) that the *vis viva* of the *incident* vibrations is equal to the sum of the *vires vivæ* of the *reflected* and *refracted* vibrations. Or, m and $m_{/}$ being the simultaneously vibrating masses of æther without and within the medium, h, h', $h_{/}$ respectively the amplitudes (which are the measures of the velocities) of the incident, reflected, and refracted vibrations;—then the law of *vis viva* is expressed by the equation

$$m(h^2-h'^2)=m_{/}h_{/}^2.$$

II. The law of equivalent vibrations, which on Maccullagh's view is expressed by

(α) . . . $h+h'=h_{\prime}$ for vibrations *perpendicular* to the plane of incidence,

and (i and r being the angles of incidence and refraction)

(β) . . . $h+h'=h_{\prime}\frac{\cos r}{\cos i}$ for vibrations *parallel* to the plane of incidence.

III. On the principle adopted by Fresnel in the second case (β), the same law would be expressed by

(α) . . . $h-h'=h_{\prime}$ for vibrations *perpendicular* to the plane of incidence,

(β) . . . $h-h'=h_{\prime}\frac{\cos r}{\cos i}$ for vibrations *parallel* to the plane of incidence.

IV. Maccullagh's hypothesis of equal density giving

$$\frac{m}{m_{\prime}}=\frac{\sin 2i}{\sin 2r}.$$

V. Fresnel's hypothesis of increased density giving

$$\frac{m}{m_{\prime}}=\frac{\sin r \cos i}{\sin i \cos r}.$$

VI. Maccullagh's hypothesis of vibrations parallel to the plane of polarization.

VII. Fresnel's hypothesis of vibrations perpendicular to the plane of polarization.

3. By different combinations of these principles, different modifications of the formulas result. Thus we have the hypotheses—

(A) combining Nos. I. II. IV. VI., whence are obtained the formulas

$$h'=\frac{\sin(i-r)}{\sin(i+r)} \; . \; . \; . \; . \; h_{\prime}=\frac{\sin 2i}{\sin(i+r)} \; . \; . \; . \; . \text{ vibrations } \textit{parallel},$$

$$k'=\frac{\tan(i-r)}{\pm\tan(i+r)} - \; . . . . k_{\prime}=\left(1+\frac{\tan(i-r)}{\pm\tan(i+r)}\right) . . . \textit{perpendicular},$$

which are Maccullagh's formulas; the double sign indicating the change at the polarizing angle.

(B) Combining Nos. I. II. V. VII., whence are obtained,

$$h'=\frac{-\sin(i-r)}{\sin(i+r)} \quad h_{\prime}=\frac{2\sin r\cos i}{\sin(i+r)} \; . \; . \; . \; . \text{ perpendicular},$$

$$k'=\frac{-\tan(i-r)}{\pm\tan(i+r)} \quad k_{\prime}=\left(1-\frac{\tan(i-r)}{\pm\tan(i+r)}\right)\frac{\cos i}{\cos r} \; . \; . \; . \; . \text{ parallel}.$$

(C) Combining Nos. I. III. V. VII., whence are obtained,

$$h' = \frac{\sin(i-r)}{\sin(i+r)} \quad h_{,} = \frac{2\sin r\cos i}{\sin(i+r)} \; \ldots\ldots \text{perpendicular},$$

$$k' = \frac{\tan(i-r)}{\pm\tan(i+r)} \quad k_{,} = \left(1 - \frac{\tan(i+r)}{\pm\tan(i+r)}\right)\frac{\cos i}{\cos r} \; \ldots\ldots \text{parallel}.$$

4. Each of these two last sets differs from Fresnel's in the *signs*. Fresnel's *original* formulas can only be produced from assuming hypothesis (B) for h, and (C) for k; or we have,—

(D) combining Nos. I. IIα. IIIβ. V. and VII., whence are obtained,

$$h' = \frac{-\sin(i-r)}{\sin(i+r)} \quad h_{,} = \frac{2\sin r\cos i}{\sin(i+r)},$$

$$k' = \frac{\tan(i-r)}{\pm\tan(i+r)} \quad k_{,} = \left(1 - \frac{\tan(i-r)}{\pm\tan(i+r)}\right)\frac{\cos i}{\cos r},$$

which are Fresnel's original formulas.

