THE

ILOSOPHY OF ARITHMETIC,

OR,

A COMPLETE ANALYSIS

OF

INTEGERS.

THE USE OF ADVANCED STUDENTS, BUT SUITED
TO THE MOST LIMITED CAPACITY.

Also, an Appendix,
CONTAINING DOMESTIC CALCULATIONS TO BE
PERFORMED MENTALLY.

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"Nothing can enter into the affections that stumbles at the threshold."
Blair.

LONDON:
J. SOUTER, SCHOOL LIBRARY,
73, ST. PAUL'S CHURCH YARD.
1826.

J. AND C. ADLARD, PRINTERS,
BARTHOLOMEW CLOSE.

PREFACE.

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THE following Lectures are the result of many years' experience in instructing British youth, of both sexes, in the invaluable science of Arithmetic. They have been repeatedly read by the Author with considerable success in one of the first finishing Establishments in the Metropolis.

1. NUMERATION and NOTATION are so explained, that they have merely to be read with attention to be clearly understood; the pupil will thereby be enabled to *read* or *write* the highest numbers with considerable ease. At the end of this section (Lecture I.), a diagram is given, showing the comparative magnitudes of numbers, from *one* to *one million* inclusive.

2. ADDITION. In this rule a method is recommended, by which much time may be saved in finding the amount of any collection of integral

numbers. In adding, we always commence at the column of units, and proceed step by step to those columns that stand in higher positions; but, to show that it is of no consequence where we begin, nor where we end, an example of three columns is given, the amount of which is found in six different ways. The *proof* by casting out the *nines*, recommended by Dr. Wallis, is shown to be a *fallible* method of ascertaining whether sums are correct.

3. Subtraction. Here several Methods are given, and the reason for carrying 1 to the next figure in the lower line when we add 10 to the preceding upper one. Two Methods of Proof are given and explained.

4. Multiplication. Under this head a Circular Table is given as high as 24 times 12, as well as simple rules, whereby a young pupil may acquire this grand engine of all calculations in one-third of the usual time: indeed, 9, 10, 11, and 12 times may be learned in one hour, provided the pupil can add and subtract with facility; then follow the *various* methods of multiplying by *the same*

number, and the advantages that may be taken in multiplying by *particular* numbers, with several Methods of Proof.

5. DIVISION. The advantages that may be taken with some divisors explained. The different ways in which the component parts of a divisor may be employed, the method of finding the true remainder, and reasons for the same. *Long Division,* the greatest difficulty which the young arithmetician has to encounter, it is presumed, is explained in so simple a manner, that half the usual time employed in this part of Division will be sufficient for him to acquire a complete knowledge of it.

The Author does not scruple to say that the various methods of *Adding, Subtracting, Multiplying,* and *Dividing,* with their several proofs, will be found worthy the attention of those who wish to be made thoroughly acquainted, not only with the numerous ways in which numbers may be employed to produce the same effect, but also with the reason for every step in each process.

6. SEQUEL TO MULTIPLICATION AND DIVISION, WITH PROOFS OF DIVISION. Here

numerous abridgments, both in Multiplication and Division, are introduced, which could not, with any degree of propriety, have been shown before. This part of the Work will require the most undivided attention on the part of the pupil, and will call forth his reasoning powers in their full force; an exercise, it will be acknowledged, that must be attended with the most beneficial effects, and may be the means of giving him a habit of thinking which will be of use to him in after-life: that it may do so, is the highest aim of the Author.

The *proofs* in Division are numerous.

The Multiplication and Division by mixed Numbers are fully explained.

In a school or family, a Lecture read occasionally by a competent person would be attended with considerable advantage.

In the APPENDIX the most simple and expeditious methods are given for calculating *mentally*; by a little attention to which, it will be no difficult matter to examine common accounts, tradesmen's bills, &c.

Should this Introduction to the Science of Arithmetic meet with encouragement, the Public may, at some future period, expect a similar elucidation of the higher branches of the science.

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27, *Earl Street East, Lisson Grove.*

The Author has lately written the following Works, which have been introduced into many of the principal Academies in the British dominions, and which may be had of his publisher, Mr. Souter; viz.

A COMPANION TO EVERY TREATISE ON ARITHMETIC.

APPENDIX AND ENLARGEMENT OF WALKER'S ARITHMETIC.

KEY TO DITTO, containing all the Solutions.

SUPERIOR SMALL-HAND COPIES, with Directions, &c.

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ANALYSIS OF INTEGERS,

&c.

CHAPTER I.

INTRODUCTION.

THE origin of Arithmetic is concealed in the most remote antiquity; and it is probable that this beautiful science did not arrive at any degree of perfection but by the slowly-progressive improvement of many ages.

Keith tells us that "the Phœnicians, the descendants of Noah, who settled on the coasts of Palestine, were the first people in the world who rendered *Navigation* subservient to commerce. It is, therefore, extremely probable that *Arithmetic* was invented by them, and that they introduced it into Egypt; and this opinion is supported by *Procleus* in his commentary on the first book of *Euclid's Elements of Geometry*." It is, indeed, evident, that, as they knew Navigation, they must have had a knowledge of Arithmetic, as the problems of the former must be performed by the rules of the latter.

But Josephus informs us that, a famine happening in Canaan, Abraham retired into Egypt, and was the first who taught the Egyptians the sciences of Arithmetic and Astronomy; and these he brought with him from Chaldea. From Egypt they were transmitted to Greece, by Pythagoras and others; and from thence to the Romans.

The Greeks were, therefore, the first among the Europeans who reduced Arithmetic to any thing like a system. They used all the letters of their alphabet to denote numbers.

The Romans followed the same method without confining themselves to the use of their alphabet in the *regular* order, as the Greeks did. Some traces of their system are still extant, since we see on almost all our clocks, and some of our watches, as well as on the chapters of our invaluable bible, the characters which they used in their numeral notation.

About 200 years after the birth of our blessed Saviour, *Claudius Ptolemeus* invented *sexagesimal* numeration, which is that in which every unit is divided into sixty equal parts, which are again subdivided into sixty other equal parts, &c. and whose whole numbers are reckoned from one to sixty. The traces of this mode of numeration are still observable in the division of hours into sixty minutes, and again of these into sixty seconds.

The Arabs afterwards introduced the figures which we now use in our calculations; yet, such is the prevailing predominance of custom, that, notwithstanding the *superiority* of their system, its substitution to sexagesimal numeration was very gradual.

Archimides, of Syracuse, celebrated for his skill in mathematics and mechanics, invented a curious scale of notation, which he employed in calculating the number of the sands.

As I have told you in the introductory part of my "*Companion to every Treatise on Arithmetic,*" *John of*

Basingstoke, archdeacon of Leicester, introduced the present mode of numeral notation about the middle of the eleventh century; he received it from the *Moors*, who brought it into *Spain;* they had it from the *Arabs;* and the Arabs had it from the *Indians.*

Arithmetic, however, needs not the assistance of distant authority or ancient history to recommend it; it requires not the aid of remote ages to point out its value, nor the help of refined modern philosophy to give it lustre; the universal consent of mankind is of itself sufficient to convince us that we cannot employ our time in the cultivation of a science which is in itself more generally useful.

