A

TREATISE

ON THE

GEOMETRICAL REPRESENTATION

OF THE

SQUARE ROOTS OF NEGATIVE QUANTITIES.

BY THE

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CHAP. I.

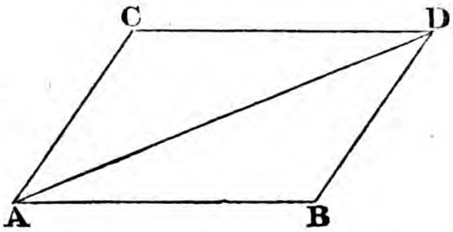
DEFINITIONS, ADDITION, SUBTRACTION, PROPORTION, MULTIPLICATION, DIVISION, FRACTIONS, AND RAISING OF POWERS.

(ART. 1.) ALL straight lines drawn in a given plane, from a given point, are represented in *length* and *direction* by Algebraic quantities; and in the following Treatise whenever the word *quantity* is used, it is to be understood as signifying a *line*.

(2.) DEF. The given point from which the straight lines are measured is called the *origin*.

(3.) DEF. The *sum* of two quantities is the diagonal of the parallelogram whose sides are the two quantities.

Thus if a represent AB in length and direction,

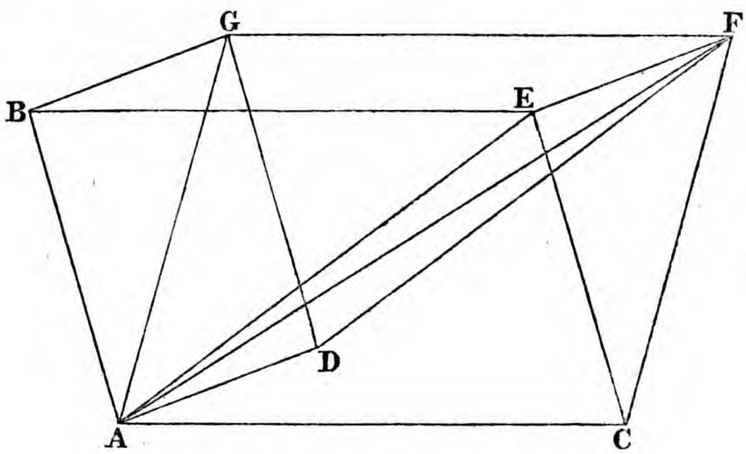


and b represent AC in length and direction; and the parallelogram $ABDC$ be completed, and the diagonal AD be drawn; $a+b$ represents AD in length and direction.

(4.) Def. *Subtraction* is the reverse of Addition, or if the sum of any two quantities a, b be c; b is called the *difference* which arises from *subtracting* a from c.

(5.) *If three quantities be added together, the sum will be the same, whatever be the order in which the quantities are added together.*

Let a, b, c be the three quantities.



Let $AB=a$, $AC=b$, $AD=c$.

Let the parallelogram *ABEC* be completed, and the diagonal *AE* be drawn

$$AE = a + b.$$

Let the parallelogram *AEFD* be completed, and the diagonal *AF* be drawn

$$AF = (a + b) + c.$$

Again, let the parallelogram *ABGD* be completed, and the diagonal *AG* be drawn

$$AG = a + c.$$

Join *CF, GF*;

$\therefore$ *BG* is parallel and equal in length to *AD*,

and *EF* also parallel and equal in length to *AD*,

BG is parallel and equal in length to *EF*;

$\therefore$ *GF* is parallel and equal in length to *BE*, and therefore to *AC*;

$\therefore$ *ACFG* is a parallelogram, and *AF* the diagonal;

$$\therefore AF = (a + c) + b;$$

$$\therefore (a + b) + c = (a + c) + b.$$

In like manner $(a + b) + c = (b + c) + a$.

(6.) *The sum of any number of quantities is the same, whatever be the order in which the quantities are added together.*

First, let there be four quantities, a, b, c, d.

By the preceding article, the sum of any three of these will be the same, whatever be the order in which they are added together.

$$\text{Let } a + b + c = s,$$
$$a + b + d = t,$$
$$a + c + d = u,$$
$$b + c + d = v.$$

To prove that $s + d = t + c = u + b = v + a$,

$$s = a + b + c$$
$$= (a + b) + c;$$
$$\therefore s + d = \{(a + b) + c\} + d$$
$$= \{(a + b) + d\} + c, \quad \text{(by Art. 5.)}$$
$$= t + c.$$

In the same manner it may be proved that

$$s + d = u + b = v + a;$$

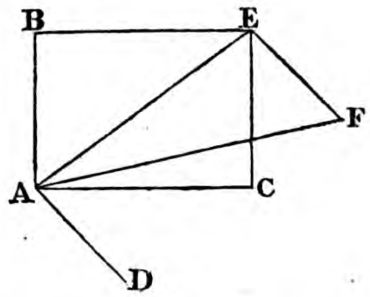
therefore if four quantities be added together, the sum will be the same, whatever be the order in which the quantities are added together.

In like manner it may be proved, if there be five quantities, &c.; therefore the sum of any number of quantities is the same, whatever be the order in which the quantities are added together.

(7.) *Let there be any number of quantities* AB, AC, AD; *and let* BE *be parallel and equal in length to* AC, *and* EF *parallel and equal in length to* AD, *and let* AF *be joined;* AF = AB + AC + AD.

For join *AE, CE.*

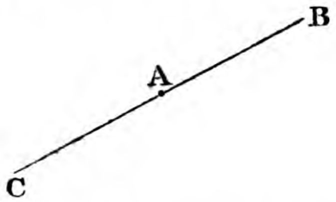
Then *ABEC* is a parallelogram, and *AE* the diagonal;



$$\therefore\ AE = AB + AC.$$

In like manner it may be proved that $AF = AE + AD$;

$$\therefore\ AF = AB + AC + AD.$$



(8.) Def. If a represent a quantity in any direction, $-a$ will represent a quantity equal in length to the former, but drawn in the *opposite* direction: thus if *AB* be represented by a, and *BA* be produced to *C*, and *AC* taken equal in length to *AB*, *AC* will be represented by $-a$.

(9.) *The difference which arises from subtracting* b *from* a = a + (− b).

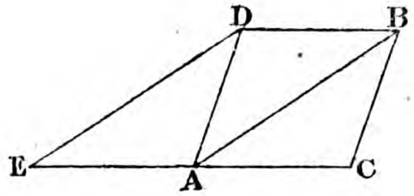
Let $AB = a$, $AC = b$.

Join *BC*, and complete the parallelogram *ACBD*,

$$AB = AC + AD,$$

$$\text{or } a = b + AD;$$

$\therefore$ AD is the difference which arises from subtracting b from a.



Again, produce CA to E, and make AE equal in length to AC, and join ED; therefore since DB is parallel and equal in length to AC, it is also parallel and equal in length to EA; therefore $EDBA$ is a parallelogram, and AD the diagonal;

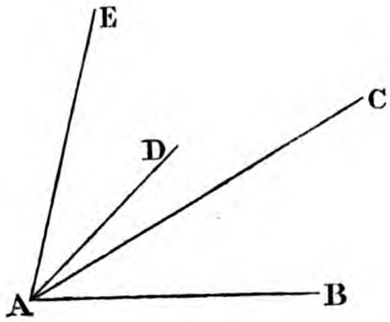
$$\therefore AD = AB + AE$$
$$= a + (-b);$$

therefore the difference which arises from subtracting b from $a = a + (-b)$.

(10.) Def. $a + (-b)$ is expressed $a - b$.

(11.) Def. Quantities drawn from the origin in a certain direction, which direction is arbitrarily assumed, are called *positive* quantities; and those drawn in the *opposite* direction are called *negative* quantities.

(12.) Def. The first of four quantities is said to have to the second the same ratio which the third has to the fourth; when the first has *in length* to the second the same ratio which the third has *in length* to the fourth, according to Euclid's definition; and also the angle at which the fourth is inclined to the third, is equal to the angle at which the second is inclined to the first, and is measured in the same direction*.



Thus let *AB* in length : *AC* :: *AD* in length : *AE*, and also let angle *DAE* be equal to angle *BAC*, and be measured in the same direction from *AD*,

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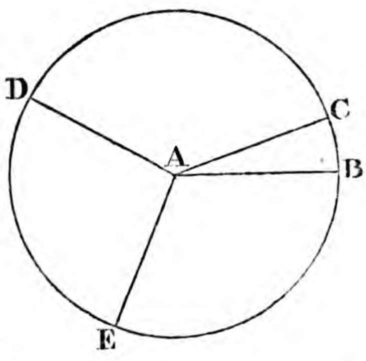
* Angles are said to be measured in the same direction, when the arcs, which subtend them, are measured in the same direction round the circle; and are said to be measured in opposite directions, when the arcs are measured in opposite directions.



Thus angle *DAE* is said to be measured from *AD* in the same direction that angle *BAC* is from *AB*, because arc *DE* is measured from *D* in the same direction round the circle that arc *BC* is from *B*; and angle *EAD* is said to be measured from *AE* in the opposite direction that angle *BAC* is from *AB*, because arc *ED* is measured from *E* in the opposite direction that arc *BC* is from *B*.

that angle *BAC* is from *AB* : then *AB* is said to have to *AC* the same ratio which *AD* has to *AE*.

(13.) *If* d *be a fourth proportional to* a, b, c; *there is no other quantity different from* d *which is also a fourth proportional to* a, b, c.

For, (by Euclid, Book v. Props. 8 and 13.)

a has in length a greater ratio to *b* than *c* has in length to any quantity greater than *d*; and *c* has in length a greater ratio to any quantity less than *d*, than *a* has in length to *b*; therefore there is no quantity greater or less in length than *d*, which is a fourth proportional to *a, b, c*.

Neither is there any quantity different *in direction* from *d*, which is a fourth proportional to *a, b, c*.

For since *d* is a fourth proportional to *a, b, c*; *d* is inclined at the same angle to *c*, which *b* is to *a*.

Let this angle be *A*.

Since *b* is inclined to *a* at the angle *A*, it is also inclined to *a* at the angles $A+360^0$, $A+2.360^0$, &c.

$$A-360^0,\ A-2.360^0,\ \&c.;$$

therefore any quantity which is equal in length to *d*, and inclined to *c* at any of the angles,

$$A,\ A+360^0,\ A+2.360^0,\ \&c.$$
$$A-360^0,\ A-2.360^0,\ \&c.$$

is by the definition a fourth proportional to *a, b, c*.

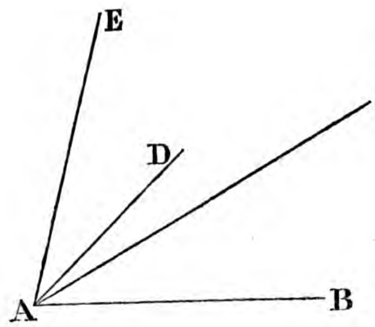
But since *d* is inclined to *c* at the angle *A*, it is also inclined to *c* at the angles

$$A + 360^0,\ A + 2.360^0,\ \&c.$$
$$A - 360^0,\ A - 2.360^0,\ \&c.;$$

therefore there is no quantity different *in direction* from *d*, which is a fourth proportional to *a, b, c*;

therefore there is no quantity different from *d*, which is a fourth proportional to *a*, *b*, *c*.

(14.) *If four quantities be proportionals, they are proportionals also when taken* inversely *or* alternately.



Let *AB* : *AC* :: *AD* : *AE*.

Then inversely *AC* : *AB* :: *AE* : *AD*,

and alternately *AB* : *AD* :: *AC* : *AE*.

For since

AB in length : *AC* :: *AD* in length : *AE*,

(by Simson's Euclid, Book v. Prop. B.)

AC in length : *AB* :: *AE* in length : *AD*.

And since angle *DAE* is equal to angle *BAC*, and is measured in the same direction from *AD*, that *BAC* is from *AB*; angle *EAD* is equal to angle *CAB*, and is measured in the same direction from *AE*, that *CAB* is from *AC*;

$$\therefore AC : AB :: AE : AD.$$

Also since

AB in length : *AC* :: *AD* in length : *AE*,

(by Euclid, Book v. Prop. 16.)

AB in length : *AD* :: *AC* in length : *AE*.

And since angle *DAE* = angle *BAC*, angle *CAE* = angle *BAD*;

$$\therefore AB : AD :: AC : AE.$$

(15.) *If* a : b :: c : d, *and* c : d :: e : f; *then* a : b :: e : f.

For since *a* in length : *b* :: *c* in length : *d*,

and *c* in length : *d* :: *e* in length : *f*,

(by Euclid, Book v. Prop. 11.)

a in length : *b* :: *e* in length : *f*.

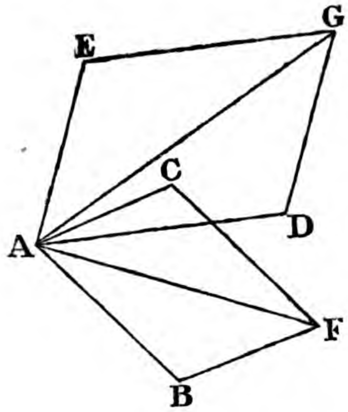
And since the angle at which *f* is inclined to *e* is equal to the angle at which *d* is inclined to *c*, and the angle at which *d* is inclined to *c* equal to the angle at which *b* is inclined to *a*; the angle at which *f* is inclined to *e* is equal to the angle at which *b* is inclined to *a*;

$$\therefore a : b :: e : f.$$

(16.) *If* a : b :: c : d, *then* a + b : b :: c + d : d.

Let $AB = a$, $AC = b$, $AD = c$, $AE = d$.

Complete the parallelograms *ABFC, ADGE,* and draw the diagonals *AF, AG*.



Then $AF = a + b$, and $AG = c + d$.

And since AB in length : AC :: AD in length : AE,

and angle DAE = angle BAC,

parallelogram $BACF$ is similar to parallelogram $DAEG$;
therefore triangle FAC is similar to triangle GAE,

therefore angle $FAC = GAE$,

and AF in length : AC :: AG in length : AE;

$\therefore$ $AF : AC :: AG : AE$,

or $a+b : b :: c+d : d$.

(17.) Def. Unity is a positive quantity arbitrarily assumed, from a comparison with which the values of other quantities are determined.

(18.) Def. If there be three quantities such that, unity is to the first as the second to the third; the third is called the *product* which arises from the *multiplication* of the second by the first.

Thus if, $1 : a :: b : c$, c is called the product which arises from the multiplication of b by a.

(19.) *If* c = *the product which arises from the multiplication of* b *by* a, c *is also equal to the product which arises from the multiplication of* a *by* b.

$$\text{For since } 1 : a :: b : c,$$
$$\text{alternando } 1 : b :: a : c;$$

therefore c is also the product which arises from the multiplication of a by b.

(20.) $$a : ac :: b : bc.$$

For (by definition of multiplication)

$$1 : c :: a : ac,$$
$$\text{and } 1 : c :: b : bc;$$

$\therefore$ (by Art. 15.) $a : ac :: b : bc$.

(21.) Cor. *Hence alternando*

$$a : b :: ac : bc.$$

(22.) *If three quantities be multiplied together, the product will be the same, whatever be the order in which they are multiplied together.*

Let a, b, c be the three quantities.

Let $ab = d$, and $cd = e$,

and let $ac = f$, and $bf = g$.

To prove that $e = g$,

$$1 : a :: b : d,$$
$$\text{and } 1 : a :: c : f;$$

$$\therefore b : d :: c : f$$
$$:: bc : bf \text{ (Art. 21.)}$$
$$:: bc : g;$$
also $b : d :: bc : dc$ (Art. 21.)
$$:: bc : e;$$
$\therefore$ (by Art. 13.) $g = e$,

or $(a \,.\, c) \,.\, b = (a \,.\, b) \,.\, c$.

In like manner it may be proved that

$$(a \,.\, c) \,.\, b = (b \,.\, c) \,.\, a;$$

therefore the product is the same, whatever be the order, in which the quantities are multiplied together.

(23.) Cor. *Hence the product of any number of quantities is the same, whatever be the order, in which the quantities are multiplied together.*

(24.) $(a + b) \,.\, c = ac + bc.$

For (by Art. 21.) $a : b :: ac : bc;$

$\therefore$ (by Art. 16.) $a + b : b :: ac + bc : bc;$

$\therefore$ alternando $a + b : ac + bc :: b : bc.$

But by definition of multiplication

$$1 : c :: b : bc;$$
$$\therefore 1 : c :: a + b : ac + bc;$$
$$\therefore (a + b)\, c = ac + bc.$$

(25.) $(a + b + c)\, d = ad + bd + cd.$

For (by preceding Article) $(a+b+c)\, d = (a+b)\, d + cd$
$$= ad + bd + cd.$$

(26.) $(a + b) . (c + d) = ac + ad + bc + bd.$

For $(a + b) . (c + d) = a . (c + d) + b . (c + d)$

$$= ac + ad + bc + bd.$$

(27.) $a \times (-b) = -ab.$

For let $ab = x$,

$$1 : a :: b : x.$$

But since the angle at which $-x$ is inclined to $-b$ is equal to the angle at which x is inclined to b,

$$b : x :: -b : -x;$$

$$\therefore 1 : a :: -b : -x;$$

$$\therefore a \times (-b) = -x = -ab.$$

(28.) $(-a) . (-b) = ab.$

For by the preceding article

$$(-a) . (-b) = -(-a) b$$

$$= -(-ab)$$

$$= ab.$$

(29.) Def. If three quantities be such that the first is to unity as the second is to the third; the first quantity is called the *quotient* which arises from the *division* of the second by the third.

Thus if $c : 1 :: a : b$; c is called the quotient which arises from the division of a by b.

(30.) **Def.** The quotient which arises from the division of a by b is thus expressed $a \div b$,

or thus $\frac{a}{b}$.

(31.) *If* $c = \frac{a}{b}$, *then* $a = bc$ *and conversely.*

$$\text{Let } c = \frac{a}{b};$$

$$\therefore c : 1 :: a : b;$$

$$\therefore \text{invertendo } 1 : c :: b : a;$$

$$\therefore a = bc.$$

In like manner the converse may be proved.

(32.) $$\frac{-a}{b} = -\left(\frac{a}{b}\right).$$

$$\text{For let } \frac{a}{b} = c;$$

$$\text{then } c : 1 :: a : b;$$

$$\therefore \text{alternando } c : a :: 1 : b;$$

$$\therefore -c : -a :: 1 : b;$$

$$\therefore \text{alternando } -c : 1 :: -a : b;$$

$$\therefore \frac{-a}{b} = -c = -\left(\frac{a}{b}\right).$$

(33.) $$\frac{a}{-b} = -\left(\frac{a}{b}\right).$$

$$\text{Let } \frac{a}{b} = c,$$

$$c : 1 :: a : b$$
$$:: -a : -b$$
$$\therefore \quad c : -a :: 1 : -b;$$
$$\therefore -c : a :: 1 : -b;$$
$$\therefore -c : 1 :: a : -b;$$
$$\therefore \frac{a}{-b} = -c = -\left(\frac{a}{b}\right).$$

(34.) $$\frac{-a}{-b} = \frac{a}{b}.$$

Let $\frac{a}{b} = c$,

$$c : 1 :: a : b$$
$$:: -a : -b;$$
$$\therefore \frac{-a}{-b} = c = \frac{a}{b}.$$

(35.) *If* a : b :: c : d, *then* $\frac{a}{b} = \frac{c}{d}$.

Let $\frac{a}{b} = x$,

$$x : 1 :: a : b$$
$$:: c : d;$$
$$\therefore \frac{c}{d} = x = \frac{a}{b}.$$

(36.) *If* $\frac{a}{b} = \frac{c}{d}$, a : b :: c : d.

For let $\frac{a}{b}$ or $\frac{c}{d} = x$,

$$x : 1 :: a : b,$$

$$\text{also } x : 1 :: c : d;$$

$$\therefore a : b :: c : d.$$

(37.) $$\frac{a}{b} = \frac{ac}{bc}.$$

For (by Art. 21.) $a : b :: ac : bc$;

$\therefore$ (by Art. 35.) $\frac{a}{b} = \frac{ac}{bc}$.

(38.) $$\frac{b}{a} + \frac{c}{a} = \frac{b+c}{a}.$$

For let $\frac{b}{a} = x$, $\frac{c}{a} = y$;

$\therefore$ (by Art. 31.) $b = ax$, and $c = ay$;

$\therefore b + c = ax + ay =$ (by Art. 24.) $a(x + y)$;

$$\therefore \frac{b+c}{a} = x + y = \frac{b}{a} + \frac{c}{a}.$$

(39.) Cor. *Hence* $\frac{b}{a} + \frac{d}{c} = \frac{bc + ad}{ac}$.

For $\frac{b}{a} = \frac{bc}{ac}$, and $\frac{d}{c} = \frac{ad}{ac}$;

$$\therefore \frac{b}{a} + \frac{d}{c} = \frac{bc + ad}{ac}.$$

(40.) $$\frac{b}{a} \times \frac{d}{c} = \frac{bd}{ac}.$$

For let $\frac{b}{a} = x$, $\frac{d}{c} = y$;

then $b = ax$ and $d = cy$;

$\therefore\ bd = ax \times cy = ac \times xy$, (by Art. 23.)

$$\therefore\ \frac{bd}{ac} = xy = \frac{b}{a} \times \frac{d}{c}.$$

(41.) $$\frac{b}{a} \div \frac{d}{c} = \frac{bc}{ad}.$$

For let $\frac{b}{a} = x$, $\frac{d}{c} = y$;

then $b = ax$, and $d = cy$;

$\therefore\ bc = (ax)\,c = (ac)\,x$,

and $ad = a\,(cy) = (ac)\,y$;

$$\therefore\ \frac{bc}{ad} = \frac{(ac)\,x}{(ac)y} = \frac{x}{y} = \frac{b}{a} \div \frac{d}{c}.$$

(42.) **Def.** Any quantity $a \times a \times a$, &c. to n factors is called the n^{th} power of a, and expressed a^n.

(43.) *If* b *be an* n^{th} *power of* a, b *is the only* n^{th} *power of* a.

For $1 : a :: a : a^2$;

therefore since a^2 is a fourth proportional to 1. a, a; it is the only fourth proportional to 1, a, a;

therefore there is only one second power of a.

In like manner since a^3 is a fourth proportional to 1, a, a^2; it is the only fourth proportional to 1, a, a^2;

therefore there is only one-third power of a;

therefore only one-fourth power of a, &c.;

therefore only one n^{th} power of a.

(44.) $$a^m \times a^n = a^{m+n}.$$

For $a^m = a \times a \times a \ldots$ to m factors,

and $a^n = a \times a \times a \ldots$ to n factors;

$\therefore \; a^m \times a^n = a \times a \times a \ldots$ to $m + n$ factors

$= a^{m+n}$.

(45.) $\dfrac{a^m}{a^n} = a^{m-n}$, *when* m *is greater than* n;

and $\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}$, *when* m *is less than* n.

For, when m is greater than n

(by Art. 44) $a^n \times a^{m-n} = a^{n+m-n} = a^m$;

$\therefore$ (by Art. 31.) $\dfrac{a^m}{a^n} = a^{m-n}$.

Again when m is less than n, $a^m \times a^{n-m} = a^n$;

$$\therefore \frac{a^m}{a^n} = \frac{a^m}{a^m \times a^{n-m}} = \text{(by Art. 37.)} \frac{1}{a^{n-m}}.$$

(46.) $(ab)^m = a^m . b^m$.

For $(ab)^m = (ab) . (ab) . (ab) \ldots .$ to m factors

$= a . a . a \ldots$ to m factors $\times b . b . b \ldots$ to m factors

$= a^m . b^m$.

(47.) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.

For $\left(\frac{a}{b}\right)^m = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \ldots .$ to m factors

$$= \frac{a \times a \times a \ldots . \text{ to } m \text{ factors}}{b \times b \times b \ldots . \text{ to } m \text{ factors}}$$

$$= \frac{a^m}{b^m}.$$

(48.) $(a^m)^n = a^{mn}$.

For $(a^m)^n = a^m \times a^m \times a^m \ldots .$ to n factors

$= a^{mn}$, (by Art. 44.)

(49.) *If* a *and* b *be positive quantities and* a *greater than* b, a^m *is greater than* b^m.

Let $a = b + c$,

$a^2 = b^2 + 2bc + c^2$;

$$\therefore\ a^2 \text{ is greater than } b^2.$$

Let $a^2 = b^2 + d$,

$$a^3 = a\,.\,(b^2 + d) = (b + c)\,.\,(b^2 + d) = b^3 + b^2c + bd + cd\,;$$

$$\therefore\ a^3 \text{ is greater than } b^3.$$

In like manner it may be proved that a^4 is greater than b^4, &c.;

$$\therefore\ a^m \text{ is greater than } b^m.$$

(50.) *If* a *and* b *be any quantities, and* a *in length greater than* b, a^{m} *is in length greater than* b^{m}.

Let c be a positive quantity equal in length to a,
and d.......................................b;

c is greater than d;

therefore by the preceding Article c^m is greater than d^m;

but c^m is in length equal to a^m, and d^m to b^m;

$\therefore\ a^m$ is in length greater than b^m.

(51.) *If* ab = c, *and* a *be inclined to unity at an angle* = A, *and* b *at an angle* = B; c *will be inclined to unity at an angle* = A + B.

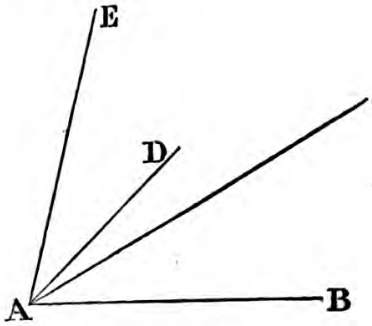
For let AB = unity, $AC = a$, $AD = b$, $AE = c$,

then angle $BAC = A$, angle $BAD = B$;

and since $c = ab$,

$$1 : a :: b : c;$$

$\therefore$ (by Art. 12.) angle DAE = angle $BAC = A$;
and angle $BAD = B$;



therefore angle $BAE = A + B$;
therefore c is inclined to unity at an angle $= A + B$.

(52.) *If* $b = a^m$, *and* a *be inclined to unity at an angle* = A; b *will be inclined to unity at an angle* = m A.

For since $a^2 = a \, . \, a$,

(by Art. 51.) a^2 is inclined to unity at an angle

$$= A + A = 2A;$$

and since $a^3 = a \, . \, a^2$, a^3 is inclined to unity at an angle

$$= A + 2A = 3A;$$

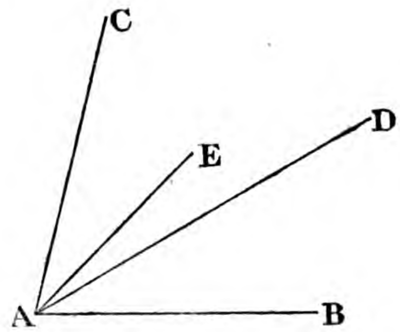
in like manner a^4 is inclined to unity at an angle

$$= 4A; \text{ \&c.};$$

$\therefore$ $b = a^m$ is inclined to unity at an angle $= mA$.

(53.) *If* a *be inclined to unity at an angle* = A, *and* b *at an angle* = B, *and* c $= \frac{a}{b}$; c *will be inclined to unity at an angle* = A − B.

Let AB = unity, angle $BAC = A$, angle $BAD = B$, $AC = a$, $AD = b$, $AE = c$;



$$\text{since } c = \frac{a}{b},$$

$$c : 1 : a : b;$$

therefore angle CAD = angle EAB;

therefore angle BAE = angle BAC − angle BAD

$$= A - B;$$

therefore c is inclined to unity at an angle $= A - B$.

CHAP. II.

ROOTS OF QUANTITIES, FRACTIONAL AND NEGATIVE INDICES.

(ART. 54.) DEF. ANY quantity, which when raised to the n^{th} power produces a quantity a, is called the n^{th} root of a, and expressed thus $\sqrt[n]{a}$.

(55.) DEF. The m^{th} power of the n^{th} root of a, or $(\sqrt[n]{a})^m$ is expressed thus $a^{\frac{m}{n}}$*.

(56.) DEF. $\frac{1}{a^m}$ is expressed thus a^{-m}, where m may be either whole or fractional.