5. With regard to the law of equivalent vibrations, it may indeed be remarked that Prof. Maccullagh in stating it, with a view to his ulterior researches on crystalline reflexion, rather *assumes* than *demonstrates* the main principle, and thus the modified form of that law (No. III.) may possibly be as open to consideration as the original form. But as *neither* form exclusively will produce Fresnel's *original* formulas, it becomes of more importance to look to some other principle which might account for Fresnel's adoption of No. III. in one case (β) and not in the other (α). That is, for vibrations *parallel* to the plane of incidence this form is avowedly adopted by him, on the ground that for that case the sign of the reflected vibration will be opposite to that of the incident at small incidences. But in the case of vibrations *perpendicular* to the plane of incidence, he not only makes no assumption of this form (IIIα.), but his result being deduced directly from the analogy of impact—if this is to be analysed up to the deduction of the same equation from the principle of the *vis viva*—it is evident (as shown in my former paper (37)) that that analysis involves necessarily the assumption of the law of equivalence No. II. Fresnel himself, however, did not so deduce the equation, but appears to have simply assumed it on the analogy of impact of elastic bodies.

6. The assumption thus made by Fresnel (which he admits to be somewhat empirical) in the one case, while no such assumption is made in the other, may possibly be accounted for on the consideration, that, in the case of vibrations parallel to the plane of incidence alone, we can have any direct application of the parallelogram of forces, from the construction of which he may possibly have been led to this inference. In the other case, where

the vibrations are all perpendicular to the plane of incidence, and parallel to the surface and to each other, no such construction can apply; nor does he seem to have extended the idea of mechanical equivalence by analogy to this case, as Maccullagh has done.

7. The crucial experiment of Professor Stokes obliges us to set aside the supposition No. VI. (see former paper (62)), and consequently (as the analysis shows) the supposition No. IV., and thus the whole hypothesis (A) or Maccullagh's formulas. Our choice then lies between the other hypotheses, or some new combination. And it remains to decide whether the *original form* of Fresnel's formulas is of necessity required by any experimental results, or whether the forms deduced on either of the hypotheses (B) or (C) will be equally applicable; in other words, whether *the difference in the signs* is of importance.

Now in fact two well-known cases of experimental results are adducible which *appear* to have a direct bearing on this question, in which light we will proceed to examine them.

Change in plane of Polarization, by Reflexion.

8. It is a result long ago ascertained by the researches of Fresnel, Arago, and Brewster (and, as far as the principal point is concerned, easily capable of verification), that if a ray previously polarized in a plane inclined at a given angle to the plane of incidence fall on a reflecting surface, then, after reflexion, *in general its plane of polarization is changed*; at incidences *less* than that of complete polarization, the new plane of polarization *deviates* on the side of the plane of incidence *opposite* to that of the original plane of polarization; at the incidence of complete polarization it *coincides* with the plane of incidence; at incidences *greater* than that of complete polarization it *deviates* on the *same side* as the original plane.

9. Now these are precisely the changes indicated by a very simple deduction of theory, derived from the *original* formulas of Fresnel. The deduction is well known, but it will be desirable to exhibit its nature explicitly as follows:—

Let the original plane of polarization (P) be inclined to that of incidence (I) by an angle (α), then after reflexion the parts of the amplitude resolved parallel and perpendicular to (I), viz.—

$$k' \sin\alpha = \frac{\tan(i-r)}{\pm\tan(i+r)} \sin\alpha \text{ and } h' \cos\alpha = \frac{-\sin(i-r)}{\sin(i+r)} \cos\alpha,$$

will by composition give a resultant ray polarized in a plane (Q) inclined to (I) by an angle (β), where

$$\tan\beta = \left(\frac{\pm k'}{-h'}\right) \tan\alpha.$$

Hence at first, the tangents having opposite signs, the arcs α and β lie in *adjacent* quadrants; at $i+r=90^\circ$ $\beta=0$; and for incidences greater, the tangents having the same sign, α and β lie in the *same* quadrant, which exactly expresses the experimental results.