In such a commercial country as Great Britain, *Arithmetic* certainly claims the highest attention: without a competent *practical* knowledge of this science, the merchant could not carry on business to any great extent with any degree of exactness. The business of the state demands a thorough knowledge of it, in almost all its branches; and there is no situation in life that does not call in the aid of this superior science: our very domestic concerns require at least a smattering of it. *Addison* says that "numbers are so much the measure of every thing that is valuable, that it is not possible to demonstrate the success of any action, or the prudence of any undertaking, without them." Besides, it is a science upon which many of the others absolutely depend. It is the key to all the treasures of mathematics, astronomy, and natural philosophy.

For by Arithmetic, the *Geometer* calculates the ratios and several dimensions of magnitudes and figures; the *Astronomer* determines the precise times of eclipses and other celestial phenomena; the *Navigator* discovers his exact place in the extensive expanse of water, the ocean, and the course and distance to his wished-for port; the *Surveyor* makes out the quantity of land in his surrounded tract; the *Merchant* regulates his various profits and exchanges, &c.; the *Usurer* settles his rates per cent.; and the useful *Mechanic* and *Labourer* come at the balance of their small pittance. What science, then, so extensively useful as Arithmetic!

An *eminent philosopher** informs us that, when scientifically treated, it need not fear a comparison with her more favoured sister *Geometry*, either in precision of ideas, in clearness or correctness of demonstration, in practical utility, or in the beautiful deduction of the most interesting truths.

The advantages of Arithmetic were, perhaps, never better stated than by Dr. Johnson, in one of his letters to Miss Susan Thrale:—"Nothing amuses more harmlessly," says he, "than computation; and nothing is more applicable to real business, and to speculative inquiries. A thousand stories which the ignorant tell, and believe, die away at once, when the computist takes them in his gripe. Numerical enquiries, my dearest girl, will give you entertainment in solitude by the practice, and reputation in public by the effect."

Sir Josiah Child, in his celebrated treatise on trade, enumerates the following circumstance that takes

* Walker.

place in general in Holland amongst the causes that render that country rich and flourishing:—"the education of their children, as well daughters as sons, all which," adds he, "be they ever so great in quality or estate, they always take care to bring up to write *good hands*, and to have the full knowledge of *arithmetic* and *merchants' accounts*."

It is almost universally known, however, that the fundamental rules of this *interesting science* is generally taught in so flimsy a way as neither to engage the attention, nor stimulate the ingenuity, of the youthful inquirer, while the field of pleasing and profitable discovery to which they naturally and directly lead, has been too often left unexplored.

Young people in general, from the whim or prejudice which any of their acquaintance may have, (which, by the bye, is nothing short of sheer indolence and a determination to learn only those things which give them least trouble,) never learn any thing well. Example is exceedingly powerful either in good or evil; and, when pupils are allowed to have their own idle inclinations gratified, it has a wonderful effect on others who witness it.

The teacher, however, who slowly and deliberately explains the nature and more immediate application of the rules upon which the higher and more intricate, as well as the most simple arithmetical operations depend, certainly performs his duty faithfully, and leaves it, without doubt, the fault of the pupils if they do not make rapid advances, as well as gain a rational knowledge of the science.

Surely nothing can be more degrading to the good sense and judgment of those pupils, who, under an able master, do not, at least, acquire a desire to traverse the beauteous fields of science, where they are always sure to be fully paid for every little advance which they may make, and where additional beauties arise on every side; and nothing more galling to the conscientious instructor, than that his pupils should lend a deaf ear to his instructions, and that, after all his best-schemed exertions, they should not have the desired effect.

Unless a powerful principle of *curiosity*, and, I might almost say, *enthusiasm*, be *possessed* and *displayed* by the pupil, the fond hope of the parent or teacher will almost always be disappointed.

Some pupils, of a *voluntary* disposition, climb the highest heights of literature, and their steady adherence to their studies depends not upon the persuasion of another. Others, again, of an *involuntary* turn, either have no inclination at all, or get others to do for them, what a very little exertion on their own part would put in their possession. The one class of pupils receive every thing as a *task*, the other as a *pleasure*. The *involuntary* pupils avoid the points that are difficult, while those very difficulties conquered by the *voluntary* pupils render the impressions lasting, and succeeding difficulties are thereby more easily surmounted.

The education of youth, says an *eminent essayist*,* is of such vast importance, and of such singular use in

* Watts.

the scene of life, that it visibly carries its own recommendation along with it; for on it, in a great measure, depend all that we hope to be; every perfection that a generous and well-disposed mind would gladly arrive at. It is this that stamps the distinction of mankind, and renders one person preferable to another; is almost the very capacity of doing well; and remarkably adorns every point of life.

Let not *prejudice*, therefore, take possession of your young minds; let no part of education be neglected; grasp at every opportunity of receiving information. Surely, if the proper use of the *voice*, the *fingers*, and the *feet*, be not considered beneath the attention of the rising generation, the acquisition of those branches that tend to *exert* the *judgment*, and *expand* the *mind*, claim *superior* attention.

I wish not to be understood as decrying the high accomplishments of music, dancing, and drawing, &c. for these are elegant refinements, and very good in their place, but certainly stand but second to the acquirements of the mind.

Before Arithmetic was collected into any thing like a regular system, every teacher arranged the rules in the ciphering-book of each pupil in a manner peculiar to his own taste; and, to this very day, prejudice in the way of arrangement obliges many to adopt this very plan, which is not only very troublesome to the master that has many pupils, but exceedingly unprofitable to the pupils themselves. All of us know by experience, that what we *write* leaves a more indelible impression upon the mind than what we *read*; therefore, I think that all pupils should themselves write a fair copy of every sum they do; and it would add to the advantage gained by such a method, if they would examine them step by step as they write them.*

There is nothing more common than for some pupils, at the commencement of a fresh rule, to imagine it quite insurmountable without elucidation on the part of the master; to such, I would recommend, first, a careful perusal of the rule before them; then, should they meet with any thing beyond their comprehension, it is the part of the master to render them every assistance.

It is astonishing what difficulties may be conquered by a willing mind. *Ferguson*, the astronomer, merely by unwearied application, attained the greatest eminence in the beautiful science of astronomy.

James Crichton,† commonly called the admirable Crichton, was also a most extraordinary character; he is said to have run through the circle of the sciences by the time he was twenty; and was then master of the twelve following languages, viz. Hebrew, Syriac, Arabic, Greek, Latin, Spanish, French, Italian, English,

* *Hamilton*, the elocutionist, says that the most expeditious method of getting any thing by heart is, first to *write* a small portion of it, and afterwards reading it twice or three times over will generally be sufficient; then another small portion, getting it in the same way, and so on. *Writing* it out in this manner seems like outlining a picture, and *reading* it carefully, the finishing part. *Cobbett*, in a Grammar of the English Language, which he has lately published, tells us, that when he was learning the French language, he first wrote each lesson neatly from beginning to end, and should he have committed a single blunder, he would burn it, and commence another.

† He was cruelly butchered by his own pupil, Vincentio de Gonzaga, son of the duke of Mantua.

Dutch, Flemish, and Sclavonian. His attainments almost exceeded credibility: he disputed in foreign universities with the most learned professors, and came off victorious. In this case, we see the effects of perseverance, and at so early an age: how great a prodigy!