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* This as a general definition of a power of any quantity is defective, for, if the p^{th} power of a be required, where p is a surd, since no fraction, whose numerator and denominator are whole numbers, will accurately represent p, this definition cannot strictly be applied. But, as the fifth definition of the fifth Book of Euclid is a correct and general definition of proportion of magnitudes, and that usually given by Algebraists a defective one, so a general definition of a power of any quantity might have been given, bearing the same analogy to the fifth definition of the fifth Book of Euclid, which the definition in Article 55 bears to the common Algebraic definition of proportion. Nevertheless, as the demonstration of the propositions, necessary to establish the properties of powers of quantities, would by this definition have been rendered tedious: the definition in Art. 55, being one in use amongst Algebraists, has been adopted, notwithstanding its imperfection as a general definition.

(57.) *If* b *be the* n^{th} *root of* a, *and* d *the* n^{th} *root of* c, *and* c *be in length equal to* a; *then shall* d *be in length equal to* b.

For if not, let one of them, d, be the greater in length.

Then since d is greater in length than b; (by Art. 50.) d^n is in length greater than b^n;

$$\text{But } d^n = c, \text{ and } b^n = a;$$

therefore c is in length greater than a, which is contrary to the hypothesis;

therefore d is in length equal to b.

(58.) COR. *Hence all the* n^{th} *roots of any quantity* a *are in length equal to one another.*

(59.) *If* a *be in length equal to* b, $a^{\frac{m}{n}}$ *is in length equal to* $b^{\frac{m}{n}}$.

For, by the preceding Article,

$$\sqrt[n]{a} \text{ is in length equal to } \sqrt[n]{b};$$

$$\therefore (\sqrt[n]{a})^m \text{ is in length equal to } (\sqrt[n]{b})^m,$$

$$\text{that is, } a^{\frac{m}{n}} \text{ is in length equal to } b^{\frac{m}{n}}.$$

(60.) *If* a *be in length equal to* b, $a^{-\frac{m}{n}}$ *is in length equal to* $b^{-\frac{m}{m}}$.

$$\text{For } a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}, \text{ and } b^{-\frac{m}{n}} = \frac{1}{b^{\frac{m}{n}}};$$

$$\therefore\ a^{-\frac{m}{n}} : b^{-\frac{m}{n}} :: \frac{1}{a^{\frac{m}{n}}} : \frac{1}{b^{\frac{m}{n}}} :: b^{\frac{m}{n}} : a^{\frac{m}{n}};$$

and by the preceding Article $b^{\frac{m}{n}}$ is in length equal to $a^{\frac{m}{n}}$;

$\therefore\ a^{-\frac{m}{n}}$ is in length equal to $b^{-\frac{m}{n}}$.

(61.) *If* b $= \sqrt[n]{a}$, b *has* n *different values.*

For let a be inclined to unity at an angle $= A$. Then a is also inclined to unity at angles

$$= A + p \,.\, 360^0,$$

where p is any whole number, either positive or negative.

Now in order that b may be an n^{th} root of a, it is necessary that b should be inclined to unity at an angle B, such that

$$nB = A + p \,.\, 360^0 \text{ (by Art. 52.)};$$

$$\therefore\ B = \frac{A + p \,.\, 360^0}{n}.$$

For p substitute successively 0, 1, 2, &c.

$n - 1$, n, $n + 1$, &c.

also -1, -2, &c.

The corresponding values of B will be respectively

$$\frac{A}{n},\ \frac{A + 360^0}{n},\ \frac{A + 2 \,.\, 360^0}{n},\ \&c.,$$

$$\frac{A+\overline{n-1}\,.\,360^0}{n},\ \frac{A+n\,.\,360^0}{n},\ \frac{A+\overline{n+1}\,.\,360^0}{n},\ \&c.,$$

$$\frac{A-360^0}{n},\ \frac{A-2\,.\,360^0}{n},\ \&c.$$

$$\text{But } \frac{A+n.360^0}{n}=\frac{A}{n}+360^0,$$

$$\text{and } \frac{A+\overline{n+1}\,.\,360^0}{n}=\frac{A+360^0}{n}+360^0;$$

therefore the value of b, which is inclined to unity at an angle $=\frac{A}{n}$, is also inclined to unity at an angle

$$=\frac{A}{n}+360^0, \text{ or } \frac{A+n\,.\,360^0}{n},$$

and all the values of b are in length equal to one another (by Art. 58.); therefore the value of b, which is inclined to unity at an angle $=\frac{A}{n}$, coincides with the value of b which is inclined to unity at an angle $=\frac{A+n\,.\,360^0}{n}$, and therefore is equal to it.

In like manner the value of b which is inclined to unity at an angle $=\frac{A+360^0}{n}$, is equal to the value of b which is inclined to unity at an angle

$$=\frac{A+\overline{n+1}\,.\,360^0}{n}.$$

Also since $\frac{A - 360^0}{n} = \frac{A + \overline{n - 1} . 360^0}{n} - 360^0$,

and $\frac{A - 2 . 360^0}{n} = \frac{A + \overline{n - 2} . 360^0}{n} - 360^0$.

The value of b, which is inclined to unity at an angle $= \frac{A + \overline{n - 1} . 360^0}{n}$, is equal to the value of b, which is inclined to unity at an angle $= \frac{A - 360^0}{n}$; and the value of b, which is inclined to unity at an angle $= \frac{A + \overline{n - 2} . 360^0}{n}$, is equal to the value of b, which is inclined to unity at an angle $= \frac{A - 2 . 360^0}{n}$; &c.;

therefore b has n different values.

(62.) Def. If a be inclined to unity at an angle $= A$, A being positive and less than 360^0; a_0 expresses a considered as inclined to unity at an angle $= A$,

a_1 .... at an angle $= A + 360^0$,

a_2 .... $= A + 2 . 360^0$,

&c. &c.

also a_{-1} .... $= A - 360^0$,

a_{-2} . . . . at an angle $= A - 2.360^0$,

&c. &c.

and generally a_p . . . . $= A + p\,360^0$,

where $p = 0$, or any whole number, either positive or negative.

Thus $\sqrt[n]{a}_0$ expresses that value of $\sqrt[n]{a}$, which arises from considering a as inclined to unity at the angle $= A$, $\sqrt[n]{a}_1$ expresses that value of $\sqrt[n]{a}$ which arises from considering a as inclined to unity at the angle $= A + 360^0$, and generally $\sqrt[n]{a}_p$ expresses that value of $\sqrt[n]{a}$ which arises from considering a as inclined to unity at the angle $= A + p.360^0$.

(63.) *If* $\left(\mathrm{a}_{\mathrm{p}}\right)^{\frac{\mathrm{m}}{\mathrm{n}}} = \mathrm{b}$, b *is inclined to unity at an angle* $= \frac{\mathrm{m}}{\mathrm{n}}\,(\mathrm{A} + \mathrm{p}.360^0)$.

For let $\sqrt[n]{a}_p = c$, then $c^m = b$ (by Art. 55.)

(By Art. 61, 62.) c is inclined to unity at an angle

$$= \frac{A + p.360^0}{n};$$

$$\text{let } \frac{A + p.360^0}{n} = C;$$

then since $b = c^m$, b is inclined to unity at an angle

$$= mC$$

$$= \frac{m}{n}\,.\,(A + p\,.\,360^0).$$

(64.) *If* $\left(\mathrm{a}\right)_{\mathrm{p}}^{-\frac{\mathrm{m}}{\mathrm{n}}} = \mathrm{b}$, b *is inclined to unity at an angle* $= -\,\frac{\mathrm{m}}{\mathrm{n}}\,.\,(\mathrm{A} + \mathrm{p}\,.\,360^0)$.

$$b = \frac{1}{\left(a\right)_p^{\frac{m}{n}}},$$

and 1 is inclined to unity at an angle = 0,

and $\left(a\right)_p^{\frac{m}{n}}$ is inclined to unity at an angle

$$= \frac{m}{n}\,(A + p\,.\,360^0) \text{ (by Art. 63.)}$$

therefore b is inclined to unity at an angle

$$= 0 - \frac{m}{n}\,(A + p\,.\,360^0) \text{ (by Art. 53.)}$$

$$= -\,\frac{m}{n}\,.\,(A + p\,.\,360^0).$$

(65.) Cor. In the two preceding Articles if a be a positive quantity, $A = 0$; therefore b is inclined to unity at an angle $= \pm\,\frac{m}{n}\,p\,.\,360^0$.

In this case; if $p = 0$, b is inclined to unity at an angle = 0, that is, is a positive quantity;

If $p = 1$, b is inclined to unity at an angle

$$= \pm \frac{m}{n} 360^0.$$

(66.) $\left(\underset{0}{a}\right)^{\frac{km}{kn}} = \left(\underset{0}{a}\right)^{\frac{m}{n}}$, a *being a positive quantity.*

For let $\underset{0}{\sqrt[n]{a}} = b$, and $\underset{0}{\sqrt[k]{b}} = c$,

then (by Art. 65.) b and c are positive quantities;

$$\text{And } a = b^n,\ b = c^k,$$

$$\therefore\ a = (c^k)^n = \text{(by Art. 48.) } c^{kn},$$

$$\therefore\ c \text{ is a value of } \sqrt[kn]{a};$$

$$\text{And } c \text{ is a positive quantity},$$

$$\therefore\ c = \underset{0}{\sqrt[kn]{a}},$$

$$\therefore\ c^k = \left(\underset{0}{\sqrt[kn]{a}}\right)^k,$$

$$\therefore\ b = \left(\underset{0}{\sqrt[kn]{a}}\right)^k,$$

$$\therefore\ b^m = \left(\underset{0}{\sqrt[kn]{a}}\right)^{km} = \left(\underset{0}{a}\right)^{\frac{km}{kn}};$$

$$\text{But } b^m = \left(\underset{0}{a}\right)^{\frac{m}{n}} \text{ (by Art. 55.)}$$

$$\therefore\ \left(\underset{0}{a}\right)^{\frac{km}{kn}} = \left(\underset{0}{a}\right)^{\frac{m}{n}}.$$

(67.) $\binom{a}{0}^{\frac{m}{n}} \times \binom{a}{0}^{\frac{p}{q}} = \binom{a}{0}^{\frac{mq+np}{nq}}$, a *being a positive quantity.*

For (by Art. 66.) $\binom{a}{0}^{\frac{m}{n}} = \binom{a}{0}^{\frac{mq}{nq}} = \binom{\sqrt[nq]{a}}{0}^{mq}$

$$\binom{a}{0}^{\frac{p}{q}} = \binom{a}{0}^{\frac{np}{nq}} = \binom{\sqrt[nq]{a}}{0}^{np};$$

$\therefore$ (by Art. 44.) $\binom{a}{0}^{\frac{m}{n}} \times \binom{a}{0}^{\frac{p}{q}} = \binom{\sqrt[nq]{a}}{0}^{mq+np}$

$$= \binom{a}{0}^{\frac{mq+np}{nq}}.$$

(68.) $\binom{a}{0}^{\frac{m}{n}} \times \binom{a}{0}^{\frac{-p}{q}} = \binom{a}{0}^{\frac{mq-np}{nq}}$, a *being a positive quantity.*

First let mq be greater than np,

$$\binom{a}{0}^{\frac{m}{n}} \times \binom{a}{0}^{\frac{-p}{q}} = \frac{\binom{a}{0}^{\frac{m}{n}}}{\binom{a}{0}^{\frac{p}{q}}} = \frac{\binom{a}{0}^{\frac{mq}{nq}}}{\binom{a}{0}^{\frac{np}{nq}}} = \frac{\binom{\sqrt[nq]{a}}{0}^{mq}}{\binom{\sqrt[nq]{a}}{0}^{np}}$$

$= \binom{\sqrt[nq]{a}}{0}^{mq-np}$ (by Art. 45.)

$$= \binom{a}{0}^{\frac{mq-np}{nq}}.$$

Next let mq be less than np,

$\binom{a}{0}^{\frac{m}{n}} \times \binom{a}{0}^{\frac{-p}{q}} = \dfrac{1}{\binom{\sqrt[nq]{a}}{0}^{np-mq}}$ (by Art. 45.)

$$\binom{a}{0}^{\frac{m}{n}} \times \binom{a}{0}^{\frac{-p}{q}} = \frac{1}{\binom{a}{0}^{\frac{np-mq}{qn}}} = \binom{a}{0}^{-\frac{np-mq}{nq}}$$

$$= \binom{a}{0}^{\frac{mq-np}{nq}}.$$

(69.) $\binom{a}{0}^{-\frac{m}{n}} \times \binom{a}{0}^{-\frac{p}{q}} = \binom{a}{0}^{\frac{-mq-np}{nq}}$, a *being a positive quantity.*

For $\binom{a}{0}^{-\frac{m}{n}} \times \binom{a}{0}^{-\frac{p}{q}} = \frac{1}{\binom{a}{0}^{\frac{m}{n}}} \times \frac{1}{\binom{a}{0}^{\frac{p}{q}}} = \frac{1}{\binom{a}{0}^{\frac{m}{n}} \times \binom{a}{0}^{\frac{p}{q}}}$

$$= \frac{1}{\binom{a}{0}^{\frac{mq+np}{nq}}} \text{ (by Art. 67.)}$$

$$= \binom{a}{0}^{\frac{-mq-np}{nq}}.$$

(70.) $\frac{\binom{a}{0}^{m}}{\binom{a}{0}^{n}} = \binom{a}{0}^{m-n}$, *where* a *is a positive quantity, and* m *and* n *either whole or fractional, positive or negative.*

For $\frac{\binom{a}{0}^{m}}{\binom{a}{0}^{n}} = \binom{a}{0}^{m} \times \binom{a}{0}^{-n}$

$$= \binom{a}{0}^{m-n} \text{ (by Art. 67, 68, 69).}$$

(71.) $\left(\underset{0}{\mathrm{a}}\right)^{\frac{m}{n}} \times \left(\underset{0}{\mathrm{b}}\right)^{\frac{m}{n}} = \left(\underset{0}{\mathrm{a}}\underset{0}{\mathrm{b}}\right)^{\frac{m}{n}}$, a *and* b *being positive quantities.*

Let $\sqrt[n]{\underset{0}{a}} = c$, $\sqrt[n]{\underset{0}{b}} = d$, then c, d are positive quantities;

$$\left(\underset{0}{a}\right)^{\frac{m}{n}} = c^m, \quad \left(\underset{0}{b}\right)^{\frac{m}{n}} = d^m;$$

$$\therefore \left(\underset{0}{a}\right)^{\frac{m}{n}} \times \left(\underset{0}{b}\right)^{\frac{m}{n}} = c^m . d^m = (c . d)^m \text{ (by Art. 46.)};$$

$$\text{Now } c^n . d^n = a . b,$$

$$\therefore (c . d)^n = ab \text{ (by Art. 46.)},$$

$\therefore$ since c, d are positive quantities $cd = \sqrt[n]{\underset{0}{a}\underset{0}{b}}$;

$$\therefore (cd)^m = \left(\underset{0}{a}\underset{0}{b}\right)^{\frac{m}{n}};$$

$$\therefore \left(\underset{0}{a}\right)^{\frac{m}{n}} \times \left(\underset{0}{b}\right)^{\frac{m}{n}} = \left(\underset{0}{a}\underset{0}{b}\right)^{\frac{m}{n}}.$$

(72.) $\dfrac{\left(\underset{0}{\mathrm{a}}\right)^{\frac{m}{n}}}{\left(\underset{0}{\mathrm{b}}\right)^{\frac{m}{n}}} = \left\{\dfrac{\underset{0}{a}}{\underset{0}{b}}\right\}^{\frac{m}{n}}$, a *and* b *being positive quantities.*

Let $\sqrt[n]{\underset{0}{a}} = c$, $\sqrt[n]{\underset{0}{b}} = d$,

$$\frac{\left(\underset{0}{a}\right)^{\frac{m}{n}}}{\left(\underset{0}{b}\right)^{\frac{m}{n}}} = \frac{c^m}{d^m} = \left(\frac{c}{d}\right)^m \text{ (by Art. 47.)}$$

$$\text{And } \left(\frac{c}{d}\right)^n = \frac{a}{b},$$

$\therefore$ since c, d are positive quantities $\frac{c}{d} = \sqrt[n]{\frac{\underset{0}{a}}{\underset{0}{b}}}$,

$$\therefore \left(\frac{c}{d}\right)^m = \left\{\frac{\underset{0}{a}}{\underset{0}{b}}\right\}^{\frac{m}{n}},$$

$$\therefore \frac{\left(\underset{0}{a}\right)^{\frac{m}{n}}}{\left(\underset{0}{b}\right)^{\frac{m}{n}}} = \left\{\frac{\underset{0}{a}}{\underset{0}{b}}\right\}^{\frac{m}{n}}.$$

(73.) $\left(\underset{0}{\mathrm{a}}\right)^{-\frac{m}{n}} \cdot \left(\underset{0}{\mathrm{b}}\right)^{-\frac{m}{n}} = \left(\underset{0}{\mathrm{a}} \cdot \underset{0}{\mathrm{b}}\right)^{-\frac{m}{n}}$, a *and* b *being positive quantities.*

For $\left(\underset{0}{a}\right)^{-\frac{m}{n}} \cdot \left(\underset{0}{b}\right)^{-\frac{m}{n}} = \frac{1}{\left(\underset{0}{a}\right)^{\frac{m}{n}} \cdot \left(\underset{0}{b}\right)^{\frac{m}{n}}}$

$$= \frac{1}{\left(\underset{0}{a} \cdot \underset{0}{b}\right)^{\frac{m}{n}}} \text{ (by Art. 71.)}$$

$$= \left(\underset{0}{a}\underset{0}{b}\right)^{-\frac{m}{n}}.$$

(74.) $\frac{\left(\underset{0}{a}\right)^{-\frac{m}{n}}}{\left(\underset{0}{b}\right)^{-\frac{m}{n}}} = \left\{\frac{\underset{0}{a}}{\underset{0}{b}}\right\}^{-\frac{m}{n}}$, a *and* b *being positive quantities.*

For $\frac{\left(\underset{0}{a}\right)^{-\frac{m}{n}}}{\left(\underset{0}{b}\right)^{-\frac{m}{n}}} = \frac{1}{\left(\underset{0}{a}\right)^{\frac{m}{n}} \cdot \left(\underset{0}{b}\right)^{-\frac{m}{n}}} = \frac{1}{\left\{\frac{\left(\underset{0}{a}\right)^{\frac{m}{n}}}{\left(\underset{0}{b}\right)^{\frac{m}{n}}}\right\}}$

$$= \frac{1}{\left\{\frac{a}{\underset{0}{}}\Big/\frac{b}{\underset{0}{}}\right\}^{\frac{m}{n}}} \text{ (by Art. 72.)}$$

$$= \left\{\frac{\underset{0}{a}}{\underset{0}{b}}\right\}^{-\frac{m}{n}}$$

(75.) *If* $\left(\underset{0}{a}\right)^{\frac{m}{n}} = b$, *and* $\left(\underset{0}{b}\right)^{\frac{p}{q}} = c$, a *being a positive quantity;* $\left(\underset{0}{a}\right)^{\frac{mp}{nq}} = c$.

For let $e = \underset{0}{\sqrt[n]{a}}$, $f = \underset{0}{\sqrt[q]{e}}$, then e and f are positive quantities,

$$a = e^n \text{ and } e = f^q,$$

$$\therefore\ a = f^{nq} \text{ (by Art. 48.)}$$

$\therefore$ since f is a positive quantity, $f = \underset{0}{\sqrt[nq]{a}}$,

$$\therefore\ \left(\underset{0}{a}\right)^{\frac{mp}{nq}} = f^{mp};$$

Again $b = \left(\underset{0}{a}\right)^{\frac{m}{n}} = e^m$,

and $e = f^q$,

$$\therefore\ b = f^{qm} = (f^m)^q,$$

$\therefore$ since f is a positive quantity, $f^m = \underset{0}{\sqrt[q]{b}}$,

$$\therefore f^{mp} = \binom{b}{0}^{\frac{p}{q}} = c,$$

$$\therefore \binom{a}{0}^{\frac{mp}{nq}} = c.$$

(76.) *If* $\binom{a}{0}^{-\frac{m}{n}} = b$, *and* $\binom{b}{0}^{\frac{p}{q}} = c$, a *being a positive quantity;* $\binom{a}{0}^{-\frac{mp}{nq}} = c$.

For let $\binom{a}{0}^{\frac{m}{n}} = e$,

$$b = \binom{a}{0}^{-\frac{m}{n}} = \frac{1}{\binom{a}{0}^{\frac{m}{n}}} = \frac{1}{e},$$

$$\therefore \binom{b}{0}^{\frac{p}{q}} = \frac{\binom{1}{0}^{\frac{p}{q}}}{\binom{e}{0}^{\frac{p}{q}}} \text{ (by Art. 72.)}$$

$$= \frac{1}{\binom{e}{0}^{\frac{p}{q}}},$$

$$\therefore c = \frac{1}{\binom{e}{0}^{\frac{p}{q}}} = \frac{1}{\binom{a}{0}^{\frac{mp}{nq}}} \text{ (by Art. 75.)}$$

$$= \binom{a}{0}^{-\frac{mp}{nq}}.$$

(77.) *If* $\binom{a}{0}^{\frac{m}{n}} = b$, *and* $\binom{b}{0}^{-\frac{p}{q}} = c$, a *being a positive quantity;* $\binom{a}{0}^{-\frac{mp}{nq}} = c$.

For let $\binom{b}{0}^{\frac{p}{q}} = e$,

then $e = \binom{a}{0}^{\frac{mp}{nq}}$ (by Art. 75.),

and $c = \binom{b}{0}^{-\frac{p}{q}} = \dfrac{1}{\binom{b}{0}^{\frac{p}{q}}} = \dfrac{1}{e} = \dfrac{1}{\binom{a}{0}^{\frac{mp}{nq}}} = \binom{a}{0}^{-\frac{mp}{nq}}$.

(78.) *If* $\binom{a}{0}^{-\frac{m}{n}} = b$, *and* $\binom{b}{0}^{-\frac{p}{q}} = c$, a *being a positive quantity;* $\binom{a}{0}^{\frac{mp}{nq}} = c$.

For let $\binom{b}{0}^{\frac{p}{q}} = e$,

then $e = \binom{a}{0}^{-\frac{mp}{nq}}$ (by Art. 76.),

and $c = \binom{b}{0}^{-\frac{p}{q}} = \dfrac{1}{e} = \dfrac{1}{\binom{a}{0}^{-\frac{mp}{nq}}} = \binom{a}{0}^{\frac{mp}{nq}}$.

(79.) *If* $\binom{a}{0}^{m} = \binom{b}{0}^{n}$, $\binom{a}{0}^{mk} = \binom{b}{0}^{nk}$, *where* a *and* b *are positive quantities, and* m, n, k *either whole or fractional, positive or negative.*

For let $c = \binom{a}{0}^{m} = \binom{b}{0}^{n}$,

then c is a positive quantity,

since $c = \binom{a}{0}^{m}$, $\binom{c}{0}^{k} = \binom{a}{0}^{mk}$ (by Art. 75, 76, 77, 78.)

$$\text{also since } c = \binom{b}{0}^{n},\ \binom{c}{0}^{k} = \binom{b}{0}^{nk};$$

$$\therefore\ \binom{a}{0}^{mk} = \binom{b}{0}^{nk}.$$

(80.) *If* $\binom{a}{0}^{m} = \binom{b}{0}^{n}$, $b = \binom{a}{0}^{\frac{m}{n}}$, *where* a *and* b *are positive quantities, and* m *and* n *either whole or fractional, positive or negative.*

$$\text{Since } \binom{b}{0}^{n} = \binom{a}{0}^{m},$$

$$\binom{b}{0}^{n\times\frac{1}{n}} = \binom{a}{0}^{m\times\frac{1}{n}} \text{ (by Art. 79.)}$$

$$\therefore\ \binom{b}{0}^{1} = \binom{a}{0}^{\frac{m}{n}},$$

$$\text{or } b = \binom{a}{0}^{\frac{m}{n}}.$$

(81.) $\binom{1}{1}^{m} \times \binom{1}{1}^{n} = \binom{1}{1}^{m+n}$, *where* m *and* n *are either whole or fractional, positive or negative.*

$$\text{For } \binom{1}{1}^{m} \text{ is in length } = 1,$$

$$\text{and } \binom{1}{1}^{n} \text{ is in length } = 1,$$

$$\therefore\ \binom{1}{1}^{m} \times \binom{1}{1}^{n} \text{ is in length } = 1;$$

Also $\binom{1}{1}^{m+n}$ is in length $= 1$,

$\therefore \binom{1}{1}^m \times \binom{1}{1}^n$ is in length $= \binom{1}{1}^{m+n}$;

Also (by Art. 65.) since 1 is a positive quantity, $\binom{1}{1}^m$ is inclined to unity at an angle $= m . 360^0$,

and $\binom{1}{1}^n$, at an angle $= n . 360^0$,

therefore (by Art. 51.) $\binom{1}{1}^m \times \binom{1}{1}^n$ is inclined to unity at an angle $= m . 360^0 + n . 360^0 = \overline{m + n} . 360^0$; Also (by Art. 65.) $\binom{1}{1}^{m+n}$ is inclined to unity at an angle $= \overline{m + n} . 360^0$;

$\therefore \binom{1}{1}^m \times \binom{1}{1}^n$ and $\binom{1}{1}^{m+n}$ are inclined to unity at the same angle;

And they have been proved to be equal in length;

$$\therefore \binom{1}{1}^m \times \binom{1}{1}^n = \binom{1}{1}^{m+n}.$$

(82.) $\dfrac{\binom{1}{1}^m}{\binom{1}{1}^n} = \binom{1}{1}^{m-n}$, *where* m *and* n *are either whole or fractional, positive or negative.*

For $\dfrac{\binom{1}{1}^m}{\binom{1}{1}^n} = \binom{1}{1}^m . \binom{1}{1}^{-n} = \binom{1}{1}^{m-n}$ (by Art. 81).

