ELEMENTS

OF

A L G E B R A:

BEING A SHORT AND PRACTICAL INTRODUCTION TO THAT USEFUL SCIENCE, ON A NEW PLAN;

INCLUDING A SIMPLIFICATION OF THE RULE FOR THE SOLUTION OF EQUATIONS OF ALL DIMENSIONS.

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"Cette Science est d'une certitude et d'une perfection admirables, comme celle d'étenduè, et elle est d'une utilité encore universelle; car il n'y a absolument aucune branche de nos connaissances qui n'en reçoive de puissans secours, et aucune classe de nos idées à la combination de laquelle elle ne contribue directement ou indirectement."

Destutt de Tracy.

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TO

DOCTOR GEORGE BIRKBECK,
F.G.S.

PRESIDENT OF THE LONDON MECHANICS' INSTITUTION,
&c. &c. &c.

DEAR DOCTOR,

Your kindness in permitting me to inscribe this little Treatise to you, I esteem not only as a testimony of your friendship, but as a proof of your ardent zeal in the Diffusion of Useful Knowledge. Perhaps I may be allowed to add, that the gratification I feel, is enhanced by the consideration, that I once had the honour of filling the Mathematical Chair in that Institution where you first threw open the portals of Science to the humblest Mechanic, and brought down Philosophy from her lofty station to the capacities and means of all classes of the People. If this elementary Treatise shall tend to serve the purpose for which it was written, it will owe to your name, the favourable aspect in which it may be viewed by all who feel interested in the progress of Science and the prosperity of Popular Institutions,

I am,

Dear Doctor,

With much regard and esteem,

Your most obedient and obliged Servant,

THE AUTHOR.

CONTENTS.

ELEMENTS

OF

ALGEBRA.

DEFINITIONS.

1. ALGEBRA is the Science of General Quantity; and, like Arithmetic, with respect to Number, consists both in the knowledge of the properties of Quantity, and of the mode of performing operations upon it. Hence, this Science was denominated, by Sir Isaac Newton, Universal Arithmetic.

2. The characters employed to denote Quantity, are the letters of the Alphabet.

3. The characters employed to denote operations, are certain conventional signs, originally used in this Science alone, but now extended to Arithmetic, Geometry, and all the other branches of the Mathematics.

CHARACTERS OR SYMBOLS OF OPERATION.

4. $=$ This character, consisting of two small parallel lines, is called the *Sign of Equality*, and is used instead of the words *equal to*. Thus, 12 pence$=$1 shilling; 2 added to $4=6$; $2a$ added to $3a=5a$.

5. $+$ This character, resembling an erect cross, is called *plus*, i. e. *more*, or the *Sign of Addition*, and is used instead of the words *added to*. Thus, $2+4=6$; $2a+3a=5a$.

6. $-$ This character, consisting of one small line, is called *minus*, i. e. *less*, or the *Sign of Subtraction*, and is used instead of the words *made less by*. Thus, $6-2=4$; $5a-2a=3a$.

7. $\times$ This character, resembling an oblique cross or the letter X, is called *into*, or the *Sign of Multiplication*, and is used instead of the words *multiplied by*. Thus, $2\times4=8$; $3a\times2=6a$.

8. The multiplication of numbers is denoted frequently by a point. Thus, $2.4=8$; $2.4.6=48$; $2.3a=6a$.

9. The multiplication of quantities, is generally denoted by simple *apposition*, i. e. by *placing the letters together*, chiefly in alphabetical order. Thus, $a\times b=ab$; $a\times b\times c=abc$.

10. The multiplication of compound quantities, is denoted by a parenthesis or bar. Thus, $a(x+y)=ax+ay$; or $a\times\overline{x+y}=ax+ay$.

11. $\div$ This character, consisting of one small line with a point above and below it, is called *by*, or the *Sign of Division*, and is used instead of the words *divided by*. Thus, $8\div2=4$; $6a\div2=3a$.

12. Division is also frequently and more commodiously denoted, by placing the dividend (*i. e.* number to be divided) above, and the divisor below, a small line in the form of a fraction. Thus, $\frac{8}{4}=2$; $\frac{6a}{2}=3a$.

13. The division of compound quantities, is denoted by a parenthesis reversed. Thus, 60)3600(=60; $x+y)ax+ay(=a$.

14. :, : :, :, This character, consisting of eight points, is called the *Sign of Proportion*, and is used instead of the signs $\div$, $=$, $\div$; Because,

15. When the quotient of any two quantities is equal to that of any other two quantities, the four quantities are said to be *proportional*. Thus, since $12\div4=6\div2$, therefore 12 : 4 : : 6 : 2. This last expression is usually read thus, *As* 12 *is to* 4, *so is* 6 *to* 2.

16. A whole number placed, in a smaller character than usual, at the right hand corner of a quantity, denotes that the power of the quantity indicated by that number, is to be found or expressed. Thus, a^1 or a denotes the first power of a; a^2, the second power or square of a; a^3, the third power or cube of a, &c. $(a+b)^2$, the second power or square of $a+b$, &c.

17. A fraction placed, in a smaller character than usual, at the right hand corner of a quantity, denotes that the root of the quantity indicated by its denominator, is to be found or expressed. Thus, $a^{\frac{1}{2}}$ denotes the square root of a; $a^{\frac{1}{3}}$, the cube root of a, &c. $(a+b)^{\frac{1}{2}}$, the square root of $a+b$, &c.

18. The whole numbers or fractions denoting the powers or roots of quantities, are called the *indices* or *exponents* of the quantities. Thus, in the quantities a^2, a^3, $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, the numbers and fractions 2, 3, $\frac{1}{2}$, $\frac{1}{3}$, are the *indices* or *exponents* of a.

19. $\surd$ This character, resembling a manuscript r, is called the *Radical Sign*, and denotes that the square root is to be found. Thus, $\sqrt{a}$ denotes the square root of a; $\sqrt{36}=6$; $\sqrt{a+b}$, the square root of $a+b$.

20. A whole number, placed in a smaller character than usual, above the Radical Sign, denotes that the root indicated by that number, is to be found or expressed. Thus, $\sqrt[3]{a}$ denotes the cube root of a; $\sqrt[4]{a}$, the fourth root of a, &c. $\sqrt[3]{a+b}$, the cube root of $a+b$, &c.

LESS COMMON SYMBOLS OF OPERATION.

21. $\backsim$ This character, resembling the letter S placed horizontally, denotes the difference of two quantities, when it is not known which of them is the greater. Thus, $a \backsim x$ denotes that x is to be subtracted from a, when a is the greater; but that a is to be subtracted from x, when x is the greater.

22. $\pm$ This character, consisting of the signs of Addition and Subtraction combined, denotes either the sum or the difference of the two quantities. Thus, $a \pm x$ denotes that x is either to be added to, or subtracted from, a.

23. $>$ This character, resembling the letter V placed on one side, with the opening to the left, is called the *Sign of Majority*, and is used instead of the words, *greater than*. Thus, $a>b$ denotes that a is greater than b.

24. $<$ This character, resembling the letter V placed on one side, with the opening to the right, is called the *Sign of Minority*, and is used instead of the words, *less than*. Thus, $x<y$ denotes that x is less than y.

25. ∞ This character, resembling the figure 8 placed on one side, is called the *Sign of Infinity*, and denotes that the quantity which is equal to it, is indefinitely great, or greater than any assignable quantity. Thus, $\frac{1}{0}=\infty$.

TERMS.

26. The object of the operations in Algebra, is, in general, to discover the value of unknown quantities, by means of their *relations* to quantities that are known.

27. An *Equation*, is any arrangement of quantities expressing these *relations*.

28. The *Known quantities*, are generally represented by the first letters of the alphabet; as, a, b, c, d, &c.

29. The *Unknown quantities*, are represented by the final letters of the alphabet; as, x, y, z.

30. The *Coefficient* of a quantity, is a number denoting how many times the quantity is to be repeated. Thus, $3a$ denotes 3 times a; $6ab$, 6 times ab; $4x$, 4 times x.

31. The *Coefficient* of an unknown quantity, is often expressed by a known quantity, instead of a number, or by both. Thus, in the quantity ax, a is the coefficient of x; in $3bz$, $3b$ is the coefficient of z.

32. The *Coefficient* of any quantity standing alone, is unity or one. Thus, the coefficient of a is 1; of $\sqrt{a}$ is 1; of xy is 1, &c.

33. *Positive* or *Affirmative quantities*, are those which have the sign *plus* before them; as, $+8a$, $+4ab$, $+7a^2$, &c.

34. *Quantities without any sign*, are always understood to be *positive*. Thus, a, $5a$, $6ab$, are the same as $+a$, $+5a$, $+6ab$.

35. *Negative quantities*, are those which have the sign *minus* before them; as, $-3a$, $-4ab$, $-7a^2$, &c.

36. *Like quantities*, consist of the same letters, with the same indices. Thus, $2a$, $3a$, $5a$, are like; as also, $7ab$, $+4ab$, $-3ab$; and $+2a^2b$, $-9a^2b$.

37. *Unlike quantities*, consist of different letters, or the same letters with different indices. Thus, $5a$, $+5b$, $-7c$, are unlike; as also, $-5a$, $+7a^2$, $3a^3$; and $7a^2b$, $+5ab^2$.

38. *Simple quantities*, consist of one term only; as, $3a$, &c.

39. *Compound quantities*, consist of several terms; as, $3a+2b$, $a+b+c$, &c.

40. *Quantities* having *like signs*, are those which are all positive, or all negative.

41. *Quantities* having *unlike signs*, are those which are partly positive and partly negative.

42. A *Binomial quantity*, consists of two terms; as, $a+b$ or $a-b$. The latter is sometimes called a residual quantity.

43. *Trinomial*, *Quadrinomial*, and *Polynomial* or *Multinomial quantities*, consist of three, four, and many terms respectively; as, $a+b+c$, trinomial; $a+b+c+d$, quadrinomial; $a+b+c+d+e$, &c. multinomial or polynomial.

44. *Factors* of quantities, are the several terms, or multipliers, of which they are composed. Thus, a, b, and c, are the factors of abc; x and $a+x$, are the factors of $ax+x^2$, &c.

45. *Composite quantities*, are such as are produced from factors; thus, $ax+x^2$, is produced from the factors $a+x$, and x; but x^2+y^2 is not producible from any factors, and therefore is not composite.

46. *Multiples* of quantities, are produced by the multiplication of the given quantities by numbers, or by other quantities. Thus, $15a$ is a multiple of $3a$ by 5; $20abc$, of $5a$ by $4bc$.

47. *Measures or divisors* of quantities, are those which will divide the quantities without leaving a remainder. Thus, $7a$ is a measure of $35a$; $4ab$, of $16abc$, &c.

48. *Commensurable quantities*, are such as have the same measure or divisor. Thus, $35a$ and $25a^2$, are commensurable, their common measure being $5a$.

49. *Incommensurable quantities*, are such as have no common measure or divisor. Thus, $7a$ and $48b$, are incommensurable.

50. The *Reciprocals* of quantities, are the quotients of unity by those quantities; the reciprocals of fractions, are those fractions inverted. Thus, the reciprocal of 2 is $\frac{1}{2}$ or .5; of 5 is $\frac{1}{5}$; of $\frac{2}{3}$ is $\frac{3}{2}$; of $\frac{a}{b}$ is $\frac{b}{a}$; &c.

51. The *Powers* of quantities, arise from the continual multiplication of quantities by themselves. Thus, a number or quantity (which is called the *Root*) multiplied by itself, produces the *square* or *second power* of that number or quantity; the square, multiplied by the root, produces the *cube*, or *third power*; the cube multiplied by the root, produces the *biquadrate*, or *fourth power*, &c. as $2\times 2=4$, the square; $4\times 2=8$, the cube; $8\times 2=16$, the fourth power, &c.

52. The *Roots* of quantities, are those from which the powers arise. Thus, the root of the square 64 is 8, because $8\times 8=64$; the root of the cube 8 is 2, because $2\times 2\times 2=8$, &c.

53. *Rational quantities*, are those which have exact roots, or which are expressed without the radical sign, or a fractional index. As, $\sqrt{64}=8$; or a, $5a^2$, $\frac{2}{3}a$, &c.

54. *Irrational quantities*, are those which have no exact roots, or which can only be expressed by the radical sign or a fractional index. As, $\sqrt{2}$ or $2^{\frac{1}{2}}$, $a^{\frac{1}{2}}$, $b^{\frac{1}{3}}$, &c.

55. A *Function* of one or more unknown quantities, is an expression or formula (little form), into which those quantities enter, either combined or not with each other, or with known quantities. Thus, x^3 and x^3+x^2, are functions of x; x^2+y^2 and x^2y+xy^2 are functions of x and y; so are $axy+bx^2$, and $(ax^2-by^3)^{\frac{1}{2}}$.

EQUATIONS.

56. The first object of Algebra, is to obtain an Equation. This is done by performing the operation required in any question, or stating the conditions of any problem, by means of known quantities given, and unknown quantities assumed. Then, by arranging the terms properly, on either side of the Sign of Equality, we obtain an Equation, expressing the relation between the known and the unknown quantities.

57. The second object of Algebra, is to resolve such equations or expressions, so as to obtain the value of the unknown quantity, in terms of the known quantities. This is done by certain rules or cases, which depend on a knowledge of the modes of performing calculations with Algebraic quantities.

58. To these two objects, the whole of the rules of Algebra have a reference, and ought to be made subservient. The rules follow in their order, with the principles on which they are founded, and the practical applications which depend upon them.

GENERAL RULE,
To obtain an Equation.

59. Denote the unknown quantities, or the quantities that are to be found, in any question, by the letters x, y, or z. Then perform with these letters and the known quantities, by means of the common signs and rules, the same operations, according to the nature of the question, that it would be requisite to perform, if the quantities were all known, to prove the truth of the statement contained in the question. The result, will give an expression or equation, from which the value of the unknown quantity may be determined.

60. In most questions, where it can be done, the best rule to be observed, is, to denote only one of the unknown quantities by x or y, and then to find an expression for the rest, from the nature of the question. Then the above general rule may be employed,

with the assistance of any other artifice that may be suggested by the nature of the question, or the rules of the science, to obtain an equation, which may be more easily solved.

61. After an equation has been obtained, certain rules and axioms are necessary, to find the value of the unknown quantity or quantities, and to disengage them from those that are known. By these rules and axioms, the unknown quantity is made to stand alone, on the left-hand side of the equation, and the known quantities, to which it is equal, on the right; the equation is then said to be resolved.

AXIOMS.

62. Some of the axioms necessary for this purpose, will be found prefixed to the Elements of Geometry, (Articles 42 to 48, inclusive,) and to those axioms the following may now be added.