An astonishing instance of undivided attention to study is exemplified in that of *Dr. Nicholas Saunderson*, who, although blind from his infancy, had a strong and vigorous mind, and so retentive a memory, that he acquired a perfect acquaintance with the dead languages, and, by hearing Euclid and Archimedes frequently read to him in Greek, became one of the most celebrated mathematicians: he, at length, was elected professor of mathematics in the university of Cambridge. What may not genius, united with perseverance, effect! How industrious, then, should we be who have the use of all our faculties! the *eye* is, certainly, the most important organ in the pursuit of knowledge; it furnishes by far the largest, and, at the same time, the most splendid share of sensible ideas. *With* the full vigour of this superior organ, we have every advantage on our side: *without it*, we are obliged to have the assiduous assistance of others. Such was Dr. Saunderson's case; yet, independently of this, he rose to the highest pitch of excellence, and that in the most difficult paths of literature!

Let us, therefore, strive to learn every thing that is useful and ornamental, and let us learn every thing *well*. *John de Witt*, an eminent Dutch statesman, was grand pensionary of Holland, and executed the multitudinous business of the state, with the greatest apparent ease: this is very easily to be accounted for, as he did but *one* thing at a time, and that one *well.*

It is the disposition of some, that, in whatever points of learning they are most at a loss, these are the points which they almost always wave: this, however, is a very wrong plan; and, by persisting in it, no degree of eminence, in any branch of education, can be attained.

As I have already said, whatever you find obscure in your studies, carefully endeavour to decipher by yourselves; and, should you not be able to meet with entire satisfaction from your own exertions, then apply for help; say what part or parts you do not understand; then will your instructors cheerfully give you every assistance.

But do not rest satisfied in that you are able, by given rules, to *sòlve* the exercises which follow: examine them sentence by sentence, *take nothing for granted,* and, that you may thoroughly understand all you do, let reason be your guide in every step. This is not only the way to gain a complete knowledge of what you pursue, but the very best means of retaining it for ever after.

It will be my endeavour, in the ensuing divisions of the science, that every operation, and each step of every operation, be prefaced with the removal of every difficulty that can possibly occur; and I do not despair that even those, who, comparatively speaking, know nothing of arithmetic, or may be disgusted by its unexplained difficulties, will, by proper attention on their part, acquire a taste for following it up to its highest branches with increased pleasure.

CHAPTER II.

LECTURE I.

ARITHMETIC is both a science and an art. As a science, it explains the properties of numbers: as an art, it teaches the method of computing by them.

It is one of those inventions, of which we often enjoy the advantages, without duly estimating their importance. Ingenious, yet simple, and highly useful, it is so familiar to us from our childhood, that it fails of exciting our admiration.

It is, however, not a matter of surprise that those who have not been taught it *scientifically*, should find no pleasure in the complex parts of it.

The rules upon which all its operations depend, are *Numeration* and *Notation*, *Addition*, *Subtraction*, *Multiplication*, and *Division*. In all calculations, some, and sometimes all, of these rules are necessarily employed.

NUMERATION is the method of *reading*, and NOTATION that of *writing*, numbers.

The following characters were used by the Romans to designate their numbers and perform their calculations,—viz.

one,	five,	ten,	fifty,	hundred,	five hundred,	thousand.
I.	V.	X.	L.	C.	D.	M.

Although it would be no difficult matter for a very young accountant to multiply 1964 by 507, agreeably to our notation; yet the same thing performed by the Roman notation, which is MDCCCCLXIV. multiplied by DVII. might puzzle even the hoary-headed veteran in the field of science. How vastly superior, then, is the mode by which we perform our calculations!

The Greeks, as I have already hinted in the Introduction, employed a notation, not unlike the Roman; yet it is truly wonderful how their mathematicians conquered the difficulties which they had to encounter in their arithmetical calculations; while we are aware that they were engaged in some of a very tedious and complicated nature.

When we examine the basis of the Arabic, or present European notation, it becomes a matter of amazement that the invention was not discovered at a much earlier period.

It proceeds upon a principle exceedingly simple and easy. The supposed reason for 10 being employed as the scale in our excellent notation, arises from the natural circumstance of our fingers; they being the readiest instruments to assist in reckoning.*

* "Nature," says the learned Goguet, "has provided us with a kind of arithmetical instrument, more generally used than is commonly imagined: I mean our fingers. Every thing inclines us to think that these were the first mathematical instruments used by man to assist him in the practice of numeration. Proteus, in Homer, counts his sea-calves by *five* and *five*; that is, by his *fingers*." And Mr. Locke tells us that "the Tououpinambos, certain American Indians, had no names for numbers above *five*: any number beyond that they made out by shewing their *fingers*, and the *fingers* of others that were present."

What is more common than for young arithmeticians, while engaged in simple addition, to count over the *half-gamut* of our notation, (if I may be allowed the expression,) one by one, from thumb to little finger; slowly indeed at first, but afterwards, from practice, enabled to run over it with a rapidity excelled only by the motion of the fingers in stopping the notes in a quick movement on a musical instrument.

The Indians of the East, the supposed inventors of this system, are, to this day, exceedingly expert in calculating by their fingers.*

This method should, however, be avoided; for it is, to say the least of it, allowing the fingers to usurp the empire of the judgment.

The characters which we employ in our notation are, cipher, one, two, three, four, five, six, seven, eight, nine,

0 1 2 3 4 5 6 7 8 9

These figures, it is evident, increase in a progression of *ones* from the cipher, and may, with no impropriety,

* In the year 1811, I was invited to an examination of one of the first public schools in Edinburgh, where the youth of both sexes are at the same time instructed in writing and arithmetic.

The master asked me to examine one of the classes, with regard to their proficiency in arithmetic: they were young ladies, and certainly performed their several calculations with a rapidity that commanded admiration. They, however, had previously practised each rule for a considerable time before the examination. Although this could reflect very little discredit on their part, yet I had reason to find fault with them for their strict adherence to the Indian method, namely, that of counting their fingers; which they certainly did with a velocity almost beyond description.

be termed the *Numerical Alphabet:* for, as no word can be formed without a letter, or a combination of letters, so no number can be represented but by one or more of the above characters.

The vowels, with their combinations, have each a particular power in regulating the sounds which we wish to represent in language, and are therefore the scale of pronunciation. In like manner the cipher, combined with the other characters of the numerical alphabet, may be made to represent any number whatever.

It is plain that, unless we employ some check on our numeration, we will be apt to lose our reckoning as we advance. What, then, is the most obvious method of securing accuracy in our calculation? Is it not to count by some fixed number, beyond which we shall not proceed?

In the scale of music, we have G, A, B, C, D, E, and F; and then proceed to G, A, B, C, D, E, and F, again, which are precisely the same as the first, with this difference—that each of the latter is just an octave higher than the former. Music is therefore reckoned by octaves, in the same way as we reckon by *tens* in Arithmetic; for, we first proceed from 1 to 10; then from one 10 to ten tens, or 100, &c. Now, ten units are but 1, in the place of tens; and 100 units make but 10 in the place of tens, and 1 in the place of hundreds. This increase by ten is, therefore, the check which we have in reckoning; for, as I have already said, ten units make one ten; ten tens make one hundred; and ten hundreds make one thousand, &c.