(83.) $\binom{1}{p}^{m} = \binom{1}{1}^{mp}$, *where* m *is either whole or fractional, positive or negative.*

For (by Art. 65.) $\binom{1}{p}^{m}$ is inclined to unity at an angle $= mp\,360^0$,

And (by same Art.) $\binom{1}{1}^{mp}$ is inclined to unity at the same angle,

$$\therefore \binom{1}{p}^{m} = \binom{1}{1}^{mp}.$$

(84.) $\binom{a}{p}^{m} = \binom{a}{0}^{m} \cdot \binom{1}{p}^{m}$, *where* m *is either whole or fractional, positive or negative.*

For $\binom{1}{p}^{m}$ is in length $= 1$,

$\therefore \binom{a}{0}^{m} \cdot \binom{1}{p}^{m}$ is in length $= \binom{a}{0}^{m}$;

And $\binom{a}{p}^{m}$ is in length $= \binom{a}{0}^{m}$;

$\therefore \binom{a}{p}^{m}$ is in length $= \binom{a}{0}^{m} \cdot \binom{1}{p}^{m}$;

Let a be inclined to unity at an angle $= A$, A being positive and less than 360^0,

then $\binom{a}{0}^{m}$ is inclined to unity at an angle $= mA$,

and $\binom{1}{p}^{m}$ .... $= mp.360^0$;

$\therefore \binom{a}{0}^{m} \cdot \binom{1}{p}^{m}$ .... $= mA + mp.360^0 = m(A + p.360^0)$;

Also $\binom{a}{p}^{m}$ .... $= m(A + p.360^0)$;

$$\therefore \binom{a}{p}^{m} = \binom{a}{0}^{m} \cdot \binom{1}{p}^{m}.$$

(85.) *Any quantity* a *may be expressed in the form* b $\binom{1}{1}^{n}$, *where* b *is a positive quantity, and* n *positive and less than unity.*

For let b be in length $= a$, and let a be inclined to unity at an angle $= A$, A being positive and less than 360^0,

Take $n = \dfrac{A}{360^0}$, then n is positive and less than unity,

And (by Art. 65.) $\binom{1}{1}^{n}$ is inclined to unity at an angle $= n.360^0$

$$= \frac{A}{360^0} . 360^0 = A;$$

$\therefore$ $b \binom{1}{1}^{n}$ is inclined to unity at an angle $= A$, that is, at the same angle that a is inclined to unity;

And $b \binom{1}{1}^{n}$ is in length equal to a;

$$\therefore\ b \binom{1}{1}^{n} = a.$$

(86.) *If* $\mathrm{a} = \mathrm{b} \binom{1}{1}^{\mathrm{n}}$, *where* b *is a positive quantity, and* n *positive and less than* 1; $\binom{\mathrm{a}}{0}^{\mathrm{m}} = \binom{\mathrm{b}}{0}^{\mathrm{m}} . \binom{1}{1}^{\mathrm{mn}}$, *where* m *may be either whole or fractional, positive or negative.*

For let a be inclined to unity at an angle $= A$, A being positive and less than 360^0, Then (by preceding Article)

$$\frac{A}{360^0} = n, \text{ or } A = n \,.\, 360^0;$$

$\binom{a}{0}^m$ is inclined to unity at an angle $= mA$;

And $\binom{1}{1}^{mn}$ . . . . $= mn.360 = mA$,

$\therefore$ since b is a positive quantity,

$\binom{b}{0}^m \binom{1}{1}^{mn}$ is inclined to unity at an angle $= mA$;

And since b is in length $= a$,

$\binom{b}{0}^n \cdot \binom{1}{1}^{mn}$ is in length $= \binom{a}{0}^m$;

$$\therefore \binom{b}{0}^m \cdot \binom{1}{1}^{mn} = \binom{a}{0}^m.$$

(87.) *If* a = b $\binom{1}{1}^n$, *where* b *is a positive quantity, and* n *positive and less than unity;* $\binom{a}{p}^m = \binom{b}{p}^m \cdot \binom{1}{1}^{mn}$, *where* m *may be either whole or fractional, positive or negative.*

For $\binom{a}{p}^m = \binom{a}{0}^m \cdot \binom{1}{p}^m$ (by Art. 84.)

$$= \binom{b}{0}^m \cdot \binom{1}{1}^{mn} \cdot \binom{1}{p}^m \text{ (by Art. 86.)}$$

$$= \binom{b}{0}^m \cdot \binom{1}{p}^m \cdot \binom{1}{1}^{mn}$$

$$= \binom{b}{p}^m \cdot \binom{1}{1}^{mn}.$$

(88.) $\binom{a}{p}^{m} \cdot \binom{a}{p}^{n} = \binom{a}{p}^{m+n}$, *where* m, n *may be either whole or fractional, positive or negative.*

For let $a = b\binom{1}{1}^{k}$, where b is a positive quantity, and k positive and less than 1,

$$\text{then } \binom{a}{p}^{m} = \binom{b}{p}^{m} \cdot \binom{1}{1}^{mk}$$

$$= \binom{b}{0}^{m} \cdot \binom{1}{p}^{m} \cdot \binom{1}{1}^{mk}$$

$$= \binom{b}{0}^{m} \cdot \binom{1}{1}^{mp} \cdot \binom{1}{1}^{mk};$$

$$\text{and } \binom{a}{p}^{n} = \binom{b}{0}^{n} \cdot \binom{1}{1}^{np} \cdot \binom{1}{1}^{nk};$$

$$\therefore \binom{a}{p}^{m} \cdot \binom{a}{p}^{n} = \binom{b}{0}^{m} \cdot \binom{b}{0}^{n} \cdot \binom{1}{1}^{mp} \cdot \binom{1}{1}^{np} \cdot \binom{1}{1}^{mk} \cdot \binom{1}{1}^{nk}$$

$$= \binom{b}{0}^{m+n} \cdot \binom{1}{1}^{\overline{m+n}.p} \cdot \binom{1}{1}^{\overline{m+n}.k} \text{ (Art.67,68,69,81.)}$$

$$= \binom{b}{p}^{m+n} \cdot \binom{1}{1}^{\overline{m+n}.k}$$

$$= \binom{a}{p}^{m+n}.$$

(89.) $\dfrac{\binom{a}{p}^{m}}{\binom{a}{p}^{n}} = \binom{a}{p}^{m-n}$, *where* m *and* n *are either whole or fractional, positive or negative.*

For $\dfrac{\binom{a}{p}^{m}}{\binom{a}{p}^{n}} = \binom{a}{p}^{m} \cdot \binom{a}{p}^{-n} = \binom{a}{p}^{m-n}$ (by Art. 88).

(90.) $$\binom{a}{p}^{0} = 1.$$

For $$\binom{a}{p}^{m-n} = \frac{\binom{a}{p}^{m}}{\binom{a}{p}^{n}},$$

Let $m = n$,

$$\binom{a}{p}^{0} = 1.$$

(91.) *If* a . b = c, *and* a *be inclined to unity at an angle* = A, *and* b *at an angle* = B; A *and* B *being positive and less than* 360^0;

$$\binom{a}{0}^{m} . \binom{b}{0}^{m} = \binom{c}{0}^{m}, \textit{ if } A + B \textit{ be less than } 360^0,$$

$$= \binom{c}{1}^{m}, \textit{ if } A + B \textit{ be greater than } 360^0,$$

where m *may be either whole or fractional, positive or negative.*

For let a be in length $= e$, and b be in length $= f$; e and f being positive quantities;

$$\therefore c \text{ is in length} = e . f;$$

$$\therefore \binom{c}{0}^{m} \text{ or } \binom{c}{1}^{m} \text{ is in length} = \left(\begin{matrix}e \\ 0\end{matrix} . \begin{matrix}f \\ 0\end{matrix}\right)^{m};$$

$$\text{And } \binom{a}{0}^{m} . \binom{b}{0}^{m} \text{ is in length} = \binom{e}{0}^{m} . \binom{f}{0}^{m}$$

$$= \left(\begin{matrix}e \\ 0\end{matrix} . \begin{matrix}f \\ 0\end{matrix}\right)^{m} \text{ (by Art. 71, 73)};$$

$$\therefore \binom{a}{0}^{m} . \binom{b}{0}^{m} \text{ is in length equal to } \binom{c}{0}^{m} \text{ or } \binom{c}{1}^{m};$$

Also $\left(\underset{0}{a}\right)^m$ is inclined to unity at an angle $= mA$,

$$\left(\underset{0}{b}\right)^m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad = mB;$$

$$\therefore \left(\underset{0}{a}\right)^m \cdot \left(\underset{0}{b}\right)^m \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = mA + mB$$

$$= m.(A + B);$$

Again $\because c = a \,.\, b$, c is inclined to unity at an angle $= A + B$,

$\therefore$ if $A + B$ be less than 360^0, $\underset{0}{c}$ represents c considered as inclined to unity at the angle $= A + B$;

$\therefore \left(\underset{0}{c}\right)^m$ is inclined to unity at an angle $= m\,(A + B)$;

$\therefore$ when $A + B$ is less than 360^0, $\left(\underset{0}{a}\right)^m \cdot \left(\underset{0}{b}\right)^m = \left(\underset{0}{c}\right)^m$;

But if $A + B$ be greater than 360^0; since $A + B$ is less than 2.360^0, $\underset{1}{c}$ represents c considered as inclined to unity at the angle $= A + B$;

$\therefore \left(\underset{1}{c}\right)^m$ is inclined to unity at an angle $= m\,.\,(A + B)$;

$\therefore$ in this case $\left(\underset{0}{a}\right)^m \cdot \left(\underset{0}{b}\right)^m = \left(\underset{1}{c}\right)^m$.

(92.) *If* a . b = c, *and* a *be inclined to unity at an angle* = A, *and* b *at an angle* = B, A *and* B *being positive and less than* 360^0;

$$\binom{a}{p}^m \cdot \binom{b}{q}^m = \binom{c}{p+q}^m, \textit{ if } A + B \textit{ be less than } 360^0,$$

$$= \binom{c}{p+q+1}^m, \ldots\ldots\ldots\textit{greater}\ldots\ldots,$$

where m *may be either whole or fractional, positive or negative.*

$$\text{For } \binom{a}{p}^m = \binom{a}{0}^m \cdot \binom{1}{p}^m = \binom{a}{0}^m \cdot \binom{1}{1}^{mp},$$

$$\binom{b}{q}^m = \ldots\ldots\ldots\ldots \binom{b}{0}^m \cdot \binom{1}{1}^{mq};$$

$$\therefore \binom{a}{p}^m \cdot \binom{b}{q}^m = \binom{a}{0}^m \cdot \binom{b}{0}^m \cdot \binom{1}{1}^{m \cdot \overline{p+q}}$$

$$\left.\begin{aligned} &= \binom{c}{0}^m \cdot \binom{1}{1}^{m \cdot \overline{p+q}}, \text{ if } A + B \text{ be less than } 360^0, \\ &= \binom{c}{1}^m \cdot \binom{1}{1}^{m \cdot \overline{p+q}}, \ldots\ldots\ldots\text{greater}\ldots\ldots\ldots \end{aligned}\right\} \text{by Art. 91.}$$

$$= \binom{c}{p+q}^m, \text{ if } A + B \text{ be less than } 360^0,$$

$$= \binom{c}{p+q+1}^m, \ldots\ldots\ldots \text{greater} \ldots\ldots$$

(93.) *If* $\frac{a}{b} = c$, *and* a *be inclined to unity at an angle* = A, *and* b *at an angle* = B; A *and* B *being positive and less than* 360^0;

$$\frac{\binom{a}{0}^m}{\binom{b}{0}^m} = \binom{c}{0}^m, \textit{ if A be greater than B}$$

$$= \binom{c}{-1}^m \text{ ....} \textit{less} \text{............}$$

where m *may be either whole or fractional, positive or negative.*

For since $\frac{a}{b} = c,\ b.c = a,$

$\therefore\ \binom{b}{0}^m . \binom{c}{0}^m$ is equal in length to $\binom{a}{0}^m$,

$\therefore\ \frac{\binom{a}{0}^m}{\binom{b}{0}^m}$ is equal in length to $\binom{c}{0}^m$ or $\binom{c}{-1}^m$;

Now $\binom{a}{0}^m$ is inclined to unity at an angle $= mA$,

$\binom{b}{0}^m$ .... $= mB$,

$\therefore\ \frac{\binom{a}{0}^m}{\binom{b}{0}^m}$ .... $= mA - mB$

$= m(A - B)$;

Again $\because\ c = \frac{a}{b}$, c is inclined to unity at an angle $= A - B$,

And if A be greater than B, $A - B$ is positive; and it is also less than 360^0, since A, B are each less than 360^0;

therefore in this case $\underset{0}{c}$ represents c considered as inclined to unity at the angle $= A - B$;

$\therefore \left(\underset{0}{c}\right)^m$ is inclined to unity at an angle $= m(A - B)$;

$$\therefore \text{ when } A \text{ is greater than } B,\ \frac{\left(\underset{0}{a}\right)^m}{\left(\underset{0}{b}\right)^m} = \left(\underset{0}{c}\right)^m.$$

But if A be less than B, $A - B$ is negative; therefore in this case $\underset{-1}{c}$ represents c considered as inclined to unity at the angle $A - B$;

$\therefore \left(\underset{-1}{c}\right)^m$ is inclined to unity at an angle $= m(A - B)$;

$$\therefore \text{ when } A \text{ is less than } B,\ \frac{\left(\underset{0}{a}\right)^m}{\left(\underset{0}{b}\right)^m} = \left(\underset{-1}{c}\right)^m.$$

(94.) Cor. *Hence if* $\frac{1}{\mathrm{b}} = \mathrm{c}$,

$\frac{1}{\left(\underset{0}{\mathrm{b}}\right)^{\mathrm{m}}} = \left(\underset{0}{\mathrm{c}}\right)^{\mathrm{m}}$, *if* b *be a positive quantity,*

$= \left(\underset{-1}{\mathrm{c}}\right)^{\mathrm{m}}$, *if* b *be not a positive quantity.*

For putting $a = 1$, in the preceding Article, $A = 0$, and if b be a positive quantity, $B = 0$;

$$\therefore A - B = 0;$$

$$\therefore\ \frac{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)^m}{\left(\begin{smallmatrix}b\\0\end{smallmatrix}\right)^m} = \left(\begin{smallmatrix}c\\0\end{smallmatrix}\right)^m, \text{ or } \frac{1}{\left(\begin{smallmatrix}b\\0\end{smallmatrix}\right)^m} = \left(\begin{smallmatrix}c\\0\end{smallmatrix}\right)^m;$$

But if b be not a positive quantity, $A - B = 0 - B$, $\therefore$ $A - B$ is negative;

$$\therefore\ \frac{1}{\left(\begin{smallmatrix}b\\0\end{smallmatrix}\right)^m} = \left(\begin{smallmatrix}c\\-1\end{smallmatrix}\right)^m.$$

(95.) *If* $\frac{a}{b} = c$, *and* a *be inclined to unity at an angle* = A, *and* b, *at an angle* = B; A, B *being positive and less than* 360°;

$$\frac{\left(\begin{smallmatrix}a\\p\end{smallmatrix}\right)^m}{\left(\begin{smallmatrix}b\\q\end{smallmatrix}\right)^m} = \left(\begin{smallmatrix}c\\p-q\end{smallmatrix}\right)^m, \textit{ if } A \textit{ be greater than } B,$$

$$= \left(\begin{smallmatrix}c\\p-q-1\end{smallmatrix}\right)^m, \ldots\ldots\ldots\ldots \textit{less} \ldots\ldots\ldots\ldots,$$

where m *may be either whole or fractional, positive or negative.*

$$\text{For } \frac{\left(\begin{smallmatrix}a\\p\end{smallmatrix}\right)^m}{\left(\begin{smallmatrix}b\\q\end{smallmatrix}\right)^m} = \frac{\left(\begin{smallmatrix}a\\0\end{smallmatrix}\right)^m \cdot \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{mp}}{\left(\begin{smallmatrix}b\\0\end{smallmatrix}\right)^m \cdot \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{mq}} = \frac{\left(\begin{smallmatrix}a\\0\end{smallmatrix}\right)^m}{\left(\begin{smallmatrix}b\\0\end{smallmatrix}\right)^m} \cdot \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{m \cdot \overline{p-q}}$$

$$\left.\begin{aligned} &= \left(\begin{smallmatrix}c\\0\end{smallmatrix}\right)^m \cdot \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{m \cdot \overline{p-q}}, \text{ if } A \text{ be greater than } B, \\ &= \left(\begin{smallmatrix}c\\-1\end{smallmatrix}\right)^m \cdot \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{m \cdot \overline{p-q}}, \ldots\ldots\ldots\ldots \text{less} \ldots\ldots\ldots\ldots, \end{aligned}\right\} \text{(by Art. 93.)}$$

$$= \left(\underset{p-q}{c}\right)^{m}, \text{ if } A \text{ be greater than } B,$$

$$= \left(\underset{p-q-1}{c}\right)^{m}, \ldots\ldots\ldots\text{less}\ldots\ldots\ldots\ldots$$

(96.) Cor. Hence, if $\frac{1}{b} = c$,

$$\frac{1}{\left(\underset{q}{b}\right)^{m}} = \left(\underset{-q}{c}\right)^{m}, \text{ if } b \text{ be a positive quantity,}$$

$$= \left(\underset{-q-1}{c}\right)^{m}, \text{ if } b \text{ be not a positive quantity.}$$

(97.) $\left(\underset{p}{a}\right)^{\frac{m}{n}} = \left(\underset{p}{a}\right)^{\frac{km}{kn}}$, *where* m, n, k *are whole numbers.*

For let c be a positive quantity equal in length to a;

$$\text{then } \left(\underset{0}{c}\right)^{\frac{m}{n}} \text{ is in length } = \left(\underset{p}{a}\right)^{\frac{m}{n}} \text{ (by Art. 59.)}$$

$$\text{and } \left(\underset{0}{c}\right)^{\frac{km}{kn}} \ldots\ldots\ldots = \left(\underset{p}{a}\right)^{\frac{km}{kn}} \ldots\ldots\ldots$$

$$\text{But } \left(\underset{0}{c}\right)^{\frac{m}{n}} = \left(\underset{0}{c}\right)^{\frac{km}{kn}} \text{ (by Art. 66),}$$

$$\therefore \left(\underset{p}{a}\right)^{\frac{m}{n}} \text{ is in length } = \left(\underset{p}{a}\right)^{\frac{km}{kn}};$$

Let a be inclined to unity at an angle $= A$, A being positive and less than 360°, then $\left(\underset{p}{a}\right)^{\frac{m}{n}}$ is inclined to unity at an angle $= \frac{m}{n}(A + p.360^0)$,

and $\left(\underset{p}{a}\right)^{\frac{km}{kn}}$ .... $= \frac{km}{kn}(A+p.360^0)$

$$= \frac{m}{n}(A+p.360^0);$$

$$\therefore \left(\underset{p}{a}\right)^{\frac{m}{n}} = \left(\underset{p}{a}\right)^{\frac{km}{kn}}.$$

(98.) *If* $a^m = b$, *and* $b^n = c$; c *is in length* $= a^{mn}$; *where* m, n *may be either whole or fractional, positive or negative.*

For let e be a positive quantity in length $= a$;

then $\left(\underset{0}{e}\right)^m$ is in length $= a^m = b$;

$\therefore c$ is in length $= \left(\underset{0}{e}\right)^{mn}$ (by Art 75, 76, 77, 78.)

But $\left(\underset{0}{e}\right)^{mn}$ is in length $= a^{mn}$,

since e is length $= a$;

$\therefore c$ is in length $= a^{mn}$.

(99.) *Let* $\left(\underset{p}{a}\right)^{\frac{k}{l}} = b$, *and* $\left(\underset{q}{b}\right)^{\frac{m}{n}} = c$, *where* $\frac{k}{l}$, $\frac{m}{n}$ *are fractions in their lowest terms; to investigate in what cases* c *is a value of* $a^{\frac{km}{ln}}$.

(By Art. 98.) c is in length $= a^{\frac{km}{ln}}$;

therefore if any value of $a^{\frac{km}{ln}}$ be inclined to unity at the same angle that c is, that value of $a^{\frac{km}{ln}}$ is equal to c;

Assume $\left(\underset{x}{a}\right)^{\frac{km}{ln}}$ to be such a value of $a^{\frac{km}{ln}}$,

and let a be inclined to unity at an angle $= A$, A being positive and less than 360^0,

then $\left(\underset{x}{a}\right)^{\frac{km}{ln}}$ is inclined to unity at an angle

$$= \frac{km}{ln}(A + x \, . \, 360^0);$$

Also since $\left(\underset{p}{a}\right)^{\frac{k}{l}} = b$, b is inclined to unity at an angle

$$= \frac{k}{l} . (A + p \, . \, 360^0);$$

Let $\frac{k}{l}$ $(A + p \, . \, 360^0) = B + r \, . \, 360^0$, where B is positive and less than 360^0, and $r = 0$, or some whole number, either positive, or negative,

$\because$ $\left(\underset{q}{b}\right)^{\frac{m}{n}} = c$, c is inclined to unity at an angle

$$= \frac{m}{n}(B + q \, . \, 360^0),$$

But $B = \frac{k}{l}(A + p.360^0) - r.360^0$,

$\therefore$ c is inclined to unity at an angle

$$= \frac{km}{ln}(A + p.360^0) + \frac{m}{n}.(q - r)\,360^0;$$

But $\left(\underset{x}{a}\right)^{\frac{km}{ln}}$ is inclined to unity at an angle

$$= \frac{km}{ln}(A + x.360^0);$$

$\therefore$ since $\left(\underset{x}{a}\right)^{\frac{km}{ln}}$ is inclined to unity at the same angle that c is,

$$\frac{km}{ln}(A + x.360^0) + y.360^0$$

$$= \frac{km}{ln}.(A + p.360^0) + \frac{m}{n}(q - r)\,360^0,$$

where y must either $= 0$, or some whole number, either positive, or negative;

$$\therefore \frac{km}{ln}.x.360^0 + y.360^0$$

$$= \frac{km}{ln}.p.360^0 + \frac{m}{n}.(q - r)\,360^0,$$

$$\therefore\ kmx + lny = kmp + lm\,(q-r),$$

$$\therefore\ kx + ln\frac{y}{m} = kp + l\,(q-r);$$

In which equation, that x may be a whole number, it is necessary that y should either $= 0$, or some multiple of m; let $y = mz$,

then $kx + lnz = kp + l\,(q-r)$,

where if k be prime to ln, integer values of x and z can always be found, which will satisfy the conditions of the equation,

but by the hypothesis k is prime to l,

$\therefore$ whenever k is prime to n, k is also prime to ln,

$\therefore$ whenever k is prime to n, c is a value of $a^{\frac{km}{ln}}$.

If k be not prime to n,

let e be the greatest common measure of k and n,

and let $k = ef$, $n = eg$,

then $efx + legz = efp + l\,(q-r)$,

$$\therefore\ fx + lgz = fp + \frac{l\,(q-r)}{e};$$

the conditions of which equation cannot be satisfied by integer values of x and z, unless $\frac{l(q-r)}{e} = 0$, or some whole number, either positive, or negative;

And $\frac{l(q-r)}{e}$ cannot $= 0$, unless $q - r = 0$,

Also since l is prime to k, l is prime to e a part of k,

$\therefore \frac{l(q-r)}{e}$ cannot = a whole number, unless $q - r$ be a multiple of e;

$\therefore$ if k be not prime to n, c is not a value of $a^{\frac{km}{ln}}$, unless, either $q - r = 0$, or the greatest common measure of k and n be also a measure of $q - r$.

(100.) Cor. 1. It will appear from a similar investigation, if $\left(\underset{p}{a}\right)^{-\frac{k}{l}} = b$, and $\left(\underset{q}{b}\right)^{-\frac{m}{n}} = c$, that c is a value of $a^{\frac{km}{ln}}$, whenever k is prime to n; and that when k is not prime to n, c is not a value of $a^{\frac{km}{ln}}$, unless either $q - r = 0$, or the greatest common measure of k and n be also a measure of $q - r$; also that if $\left(\underset{p}{a}\right)^{-\frac{k}{l}} = b$, and $\left(\underset{q}{b}\right)^{\frac{m}{n}} = c$, or $\left(\underset{p}{a}\right)^{\frac{k}{l}} = b$, and $\left(\underset{q}{b}\right)^{-\frac{m}{n}} = c$, c is a value of $a^{-\frac{km}{ln}}$ in the same cases.

(101.) Cor. 2. Let $\left(\underset{p}{a}\right)^s = b$, and let a be inclined to unity at an angle $= A$, A being positive and less than 360^0, and let $B + q.360^0 = s(A + p.360^0)$, where B is positive and less than 360^0, and $q = 0$ or some whole number either positive or negative, and let $\left(\underset{q}{b}\right)^t = c$, and let s and t be either whole or fractional, positive or negative; then $\left(\underset{p}{a}\right)^{st} = c$.

First let s and t be both positive whole numbers,

then $c = a^{st}$ (by Art. 48),

also a^{st} has no other value different from c (by Art. 43.)

$$\therefore \left(\underset{p}{a}\right)^{st} = c.$$

Next let s and t be both positive, but one or both fractions.

$$\text{Let } s = \frac{k}{l},\ t = \frac{m}{n},$$

$$\text{then } B + q.360^0 = \frac{k}{l}.(A + p.360^0);$$

$$\text{But } B + r.360^0 = \frac{k}{l}(A + p360^0) \text{ (by Art. 99)},$$

$$\therefore q = r, \text{ or } q - r = 0;$$

$\therefore$ the equation $kx + lnz = kp + l(q - r)$ (in Art. 99)

becomes $kx + lnz = kp$;

where if we make $z = 0$, and $x = p$, the conditions of the equation are answered;

$$\therefore\ c = \left(\underset{p}{a}\right)^{\frac{km}{ln}} = \left(\underset{p}{a}\right)^{st}.$$

Next let s be negative and t positive, or s positive and t negative, or both s and t negative.

These cases may be demonstrated by means of Art. 100, in the same manner that the preceding case was demonstrated by means of Art. 99; therefore whether s and t be whole or fractional, positive or negative,

$$c = \left(\underset{p}{a}\right)^{st}.$$

(102.) *If* $\left(\underset{p}{\mathrm{a}}\right)^{\mathrm{m}} = \mathrm{b}$, *where* m *is either whole or fractional, positive or negative, and* a *be inclined to unity at an angle* = A, A *being positive and less than* 360°, *and* $\mathrm{m}.(\mathrm{A} + \mathrm{p}.360^{0}) = \mathrm{B} + \mathrm{q}.360^{0}$, *where* B *is positive and less than* 360°, *and* q = 0, *or some whole number, either positive or negative;*

$$\textit{then}\ \left(\underset{q}{\mathrm{b}}\right)^{\frac{1}{\mathrm{m}}} = \mathrm{a}.$$

For let $\left(\underset{q}{b}\right)^{\frac{1}{m}} = c,$

then $c = \left(\underset{p}{a}\right)^{\frac{m}{m}}$ (by Art. 101.)

$= \left(\underset{p}{a}\right)^{1} = a;$

$\therefore \left(\underset{q}{b}\right)^{\frac{1}{m}} = a.$

(103.) *If* c *be a value of* $a^{\frac{k}{l}}$ *and also a value of* $b^{\frac{m}{n}}$, *where* $\frac{k}{l}$, $\frac{m}{n}$ *are fractions in their lowest terms; then if* k *be prime to* m, b *is a value of* $a^{\frac{kn}{lm}}$.

For since c is a value of $b^{\frac{m}{n}}$, b is a value of $c^{\frac{n}{m}}$ (by Art. 102.)

$\therefore$ since k is prime m, b is a value of $a^{\frac{kn}{lm}}$ (by Art. 99).

(104.) Cor. In like manner it may be proved, if c be a value of $a^{-\frac{k}{l}}$ and also a value of $b^{\frac{m}{n}}$, where $\frac{k}{l}$, $\frac{m}{n}$ are fractions in their lowest terms; that, if k be prime to m, b is a value of $a^{-\frac{kn}{lm}}$.

(105.) *The values of the square root of* -1 *are inclined to unity at angles* $= 90^0$ *and* 270^0.

For (by Art. 61.) $\sqrt[2]{-1}$ has two values, viz. $\left(-\underset{0}{1}\right)^{\frac{1}{2}}$, and $\left(-\underset{1}{1}\right)^{\frac{1}{2}}$;

Now -1 is inclined to unity at an angle $= 180^0$,

$\therefore \left(-\underset{0}{1}\right)^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots = \frac{1}{2}(180^0) = 90^0$,

and $\left(-\underset{1}{1}\right)^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots = \frac{1}{2}(180^0 + 360^0)$

$= 270^0$.

(106.) *If* $\left(-\underset{0}{1}\right)^{\frac{1}{2}}$ *be represented by* $+\sqrt{-1}$, $\left(-\underset{1}{1}\right)^{\frac{1}{2}}$ *will be represented by* $-\sqrt{-1}$.

For by the preceding Article

$\left(-\underset{0}{1}\right)^{\frac{1}{2}}$ is inclined to unity at an angle $= 90^0$,

$\left(-\underset{1}{1}\right)^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots = 270^0$,

$\therefore \left(-\underset{1}{1}\right)^{\frac{1}{2}}$ is inclined to $\left(-\underset{0}{1}\right)^{\frac{1}{2}}$ at an angle

$= 270^0 - 90^0 = 180^0$;

$\therefore$ since $\left(-\underset{0}{1}\right)^{\frac{1}{2}}$ is represented by $+\sqrt{-1}$, $\left(-\underset{1}{1}\right)^{\frac{1}{2}}$ will be represented by $-\sqrt{-1}$ (by Art. 8).

(107.) *The values of* $1^{\frac{1}{4}}$ *are* 1, $+\sqrt{-1}$, -1, $-\sqrt{-1}$.

For $\left(\underset{0}{1}\right)^{\frac{1}{4}} = 1$,

$\binom{1}{1}^{\frac{1}{4}}$ is inclined to unity at an angle $= \dfrac{360^0}{4} = 90^0$,

$\binom{1}{2}^{\frac{1}{4}}$ .... $= \dfrac{2.360^0}{4} = 180^0$,

$\binom{1}{3}^{\frac{1}{4}}$ .... $= \dfrac{3.360^0}{4} = 270^0$;

But $\sqrt{-1}$ .... $= 90^0$,

-1 .... $= 180^0$,

$-\sqrt{-1}$ .... $= 270^0$;

$\therefore \binom{1}{0}^{\frac{1}{4}} = 1, \binom{1}{1}^{\frac{1}{4}} = \sqrt{-1}, \binom{1}{2}^{\frac{1}{4}} = -1, \binom{1}{3}^{\frac{1}{4}} = -\sqrt{-1}.$

(108.) $\binom{1}{1}^{-\frac{1}{4}} = -\sqrt{-1}.$

For (by Art. 65.)