10. It should be remarked that this is the reasoning adopted by Mr. Airy (§ 132), and that the undeniable conclusion follows simply by virtue of the *symbols*, without the introduction of any extraneous construction or subsidiary consideration whatever.

11. Now Dr. Lloyd (in his 'Lectures on the Wave Theory,' part 2, p. 35), after stating the facts, gives the same deduction from theory, using *Maccullagh's* formulas; but with due caution makes the inference only so far as to show the *existence* of a deviation to the same *amount, without expressing in which direction.* But according to *these* formulas, tan β is *at first positive*, and changes to *negative* at the polarizing incidence. Hence as to the *direction* of the deviation, *if the former reasoning be correct*, then in this case, *on the same grounds*, the symbols would indicate that the deviation at incidences *less* than that of polarization must be on the *same* side as the original plane, and *after* that incidence, on the *opposite* side; which is the *reverse* of the *former* conclusion, and of *the fact.*

12. This, however, would appear to be only an additional reason to that already assigned for the rejection of Maccullagh's theory. But it is easily seen that precisely the same remarks apply if, instead of Maccullagh's formulas, we adopt either of the new forms of Fresnel's, viz.—

(B) which gives $\left(\dfrac{\mp k'}{-h'}\right)$,

or

(C) which gives $\left(\dfrac{\pm k'}{+h}\right)$.

The result of experiment would therefore *seem* equally decisive against the hypothesis (A), and against (B) and (C) applied to *both* h' and k', and would leave us to the sole adoption of Fresnel's original formulas (D). But before coming to this conclusion, we must advert to another experimental case in which similar considerations are involved.

Dr. Lloyd's Interference Experiment.

13. In an experiment long ago devised by Dr. Lloyd, a *direct* ray interferes with one *reflected* at a *very oblique* incidence from a plate of glass; giving a set of stripes *resembling* generally one half of those formed in the experiment of Fresnel when *two* streams of *direct* light interfere (Mem. Roy. Irish Acad. vol. xvii. 1834).

In this experiment, it is first to be remarked that the first dark band is *intensely black,* proving the *equal* intensity of the interfering rays; in agreement with theory, which at the extreme incidence gives, on either hypothesis, reflected light of equal intensity with the incident (see former paper (§ 58)).

14. The stripes, however, present this peculiarity: *on ordinary suppositions,* since the rays at the extreme limit must be in *accordance,* there would first be *half a bright band* (reckoning from the edge) followed by a whole dark, then a whole bright band, and so on; whereas, in fact, the first bright band is observed to be of the *entire breadth* of an interval, as if the system commenced from a point of *discordance* at the extreme limit, or as if the whole were *shifted* from the edge through a distance equal to *half the interval* of two bands. This was found to be equally the case whether the light was previously polarized in a plane *parallel* or *perpendicular* to the plane of incidence. In other words, the fact indicates a difference in phase of 180° between the incident and the reflected ray at the limit, and this equally in each case of the direction of polarization.

15. Now in Fresnel's original formulas for vibrations perpendicular to the plane of incidence, a difference of *sign,* indicative of this difference of *phase,* occurs at all incidences in the formulas for h', and for vibrations *parallel* to the plane of incidence at all incidences greater than that of complete polarization in the formulas for k'; that is, at very *oblique* incidences for rays polarized in one plane equally with those in the other; in exact agreement with the observed fact.