63. If equals be multiplied by the same quantity, the products are equal.

64. If equals be divided by the same quantity, the quotients are equal.

65. If equals be raised to the same power, the powers will be equal.

66. If the same roots of equals be extracted, the roots will be equal.

ADDITION.

67. The first rule necessary to the resolution of equations, is *Addition*, which signifies the collecting of quantities together, according to the nature of their signs, and the result is called their *Sum*.

Rules.

68. If the quantities are like, and have like signs, their sum is found by adding their coefficients; to the amount, the same sign is prefixed, and the same letters are annexed, that belong to the quantities.

69. If the quantities are like, and have unlike signs, their sum is found by taking the difference between the sum of the positive coefficients, and the sum of the negative coefficients; the sign of the greatest sum being prefixed to the difference, and the letters annexed as before.

70. The reason of this is, that when a positive quantity is added to a negative quantity of the same kind, their sum is nothing, because the one destroys the other.

71. If the quantities are unlike, in the letters, or in the powers of the same letters, their sum is found by merely arranging them in a line (generally in the order of the alphabet) one after another, with their proper signs.

Examples.

$$\begin{array}{r} 3a \\ 5a \\ a \\ 7a \\ 14a \\ \hline 30a \end{array} \qquad \begin{array}{r} -4ax \\ -ax \\ -5ax \\ -8ax \\ -7ax \\ \hline -25ax \end{array} \qquad \begin{array}{r} 3b+4y \\ 6b+8y \\ b+3y \\ 7b+y \\ 10b+4y \\ \hline 27b+20y \end{array}$$

$$\begin{array}{r} -3a \\ +7a \\ -7a \\ +10a \\ \hline +7a \end{array} \qquad \begin{array}{r} -7a+3x^2 \\ +2a-5x^2 \\ +\ a-\ x^2 \\ -3a+3x^2 \\ \hline -7a\quad * \end{array} \qquad \begin{array}{r} 5b+4xy \\ -5b-5xy \\ +6b+6xy \\ -6b-8xy \\ \hline *\ -3xy \end{array}$$

$$\begin{array}{c} a \\ b \\ c \\ d \\ \hline a+b+c+d \end{array} \qquad \begin{array}{l} 4ay \\ 5ax \\ -6ay \\ -5ax \\ \hline -2ay \end{array} \qquad \begin{array}{l} 8a^2x^2-3ax \\ 7ax\quad -6xy \\ 9xy\quad -5ax \\ 3a^2x^2+\ xy \\ \hline 11a^2x^2-ax+4xy \end{array}$$

72. If the quantities have letters for their coefficients, instead of numbers, these coefficients are to be placed one after another, with their proper signs, and within a parenthesis; while the common quantity to which they belong, is to be placed without the parenthesis, and generally on its right hand side.

Examples.

$$\begin{array}{c} ax \\ bx \\ cx \\ dx \\ \hline (a+b+c+d)x \end{array} \qquad \begin{array}{c} ax^2-5x \\ -bx^2+4x \\ +cx^2-dx \\ -\ x^2+ax \\ \hline (a-b+c-1)x^2+(a-d-1)x \end{array}$$

Exercises.

1. Add together $2ay$, $5ay$, $4ay$, $7ay$, and $16ay$.

2. Find the sum of $-2by^2$, $-6by^2$, $-by^2$, $-8by^2$, and $-by^2$.

3. Add together $a-2x^2$, $a-6x^2$, $4a-x^2$, $3a-5x^2$, and $7a-x^2$.

4. What is the sum of $-2a^2$, $-3a^2$, $-8a^2$, $+10a^2$, and $+16a^2$.

5. Add together $2ay-7$, $-ay+8$, $+2ay-9$, $-3ay-11$, and $+12ay+13$.

6. Find the sum of $-3ab+7x$, $+3ab-10x$, $+3ab-6x$, $-ab+2x$, and $+2ab+7x$.

7. Find the sum of x, y, and z.

8. Find the sum of $+3x^2y$, $-2xy^2$, $-3y^2x$, $-8x^2y$, and $+2xy^2$.

9. Find the sum of $10b^2$, $-b^2$, 50, a^2x^2, $-3a^2x$, $+2a^2x^2$, $+2a^2x$, and $+120$.

10. Required the sum of $\frac{1}{2}(a+b)$ and $\frac{1}{2}(a-b)$.

11. Add $3a+2b-5$, $a+5b-c$, and $6a-2c+3$, together.

12. Add x^3+ax^2+bx+2 and x^3+cx^2+dx-1 together.

13. Add x^3, x^2, $x^{\frac{1}{2}}$, $-3x^3$, $+x^{\frac{1}{2}}$, $-3x+2x^2$, and $x^4-x^3-x^2-x-1$, together.

EQUATIONS TO BE RESOLVED BY ADDITION.

Example.

Given $2x+3x+4x+5x=70$, to find the value of x.

Here, $2x+3x+4x+5x=14x$, by addition;
Hence, $14x=70$; which being divided by 14,
Gives, $x=5$ (Art. 64).

Exercises.

1. Given $3y+4\frac{1}{2}y+5y+\frac{1}{2}y=260$, to find the value of y.

2. Given $140z+13z+16z+z=350$, to find the value of z.

3. Given $15\frac{1}{4}x+8\frac{3}{4}x=24$, to find the value of x.

73. In the following questions, the general rule (Arts. 59 and 60) must be employed, to which this important rule may be added: Assume that quantity first, which appears to be most independent of the other quantities, and then find expressions for them in terms of the first.

Example.

Divide £300 amongst A, B, and C, so that A may have twice as much as B, and C as much as A and B together.

$$\text{Let B's share} = x,$$
$$\text{Then, A's share} = 2x,$$
$$\text{And C's share} = 3x,$$

Hence, the sum of the shares $=6x=£300$ by the question;

Therefore, $x=£50$ (Art. 64).

Hence, B's share is £50, A's share is $£50\times 2=£100$, and C's share is $£50+100=£150$.

Exercises.

1. A's age is double of B's, and B's is triple of C's, and the sum of all their ages is 140; what is the age of each?

2. My purse and money are together worth 20*s.*, and the money is worth 7 times as much as the purse; how much is in it?

3. A person bought a chaise, horse, and harness, for £60; the price of the horse was twice the price of the harness, and the price of the chaise was twice the price of both; what was the price of each?

4. Two travellers set out at the same time from London and York, the distance of which is 197 miles; one of them travels 16 miles a-day, and the other 14 miles; when will they meet?

5. Divide the number 36 into three such parts, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, shall be all equal to each other.

6. A, B, and C bought a hive for 20*s.*; A agrees to pay $\frac{1}{2}$, B $\frac{1}{3}$, and C $\frac{1}{4}$; what is the proportional share of each?

SUBTRACTION.

74. The second rule necessary for the resolution of equations, is *Subtraction*, which signifies the finding of the difference between any two quantities, according to the nature of their signs; and the result is called the *remainder*, or the *difference*.

Rules.

75. If the quantities are like, and have like signs, their difference is found by subtracting their coefficients; to the remainder, the

same sign if the upper quantity is *greater*, or the *contrary* sign (77), if *less*, is prefixed, and the same letters are annexed, that belong to the quantities.

76. If the quantities are like, and have unlike signs, their difference is found by adding their coefficients, and to the sum, prefixing the sign of the upper quantity, and annexing the letters as before.

77. The reason of this is, that when a positive quantity is subtracted from a negative quantity, their difference is equal to the sum of both considered as negative; because subtracting a positive quantity, is the same as adding a negative quantity. In like manner, when a negative quantity is subtracted from a positive quantity, their difference is equal to the sum of both considered as positive; because subtracting a negative quantity is the same as adding a positive quantity.

78. If the quantities are unlike, their difference is found by arranging the quantities to be subtracted with all their signs changed (namely, from *plus* to *minus*, and from *minus* to *plus*,) one after another, along with the quantities from which they are to be subtracted, in alphabetical order.

79. These rules are all comprehended in the following one: Conceive the quantities *to be subtracted*, which are generally placed in the *lower line*, to have all their signs *changed*, and add according to the rules in Arts. 68, 69, 71, and 72.

Examples.

$$\begin{array}{r} 7a+8d \\ 4a+3d \\ \hline 3a+5d \end{array} \qquad \begin{array}{r} 4a-2b \\ 12a-6b \\ \hline -8a+4b \end{array} \qquad \begin{array}{r} 3a-17x \\ 3a-10x \\ \hline * -7x \end{array}$$

$$\begin{array}{r} -4x-3y \\ +4x-3y \\ \hline -8x \quad * \end{array} \qquad \begin{array}{r} 6x^2-13z \\ -6x^2+5z \\ \hline +12x^2-18z \end{array} \qquad \begin{array}{r} 7ab+35x \\ -6ab-5x \\ \hline +13ab+40x \end{array}$$

$$\begin{array}{r} 5x^2y-3z-4z^2 \\ 6x^2y-5x+7y \\ \hline -x^2y+5x-7y-3z-4z^2 \end{array} \qquad \begin{array}{r} a+d \\ b-c \\ \hline a-b+c+d \end{array}$$

80. If the quantities have *literal* instead of *numeral* coefficients, these coefficients are to be subtracted, that is, placed one after another within a parenthesis, according to the directions given in rule (Art. 78), and the common quantity is to be placed on the right hand side of it.

Examples.

$$\begin{array}{c} ax^2+bx \\ cx^2-dx \\ \hline (a-c)x^2+(b+d)x \end{array} \qquad \begin{array}{c} ax^3-bx^2-x \\ -cx^3+dx^2-ex \\ \hline (a+c)x^3-(b+d)x^2+(e-1)x \end{array}$$

Exercises.

1. Subtract $3xy-8x-7$ from $5xy+8x-2$.
2. Take $-xy+12$ from $5xy-18$.
3. Find the difference between $-y^2+3y+2$ and $8y^2-2y-5$.
4. Take $3a-2x-y+7$ from $4a-3x-3y+8$.
5. Subtract $-b^2-7ax+5x^2$ from $9b^2-2ax+13x^2$.
6. Take $6x^2-8x-x^{\frac{1}{2}}$ from $5x^2-4y+x^{\frac{1}{2}}$.
7. Find the difference of $5a^2-7ax-y^2$ and $20a^2-2ab-z^2$.
8. Subtract $x^3-2ax+a^3$ from $x^3+2ax-a^3$.
9. Subtract $d+e+f$ from $a+b+c$.
10. Take $bx^2+ex-2d$ from ax^3-bx^2+cx+d.
11. What is the difference between $\frac{1}{2}(a+b)$ and $\frac{1}{2}(a-b)$.

EQUATIONS TO BE RESOLVED BY SUBTRACTION.

Example.

Given $16y-3y=36-10$, to find the value of y.

Here, $16y-3y=13y$, and $36-10=26$, by Subtraction;
Hence, $13y=26$; which, being divided by 13,
Gives $y=2$ (Art. 64).

Exercises.

1. Given $50z-10z-5z-3z-2z=360$, to find the value of z.
2. Given $7y-2y-1\frac{1}{2}y-\frac{1}{2}y=19$, to find the value of y.
3. Given $30x-10x-5x=75-40-5$ to find the value of x.

81. In resolving questions where fractions occur, it is sometimes more convenient to assume an unknown quantity with a coefficient which is a multiple of all the denominators of those fractions.

Example.

After paying away $\frac{1}{4}$ of my money, and then $\frac{1}{5}$ of the remainder, I had 72 guineas left; what had I at first?

Here let the money$=20y$, because $4\times5=20$.

Then $\frac{1}{4}$ of it$=\ 5y$

The remainder$=15y$

$\frac{1}{5}$ of which is$=\ 3y$

Money left$=12y=72$ gs. by the question.

Therefore, $y=6$; (Art. 64.)

Hence, the money$=20y=120$ guineas (Art. 63).

Exercises.

1. A post is $\frac{1}{4}$ of its length in the mud, $\frac{1}{3}$ in the water, and 10 feet above the water; what is its whole length?

2. What number is that whose $\frac{1}{3}$ part exceeds its $\frac{1}{5}$ part by 72.

3. A line of a certain length is divided into two parts, so that the one is $\frac{3}{4}$ of the other, and their difference is 12 inches; what is the length of the line, and of each of its parts?

4. A captain being asked how many soldiers he had in his company, replied, $\frac{1}{2}$ of them are in the camp, $\frac{1}{3}$ in the trenches, $\frac{1}{8}$ in the hospital, and four in prison; of how many soldiers did the company consist?

RESOLUTION OF EQUATIONS.

CASE. I. *Rule of Transposition.*

82. Any quantity may be transposed, that is, carried from one side of an equation to the other, by changing its sign, and the two sides of the equation will still be equal. This may be proved either by addition or by subtraction.

Thus, let $z+a=b$; by adding $-a$ to, or subtracting $+a$ from, both sides of the equation, it becomes $z=b-a$ (Arts. 43 and 44 Geometry).

Again, let $z+a-b+c=d$; by adding $-a+b-c$ to, or subtracting $+a-b+c$ from, both sides, the equation becomes $z=d-a+b-c$ (43 and 44 Geometry).

83. In these examples, it is obvious that the known quantities, with their signs changed, are added to both sides of the equation, for the purpose of exterminating, or taking them away from the left hand side, so that the unknown quantity z, may stand alone. Hence, the reason of the foregoing rule (82) is manifest.

CASE II. ***Rule of Changing Signs.***

84. All the quantities of any equation, may have their signs changed, and the two sides will still be equal. This is obvious, for if two quantities are equal when negative, they will also be equal when positive, and *vice versa.*

Thus, let $a-y=b-c$; then by changing signs, the equation becomes, $y-a=c-b$; hence, by transposition, $y=a+c-b$; (82.)

85. It may also be remarked, that if the same quantity with the same sign, be found on both sides of an equation, it may be taken away from both. (44 Geometry.)

Examples.

1. Given $y-4+6=8$, to find the value of y.

Here, $y=8+4-6$; (82.)
And $8+4-6=6$;
Therefore, $y=6$.

2. Given $4y-8=3y+20$, to find x.

Here, $4y-3y=20+8$; (82.)
Also, $4y-3y=y$; and $20+8=28$;
Therefore, $y=28$.

3. A person was desirous of giving 3 pence a-piece to some paupers, but found that he had not enough of money by 8 pence; he therefore gave them 2 pence a-piece, and had 3 pence remaining; required the number of paupers?

Let z=number of paupers,
Then, $3z-8$=the person's money,
Also, $2z+3$=the person's money;
Whence, $3z-8=2z+3$; (Art. 42, Geom.)
Hence, $3z-2z=3+8$; (82.)
Therefore, $z=11$, the number of paupers.

Exercises.