$\binom{1}{1}^{-\frac{1}{4}}$ is inclined to unity at an angle $= -\frac{1}{4}\, 360^0$,

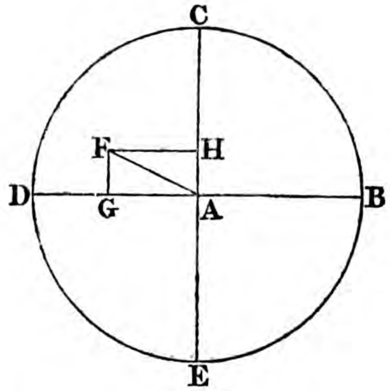
and $\binom{1}{3}^{\frac{1}{4}}$ .... $= \frac{3}{4}\, 360^0$

.... $= 360^0 - \frac{1}{4} 360^0$,

$\therefore \binom{1}{1}^{-\frac{1}{4}}$ coincides with $\binom{1}{3}^{\frac{1}{4}}$,

$\therefore \binom{1}{1}^{-\frac{1}{4}} = -\sqrt{-1}$ (by Art. 107).

(109.) *Any quantity may be expressed in the form* $\pm a \pm b\sqrt{-1}$, *where* a, b *are positive quantities.*



Let A be the origin, and $AB =$ unity;

From centre A with radius AB describe circle $BCDE$, and produce BA to D, and draw EAC perpendicular to DB;

Let BC be the direction in which positive angles are measured,

then $AC = \sqrt{-1}$, $AD = -1$, $AE = -\sqrt{-1}$;

Let c be the given quantity;

Draw $AF = c$, and draw FG perpendicular to DA, and FH perpendicular to AC;

Since $AGFH$ is a parallelogram,

$$AF = AG + AH \text{ (by Art. 3.)}$$

$$\text{or } c = AG + AH;$$

Let a be a positive quantity in length $= AG$,

b .... $= AH$;

$$\text{Then } AB : a :: AD : AG,$$

$$\text{or } 1 : a :: -1 : AG,$$

$$\therefore AG = a \times -1 = -a;$$

$$\text{And } AB : b :: AC : AH,$$

$$\text{or } 1 : b :: \sqrt{-1} : AH,$$

$$\therefore AH = b\sqrt{-1};$$

$$\therefore c = -a + b\sqrt{-1};$$

In like manner it may be proved in any other case.

(110.) Cor. 1. Let c be inclined to unity at an angle $= C$, C being positive and less than 360^0;

Then if C be less than 90^0,

c is of the form $+a + b\sqrt{-1}$;

if C be greater than 90^0 and less than 180^0,

c is of the form $-a + b\sqrt{-1}$;

if C be greater than 180^0 and less than 270^0,

c is of the form $-a - b\sqrt{-1}$;

if C be greater than 270^0,

c is of the form $+a - b\sqrt{-1}$;

(111.) Cor. 2. Hence if $c = e\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^n$, e being a positive quantity, and n positive and less than 1;

Then if n be less than $\frac{1}{4}$,

c is of the form $+a + b\sqrt{-1}$;

if n be greater than $\frac{1}{4}$ and less than $\frac{1}{2}$,

c is of the form $-a + b\sqrt{-1}$;

if n be greater than $\frac{1}{2}$ and less than $\frac{3}{4}$,

c is of the form $-a - b\sqrt{-1}$;

if n be greater than $\frac{3}{4}$,

c is of the form $+a - b\sqrt{-1}$.

(112.) Cor. 3. When n is less than $\frac{1}{4}$,

as n increases, $\frac{b}{a}$ increases; and conversely;

when n is greater than $\frac{1}{4}$ and less than $\frac{1}{2}$,

as n increases, $\frac{b}{a}$ decreases; and conversely;

when n is greater than $\frac{1}{2}$ and less than $\frac{3}{4}$,

as n increases, $\frac{b}{a}$ increases; and conversely;

when n is greater than $\frac{3}{4}$,

as n increases, $\frac{b}{a}$ decreases; and conversely.

(113.) *To express the values of* $1^{\frac{1}{6}}$ *in the form* $\pm a \pm b\sqrt{-1}$, *where* a, b *are positive quantities.*

The values of

$$1^{\frac{1}{6}} \text{ are } \binom{1}{0}^{\frac{1}{6}},\ \binom{1}{1}^{\frac{1}{6}},\ \binom{1}{2}^{\frac{1}{6}},\ \binom{1}{3}^{\frac{1}{6}},\ \binom{1}{4}^{\frac{1}{6}},\ \binom{1}{5}^{\frac{1}{6}};$$

$$\text{of which } \binom{1}{0}^{\frac{1}{6}} = 1;$$

And since $\frac{1}{6}$ is less than $\frac{1}{4}$,

$$\binom{1}{1}^{\frac{1}{6}} \text{ is of the form } a + b\sqrt{-1};$$

$$\binom{1}{2}^{\frac{1}{6}} = \binom{1}{1}^{\frac{2}{6}} = \binom{1}{1}^{\frac{1}{3}},$$

$\therefore$ since $\frac{1}{3}$ is greater than $\frac{1}{4}$ and less than $\frac{1}{2}$,

$$\binom{1}{2}^{\frac{1}{6}} \text{ is of the form } -a + b\sqrt{-1};$$

$$\binom{1}{3}^{\frac{1}{6}} = \binom{1}{1}^{\frac{3}{6}} = \binom{1}{1}^{\frac{1}{2}} = -1;$$

$$\binom{1}{4}^{\frac{1}{6}} = \binom{1}{1}^{\frac{4}{6}} = \binom{1}{1}^{\frac{2}{3}},$$

$\therefore$ since $\frac{2}{3}$ is greater than $\frac{1}{2}$ and less than $\frac{3}{4}$,

$$\binom{1}{4}^{\frac{1}{6}} \text{ is of the form } -a,\ -b\sqrt{-1};$$

$$\binom{1}{5}^{\frac{1}{6}} = \binom{1}{1}^{\frac{5}{6}},$$

$\therefore$ since $\frac{5}{6}$ is greater than $\frac{3}{4}$,

$\left(1\atop 5\right)^{\frac{1}{6}}$ is of the form $+\, a - b\sqrt{-1}$;

$$\text{Let } 1^{\frac{1}{6}} = x,$$

$$\text{then } x^6 = 1, \text{ or } x^6 - 1 = 0,$$

From the solution of this equation we obtain the following values of x;

$$1,\ -\tfrac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1},\ -\tfrac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1},$$

$$-1,\ \tfrac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1},\ \tfrac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1};$$

$$\therefore \left(1\atop 0\right)^{\frac{1}{6}} = 1,$$

$$\left(1\atop 1\right)^{\frac{1}{6}} = \tfrac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1},$$

$$\left(1\atop 2\right)^{\frac{1}{6}} = -\tfrac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1},$$

$$\left(1\atop 3\right)^{\frac{1}{6}} = -1,$$

$$\left(1\atop 4\right)^{\frac{1}{6}} = -\tfrac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1},$$

$$\left(1\atop 5\right)^{\frac{1}{6}} = \tfrac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1}.$$

(114.) *If* $a + b \binom{1}{1}^m + c \binom{1}{1}^n = f \binom{1}{1}^p$, *and* a, b, c, f *be positive quantities*,

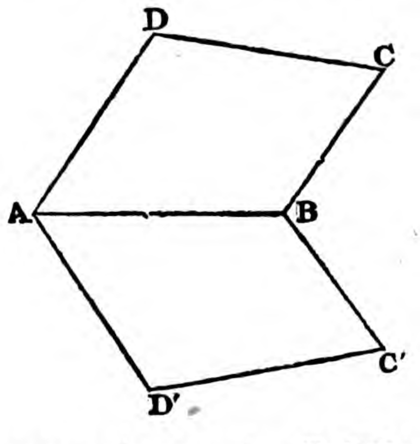
$$a + b \binom{1}{-1}^m + c \binom{1}{-1}^n = f \binom{1}{-1}^p.$$

Let A be the origin, and $AB = a$.

From B draw BC parallel and equal in length to the line which represents $b \binom{1}{1}^m$, and from C draw CD parallel and equal in length to the line which represents $c \binom{1}{1}^n$, and join AD;

$$AD = a + b \binom{1}{1}^m + c \binom{1}{1}^n \text{ (by Art. 7.)}$$
$$= f \binom{1}{1}^p;$$

On the other side of AB draw BC' making angle $ABC' = ABC$ and make BC' in length $= BC$,



In like manner make angle $BC'D'$ = angle BCD, and $C'D'$ in length $= CD$,

Join AD';

Since BC' is equal in length to BC, and angle ABC' equal to ABC, but in the opposite direction, and BC parallel and equal in length to the line which represents $b\binom{1}{1}^m$; BC' will be parallel and equal in length to the line which represents $b\binom{1}{-1}^m$;

In like manner $C'D'$ is parallel and equal in length to the line which represents $c\binom{1}{-1}^n$;

$$\therefore\ AD' = a + b\binom{1}{-1}^m + c\binom{1}{-1}^n;$$

But since figures $ABCD$, $ABC'D'$ are similar and equal, AD' is equal in length to AD, and angle BAD' equal to BAD, but in the opposite direction,

$$\therefore \text{ since } AD = f\binom{1}{1}^p,\quad AD' = f\binom{1}{-1}^p;$$

$$\therefore\ a + b\binom{1}{-1}^m + c\binom{1}{-1}^n = f\binom{1}{-1}^p.$$

(115.) *If* $\mathrm{a} + \mathrm{b}\binom{1}{1}^{\mathrm{m}} + \mathrm{c}\binom{1}{1}^{\mathrm{n}} = \mathrm{f}\binom{1}{1}^{\mathrm{p}}$, *and* a, b, c, f *be either positive or negative quantities;*

$$\mathrm{a} + \mathrm{b}\binom{1}{-1}^{\mathrm{m}} + \mathrm{c}\binom{1}{-1}^{\mathrm{n}} = \mathrm{f}\binom{1}{-1}^{\mathrm{p}}.$$

For let one of the quantities b be negative, let $b = -g$;

$$\text{then } a - g\binom{1}{1}^m + c\binom{1}{1}^n = f\binom{1}{1}^p;$$

$$\text{Now } -g = g\binom{1}{1}^{\frac{1}{2}},$$

$$\therefore\ -g\binom{1}{1}^{m} = g\binom{1}{1}^{\frac{1}{2}}\cdot\binom{1}{1}^{m} = g\binom{1}{1}^{m+\frac{1}{2}},$$

$$\therefore\ a + g\binom{1}{1}^{m+\frac{1}{2}} + c\binom{1}{1}^{n} = f\binom{1}{1}^{p},$$

$$\therefore\ a + g\binom{1}{-1}^{m+\frac{1}{2}} + c\binom{1}{-1}^{n} = f\binom{1}{-1}^{p} \text{ (by Art. 114.)}$$

$$\text{But } g\binom{1}{-1}^{m+\frac{1}{2}} = g\binom{1}{-1}^{\frac{1}{2}}\cdot\binom{1}{-1}^{m} = -g\binom{1}{-1}^{m},$$

$$\therefore\ a - g\binom{1}{-1}^{m} + c\binom{1}{-1}^{n} = f\binom{1}{-1}^{p},$$

$$\text{or } a + b\binom{1}{-1}^{m} + c\binom{1}{-1}^{n} = f\binom{1}{-1}^{p};$$

In like manner it may be proved if any other of the quantities be negative.

(116.) *If* $\mathrm{a} + \mathrm{b}\binom{1}{1}^{\mathrm{m}} + \mathrm{c}\binom{1}{1}^{\mathrm{n}} = \mathrm{f}\binom{1}{1}^{\mathrm{p}}$, *and* a, b, c, f, *be either positive or negative quantities,*

$$\mathrm{a} + \mathrm{b}\binom{1}{1}^{-\mathrm{m}} + \mathrm{c}\binom{1}{1}^{-\mathrm{n}} = \mathrm{f}\binom{1}{1}^{-\mathrm{p}}.$$

$$\text{For } a + b\binom{1}{-1}^{m} + c\binom{1}{-1}^{n} = f\binom{1}{-1}^{p} \text{ (by Art. 115.)}$$

$$\text{And } \binom{1}{-1}^{m} = \binom{1}{1}^{-m},$$

$$\therefore\ a + b\binom{1}{1}^{-m} + c\binom{1}{1}^{-n} = f\binom{1}{1}^{-p}.$$

(117.) *If* $a + b\sqrt{-1} + c\binom{1}{1}^{n} = f\binom{1}{1}^{p}$, *and* a, b, c, f *be either positive or negative quantities,*

$$a - b\sqrt{-1} + c\binom{1}{1}^{-n} = f\binom{1}{1}^{-p}.$$

For $+\sqrt{-1} = \binom{1}{1}^{\frac{1}{4}}$ (by Art. 107.)

$$\therefore\ a + b\binom{1}{1}^{\frac{1}{4}} + c\binom{1}{1}^{n} = f\binom{1}{1}^{p},$$

$$\therefore\ a + b\binom{1}{1}^{-\frac{1}{4}} + c\binom{1}{1}^{-n} = f\binom{1}{1}^{-p};$$

But $\binom{1}{1}^{-\frac{1}{4}} = -\sqrt{-1}$ (by Art. 108.)

$$\therefore\ a - b\sqrt{-1} + c\binom{1}{1}^{-n} = f\binom{1}{1}^{-p}.$$

(118.) *If* $a + b\sqrt{-1} = e + f\sqrt{-1}$, *and* a, b, e, f, *be either positive or negative quantities;* $a = e$ *and* $b = f$.

For since $a + b\sqrt{-1} = e + f\sqrt{-1}$,

$a - b\sqrt{-1} = e - f\sqrt{-1}$ (by Art. 117.)

$\therefore$ by addition and subtraction,

$$2a = 2e,$$

and $2b\sqrt{-1} = 2f\sqrt{-1}$;

$$\therefore\ a = e,$$

and $b = f$.

(119.) *If* $a = \pm b \pm c\sqrt{-1}$, *where* b *and* c *are positive quantities,* a *will be in length* $= \sqrt{b^2 + c^2}$.

For let $a = e \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^n$, where e is a positive quantity,

then $e \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^n = \pm b \pm c\sqrt{-1}$,

$\therefore e \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-n} = \pm b \mp c\sqrt{-1}$ (by Art. 117.)

$\therefore$ by multiplication,

$$e^2 = b^2 + c^2,$$

$$e = \sqrt{b^2 + c^3},$$

$\therefore a$ is in length $= \sqrt{b^2 + c^2}$.

(120.) *Let* $a = b \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^n$, *where* b *is a positive quantity, and* n *positive and less than* 1; *then if* n *be either not greater than* $\frac{1}{4}$, *or not less than* $\frac{3}{4}$, $a + 1$ *is greater in length than* 1.

For since $a = b \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^n$, and n is either not greater than $\frac{1}{4}$ or not less than $\frac{3}{4}$, a is of the form $e \pm f\sqrt{-1}$ (by Art. 111), where $e = 0$, or some positive quantity, and f is a positive quantity;

$$\therefore a + 1 = 1 + e \pm f\sqrt{-1},$$

$\therefore$ $a + 1$ is in length equal to $\sqrt{(1+e)^2 + f^2}$;

$\therefore$ $a + 1$ is in length greater than 1.

(121.) *Let* a = b $\left(\frac{1}{1}\right)^n$, *where* b *is a positive qnantity, and* n *not less than* $\frac{1}{4}$ *and not greater than* $\frac{3}{4}$; *then* a − 1 *is in length greater than* 1.

For a is of the form $-e \pm f\sqrt{-1}$ (by Art. 111), where e and f either = 0, or are positive quantities;

$$\therefore\ a - 1 = -1 - e \pm f\sqrt{-1},$$

$\therefore$ $a - 1$ is in length equal to $\sqrt{(1+e)^2 + f^2}$;

$\therefore$ $a - 1$ is in length greater than 1.

(122.) *Let* a = b $\left(\frac{1}{1}\right)^n$, *where* b *is a positive quantity, and* n *positive and less than* 1; *then if* n *be either less than* $\frac{1}{4}$, *or greater than* $\frac{3}{4}$, $\frac{a-1}{a+1}$ *is in length less than* 1; *but if* n *be greater than* $\frac{1}{4}$, *and less than* $\frac{3}{4}$, $\frac{a-1}{a+1}$ *is in length greater than* 1.

First let n be either less than $\frac{1}{4}$, or greater than $\frac{3}{4}$;

Then a is of the form $e \pm f\sqrt{-1}$, where e and f are positive quantities;

$\therefore$ $a-1$ is in length equal to $\sqrt{(e-1)^2+f^2}$,

and $a+1$ .... $\sqrt{(e+1)^2+f^2}$,

$\therefore$ $\dfrac{a-1}{a+1}$ .... $\dfrac{\sqrt{(e-1)^2+f^2}}{\sqrt{(e+1)^2+f^2}}$;

$\therefore$ $\dfrac{a-1}{a+1}$ is in length less than 1.

Next let n be greater than $\frac{1}{4}$, and less than $\frac{3}{4}$,

Then a is of the form $-e \pm f\sqrt{-1}$,

$\therefore$ $\dfrac{a-1}{a+1}$ is in length equal to $\dfrac{\sqrt{(e+1)^2+f^2}}{\sqrt{(e-1)^2+f^2}}$;

$\therefore$ $\dfrac{a-1}{a+1}$ is in length greater than 1.

(123.) $\binom{1}{1}^{n} + \binom{1}{1}^{-n}$ *is equal to a positive quantity, when* n *is positive and less than* $\frac{1}{4}$.

For if n be positive and less than $\frac{1}{4}$, $\binom{1}{1}^{n}$ is of the form $a + b\sqrt{-1}$, where a and b are positive quantities,

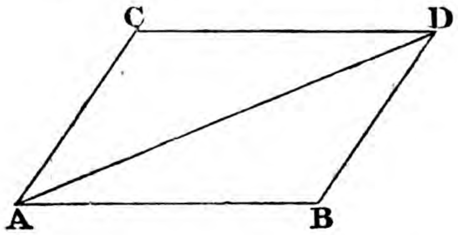
$$\therefore \binom{1}{1}^{-n} = a - b\sqrt{-1} \text{ (by Art. 117)},$$

$$\therefore \binom{1}{1}^{n} + \binom{1}{1}^{-n} = 2a.$$

(124.) *Let* a = b $\binom{1}{1}^{m}$ *and* a + 1 = c $\binom{1}{1}^{p}$, *where* b *and* c *are positive quantities, and* m *and* p *positive and less than* 1;

Then if m *be less than* $\frac{1}{2}$, p *will be less than* m,
........*greater*......., ...*greater*.......

First let m be less than $\frac{1}{2}$,



And let AB = unity, $AC = a$, and let the parallelogram $ABDC$ be completed, and the diagonal AD be drawn;

$$\text{Then } AD = a + 1 = c \binom{1}{1}^{p};$$

$$\therefore \text{ angle } BAD = p \,.\, 360^\circ;$$

$$\text{and angle } BAC = m \,.\, 360^\circ;$$

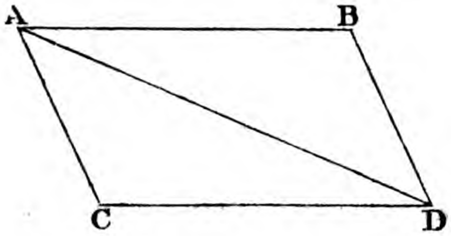
$$\therefore p \text{ is less than } m.$$

Next let m be greater than $\frac{1}{2}$,

The same construction being made, AC falls on the other side of AB;

$\therefore$ angle $BAD = \overline{1 - p}\,.\,360^0$,

.... $BAC = \overline{1 - m}\,.\,360^0$;



$\therefore$ $1 - m$ is greater than $1 - p$;

$\therefore$ p is greater than m.

CHAP. III.

BINOMIAL THEOREM, EXPANSION OF a^x IN A SERIES ARRANGED ACCORDING TO THE POWERS OF x, DIFFERENTIATION OF a^x.

(125.) *If* a = 1 + b, *it is proved in the binomial theorem that*

$$a^m = 1 + mb + \frac{m.\overline{m-1}}{1.2}b^2 + \frac{m.\overline{m-1}.\overline{m-2}}{1.2.3}b^3 + \&c.$$

which is a converging series when b *is in length less than unity;*

Now if m *be not a whole number,* a^m *has many values;*

To investigate which of the values of a^m *is represented by the series*

$$1 + mb + \frac{m.\overline{m-1}}{1.2}b^2 + \frac{m.\overline{m-1.m-2}}{1.2.3}b^3 + \&c.$$

First let *a* be a positive quantity,

then $\left(\underset{0}{a}\right)^m$ is also a positive quantity;

$\therefore \left(\begin{smallmatrix}a\\0\end{smallmatrix}\right)^m$ is that value of a^m which is represented by the series

$$1 + mb + \frac{m\,.\,\overline{m-1}}{1\,.\,2} b^2 + \&c.$$

Next let $a = c \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^n$, where c is a positive quantity and n positive and less than $\frac{1}{4}$;

When $b = 0$, the series

$$1 + mb + \frac{m\,.\,\overline{m-1}}{1\,.\,2} b^2 + \&c. = 1\,;$$

$\therefore$ when $b = 0$, that value of a^m which is represented by $1 + mb + \frac{m\,.\,\overline{m-1}}{1\,.\,2} b^2 + \&c.$ must $= 1$;

But 1 is inclined to unity at an angle $= 0$;

$\therefore$ when $b = 0$, that value of a^m which is represented by $1 + mb + \frac{m\,.\,\overline{m-1}}{1\,.\,2} b^2 + \&c.$ must be inclined to unity at an angle $= 0$;

Now (by Art. 86) $\left(\begin{smallmatrix}a\\0\end{smallmatrix}\right)^m = \left(\begin{smallmatrix}c\\0\end{smallmatrix}\right)^m \cdot \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{mn}$,

$\therefore \left(\begin{smallmatrix}a\\0\end{smallmatrix}\right)^m$ is inclined to unity at an angle $= mn\,.\,360^\circ$,

And a is of the form $e + f\sqrt{-1}$ (by Art. 111), where e and f are positive quantities,

$$\therefore\ b = e - 1 + f\sqrt{-1};$$

$\therefore$ as b decreases, e approaches to 1 and f to 0,

$$\therefore\ \frac{f}{e} \text{ decreases,}$$

$$\therefore \text{ (by Art. 112.) } n \text{ decreases,}$$

And when $b = 0$, $f = 0$, and $e = 1$,

$$\therefore\ \frac{f}{e} = 0, \quad \therefore\ n = 0,$$

$$\therefore\ mn\, 360^0 = 0,$$

$\therefore$ when $b = 0$, $\left(\underset{0}{a}\right)^m$ is inclined to unity at an angle $= 0$;

$\therefore$ $\left(\underset{0}{a}\right)^m$ is that value of a^m which $= 1 + mb + \frac{m.\overline{m-1}}{1.2}b^2 + \&c.$

Next let n be not less than $\frac{1}{4}$ and not greater than $\frac{3}{4}$;

In this case, since $b = a - 1$, b is always in length greater than 1 (by Art. 121.)

$\therefore$ the series $1 + mb + \frac{m.\overline{m-1}}{1.2}b^2 + \&c.$ is not a converging series, and therefore it does not represent any value of a^m.

Next let n be greater than $\frac{3}{4}$ and less than 1;

Then a is of the form $e - f\sqrt{-1}$,

And $b = e - 1 - f\sqrt{-1}$;

$\therefore$ as b decreases, $\frac{f}{e}$ decreases,

$\therefore$ (by Art. 112.) n increases,

And when $b = 0$, $\frac{f}{e} = 0$, and $n = 1$;

Now $\left(\begin{smallmatrix}a\\-1\end{smallmatrix}\right)^m = \left(\begin{smallmatrix}a\\0\end{smallmatrix}\right)^m \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)^m = \left(\begin{smallmatrix}a\\0\end{smallmatrix}\right)^m \cdot \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-m}$

$= \left(\begin{smallmatrix}c\\0\end{smallmatrix}\right)^m \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{mn} \cdot \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-m}$

$= \left(\begin{smallmatrix}c\\0\end{smallmatrix}\right)^m \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{m.(n-1)}$

$\therefore$ $\left(\begin{smallmatrix}a\\-1\end{smallmatrix}\right)^m$ is inclined to unity at an angle

$= m \,.\, \overline{n-1} \,.\, 360^0$;

But when $b = 0$, $n = 1$; $\therefore$ $m \,.\, \overline{n-1} \,.\, 360^0 = 0$,

$\therefore$ when $b = 0$, $\left(\begin{smallmatrix}a\\-1\end{smallmatrix}\right)^m$ is inclined to unity at an angle $= 0$;

$\therefore$ in this case $\left(\begin{smallmatrix}a\\-1\end{smallmatrix}\right)^m$ is that value of a^m which

$$= 1 + mb + \frac{m \,.\, \overline{m-1}}{1 \,.\, 2} b^2 + \&c.$$

(126.) *Let* $a = c\binom{1}{1}^{n}$, *where* c *is a positive quantity, and* n *positive and less than* 1, *and let* $a - 1 = b$, *where* b *is in length less than unity;*

Then, if n *be less than* $\frac{1}{4}$,

$$\binom{a}{p}^{m} = \binom{1}{1}^{pm}.\left\{1 + mb + \frac{m.\overline{m-1}}{1.2}b^2 + \&c.\right\};$$

If n *be greater than* $\frac{3}{4}$,

$$\binom{a}{p}^{m} = \binom{1}{1}^{\overline{p+1}.m}.\left\{1 + mb + \frac{m.\overline{m-1}}{1.2}b^2 + \&c.\right\}.$$

First let n be less than $\frac{1}{4}$;

Then (by Art. 125),

$$\binom{a}{0}^{m} = 1 + mb + \frac{m.\overline{m-1}}{1.2}b^2 + \&c.$$

But $\binom{a}{p}^{m} = \binom{1}{p}^{m}.\binom{a}{0}^{m} = \binom{1}{1}^{mp}.\binom{a}{0}^{m}$;

$$\therefore \binom{a}{p}^{m} = \binom{1}{1}^{pm}.\left\{1 + mb + \frac{m.\overline{m-1}}{1.2}b^2 + \&c.\right\}.$$

Next let n be greater than $\frac{3}{4}$,

(By Art. 125), $\binom{a}{-1}^{m} = 1 + mb + \frac{m.\overline{m-1}}{1.2}b^2 + \&c.$

$$\therefore \binom{a}{0}^{m}.\binom{1}{1}^{-m} = 1 + mb + \frac{m.\overline{m-1}}{1.2}b^2 + \&c.$$

$$\therefore \left(\begin{smallmatrix}a\\0\end{smallmatrix}\right)^m = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^m \left\{1 + mb + \frac{m.\overline{m-1}}{1.2} b^2 + \&c.\right\},$$

$$\therefore \left(\begin{smallmatrix}a\\p\end{smallmatrix}\right)^m = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\overline{p+1}.m} \left\{1 + mb + \frac{m.\overline{m-1}}{1.2} b^2 + \&c.\right\}.$$

(127.) *Let* $1 + Ax + Bx^2 + Cx^3$ *be a series such that*

$$(1 + Ax + Bx^2 + \&c.).(1 + Ay + By^2 + \&c.)$$
$$= 1 + A.(x + y) + B(x + y)^2 + \&c.$$

whatever be the values of x *and* y;

To find the law of the series.

By multiplication,

$$(1 + Ax + Bx^2 + \&c.).(1 + Ay + By^2 + \&c.) =$$

$$= \begin{cases} 1 + Ax + Bx^2 + Cx^3 + \&c. \\ \quad + Ay + A^2xy + ABx^2y + \&c. \\ \qquad + By^2 + ABxy^2 + \&c. \\ \qquad\quad + Cy^3 + \&c. \end{cases}$$

And by expansion,

$$1 + A(x + y) + B(x + y)^2 + \&c. =$$

$$= \left\{ 1 + \left\{\begin{matrix} Ax \\ + Ay \end{matrix}\right\} + \left\{\begin{matrix} Bx^2 \\ + 2Bxy \\ + By^2 \end{matrix}\right\} + \left\{\begin{matrix} Cx^3 \\ + 3Cx^2y \\ + 3Cxy^2 \\ + Cy^3 \end{matrix}\right\} + \&c. \right.$$

∴ equating the coefficients of the combinations of the like powers of x and y,

$$2B = A^2; \;\therefore\; B = \frac{A^2}{2},$$

$$3C = AB; \;\therefore\; C = \frac{AB}{3} = \frac{A^3}{2.3};$$

$\therefore\; 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \&c.$ is a series such that

$$\left(1 + Ax + \frac{A^2x^2}{1.2} + \&c.\right).\left(1 + Ay + \frac{A^2y^2}{1.2} + \&c.\right)$$

$$= 1 + A(x+y) + \frac{A^2(x+y)^2}{1.2} + \&c.$$

whatever be the values of x, y, or A.