How easy, then, in this case, to reckon any number, even of the greatest magnitude!

Yet, although beginners easily conceive what is meant by 2, 3, or 4, they may have no distinct idea of two or three hundred; and one thousand pebbles placed in a row would to them be a very tiresome task to reckon, one by one: but, should they be arranged, first in rows each containing ten, there would be one hundred such rows; and these again arranged in ten rows, each containing ten tens, the whole number of pebbles would, even by them, be very easily reckoned. Thus, by classing numbers in this way, agreeably to the scale of our numeration, every difficulty is at once removed.

Every figure has two values: first, its *real* or *permanent* value; secondly, its *local* or *changeable* value. Thus, 5 standing alone signifies five units, which is its *real* or *permanent* value; but, if any figure be put on the right of it, it instantly becomes ten times 5, or 50; place another on its right, and it is thereby ten times 50, or one hundred times 5, which are 500: this is its *local* or *changeable* value. From this it is evident that a figure standing singly, or on the right of others, signifies so many *units*; in the second place, so many *tens*; in the third place, so many *hundreds*; and in the fourth place, so many *thousands*, &c.

The cipher has no value of its own, but, placed on the right of any number, it increases the other in a tenfold proportion; and neither increases nor diminishes a number by being placed on its left.

Thus 8 = eight units or ones;
80 = eight tens, or eighty units;
800 = eight hundred, or eighty tens;
and 08 = no tens, and eight units;
008 = no tens, no hundreds, and eight units.

It is evident, then, in this case, that a cipher, placed in the right of any number, increases it in a tenfold proportion, and two ciphers annexed increase it a hundred-fold, &c.; but, ciphers placed on the left of integers, neither increase nor diminish them. Figures, therefore, *increase* in a ten-fold proportion from right to left, and *decrease* in the same proportion from left to right.

The following rule will be found of use in *reading* any number:—First, divide the number into periods of six figures each from the right, by means of a perpendicular line, and these again into half-periods with a comma.

The figure on the right of each division, whether period or half-period, represents so many hundreds; the second, so many tens; and the third, so many units of the class wherein they are situated; which will be very clearly seen from the following scheme:

Trillions.	Billions.	Millions.	Units.
678,943	678,943	678,943	678,943

Here the 9, which is the first on the right of the division in the first half-period, is 9 hundred; 4 is 4 tens, or 40, and 3 is 3 units; then the 6, which is the first on the right of the whole period, is 6 hundred, but these are 600 thousand; 7 is 70 thousand, and 8 is 8 thousand, &c.

From this it is evident that, in every whole-period, you have so many units, tens, hundreds, thousands, tens of thousands, and hundreds of thousands. Thus, in the first you have 678 thousand, 943 *units*; in the second you have 678 thousand, 943 *millions*; in the third, 678 thousand, 943 *billions*; and in the fourth, 678 thousand, 943 *trillions*. Here it may be observed, that each period in any number is repeated in precisely the same manner, with this difference, that, at the end of each, the period to which it belongs is repeated.

There is no definite number of periods in figures, but seldom more than the following,—namely, units, millions, billions, trillions, quadrillions, quintillions, sextillions, septillions, octillions, nonillions, and decillions.

NOTATION, which is the next part of this section, and which is the method of *writing* a number, may be easily managed by the following rule:—Put down a cipher for every place which the number contains; then put each significant figure in its proper position, by rubbing out the cipher which occupies its place. Thus, suppose it were required to write seven thousand million, three hundred and twenty-one thousand, and fifty-six; you then write down a cipher for each step in this number, till you come to thousands of millions, in the following manner:

th. of m.	h. of m.	t. of m.	m.	h. of th.	t. of th.	th.	h.	t.	u.
0	0	0	0	0	0	0	0	0	0
7 ,	0	0	0 |	3	2	1 ,	0	5	6

Nothing can be more simple than this method of expressing a number; another example is, therefore, unnecessary. But, since there is a difficulty in the apprehension of very high numbers, it may not be superfluous to shew you the value of *million*, *billion*, *trillion*, and *quadrillion*.

Suppose a person were engaged in counting *sovereigns*, and counted a hundred in a minute, continuing at it twelve hours each day, it would be 13 days, 10 hours, 40 minutes, before a *million* could be counted.

A thousand persons, in counting *a billion*, would be 38 years, 18 days, 10 hours, 40 minutes, in doing it.

If we suppose the whole population of England and Wales, which at present is 9500,000 souls, to be employed in counting a *trillion*, they could not perform it in less than 4005 years, 163 days, 3 hours, 38 minutes; which is just 1 year, 163 days, 3 hours, 38 minutes, more than the time which elapsed from the Creation of the World to the commencement of the Christian era.

If the whole of the inhabitants of Europe, Asia, Africa, and America, Australasia, Polynesia, and the Isles of the Pacific Ocean, amounting to 700500,000 human beings, were employed in counting a *quadrillion*, they would be employed no less than 54320,842 years, 291 days, 6 hours, 28 minutes; and, for an individual to count even the number of years which they would take to reckon a quadrillion, he could not do it in less than 2 years, 24 days, 5 hours, 28 minutes.

In fact, it is only by such calculations as the above that we can have any conception of numbers so immense!

The relative magnitudes of numbers, from 1 to one million inclusive, are clearly exhibited to the eye in the following Diagram.

From what has been said on Numeration and Notation, it is obvious that there is at least as great a simplicity in *reading* and *writing* numbers, as there is in reading and writing monosyllables: yet how few, even of those who are supposed to have finished their numerical studies, know even these, the first two necessary steps, and which are the most simple parts of Arithmetic!

It certainly does not argue greater ignorance in writing o, n, e, for *ten*, than it does in writing 0, 1, for 10. Yet I was very much surprised lately, in opening a Hymn-book in church, which belonged to a seemingly respectable and accomplished person, with the Roman characters obliterated by a pencil, and the Arabian numerals written underneath; and, what was still more astonishing, from the 100th Hymn, 1001, 1002, 1003, &c. were invariably written, instead of 101, 102, 103, &c. Now, the least acquaintance with the part of the subject which we have just finished, would have prevented so glaring a display of ignorance in numbers.

CHAPTER III.

LECTURE II.

SIMPLE ADDITION is the method of collecting two, or more, integral numbers into one sum.

Rule.—First arrange the several numbers in such a way, that digits of the same name may stand perpendicularly to each other: that is, units under units, tens under tens, and hundreds under hundreds, &c.; then draw a line under them, and first add the units' place, setting down the right-hand figure of the sum, and carrying the left-hand figure or figures to the next superior place, which is the tens column; then find the amount of this second column, setting down the right-hand figure of the sum, and carrying the left to the next superior place, as before; and so on with all the other columns, observing to set down the full amount of the last.

Setting down the right-hand figure and carrying the left, is precisely the same as carrying one for every ten, and setting down the remainder, as the common rule directs. Let us add

738
964
638
and 596 together;

The sum is 2936. Here the amount of the units' place is 26 = 2 tens and 6 units: you, therefore, set down the 6 units under the units' column, and carry the 2 tens to the place of tens. In the same manner you proceed with the amount of the tens, which is 23 = 2 hundreds and 3 tens: you, therefore, place the 3 tens under the tens' column, and carry the 2 hundreds to the position of hundreds. Lastly, the amount of the last column is 29, which you set down in full.