1. Given $3z-2+24=31$, to find the value of z.
2. Given $4-9y=14-11y$, to find the value of y.
3. Given $y+18=3y-5$, to find the value of y.
4. Given $40-6z-16=120-14z$, to find the value of z.
5. A shepherd being asked how many sheep were in his flock, said, if I had as many more, half as many more, and 7 sheep and a half, I should just have 500; how many had he?

6. In a mixture of copper, tin, and lead, $\frac{1}{2}$ of the whole *minus* 16 lb. was copper; $\frac{1}{3}$ of the whole *minus* 12 lb. was tin; and $\frac{1}{4}$ of the whole *plus* 4 lb. was lead; how much of each metal was in the mixture?

7. A fish was caught, the tail of which weighed 9 lb. Its head weighed as much as its tail and half its body; and its body weighed as much as its head and tail; what was the whole weight of the fish?

8. A person at a tavern borrowed as much money as he had in his pocket, and spent 1*s*; he then went to a second tavern, borrowed as much as he had remaining, and spent another 1*s*; in this manner, he went to a third and a fourth tavern, and after spending a shilling at the last, he had nothing left; how much money had he at first?

9. In a mixture of spirits and water, $\frac{1}{2}$ of the whole plus 25 gallons was spirits, and $\frac{1}{3}$ minus 5 gallons was water; how many gallons of spirits and how many of water, were in the mixture.

10. At an election 1296 persons voted, and the successful candidate had a majority of 120; how many voted for each?

MULTIPLICATION.

86. The third rule necessary for the resolution of equations, is *Multiplication*, which signifies the finding the product of two or more quantities according to the nature of their signs; the quantities to be multiplied together, are called the *Factors*, of which the one is called the *Multiplicand*, and the other the *Multiplier;* the result is called the *Product.*

Rules.

87. When the quantities have like signs, the sign of their product is always *plus.* For, when a positive quantity is multiplied by a positive quantity, the former is to be added as often as there are units in the latter, because the sign of the multiplier is *plus.* And, when a negative quantity is multiplied by a negative quantity, the former is to be subtracted as often as there are units in the latter, because the sign of the multiplier is *minus*, and subtracting a negative quantity is the same as adding a positive one. (Art. 77.)

88. When the quantities have unlike signs, the sign of their product is always *minus.* For, when a negative quantity is mul-

tiplied by a positive quantity, the former is to be added as often as there are units in the latter, because the sign of the multiplier is *plus*. And, when a positive quantity is multiplied by a negative quantity, the former is to be subtracted as often as there are units in the latter; because the sign of the multiplier is *minus*, and subtracting a positive quantity is the same as adding a negative one (Art. 77).

89. The coefficients of the factors are always to be multiplied together, to obtain the *coefficient* of the product.

90. When the quantities consist of the *same* letters, with the same, or with different indices, their product is found by *adding* together their *indices*, and making the sum the index of the common letter.

91. When the quantities consist of different letters, their product is expressed by *placing them together*, in alphabetical order. (Art. 9.)

92. Whether the factors be simple or compound quantities, every term of the multiplicand must be multiplied by every term of the multiplier; and if the latter consists of more than one term, its products will consist of several lines, which must be all added together to obtain the complete product, as in arithmetic. The operation in such cases, generally begins at the left, and proceeds towards the right.

Examples.

$a\times a=a^2$. $-a^2\times -a=a^3$. $-a^2\times a^3=-a^5$.

$12a\times -3b=-36ab$. $+3a\times +4a=+12a^2$.

$-6ab\times -4cd=+24abcd$. $2x^2\times 3x^3=6x^5$.

$$\begin{array}{l}3a-2b\\4a\\\hline 18a-8b\end{array}\qquad\begin{array}{l}13a^2-c^2b\\-2a\\\hline -26a^3+2ac^2b\end{array}\qquad\begin{array}{l}a^2-2n+1\\4e\\\hline 4a^2e-8en+4e\end{array}$$

$$\begin{array}{l}a+b\\a+b\\\hline a^2+ab\\\quad +ab+b^2\\\hline a^2+2ab+b^2\end{array}\qquad\begin{array}{l}a^2-ab+b^2\\a+b\\\hline a^3-a^2b+ab^2\\\quad +a^2b-ab^2+b^3\\\hline a^3\quad * \quad * \quad +b^3\end{array}$$

$$\begin{array}{l} 5x+4y+1 \\ 3x-2y-1 \\ \hline 15x^2+12xy+3x \\ \quad -10xy-8y^2-2y \\ \qquad -5x-4y-1 \\ \hline 15x^2-2xy-2x-8y^2-6y-1 \\ \hline \hline \end{array} \qquad \begin{array}{l} a+b+c \\ a+b+c \\ \hline a^2+ab+ac \\ \quad +ab+b^2+bc \\ \qquad +ac+bc+c^2 \\ \hline a^2+2ab+2ac+b^2+2bc+c^2 \\ \hline \hline \end{array}$$

93. When the quantities to be multiplied have literal coefficients or indices, the coefficients are to be multiplied, and the indices added, in the same manner as other quantities.

Examples.

$$\begin{array}{l} ax^2+bx \\ cx^2-dx \\ \hline acx^4+bcx^3 \\ \quad -adx^3-bdx^2 \\ \hline acx^4+(bc-ad)x^3-bdx^2 \\ \hline \end{array} \qquad \begin{array}{l} x^n+y^m \\ x^m+y^n \\ \hline x^{m+n}+x^m y^m \\ \quad +x^n y^n+y^{m+n} \\ \hline x^{m+n}+x^m y^m+x^n y^n+y^{m+n} \\ \hline \end{array}$$

Exercises.

1. Multiply $-6x$ by $5a$; $7ab$ by $-5ac$; $-4a^2b^2$ by $-6a^3b^3$; $-6xyz$ by $+ay^2z$; and $-a^2xy$ by $2xy^2$.
2. Multiply $35x^2-7a$ by $-2x$; and $25xy+3a^2$ by $13x^2$.
3. Multiply $7a^2-3b+4c-8d+7e$ by $-11ax$.
4. Multiply $x-y$ by $x+y$; and x^2+xy-y^2 by $x-y$.
5. What is the product of $a^3+a^2b+ab^2+b^3$ and $a-b$.
6. Find the product of a^2+ab+b^2 and a^2-ab+b^2.
7. Multiply $3a^2-2ad+5$ and $a^2+2ad-3$ together.
8. Multiply $2a^2-3ax+4x^2$ by $5a^2-6ax-2x^2$.
9. What is the product of $a^5+a^4+a^3+a^2+a+1$ by $a-1$.
10. Find the product of x^3-ax^2+bx-c and x^2-dx+c.
11. Multiply ayz by $a^{-1}y^{-1}x^{-1}$, and $a^n x^2$ by $a^2 x^n$.

Equations solvable by Multiplication.

Example.

A person on his wedding day, was thrice as old as his wife; but on the fifteenth anniversary of that event, it happened that he was only twice as old; what were the ages of the bride and bridegroom?

Let x=bride's age,
Then, $3x$=bridegroom's age, per question;
Hence, $x+15$=the wife's age, per question,
And $3x+15$=the husband's age, per question;
Therefore, $3x+15=2\times(x+15)$ per question;
Now, $2\times(x+15)=2x+30$, by multiplication;
Hence, $3x+15=2x+30$;
And $3x-2x=30-15$, by transposition;
Therefore, $x=15$, the bride's age;
And $3x=45$, the bridegroom's age.

Exercises.

1. Two persons, A and B lay out equal sums of money in trade; A gains £126, and B loses £87; A's money is now double of B's; how much was the equal sum laid out?

2. Divide the number 75 into two such parts, that three times the greater shall exceed seven times the less by 15.

3. A rectangular board is 9 inches in breadth, and equal to a square foot in area; how much greater than 12 inches is it in length?

4. Two boards are equal in area; the breadth of the one is 18 inches, and that of the other 16 inches, and the difference of their lengths is 3 inches; what is the length of each, and the area of both?

5. Six gamesters, A, B, C, D, E, and F, sat down to play, with the following sums on the table: A £68, B £73, C £83, D £86, E £84, and F £61. A constable happening to make his appearance amongst them, each of the fellows seized as much as he could, and made his escape. When they met again after the hurry was over, it appeared that if each were to give up the following parts of what he had seized in the scramble, namely, A $\frac{1}{2}$, B $\frac{2}{3}$, C $\frac{3}{4}$, D $\frac{4}{5}$, E $\frac{5}{6}$, and F $\frac{6}{7}$, and the amount were equally divided among them, then every one would have his own money again; how much did each man seize?

DIVISION.

94. The fourth rule necessary for the resolution of equations, is *Division*, which signifies the finding how often one quantity is

contained in another, regard being had to the nature of the signs; the quantity to be divided is called the *Dividend*, the quantity by which it is to be divided is called the *Divisor*, and the result is called the *Quotient*.

Rules.

95. When the quantities have like signs, the sign of the quotient is always *plus*. For, when a positive quantity is multiplied by a positive quantity, the product is positive (87); hence, when that product is divided by a positive quantity, the quotient must be positive. And, when a negative quantity is multiplied by a *positive* quantity, the product is negative (88); hence, when that product is divided by the negative quantity, the quotient must be *positive*.

96. When the quantities have unlike signs, the sign of the quotient is always *minus*. For, when a *negative* quantity is multiplied by a positive quantity, the product is negative (88); hence, when that product is divided by the positive quantity, the quotient must be *negative*.

97. The coefficient of the dividend is always to be divided by the coefficient of the divisor, to obtain the *coefficient* of the quotient.

98. When the quantities consist of the *same* letters, with different indices, their quotient is found by *subtracting* the *index* of the divisor from that of the dividend, and making their difference the index of the common letter.

99. If the quantities have the same index, their quotient is unity; hence, such quantities cancel each other, but the quotient of their coefficients, or of the other quantities is to be found.

100. When the quantities consist of different letters, their quotient is expressed by placing them in the form of a fraction, with the dividend for numerator, and the divisor for denominator (12).

101. When the divisor is a simple quantity, and the dividend either simple or compound, it is to be divided by the divisor according to the above rules.

102. When both the divisor and dividend are compound, then the first term of the dividend is to be divided by the first term of the divisor, and the result is the first term of the quotient. All the terms of the divisor are then multiplied by this first term of the quotient, the product is placed under the similar and corresponding terms in the dividend, subtracted from them, and the next term of the dividend brought down to the remainder; the division is then carried on as before, like common arithmetic.

Examples.

$a^2 \div a = a.$ $\qquad \dfrac{a^3}{-a^2} = -a.$ $\qquad \dfrac{-a}{a} = -1.$ $\qquad \dfrac{a^7}{a^3} = a^4.$

$\dfrac{12ax^2}{-3x} = -4ax.$ $\qquad \dfrac{36ab}{3a} = 12b.$ $\qquad \dfrac{-14a^2x^2}{-3ax} = +4\tfrac{2}{3}ax.$

$$5a)10ab-15ax \atop \overline{2b-3x} \qquad -6x)30ax-18x^2 \atop \overline{-5a+3x} \qquad 7a)14a^3-21ab-7ac \atop \overline{2a^2-3b-c}$$

$$\begin{array}{l}
x+y)x^2+2xy+y^2(x+y \\
\qquad x^2+xy \\
\qquad \overline{} \\
\qquad * \quad xy+y^2 \\
\qquad \quad xy+y^2 \\
\qquad \quad \overline{} \\
\qquad \quad * \quad *
\end{array}
\qquad
\begin{array}{l}
a-b)a^2-b^2(a+b \\
\qquad a^2-ab \\
\qquad \overline{} \\
\qquad * +ab-b^2 \\
\qquad +ab-b^2 \\
\qquad \overline{} \\
\qquad \quad * \quad *
\end{array}$$

$$\begin{array}{l}
a+b)a^4-b^4(a^3-a^2b+ab^2-b^3 \\
\qquad a^4+a^3b \\
\qquad \overline{} \\
\qquad * -a^3b-b^4 \\
\qquad -a^3b-a^2b^2 \\
\qquad \overline{} \\
\qquad \qquad * +a^2b^2-b^4 \\
\qquad \qquad +a^2b^2+ab^3 \\
\qquad \qquad \overline{} \\
\qquad \qquad \qquad * -ab^3-b^4 \\
\qquad \qquad \qquad -ab^3-b^4 \\
\qquad \qquad \qquad \overline{} \\
\qquad \qquad \qquad \quad * \quad *
\end{array}$$

$$\begin{array}{l}
2a^2-3ab+b^2)4a^4-9a^2b^2+6ab^3-b^4(2a^2+3ab-b^2 \\
\qquad \qquad 4a^4-6a^3b+2a^2b^2 \\
\qquad \qquad \overline{} \\
\qquad \qquad * +6a^3b-11a^2b^2+6ab^3 \\
\qquad \qquad +6a^3b-9a^2b^2+3ab^3 \\
\qquad \qquad \overline{} \\
\qquad \qquad \qquad * \quad -2a^2b^2+3ab^3-b^4 \\
\qquad \qquad \qquad \quad -2a^2b^2+3ab^3-b^4 \\
\qquad \qquad \qquad \quad \overline{} \\
\qquad \qquad \qquad \qquad * \qquad * \qquad *
\end{array}$$

103. When the quantities to be divided have literal coefficients or indices, the coefficients are to be divided and the indices subtracted, in the same manner as other quantities.

Examples.

$$\begin{array}{l} ax+b)acx^2+(bc+ad)x+bd(cx+d \\ \quad\quad acx^2+bcx \\ \hline \quad\quad\quad * \quad\quad +adx+bd \\ \quad\quad\quad\quad\quad\;\; +adx+bd \\ \hline \quad\quad\quad\quad\quad\quad\;\; * \quad\quad * \end{array}$$

$$\frac{x^m+x^n y^n}{x^n}=x^{m-n}+y^n$$

Exercises.

1. Divide $16a^2$ by $8a$, and $-12a^3x^2$ by $4a^2x^2$.
2. Divide $-15ay^2$ by $-3ay$, and $+18ax^2y$ by $8ax$.
3. Divide $3a^3+6a^2+3ab-15a$ by $3a$.
4. Divide $3abc+12abz-9a^2b$ by $3ab$.
5. Divide $15a^2bc-12acx^2+5ac$ by $-5ac$.
6. Divide $a^2-2az+z^2$ by $a-z$, and $2y^3-19y^2+26y-17$ by $y-8$.
7. What is the quotient of $a^3+5a^2x+5ax^2+x^3$ by $a+x$.
8. Find the quotient of x^5+1 by $x+1$, and of x^6-1 by $x-1$.
9. Divide $48a^3-76a^2b-64ab^2+105b^3$ by $2a-3b$.
10. Divide $4e^4-9e^2+6e-3$ by $2e^2+3e-1$.
11. Divide x^6-a^6 by $x-a$, and $a^5+4a^2b+8b^4$ by $a+2b$.
12. Divide abc by $a^{-1}b^{-1}c^{-1}$, and a^nb^2 by a^2b^n.