(128.) *Let* $a = c\left({1 \atop 1}\right)^n$, *where* c *is a positive quantity, and* n *positive and less than* 1, *and let* $a - 1$ *be in length less than unity, and* $A = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.$

Then if n *be less than* $\frac{1}{4}$,

$$\left({a \atop 0}\right)^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \&c.$$

If n *be greater than* $\frac{3}{4}$,

$$\left({a \atop -1}\right)^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \&c.$$

For it is proved in the Appendix to Woodhouse's Trigonometry, that

$$a^x = 1 + Ax + \frac{A^2 x^2}{1 \cdot 2} + \frac{A^3 x^3}{1 \cdot 2 \cdot 3} + \&c.$$

And the proof given by Woodhouse depends upon the binomial theorem;

∴ (by Art. 125.) If n be less than $\frac{1}{4}$,

$$\binom{a}{0}^x = 1 + Ax + \frac{A^2 x^2}{1 \cdot 2} + \frac{A^3 x^3}{1 \cdot 2 \cdot 3};$$

If n be greater than $\frac{3}{4}$,

$$\binom{a}{-1}^x = 1 + Ax + \frac{A^2 x^2}{1 \cdot 2} + \frac{A^3 x^3}{1 \cdot 2 \cdot 3}.$$

(129.) *To express* $\binom{1}{1}^x$ *in a series of the form* $a + bx + cx^2 + \&c.$

Let $\binom{1}{1}^x = a + bx + cx^2 + \&c.$

Let $x = 0$, then $\binom{1}{1}^0 = a$,

$\therefore 1 = a$;

$\therefore \binom{1}{1}^x = 1 + bx + cx^2 + \&c.$

Now $\binom{1}{1}^x \cdot \binom{1}{1}^y = \binom{1}{1}^{x+y}$;

$\therefore\ 1 + bx + cx^2 + \&c.$ is a series such that

$$(1 + bx + cx^2 + \&c.)\,.\,(1 + by + cy^2 + \&c.)$$

$$= 1 + b\,.\,(x + y) + c\,(x + y)^2 + \&c.$$

$$\therefore \text{ (by Art. 127.) } c = \frac{b^2}{2}, \ \&c.$$

$$\therefore \left(\begin{matrix}1\\1\end{matrix}\right)^x = 1 + bx + \frac{b^2x^2}{1\,.\,2} + \&c.$$

To find the value of b,

Since x may be any value, let x be positive and less than $\frac{1}{4}$;

$$\text{And let } \left(\begin{matrix}1\\1\end{matrix}\right)^x = \frac{1 + a}{1 - a},$$

$$\text{Also let } \left(\begin{matrix}1\\1\end{matrix}\right)^x = k,\ 1 + a = m,\ 1 - a = n,$$

$$\text{then } \frac{m}{n} = k;$$

and $a = \dfrac{\left(\begin{matrix}1\\1\end{matrix}\right)^x - 1}{\left(\begin{matrix}1\\1\end{matrix}\right)^x + 1}$, where since x is positive and less than $\frac{1}{4}$, a is in length less than unity (by Art. 122),

$$\text{and } m = 1 + \frac{\left(\begin{matrix}1\\1\end{matrix}\right)^x - 1}{\left(\begin{matrix}1\\1\end{matrix}\right)^x + 1} = \frac{2\left(\begin{matrix}1\\1\end{matrix}\right)^x}{\left(\begin{matrix}1\\1\end{matrix}\right)^x + 1} = \frac{2\left(\begin{matrix}1\\1\end{matrix}\right)^{\frac{x}{2}}}{\left(\begin{matrix}1\\1\end{matrix}\right)^{\frac{x}{2}} + \left(\begin{matrix}1\\1\end{matrix}\right)^{-\frac{x}{2}}};$$

$$\text{Let } \frac{2}{\binom{1}{1}^{\frac{x}{2}} + \binom{1}{1}^{-\frac{x}{2}}} = g,$$

then g is a positive quantity (by Art. 123.)

$$\therefore\ m = g\binom{1}{1}^{\frac{x}{2}},$$

$$\therefore\ n = \frac{g\binom{1}{1}^{\frac{x}{2}}}{\binom{1}{1}^{x}} = g\binom{1}{1}^{-\frac{x}{2}} = g\binom{1}{1}^{1-\frac{x}{2}};$$

$\therefore$ m is inclined to unity at an angle $= \frac{x}{2}\,360^0$,

n .... $= \left(1 - \frac{x}{2}\right).360^0$;

And $1 - \frac{x}{2}$ is greater than $\frac{x}{2}$, since x is less than $\frac{1}{4}$,

$$\therefore\ \text{(by Art. 93.)}\ \frac{\binom{m}{0}^y}{\binom{n}{0}^y} = \binom{k}{-1}^y = \binom{k}{0}^y . \binom{1}{1}^{-y};$$

$$\therefore\ \binom{k}{0}^y = \frac{\binom{m}{0}^y}{\binom{n}{0}^y . \binom{1}{1}^{-y}} = \frac{\binom{m}{0}^y}{\binom{n}{-1}^y};$$

But $k = \binom{1}{1}^x$,

$\therefore$ since x is positive and less than 1,

$\binom{k}{0}^y = \binom{1}{1}^{xy}$ (by Art. 86.)

$$\therefore \binom{1}{1}^{xy} = \frac{\binom{m}{0}^y}{\binom{n}{-1}^y} = \binom{m}{0}^y \cdot \binom{n}{-1}^{-y};$$

Since $m = g\binom{1}{1}^{\frac{x}{2}}$, and $\frac{x}{2}$ is positive and less than $\frac{1}{4}$,

$$\binom{m}{0}^y = 1 + My + \frac{M^2y^2}{1.2} + \&c. \text{ (by Art. 128)},$$

where $M = (m-1) - \frac{1}{2}(m-1)^2 + \frac{1}{3}(m-1)^3 - \&c.$

And since $n = g\binom{1}{1}^{1-\frac{x}{2}}$, and $1 - \frac{x}{2}$ is greater than $\frac{3}{4}$ and less than 1,

$$\binom{n}{-1}^{-y} = 1 - Ny + \frac{N^2y^2}{1.2} - \&c. \text{ (by Art. 128)},$$

where $N = (n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \&c.$

$$\therefore \binom{1}{1}^{xy} = \left(1 + My + \frac{M^2y^2}{1.2} + \&c.\right).\left(1 - Ny + \frac{N^2y^2}{1.2} - \&c.\right)$$

$$= 1 + (M-N)y + \frac{(M-N)^2y^2}{1.2} + \&c. \text{ (by Art. 127.)}$$

$$\therefore 1 + bxy + \frac{b^2x^2y^2}{1.2} + \&c.$$

$$= 1 + (M-N)y + \frac{(M-N)^2y^2}{1.2} + \&c.$$

$$\therefore bx = M - N,$$

$$\therefore b = \frac{1}{x}.(M-N);$$

Now $m = 1 + a$, $n = 1 - a$,

$\therefore m - 1 = a$, $n - 1 = -a$;

$$\therefore M = a - \tfrac{1}{2}a^2 + \tfrac{1}{3}a^3 - \tfrac{1}{4}a^4 + \tfrac{1}{5}a^5 - \&c.$$

$$N = -a - \tfrac{1}{2}a^2 - \tfrac{1}{3}a^3 - \tfrac{1}{4}a^4 - \tfrac{1}{5}a^5 - \&c.$$

$$\therefore M - N = 2\,\{a + \tfrac{1}{3}a^3 + \tfrac{1}{5}a^5 + \&c.\}$$

$$\therefore b = \frac{2}{x}.\{a + \tfrac{1}{3}a^3 + \tfrac{1}{5}a^5 + \&c.\}$$

Let $x = \frac{1}{6}$,

Then $\left(\frac{1}{1}\right)^{\frac{1}{6}} = \frac{1}{2} + \frac{\sqrt{3}}{2}.\sqrt{-1}$ (by Art. 113.)

$$\therefore a = \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1} - 1}{\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1} + 1} = \frac{-1 + \sqrt{3}.\sqrt{-1}}{3 + \sqrt{3}.\sqrt{-1}}$$

$$= \frac{(\sqrt{3} + \sqrt{-1})\sqrt{-1}}{3 + \sqrt{3}.\sqrt{-1}} = \frac{\sqrt{-1}}{\sqrt{3}};$$

$$\therefore b = 2.6.\left\{\frac{\sqrt{-1}}{\sqrt{3}} + \tfrac{1}{3}.\left(\frac{\sqrt{-1}}{\sqrt{3}}\right)^3 + \tfrac{1}{5}.\left(\frac{\sqrt{-1}}{\sqrt{3}}\right)^5 + \&c.\right\}$$

$$= 12\sqrt{-1}\left\{\frac{1}{\sqrt{3}} - \tfrac{1}{3}.\left(\frac{1}{\sqrt{3}}\right)^3 + \tfrac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5 - \&c.\right\},$$

$$\text{Let } 12\left\{\frac{1}{\sqrt{3}} - \tfrac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 + \tfrac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5 - \&c.\right\} = c,$$

then $b = c\sqrt{-1}$, where c is a positive quantity;

$$\therefore \binom{1}{1}^x = 1 + cx\sqrt{-1} - \frac{c^2x^2}{1\,.\,2} - \frac{c^3x^3}{1\,.\,2\,.\,3}\sqrt{-1} + \&c.$$

(130.) Cor. 1. $\binom{1}{p}^x$

$$= 1 + pcx\sqrt{-1} - \frac{p^2c^2x^2}{1\,.\,2} - \frac{p^3c^3x^3}{1\,.\,2\,.\,3}\sqrt{-1} + \&c.$$

For $\binom{1}{p}^x = \binom{1}{1}^{px}$

$$= 1 + pcx\sqrt{-1} - \frac{p^2c^2x^2}{1\,.\,2} - \frac{p^3c^3x^3}{1\,.\,2\,.\,3}\sqrt{-1} + \&c.$$

(131.) Cor. 2. If a be a positive quantity

$$\binom{a}{p}^x = 1 + (A + pc\sqrt{-1})x + \frac{(A + pc\sqrt{-1})^2x^2}{1\,.\,2}$$

$$+ \frac{(A + pc\sqrt{-1})^3x^3}{1\,.\,2\,.\,3} + \&c.$$

$$\text{where } A = 2\left\{\frac{a-1}{a+1} + \tfrac{1}{3}\left(\frac{a-1}{a+1}\right)^3 + \tfrac{1}{5}\left(\frac{a-1}{a+1}\right)^5 + \&c.\right\}.$$

For by the Appendix to Woodhouse's Trigonometry,

$$\binom{a}{0}^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \&c.$$

where $A = 2\left\{\frac{a-1}{a+1} + \frac{1}{3}\left(\frac{a-1}{a+1}\right)^3 + \frac{1}{5}\left(\frac{a-1}{a+1}\right)^5 + \&c.\right\}$

$$\therefore \binom{a}{p}^x = \left(1 + Ax + \frac{A^2x^2}{1.2} + \&c.\right).\binom{1}{p}^x$$

$$= \left(1+Ax+\frac{A^2x^2}{1.2}+\&c.\right).\left(1+pcx\sqrt{-1}+\frac{(pcx\sqrt{-1})^2}{1.2}+\&c.\right)$$

$$= 1+(A+pc\sqrt{-1})x+\frac{(A+pc\sqrt{-1})^2x^2}{1.2}+\&c. \text{ (by Art. 127.)}$$

(132.) *Let* a = b $\binom{1}{1}^{\text{p}}$, *where* b *is a positive quantity, and* p *positive and less than* 1;

Then if p *be less than* $\frac{1}{4}$,

$$\binom{\text{a}}{0}^{\text{x}} = 1 + \text{Ax} + \frac{\text{A}^2\text{x}^2}{1.2} + \&c.$$

Then if p *be greater than* $\frac{3}{4}$,

$$\binom{\text{a}}{-1}^{\text{x}} = 1 + \text{Ax} + \frac{\text{A}^2\text{x}^2}{1.2} + \&c.$$

where $\text{A} = 2\left\{\frac{\text{a}-1}{\text{a}+1} + \frac{1}{3}\left(\frac{\text{a}-1}{\text{a}+1}\right)^3 + \frac{1}{5}\left(\frac{\text{a}-1}{\text{a}+1}\right)^5 + \&c.\right\}$

For let $a = \frac{1+e}{1-e}$, and let $1 + e = m$, $1 - e = n$,

$$\text{then } e = \frac{a-1}{a+1}, \quad m = \frac{2a}{a+1}, \quad n = \frac{2}{a+1};$$

Let $a + 1 = f\binom{1}{1}^q$, where f is a positive quantity, and q positive and less than 1,

$$\text{then } m = \frac{2b\binom{1}{1}^p}{f\binom{1}{1}^q} = \frac{2b}{f}\cdot\binom{1}{1}^{p-q},$$

$$n = \frac{2}{f\binom{1}{1}^q} = \frac{2}{f}\cdot\binom{1}{1}^{-q} = \frac{2}{f}\cdot\binom{1}{1}^{1-q};$$

First let p be less than $\frac{1}{4}$,

Then q is less than p (by Art. 124),

$\therefore$ $p - q$ is positive and less than $\frac{1}{4}$, and $1 - q$ is greater than $\frac{3}{4}$ and less than 1;

Also m is inclined to unity at an angle $= \overline{p-q}\,.\,360^\circ$,

n.............................. $= \overline{1-q}\,.\,360^\circ$,

And $p - q$ is less than $1 - q$,

$$\therefore \binom{a}{-1}^x = \frac{\binom{m}{0}^x}{\binom{n}{0}^x} \text{ (by Art. 93.)}$$

$$\text{or } \binom{a}{0}^x\cdot\binom{1}{-1}^x = \frac{\binom{m}{0}^x}{\binom{n}{0}^x},$$

$$\therefore \left(\begin{smallmatrix}a\\0\end{smallmatrix}\right)^x = \frac{\left(\begin{smallmatrix}m\\0\end{smallmatrix}\right)^x}{\left(\begin{smallmatrix}m\\0\end{smallmatrix}\right)^x \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)^x} = \frac{\left(\begin{smallmatrix}m\\0\end{smallmatrix}\right)^x}{\left(\begin{smallmatrix}n\\-1\end{smallmatrix}\right)^x} = \left(\begin{smallmatrix}m\\0\end{smallmatrix}\right)^x \cdot \left(\begin{smallmatrix}n\\-1\end{smallmatrix}\right)^{-x}$$

$$= \left(1 + Mx + \frac{M^2 x^2}{1.2} + \&c.\right).$$

$$\left(1 - Nx + \frac{N^2 x^2}{1.2} - \&c.\right) \text{ (by Art. 128.)}$$

where $M = (m-1) - \frac{1}{2}(m-1)^2 + \frac{1}{3}(m-1)^3 - \&c.$

$$N = (n-1) - \tfrac{1}{2}(n-1)^2 + \tfrac{1}{3}(n-1)^3 - \&c.$$

$$\therefore \left(\begin{smallmatrix}a\\0\end{smallmatrix}\right)^x = 1 + (M-N)\,x + \frac{(M-N)^2 x^2}{1.2} + \&c.$$

But $m = 1 + e$, $n = 1 - e$;

$$\therefore M = e - \tfrac{1}{2}e^2 + \tfrac{1}{3}e^3 - \tfrac{1}{4}e^4 + \tfrac{1}{5}e^5 - \&c.$$

$$N = -e - \tfrac{1}{2}e^2 - \tfrac{1}{3}e^3 - \tfrac{1}{4}e^4 - \tfrac{1}{5}e^5 - \&c.$$

$$\therefore M - N = 2\left\{e + \tfrac{1}{3}e^3 + \tfrac{1}{5}e^5 + \&c.\right\}$$

$$= 2\left\{\frac{a-1}{a+1} + \frac{1}{3}\left(\frac{a-1}{a+1}\right)^3 + \frac{1}{5}\left(\frac{a-1}{a+1}\right)^5 + \&c.\right\};$$

$$\therefore \left(\begin{smallmatrix}a\\0\end{smallmatrix}\right)^x = 1 + Ax + \frac{A^2 x^2}{1.2} + \&c.$$

where $A = 2\left\{\frac{a-1}{a+1} + \frac{1}{3}\left(\frac{a-1}{a+1}\right)^3 + \frac{1}{5}\left(\frac{a-1}{a+1}\right)^5 + \&c.\right\}$

which is a converging series (by Art. 122).

Next let p be greater than $\frac{3}{4}$,

Then q is greater than p (by Art. 124.)

$\therefore\ p-q$ is negative and between $-\frac{1}{4}$ and 0;

$$\text{But } m = \frac{2b}{f}\cdot\binom{1}{1}^{p-q} = \frac{2b}{f}\cdot\binom{1}{1}^{1+p-q},$$

where $1+p-q$ is between $\frac{3}{4}$ and 1;

Also $1-q$ is between 0 and $\frac{1}{4}$;

And m is inclined to unity at an angle $= \overline{1+p-q}\,.\,360^0$,

n .... $= \overline{1-q}\,.\,360^0$,

And $1+p-q$ is greater than $1-q$,

$$\therefore\ \binom{a}{0}^x = \frac{\binom{m}{0}^x}{\binom{n}{0}^x} \text{ (by Art. 93);}$$

$$\therefore\ \binom{a}{-1}^x = \frac{\binom{m}{-1}^x}{\binom{n}{0}^x} = \binom{m}{-1}^x\cdot\binom{n}{0}^{-x}$$

$$= \left(1 + Mx + \frac{M^2x^2}{1\,.\,2} + \&c.\right).$$

$$\left(1 - Nx + \frac{N^2x^2}{1\,.\,2} - \&c.\right) \text{ (by Art. 128.)}$$

$$= 1 + (M-N)\,x + \frac{(M-N)^2\,x^2}{1\,.\,2} + \&c.$$

$$= 1 + Ax + \frac{A^2x^2}{1\,.\,2} + \&c.$$

(133.) *Let* $a = b \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^p$, *where* b *is a positive quantity, and* p *positive and less than* 1; *then if* p *be greater than* $\frac{1}{4}$ *and less than* $\frac{3}{4}$,

$$\left(\begin{smallmatrix}a\\0\end{smallmatrix}\right)^x = 1 + \left(A' + \frac{c}{2}\sqrt{-1}\right)x + \frac{\left(A' + \frac{c}{2}\sqrt{-1}\right)^2 x^2}{1 \cdot 2} + \&c.$$

where $A' = 2\left\{\frac{a+1}{a-1} + \frac{1}{3}\left(\frac{a+1}{a-1}\right)^3 + \frac{1}{5}\left(\frac{a+1}{a-1}\right)^5 + \&c.\right\}$

and c *is of the same value as in Art.* 129.

For first let p be less than $\frac{1}{2}$.

Let $f = -a$,

then $f = -1 \cdot a = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{1}{2}} \cdot a = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{1}{2}} b \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^p = b \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{p+\frac{1}{2}}$,

where since p is greater than $\frac{1}{4}$ and less than $\frac{1}{2}$,

$$p + \tfrac{1}{2} \ldots\ldots\ldots\ldots \tfrac{3}{4} \ldots\ldots\ldots\ldots 1;$$

$$\therefore \left(\begin{smallmatrix}f\\-1\end{smallmatrix}\right)^x = 1 + A'x + \frac{A'^2 x^2}{1 \cdot 2} + \&c.$$

where $A' = 2\left\{\frac{f-1}{f+1} + \frac{1}{3}\left(\frac{f-1}{f+1}\right)^3 + \frac{1}{5}\left(\frac{f-1}{f+1}\right)^5 + \&c.\right\}$
(by Art. 132.)

$$\text{But } \frac{f-1}{f+1} = \frac{-a-1}{-a+1} = \frac{a+1}{a-1}$$

a proper fraction (by Art. 122.)

$$\therefore\ A' = 2\left\{\frac{a+1}{a-1} + \tfrac{1}{3}\left(\frac{a+1}{a-1}\right)^3 + \tfrac{1}{5}\left(\frac{a+1}{a-1}\right)^5 + \&c.\right\}$$

But $f = -1 \,.\, a$,

$$\therefore\ \binom{f}{0}^x = \binom{-1}{0}^x \cdot \binom{a}{0}^x \text{ (by Art. 91.)}$$

$$= \binom{1}{1}^{\frac{x}{2}} \cdot \binom{a}{0}^x \text{ (by Art. 86.)}$$

$$\therefore\ \binom{f}{-1}^x = \binom{1}{1}^{\frac{x}{2}} \cdot \binom{a}{0}^x \cdot \binom{1}{1}^{-x} = \binom{a}{0}^x \cdot \binom{1}{1}^{-\frac{x}{2}},$$

$$\therefore\ \binom{a}{0}^x = \binom{f}{-1} \cdot \binom{1}{1}^{\frac{x}{2}}$$

$$= \left(1 + A'x + \frac{A'^2x^2}{1\,.\,2} + \&c.\right).$$

$$\left\{1 + \frac{cx}{2}\sqrt{-1} + \frac{\left(c\,\frac{x}{2}\sqrt{-1}\right)^2}{1\,.\,2} + \&c.\right\}$$

$$= 1 + \left(A' + \frac{c}{2}\sqrt{-1}\right)x + \frac{\left(A' + \frac{c}{2}\sqrt{-1}\right)^2 x^2}{1\,.\,2} + \&c.$$

Next let p be greater than $\frac{1}{2}$;

As in the first case, let $f = -a$,

then $f = b\binom{1}{1}^{p+\frac{1}{2}} = b\binom{1}{1}^{p+\frac{1}{2}-1} = b\binom{1}{1}^{p-\frac{1}{2}}$,

where since p is greater than $\frac{1}{2}$ and less than $\frac{3}{4}$,

$p - \frac{1}{2}$. . . . . .0. . . . .$\frac{1}{4}$;

$$\therefore \left(\underset{0}{f}\right)^x = 1 + A'x + \frac{A'^2 x^2}{1 \cdot 2} + \&c.$$

But $f = -1 \cdot a$;

$\therefore$ in this case $\left(\underset{1}{f}\right)^x = \left(-\underset{0}{1}\right)^x \cdot \left(\underset{0}{a}\right)^x$ (by Art. 91.)

$$= \left(\underset{1}{1}\right)^{\frac{x}{2}} \cdot \left(\underset{0}{a}\right)^x;$$

$$\therefore \left(\underset{0}{a}\right)^x = \left(\underset{1}{f}\right)^x \cdot \left(\underset{1}{1}\right)^{-\frac{x}{2}} = \left(\underset{0}{f}\right)^x \cdot \left(\underset{1}{1}\right)^{\frac{x}{2}}$$

$$= \left(1 + A'x + \frac{A'^2 x^2}{1 \cdot 2} + \&c.\right) \cdot \left(\underset{1}{1}\right)^{\frac{x}{2}}$$

$$= 1 + \left(A' + \frac{c}{2}\sqrt{-1}\right)x + \frac{\left(A' + \frac{c}{2}\sqrt{-1}\right)^2 x^2}{1 \cdot 2} + \&c.$$

(134.) Cor. Let $a = b\left(\underset{1}{1}\right)^p$, where b is a positive quantity and p positive and less than unity, and

let $A = 2\left\{\frac{a-1}{a+1} + \frac{1}{3}\left(\frac{a-1}{a+1}\right)^3 + \frac{1}{5}\left(\frac{a-1}{a+1}\right)^5 + \&c.\right\}$,

$$A' = 2\left\{\frac{a+1}{a-1} + \frac{1}{3}\left(\frac{a+1}{a-1}\right)^3 + \frac{1}{5}\left(\frac{a+1}{a-1}\right)^5 + \&c.\right\},$$

and let c be of the same value as in Art. 129.

Then if p be less than $\frac{1}{4}$,

$$\left(\underset{q}{a}\right)^x = 1 + (A + qc\sqrt{-1})\,x + \frac{(A + qc\sqrt{-1})\,x^2}{1 \cdot 2} + \&c.$$

If p be greater than $\frac{1}{4}$ and less than $\frac{3}{4}$,

$$\binom{a}{q}^x = 1 + (A' + \overline{q+\tfrac{1}{2}}.c\sqrt{-1})x + \frac{(A' + \overline{q+\tfrac{1}{2}}.c\sqrt{-1})x^2}{1.2} + \&c.$$

If p be greater than $\frac{3}{4}$,

$$\binom{a}{q}^x = 1 + (A + \overline{q+1}.c\sqrt{-1})\,x + \frac{(A + \overline{q+1}.c\sqrt{-1})^2}{1.2}x^2 + \&c.$$

For first let p be less than $\frac{1}{4}$,

$$\text{Then } \binom{a}{q}^x = \binom{a}{0}^x \cdot \binom{1}{q}^x = \left(1 + Ax + \frac{A^2x^2}{1.2} + \&c.\right).$$

$$\left(1 + qcx\sqrt{-1} + \frac{(qcx\sqrt{-1})^2}{1.2} + \&c.\right)$$

$$= 1 + (A + qc\sqrt{-1})\,x + \frac{(A + qc\sqrt{-1})^2x^2}{1.2} + \&c.$$

Next let p be greater than $\frac{1}{4}$ and less than $\frac{3}{4}$,

$$\text{Then } \binom{a}{q}^x = \binom{a}{0}^x \cdot \binom{1}{q}^x =$$

$$\left\{1 + \left(A' + \frac{c}{2}\sqrt{-1}\right)x + \frac{\left(A' + \frac{c}{2}\sqrt{-1}\right)x^2}{1.2} + \&c.\right\}.$$

$$\left(1 + qcx\sqrt{-1} + \frac{(qcx\sqrt{-1})^2}{1.2} + \&c.\right)$$

$$= 1 + (A' + \overline{q+\tfrac{1}{2}}.c\sqrt{-1})\,x + \frac{(A' + \overline{q+\tfrac{1}{2}}.c\sqrt{-1})x^2}{1.2} + \&c.$$

Next let p be greater than $\frac{3}{4}$,

Then $\binom{a}{q}^x = \binom{a}{0}^x \cdot \binom{1}{q}^x$

$$= \binom{a}{-1}^x \cdot \binom{1}{1}^x \cdot \binom{1}{q}^x = \binom{a}{-1}^x \cdot \binom{1}{1}^{\overline{q+1}.x}$$

$$= \left(1 + Ax + \frac{A^2x^2}{1.2} + \&c.\right).$$

$$\left(1 + \overline{q+1}.cx\sqrt{-1} + \frac{(\overline{q+1}.cx\sqrt{-1})^2}{1.2} + \&c.\right)$$

$$= 1 + (A + \overline{q+1}.c\sqrt{-1})\,x + \frac{(A+\overline{q+1}.c\sqrt{-1})x^2}{1.2} + \&c.$$

(135.) *Let* $\mathrm{a} = \mathrm{b}\binom{1}{1}^{\mathrm{p}}$, *where* b *is a positive quantity, and* p *positive and less than* 1,

Then $\binom{\mathrm{a}}{\mathrm{q}}^{\mathrm{x}}$

$$= (\mathrm{B} + \overline{\mathrm{p}+\mathrm{q}}.\mathrm{c}\sqrt{-1})\,\mathrm{x} + \frac{(\mathrm{B} + \overline{\mathrm{p}+\mathrm{q}}.\mathrm{c}\sqrt{-1})^2\,\mathrm{x}^2}{1.2} + \&c.$$

where $\mathrm{B} = 2\left\{\frac{\mathrm{b}-1}{\mathrm{b}+1} + \frac{1}{3}\left(\frac{\mathrm{b}-1}{\mathrm{b}+1}\right)^3 + \frac{1}{5}\left(\frac{\mathrm{b}-1}{\mathrm{b}+1}\right)^5 + \&c.\right\}$,

and c *is of the same value as in Art.* 129.