You should, while adding the columns, endeavour to spend the least possible time. It is unnecessary, in the first column of the above example, to say, 6 and 8 are 14; 14 and 4 are 18; and 18 and 8 are 26. Here there is a very great waste of words, and consequently of time; for it requires no less assistance than twenty-one syllables to find the sum of four figures.

And to say, 6 and 8 are 14, and 4 are 18, and 8 are 26, requires less time, there being but seventeen syllables employed. But it may be added in a much shorter period than either of the above methods; by simply telling, at each step, the *amount* that every additional digit makes, without naming the digit which produces such amount, thus, 6, 14, 18, 26. This is done in eight syllables.

Adding one column by the first method, would very nearly occupy as much time as three columns of the same length would by the last method; and adding one column by the second method, would require more than twice the time by the last method.

The last method should, therefore, have a decided preference to either of the other two; as it gives less trouble,—is more easily performed,—is less bewildering to the young learner, and does not hinder the rapid flight of the more advanced; besides, it is more elegant.

In Addition, as you have seen in the foregoing example, you take the sum of the digits in the right-hand column, or place of units; proceeding in the same manner with each succeeding column to the left, observing to set down the full amount of the last, or left-hand column.

Although this method is certainly the most expeditious way of adding integers, yet there is no absolute necessity for commencing at the right, or place of units, and proceeding to the left, or highest place. It is of no consequence where you begin, or where you end; only great care must be taken, when you do not proceed from right to left, to give the sum of each column its proper name and place.

Suppose, contrary to the usual way, we commence in the above example at the left, and proceed toward the right, the amount then of the first column is 27; but, as this sum is the amount of the third position, it is therefore 27 hundreds, or 2 thousand 7 hundred; the amount of the next is 21 = 21 tens, or 210; and the sum of the last is 26 units. Here, then, we have

first,	2,700	
then	210	
and lastly	26,	the sum of which
is	2936	as before.

And, as a proof that it is of no consequence whatever where we begin our addition, we have but to arrange the above sums in any way we choose, or in all the ways that it is possible to place three different things, and we shall find equal results. Thus—

	2700		2700		210
(1)	210	(2)	26	(3)	2700
	26		210		26
	2936		2936		2936

	210		26		26
(4)	26	(5)	2700	(6)	210
	2700		210		2700
	2936		2936		2936

It perhaps may appear more clearly to the eye in the following form,—namely, to omit the ciphers; putting down only the amount of each column immediately below itself. Thus

	738		738		738	
(1)	964	(2)	964	(3)	964	
	638		638		638	
	596		596		596	
	27		21		26	
	21		27		21	
	26		26		27	
	2936		2936		2936	&c.

Now we see that, by every different way that it is possible to invent, we have precisely the same results. The inference, then, that we derive from this is, that it neither signifies where we begin nor where we end; but, as proceeding from right to left saves us the trouble of writing down the intrinsic amount of each column separately, and afterwards finding their amount, as we have done in the above cases, as well as being better adapted for the combination of the several columns as we proceed, it is certainly preferable to any other method. This is all that is necessary to be said on adding simple numbers.

Proof.—Dr. Wallis gives the following rule to prove whether the total of two or more integral numbers be correct:—" Add the figures in the uppermost row together, reject the nines contained in their sum, and set the excess directly even with the figures in the row. Do the same with each row, and set all the excesses of nine together in a line, and find their sum. Then, if the excess of nines in this sum (found as before) be equal to the excess of nines in the total sum, the work is right."

Proof by the above Rule.

738	. .	0	Remainders after rejecting the nines.
964	. .	1	
638	. .	8	
596	. .	2	
2936	. .	2	Proof.

The excess of nines in the first row is 0; in the second, 1; in the third, 8; and in the fourth, 2; the amount of these excesses is 11, and the excess above nine in 11 is 2; and we find the excess of nines in the total (2936) to be 2 also. This is the proof of the correctness of the total, agreeably to the rule.

But, with due deference to the superior talent of Dr. Wallis, I beg leave to say, that, although his rule certainly will prove a sum to be correct that *is so* before the application of the rule, as in the above case, yet there is a possibility of this very rule proving a sum to be *right* that is *palpably wrong*.

Fallibility of the Doctor's Rule.

We found the excess above nine in the sum of the several excesses to be 2: now, by the rule, it is necessary that the excess of the total be 2 also; which it certainly is, and is, as I said before, *proving the sum to be correct, because it is so.*

But suppose that, instead of 29 (the amount of the last column), the young pupil should invert the order of the number, (which, by the by, is no uncommon case with a beginner,) we should then, instead of 2936, have 9236. Now, the excess above nine in either case is precisely the same, namely 2. From this it is evident that 9236, or, of course, any other number whatever having the same excess, would, agreeably to the Doctor's rule, naturally enough, be thought correct. But our previous results, in the different methods of performing *this* example, show at once the fallacy of that rule, which can thus prove a sum to be *right* which may be *evidently wrong.*

The best method of Proof is, first carefully to add each column upwards, and then downwards; and, if the amount be the same each way, it is very likely to be correct. Thus, in adding the first column upwards, we say 6, 14, 18, 26; and, in adding it downwards, we say 8, 12, 20, 26; so that, 26 being the sum either way, we conclude that this amount is correct. There is undoubtedly less risk in a person *telling* wrong who counts a sum of money *twice*, than there is with one who does it by a method which *may* or *may not*, according to circumstances, be correct.

CHAPTER IV.

LECTURE II. CONTINUED.

SIMPLE SUBTRACTION is the method of finding the difference between a greater and a less integral number, the greater of which is called the *minuend*, and the less the *subtrahend*.

In Subtraction, as in addition of simple numbers, units must be placed under units, and tens under tens, &c. and, when the digits in the minuend, or greater number, are greater than those of the subtrahend, or less number, we simply write the difference below the subtrahend, with a line between them, in the following way:—

5467893 minuend.
2154321 subtrahend.

3313572 difference, or remainder.

1 from 3, and 2 remain; 2 from 9, and 7 remain; 3 from 8, and 5 remain, &c. But, as it is best to be as laconic as possible, the word *remain* may be left out altogether; thus—1 from 3, and 2; 2 from 9, and 7; 3 from 8, and 5, &c. This is the usual way of subtracting a less digit from a greater.

But I think that the young pupil might sooner master this part of subtraction, by asking how many must be added to the lower digit to make it equal to the upper one. The answer to this query would just be the difference between the two digits. Thus, How many added to 1 will make 3? Answer, 2: then 2 is the difference between 1 and 3.—Again, How many added to 2 will produce 9? Answer, 7: then 7 is the difference between 2 and 9, &c.