Equations solvable by Division.

Example.

A servant agreed to live with his master for £8 a-year and a livery, but left him at the end of 7 months, and received only £2 : 13 : 4 and the livery; what was its value?

Here, let $12x$=the livery, £8=1920d. and £2 : 13 : 4=640d.

Then, $12x+1920$=a year's wages.

And $\dfrac{12x+1920}{12}=x+160$=a month's wages;

Hence, $(x+160)\times 7=7x+1120$=seven months' wages;

But $12x+640$=seven months' wages by the question;

Hence, $12x+640=7x+1120$ (42 Geometry);

And $12x-7x=1120-640$, by transposition;

Therefore, $5x=480$, and $x=96$ (64);

Hence, $12x=1152d.$=£4 : 16, the livery.

Exercises.

1. A person left £560 between his son and daughter, to be divided in such a manner, that, for every half crown the son receives, the daughter shall receive a shilling; what were the shares of each?

2. Two Persons A and B, have both the same income; A saves $\frac{1}{5}$ of his yearly, but B by spending £50 a year more than A, at the end of 4 years finds himself £100 in debt; what is their income?

3. A person has two horses, and a saddle worth £50; if the saddle be put on the first horse, it will make its value double that of the second; but if it be put on the second, it will make its value triple that of the first; what is the value of each horse?

4. A person in play lost a fourth of his money, and then won back 3*s*; he again lost a third of what he now had, and then won back 12*s*; lastly he lost a seventh of what he then had, and found he had 24*s*. remaining; what had he at first?

5. It is required to divide the number 90 into four such parts, that if the first be increased by 2, the second diminished by 2, the third multipled by 2, and the fourth divided by 2, the sum, difference, product, and quotient, shall all be equal?

6. The quotient and remainder of a number in division, are each 21; and the divisor is 7 less than their sum; what is the dividend or number to be divided?

RESOLUTION OF EQUATIONS.

Case III. *Rule for clearing of Factors or Divisors.*

104. If the unknown quantity in any equation be affected with a factor, multiplier, or coefficient, it may be taken away by dividing all the rest of the terms by it; and if the unknown quantity be affected with a divisor, it may be taken away by multiplying all the other terms by it. The first part of this rule evidently depends on Art. 64; and the second on Art. 63, and the obvious principle, that if a quantity be multiplied and then divided by the same quantity or number, it will remain the same.

Examples.

1. Given $ax=3ab-c$ to find the value of x.

$$a)ax=3ab-c$$

$$x=3b-\frac{c}{a}$$

2. Given $6ax^3-12abx^2=3ax^3+6ax^2$, to find the value of x.

$$3ax^2)\underline{6ax^3-12abx^2=3ax^3+6ax^2}$$

$$2x-4b=x+2$$

Hence, $2x-x=4b+2$, by transposition;

Therefore, $x=4b+2$.

3. Given $\frac{x}{a}+c=b$, to find the value of x.

Here, $(\frac{x}{a}+c=b)\times a=x+ac=ab$;

And $x=ab-ac$, by transposition.

4. Given $x+a=\frac{x^2}{x+a}$, to find the value of x.

Here, $(x+a)\times(x+a)=x^2$, by the rule;

Hence, $x^2+2ax+a^2=x^2$, by multiplying;

Therefore, $2ax+a^2=0$ (85);

Hence, $2ax=-a^2$, by transposition;

Therefore, $x=-\frac{a}{2}$, by the rule.

105. *Cor.* From this rule it is also obvious, that if the same number or quantity be found in all the terms of an equation, either as a multiplier or divisor, that it may be cancelled from each term, and both sides of the equation will still be equal.

Examples.

Thus, if $ax=ab+ac$, then $x=b+c$ by cancelling a.

And if $\frac{z}{y}=\frac{b}{y}+\frac{c}{y}$, then $z=b+c$ by cancelling y.

Exercises.

1. Given $5ay-3b=2ay+c$, to find the value of y.
2. Given $\frac{y}{3a}+5b=6c$, to find the value of y.
3. Given $\frac{2y}{1+b}=1-b$, to find the value of y.
4. Given $\frac{3ay+by}{a-b}=3a+b$, to find the value of y.

106. *Note.* In this last equation, as in many others, where the unknown quantity is affected by different coefficients, it is usual to put them within a parenthesis, and then to annex the unknown quantity. Thus, $3ay+by=(3a+b)y$. (10.)

ALGEBRAIC FRACTIONS.

107. A knowledge of Algebraic fractions being necessary to the resolution of equations, though the rules of operation are nearly the same as in Arithmetic, yet they are here repeated shortly, for the sake of reference and connection.

CASE I. *To find the greatest common measure of a Fraction.*

108. *Rule.* Divide the greater term by the less, the less by the remainder, this remainder by the next, and so on successively, till there be no remainder left; the last divisor is the greatest common measure.

109. *Note.* 1. If, previous to the division, or during any step of its progress, there exist any common factor in all the terms of the divisor, cancel it; that is, divide all those terms by it, before farther operation. 2. If any of the divisors become negative, they may be made positive without affecting the result. 3. If the first term of the dividend does not contain the first term of the divisor exactly, in those cases where there is a common measure, the dividend may be multiplied by any number or quantity that will render its first term exactly divisible by the first term of the divisor.

CASE II. *To reduce a Fraction to its lowest terms.*

110. *Rule.* Find the greatest common measure (108), and divide both numerator and denominator by it, the quotients will be the fraction in its lowest terms.

Examples.

2. Find the greatest common measure and lowest terms of the Fraction $\frac{y^4-1}{y^5+y^3}$.

$$y^4-1)y^5+y^3(y$$
$$y^5-y$$
$$y)y^3+y$$

Greatest common measure, $y^2+1)y^4-1(y^2-1$

$$y^4+y^2$$
$$-y^2-1$$
$$-y^2-1$$
$$* \quad *$$

$y^2+1)\frac{y^4-1}{y^5+y^3}=\frac{y^2-1}{y^3}$, lowest terms.

2. Find the greatest common measure and lowest terms of the fraction, $\dfrac{3a^2-2a-1}{4a^3-2a^2-3a+1}$.

$$
\begin{array}{l}
3a^2-2a-1)4a^3-2a^2-3a+1(4a+2 \\
\qquad\qquad\qquad 3 \\
\hline
\qquad\qquad\qquad 12a^3-6a^2-9a+3 \\
\qquad\qquad\qquad 12a^3-8a^2-4a \\
\hline
\qquad\qquad\qquad\qquad 2a^2-5a+3 \\
\qquad\qquad\qquad\qquad 3 \\
\hline
\qquad\qquad\qquad\qquad 6a^2-15a+9 \\
\qquad\qquad\qquad\qquad 6a^2-4a-2 \\
\hline
\qquad\qquad\qquad -11)-11a+11 \\
\end{array}
$$

$$
\begin{array}{l}
\text{Greatest common measure, } a-1)3a^2-2a-1(3a+1 \\
\qquad\qquad\qquad\qquad\qquad\qquad 3a^2-3a \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\quad +a-1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad +a-1 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\quad * \quad *
\end{array}
$$

$$a-1)\frac{3a^2-2a-1}{4a^3-2a^2-3a+1}=\frac{3a+1}{4a^2+2a-1}, \text{ lowest terms.}$$

Exercises.

Find the greatest common measure and lowest terms of the fractions, $\dfrac{x^3-a^3}{x^4-a^4}$, $\dfrac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$, $\dfrac{x^4+a^2x^2+a^4}{x^4+ax^3-a^3x-a^4}$, $\dfrac{2x^3-16x-6}{3x^3-24x-9}$, and $\dfrac{7a^2-23ab+6b^2}{5a^3-18a^2b+11ab^2-6b^3}$.

CASE III. *To reduce a whole or mixed quantity to an improper Fraction.*

111. *Rule I.* A whole quantity is made an improper fraction, by putting unity for its denominator.

112. *Rule II.* When the quantity is mixed, that is, partly whole and partly fractional, multiply the whole or integral quantity, by the denominator of the fraction, and to the product, *add* the numerator when the *fraction is affirmative,* or *subtract* it, when *negative;* place the denominator under the result, for the improper fraction required.

Examples.

1. Reduce $3\frac{5}{8}$ and $a+\frac{b}{c}$ to improper fractions.

Here, $3\frac{5}{8}=\frac{29}{8}$; because $(3\times 8)+5=29$.

And $a+\frac{b}{c}=\frac{ac+b}{c}$; because $(a\times c)+b=ac+b$.

2. Reduce $x-\frac{a^2-x^2}{x}$ and $1+b+\frac{a^2-bc}{1-c}$ to improper fractions

Here, $x-\frac{a^2-x^2}{x}=\frac{2x^2-a^2}{x}$; because $(x\times x)-a^2+x^2=2x^2-a^2$. (76.)

And $1+b+\frac{a^2+bc}{1-c}=\frac{1+b-c+a^2}{1-c}$; because $\overline{(1+b)}\times\overline{(1-c)}+a^2+bc=1+b-c-bc+a^2+bc=1+b-c+a^2$.

Exercises.

1. Reduce $1-\frac{2x}{a}$ and $5a+\frac{3x-b}{a}$ to improper fractions.

2. Reduce $x-\frac{ax+x^2}{2a}$ and $5+\frac{2x-7}{3x}$ to improper fractions.

3. Reduce $1-\frac{x-a-1}{a}$ and $1+2x+\frac{x-3}{5x}$ to improper fractions

CASE IV. *To reduce an improper Fraction to a whole or mixed quantity.*

113. *Rule.* Divide the numerator by the denominator, and the quotient will be the quantity required. If there be a remainder, place the denominator under it, and annex it to the quotient, with its proper sign.

Examples.

1. Reduce $\frac{27}{5}$ and $\frac{ax+x^2}{x}$ to whole or mixed quantities.

Here, $\frac{27}{5}=5\frac{2}{5}$; and $\frac{ax+x^2}{x}=a+x$.

2. Reduce $\frac{a^2-y^2}{a+y}$ and $\frac{a^3-b^3}{a-b}$ to whole or mixed quantities.

Here, $\frac{a^2+y^2}{a+y}=a-y+\frac{2y^2}{a+y}$; and $\frac{a^3-b^3}{a-b}=a^2+ab+b^2$.

Exercises.

1. Reduce $\frac{ab-2a^2}{ab}$ and $\frac{a^2-b^2}{a-b}$ to whole or mixed quantities.

2. Reduce $\frac{10x^2-5x+3}{5x}$ and $\frac{x^4-y^4}{x^2+y^2}$ to whole or mixed quantities.

3. Reduce $\frac{a^3+3a^2z+3az^2+z^3}{a^2+2az+z^2}$ to a whole quantity.

CASE V. *To reduce Fractions to others of equal value, having a common denominator.*

114. *Rule.* Multiply each fraction by the denominators of all the rest successively, and the products will be the equivalent fractions required.

115. The reason of this rule is, that if both the numerator and denominator of a fraction be multiplied by the same number or quantity, its value remains the same.

Examples.

1. Reduce the fractions $\frac{1}{2}$ and $\frac{2}{3}$, and the fractions $\frac{a}{b}$ and $\frac{c}{d}$ to others, having a common denominator.

$$\frac{1}{2}\times 3=\frac{3}{6} \qquad\qquad \frac{a}{b}\times d=\frac{ad}{bd}$$

$$\frac{1}{3}\times 2=\frac{2}{6}. \qquad\qquad \frac{c}{d}\times b=\frac{bc}{bd}.$$

2. Reduce $\frac{a+b}{c}$, $\frac{d}{a+b}$, and $\frac{3x}{4y}$, to fractions having a common denominator.

$$\frac{a+b}{c}\times a+b\times 4y=\frac{4a^2y+8aby+4b^2y}{4acy+4bcy}$$

$$\frac{d}{a+b}\times c\times 4y\quad=\frac{4cdy}{4acy+4bcy}$$

$$\frac{3x}{4y}\times c\times a+b\quad=\frac{3acx+3bcx}{4acy+4bcy}$$

Exercises.

1. Reduce $\frac{2x}{a}$ and $\frac{c}{d}$ to fractions having a common denominator.

2. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$, and $\frac{a+d}{5x}$, to fractions having a common denominator.

3. Reduce $\frac{a}{2}$, $\frac{3b}{3c}$, and $\frac{a+x}{a-x}$, to fractions having a common denominator.

4. Reduce a and $\frac{a+d}{2}$ to fractions having a common denominator.

CASE VI. *To add Fractions.*

116. *Rule.* Reduce the fractions to others having a common denominator, if their denominators are not the same; then add the numerators, and under the sum, place the common denominator, for the sum of the fractions required. If the quantities are mixed, it is easiest, in general, to add the whole quantities by themselves, and prefix their sum to the sum of the fractions, as found by the rule.

Examples.

1. Add $\frac{x}{2}$, $\frac{x}{3}$, and $\frac{x}{4}$, together.

Numerators.

$\frac{x}{2}\times 3\times 4=\frac{12x}{24}$ .... $12x$

$\frac{x}{3}\times 2\times 4=\frac{8x}{24}$ .... $8x$

$\frac{x}{4}\times 2\times 3=\frac{6x}{24}$ .... $6x$

Sum of the fractions $=\frac{26x}{24}=1\frac{1}{12}x$.

2. Find the sum of $x+\frac{a}{b}$, $y+\frac{c}{d}$, and $z-\frac{e}{f}$.

$$\begin{array}{ll} \text{Here, } x & \text{And} +\frac{a}{b}\times d\times f=+\frac{adf}{bdf} \\ \quad\;\; y & \\ \quad\;\; z & \quad\;\; +\frac{c}{d}\times b\times f=+\frac{bcf}{bdf} \\ \hline x+y+z & \quad\;\; -\frac{e}{f}\times b\times d=-\frac{bde}{bdf} \\ \hline \end{array}$$

Now, $adf+bcf-bde$=sum of numerators.

Hence, $x+y+z+\frac{adf+bcf-bde}{bdf}$=sum required.

Exercises.

1. Add together the fractions, $\frac{4a}{7}$ and $\frac{b-2}{5}$.

2. Find the sum of the fractions, $\frac{a}{4}$, $\frac{a}{5}$, and $\frac{a}{6}$.

3. Find the sum of $2a$, $3a+\frac{2b}{5}$, and $a-\frac{8b}{9}$.

4. What is the sum of $5a+\frac{a-2}{3}$ and $4a-\frac{2a-3}{5a}$.

5. Find the sum of $5x$, $\frac{2a}{3x^2}$, and $\frac{a+2x}{4x}$.

CASE VII. *To subtract one Fraction from another.*

117. *Rule.* Reduce the fractions to others having a common denominator, if not so given; then subtract the numerator of the one from the numerator of the other, and under the difference, place the common denominator, for the difference of the fractions required. If the quantities are mixed, find the difference of the whole quantities, and prefix it to that of the fractions as found by the rule.