For $\binom{b}{0}^x = 1 + Bx + \frac{B^2x^2}{1.2} + \&c.$

and $\left(\begin{smallmatrix}a\\0\end{smallmatrix}\right)^x = \left(\begin{smallmatrix}b\\0\end{smallmatrix}\right)^x \cdot \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{px}$;

$$\therefore \left(\begin{smallmatrix}a\\q\end{smallmatrix}\right)^x = \left(\begin{smallmatrix}b\\0\end{smallmatrix}\right)^x \cdot \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{px} \cdot \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{qx} = \left(\begin{smallmatrix}b\\0\end{smallmatrix}\right)^x \cdot \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\overline{p+q}.x}$$

$$= \left(1 + Bx + \frac{B^2x^2}{1.2} + \&c.\right).$$

$$\left(1 + \overline{p+q}.cx\sqrt{-1} + \frac{(\overline{p+q}.cx\sqrt{-1}^2)}{1.2} + \&c.\right)$$

$$= 1 + (B + \overline{p+q}.c\sqrt{-1})\,x + \frac{(B+\overline{p+q}.c\sqrt{-1})^2x^2}{1.2} + \&c.$$

(136.) *To find the differential of* $\left(\begin{smallmatrix}a\\q\end{smallmatrix}\right)^x$.

Let $a = b\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^p$, where b is a positive quantity, and p positive and less than 1;

First let p be less than $\frac{1}{4}$,

$$\text{Let } \left(\begin{smallmatrix}a\\q\end{smallmatrix}\right)^x = u,$$

$$\left(\begin{smallmatrix}a\\q\end{smallmatrix}\right)^{x+h} = u',$$

$$u' - u = \left(\begin{smallmatrix}a\\q\end{smallmatrix}\right)^{x+h} - \left(\begin{smallmatrix}a\\q\end{smallmatrix}\right)^x = \left(\begin{smallmatrix}a\\q\end{smallmatrix}\right)^x \cdot \left\{\left(\begin{smallmatrix}a\\q\end{smallmatrix}\right)^h - 1\right\}$$

$$= \left(\begin{smallmatrix}a\\q\end{smallmatrix}\right)^x\left\{1 + (A + qc\sqrt{-1})h + \frac{(A+qc\sqrt{-1})^2h^2}{1.2} + \&c. - 1\right\},$$

$$\therefore \frac{u' - u}{h}$$

$$= u\left\{A + qc\sqrt{-1} + \frac{(A + qc\sqrt{-1})^2 h}{1 . 2} + \&c.\right\};$$

$$\therefore \frac{du}{dx} = u\,(A + qc\sqrt{-1}).$$

Next let p be greater than $\frac{1}{4}$ and less than $\frac{3}{4}$,

It may be proved nearly as in the first case, that is this case,

$$\frac{du}{dx} = u\,(A' + \overline{q + \tfrac{1}{2}} \,.\, c\sqrt{-1}).$$

Next let p be greater than $\frac{3}{4}$,

In this case,

$$\frac{du}{dx} = u\,(A + \overline{q + 1} \,.\, c\sqrt{-1}).$$

Or if $B = 2\left\{\frac{b-1}{b+1} + \frac{1}{3}\left(\frac{b-1}{b+1}\right)^3 + \frac{1}{5}\left(\frac{b-1}{b+1}\right)^5 + \&c.\right\}$,

it may be proved in all cases that

$$\frac{du}{dx} = u\,(B + \overline{p + q} \,.\, c\sqrt{-1}).$$

(137.) Cor. Hence if $u = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^x$, $\frac{du}{dx} = u\epsilon\sqrt{-1}$.

(138.) *To integrate* $\frac{du}{dx} = (a + b\sqrt{-1})\,u.$

Let $u = C.\binom{e}{0}^{mx}.\binom{1}{1}^{nx}$, where e is the base of the hyperbolic logarithms,

$$\text{then } \frac{du}{dx}$$

$$= C.\binom{e}{0}^{mx}.m.\binom{1}{1}^{nx} + C.\binom{e}{0}^{mx}.\binom{1}{1}^{nx}.nc\sqrt{-1}$$

$$= C.\binom{e}{0}^{mx}.\binom{1}{1}^{nx}.(m + nc\sqrt{-1})$$

$$= u\,(m + nc\sqrt{-1});$$

$$\therefore\ m + nc\sqrt{-1} = a + b\sqrt{-1},$$

$$\therefore\ \text{(by Art. 118.)}\ m = a,\ \ nc = b,$$

$$\therefore\ n = \frac{b}{c},$$

$$\therefore\ u = C.\binom{e}{0}^{ax}.\binom{1}{1}^{\frac{b}{c}x},$$

where C is an arbitrary constant.

$$\text{If } b = 0,\ \frac{du}{dx} = au,$$

$$\text{and } u = C.\binom{e}{0}^{ax},$$

$$\therefore\ ax = \text{hyp. log. } u - \text{hyp. log. } C.$$

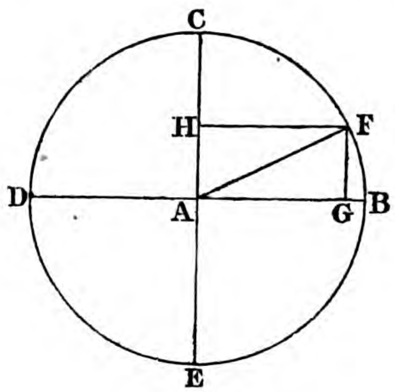
$$\text{If } a = 0,\ \frac{du}{dx} = (b\sqrt{-1})\,u,$$

$$\text{and } u = C.\left(\begin{matrix}1\\1\end{matrix}\right)^{\frac{b}{c}x}.$$

CHAP. IV.

EXAMPLES ILLUSTRATIVE OF THE PRINCIPLES ESTABLISHED IN THE PRECEDING CHAPTERS.

(139.) *To express the sines, cosines, &c. of arcs in terms of the powers of unity.*



Let AB = unity, $BCDE$ be a circle described with radius AB;

Let BA be produced to D, and through A let EAC be drawn perpendicular to DB;

Let BF be a given arc;

Join AF, and draw FG perpendicular to AB, and FH perpendicular to AC;

Let $BF = \theta$, and 2π = whole circumference,

$$\text{Then } AF = \binom{1}{1}^{\frac{\theta}{2\pi}},$$

$$AG = \cos\theta,\ AH = \sin\theta\sqrt{-1},$$

$$\therefore \binom{1}{1}^{\frac{\theta}{2\pi}} = \cos\theta + \sin\theta\sqrt{-1},$$

$$\therefore \binom{1}{1}^{-\frac{\theta}{2\pi}} = \cos\theta - \sin\theta\sqrt{-1} \text{ (by Art. 117.)}$$

$$\therefore \text{by addition } \binom{1}{1}^{\frac{\theta}{2\pi}} + \binom{1}{1}^{-\frac{\theta}{2\pi}} = 2.\cos\theta,$$

$$\text{By subtraction } \binom{1}{1}^{\frac{\theta}{2\pi}} - \binom{1}{1}^{-\frac{\theta}{2\pi}} = 2.\sin\theta\sqrt{-1},$$

$$\therefore \cos\theta = \frac{\binom{1}{1}^{\frac{\theta}{2\pi}} + \binom{1}{1}^{-\frac{\theta}{2\pi}}}{2},$$

$$\sin\theta = \frac{\binom{1}{1}^{\frac{\theta}{2\pi}} - \binom{1}{1}^{-\frac{\theta}{2\pi}}}{2\sqrt{-1}};$$

In like manner it may be proved that

$$\sec\theta\binom{1}{1}^{\frac{\theta}{2\pi}} = 1 + \tan\theta\sqrt{-1},$$

$$\therefore \sec\theta\binom{1}{1}^{-\frac{\theta}{2\pi}} = 1 - \tan\theta\sqrt{-1},$$

$\therefore$ by addition $\sec\theta\left\{\binom{1}{1}^{\frac{\theta}{2\pi}}+\binom{1}{1}^{-\frac{\theta}{2\pi}}\right\}=2,$

by subtraction $\sec\theta\left\{\binom{1}{1}^{\frac{\theta}{2\pi}}-\binom{1}{1}^{-\frac{\theta}{2\pi}}\right\}=2\tan\theta\sqrt{-1},$

$$\therefore \sec\theta=\frac{2}{\binom{1}{1}^{\frac{\theta}{2\pi}}+\binom{1}{1}^{-\frac{\theta}{2\pi}}},$$

$$\therefore 2\tan\theta\sqrt{-1}=\frac{2}{\binom{1}{1}^{\frac{\theta}{2\pi}}+\binom{1}{1}^{-\frac{\theta}{2\pi}}}\cdot\left\{\binom{1}{1}^{\frac{\theta}{2\pi}}-\binom{1}{1}^{-\frac{\theta}{2\pi}}\right\},$$

$$\therefore \tan\theta=\frac{1}{\sqrt{-1}}\cdot\frac{\binom{1}{1}^{\frac{2\theta}{2\pi}}-1}{\binom{1}{1}^{\frac{2\theta}{2\pi}}+1};$$

$$\text{Cosec }\theta=\sec\left(\frac{\pi}{2}-\theta\right)$$

$$=\frac{2}{\binom{1}{1}^{\frac{1}{4}-\frac{\theta}{2\pi}}+\binom{1}{1}^{-\frac{1}{4}+\frac{\theta}{2\pi}}}=\frac{2\binom{1}{1}^{\frac{1}{4}}}{\binom{1}{1}^{\frac{1}{2}-\frac{\theta}{2\pi}}+\binom{1}{1}^{\frac{\theta}{2\pi}}}$$

$$=\frac{2\binom{1}{1}^{\frac{1}{4}}}{\binom{1}{1}^{\frac{\theta}{2\pi}}+\binom{1}{1}^{\frac{1}{2}}\cdot\binom{1}{1}^{-\frac{\theta}{2\pi}}}=\frac{2\sqrt{-1}}{\binom{1}{1}^{\frac{\theta}{2\pi}}-\binom{1}{1}^{-\frac{\theta}{2\pi}}};$$

$$\text{Cotan }\theta=\tan\left(\frac{\pi}{2}-\theta\right)$$

$$\therefore \text{cotan}\ \theta = \frac{1}{\sqrt{-1}} \cdot \frac{\left(\frac{1}{1}\right)^{\frac{\pi - 2\theta}{2\pi}} - 1}{\left(\frac{1}{1}\right)^{\frac{\pi - 2\theta}{2\pi}} + 1}$$

$$= \frac{1}{\sqrt{-1}} \cdot \frac{\left(\frac{1}{1}\right)^{\frac{1}{2}} \cdot \left(\frac{1}{1}\right)^{-\frac{2\theta}{2\pi}} - 1}{\left(\frac{1}{1}\right)^{\frac{1}{2}} \cdot \left(\frac{1}{1}\right)^{-\frac{2\theta}{2\pi}} + 1}$$

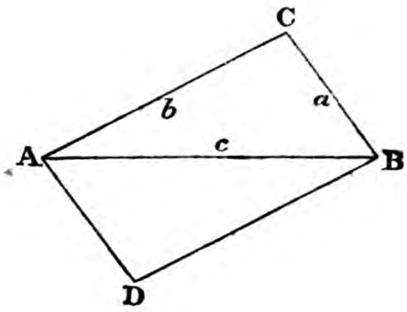
$$= \frac{1}{\sqrt{-1}} \cdot \frac{-\left(\frac{1}{1}\right)^{-\frac{2\theta}{2\pi}} - 1}{-\left(\frac{1}{1}\right)^{-\frac{2\theta}{2\pi}} + 1}$$

$$= \frac{-1}{\sqrt{-1}} \cdot \frac{1 + \left(\frac{1}{1}\right)^{\frac{2\theta}{2\pi}}}{-1 + \left(\frac{1}{1}\right)^{\frac{2\theta}{2\pi}}}$$

$$= \sqrt{-1} \cdot \frac{\left(\frac{1}{1}\right)^{\frac{2\theta}{2\pi}} + 1}{\left(\frac{1}{1}\right)^{\frac{2\theta}{2\pi}} - 1}.$$

(140.) Def. When θ is said to be equal to any angle, it is to be understood that θ is equal to the arc which subtends that angle to a radius equal unity; thus if θ be said to be equal to angle BAF in figure Art. 139, it is to be understood that $\theta = BF$.

(141.) *To express the relations of the sides and angles of triangles by means of powers of unity.*



Let ABC be a triangle, and let its angles be $= A, B, C$, and the opposite sides be in length respectively equal to a, b, c;

Let AB be in the positive direction, and let the parallelogram $ACBD$ be completed, and let BAC be a positive angle;

Then $BAD = -B$, and AD is in length $= a$;

$$\therefore AC = b \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{A}{2\pi}}, \; AD = a \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-\frac{B}{2\pi}};$$

$$\therefore c = b \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{A}{2\pi}} + a \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-\frac{B}{2\pi}};$$

In like manner it may be proved that

$$b = a \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{C}{2\pi}} + c \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-\frac{A}{2\pi}};$$

In which two equations, together with the consideration that A, B, C being angles must be each less than π, are contained all the properties of triangles.

(142.) Ex. 1. *To prove that the three angles of any triangle are together equal to π.*

Since $b = a \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{C}{2\pi}} + c \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-\frac{A}{2\pi}}$,

by multiplying both sides of the equation by $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{A}{2\pi}}$,

$$b \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{A}{2\pi}} = a \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{A+C}{2\pi}} + c,$$

But $b \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{A}{2\pi}} + a \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-\frac{B}{2\pi}} = c$,

$\therefore$ by subtraction $-a \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-\frac{B}{2\pi}} = a \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{A+C}{2\pi}}$,

$$\therefore -1 = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{A+B+C}{2\pi}};$$

Now $-1 = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{p+\frac{1}{2}}$, where p either $= 0$, or some whole number either positive or negative;

$$\therefore p + \tfrac{1}{2} = \frac{A + B + C}{2\pi};$$

Since $\frac{A+B+C}{2\pi}$ is positive, $p+\frac{1}{2}$ is positive,

$\therefore$ p cannot be a negative whole number;

And since, A, B, C, are each less than π,

$$\frac{A+B+C}{2\pi} \text{ is less than } \frac{3}{2},$$

$\therefore$ p cannot be a positive whole number;

$$\therefore p = 0;$$

$$\therefore \tfrac{1}{2} = \frac{A+B+C}{2\pi},$$

$$\therefore A + B + C = \pi.$$

(143.) Ex. 2. *To prove that the sides of triangles are to one another as the sines of the opposite angles.*

First to prove that

$$b : c :: \sin B : \sin C,$$

$$c = b\binom{1}{1}^{\frac{A}{2\pi}} + a\binom{1}{1}^{-\frac{B}{2\pi}}$$

$$= b\binom{1}{1}^{\frac{2\pi-(B+C)}{2\pi}} + a\binom{1}{1}^{-\frac{B}{2\pi}}$$

$$= b\cdot\binom{1}{1}^{\frac{1}{2}}\cdot\binom{1}{1}^{-\frac{B+C}{2\pi}} + a\binom{1}{1}^{-\frac{B}{2\pi}}$$

$$= - b \binom{1}{1}^{-\frac{B+C}{2\pi}} + a \binom{1}{1}^{-\frac{B}{2\pi}},$$

$$\therefore c \binom{1}{1}^{\frac{B}{2\pi}} = - b \binom{1}{1}^{-\frac{C}{2\pi}} + a,$$

$$\therefore a = c \binom{1}{1}^{\frac{B}{2\pi}} + b \binom{1}{1}^{-\frac{C}{2\pi}},$$

$$\therefore a = c \binom{1}{1}^{-\frac{B}{2\pi}} + b \binom{1}{1}^{\frac{C}{2\pi}} \text{ (by Art. 116.)}$$

$\therefore$ by subtraction

$$0 = c \left\{ \binom{1}{1}^{\frac{B}{2\pi}} - \binom{1}{1}^{-\frac{B}{2\pi}} \right\} - b \left\{ \binom{1}{1}^{\frac{C}{2\pi}} - \binom{1}{1}^{-\frac{C}{2\pi}} \right\},$$

$$0 = c \,.\, 2 \sin B \sqrt{-1} - b \,.\, 2 \sin C \sqrt{-1},$$

$$0 = c \,.\sin B - b \sin C,$$

$$\therefore b : c :: \sin B : \sin C.$$

It may be proved in like manner that

$$a : b :: \sin A : \sin B,$$

and $a : c :: \sin A : \sin C.$

(144.) Ex. 3. *To prove that* $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

$$c = b \binom{1}{1}^{\frac{A}{2\pi}} + a \binom{1}{1}^{-\frac{B}{2\pi}},$$

$$\therefore\ a\left(\tfrac{1}{1}\right)^{-\frac{B}{2\pi}} = c - b\left(\tfrac{1}{1}\right)^{\frac{A}{2\pi}},$$

$$\therefore\ a\left(\tfrac{1}{1}\right)^{\frac{B}{2\pi}} = c - b\left(\tfrac{1}{1}\right)^{-\frac{A}{2\pi}},$$

$\therefore$ by multiplication

$$a^2 = c^2 - bc\left\{\left(\tfrac{1}{1}\right)^{\frac{A}{2\pi}} + \left(\tfrac{1}{1}\right)^{-\frac{A}{2\pi}}\right\} + b^2$$

$$= c^2 - bc.2\cos A + b^2,$$

$$\therefore\ \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

In like manner any other properties of triangles may be established.

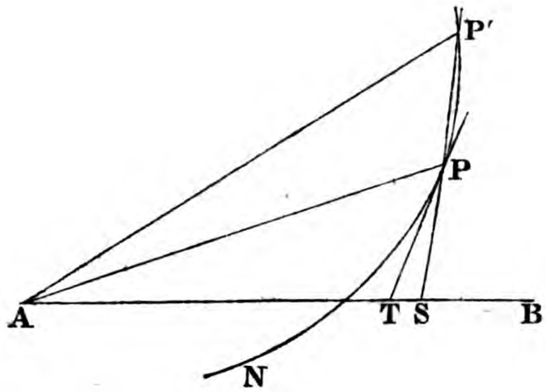
(145.) Cor. *Hence all the properties of triangles are contained in the two equations*

$$c = b\left(\tfrac{1}{1}\right)^{\frac{A}{2\pi}} + a\left(\tfrac{1}{1}\right)^{-\frac{B}{2\pi}},$$

$$A + B + C = \pi.$$

For the equation $b = a\left(\tfrac{1}{1}\right)^{\frac{C}{2\pi}} + c\left(\tfrac{1}{1}\right)^{-\frac{A}{2\pi}}$ may be deduced from these two equations in the same manner that the equation $a = c\left(\tfrac{1}{1}\right)^{\frac{B}{2\pi}} + b\left(\tfrac{1}{1}\right)^{-\frac{C}{2\pi}}$ was deduced from them in Art. 143.

(146.) *To find the lengths of curves and the position of their tangents.*



Let NPP' be any curve,

A the origin, AB the positive direction,

AP, AP' two radii, PT a tangent at P;

Let $P'P$ be joined and produced to S;

Let AP be represented in length and direction by ρ,

Let AP' be represented in length and direction by ρ'

Let NP be in length $= s$,

......NP'.................. $= s'$,

......chord PP'......... $= k$,

Let angle $BAP = \theta$,

..............$APT = \phi$,

.... $TPS = \delta$;

Then angle $BTP = \theta + \phi$,

and ....$BSP = \theta + \phi + \delta$;

$$\therefore \rho' - \rho = k \cdot \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\theta+\phi+\delta}{2\pi}},$$

$$\therefore \frac{\rho' - \rho}{s' - s} = \frac{k}{s' - s} \cdot \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\theta+\phi+\delta}{2\pi}};$$

Now let PP' be diminished *sine limite*,

Then limit of $\frac{k}{s' - s} = 1$ (Newton, Lemma 7.)

and limit of $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\theta+\phi+\delta}{2\pi}} = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\theta+\phi}{2\pi}}$ (Newton, Lemma 6.)

$$\therefore \frac{d\rho}{ds} = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\theta+\phi}{2\pi}};$$

From which equation the differential expressions for determining the lengths of curves and the position of their tangents may be deduced.

(147.) Ex. 1. *Let the curve be a circle, and A its centre.*

$\rho = r\left(\frac{1}{1}\right)^{\frac{\theta}{2\pi}}$, where r is a constant quantity,

$$\therefore \frac{d\rho}{ds} = r\left(\frac{1}{1}\right)^{\frac{\theta}{2\pi}} \cdot \frac{c}{2\pi} \cdot \sqrt{-1} \cdot \frac{d\theta}{ds} \text{ (by Art. 137.)}$$

$$\therefore r\left(\frac{1}{1}\right)^{\frac{\theta}{2\pi}} \cdot \frac{c}{2\pi} \cdot \sqrt{-1}\ \frac{d\theta}{ds} = \left(\frac{1}{1}\right)^{\frac{\theta+\phi}{2\pi}},$$

$$\therefore r \cdot \frac{c}{2\pi}\sqrt{-1}\ \frac{d\theta}{ds} = \left(\frac{1}{1}\right)^{\frac{\phi}{2\pi}},$$

$$\therefore -r\ \frac{c}{2\pi} \cdot \sqrt{-1} \cdot \frac{d\theta}{ds} = \left(\frac{1}{1}\right)^{-\frac{\phi}{2\pi}},$$

$$\therefore \text{by division } -1 = \left(\frac{1}{1}\right)^{\frac{2\phi}{2\pi}},$$

$$\therefore \frac{2\phi}{2\pi} = \tfrac{1}{2},$$

$$\therefore \phi = \frac{\pi}{2},$$

that is, the tangent is at right angles to the radius;

$$\therefore \left(\frac{1}{1}\right)^{\frac{\phi}{2\pi}} = \left(\frac{1}{1}\right)^{\frac{1}{4}} = +\sqrt{-1},$$

$$\therefore r \cdot \frac{c}{2\pi}\sqrt{-1}\ \frac{d\theta}{ds} = \sqrt{-1},$$

$$\therefore r \cdot \frac{c}{2\pi}\ \frac{d\theta}{ds} = 1,$$

$$\therefore\ s = r \cdot \frac{c}{2\pi}\theta + \text{const.}$$

When $\theta = 0$, let $s = 0$, then const. $= 0$,

$$\therefore\ s = r \cdot \frac{c}{2\pi}\theta.$$

(148.) Cor. 1. Let $r = 1$, then $s =$ arc which subtends angle BAP to radius $= 1$, that is, $s = \theta$,

$$\therefore\ \frac{c}{2\pi} = 1,$$

$$\therefore\ 2\pi = c =$$

$$12 \cdot \left\{\frac{1}{\sqrt{3}} - \tfrac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 + \tfrac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5 - \&c.\right\} \text{ (Art. 129).}$$

(149.) Cor. 2. Hence $s = r\theta$.

(150.) Ex. 2. *Let the curve be any spiral,* A *its pole.*

Here $\rho = r\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\theta}{2\pi}}$, where r is a variable quantity;

$$\therefore\ \rho = r\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\theta}{c}} \text{ (by Art. 148.)}$$

$$\therefore\ \frac{d\rho}{ds} = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\theta}{c}} \cdot \frac{dr}{ds} + r\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\theta}{c}} \cdot \frac{d\theta}{ds}\sqrt{-1},$$

$$\therefore \left(\frac{1}{1}\right)^{\frac{\theta}{c}} \cdot \frac{dr}{ds} + r\left(\frac{1}{1}\right)^{\frac{\theta}{c}} \cdot \frac{d\theta}{ds}\sqrt{-1} = \left(\frac{1}{1}\right)^{\frac{\theta+\phi}{c}},$$

$$\therefore \frac{dr}{ds} + r\frac{d\theta}{ds}\sqrt{-1} = \left(\frac{1}{1}\right)^{\frac{\phi}{c}},$$

$$\therefore \frac{dr}{ds} - r\frac{d\theta}{ds}\sqrt{-1} = \left(\frac{1}{1}\right)^{-\frac{\phi}{c}},$$

$\therefore$ by multiplication,

$$\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 = 1,$$

which is the differential equation for the length of a spiral.

Also by addition and subtraction,

$$2\frac{dr}{ds} = \left(\frac{1}{1}\right)^{\frac{\phi}{c}} + \left(\frac{1}{1}\right)^{-\frac{\phi}{c}} = 2\cos\phi,$$

$$2r\frac{d\theta}{ds}\sqrt{-1} = \left(\frac{1}{1}\right)^{\frac{\phi}{c}} - \left(\frac{1}{1}\right)^{-\frac{\phi}{c}} = 2\sin\phi\sqrt{-1},$$

$$\left.\begin{aligned} \therefore \cos\phi &= \frac{dr}{ds} \\ \sin\phi &= r\cdot\frac{d\theta}{ds} \\ \tan\phi &= r\frac{d\theta}{dr} \end{aligned}\right\}$$ equations for determining the position of the tangent.

(151.) Ex. 3. *Let the curve be referred to rectangular co-ordinates.*

Let A be the origin,

...... x be measured in the positive direction,

...... y .... in the direction of $+\sqrt{-1}$;

$$\text{then } \rho = x + y\sqrt{-1},$$

$$\therefore \frac{d\rho}{ds} = \frac{dx}{ds} + \frac{dy}{dx}\sqrt{-1},$$

$$\therefore \frac{dx}{ds} + \frac{dy}{ds}\sqrt{-1} = \binom{1}{1}^{\frac{\theta+\phi}{c}},$$

$$\therefore \frac{dx}{ds} - \frac{dy}{ds}\sqrt{-1} = \binom{1}{1}^{-\frac{\theta+\phi}{c}},$$

$\therefore$ by multiplication,

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1,$$

the differential equation for the length of a curve.

Also by addition and subtraction,

$$2\,\frac{dx}{ds} = \binom{1}{1}^{\frac{\theta+\phi}{c}} + \binom{1}{1}^{-\frac{\theta+\phi}{c}} = 2\cos(\theta+\phi),$$

$$2\,\frac{dy}{ds}\sqrt{-1} = \binom{1}{1}^{\frac{\theta+\phi}{c}} - \binom{1}{1}^{-\frac{\theta+\phi}{c}} = 2\sin(\theta+\phi)\sqrt{-1},$$

$$\therefore \cos(\theta+\phi) = \frac{dx}{ds},$$

$$\sin(\theta+\phi) = \frac{dy}{ds},$$

$$\tan(\theta+\phi) = \frac{dy}{dx};$$

And $\theta+\phi$ = angle *BTP* (Art. 146.)

. . . . = the angle at which the tangent cuts the axis;

∴ the differential equations for determining the position of the tangent have been found.

(152.) $Sin\ x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.$

$$cos\ x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.$$

$$\text{For } \sin x = \frac{\left(\frac{1}{1}\right)^{\frac{x}{c}} - \left(\frac{1}{1}\right)^{-\frac{x}{c}}}{2\sqrt{-1}}$$

$$= \frac{1}{2\sqrt{-1}} \cdot \left\{ \begin{array}{l} 1 + x\sqrt{-1} - \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3} + \frac{x^4}{1.2.3.4} + \&c. \\ -1 + x\sqrt{-1} + \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3} - \frac{x^4}{1.2.3.4} + \&c. \end{array} \right\}$$

$$= x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.$$

$$\cos\ x = \frac{\binom{1}{1}^{\frac{x}{c}} + \binom{1}{1}^{-\frac{x}{c}}}{2}$$

$$= 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.$$

(153.) DEF. One *second* is assumed as the unit of time; and any other time t'' is represented by a line measured in the positive direction, and bearing the same ratio to unity which t'' does to 1″ (see Art. 1).

(154.) DEF. The degree of swiftness or slowness with which a body is moving at any time is called its *velocity* at that time: and the velocity is thus *measured*, let t be the time, at the end of which the velocity is required, and s the length of the path described by the body in time t (the path being either a straight line or a curve, and the motion of the body in that path being either uniform, or accelerated, or retarded), let s' be the length of the path described in the time $t + h$; and let $V : 1 ::$ limiting ratio of $s' - s : h$, when h is diminished *sine limite;* then V is a line which *in length measures* the *velocity* at the end of time t; also let the direction in which the body is moving at the end of time t be inclined to unity at an angle $= \alpha$, and let $u = V . \left(\frac{1}{1}\right)^{\frac{\alpha}{c}}$, then u is a line which in *length* and *direction measures* the *velocity* at the end of the time t.

(155.) Cor. 1. Hence $V = \frac{ds}{dt}$, $u = \frac{ds}{dt} \cdot \left(\frac{1}{1}\right)^{\frac{\alpha}{c}}$.