16. This is the argument expressly adduced by Dr. Lloyd, derived *directly,* without any subsidiary considerations, from the indications of the symbols; and it has, I believe, been generally received as perfectly clear and conclusive. It should be remembered that Dr. Lloyd's paper was communicated and published in 1834, and refers to Fresnel's formulas *in the form in which he originally gave them,* and before any modifications of the theory had been contemplated. Prof. Maccullagh's new views and formulas were first communicated to the Royal Irish Academy in 1837, and published in its Memoirs in 1838. Thus Dr. Lloyd's reasoning is of necessity wholly independent of any speculations on these newer principles, which, if applied to Fresnel's theory, give the formulas with different *signs.*

17. Now so far as the mere symbols are concerned, we find in Fresnel's formulas (B) this difference of sign, at great incidences, occurring only for h'; and in the formulas (C) (as is the case also in Maccullagh's formulas) only for k'; while in (B) for k', and in (C) for h', no such difference occurs.

This result would therefore *seem* decisive in favour of Fresnel's *original* formulas (D), to the *exclusion* not only of Maccullagh's, but of Fresnel's, in the entire forms (B) and (C).

18. The reasoning both in this case and the former (9), (10), is indeed of a nature apparently so obvious, that I should not have thought it necessary to state it in detail, were it not that some considerations *have been suggested* which seem to set it aside, or at any rate to require a closer review of its meaning;—and which may be stated as follows:—

19. For vibrations *parallel* to the plane of incidence, if we make a construction of the course of the rays, then, at an incidence very near the perpendicular, the vibrations k and k' respectively perpendicular to the incident and reflected rays will lie very nearly in one line; and if we suppose them in the *same* direction, then in passing to an incidence extremely oblique, they will come into directions *opposite* to each other, owing merely to the *position* the rays have now assumed.

Hence it is inferred, whatever relative directions of the vibrations we express by the signs + or − at $i=0$ will be reversed when we pass to $i=90°$. And thus, for example, in the formula (Bk'), if at $i=0°$ we suppose $-k'$ and $+k$ to express *opposite* directions, then at $i=90°$ the *same* signs would express *accordant* directions. But in fact at $i=90°$ we have $+k'$ and $+k$, which on this principle therefore now express opposing directions.

20. In the case of Fresnel's and Arago's result (see above, (9)) of the change in the plane of polarization after reflexion, it should be remarked that the reasoning turns wholly upon the *signs* of the resulting equation for the *tangents* of the two related arcs; and these are derived from the relative *signs* of the quantities which enter into the numerator and denominator *simply as algebraic quantities, and without any reference whatever to the interpretation* of those signs as expressive of *difference of phase, or any other physical conditions.* The resulting relation is a purely trigonometrical one, and the direction in which the arcs are to be relatively measured would be just the same whatever might be the physical theory to which they were to apply. *This* case therefore can in no way be affected by any theory of the change of direction due to the position of the vibrations. Here, then, Fresnel's original formulas exclusively apply.

21. The case then stands thus:—From the experiment of Prof. Stokes, Maccullagh's formulas are set aside. From the experiments of Lloyd and Arago, so far as relates to vibrations *perpendicular* to the plane of incidence, Fresnel's formula (Ch') is set aside, since it does not apply directly; and for such vibrations the subsidiary theory as to the signs is inapplicable; while Fresnel's formula (Bh') fully agrees with experiment without any subsidiary explanation. *This formula then must necessarily be adopted exclusively.*

22. It is only, then, in the case of vibrations *parallel* to the plane of incidence that the question remains between Fresnel's formula (Ck'), which applies *directly* to the experimental results, but rests on an hypothesis (No. III.) of equivalent vibrations open to question,—and his formula (Bk'), which rests on Maccullagh's law of equivalent vibrations (No. II.), but does not apply to the experimental facts without a subsidiary interpretation as to the *signs*.