Examples.

1. Find the difference between $\frac{a}{b}$ and $\frac{c}{d}$.

$$\begin{array}{ll} & \textbf{Numerators.} \\ \frac{a}{b}\times d=\frac{ad}{bd}\text{..............} & ad \\ \frac{c}{d}\times b=\frac{bc}{bd}\text{..............} & bc \\ \hline \end{array}$$

Difference of the fractions, $\frac{ad-bc}{bd}$.

2. Find the difference between $a+\frac{x-a}{2b}$ and $2a-\frac{2a-4x}{3c}$.

Here, $2a$ And $-\frac{2a-4x}{3c}\times 2b=-\frac{4ab-8bx}{6bc}$

$\underline{a}$ $+\frac{x-a}{2b}\times 3c=+\frac{3cx-3ac}{6bc}$

$\underline{a}$

Now, $-(4ab-8bx)-(3cx-3ac)=-(4ab-8bx+3cx-3ac)$ difference of the numerators;

Hence, $a-\frac{4ab-8bx+3cx-3ac}{6bc}$=difference required.

Exercises.

1. Find the difference of $\frac{12x}{7}$ and $\frac{3x}{5}$.
2. Find the difference of $15y$ and $\frac{1+2y}{8}$.
3. What is the difference between $\frac{ax}{b-c}$ and $\frac{ax}{b+c}$.
4. Subtract $x-\frac{x-a}{c}$ from $x+\frac{x}{2b}$.
5. Take $a+\frac{a-x}{a(a+x)}$ from $ax+\frac{a+x}{a(a-x)}$.

CASE VIII. *To multiply Fractions together.*

118. ***Rule.*** Multiply the numerators together for the numerator of the product, and the denominators together for the denominator of the product.

119. To multiply a fraction by a whole number, or quantity. Multiply the numerator only by the multiplier, or divide the denominator by it; if the denominator of the fraction be the *same* as the multiplier, the product is the *numerator*.

120. When the quantities are mixed, reduce them to improper fractions, and then multiply according to the rule (118).

Examples.

1. Multiply $\frac{a}{b}$ by $\frac{c}{d}$. Here, $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$, product.

2. Multiply $\frac{x}{a}$ and $\frac{a+x}{a-x}$ together.

Here, $\frac{x}{a}\times\frac{a+x}{a-x}=\frac{ax+x^2}{a^2-ax}$, product.

3. Multiply $\frac{x}{4}$ by 4, and $\frac{x}{12}$ by 3.

Here, $\frac{x}{4}\times\frac{4}{1}=\frac{4x}{4}=x$; or $\frac{x}{4\div4}=\frac{x}{1}=x$.

Also, $\frac{x}{12}\times\frac{3}{1}=\frac{3x}{12}=\frac{x}{4}$; or $\frac{x}{12\div3}=\frac{x}{4}$.

Exercises.

1. Find the product of $\frac{2a}{5}$ and $\frac{3b^2}{2x}$.
2. Find the product of $\frac{2x}{3}$, $\frac{4x^2}{7}$, and $\frac{a}{a+x}$.
3. Multiply $3x$, $\frac{x+1}{2a}$, and $\frac{x-1}{a+b}$, together.
4. What is the product of $2a+\frac{bx}{a}$ and $3a-\frac{b}{ax}$.
5. Multiply $\frac{a^2-x^2}{a+b}$, $\frac{a^2-b^2}{ax+x^2}$, and $a+\frac{ax}{a-x}$, together.

CASE IX. *To divide one Fraction by another.*

121. ***Rule.*** Multiply the dividend by the reciprocal of the divisor (50), the product is the quotient required.

122. To divide a fraction by a whole number or quantity. Divide the numerator of the fraction by the divisor, or multiply the denominator by it.

123. When the quantities are mixed, reduce them to improper fractions, and then divide according to the rule (121).

Examples.

1. Divide $\frac{a}{b}$ by $\frac{c}{d}$. Here, $\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$, quotient.

2. Divide $\frac{x+a}{x-b}$ by $\frac{x+b}{5x+a}$.

Here, $\frac{x+a}{x-b}\times\frac{5x+a}{x+b}=\frac{5x^2+6ax+a^2}{x^2-b^2}$, quotient.

3. Divide $\frac{12x}{a}$ by 3, and $\frac{4a}{4b}$ by $4a$.

Here, $\frac{12x}{a} \div \frac{3}{1} = \frac{4x}{a}$; or, $\frac{12x}{a \times 3} = \frac{12x}{3a} = \frac{4x}{a}$.

And $\frac{4a}{4b} \div \frac{4a}{1} = \frac{1}{4b}$; or, $\frac{4a}{4b \times 4a} = \frac{4a}{16ab} = \frac{1}{4b}$.

Exercises.

1. Divide $\frac{7a}{5}$ by $\frac{3}{a}$, and $\frac{4a^2}{7}$ by $5b$.

2. What is the quotient of $\frac{x+1}{6}$ and $\frac{2x}{3}$.

3. Find the quotient of $\frac{a}{1-a}$ and $\frac{a}{5}$.

4. Divide $\frac{2ax+x^2}{c^3-x^3}$ by $\frac{x}{c-x}$, and a by b.

5. What is the quotient of $a+\frac{a}{1-a}$ by $b-\frac{a}{1-a}$.

6. Find the sum, difference, product, and quotient of $\frac{4a}{5b}$ and $\frac{6c}{7d}$.

RESOLUTION OF EQUATIONS.

CASE IV. *Rule for clearing of Fractions.*

124. An equation is cleared of fractions by multiplying all the terms by each of the denominators successively, or by a multiple of all the denominators (63 and 119).

Examples.

1. Thus, if $\frac{y}{a}+\frac{y}{b}+\frac{y}{c}=d$; then, multiplying by a,

Gives $y+\frac{ay}{b}+\frac{ay}{c}=ad$; again, multiplying by b,

Gives $by+ay+\frac{aby}{c}=abd$; lastly, multiplying by c,

Gives $bcy+acy+aby=abcd$;

Whence, $(bc+ac+ab)y=abcd$;

And $y=\frac{abcd}{bc+ac+ab}$.

2. Also, if $\frac{y-3}{2}+\frac{y}{3}=20-\frac{y-19}{2}$; then, multiplying by 6, which is a multiple of the denominators (46), gives $3y-9+2y=120-3y+57$; whence, $3y+2y+3y=120+57+9$ (82); hence, $8y=186$, and $y=23\frac{1}{4}$.

Exercises.

1. Given $x+x+\frac{x}{2}+\frac{x}{3}=11$, to find the value of x.

2. Given $2y-\frac{y}{2}+1=5y-2$, to find the value of y.

3. Given $\frac{z}{2}+\frac{z}{3}-\frac{z}{4}=\frac{7}{10}$, to find the value of z.

4. Given $\frac{x+3}{2}+\frac{x}{3}=1-\frac{x-5}{4}$, to find the value of x.

5. Given $\frac{ax-b}{4}+\frac{a}{3}=\frac{bx}{2}-\frac{bx-a}{3}$, to find x.

6. Given $\frac{a}{1+x}+a=b$, to find the value of x.

7. Divide a line of 15 inches length, into three such parts, that the first shall be the half of the second, and the second the half of the third.

8. Divide 100 into two such parts, that if a third of the one be taken from a fourth of the other, the remainder shall be 11?

9. What number is that of which a half and three fourths are equal to 1?

10. What number is that of which $\frac{2}{3}$ of $\frac{3}{4}$ plus $\frac{1}{2}$ of $\frac{5}{6}$ of it, are equal to 11?

11. The triple, the half, and the fourth of a certain number, are equal to 104; what is the number?

12. A man and his wife usually drank a cask of beer in 12 days, but when the man was absent it lasted the wife 30 days; how long would the man alone take to drink it?

13. A cistern has three pipes, which will fill it in 3, 5, and 7 hours respectively; in what time will they fill it, if they be all set a-running at once?

14. Two persons can perform a piece of work in the times a and b respectively; in what time can they perform it both working together.

Note. Several of the questions formerly proposed, may be solved by this rule without supposing a quantity with a particular coefficient, as was done in some of the preceding cases.

INVOLUTION.

125. Involution signifies the raising of Powers; and is performed by the multiplication of a quantity, first into itself, and then into each successive product; (51).

126. TABLE of POWERS.

Roots or 1st Powers,	1	2	3	4	5	6	7	8	9
Squares or 2d Powers,	1	4	9	16	25	36	49	64	81
Cubes or 3d Powers,	1	8	27	64	125	216	343	512	729
Biquadrates or 4th Powers,	1	16	81	256	625	1296	2401	4096	6561
5th Powers,	1	32	243	1024	3125	7776	16807	32768	59049
6th Powers,	1	64	729	4096	15625	46656	117649	262144	531441
7th Powers,	1	128	2187	16384	78125	279936	823543	2097152	4782969
8th Powers,	1	256	6561	65536	390625	1679616	5764801	16777216	43046721
9th Powers,	1	512	19683	262144	1953125	10077696	40353607	134217728	387420489
10th Powers,	1	1024	59049	1048576	9765625	60466176	282475249	1073741824	3486784401

Rules.

127. To find any power of a simple or compound quantity. Multiply the quantity into itself, and into each successive product, as many times as are denoted by the index of the power required minus one, (because the first power of the given quantity is found without multiplication, being the root,) and the result is the power required.

128. To find any power of a simple quantity, it is most convenient, first, to find the power of the numeral coefficient, if there be any, either by the table or by the preceding rule; then to multiply the index of the quantity by the index of the required power, making the product the new index of the quantity, and prefixing the power of the coefficient for the answer.

129. When the sign of the root is *plus*, the signs of all the powers will be *plus* (87). When the sign of the root is *minus*, the signs of all the even powers, *i. e.* the 2d, 4th, 6th, &c. will be *plus;* but the signs of all the odd powers, *i. e.* the 3d, 5th, 7th, &c. will be *minus* (88).

130. To find any power of a fraction. Find the required powers both of the numerator and denominator, and they will be the numerator and denominator of the answer.

Examples.

1. Find the powers of a and a^2, from the 1st to the 5th inclusive.

a=1st power or root	a^2=1st power
a^2=2d power or square	a^4=2d power
a^3=3d power or cube	a^6=3d power
a^4=4th power or biquadrate	a^8=4th power
a^5=5th power.	a^{10}=5th power.

2. Find the fourth powers of $-3a$ and $-2ax^2$.

Here $(-3a)^4=+81a^4$; and $(-2ax^2)^4=+16a^4x^8$.

Or, $-3a$ root.	And $-2ax^2$ 1st power.
$-3a$	$-2ax^2$
$+9a^2$ square.	$+4a^2x^4$ 2d power.
$-3a$	$-2ax^2$
$-27a^3$ cube.	$-8a^3x^6$ 3d power.
$-3a$	$-2ax^2$
$+81a^4$ biquadrate.	$+16a^4x^8$ 4th power.

3. Find the cubes of $\frac{x}{a}$ and $\frac{2ax^2}{3b}$.

Here, $(\frac{x}{a})^3 = \frac{x^3}{a^3}$; and $(-\frac{2ax^2}{3b})^3 = -\frac{8a^3x^6}{27b^3}$.

4. Find the cubes of $x-a$ and $2x+3b$.

$x-a$ root.	$2x+3b$
$x-a$	$2x+3b$
x^2-ax	$4x^2+6bx$
$\quad -ax+a^2$	$\quad +6bx+9b^2$
$x^2-2ax+a^2$ square.	$4x^2+12bx+9b^2$ square.
$x-a$	$2x+3b$
$x^3-2ax^2+a^2x$	$8x^3+24bx^2+18b^2x$
$\quad -ax^2+2a^2x-a^3$	$\quad +12bx^2+36b^2x+27b^3$
$x^3-3ax^2+3a^2x-a^3$ cube.	$8x^3+36bx^2+54b^2x+27b^3$ cube.

Exercises.

1. Find the cubes of $2a^2$, $3a$, $5a^2$, and $7ay$.
2. Find the biquadrates of $2a^2x$ and $7a^3x^2$.
3. Find the cubes of $-\frac{2}{3}x^2y^3$, $\frac{a^2b}{5c}$ and $-\frac{3a^2x}{8b^2}$.
4. Find the fourth power of $a+b$ and the fifth power of $a-y$.
5. Find the eighth powers of $1+x$ and ay^2z^3.

Sir Isaac Newton's Rule,

For finding any Power of a Binomial.

131. Formula I. $(a+b)^n = a^n + na^{n-1}b + n.\frac{n-1}{2}a^{n-2}b^2 + n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}b^3 +$, &c.......$+b^n$.

132. Formula II. $(a-b)^n = a^n - na^{n-1}b + n.\frac{n-1}{2}a^{n-2}b^2 - n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}b^3 +$ &c.......$\pm b^n$.

133. *Rule.* 1. The number of terms in any required integral power of a binomial, is always one more than the number denoted by its index. 2. The first term is the required power of the first quantity in the binomial, and the last term is the same power of the second quantity. 3. The first quantity then decreases by one in its index in the intermediate terms, from the second term, in which its index is one less than that of the first term, to the last

term but one, in which it is unity. 4. But the second quantity increases by one in its index in the intermediate terms, from the second term in which its index is unity, to the last term but one in which it is one less than that of the last term. 5. The sum of the indices of both quantities in any term is equal to the index of the required power. 6. When the coefficients of the binomial are unity, the coefficients of the first and last terms of the power are always unity. 7. The coefficient of the second term is always the index of the first term. 8. The coefficient of every succeeding term is found by multiplying the coefficient of the preceding term by the index of the first quantity in it, and dividing the product by the index of the second quantity in the required term; that is, to find the coefficient of the third term, multiply the coefficient of the second term by the index of the first quantity in that term, and divide the product by the index of the second quantity in the third term, &c. 9. The sum of the coefficients of all the terms is equal to that power of the number 2, denoted by the index of the required power. 10. When the signs of both terms of the binomial are plus, the signs of all the terms of the power, will be plus, and vice versa. But when the sign of the one is plus, and that of the other minus, then the signs of the terms in the power, will be plus and minus alternately.

Examples.