(156.) Cor. 2. Let NP in figure of Art. 146, be the path of the body;

$$\text{then } \alpha = \phi + \theta, \therefore u = \frac{ds}{dt} \cdot \left(\frac{1}{1}\right)^{\frac{\phi + \theta}{c}};$$

$$\text{But } \left(\frac{1}{1}\right)^{\frac{\phi + \theta}{c}} = \frac{d\rho}{ds},$$

$$\therefore u = \frac{ds}{dt} \cdot \frac{d\rho}{ds} = \frac{d\rho}{dt};$$

$$\text{Also } V^2 = \left(\frac{ds}{dt}\right)^2,$$

$$\text{and } 1 = \left(\frac{dr}{ds}\right)^2 + r^2 \left(\frac{d\theta}{ds}\right)^2 \text{ (Art. 150.)}$$

$$\therefore V^2 = \left(\frac{dr}{ds}\right)^2 \cdot \left(\frac{ds}{dt}\right)^2 + r^2 \left(\frac{d\theta}{ds}\right)^2 \cdot \left(\frac{ds}{dt}\right)^2$$

$$= \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2;$$

Also $1 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2$ (Art. 151.)

$$\therefore V^2 = \left(\frac{dx}{ds}\right)^2 \cdot \left(\frac{ds}{dt}\right)^2 + \left(\frac{dy}{ds}\right)^2 \cdot \left(\frac{ds}{dt}\right)^2$$

$$= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2.$$

(157.) Def. If the velocity of a body be instantaneously changed either in length or direction, the cause which produces this effect is called an *impulsive force* or an *impulse;* but if the velocity be gradually changed, the cause which produces this effect is called a *continued force.*

(158.) Def. A continued force, when its effects are estimated with respect only to the change produced in the velocity, without regard to the magnitude of the body moved, is called an *accelerating force;* and is thus *measured,* let u represent in *length* and *direction* the velocity of the body at the end of time t, u', at the end of time $t + h$, and let $f : 1 ::$ limiting ratio $u' - u : h$, when h is diminished *sine limite;* then f is a line which represents in *length* and *direction* the accelerating force at the end of time t.

(159.) Cor. Hence $f = \dfrac{du}{dt} = \dfrac{d^2\rho}{dt^2}$.

(160.) *When a body revolves in a circle in consequence of the action of a force tending to the centre, to determine the velocity and the periodic time.*

Let ρ represent the radius in length and direction at end of time t,

Let r be a positive quantity in length $= \rho$,

Let ρ be inclined to unity at an angle $= \theta$,

$$\text{then } \rho = r \left(\tfrac{1}{1}\right)^{\frac{\theta}{c}},$$

$$\text{force} = \frac{d^2\rho}{dt^2},$$

And this force tends to the centre,

$\therefore \dfrac{d^2\rho}{dt^2}$ is a line measured in the opposite direction to that in which ρ is measured,

$\therefore \dfrac{d^2\rho}{dt^2}$ is inclined to unity at an angle $= \pi + \theta$,

Let $\dfrac{d^2\rho}{dt^2} = P \cdot \left(\tfrac{1}{1}\right)^{\frac{\pi+\theta}{c}}$, P is a positive quantity;

$$\text{now } \rho = r \left(\tfrac{1}{1}\right)^{\frac{\theta}{c}},$$

$\therefore$ since r is constant in a circle,

$$\frac{d\rho}{dt} = r\left(\frac{1}{1}\right)^{\frac{\theta}{c}} \cdot \frac{d\theta}{dt}\sqrt{-1},$$

$$\therefore \frac{d^2\rho}{dt^2} = -r\left(\frac{1}{1}\right)^{\frac{\theta}{c}}\left(\frac{d\theta}{dt}\right)^2 + r\left(\frac{1}{1}\right)^{\frac{\theta}{c}}\frac{d^2\theta}{dt^2}\cdot\sqrt{-1},$$

$$\therefore -r\left(\frac{1}{1}\right)^{\frac{\theta}{c}}\cdot\left(\frac{d\theta}{dt}\right)^2 + r\left(\frac{1}{1}\right)^{\frac{\theta}{c}}\frac{d^2\theta}{dt^2}\sqrt{-1} = P\cdot\left(\frac{1}{1}\right)^{\frac{\pi+\theta}{c}},$$

$$\therefore -r\left(\frac{d\theta}{dt}\right)^2 + r\frac{d^2\theta}{dt^2}\sqrt{-1} = P\cdot\left(\frac{1}{1}\right)^{\frac{\pi}{c}} = -P;$$

$\therefore$ we have the two following equations,

$$\left.\begin{array}{r} -r\left(\dfrac{d\theta}{dt}\right)^2 = -P \\ r\dfrac{d^2\theta}{dt^2} = 0 \end{array}\right\};$$

$$\therefore \frac{d^2\theta}{dt^2} = 0,$$

$$\therefore \frac{d\theta}{dt} = \text{const.} = C;$$

$\therefore$ $P = r\,.\,C^2$, a constant quantity;

And $\dfrac{d\theta}{dt} = \sqrt{\dfrac{P}{r}}$,

$$\therefore t = \theta\sqrt{\frac{r}{P}} + \text{const.}$$

when $\theta = 0$, let $t = 0$, then const. $= 0$;

$$\therefore t = \theta \sqrt{\frac{r}{P}};$$

Let $\theta = 2\pi$,

then periodic time $= 2\pi \sqrt{\frac{r}{P}}$;

Let V be in length equal to the velocity,

then generally $V^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2$ (by Art 156.)

$\therefore$ in a circle $V^2 = r^2 \left(\frac{d\theta}{dt}\right)^2 = rP$,

$$\therefore V = \sqrt{rP}.$$

(161.) *When a body is acted upon by a centripetal force, to find the sectorial area described round the centre, and the velocity.*

As in Art. 160. $\frac{d^2\rho}{dt^2} = P \cdot \left(\frac{1}{1}\right)^{\frac{\pi+\theta}{c}}$,

$$\text{and } \rho = r \left(\frac{1}{1}\right)^{\frac{\theta}{c}};$$

$$\therefore \frac{d\rho}{dt} = \frac{dr}{dt} \cdot \left(\frac{1}{1}\right)^{\frac{\theta}{c}} + r \left(\frac{1}{1}\right)^{\frac{\theta}{c}} \frac{d\theta}{dt} \sqrt{-1},$$

$$\therefore \frac{d^2\rho}{dt^2} = \frac{d^2r}{dt^2}\cdot\left(\begin{matrix}1\\1\end{matrix}\right)^{\frac{\theta}{c}} + 2\frac{dr}{dt}\cdot\frac{d\theta}{dt}\cdot\left(\begin{matrix}1\\1\end{matrix}\right)^{\frac{\theta}{c}}\sqrt{-1}$$

$$-r\left(\frac{d\theta}{dt}\right)^2\cdot\left(\begin{matrix}1\\1\end{matrix}\right)^{\frac{\theta}{c}} + r\frac{d^2\theta}{dt^2}\cdot\left(\begin{matrix}1\\1\end{matrix}\right)^{\frac{\theta}{c}}\sqrt{-1},$$

$$\therefore P\cdot\left(\begin{matrix}1\\1\end{matrix}\right)^{\frac{\pi+\theta}{c}} = \frac{d^2r}{dt^2}\cdot\left(\begin{matrix}1\\1\end{matrix}\right)^{\frac{\theta}{c}} + 2\frac{dr}{dt}\cdot\frac{d\theta}{dt}\cdot\left(\begin{matrix}1\\1\end{matrix}\right)^{\frac{\theta}{c}}\sqrt{-1}$$

$$-r\left(\frac{d\theta}{dt}\right)^2\cdot\left(\begin{matrix}1\\1\end{matrix}\right)^{\frac{\theta}{c}} + r\frac{d^2\theta}{dt^2}\cdot\left(\begin{matrix}1\\1\end{matrix}\right)^{\frac{\theta}{c}}\sqrt{-1},$$

$$\therefore P\cdot\left(\begin{matrix}1\\1\end{matrix}\right)^{\frac{\pi}{c}} = \frac{d^2r}{dt^2} + 2\frac{dr}{dt}\cdot\frac{d\theta}{dt}\cdot\sqrt{-1}$$

$$-r\left(\frac{d\theta}{dt}\right)^2 + r\frac{d^2\theta}{dt^2}\cdot\sqrt{-1};$$

but $P\left(\begin{matrix}1\\1\end{matrix}\right)^{\frac{\pi}{c}} = -P,$

$$\therefore -P = \frac{d^2r}{dt^2} + 2\frac{dr}{dt}\cdot\frac{d\theta}{dt}\cdot\sqrt{-1}$$

$$-r\left(\frac{d\theta}{dt}\right)^2 + r\frac{d^2\theta}{dt^2}\cdot\sqrt{-1};$$

∴ we have the two following equations,

$$\left.\begin{aligned}&(1)\quad \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P\\ &(2)\quad 2\frac{dr}{dt}\cdot\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0\end{aligned}\right\},$$

multiplying all the terms of (2) by r, we have

$$2r\frac{dr}{dt}\cdot\frac{d\theta}{dt}+r^2\frac{d^2\theta}{dt^2}=0,$$

$\therefore$ integrating, $r^2\frac{d\theta}{dt}=h$, a constant quantity,

$$\therefore \int r^2 d\theta = ht + \text{const.}$$

when $\int r^2 d\theta = 0$, let $t = t'$,

then $\int r^2 d\theta = h\ (t-t')$,

$$\therefore \int \frac{r^2 d\theta}{2} = \frac{h}{2}(t-t'),$$

or sectorial area $= \frac{h}{2}\ (t-t')$.

Now multiplying every term of (1) by $2.\frac{dr}{dt}$ and every term of (2) by $2r\frac{d\theta}{dt}$, we have

$$\left.\begin{array}{l} 2\frac{dr}{dt}\cdot\frac{d^2r}{dt^2}-2r\frac{dr}{dt}\cdot\left(\frac{d\theta}{dt}\right)^2 \ldots\ldots\ldots\ldots = -2P\frac{dr}{dt} \\ \ldots\ldots\ldots\ldots 4r\frac{dr}{dt}\cdot\left(\frac{d\theta}{dt}\right)^2+2r^2\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2}=0 \end{array}\right\};$$

$\therefore$ by addition,

$$2\frac{dr}{dt}\cdot\frac{d^2r}{dt^2}+2r\frac{dr}{dt}\cdot\left(\frac{d\theta}{dt}\right)^2+2r^2\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2}=-2P\frac{dr}{dt};$$

$\therefore$ integrating, $\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \ldots\ldots = \int -2P\,dr,$

$\therefore$ (by Art. 156.) $V^2 \ldots\ldots\ldots\ldots\ldots\ldots = \int -2P\,dr.$

(162.) Cor. 1. From equation (1),

$$\frac{d^2r}{dt^2} = r\left(\frac{d\theta}{dt}\right)^2 - P$$

$= \text{centrifugal force} - \text{centripetal force}.$

(163.) Cor. 2. Since $r^2\dfrac{d\theta}{dt} = h$,

$$\left(\frac{dt}{d\theta}\right)^2 = \frac{r^4}{h^2};$$

and $\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = V^2$,

$$\therefore \left(\frac{dr}{d\theta}\right)^2 + r^2 = \frac{r^4V^2}{h^2},$$

$$\therefore V^2 = h^2\left\{\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^2}\right\}.$$

(164.) Cor. 3. Let $u = \dfrac{1}{r}$,

then $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\cdot\dfrac{dr}{d\theta}$,

$$\therefore V^2 = h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\}.$$

(165.) Cor. 4. Since $V^2 = \int -2P\,dr$,

$$V^2 = \int + \frac{2P\,du}{u^2},$$

$$\therefore h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = \int \frac{2P\,du}{u^2},$$

$$\therefore \text{differentiating, } h^2\left\{2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2} + 2u\frac{du}{d\theta}\right\} = \frac{2P}{u^2}\frac{du}{d\theta},$$

$$\therefore h^2\left\{\frac{d^2u}{d\theta^2} + u\right\} = \frac{P}{u^2},$$

$$\therefore \frac{d^2u}{d\theta^2} + u - \frac{P}{h^2u^2} = 0.$$

(166.) *Let* $P \propto \frac{1}{r^2}$, *to find the equation to the curve described.*

$$\text{Let } P = \frac{m}{r^2} = mu^2;$$

$$\therefore \frac{d^2u}{d\theta^2} + u - \frac{m}{h^2} = 0, \text{ (by Art. 165.)}$$

$$\text{Let } u - \frac{m}{h^2} = x,$$

$$\therefore \frac{du}{d\theta} = \frac{dx}{d\theta},$$

$$\frac{d^2u}{d\theta^2} = \frac{d^2x}{d\theta^2},$$

$$\therefore \frac{d^2x}{d\theta^2} + x = 0;$$

Let $\binom{e}{0}^{k\theta} \cdot \binom{1}{1}^{\frac{n\theta}{c}}$ represent a particular value of x, where e is the base of the hyperbolic logarithms,

$$\text{then } \frac{dx}{d\theta} = \binom{e}{0}^{k\theta} \cdot \binom{1}{1}^{\frac{n\theta}{c}} (k + n\sqrt{-1}),$$

$$\frac{d^2x}{d\theta^2} = \binom{e}{0}^{k\theta} \cdot \binom{1}{1}^{\frac{n\theta}{c}} \cdot (k + n\sqrt{-1})^2;$$

$$\therefore \binom{e}{0}^{k\theta} \cdot \binom{1}{1}^{\frac{n\theta}{c}} \cdot (k + n\sqrt{-1})^2 + \binom{e}{0}^{k\theta} \cdot \binom{1}{1}^{\frac{n\theta}{c}} = 0,$$

$$\therefore (k + n\sqrt{-1})^2 + 1 = 0,$$

$$\therefore k + n\sqrt{-1} = \pm\sqrt{-1},$$

$$\therefore k = 0, \quad n = \pm 1;$$

$\therefore$ the general value is,

$$x = C\binom{1}{1}^{\frac{\theta}{c}} + C'\binom{1}{1}^{-\frac{\theta}{c}};$$

Let $C = a\binom{1}{1}^{\frac{\alpha}{c}}$, $C' = b\binom{1}{1}^{\frac{\beta}{c}}$, where a, b are positive quantities and α, β positive and less than 2π;

$$\text{then } x = a\binom{1}{1}^{\frac{\alpha+\theta}{c}} + b\binom{1}{1}^{\frac{\beta-\theta}{c}},$$

$$x = a\left(\frac{1}{1}\right)^{\frac{\alpha+\theta}{c}} + b\left(\frac{1}{1}\right)^{\frac{\beta-\theta}{c}}$$

$$\therefore\ x = a\left(\frac{1}{1}\right)^{-\frac{\alpha+\theta}{c}} + b\left(\frac{1}{1}\right)^{-\frac{\beta-\theta}{c}};$$

∴ by subtraction,

$$0 = a.\left(\frac{1}{1}\right)^{\frac{\alpha+\theta}{c}} - a.\left(\frac{1}{1}\right)^{-\frac{\alpha+\theta}{c}} + b.\left(\frac{1}{1}\right)^{\frac{\beta-\theta}{c}} - b\left(\frac{1}{1}\right)^{-\frac{\beta-\theta}{c}}$$

$$= \left\{a\left(\frac{1}{1}\right)^{\frac{\alpha}{c}} - b\left(\frac{1}{1}\right)^{-\frac{\beta}{c}}\right\}.\left(\frac{1}{1}\right)^{\frac{\theta}{c}} - \left\{a\left(\frac{1}{1}\right)^{-\frac{\alpha}{c}} - b.\left(\frac{1}{1}\right)^{\frac{\beta}{c}}\right\}.\left(\frac{1}{1}\right)^{-\frac{\theta}{c}},$$

$$\therefore\ 0 = \left\{a\left(\frac{1}{1}\right)^{\frac{\alpha}{c}} - b\left(\frac{1}{1}\right)^{-\frac{\beta}{c}}\right\}.\left(\frac{1}{1}\right)^{\frac{2\theta}{c}} - \left\{a.\left(\frac{1}{1}\right)^{-\frac{\alpha}{c}} - b.\left(\frac{1}{1}\right)^{\frac{\beta}{c}}\right\};$$

∴ since θ is variable, we have the two following equations,

$$\left.\begin{array}{l} a\left(\frac{1}{1}\right)^{\frac{\alpha}{c}} - b\left(\frac{1}{1}\right)^{-\frac{\beta}{c}} = 0 \\ a\left(\frac{1}{1}\right)^{-\frac{\alpha}{c}} - b\left(\frac{1}{1}\right)^{\frac{\beta}{c}} = 0 \end{array}\right\};$$

from either of which we deduce

$$b = a$$

$$\left(\frac{1}{1}\right)^{\frac{\beta}{c}} = \left(\frac{1}{1}\right)^{-\frac{\alpha}{c}},$$

$\therefore$ substituting these values for b and $\binom{1}{1}^{\frac{\beta}{c}}$,

$$x = a\binom{1}{1}^{\frac{\alpha+\theta}{c}} + a\binom{1}{1}^{-\frac{\alpha+\theta}{c}},$$

$$= a \,.\, 2 \cos(\alpha + \theta);$$

$$\therefore\ u - \frac{m}{h^2} = 2a \cos(\alpha + \theta),$$

$$\therefore\ \frac{1}{r} - \frac{m}{h^2} = 2a \cos(\alpha + \theta),$$

equation to the curve described.

(167.) *To solve a cubic equation* $x^3 - qx + r = 0$, *when* $\frac{q^3}{27}$ *is greater than* $\frac{r^2}{4}$.

By Cardan's rule,

$$x = \left(-\frac{r}{2} + \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{r}{2} - \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}\right)^{\frac{1}{3}}$$

$$= \left\{-\frac{r}{2} + \left(\frac{q^3}{27} - \frac{r^2}{4}\right)^{\frac{1}{2}}\sqrt{-1}\right\}^{\frac{1}{3}}$$

$$+ \left\{-\frac{r}{2} - \left(\frac{q^3}{27} - \frac{r^2}{4}\right)^{\frac{1}{2}}\sqrt{-1}\right\}^{\frac{1}{3}};$$

$$\text{Let } \left\{-\frac{r}{2}+\left(\frac{q^3}{27}-\frac{r^2}{4}\right)^{\frac{1}{2}}\sqrt{-1}\right\}^{\frac{1}{3}}=f\binom{1}{1}^{\frac{\theta}{c}},$$

where f is a positive quantity, and $\frac{\theta}{c}$ positive and less than 1;

$$\text{Then } \left\{-\frac{r}{2}-\left(\frac{q^3}{27}-\frac{r^2}{4}\right)^{\frac{1}{2}}\sqrt{-1}\right\}^{\frac{1}{3}}=f\binom{1}{1}^{-\frac{\theta}{c}},$$

$$\therefore\ x=f\binom{1}{1}^{\frac{\theta}{c}}+f\binom{1}{1}^{-\frac{\theta}{c}}=f.\,2\cos\theta;$$

By multiplication,

$$\left(\frac{r^2}{4}+\frac{q^3}{27}-\frac{r^2}{4}\right)^{\frac{1}{3}}=f^2;$$

$$\therefore f=\sqrt{\frac{q}{3}};$$

$$\therefore\ x=2\sqrt{\frac{q}{3}}\,.\cos\theta;$$

$$\text{Also } -\frac{r}{2}+\left(\frac{q^3}{27}-\frac{r^2}{4}\right)^{\frac{1}{2}}.\sqrt{-1}=f^3\binom{1}{1}^{\frac{3\theta}{c}},$$

$$\text{And } -\frac{r}{2}-\left(\frac{q^3}{27}-\frac{r^2}{4}\right)^{\frac{1}{2}}.\sqrt{-1}=f^3\binom{1}{1}^{-\frac{3\theta}{c}};$$

$$\therefore\ -r\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots=f^3\left\{\binom{1}{1}^{\frac{3\theta}{c}}+\binom{1}{1}^{-\frac{3\theta}{c}}\right\}$$

$$=f^3\,.\,2\cos(3\theta);$$

$$\therefore \cos(3\theta) = -\frac{r}{2f^3} = -\frac{r}{2\left(\frac{q}{3}\right)^{\frac{3}{2}}} = -\frac{r\sqrt{27}}{2\sqrt{q^3}}.$$

Let $-\dfrac{r\sqrt{27}}{2\sqrt{q^3}} = \cos\phi$, where ϕ is positive and less than 560°,

Then ϕ is known from trigonometrical tables;

And since $-\dfrac{r\sqrt{27}}{2\sqrt{q^3}} = \cos\phi$,

$$-\frac{r\sqrt{27}}{2\sqrt{q^3}} = \cos(360^\circ + \phi),$$

$$= \cos(2.360^\circ + \phi),$$

$$\therefore \cos(3\theta) = \cos\phi = \cos(360^\circ + \phi) = \cos(2.360^\circ + \phi);$$

$\therefore$ the values of θ are

$$\frac{\phi}{3},\ 120^\circ + \frac{\phi}{3},\ 240^\circ + \frac{\phi}{3};$$

$\therefore$ the values of x are

$$2\sqrt{\frac{q}{3}}.\cos\left(\frac{\phi}{3}\right),\ 2\sqrt{\frac{q}{3}}.\cos\left(120^\circ + \frac{\phi}{3}\right),$$

$$2\sqrt{\frac{q}{3}}.\cos\left(240^\circ + \frac{\phi}{3}\right),$$

which may be found from trigonometrical tables.

(168.) *To find the integral of* $\frac{dx}{x}$.

Let $x = \binom{e}{0}^{my} \cdot \binom{1}{1}^{\frac{ny}{c}}$,

Then $\frac{dx}{dy} = \binom{e}{0}^{my} \cdot \binom{1}{1}^{\frac{ny}{c}} \cdot (m + n\sqrt{-1})$,

$$= x \,.\, (m + n\sqrt{-1}),$$

$$\therefore \frac{1}{x} \cdot \frac{dx}{dy} = m + n\sqrt{-1},$$

$$\therefore \int \frac{dx}{x} = my + ny\sqrt{-1};$$

Now x is in length $= \binom{e}{0}^{my}$,

$\therefore$ my is the hyperbolic logarithm of the length of x;

And ny is the angle, at which x is inclined to unity,

$\therefore$ if this angle be called *angle of* x,

$\int \frac{dx}{x} =$ hyp. log. of length of $x + \sqrt{-1}$. angle of x.

(169.) *To find the integral of* $\frac{2a\,dx}{a^2 + x^2}$.

$$\frac{2a}{a^2+x^2} = \frac{1}{a+x\sqrt{-1}} + \frac{1}{a-x\sqrt{-1}},$$

$$\therefore \int \frac{2a\,dx}{a^2+x^2} = \int \frac{dx}{a+x\sqrt{-1}} + \int \frac{dx}{a-x\sqrt{-1}}$$

$$= \frac{1}{\sqrt{-1}} \cdot \int \frac{dx\sqrt{-1}}{a+x\sqrt{-1}} - \frac{1}{\sqrt{-1}} \cdot \int \frac{-dx\sqrt{-1}}{a-x\sqrt{-1}}$$

$$= \frac{1}{\sqrt{-1}} \left\{ \text{hyp. log. of length of } \overline{a+x\sqrt{-1}} \right.$$

$$\left. + \sqrt{-1} \,.\, \text{angle of } \overline{a+x\sqrt{-1}} \right\}$$

$$- \frac{1}{\sqrt{-1}} \left\{ \text{hyp. log. of length of } \overline{a-x\sqrt{-1}} \right.$$

$$\left. + \sqrt{-1} \,.\, \text{angle of } .\ \overline{a-x\sqrt{-1}} \right\}$$

$$= \frac{1}{\sqrt{-1}} \cdot \left\{ \text{hyp. log. of length of } \frac{a+x\sqrt{-1}}{a-x\sqrt{-1}} \right.$$

$$\left. + \sqrt{-1} \,.\, \text{angle of } \frac{a+x\sqrt{-1}}{a-x\sqrt{-1}} \right\}$$

$$= \frac{1}{\sqrt{-1}} \cdot \left\{ \text{hyp. log. of length of } \frac{a+x\sqrt{-1}}{a-x\sqrt{-1}} \right\}$$

$$+ \text{angle of } \frac{a+x\sqrt{-1}}{a-x\sqrt{-1}}.$$

(170.) Cor. *Hence, if* a *and* x *be positive quantities, and* v *be an angle whose tangent is* $\frac{x}{a}$;

$$\int \frac{2a\,dx}{a^2+x^2} = 2v.$$

For in this case $\frac{a+x\sqrt{-1}}{a-x\sqrt{-1}}$ is in length = 1,

$\therefore$ hyp. log. of length of $\frac{a+x\sqrt{-1}}{a-x\sqrt{-1}} = 0$;

And angle of $\frac{a+x\sqrt{-1}}{a-x\sqrt{-1}}$ = angle of $\frac{(a+x\sqrt{-1})^2}{a^2+x^2}$,

= angle of $\left(1+\frac{x}{a}\sqrt{-1}\right)^2$,

= 2.angle of $\left(1+\frac{x}{a}\sqrt{-1}\right)$,

$= 2v$;

$$\therefore \int \frac{2a\,dx}{a^2+x^2} = 2v.$$

(171.) *To find the integral of* $\frac{dx}{\pm\sqrt{x^2+a^2}}$.

Let $\pm\sqrt{x^2+a^2} = y$,

then $x^2+a^2 = y^2$,

$$\text{and } \frac{dy}{dx} = \frac{x}{y},$$

$$\therefore \frac{dy + dx}{dx} = \frac{x+y}{y},$$

$$\therefore \int \frac{dx}{y} = \int \frac{dx+dy}{x+y},$$

$$\text{or} \int \frac{dx}{\pm\sqrt{x^2+a^2}} = \int \frac{dx+dy}{x+y}$$

$$= \text{hyp. log. of length of } (x+y)$$

$$+ \sqrt{-1}\,.\text{angle of } (x+y)$$

$$= \text{hyp. log. of length of } (x \pm \sqrt{x^2+a^2})$$

$$+ \sqrt{-1}\,.\text{angle of } (x \pm \sqrt{x^2+a^2}).$$

(172.) COR. In like manner the integral of

$\dfrac{dx}{\pm\sqrt{x^2+2ax}}$ will be found to be

$$\text{hyp. log. of length of } (x+a \pm \sqrt{x^2+2ax})$$

$$+ \sqrt{-1}\,.\text{angle of } (x+a \pm \sqrt{x^2+2ax});$$

And by similar investigations the integrals of other logarithmic and circular functions may be obtained.

(173.) *When a body is moving about a centre of force as in Art.* 161; *a radius vector, and the velocity of the body at the extremity of that radius vector being known both in length and direction; to find the value of the constant quantity* h.