23. Now any subsidiary construction, if necessary for the right interpretation of a formula, only shows that *that formula is symbolically incomplete*, and does not include the expression of the *whole* case it is designed to represent. This, then, would be an additional reason for rejecting that formula and adopting the former in preference.

24. But this construction (19) is in itself not free from question and difficulty. For let us only consider the case at a point *upon the surface*: here we have an incident *rectilinear* vibration in a determinate direction; but at its point of incidence it gives rise to a nascent *circular* wave, in which it is impossible to say that the vibration is more in one direction than another; it has no rectilinear direction till we come to take the common tangent (as in the ordinary explanation of reflexion) to two successive waves; and there is nothing to determine the direction of vibration at the points coinciding with this tangent. We cannot therefore make any assumption as to its constancy at different incidences; there is nothing to show that it may not change from one incidence to another; and we know that it does undergo such a change at the incidence of complete polarization.

25. On the other hand, the formula (Ck') applies *directly* to the phænomena without the aid of any subsidiary construction whatever, *unless indeed the whole principle of the reasoning hitherto adopted be contested.*

But if we may still accept the reasoning of Fresnel and Arago, of Airy and Lloyd, as valid, then it follows that *both* Fresnel's *original* formulas, and they *alone,* will apply directly to all the experimental results without any extraneous considerations. But one of them (k') rests on an assumption as to the law of equivalent vibrations *different* from the other (h'); and the question remains, whether that difference of assumption can be justified, or whether any other view of the theoretical principles can be found to lead to the same results.

I have thus stated in full detail the difficulties of this important case; and will only add, that I shall look with great interest to any attempts at removing them, which I hope this representation may be the means of eliciting from those mathematicians who have attended to the subject.

From the PHILOSOPHICAL MAGAZINE for October 1856.

ON THE DEMONSTRATION OF FRESNEL'S FORMULAS FOR REFLECTED AND REFRACTED LIGHT.

No. III.

BY

THE REV. BADEN POWELL, M.A., F.R.S. ETC.,

SAVILIAN PROFESSOR OF GEOMETRY IN THE UNIVERSITY OF OXFORD.

1. IN two former papers (Phil. Mag. July and August 1856), especially in the last paper, I have shown that, on received principles, the *original* formulas of Fresnel are apparently necessary for the application of theory to certain experimental results, to the exclusion of some *newer* modifications, though deduced on more systematic theoretical grounds.

In opposition to this, however, another view has been suggested (as there mentioned), which, if true, would set aside all the reasoning hitherto adopted on the subject, but which to me seems open to great doubt in itself.

It is, however, clear that Fresnel's original formulas cannot *both* be deduced on any common principle hitherto proposed, it being, as far as yet appears, necessary to assume a separate hypothesis for each of the two cases, and these not apparently reconcileable with each other.

2. If the considerations I have adduced in my second paper (§§ 24, 25) be regarded as well founded, it becomes highly important to find some mode of deducing both Fresnel's original formulas on a common principle.

But whether the arguments I have advanced be thought valid or not, it must still be allowed, on all hands, to be a matter of some interest if possible to suggest a proof free from the objections mentioned.

Since writing those papers, it has appeared to me that this may be effected, provided the following considerations be admitted relative to the law of equivalent vibrations, which (as before hinted) appears to be the doubtful element in the former investigations.

3. In the case (α) of vibrations *perpendicular* to the plane of incidence, and where the incident, reflected and refracted, vibrations are all parallel to each other and to the surface, there is no difficulty. Here there is no geometrical construction from which to find the relation of the amplitudes. In this case the proof of equivalence depends directly on mechanical considerations alone, agreeably to the reasoning referred to before (first paper, § 26). Here h, h', and $h_{\prime}$ being the mechanical values of the amplitudes, we have simply for the law of equivalence,

$$h_{\prime}=h+h'.$$

4. In the case (β) of vibrations *parallel* to the plane of incidence, we have the obvious geometrical construction of the triangle formed by the known directions of the rays, and consequently of the amplitudes at right angles to them, and in the same plane (first paper, § 23). In this triangle, the angles being $(i-r)$, $(i+r)$, and $(\pi-2i)$, the ratios of the sides will be (as before)

$$\frac{h'}{h}=\frac{\sin(i-r)}{\sin(i+r)}, \quad \frac{h_{\prime}}{h}=\frac{\sin 2i}{\sin(i+r)}.$$

5. On Maccullagh's theory of *equal* densities, it will be remembered that this construction represents the mechanical equivalence, and the sides thus give the values of the amplitudes.