1. Find the 6th power of $a+x$, and the 7th power of $a-x$.

Here, the terms of the 6th power of $a+x$ without the coefficients, are, $a^6,\ a^5x,\ a^4x^2,\ a^3x^3,\ a^2x^4,\ ax^5,\ x^6$;

And the coefficients of each of the terms, without the quantities, are, $1,\ 6,\ \frac{6.5}{2}=15,\ \frac{15.4}{3}=20,\ \frac{20.3}{4}=15, \frac{15.2}{5}=6,\ 1$; Hence, $(a+x)^6=a^6+6a^5x+15a^4x^2+20a^3x^3+15a^2x^4+6ax^5+x^6$.

Again, the terms of the 7th power of $a-x$, without the coefficients, are, $a^7,\ a^6x,\ a^5x^2,\ a^4x^3,\ a^3x^4,\ a^2x^5,\ ax^6,\ x^7$; And the coefficients are, $1,\ 7,\ \frac{7.6}{2}=21,\ \frac{21.5}{3}=35,\ \frac{35.4}{4}=35,$ $\frac{35.3}{5}=21,\ \frac{21.2}{6}=7,\ 1$; Hence, $(a-x)^7=a^7-7a^6x+21a^5x^2-35a^4x^3+35a^3x^4-21a^2x^5+7ax^6-x^7$.

134. This rule may also be employed to find any power of a trinomial, quadrinomial, &c. by putting any two or three of the terms within a parenthesis, and then expanding the expression in the same manner as if it were a binomial.

2. Find the 4th power of $a+b+c$.

Here, $a+b+c=(a+b)+c$, therefore, by the rule, $((a+b)+c)^4=(a+b)^4+4(a+b)^3c+6(a+b)^2c^2+4(a+b)c^3+c^4$.

But, $(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$; and $4(a+b)^3c=4c\ (a^3+3a^2b+3ab^2+b^3)=4a^3c+12a^2bc+12ab^2c+4b^3c$; also, $6(a+b)^2c^2=6c^2(a^2+2ab+b^2)=6a^2c^2+12abc^2+6b^2c^2$; and $4(a+b)c^3=4ac^3+4bc^3$; whence, collecting all the terms, $(a+b+c)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4+4a^3c+12a^2bc+12ab^2c+4b^3c+6a^2c^2+12abc^2+6b^2c^2+4ac^3+4bc^3+c^4$.

Exercises.

1. Find all the powers of $a+y$, from the 2d to the 8th.
2. Find the 10th power of $a-y$, the 5th power of $a-b+c$, and the 6th power of $2+x$.
3. Find the 4th power of $2x+3z$, and the cube of $a+b+c+d$.
4. Find the 10th power of 12, or $10+2$.

EVOLUTION.

135. Evolution, is the extracting of Roots, or the method of finding the Root or 1st Power of any given Power or Quantity, (52).

136. TABLE of ROOTS.

Nos.	Square Roots.	Cube Roots.	Biquadrate Roots.	Fifth Roots.
1	1.0000000	1.0000000	1.00000	1.00000
2	1.4142136	1.2599210	1.18920	1.14869
3	1.7320508	1.4422496	1.31607	1.24573
4	2.0000000	1.5874011	1.41421	1.31950
5	2.2360680	1.7099759	1.49535	1.37973
6	2.4494897	1.8171206	1.56508	1.43096
7	2.6457513	1.9129312	1.62658	1.47577
8	2.8284271	2.0000000	1.68179	1.51571
9	3.0000000	2.0800837	1.73205	1.55184
10	3.1622777	2.1544347	1.77828	1.58489

Rules.

137. To find any root of a simple quantity. Find the root of the coefficient, by the tables (126, or 136), or by the common arithmetical rule, (141), then multiply the index of the given quantity, by the index of the required root, or divide by its denominator, making the quotient the index of the quantity, and prefixing the root of the coefficient, for the answer.

138. When the sign of the quantity is *plus*, the signs of all the even roots may be either *plus* or *minus*, (87). When the sign of the quantity is *minus*, the signs of all the odd roots, will be *minus*; but the signs of all the even roots, are impossible, that is, their roots cannot be found; for example, $\sqrt{-a^2}$ is impossible, because no quantity, either positive or negative, can be found, whose square is equal to $-a^2$.

139. The root of a fraction is found by extracting the roots both of the numerator and the denominator.

140. Those numbers or quantities whose roots cannot be exactly determined, are called surds, or irrational quantities (54); of this kind of numbers the preceding table is chiefly composed.*

Examples.

1. Find the square root of $16x^2$, and the cube root of $27x^3$.
   Here, $\sqrt{16x^2}=4x$; and $\sqrt[3]{27x^3}=3x$.
2. Find the 4th root of $16x^4$, and the 5th root of $10000\ a^{10}$.
   Here, $\sqrt[4]{16x^4}=2x$; and $\sqrt[5]{100000a^{10}}=10a^2$.
3. Find the square roots of $2a^6$, and $\dfrac{25a^2}{64b^8}$; and the cube roots of $-3a^6$, and $-\dfrac{8a^3b^3}{27c^3}$.

   Here, $\sqrt{2a^6}=1.4142136a^3$; and $\sqrt{\dfrac{25a^2}{64b^8}}=\dfrac{5a}{8b^4}$;

   Also, $\sqrt[3]{-3a^6}=-1.4422a^2$; and $\sqrt[3]{-\dfrac{8a^3x^3}{27c^3}}=\dfrac{2ax}{3c}$.

Exercises.

1. Find the square roots of $4a^2x^6$, $5a^2y^2$, and $\dfrac{4a^4}{9x^2y^2}$.
2. Find the cube roots of $-125a^3x^6$, $216x^3y^9$, and $\dfrac{8a^3}{125x^6}$
3. Find the 4th root of $256a^4x^5$, and the 5th root of $\dfrac{32a^5x^{10}}{243}$.

* The reader will find in a very useful work, "Barlow's Mathematical Tables," the squares, cubes, square and cube roots, &c. of all numbers, from 1 to 10,000.

141. To extract the square root of a compound quantity. Arrange the terms according to the order of their powers, as in Involution. Find the square root of the first term, and place it in the quotient; subtract the square of this root from the first term, and bring down the next two terms, to the remainder, if any, for a dividend; double the quotient for a divisor, and divide the first term of the dividend by the first term of the divisor, for the next term of the quotient, place this term also on the right of the divisor; multiply the divisor thus increased, by this term, and subtract the product from the dividend; to the remainder, bring down the next two terms, and proceed as before.

142. This rule is founded on the theorem, that the square of a binomial, is equal to the sum of the squares of the two terms, and twice their product; for $(a+b)^2=a^2+2ab+b^2$.

143. If there be a remainder after all the terms are brought down, the quantity is a surd, and the extraction may be carried on till as many terms of the root be obtained as are necessary to determine their law of continuation.

Examples.

1. Extract the square root of $a^2-2ab+b^2$; and of $x^4-4x^3+6x^2-4x+1$.

$$\begin{array}{ll} & a^2+2ab+b^2(a+b \\ & \underline{a^2} \qquad\qquad \text{root.} \\ 2a+b) & +2ab+b^2 \\ & \underline{+2ab+b^2} \\ & \quad * \quad * \end{array} \qquad \begin{array}{ll} & x^4-4x^3+6x^2-4x+1(x^2-2x+1 \\ & \underline{x^4} \qquad\qquad\qquad\qquad \text{root.} \\ 2x^2-2x) & -4x^3+6x^2 \\ & \underline{-4x^3+4x^2} \\ 2x^2-4x+1) & +2x^2-4x+1 \\ & \underline{+2x^2-4x+1} \\ & \quad * \quad * \quad * \end{array}$$

Exercises.

1. Find the square root of $a^2-2ax+x^2$, and of $a^2+2ab+2ac+b^2+2bc+c^2$.

2. Find the square root of $a^4+4a^3x+6a^2x^2+4ax^3+x^4$, and of $x^4-2x^3+\frac{3}{2}x^2-\frac{1}{2}x+16$.

3. Extract the square root of $x^6+4x^5+10x^4+20x^3+25x^2+24x+16$; of a^2+b; and of 2, or $1+1$.

144. To find any root of a compound quantity. Find the root of the first term, and place it in the quotient, subtract the power of the root denoted, by the denominator of its index, from the first term; and, to the remainder bring down the second term for a dividend.

Raise the quotient to such a power as is denoted by the denominator of the index of the root minus one, multiply this power by that denominator, and divide the dividend by the product for the next term of the root; that is, divide by twice the quotient for the square root; thrice the square of the quotient for the cube root; four times the cube of it for the fourth root; &c. Involve the whole of the quotient to the same power as before, and subtract it from the given quantity or top line; divide the first term of the remainder by the same divisor as before, and proceed in this manner till there be no remainder left, or as far as may be considered necessary, if the quantity be a surd, the quotient is the root required.

145. The roots of many quantities may be found by inspection, on referring to the rule (133); thus, we find that the cube root of x^3+3x^2+3x+1 is $x+1$, because $(x+1)^3=x^3+3x^2+3x+1$.

Example. Find the cube root of $x^6+6x^5-40x^3+96x-64$.

$$x^6+6x^5-40x^3+96x-64(x^2+2x-4 \text{ cube root.}$$

$$(x^2)^3=x^6$$

$$(x^2)^2\times 3=3x^4)+6x^5$$

$$(x^2+2x)^3= \quad x^6+6x^5+12x^4+8x^3$$

$$3x^4) \quad -12x^4$$

$$(x^2+2x-4)^3=x^6+6x^5-40x^3+96x-64.$$

$$* \quad * \quad * \quad *$$

Exercises.

1. Find the cube root of $a^3+3a^2b+3ab^2+b^3$, and the fourth root of $x^4+4x^3+6x^2+4x+1$.

2. Find the fifth root of $32x^5-80x^4+80x^3-40x^2+10x-1$, and the fourth root of $a^4+4a^3b+6a^2b^2+4ab^3+b^4+4a^3b+12a^2bc+12ab^2c+4b^3c+6a^2c^2+12abc^2+6b^2c^2+4ac^3+4bc^3+c^4$.

RESOLUTION OF EQUATIONS.

CASE V. *Rule for clearing of Powers and Surds.*

146. If after the usual operations are performed, as directed in the preceding cases, any power of the unknown quantity occurs in any equation, its value will be found by extracting the corre-

sponding root of both sides of the equation; and if any root of the unknown quantity occurs, its value will be found by raising both sides of the equation to the corresponding power.

Examples.

1. Given $3x^2—16=32$, and $ax^3+b=c$, to find the values of x.

Here, $3x^2=32+16=48$ (82); hence, $x^2=16$ (104); and, $x=4$, by extracting the square root of both sides of the equation.

Again, $ax^3=c—b$ (82); hence, $x^3=\frac{c—b}{a}$ (104); and, $x=\sqrt[3]{\frac{c—b}{a}}$, by extracting the cube root.

2. Given $\sqrt{x—2}=3$, and $\sqrt[3]{2x+7}=3$, to find the values of x.

Here, $(\sqrt{x—2})^2=3^2$, or $(x—2)^{\frac{1}{2}}\times(x—2)^{\frac{1}{2}}=x—2=9$, by squaring; whence, $x=9+2=11$.

Again, $(\sqrt[3]{2x+7})^3=3^3$, or $(2x+7)^{\frac{1}{3}\times 3}=2x+7=27$, by cubing; whence, $2x=27—7=20$; and $x=10$.

147. If the side of the equation which contains the unknown quantity be a complete power, its value may sometimes be found by extracting the corresponding root of both sides of the equation; and the equation may always be reduced to one containing a lower power of the unknown quantity.

Example. Given $x^2+6x+9=25$, to find the value of x.

Here, $\sqrt{x^2+6x+9}=x+3$, and $\sqrt{25}=5$; whence, $x+3=5$, or $x=5—3=2$.

Exercises.

1. Given $5x^2—15=2x^2+12$, to find the value of x.
2. Given $3x^{\frac{1}{2}}+4=19$, to find the value of x.
3. Given $\sqrt{ax+b^2}=a+b$, to find the value of x.
4. Given $2+\sqrt{3x}=\sqrt{4+5x}$, to find the value of x.
5. Given $\sqrt{a^2+x^2}=\sqrt[4]{b^4+x^4}$, to find the value of x.
6. Given $\sqrt{a}+\sqrt{y}=\sqrt{ay}$, to find the value of y.
7. Given $\sqrt{z}+\sqrt{6+z}=\frac{12}{\sqrt{6+z}}$, to find the value of z.
8. Given $\sqrt{a+y}+\sqrt{a—y}=b$, to find the value of y.
9. Given $a+x=\sqrt{a^2+x\sqrt{b^2+x^2}}$, to find the value of x.
10. Given $\sqrt{\frac{x+1}{x—1}}+\sqrt{\frac{x—1}{x+1}}=a$, to find the value of x.
11. Given $\sqrt{x+a}=c—\sqrt{x+b}$, to find the value of x.

PROPORTION.

148. Proportion is of two kinds, arithmetical and geometrical.

149. Four quantities are said to be in arithmetical proportion, when the difference between the first and second, is equal to the difference between the third and fourth. Thus, 3, 7, 12, 16, and a, $a+b$, c, $c+b$, are quantities in arithmetical proportion.

150. Three quantities are said to be in arithmetical proportion, when the difference between the first and second, is equal to the difference between the second and third. Thus, 2, 5, 8, and a, $a+b$, $a+2b$, are quantities in arithmetical proportion.

From these definitions, the following theorems and problems relative to arithmetical proportion are derived:

151. If four quantities be in arithmetical proportion, the sum of the two extremes, or first and last terms, is equal to the sum of the two means or middle terms. For, if $a-b=c-d$ (149); then, $a+d=b+c$ (82).

152. Hence, if any three terms be given, the fourth may be found by the following rule: Add the means together, and subtract the one extreme, the remainder will be the other extreme. For, if $a-b=c-d$; then, $a+d=b+c$ (151), and $a=b+c-d$, or $d=b+c-a$ (82).

153. Again, if three quantities be in arithmetical proportion, the sum of the extremes is double of the middle term. For, if $a-b=b-c$ (150); then, $a+c=2b$ (82).

154. Hence, an arithmetical mean between any two quantities, may be found by the following rule: Add the two quantities and take half their sum, the quotient will be the mean required. For, if $a-b=b-c$, then $2b=a+c$ (153), and $b=\frac{a+c}{2}$ (104).

155. Four quantities are said to be in geometrical proportion, when the first is the same multiple or part of the second, that the third is of the fourth. Thus, 2, 8, 3, 12, and a, ar, b, br, are quantities in geometrical proportion.

156. Three quantities are said to be in geometrical proportion, when the first is the same multiple or part of the second, that the second is of the third. Thus, 2, 4, 8, and a, ar, ar^2, are quantities in geometrical proportion.

157. Four quantities are *directly* proportional, when the first has the same ratio to the second, which the third has to the fourth. Thus, 2 : 8 : : 3 : 12, are directly proportional.