The above example is among the simplest cases that can possibly occur in Subtraction, as there is no carrying introduced in it. But, when any figure in the lower line is greater than its corresponding figure in the upper line, we then, agreeably to the common rule, add 10 to the upper figure, subtract the lower one from the amount, set down the difference, and carry 1 to the next figure in the lower line. This will be illustrated in the following example:

$$\begin{array}{r} 941189 \\ 561984 \\ \hline 379205 \end{array}$$

4 from 9, and 5, or 4 wants 5 to equal 9; 8 wants 0 to equal 8. The next position is the case in point. We cannot take 9 from 1, nor can we ask, how many added to 9 will make 1? No; we must add 10 to the 1, and say 9 from 11, which leaves a remainder of 2. We now carry 1 to the next figure in the lower line, for the 10 which we added to the upper figure: and the reason for this is very evident from the following consideration,—that, if we take any less number from another which is greater, we shall have a certain remainder; and, if we add 10, or any other number to each of them, the remainder will still be the same.

Thus— 12 from 24 = 12
(adding 10 to each) 22 from 34 = 12
(adding 50 to each) 62 from 74 = 12 &c.

Thus we see that, let us add any number whatever to each of the numbers in subtraction, that the remainder must still be the same as if no addition had been made; and this is a sufficient reason for adding 1 to the next position in the lower line, for the 10 added to the top figure on the right of it; for 10 in any position are but equivalent to 1 in the next position towards the left.*

Another method, and I think a better, is to add 10 to the less digit, by which means the digit on its left will be diminished by 1: this will, of course, preclude the necessity of carrying 1 to the next figure in the lower line; so that by this means we shall have no carrying at all, thereby rendering subtraction less formidable to the young pupil.

If in this case there should be ciphers in the minuend, conceive them all to be *nines*, and the significant figure which precedes them lessened by 1. I shall explain this by the following example:

7940076 minuend.
2150679 subtrahend.

5789397 difference.

We here say 9 from 16, and 7, and then not 8 from 17, as by the common method, but 7 from 16; for, in adding 10 to the first position, we necessarily lessened the

* See Russell's Companion to Arithmetic.

preceding digit by 1, the 7 therefore became 6: consequently we say 7 from 16, and 9.

Then 6 from 10, lessened by 1 for the 10 borrowed, and there remains 3; and 0 from 10, lessened by 1, and there remains 9; and 5 from 14, lessened by 1, and there remains 8, &c. Or, agreeably to the above remark, conceive the two ciphers nines, and the digit which precedes them lessened by 1, then we have but to say 6 from 9, and 3; 0 from 9, and 9; 5 from 13, and 8, &c.

Proof.—There are two methods of proving subtraction: thus, if we add the difference between any two numbers to the less, it is obvious that their sum will be equal to the greater; which is the same as when a person, after having paid a certain portion of a debt, finds a balance which he still owes, and which, if added to the sum which he has already paid, would cancel the whole; or (which is the same thing) their sum would just be equal to the whole of the debt. Thus—

	7940076	= the greater number, or debt;
and	2150679	= the less number, or sum paid;
then	5789397	= the difference, or balance unpaid;
and	7940076	= the greater number, or debt.

And it is also obvious that the difference between any two numbers taken from the greater, will leave the less, or (which is the same thing) a balance which is owing, taken from the whole debt, will leave the sum which was formerly paid, thus—

	7940076	=	the greater, or debt;
and	2150679	=	the less, or sum formerly paid;
then	5789397	=	the difference, or balance owing;
and	2150679	=	the less, or sum formerly paid.

Addition and Subtraction of simple numbers have, it is presumed, been so fully and clearly explained, that further elucidation on these heads is unnecessary.

CHAPTER V.

LECTURE III.

SIMPLE MULTIPLICATION is a compendious way of performing simple Addition; for the number by which any other number is to be multiplied, is the number of times that that number is to be repeated. Thus $10 \times 5 = 50$, and $10 + 10 + 10 + 10 + 10 = 50$ also.

As a proof that Multiplication is a short method of performing Addition, suppose we had to add 123,456,789 twenty times repeated, a tolerably expert adder would take ten minutes to perform it: whereas, the same thing could be done by Multiplication in about half a minute.

The number to be multiplied is called the *multiplicand;* that by which we multiply, the *multiplier;* and the result is the *product.* The multiplicand and multiplier are often called the *factors* of their product; and these numbers as frequently get the name of *component parts.* Thus, in the above example, 10 is the multiplicand, and 5 the multiplier; and both of these are factors, or component parts, of 50; and 50 is the product: 50 is, of course, a *composite number*, since the product of two or more numbers can produce it, for $5 \times 5 \times 2$, as well as $10 \times 5 = 50$. (See Russell's "Companion to Arithmetic.")

Any product is said to be a *multiple* of either factor; and either factor is said to be a *sub-multiple* of the product: thus, 50 is a multiple of 10 or of 5; and 10 or 5 is a sub-multiple of 50.

A few remarks on the multiplication table will be of use in this place:—

First, any number multiplied by 10 is that number with a cipher annexed:—

Thus 10 times 1 = 1 and a cipher, or 10
10 times 7 = 7 and a cipher, or 70
10 times 12 = 12 and a cipher, or 120
And 10 times 71 = 71 and a cipher, or 710, &c.

Secondly, 100 times any number is produced by annexing *two* ciphers; 1,000 times, by annexing *three* ciphers; and 1,000,000 times, by annexing *six* ciphers, &c.—

Thus 100 times 123 = 12,300
1,000 times 123 = 123,000
And 1,000,000 times 123 = 123,000,000, &c.

Thirdly, since 10 times any number is so easily produced, 5 times any number may be very readily found by *halving* 10 times the same:—

Thus 5 times 4 = ½ of 40, which is 20
5 times 5 = ½ of 50, which is 25
And 5 times 24 = ½ of 240, which is 120, &c.

We may also observe, that 5 times any *odd* number has 5 for the units' place of the product, as in 5 times 5, which are 25; and that 5 times any *even* number has a cipher in the units' place, as in 5 times 4, which are 20: so that, in the table, we have 5 and 0 alternately in the products.

Fourthly, in 6 times any of the even numbers, we have half of the even number for the tens' place of the product, and the number itself for the units' place:—

Thus 6 times 2 = ½ of 2 for tens, and 2 for units = 12
6 times 4 = ½ of 4 for tens, and 4 for units = 24
6 times 6 = ½ of 6 for tens, and 6 for units = 36
And 6 times 8 = ½ of 8 for tens, and 8 for units = 48
&c.

The product of the odd numbers in 6 times may be found in this way:—annex a cipher to the number which is to be multiplied; to its half add the number to be multiplied.

Thus 6 times 3 = 3, and ½ of 30 = 18
And 6 times 4 = 4, and ½ of 40 = 24, &c.

The product of the even numbers may also be found in this way:—

Thus 6 times 2 = ½ of 20 + 2 = 12
And 6 times 6 = ½ of 60 + 6 = 36, &c.

But the first method is certainly simplest.

Fifthly, 7 times any number may be found thus :— annex a cipher; halve it, as in the last case, to which add twice the number :—

Thus 7 times 2 = twice 2 + $\frac{1}{2}$ of 20 = 14
7 times 3 = twice 3 + $\frac{1}{2}$ of 30 = 21
7 times 7 = twice 7 + $\frac{1}{2}$ of 70 = 49
And 7 times 12 = twice 12 + $\frac{1}{2}$ of 120 = 84, &c.