The close agreement of these values with those for the amplitudes in case (α) deduced on the hypothesis of *increased* density, —to which no such construction can apply,—has been already remarked (first paper, §§ 25, 41).

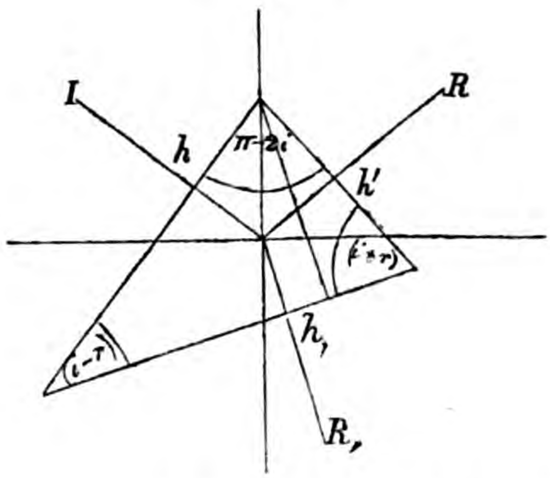
But these values cannot be those of the amplitudes in case (β) on the hypothesis of increased density, since experimental results essentially require a different relation in this case from that in case (α).

6. From the triangle, however, we have this obvious and simple relation of equivalence,

$$h_{\prime}=h\cos(i-r)+h'\cos(i+r);$$

or expressing these components parallel to $h_{\prime}$ by (h) (h'), we have

$$h_{\prime}=(h)+(h').$$



But in this construction, it must be borne in mind that if we assume the side $h_{\prime}$ as the value of the refracted amplitude, it is represented geometrically on the same scale with the other sides by which we measure the components; and for its actual value it is necessary to take into account its diminution in the more retarding medium in the ratio of the refractive index; or if $(h_{\prime})$ be this physical value, we must take

$$(h_{\prime})=h_{\prime}\frac{1}{\mu}=\big((h)+(h')\big)\frac{1}{\mu}.$$

Thus both cases of equivalence may be included in one expression (θ being the inclination of the plane of vibration to that of incidence),

$$\big((h)+(h')\big)\left(\sin\theta+\cos\theta\frac{1}{\mu}\right)=(h_{\prime}),$$

where $\theta=90°$ gives case (α), and $\theta=0°$ gives case (β).

7. If this be admitted as the true expression of the law of equivalence, we can deduce *both Fresnel's original formulas* from *this* law of equivalence combined with that of *vis viva*, viz.

$$(h^2-h'^2)m=h_{\prime}^2m_{\prime}.$$

(α) For vibrations *perpendicular* to the plane of incidence, the law of equivalence, $(h+h')=h_{\prime}$, combined with the law of *vis viva*, gives

$$(h-h')\frac{m}{m_{\prime}}=\frac{h_{\prime}^2}{h+h_{\prime}}=h+h_{\prime}.$$

But on the hypothesis of increased density or retardation, as before (first paper, § 17),

$$\frac{m}{m_{\prime}}=\frac{\sin r\cos i}{\sin i\cos r}$$

$$(h-h')\sin r\cos i=(h+h')\sin i\cos r,$$

or

$$h(\sin r\cos i-\sin i\cos r)=h'(\sin r\cos i+\sin i\cos r)$$

$$\frac{h'}{h}=\frac{-\big(\sin(i-r)\big)}{\sin(i+r)},\quad \frac{h_{\prime}}{h}=\frac{2\sin r\cos i}{\sin(i+r)}.$$