158. Four quantities are *inversely* proportional, when the first has the same ratio to the second, which the reciprocal of the third has to the reciprocal of the fourth. Thus, 2, 8, 12, 3, are inversely proportional; for, $2 : 8 :: \frac{1}{12} : \frac{1}{3}$.

159. The nature of geometrical proportion being fully explained in the fifth Book of the Geometry, all that is necessary here, is to state the practical results. The common Rule of Three is founded on the following theorems:

160. If four quantities be in geometrical proportion, the product of the extremes, or first and last terms, is equal to that of the means, or two middle terms. For, if $a : b :: c : d$; then, $\frac{a}{b}=\frac{c}{d}$ (15), and by clearing this equation of fractions (124), we have $ad=bc$.

161. Hence, if any three terms be given, the fourth may be found by the following rule: Multiply the means together, and divide by the one extreme, and it will give the other extreme. For, if $a : b :: c : d$; then, $ad=bc$ (160), and $d=\frac{bc}{a}$, or $a=\frac{bc}{d}$ (104).

162. Again, if the product of any two quantities be equal to the product of any other two quantities, the four quantities are proportional, that is, the first two may be made the extremes, and the other two the means, and vice versa. Thus, if $ad=bc$, then dividing this equation by bd, we have $\frac{a}{b}=\frac{c}{d}$ (64); whence, $a : b :: c : d$ (15). And, if $3\times 4=6\times 2$, then $3 : 6 :: 2 : 4$.

163. The following table contains an exemplification of the most useful properties of proportionals, as demonstrated in the fifth book of Euclid, at one view.

If $a : b :: c : d$, and $6 : 4 :: 3 : 2$, directly;
Then, $b : a :: d : c$, and $4 : 6 :: 2 : 3$, by inversion;
$a : c :: b : d$, and $6 : 3 :: 4 : 2$, alternately;
$a+b : b :: c+d : d$, and $10 : 4 :: 5 : 2$, by composition;
$a-b : b :: c-d : d$, and $2 : 4 :: 1 : 2$, by division;
$a : a-b :: c : c-d$, and $6 : 2 :: 3 : 1$, by conversion;
$a+b : a-b :: c+d : c-d$, and $10 : 2 :: 5 : 1$, by equality.

In all these cases, the product of the extremes is equal to the product of the means.

164. If three quantities be in geometrical proportion, the product of the extremes, or first and last terms, is equal to the square

of the mean, or middle term. For, if $a : b :: b : c$, then $\frac{a}{b}=\frac{b}{c}$ (15), and clearing of fractions, we have $ac=b^2$ (124).

165. Hence, a geometrical mean between any two quantities, may be found by the following rule: Multiply the two quantities together and extract the square root of their product, the result will be the mean required. For, if $a : b :: b : c$, then $b^2=ac$ (160), and $b=\sqrt{ac}$ (146).

Exercises.

1. Find the fourth arithmetical proportionals to 3, 6, 8, and to 97, 63, 48.
2. Find the arithmetical means between 5 and 13, and between 130 and 480.
3. What are the fourth geometrical proportionals to 8, 72, 13, and to 156, 133, 121?
4. What are the geometrical means between 4 and 9, and between 53 and 477?

RESOLUTION OF EQUATIONS.

CASE VI. *Rule for converting an Analogy into an Equation.*

166. Make the product of the extremes equal to the product of the means (160).

Example. Given $5 : 15 :: 6 : x^2$, to find the value of x.

Here, $5x^2=90$; hence, $x^2=18$, and $x=\sqrt{18}=\sqrt{9}\times 2$, or $x=3\sqrt{2}$ (146).

Exercises.

1. Given $3x : 16 : 5 : 6$, to find the value of x.
2. Given $12-x : \frac{x}{2} :: 4 : 1$, to find the value of x.

CASE VII. *Rules for determining the values of two Unknown Quantities.*

167. When there are two unknown quantities in any question, it requires two independent equations to determine their values, which is performed by reducing them to one, by any of the three following rules:

168. *Rule I.* Find the value of that unknown quantity, which appears to be the least involved in each equation, by the preceding cases; then make these two values equal to each other (42 Geometry), and this new equation having only one unknown quantity may be solved as before; or,

169. *Rule II.* Find the value of one of the unknown quantities in that equation in which it is least involved, and substitute this value for that unknown quantity in the other equation, which will then have only one unknown quantity, and may be solved as in the preceding cases; or,

170. *Rule III.* Multiply or divide one or both of the equations by any other number or quantity that will make the coefficients of one of the unknown quantities the same in both; then, when the signs of this unknown quantity in each are contrary, add, or when they are the same, subtract the two equations, and the remainder will be an equation containing only one unknown quantity, whose value may be found as formerly.

Examples.

1. Given $2x+3y=23$, and $5x-2y=10$, to find the values of x and y.

By Rule I. From the first equation, $x=\frac{23-3y}{2}$, and from the second, $x=\frac{10+2y}{5}$ (82 and 104). Therefore, $\frac{10+2y}{5}=\frac{23-3y}{2}$ (168), and $20+4y=115-15y$, or $19y=95$ (124); whence, $y=5$, and $x=\frac{23-15}{2}=4$.

By Rule II. From the first equation, $y=\frac{23-2x}{3}$; hence, $5x-2(\frac{23-2x}{3})=10$, or $5x-\frac{46+4x}{3}=10$ (169); whence, $15x-46+4x=30$, or $19x=76$ (124); therefore, $x=4$, and $y=\frac{23-8}{3}=5$.

By Rule III. From the first equation, $10x+15y=115$, by multiplying by 5 the coefficient of x in the second; and from the second equation, $10x-4y=20$, by multiplying by 2 the coefficient of x in the first. Then, subtracting the latter of these *new* equations from the former, gives $19y=95$ (170); whence, $y=5$, and x, may be found from either of the equations as above.

Exercises.

1. Given $4x+y=31$, and $4y+x=19$, to find the values of x and y.

2. Given $2x+3y=16$, and $3x-2y=11$, to find the values of x and y.

3. Given $\frac{2x}{5}+\frac{3y}{4}=\frac{9}{20}$, and $\frac{3x}{4}+\frac{2y}{5}=\frac{61}{120}$, to find the values of x and y.

4. Given $\frac{y}{7}+7x=99$, and $\frac{x}{7}+7y=51$, to find the values of x and y.

5. Given $\frac{u}{2}-12=\frac{z}{4}+8$, and $\frac{u+z}{5}+\frac{u}{3}-8=\frac{2z-u}{4}+27$, to find the values of u and z.

6. Given $x+y=s$, and $x^2-y^2=d$, to find the values of x and y.

7. Given $x : y :: a : b$, and $x^2+y^2=d$, to find the values of x and y.

8. Given $x^2+xy=a$, and $y^2+xy=b$, to find the values of x and y.

9. There is a certain number, consisting of two places of figures, which is equal to four times the sum of its digits; and if 18 be added to it, the digits will be inverted; what is the number?

10. What two numbers are those, whose difference, sum, and product, are to each other, as the numbers 2, 3, and 5, respectively?

11. If A gives B 5*s.* of his money, B will have twice as much as A has left; and if B gives A 5*s.* of his money, A will have three times as much as B has left; how much has each?

12. When a company at a tavern came to pay their reckoning, they found, that if there had been three persons more, they would have had a shilling each less to pay; and if there had been two persons less, they would have had a shilling each more to pay; what was the number in the company, and the share of each?

13. A bill of £120 was paid in guineas and moidores, and the number of pieces of both kinds of money used was just 100; how many were there of each, supposing the moidore worth 27*s.*?

14. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3, but 2 of the greyhound's leaps are as much as 3 of the hare's; how many leaps must the greyhound take to catch the hare.

15. A composition of tin and copper containing 100 cubic inches weighed 505 ounces; how many ounces were there of each, supposing the weight of a cubic inch of copper $5\frac{1}{4}$ ounces, and that of a cubic inch of tin $4\frac{1}{4}$ ounces.

CASE VIII. *Rules for determining the values of three or more Unknown Quantities.*

171. When there are three or more unknown quantities, it requires the same number of independent equations to determine their values, which is performed by any of the following rules:

172. *Rule I.* Find the values of one of the unknown quantities in each of the given equations, as if the rest were known; then, make the first of these values equal to the second, and either the first or second equal to the third, and they will give two new equations having only two unknown quantities (42 Geometry), whose values may then be found by the foregoing case; or,

173. *Rule II.* Multiply each of the equations by such numbers or quantities as will make the coefficients of one of the unknown quantities the same in all; then, subtract any two of these new equations from the third, or add them together according to the nature of the signs, and they will give only two equations, which may be resolved by the former rules.

174. In the same manner, may four, five, or any number of unknown quantities be exterminated, having the same number of independent equations.

Examples.

Given $2x+4y-3z=22$, $4x-2y+5z=18$, and $6x+7y-z=63$, to find the values of x, y, and z.

By Rule I. From the first equation, $x=\frac{22-4y+3z}{2}$, from the second, $x=\frac{18+2y-5z}{4}$, and from the third, $x=\frac{63-7y+z}{6}$. Whence, $\frac{18+2y-5z}{4}=\frac{22-4y+3z}{2}$, or $10y-11z=26$; and $\frac{18+2y-5z}{4}=\frac{63-7y+z}{6}$, or $20y-17z=72$. From the first of these new equations, $y=\frac{26+11z}{10}$, and from the second, $y=\frac{72+17z}{20}$; whence, $\frac{26+11z}{10}=\frac{72+17z}{20}$, or $5z=20$; whence, $z=4$; and consequently $y=7$, and $x=3$.

By Rule II. Multiplying the first equation by 6, the second by 3, and the third by 2, gives $12x+24y-18z=132$; $12x-6y+15z=54$, and $12x+14y-2z=126$. Subtracting the second of these new equations from the first and third, gives $30y-33z$

$=78$, and $20y-17z=72$. Again, multiplying the first of these equations by 2, and the second by 3, gives $60y-66z=156$, and $60y-51z=216$. Lastly, subtracting the former of these two equations from the latter, gives $15z=60$, whence $z=4$. Now, $20y=72+17z=72+68=140$, whence $y=7$; and $2x=22-4y+3z=22-28+12=6$, whence, $x=3$.

Exercises.

1. Given $x+y+z=18$, $x+2y+3z=38$, and $x+3y+4z=51$, to find the values of x, y, and z.

2. Given $7x+5y+2z=79$, $8x-7y+9z=-4$, and $x+4y+5z=55$, to find the values of x, y, and z.

3. Given $x+\frac{1}{2}y+\frac{1}{3}z=32$, $\frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15$, and $\frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12$, to find the values of x, y, and z.

4. Given $x+y=a$, $x+z=b$, and $y+z=c$, to find the values of x, y, and z.

5. Given $aey=a+e+y$, $2a+2e=y$, and $ay-a=e$, to find the values of a, e, and y.

6. Given $xy+uz=390$, $xz+uy=339$, $ux+yz=290$, and $uxyz=19800$, to find the English word formed by the letters whose places in the alphabet are denoted by the values of u, x, y, and z.

7. If A and B together can perform a piece of work in 8 days; A and C together in 9 days; and B and C together in 10 days; how many days will it take each person singly to perform the same work?

8. Three persons, A, B, and C, play together. In the first game, A loses to each of the other two, as much money as each of them has. In the next game, B loses to each of the other two as much money as they then had. Lastly, in the third game, A and B gain each from C as much money as they had before. On leaving off, they find that each has an equal sum, namely, 24 guineas. With what money did each sit down to play?

9. Given $ax+by+cz=d$, $ex+fy+gz=h$, and $ix+ky+lz=m$, to find the values of x, y, and z.

175. The solution of this question will afford general formulæ for all questions of the same kind; and, if of such of the preceding questions as are of the same form, the numerical

coefficients and absolute terms be substituted for the literal ones in these formulæ, the values of the unknown quantities will be found without any other operation than what they indicate, provided due attention be paid to the signs.

RESOLUTION OF ADFECTED QUADRATIC EQUATIONS.

176. An adfected Quadratic Equation is one which contains both the first and second power of the unknown quantity: as, $x^2+2x=15$.

177. *Rule.* In every adfected quadratic equation, where the coefficient of the first term is unity, the unknown quantity is equal to the half of the coefficient of the second term with its sign changed, plus or minus the square root of the sum of the square of this half coefficient and the absolute term.

Demonstration. Let $x^2+ax=b$, be the general form of quadratics. Then adding $\frac{1}{4}a^2$ to both sides of the equation, gives $x^2+ax+\frac{1}{4}a^2=\frac{1}{4}a^2+b$; now, extracting the square root of both sides of this equation, gives $x+\frac{1}{2}a=\surd(\frac{1}{4}a^2+b)$; whence, $x=-\frac{1}{2}a\pm\surd(\frac{1}{4}a^2+b)$. In the same manner it may be shown, in the equation $x^2-ax=b$, that $x=\frac{1}{2}a\pm\surd(\frac{1}{4}a^2+b)$; in the equation $x^2-ax=-b$, that $x=\frac{1}{2}a\pm\surd(\frac{1}{4}a^2-b)$; and in the equation $x^2+ax=-b$, that $x=-\frac{1}{2}a\pm\surd(\frac{1}{4}a^2-b.)$ These four formulæ include all the possible cases.

178. Before this rule can be applied to the solution of quadratic equations, they must be reduced to their simplest forms either by the preceding rules, or by any easy substitution that suggests itself.

Examples.

1. Given $x^2+2x=15$, to find the value of x. Here, according to the rule, $x=-1\pm\surd\overline{1+15}$, whence, $x=3$, or, -5.

2. Given $10x^2-8x+6=318$, to find the values of x. Here, by transposing, we have $10x^2-8x=312$; and dividing this equation by 10, it becomes $x^2-\frac{4}{5}x=\frac{156}{5}$;

whence by the rule, $x=\frac{2}{5}\pm\sqrt{\frac{4}{25}+\frac{156}{5}}$, or $x=\frac{2}{5}\pm\sqrt{\frac{784}{25}}$; and lastly, $x=\frac{2}{5}\pm\frac{28}{5}=6$, or $-\frac{26}{5}$.

179. All equations containing two powers of the unknown quantity, and having the index of the one power just double that of the other, may be solved by the rule for quadratics.

3. Given $x^4-3x^2=54$, to find the value of x. Here let $x^2=y$, then $x^4=y^2$, and the equation becomes $y^2-3y=54$; whence, $y=\frac{3}{2}\pm\sqrt{\frac{9}{4}+54}=\frac{3}{2}\pm\sqrt{\frac{225}{4}}=\frac{3}{2}\pm\frac{15}{2}$; therefore, $y=9$, or -6, and consequently, $x=\sqrt{9}=3$, or $x=\sqrt{-6}$.