It may be remarked, that 7 times 3, 6, 9, or 12, gives us the third part of the number to be multiplied as the units' place, and twice this for the tens of the product:—

Thus 7 times 3 = $\frac{1}{3}$ of 3, or 1 for the units, and twice 1 or 2 for tens = 21
7 times 6 = $\frac{1}{3}$ of 6, or 2 for units, and twice 2 or 4 for tens = 42
7 times 9 = $\frac{1}{3}$ of 9, or 3 for units, and twice 3 or 6 for tens = 63
And 7 times 12 = $\frac{1}{3}$ of 12, or 4 for units, and twice 4 or 8 for tens = 84

It may also be observed, that the sum of the digits in the above products is just the number multiplied by 7, and the units just half of the tens :—

Thus 7 times 3 = 21 and 2 + 1 = 3
7 times 6 = 42 and 4 + 2 = 6
7 times 9 = 63 and 6 + 3 = 9
And 7 times 12 = 84 and 8 + 4 = 12

Sixthly, 8 times any number is equal to 10 times the number minus twice the same :—

Thus 8 times 2 = 20 — twice 2 = 16
8 times 3 = 30 — twice 3 = 24
8 times 4 = 40 — twice 4 = 32
And 8 times 7 = 70 — twice 7 = 56, &c.

Seventhly, from your knowledge of 10 times, the product of 9 times any number may be very easily ascertained, for 9 times any number is just 10 times less once the same:—

Thus 9 times 2 = 20 — 2, or 18
9 times 3 = 30 — 3, or 27
9 times 4 = 40 — 4, or 36
9 times 7 = 70 — 7, or 63
9 times 9 = 90 — 9, or 81
And 9 times 12 = 120 — 12, or 108, &c.

It is rather remarkable, that the sum of the digits in 9 times amounts to 9: thus 9 times 2 are 18, 1 and 8 are 9; 9 times 3 are 27, 2 and 7 are 9; 9 times 4 are 36, 3 and 6 are 9, &c.

It is evident from the above products, that the tens' place of each is just 1 less than the digit multiplied by 9, and the units' place is the difference between the tens' place, and 9:—

Thus 9 times 2 = 1 ten and 8 units = 18
9 times 3 = 2 tens and 7 units = 27
9 times 4 = 3 tens and 6 units = 36
9 times 5 = 4 tens and 5 units = 45
9 times 6 = 5 tens and 4 units = 54
9 times 7 = 6 tens and 3 units = 63
9 times 8 = 7 tens and 2 units = 72
9 times 9 = 8 tens and 1 unit = 81

Eighthly, 11 times any number is also easily discovered, for 11 times any number is 10 times plus once the same:—

Thus 11 times 2 = 20 + 2 = 22
11 times 5 = 50 + 5 = 55
And 11 times 9 = 90 + 9 = 99, &c.

Or it may be found in a much simpler way—thus: 11 times any of the 9 digits is just 2 of the digit to be multiplied, the one considered as the units, and the other as the tens of the product:—

Thus 11 times 3 = 3,3, or 33
11 times 4 = 4,4, or 44
And 11 times 6 = 6,6, or 66, &c.

But when the number to be multiplied is above 12, and not beyond 99, it may be managed in the following way:—Set down the number to be multiplied, keeping the digits of it a little detached from each other, and between them place their sum: this will produce the product: but this can as easily be reckoned mentally as on your slates. Thus in 11 times 15:—imagine you see the digits of 15, which are 1 and 5, a little apart, and between them place the sum of 1 and 5, which is 6; the product is, therefore, 1,6,5, or 165. In 11 times 24, we place between the two digits, 2 and 4, the sum of 2 and 4, which is 6; we, therefore, have 2,6,4, or 264, which is the product.

By the same rule:—

11 times 17 = 187
11 times 25 = 275
And 11 times 36 = 396, &c.

But, should it happen that the sum of the two digits amounts to more than 9, we make the left hand digit one more: thus in 11 times 19, we have the sum of 1 and 9, which is 10, for the middle number; but, as our numeration increases in a ten-fold proportion from right to left, and as this sum of 10 occupies the second or tens' place, it is equal to a hundred, or one of the next place; this 1 must, therefore, be added to the digit on the left; and, as the sum was no more than 10, we put a cipher in the middle: the product is, therefore, 209:—

Consequently 11 times 28 = 308
11 times 29 = 429
11 times 98 = 1078, &c.

Ninthly, as 12 times any number is equal to 10 times the number added to twice the same, or equal to 11 times any number added to once the same, we have:—

1st. 12 times 1 = 10 + twice 1 = 12
12 times 3 = 30 + twice 3 = 36
12 times 7 = 70 + twice 7 = 84
And 12 times 9 = 90 + twice 9 = 108, &c.

2dly. 12 times 2 = 22 + 2 = 24
12 times 3 = 33 + 3 = 36
12 times 7 = 77 + 7 = 84
12 times 9 = 99 + 9 = 108, &c.

This method is preferable to the first.

These are remarks which may be of use to the advanced pupil, and of singular service to the beginner, while learning this grand foundation of every calculation.

CHAPTER VI.

LECTURE III. CONTINUED.

The product of any two numbers will be the same whichever of them you make the multiplier. Thus $7 \times 4 = 28$, and $4 \times 7 = 28$ also. Many conceive this to be so self-evident as not to require proof; but they are certainly deceived by their familiarity with the fact. It is by no means evident, at least to me, at first sight, that the sum of 4 sevens and that of 7 fours is the same, or, which is the same thing, that $7 + 7 + 7 + 7 = 4 + 4 + 4 + 4 + 4 + 4 + 4$. The only proof that I can adduce is the following, which I conceive to be so very simple, that an infant may almost be persuaded of its truth. Take 4 rows, each containing 7 counters, and you will find their sum equal to 28. Take also 7 rows, each containing 4 counters, and you will also find their sum 28, as before.

Thus 4 rows, each of 7 counters:—

$$
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & = & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & = & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & = & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & = & 7 \\
\hline
\multicolumn{7}{c}{4+4+4+4+4+4+4} & = & 28
\end{array}
$$

And 7 rows, each containing 4 counters :—

0	0	0	0	=	4
0	0	0	0	=	4
0	0	0	0	=	4
0	0	0	0	=	4
0	0	0	0	=	4
0	0	0	0	=	4
0	0	0	0	=	4
7 +	7 +	7 +	7	=	28

From the above schemes it is evident, even from ocular demonstration, that 7×4 and 4×7 are the same, for, count each scheme whichever way you will, still you find the same result. This will hold true with any other factors whatever: thus 6 times 9 are the same as 9 times 6, and 12 times 5 the same as 6 times 10, &c.

The product of any number by 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12, will, from a thorough knowledge of the multiplication table, be as easily found as adding 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12 numbers together for the purpose of finding their sum.

Thus 1742
Multiplied by 6

10452 product.

Here we say 6 times 2 are 12; put down 2, and carry 1, which is precisely the same as if we had added 6 twos, or 2 sixes, together; then, 6 times 4 are 24 and

1 are twenty-five, 5 and carry two; 6 times 7 are 42, and 2 are 44, 4 and carry 4; and 6 times 1 are 6, and 4 are 10. It is clearly evident from this, that, with a *perfect* knowledge of *the table*, Multiplication will be found, not only more expeditious, but even simpler than Addition.