8. (β) For vibrations *parallel* to the plane of incidence, the law of equivalence (writing $kk'k_{\prime}$ instead of (h) (h') $(h_{\prime})$),

$$(k+k')\frac{1}{\mu}=k_{\prime},$$

combined with the law of *vis viva*, gives

$$(k-k')\frac{m}{m_{\prime}}=\frac{k_{\prime}^{2}}{k+k'}=(k+k')\frac{1}{\mu^{2}}.$$

But

$$\frac{m}{m_{\prime}}\mu^{2}=\frac{\sin r\cos i}{\sin i\cos r}\cdot\frac{\sin^{2}i}{\sin^{2}r}=\frac{\sin 2i}{\sin 2r}$$

$$(k-k')\sin 2i=(k+k')\sin 2r,$$

or

$$k(\sin 2i-\sin 2r)=k'(\sin 2i+\sin 2r)$$

$$\frac{k'}{k}=\frac{\sin 2i-\sin 2r}{\sin 2i+\sin 2r}=\frac{\tan(i-r)}{\tan(i+r)}$$

$$k_{\prime}=(k+k')\frac{1}{\mu}2\sin 2i\frac{1}{\mu}=4\sin r\cos i$$

$$\frac{k_{\prime}}{k}=\frac{4\sin r\cos i}{\sin 2i+\sin 2r}=\left(1-\frac{\tan(i-r)}{\tan(i+r)}\right)\frac{\cos i}{\cos r}.$$

9. These amplitudes are portions of the sides of the triangle taken in the ratio of the components; and if in that ratio before given we substitute the value of $\frac{h'}{h}$, we find

$$\frac{(h')}{(h)}=\frac{\sin(i-r)\cos(i+r)}{\sin(i+r)\cos(i-r)}=\frac{\sin 2i-\sin 2r}{\sin 2i+\sin 2r},$$

which coincides with the value just deduced on the principle of the *vis viva*.

Thus for both cases we have Fresnel's *original* formulas.

10. It is also worthy of remark, that as the investigation of case (α) is obviously equivalent to the direct application of the formula for impact, viz.

$$v = \frac{m - m_{\prime}}{m + m_{\prime}}, \quad v_{\prime} = \frac{2m}{m + m_{\prime}},$$

so on the principle here adopted for case (β), it is easily seen that the same is true, if here, instead of m $m_{\prime}$, we take (as above (8))

$$(m) = \frac{m \sin i}{\sin r} = \sin 2i \text{ and } (m_{\prime}) = \frac{m_{\prime} \sin r}{\sin i} = \sin 2r.$$

The result is the same as would be obtained directly from the equations

$$v = \frac{(m) - (m_{\prime})}{(m) + (m_{\prime})}, \quad v_{\prime} = \frac{2(m)}{(m) + (m_{\prime})}.$$

11. It is also remarkable that this result is the same as if in case (α) we had supposed increased density, but in case (β) equal densities.

It may also be observed, that in the investigations of Fresnel and Maccullagh the analogy of impact is referred to only in the case of vibrations *perpendicular* to the plane of incidence, where the mechanical analogy is not apparent; whereas in the present investigation we see that that analogy applies also to the case of vibrations *parallel* to the plane of incidence, which is the actual case in ordinary mechanical impact.

Erratum in first Paper.

§ 37, line 2, *dele* " or direction of vibrations."

which is the same as

$$\sin \frac{2\pi}{\lambda}\left(vt - x \pm \frac{\lambda}{2}\right) + a)$$

or the one wave differs from the other by half a wavelength & the same with between from $\frac{3\lambda}{2}$ $\frac{5\lambda}{2}$ &c

~~$a \sin \frac{2\pi}{\lambda}(vt - x + a) \pm$~~

taken a sin ± a ±