4. Given $x^{\frac{1}{2}}-4x^{\frac{1}{4}}=-4$, to find the value of x. Here, let $x=y^4$, then $x^{\frac{1}{2}}=y^2$, and $x^{\frac{1}{4}}=y$; and the equation becomes $y^2-4y=-4$; whence, $y=2\pm\sqrt{4-4}=2$, and consequently, $x=16$.

Exercises.

1. Given $x^2+6x+4=59$, to find the value of x.
2. Given $2x^2+12x+36=356$, to find the value of x.
3. Given $4x=\frac{36-x}{x}+46$, to find the value of x.
4. Given $5x-\frac{3x-3}{x-3}=2x+\frac{3x-6}{2}$, to find the value of x.
5. Given $ax^2-bx+c=d$, to find the value of x.
6. Given $\frac{1}{2}x-\frac{1}{3}\sqrt{x}=22\frac{1}{6}$, to find the value of x.
7. Given $\sqrt{(10+x)}-\sqrt[4]{(10+x)}=2$, to find the value of x.
8. Given $x^6+20x^3-10=59$, to find the value of x.
9. Given $\sqrt{(1+x-x^2)}-2(1+x-x^2)=\frac{1}{9}$, to find the value of x.
10. Given $x^{\frac{3}{2}}+x^{\frac{5}{2}}=6x^{\frac{1}{2}}$, to find the value of x.
11. Given $\sqrt[3]{(x^3-a^3)}=x-b$, to find the value of x.
12. Given $\sqrt{(4+\sqrt{[2x^3+x^2]})}=\frac{x+4}{2}$, to find the value of x.
13. Find two numbers such that their sum shall be 10, and the sum of their squares 58.

14. Find two numbers such that their difference shall be 8, and their product 240.

15. Find two numbers such that their sum, product, and difference of their squares, shall be equal.

16. Find a number such, that if it be subtracted from 10, and the remainder be multiplied by the number itself, the product shall be 21.

17. Divide the number 40 into two such parts, that the sum of their squares shall be 818.

18. Divide the number 24 into two such parts, that their product shall be equal to 35 times their difference.

19. A merchant bought a quantity of cloth for £33 15*s*., which he sold again at £2 8*s*. per piece; and gained as much as one piece cost him; required the number of pieces.

20. Divide a line of 20 inches in length into two such parts, that the product of the whole and one of the parts shall be equal to the square of the other.

21. Find two numbers such, that their product shall be equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes.

22. Find four numbers in geometrical progression, such, that their sum shall be 15, and the sum of their squares 85.

23. The sum of two numbers is 11, and the sum of their fifth powers is 17831; required the numbers.

24. Find four numbers in arithmetical progression, such, that their common difference shall be 4, and their continued product 176985.

26. When a gentleman entered upon his estate, it consisted of 1000 acres, rented at 5*s*. per acre; he increased it by yearly acquisitions 10 acres, and also raised the rent 1*s*. per acre every year, till his whole rent amounted to £7000 per annum. Required the number of years he took to raise it to this rent, the number of acres he possessed in all, and the rent per acre at last.

26. What three numbers are those, of which the third is equal to the difference between the first and second, the square of the third equal to their sum, and its cube equal to their product.

27. Find the side of a square inscribed in a semicircle whose diameter is d or 10.

28. Given the hypothenuse of a right angled triangle (13), and the difference between the other two sides (7), to find those sides.

29. Required the area of a right angled triangle, whose hypothenuse is x^{3x}, and the base and perpendicular x^{2x} and x^x respectively.

30. Given the perimeter of a rhombus (12), and the sum of its two diagonals (8), to find the diagonals.

31. Given the perpendicular (300) of a plane triangle, the sum of the two sides (1155), and the difference of the segments of the base (495), to find the base and the sides.

32. Given the base of a plane triangle (15), its area (45), and the ratio of its other two sides (2 : 3), to find the sides.

33. Given the hypothenuse of a right angled triangle (10), and the difference of two lines drawn from its extremities to the centre of the inscribed circle (2), to determine the base and perpendicular.

RESOLUTION OF EQUATIONS OF ALL DIMENSIONS.

180. *Rule.* 1. Arrange the terms of the given equation, whether quadratic, cubic, biquadratic, or any higher dimension, in the order of their powers, beginning with the highest, and place the numerical or absolute term on the right of the sign of equality, and all the other terms on the left. 2. Reduce the equation, if necessary, so that the coefficient of the first term shall be unity; and supply the want of any term in the regular series, putting zero for its coefficient. 3. Divide the absolute term into periods of as many figures each as there are units in the index of the first term, if necessary; and mark out a place for the quotient on the right. 4. Find by trial the first figure of the required root of the equation, and place it in the quotient. 5. Add this first figure to the coefficient of the second term and to each successive sum, as often as there are units in the index of the first term. 6. Multiply each of these sums, except the last, by the first figure, and add the products to the coefficient of the third term and to each successive sum. 7. Proceed in this manner to the coefficient of the last term, under which by this process will be found two sums; the first of which is the proper divisor

for the first figure of the root, and the second the trial divisor for the next. 8. Multiply the first divisor by the first figure of the root, and subtract the product from the first period of the absolute term, bringing down the next period to the remainder for a dividend. 9. By means of the trial divisor, find the second figure of the root, making some allowance for its increment. 10. Add this second figure to the last sum under the second term, and to each successive sum, in the same manner as was done with the first figure; proceed to find the successive products and sums as in finding the proper divisor for that figure, till the proper divisor for the second figure of the root be found. 11. Multiply this divisor by the second figure in the root, and subtract the product from the dividend, bringing down the third period, if any, to the remainder for a new dividend. 12. Proceed in the same manner till the process terminate without a remainder, or till as many figures of the root be found as are required.

Examples.

1. Given $x^2+2x=18$, to find the value of x.

$$\begin{array}{rcll}
3 & & & \text{Root} \\
x^2+2x & = & 18 & \text{(3.3589, value of } x. \\
\overline{5\times 3} & = & \underline{15} & \\
3 & & 3 & \\
\overline{8.3\times 3} & = & \underline{2.49} & \\
.3 & & .51 & \\
\overline{8.65\times 5} & = & \underline{.4325} & \\
5 & & .0775 & \\
\overline{8.708\times 8} & = & \underline{.069664} & \\
8 & & .007836 & \\
\overline{8.7169\times 9} & = & \underline{.00784521} &
\end{array}$$

The other value of x will be found in the same manner to be -5.3589; which may be proved by the rule for quadratics.

2. Given $x^3+9x^2+4x=80$, to find the value of x.

2			Root
$x^3 + 9x^2$	$+ 4x$	$= 80$	(2.4721359
11×2	$= 22$		value of x.
2	26×2	$= 52$	
13×2	$= 26$	28	
2	52		
15.4×4	$= 6.16$		
.4	58.16×4	$= 23.264$	
15.8×4	$= 6.32$	4.736	
.4	64.48		
16.27×7	$= 1.1389$		
7	65.6189×7	$= 4.593323$	
16.34×7	$= 1.1438$	.142677	
7	66.7627		
16.412×2	$= .032824$		
2	66.795524×2	$= .133591048$	
16.414×2	$= .032828$	.009085952	
2	66.828352		
16.4161×1	$= .00164161$		
1	66.82999361×1	$= .006682999361$	
16.4162×1	$= .00164162$	.002402952639	
1	66.83163523		
16.41633×3	$= .0004924899$		
	66.8321277199×3	$= .002004963831597$	
		.000397988807403	

The other two values of x may be found in the same manner, the one being -5, and the other -6.4721359; which may be proved by transposing the terms, so that they shall be all equal to zero, and dividing them by $x-2.4721359$; whence we obtain the quadratic $x^2+11.4721359x=-32.360679$, which gives the above values.

3. Given $x^4-8x^3+14x^2+4x=8$ to find the value of x.

2				
$x^4 - 8x^3$	$+14x^2$	$+ \quad 4x$	$= \quad 8$	Root (2.732050
-6×2	$=-12$			value of x.
2	2×2	$= \quad 4$		
-4×2	$=- \quad 8$	8×2	$= \quad 16$	
2	$- \quad 6 \times 2$	$= -12$	$- \quad 8$	
-2×2	$=- \quad 4$	$- \quad 4$		
2	-10			
0.7×7	$= \quad .49$			
.7	$- \quad 9.51 \times 7$	$= - \quad 6.657$		
1.4×7	$= \quad .98$	-10.657×7	$=- \quad 7.4599$	
.7	$- \quad 8.53 \times 7$	$= - \quad 5.971$	$- \quad .5401$	
2.1×7	$= \quad 1.47$	-16.628		
.7	$- \quad 7.06$			
2.83×3	$= \quad .0849$			
3	$- \quad 6.9751 \times 3$	$= - \quad .209253$		
2.86×3	$= \quad .0858$	-16.837253×3	$=- \quad .50511759$	
3	$- \quad 6.8893 \times 3$	$= - \quad .206679$	$- \quad .03498241$	
2.89×3	$= \quad 867$	-17.043932		
3	$- \quad 6.8026$			
2.922×2	$= \quad .005844$			
	$- \quad 6.796756 \times 2$	$= - \quad .013593512$		
		-17.057525512×2	$=-.034115051024$	

4. Given $x^5+x^3+x=183$, to find the value of x.

	2					Root
$x^5 +$	$0x^4$	$+ 1x^3$	$+ 0x^2$	$+ 1x$	$=180($	2.747
	2×2	$= 4$				value of x.
	2	5×2	$= 10$			
	4×2	$= 8$	10×2	$= 20$		
	2	13×2	$= 26$	21×2	$= 42$	
	6×2	$=12$	36×2	$= 72$	138	
	2	25×2	$= 50$	93		
	8×2	$=16$	86			
	2	41				
	10.7×7	$= 7.49$				
	.7	48.49×7	$= 33.943$			
	11.4×7	$= 7.98$	119.943×7	$= 83.9601$		
	.7	56.47×7	$= 39.529$	176.9601×7	$=123.87207$	
	12.1×7	$= 8.47$	159.472×7	$= 111.6304$	14.12793	
	.7	64.94×7	$= 45.458$	288.5905		
	12.8×7	$= 8.96$	204.930			
	.7	73.90				
	13.5					

The other roots of the two preceding equations may be found in the same way as we have here shown to as great length as the size of the page would admit. A contracted mode of carrying on the operation, after the first three or four figures are obtained, might have been given; but the student who is acquainted with the mode of contracting division in decimal arithmetic, will find no difficulty in applying it here; besides it is of more importance to exhibit the operation at full length to a learner, in order that he may have a clear idea of the method.

Exercises.

1. Given $x^2+20x=100$, to find the value of x.
2. Given $x^3+x^2+x=100$, to find the value of x.
3. Given $x^3+x^2=500$, to find the value of x.
4. Given $x^3-48x^2=-200$, to find the value of x.
5. Given $x^3+9x=6$, to find the value of x.
6. Given $x^3-17x^2+54x=360$, to find the value of x.
7. Given $x^4-17x^2-20x=6$, to find the value of x.
8. Given $x^4-27x^3+162x^2+356x=1200$, to find the value of x.
9. Given $x^4-27x^3+162x^2+356x=1200$, to find the value of x.
10. Given $x^5+6x^4-10x^3-112x^2-207x=110$, to find the value of x.
11. Given $x^5+2x^4+3x^3+4x^2+5x=54321$, to find the value of x.
12. Given $x^{10}-x^7=1500$, to find the value of x.
13. The difference of two numbers is 8, and the difference of their fourth powers is 14560; required the numbers.
14. Given $x^3y+y^3x=3$, and $x^6y^2+y^6x^2=7$, to find the values of x and y.
15. Given $x^2+yz=920$, $y^2+xz=980$, and $z^2+xy=1000$, to find the values of x, y, and z.
16. The lengths of two lines that bisect the acute angles of a right angled triangle, and terminate in the opposite sides, being 40 and 50 respectively, it is required to determine the three sides of the triangle.

181. The method given in the preceding rule for the solution of equations of all dimensions, supersedes the necessity of giving in this short elementary treatise, the old methods of solving cubic and biquadratic equations by the rules of Cardan, Tartaglia, Euler, and others, as well as the various rules for approximating to the roots of equations, which are to be found in all the larger works on Algebra. Such works may be consulted with advantage after the student has made himself master of this introduction. The treatise of Euler may be recommended as one of the best extant for a learner.

182. The demonstration of this rule has also been omitted, as well as that of the Binomial Theorem, on account of the limits to which the Author has confined himself in this treatise. A larger work, which is intended as a sequel to the present, will include not only these demonstrations, but several others on a more elementary plan than has hitherto been attempted.

183. The solution of equations of all degrees by the method from which this rule was derived, is generally ascribed to a Mr. Holdred, of London, who published a tract on the subject in 1820. A similar method, by Mr. Horner of Bath, appeared in the *Philosophical Transactions for* 1819. It is not given, however, to one individual to accomplish the work of ages. For while we do not dispute the originality of either of these authors, we claim the priority of the invention for a Scotsman, of the name of Halbert, (schoolmaster at Auchinleck), in as far as regards the solution of equations of the *third* degree. While Mr. Bonnycastle in his elementary treatise asserted so late as 1818, that the solution of the *Irreducible Case* of cubic equations, except by means of a table of sines, or by infinite series, had hitherto baffled the united efforts of the most celebrated mathematicians of Europe; the rule for solving cubic equations of all kinds, whether *reducible* or *irreducible*, had been given by Mr. Halbert so far back as 1789, in his treatise on Arithmetic, published at Paisley in that year. The inventor, after giving his rule and a great variety of examples, says, —"So that I reckon this method a valuable discovery, when compared with the jargon we meet with in other authors, about *Transmutations, Limitations*, and *Approximations*, and what brings us never the nearer our purpose."

184. The step from the solution of the *irreducible* case of cubics to that of equations of all degrees, was evidently very easy from the nature of the rule there given. Besides, it is a singular circumstance, that Mr. Holdred is said to have been in possession of his method for a length of time previous to publication, which tallies almost exactly with the date of Mr. Halbert's treatise. Such are the facts respecting this invention, and we now leave the mathematical world to draw its own conclusions, and award the honour to whomsoever it is due. In his next publication, the author may be induced to unfold this subject a little more than it is possible for him to do in the present, without encroaching on his prescribed limits.

THE END.

ERRATA.

Page	Line	
7	40	for *kind* read *amount.*
14	17	for x read y.
16	31	for $18a-8b$ read $12a^2-8ab$, in some copies.
39	28	insert — before $\frac{2ax}{3c}$.
40	21	for $-2ab$ read $+2ab$.
—	33	for 16 read $\frac{1}{16}$.

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