Let $V . \left(\frac{1}{1}\right)^{\frac{\phi}{c}}$ be the velocity,

then $$V . \left(\frac{1}{1}\right)^{\frac{\phi}{c}} = \frac{d\rho}{dt}$$

$$= \left(\frac{1}{1}\right)^{\frac{\theta}{c}} . \frac{dr}{dt} + r \left(\frac{1}{1}\right)^{\frac{\theta}{c}} . \frac{d\theta}{dt} . \sqrt{-1},$$

$$\therefore V . \left(\frac{1}{1}\right)^{\frac{\phi - \theta}{c}} = \frac{dr}{dt} + r \frac{d\theta}{dt} \sqrt{-1},$$

or $$V . \cos(\phi - \theta) + V . \sin(\phi - \theta) . \sqrt{-1}$$

$$= \frac{dr}{dt} + r . \frac{d\theta}{dt} . \sqrt{-1},$$

$$\therefore V \sin(\phi - \theta) = r . \frac{d\theta}{dt};$$

But $$h = r^2 . \frac{d\theta}{dt},$$

$$\therefore h = r V . \sin(\phi - \theta).$$

S

(174.) *Let a body be projected from the extremity of a given radius vector, with a given velocity, in a direction perpendicular to the radius vector, and let it be acted upon by a force tending to the origin of the radii vectors, and varying inversely as the square of the distance; to find the equation to the curve described by the body, and the time of describing any part of that curve.*

The equation to the curve described in this case has already been found in Article 166; but, in order further to illustrate the principles established in the preceding chapters, another method of finding this equation is here given;

Let the given radius vector be in the positive direction, and let it $= l$,

Let the velocity of a body revolving in a circle at the distance l with the same centripetal force be in length $= V$,

And let the velocity of projection be in length $= nV$;

Then velocity of projection $= nV\sqrt{-1}$;

Also $V = \sqrt{P.l}$, (by Art. 160.)

$$= \sqrt{\frac{ml}{l^2}} = \sqrt{\frac{m}{l}},$$

$$\therefore\ m = lV^2,$$

Also $h = l.nV$ (by Art. 173);

Now $\dfrac{d^2\rho}{dt^2} = -\dfrac{m}{r^2}.\left(\dfrac{1}{1}\right)^{\frac{\theta}{c}} = -\dfrac{l.V^2}{r^2}\left(\dfrac{1}{1}\right)^{\frac{\theta}{c}},$

and $\dfrac{1}{r^2} = \dfrac{1}{h}.\dfrac{d\theta}{dt} = \dfrac{1}{lnV}.\dfrac{d\theta}{dt},$

$$\therefore\ \frac{d^2\rho}{dt^2} = -\frac{V}{n}.\left(\frac{1}{1}\right)^{\frac{\theta}{c}}.\frac{d\theta}{dt},$$

$$\therefore\ \frac{d\rho}{dt} = -\frac{V}{n}.\left(\frac{1}{1}\right)^{\frac{\theta}{c}}.\frac{1}{\sqrt{-1}} + \text{const.}$$

$$= \frac{V}{n}.\left(\frac{1}{1}\right)^{\frac{\theta}{c}}.\sqrt{-1} + \text{const.}$$

when $\theta = 0$, $\dfrac{d\rho}{dt} = nV.\sqrt{-1},$

$$\therefore\ nV\sqrt{-1} = \frac{V}{n}\sqrt{-1} + \text{const.},$$

$$\therefore\ \text{const.} = \frac{n^2-1}{n}.V.\sqrt{-1},$$

Let $n^2 - 1 = e,$

then const. $= \dfrac{eV}{n}\sqrt{-1};$

$$\therefore \frac{d\rho}{dt} = \frac{V}{n}\sqrt{-1}\,.\left\{e + \left(\begin{matrix}1\\1\end{matrix}\right)^{\frac{\theta}{c}}\right\};$$

Now $\dfrac{1}{r^2} = \dfrac{1}{lnV}\cdot\dfrac{d\theta}{dt}$,

$$\therefore \frac{1}{\rho^2} = \frac{1}{lnV}\cdot\left(\begin{matrix}1\\1\end{matrix}\right)^{-\frac{2\theta}{c}}\frac{d\theta}{dt},$$

$$\therefore \frac{1}{\rho^2}\cdot\frac{d\rho}{dt} = \frac{\sqrt{-1}}{n^2.l}\cdot\left\{e + \left(\begin{matrix}1\\1\end{matrix}\right)^{\frac{\theta}{c}}\right\}\cdot\left(\begin{matrix}1\\1\end{matrix}\right)^{-\frac{2\theta}{c}}\cdot\frac{d\theta}{dt}$$

$$= \frac{\sqrt{-1}}{1+e.l}\cdot\left\{e\left(\begin{matrix}1\\1\end{matrix}\right)^{-\frac{2\theta}{c}}\cdot\frac{d\theta}{dt} + \left(\begin{matrix}1\\1\end{matrix}\right)^{-\frac{\theta}{c}}\frac{d\theta}{dt}\right\},$$

$$\therefore -\frac{1}{\rho} = \frac{\sqrt{-1}}{1+e.l}\cdot\left\{\frac{e\left(\begin{matrix}1\\1\end{matrix}\right)^{-\frac{2\theta}{c}}}{-2\sqrt{-1}} + \frac{\left(\begin{matrix}1\\1\end{matrix}\right)^{-\frac{\theta}{c}}}{-\sqrt{-1}}\right\} + \text{const.}$$

$$= -\frac{1}{2.(1+e).l}\left\{e\left(\begin{matrix}1\\1\end{matrix}\right)^{-\frac{2\theta}{c}} + 2\left(\begin{matrix}1\\1\end{matrix}\right)^{-\frac{\theta}{c}}\right\} + \text{const.},$$

When $\theta = 0$, $\rho = l$,

$$\therefore -\frac{1}{l} = -\frac{e+2}{2(1+e).l} + \text{const.},$$

$$\therefore \text{const.} = \frac{e+2-2-2e}{2(1+e).l} = -\frac{e}{2(1+e).l},$$

$$\therefore \frac{1}{\rho} = \frac{1}{2(1+e)\,l}\cdot\left\{e + e\left(\begin{matrix}1\\1\end{matrix}\right)^{-\frac{2\theta}{c}} + 2\left(\begin{matrix}1\\1\end{matrix}\right)^{-\frac{\theta}{c}}\right\},$$

$$\therefore \frac{1}{r} = \frac{1}{2(1+e)l} \cdot \left\{ e \binom{1}{1}^{\frac{\theta}{c}} + e \binom{1}{1}^{-\frac{\theta}{c}} + 2 \right\}$$

$$= \frac{1}{2(1+e)l} \cdot (2e \cos\theta + 2)$$

$$= \frac{1}{(1+e)l} \cdot (e \cos\theta + 1),$$

Let $a = \dfrac{l}{1-e}$,

Then $\dfrac{1}{r} = \dfrac{1 + e\cos\theta}{a.(1-e^2)}$, equation to curve described.

Next to find the time of describing any part of this curve,

$$\frac{1}{\rho} = \frac{2}{2(1+e)l} \cdot \left\{ e + e \binom{1}{1}^{-\frac{2\theta}{c}} + 2 \binom{1}{1}^{-\frac{\theta}{c}} \right\}$$

$$= \frac{1}{2(1-e^2).a} \left\{ e + e \binom{1}{1}^{-\frac{2\theta}{c}} + 2 \binom{1}{1}^{-\frac{\theta}{c}} \right\},$$

$$\therefore \frac{2(1-e^2)}{e} \cdot \frac{a}{\rho} = 1 + \binom{1}{1}^{-\frac{2\theta}{c}} + \frac{2}{e} \cdot \binom{1}{1}^{-\frac{\theta}{c}},$$

$$\therefore \binom{1}{1}^{-\frac{2\theta}{c}} + \frac{2}{e} \cdot \binom{1}{1}^{-\frac{\theta}{c}} + \frac{1}{e^2} = \frac{2.(1-e^2)}{e} \frac{a}{\rho} + \frac{1}{e^2} - 1$$

$$= \frac{2(1-e^2)}{e} \cdot \frac{a}{\rho} + \frac{1-e^2}{e^2}$$

$$\therefore \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-\frac{2\theta}{c}} + \frac{2}{e}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-\frac{\theta}{c}} + \frac{1}{e^2} = \frac{(\rho+2ae).(1-e^2)}{\rho e^2},$$

$$\therefore \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-\frac{\theta}{c}} + \frac{1}{e} = \pm\,\frac{(\rho+2ae)^{\frac{1}{2}}.(1-e^2)^{\frac{1}{2}}}{e\rho^{\frac{1}{2}}},$$

$$\therefore \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-\frac{\theta}{c}} = \pm\,\frac{(\rho+2ae)^{\frac{1}{2}}.(1-e^2)^{\frac{1}{2}}}{e\rho^{\frac{1}{2}}} - \frac{1}{e};$$

Now $\dfrac{d\rho}{dt} = \dfrac{V}{n}\sqrt{-1}.\left\{e + \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\theta}{c}}\right\}$,

$$\therefore \frac{dt}{d\rho} = \frac{n}{V\sqrt{-1}}.\frac{1}{e+\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\theta}{c}}}$$

$$= \frac{n}{V\sqrt{-1}}.\frac{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-\frac{\theta}{c}}}{e\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-\frac{\theta}{c}}+1}$$

$$= \frac{n}{eV\sqrt{-1}}.\frac{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-\frac{\theta}{c}}}{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-\frac{\theta}{c}}+\frac{1}{e}}$$

$$= \frac{n}{eV\sqrt{-1}}.\left\{1 \mp \frac{\rho^{\frac{1}{2}}}{(\rho+2ae)^{\frac{1}{2}}.(1-e^2)^{\frac{1}{2}}}\right\},$$

$$\therefore t = \frac{\sqrt{1+e}}{eV.\sqrt{1-e^2}.\sqrt{-1}}.\left\{(1-e^2)^{\frac{1}{2}}\rho \mp \int\frac{\rho^{\frac{1}{2}}d\rho}{(\rho+2ae)^{\frac{1}{2}}}\right\} + \text{const.},$$

$$\therefore t = \frac{1}{eV.\sqrt{1-e}.\sqrt{-1}}.\left\{(1-e^2)^{\frac{1}{2}}\rho \mp \int \frac{\rho^{\frac{1}{2}} d\rho}{(\rho+2ae)^{\frac{1}{2}}}\right\} + \text{const.};$$

Now $\int \frac{\rho^{\frac{1}{2}} d\rho}{(\rho + 2ae)^{\frac{1}{2}}}$

$$= 2\rho^{\frac{1}{2}} (\rho + 2ae)^{\frac{1}{2}} - \int (\rho + 2ae)^{\frac{1}{2}} . \frac{d\rho}{\rho^{\frac{1}{2}}}$$

$$= 2\rho^{\frac{1}{2}} (\rho + 2ae)^{\frac{1}{2}} - \int \frac{\rho^{\frac{1}{2}} d\rho}{(\rho+2ae)^{\frac{1}{2}}} - \int \frac{2ae\,d\rho}{\rho^{\frac{1}{2}} (\rho+2ae)^{\frac{1}{2}}},$$

$$\therefore \int \frac{\rho^{\frac{1}{2}} d\rho}{(\rho + 2ae)^{\frac{1}{2}}} = \rho^{\frac{1}{2}}.(\rho + 2ae)^{\frac{1}{2}} - \int \frac{ae\,d\rho}{\rho^{\frac{1}{2}}.(\rho + 2ae)^{\frac{1}{2}}},$$

$$\therefore \mp \int \frac{\rho^{\frac{1}{2}} d\rho}{(\rho+2ae)^{\frac{1}{2}}} = \mp (\rho^2 + 2ae\rho)^{\frac{1}{2}} \pm ae \int \frac{d\rho}{(\rho^2 + 2ae\rho)^{\frac{1}{2}}}$$

$$= \mp (\rho^2 + 2ae\rho)^{\frac{1}{2}} + ae \int \frac{d\rho}{\pm (\rho^2 + 2ae\rho)^{\frac{1}{2}}}$$

$$= \mp (\rho^2 + 2ae\rho)^{\frac{1}{2}}$$

$+ ae.$hyp. log. of length of $(\rho + ae \pm \sqrt{\rho^2 + 2ae\rho})$

$+ ae\sqrt{-1}$. angle of $(\rho + ae \pm \sqrt{\rho^2 + 2ae\rho})$;

$$\therefore t = \frac{1}{eV.\sqrt{1-e}.\sqrt{-1}} . \left\{ (1-e^2)^{\frac{1}{2}}.\rho \mp (\rho^2 + 2ae\rho)^{\frac{1}{2}} \right.$$

$+ ae.$hyp. log. of length of $(\rho + ae \pm \sqrt{\rho^2 + 2ae\rho})$

$+ ae\sqrt{-1}$. angle of $(\rho + ae \pm \sqrt{\rho^2 + 2ae\rho}) \Big\}$ + const.;

$$\text{Now } \pm \frac{(\rho+2ae)^{\frac{1}{2}}.(1-e^2)^{\frac{1}{2}}}{e\rho^{\frac{1}{2}}} = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-\frac{\theta}{c}} + \frac{1}{e},$$

$$\therefore \ \pm (\rho^2+2ae\rho)^{\frac{1}{2}} = \rho . \frac{1+e\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{-\frac{\theta}{c}}}{(1-e^2)^{\frac{1}{2}}};$$

When $t=0$, $\theta=0$, $\rho=\overline{1-e}.a$,

$$\text{and } \pm(\rho^2+2ae\rho)^{\frac{1}{2}} = \frac{(1-e).a.(1+e)}{(1-e^2)^{\frac{1}{2}}} = a.(1-e^2)^{\frac{1}{2}};$$

$$\therefore 0 = \frac{1}{eV.\sqrt{1-e}.\sqrt{-1}}.\Big\{(1-e^2)^{\frac{1}{2}}.(1-e)a - a.(1-e^2)^{\frac{1}{2}}$$

$$+ ae.\text{hyp. log. of length of } (a+a\sqrt{1-e^2})$$

$$+ ae.\sqrt{-1}.\text{angle of } (a+a\sqrt{1-e^2})\Big\} + \text{const.}$$

$$= \frac{1}{eV\sqrt{1-e}.\sqrt{-1}}.\Big\{-(1-e^2)^{\frac{1}{2}}.ae$$

$$+ ae.\text{hyp. log. of length of } (a+a\sqrt{1-e^2})$$

$$+ ae.\sqrt{-1}.\text{angle of } (a+a\sqrt{1-e^2})\Big\} + \text{const.},$$

$$\therefore t = \frac{1}{eV\sqrt{1-e}.\sqrt{-1}}.\Big\{\sqrt{1-e^2}.(\rho+ae) \mp (\rho^2+2ae\rho)^{\frac{1}{2}}$$

$$+ ae.\text{hyp. log. of length of } \frac{\rho+ae \pm \sqrt{\rho^2+2ae\rho}}{a+a\sqrt{1-e^2}}$$

$$+ ae.\sqrt{-1}.\text{angle of } \frac{\rho+ae \pm \sqrt{\rho^2+2ae\rho}}{a+a\sqrt{1-e^2}}\Big\},$$

an equation which gives t in terms of ρ, where that value of $\pm\sqrt{\rho^2+2ae\rho}$ must be used, which satisfies the conditions of the equation

$$\frac{\pm(\rho+2ae)^{\frac{1}{2}}}{e\rho^{\frac{1}{2}}}(1-e^2)^{\frac{1}{2}} = \binom{1}{1}^{-\frac{\theta}{c}} + \frac{1}{e}.$$

(175.) *To find the time* t *in terms of the angle θ.*

$$\mp\sqrt{\rho^2+2a\rho} = -\rho.\frac{1+e\binom{1}{1}^{-\frac{\theta}{c}}}{(1-e^2)^{\frac{1}{2}}}$$

$$= -\frac{r.\binom{1}{1}^{\frac{\theta}{c}}+er}{(1-e^2)^{\frac{1}{2}}},$$

$$\therefore \sqrt{1-e^2}.(\rho+ae) \mp (\rho^2+2ae\rho)^{\frac{1}{2}}$$

$$= \frac{(1-e^2).\left\{r\binom{1}{1}^{\frac{\theta}{c}}+ae\right\} - r.\binom{1}{1}^{\frac{\theta}{c}} - er}{(1-e^2)^{\frac{1}{2}}}$$

$$= \frac{e}{(1-e^2)^{\frac{1}{2}}}.\left\{(1-e^2).a - r.\left[1+e\binom{1}{1}^{\frac{\theta}{c}}\right]\right\}$$

$$= \frac{e}{(1-e^2)^{\frac{1}{2}}}.\left\{(1-e^2)a - \frac{(1-e^2).a.\left\{1+e\binom{1}{1}^{\frac{\theta}{c}}\right\}}{1+e\cos\theta}\right\}$$

$$\therefore \sqrt{1-e^2}\,.\,(\rho+ae) \mp \sqrt{\rho^2+2ae\rho}$$

$$= ae\,.\,(1-e^2)^{\frac{1}{2}}\,.\,\frac{1+e\cos\theta-1-e\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\theta}{c}}}{1+e\cos\theta}$$

$$= ae^2\,(1-e^2)^{\frac{1}{2}}\,.\,\frac{\cos\theta-\cos\theta-\sin\theta\sqrt{-1}}{1+e\cos\theta}$$

$$= -\frac{ae^2\,.\,(1-e^2)^{\frac{1}{2}}\,.\,\sin\theta\sqrt{-1}}{1+e\cos\theta};$$

Also $\rho + ae \pm \sqrt{\rho^2 + 2ae\rho}$

$$= r\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\theta}{c}} + ae + \frac{r\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\theta}{c}} + er}{(1-e^2)^{\frac{1}{2}}}$$

$$= ae + r\,\frac{(\sqrt{1-e^2}+1)\,.\,\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\theta}{c}} + e}{\sqrt{1-e^2}}$$

$$= ae + a\,.\,\sqrt{1-e^2}\,.\,\frac{(\sqrt{1-e^2}+1)\,.\,\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\theta}{c}} + e}{1+e\cos\theta}$$

$$= a\,.\,\frac{e+e^2\cos\theta+(1-e^2+\sqrt{1-e^2})\,.\,\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^{\frac{\theta}{c}} + e\sqrt{1-e^2}}{1+e\cos\theta}$$

$$= a\,.\,\frac{e+e\sqrt{1-e^2}+e^2\cos\theta+(1-e^2+\sqrt{1-e^2})\,.\,(\cos\theta+\sin\theta\sqrt{-1})}{1+e\cos\theta}$$

$$= a\,.\,\frac{e+e\sqrt{1-e^2}+(1+\sqrt{1-e^2})\,.\cos\theta+(1-e^2+\sqrt{1-e^2})\sin\theta\sqrt{-1}}{1+e\cos\theta}$$

$$= a.(1+\sqrt{1-e^2}).\frac{e+\cos\theta+\sqrt{1-e^2}.\sin\theta\sqrt{-1}}{1+e\cos\theta},$$

$$\therefore \frac{\rho+ae\pm\sqrt{\rho^2+2ae\rho}}{a+a\sqrt{1-e^2}}=\frac{e+\cos\theta+\sqrt{1-e^2}.\sin\theta\sqrt{-1}}{1+e\cos\theta};$$

$$\therefore t=\frac{1}{e.V.\sqrt{1-e}.\sqrt{-1}}.\left\{ae.\text{hyp. log. of length of}\right.$$

$$\frac{e+\cos\theta+\sqrt{1-e^2}.\sin\theta.\sqrt{-1}}{1+e\cos\theta}$$

$$+\,ae.\sqrt{-1}\text{ angle of }\frac{e+\cos\theta+\sqrt{1-e^2}.\sin\theta.\sqrt{-1}}{1+e\cos\theta}$$

$$\left.-\frac{ae^2\sqrt{1-e^2}.\sin\theta.\sqrt{-1}}{1+e\cos\theta}\right\}$$

$$=\frac{a}{V.\sqrt{1-e}}.\left\{\frac{1}{\sqrt{-1}}.\text{hyp. log. of length of}\right.$$

$$\frac{e+\cos\theta+\sqrt{1-e^2}.\sin\theta\sqrt{-1}}{1+e\cos\theta}$$

$$+\text{ angle of }\frac{e+\cos\theta+\sqrt{1-e^2}.\sin\theta.\sqrt{-1}}{1+e\cos\theta}$$

$$\left.-\frac{e\sqrt{1-e^2}.\sin\theta}{1+e\cos\theta}\right\};$$

now $V=\sqrt{\frac{m}{l}}$,

$$\therefore V.\sqrt{1-e} = \sqrt{\frac{m.\overline{1-e}}{l}} = \sqrt{\frac{m}{a}};$$

$$\therefore \frac{a}{V\sqrt{1-e}} = \frac{a^{\frac{3}{2}}}{\sqrt{m}},$$

$$\therefore t = \frac{a^{\frac{3}{2}}}{\sqrt{m}}.\left\{\frac{1}{\sqrt{-1}}.\text{hyp. log. of length of}\right.$$

$$\frac{e+\cos\theta+\sqrt{1-e^2}.\sin\theta\sqrt{-1}}{1+e\cos\theta}$$

$$+ \text{ angle of } \frac{e+\cos\theta+\sqrt{1-e^2}.\sin\theta.\sqrt{-1}}{1+e\cos\theta}$$

$$\left.- \frac{e(1-e^2)^{\frac{1}{2}}.\sin\theta}{1+e\cos\theta}\right\},$$

an equation which gives the value of t in terms of θ.

(176.) *To determine what this value of* t *becomes according as different values are given to* n.

First let n be less than 1,

Then e is negative,

Let $e = -e'$,

then $t = \frac{a^{\frac{3}{2}}}{\sqrt{m}} \left\{ \frac{1}{\sqrt{-1}} \right.$. hyp. log of length of

$$\frac{-e' + \cos\theta + \sqrt{1-e'^2}.\sin\theta\sqrt{-1}}{1 - e'\cos\theta}$$

$$+ \text{ angle of } \frac{-e' + \cos\theta + \sqrt{1-e'^2}.\sin\theta\sqrt{-1}}{1 - e'\cos\theta}$$

$$\left. + \frac{e'\sqrt{1-e'^2}.\sin\theta}{1 - e'\cos\theta} \right\},$$

Now length of $\frac{-e' + \cos\theta + \sqrt{1-e'^2}.\sin\theta\sqrt{-1}}{1 - e'\cos\theta}$

$$= \frac{\{(-e' + \cos\theta)^2 + (1 - e'^2).\sin^2\theta\}^{\frac{1}{2}}}{1 - e'\cos\theta} = 1,$$

$\therefore$ hyp. log. of length of

$$\frac{-e' + \cos\theta + \sqrt{1-e'^2}.\sin\theta\sqrt{-1}}{1 - e'\cos\theta} = 0,$$

and angle of $\frac{-e' + \cos\theta + \sqrt{1-e'^2}.\sin\theta\sqrt{-1}}{1 - e'\cos\theta}$,

$$= \text{angle, whose cosine} = \frac{-e' + \cos\theta}{1 - e'\cos\theta},$$

$$\text{and whose sine} = \frac{\sqrt{1-e'^2}.\sin\theta}{1 - e'\cos\theta},$$

Let this angle $= u$,

$$\text{Then } t = \frac{a^{\frac{3}{2}}}{\sqrt{m}} \cdot \left\{ u + e' \sin u \right\}.$$

Next let $n = 1$,

Then $e = 0$,

$$\therefore\ t = \frac{a^{\frac{3}{2}}}{\sqrt{m}} \left\{ \frac{1}{\sqrt{-1}} . \text{hyp. log. of length of } (\cos\theta + \sin\theta\sqrt{-1}) + \text{angle of } (\cos\theta + \sin\theta\sqrt{-1}) \right\}$$

$$= \frac{a^{\frac{3}{2}}}{\sqrt{m}} \cdot \left\{ \frac{1}{\sqrt{-1}} . \text{hyp. log. of length of } \left({1 \atop 1}\right)^{\frac{\theta}{c}} + \text{angle of } \left({1 \atop 1}\right)^{\frac{\theta}{c}} \right\}$$

$$= \frac{a^{\frac{3}{2}}}{\sqrt{m}} . \theta.$$

Next let n be greater than 1 and less than $\sqrt{2}$,

Then e is positive and less than 1;

Let u be an angle whose cosine $= \dfrac{e + \cos\theta}{1 + e\cos\theta}$, Then it may be proved nearly as in the first case that

$$t = \frac{a^{\frac{3}{2}}}{\sqrt{m}} \cdot \left\{ u - e \sin u \right\}.$$

Next let $n = \sqrt{2}$,

Then $e = 1$;

Here we must find t by the method of determining the value of fractions, whose numerators and denominators are evanescent;

By the preceding case, when e is less than 1,

$$t = \frac{a^{\frac{3}{2}}}{\sqrt{m}} \cdot \left\{ u - e \sin u \right\}$$

$$= \frac{l^{\frac{3}{2}}}{\sqrt{m}} \cdot \frac{u - e \sin u}{(1-e)^{\frac{3}{2}}}$$

$$= \frac{l^{\frac{3}{2}}}{\sqrt{m}} \cdot \frac{\sin u + \frac{1}{2} \cdot \frac{1}{3} \sin^3 u + \frac{1.3}{2.4} \cdot \frac{1}{5} \sin^5 u + \&c. - e \sin u}{(1-e)^{\frac{3}{2}}}$$

$$= \frac{l^{\frac{3}{2}}}{\sqrt{m}} \cdot \frac{(1-e) \sin u + \frac{1}{6} \cdot \sin^3 u + \frac{3}{40} \cdot \sin^5 u + \&c.}{(1-e)^{\frac{3}{2}}};$$

Now $\sin u = \dfrac{\sqrt{1-e^2} . \sin \theta}{1 + e . \cos \theta} = \dfrac{\sqrt{1-e} . \sqrt{1+e} . \sin \theta}{1 + e \cos \theta}$,

$$\therefore\ t = \frac{l^{\frac{3}{2}}}{\sqrt{m}} \cdot \left\{ \frac{\sqrt{1+e}.\sin\theta}{1+e\cos\theta} + \tfrac{1}{6}\frac{(1+e)^{\frac{3}{2}}\sin^3\theta}{(1+e\cos\theta)^3} \right.$$

$$\left. + \frac{3}{40}.(1-e).\frac{(1+e)^{\frac{5}{2}}\sin^5\theta}{(1+e\cos\theta)^5} + \&c. \right\};$$

Let $e = 1$,

$$\text{Then } t = \frac{l^{\frac{3}{2}}}{\sqrt{m}}.\left\{\sqrt{2}.\frac{\sin\theta}{1+\cos\theta} + \frac{2^{\frac{3}{2}}}{6}.\frac{\sin^3\theta}{(1+\cos\theta)^3}\right\}$$

$$= \frac{l^{\frac{3}{2}}\sqrt{2}}{\sqrt{m}}.\left\{\frac{\sin\theta}{1+\cos\theta} + \tfrac{1}{3}.\frac{\sin^3\theta}{(1+\cos\theta)^3}\right\}$$

$$= \frac{l^{\frac{3}{2}}\sqrt{2}}{\sqrt{m}}.\left\{\tan\frac{\theta}{2} + \tfrac{1}{3}.\tan^3\frac{\theta}{2}\right\}.$$

Next let n be greater than $\sqrt{2}$,

Then e is greater than 1,

$$\text{In this case } a^{\frac{3}{2}} = \left(\frac{l}{1-e}\right)^{\frac{3}{2}} = \left(\frac{l}{-(e-1)}\right)^{\frac{3}{2}}$$

$$= \frac{l^{\frac{3}{2}}}{-\sqrt{-1}.(e-1)^{\frac{3}{2}}} = \frac{l^{\frac{3}{2}}\sqrt{-1}}{(e-1)^{\frac{3}{2}}};$$

Let $\dfrac{l}{e-1} = a'$,

Then $a^{\frac{3}{2}} = a'^{\frac{3}{2}}.\sqrt{-1}$;

$$\therefore\ t = \frac{a'^{\frac{3}{2}}\sqrt{-1}}{\sqrt{m}}.\left\{\frac{1}{\sqrt{-1}}.\text{hyp. log. of length of}\right.$$

$$\frac{e+\cos\theta+\sqrt{e^2-1}\,.\sqrt{-1}\,.\sin\theta\,.\sqrt{-1}}{1+e\cos\theta}$$

$$+\ \text{angle of}\ \frac{e+\cos\theta+\sqrt{e^2-1}\,.\sqrt{-1}\,.\sin\theta\,.\sqrt{-1}}{1+e\cos\theta}$$

$$\left.\frac{-e\sqrt{e^2-1}\,.\sqrt{-1}\,.\sin\theta}{1+e\cos\theta}\right\}$$

$$= \frac{a'^{\frac{3}{2}}}{\sqrt{m}}.\left\{\text{hyp. log. of length of}\ \frac{e+\cos\theta-\sqrt{e^2-1}\,.\sin\theta}{1+e\cos\theta}\right.$$

$$+\ \sqrt{-1}\,.\text{angle of}\ \frac{e+\cos\theta-\sqrt{e^2-1}\,.\sin\theta}{1+e\cos\theta}$$

$$\left.+\ \frac{e\sqrt{e^2-1}\,.\sin\theta}{1+e\cos\theta}\right\};$$

Now $\frac{e+\cos\theta-\sqrt{e^2-1}\,.\sin\theta}{1+e\cos\theta}$ is a positive quantity,

$$\therefore\ \text{hyp. log. of length of}\ \frac{e+\cos\theta-\sqrt{e^2-1}\,.\sin\theta}{1+e\cos\theta}$$

$$= \text{hyp. log.}\ \frac{e+\cos\theta-\sqrt{e^2-1}\,.\sin\theta}{1+e\cos\theta},$$

$$\text{and angle of}\ \frac{e+\cos\theta-\sqrt{e^2-1}\,.\sin\theta}{1+e\cos\theta} = 0;$$

$$\therefore\ t = \frac{a'^{\frac{3}{2}}}{\sqrt{m}} \cdot \left\{ \text{hyp. log.}\ \frac{e + \cos\theta - \sqrt{e^2 - 1}\,.\sin\theta}{1 + e\cos\theta} \right.$$

$$\left. + \frac{e\sqrt{e^2 - 1}\,.\sin\theta}{1 + e\cos\theta} \right\}.$$