Multipliers above 12 may be managed in various ways: thus, if 15 be the multiplier, we may multiply the top line, or multiplicand, by 12, and by the difference between 12 and 15, which is 3: these two sums, added together, will be the required product.

Let us multiply 7896 by 15:—

1st.	7896 × 3	
	12	
	94752	product by 12
	23688	ditto by 3
	118440	ditto by 15

Or 10 times the top line, added to 5 times the same will produce the same result:—

Thus	2d.	7896 × 5	
½ 78960		10	
39480			
		78960	the product by 10
118440		39480	ditto by 5
		118,440	ditto by 15 as before.

Or we may first multiply by 5, then by 10, and add the two sums together for the product, thus—

```
3d.      7896 × 10
            5
        ------
        39480 product by  5
        78960  ditto  by 10
       -------        --
       118,440 ditto  by 15 as before.
```

From this we may observe that, as 10 times any number always ends with a cipher (see p. 37); and, as a cipher is of no value in addition, this gives rise to the rule for commencing each succeeding product in place more to the left, omitting the cipher; or, which is the same thing, beginning the product of each significant figure in the multiplier immediately below itself. Thus—

```
4th.     7896
           15
        -----
        39480
        7896
       ------
       118,440 as before.
```

Here we see the product of the 5 begun below itself, and that of the 1, in the second place, which is the position in which *it* stands; the cipher omitted having no effect whatever on the addition.

As 15 is a composite number, and its component parts 3 and 5, or 5 and 3, it may be managed thus:—

5th.	7896	Or	7896
	3		5
	23688		39480
	5		3
	118,440 as before		118,440 as before.

If we commence the product of each digit immediately below itself, it does not signify which digit we begin or end with: thus—

```
6th.   7896
         15
       ----
       7896
      39480
     ------
     118,440 as before.
```

And to multiply by a number containing two places, when unity, or 1, is the *left hand* figure, we have but to multiply by the other digit, and to the product of every figure in the multiplicand add its *right hand* figure, to produce the product in one line, thus—

```
7th.   7896
         15
     ------
     118,440 as before.
```

Here we say $6 \times 5 = 30$; but, as there is no figure to the right of 6 in the multiplicand, we put down a cipher, and carry 3; $9 \times 5 + 3 +$ the right hand figure to 9, which is 6, equal to 54, 4, and carry 5; $8 \times 5 + 5 + 9 = 54$, 4, and carry 5; $7 \times 5 + 5 + 8 = 48$, 8, and carry 4; then this 4, added the remaining right hand figure, produces 11, which we also put down in the product. From a review of method 4th, (which see,) the reason of this is perfectly evident. By this method we can multiply simple numbers by 13, 14, 15, 16, 17, 18, and 19, in one line, with the greatest ease.

When the *right hand* digit of two places is unity, the product mny also be produced in one line, by first putting down the units' plåce of the multiplicand, and then adding to the product of each succeeding place in the multiplicand its *left hand* figure, thus—

8th. Multiply 7896
By 31

244,776

First we put down the units' place of the multiplicand, which is 6; then $\overline{6 \times 3} + 9 = 27$, 7, and carry 2; $\overline{9 \times 3 + 2} + 8 = 37$, 7, and carry 3; $\overline{8 \times 3 + 3} + 7 = 34$, 4, and carry 3; and $\overline{7 \times 3 + 3} = 24$. This process is evident from the 6th method, (which see.) After the same manner we may very easily multiply simple numbers by 21, 31, 41, 51, 61, 71, 81, and 91, in one line.

This gives rise to a very simple method of multiplying by 11, for we have only to set down the units' place of the multiplicand, and afterwards add the tens and units, then the hundreds and tens, &c. till all the places of the multiplicand are exhausted, thus—

To multiply 7896
By 11

86,856 the product.

Here we put down 6; then $9 + 6 = 15$, 5 and carry 1; $\overline{1 + 8} + 9 = 18$, 8 and carry 1; $\overline{1 + 7} + 8 = 16$, 6 and carry 1; $1 + 7 = 8$. This is the same as if to 10 times the multiplicand we added once the same.

From what has been said of 9 times, in the observations on the multiplication table, (p. 39.) the multiplication of any simple number by 9 may be managed in the following way, thus—

To multiply 7896
By 9

71064 the product.

We suppose a cipher annexed to the multiplicand, which is, in fact, multiplying it by 10, then subtract from right to left, as follows:—6 from 10, and 4; $\overline{1+9} = 10$, 10 from 16, and 6; $\overline{1 + 8} = 9$, 9 from 9, and 0; 7 from 8, and 1; and then, as there is no figure to take from the 7, we put it down on the left of the other remainders.

This is the same as if from 10 times the multiplicand we subtract once the same, which, of course, leaves 9 times the multiplicand.

Therefore, to multiply by any number consisting wholly of nines, we have but to annex a cipher to the multiplicand for each 9 in the multiplier, and subtract the multiplicand from it. Thus, to multiply 7896 by 999, we say—

From	7896000	the product	by	1000
Take	7896	ditto	by	1
And there remains	7888104	ditto	by	999

From what has been said in p. 37, 1000 times any number is that number with three ciphers annexed, and, if from 1000 times any number we subtract once the same, it must leave a difference equal to the product by 999, since $1000 - 1 = 999$.

This method, as I said before, will hold good with any number of nines.

Let us now multiply	7896	by 999,999		
Then from	7896000000	product	by	1,000,000
Take	7896	ditto	by	1
There remains	7,895 | 992,104	ditto	by	999,999

Should there be any other digit in the multiplier besides nines, we have first to find the product, as if they were all nines, and from this result take the product of the multiplicand by the difference between such digit and 9, observing to place the sum to be subtracted immediately under the place where the digit stands:—thus, to multiply 7896 by 919:

From 7896000
Take 7896 × 8, the difference between 1 and 9

And from 7,888,104 product by 999
Take 63168 ditto by 80

And there remains 7,256,424 ditto by 919

Secondly, to multiply 7896 by 199, we say—

From 7896000
Take 7896 × 8

And from 7,888,104 product by 999
Take 6,316,8 ditto by 800

And there remains 1,571,304 ditto by 199

Thirdly, to multiply 7896 by 991, we say—

From 7896000
Take 7896 × 8

And from 7888104 product by 999
Take 63168 ditto by 8

And there remains 7824936 ditto by 991

But when 1 happens to be on the extreme right or left of the multiplier, it is, perhaps, better to find the product by the number of nines, and add the multiplicand, placed under the position in which it stands in the multiplier. Thus, to multiply 7896 by 199:—

From	789600		
Take	7896		
And to	781704	product by	99
Add	7896	ditto by	100
And we have	1,571,304	ditto by	199 as before.

Secondly, to multiply 7896 by 991 :—

From	789600		
Take	7896		
And to	781,704	product by	99
Add	7896	ditto by	1
And we have	7,824,936	ditto by	991 as before

Thirdly, from what has been said there can be no difficulty of multiplying by any number of nines having unity both on right and left. Thus, to multiply 7896 by 19991, we say—

From	7896000		
Take	7896		
	7888104	product by	999
	7896	ditto by	1
	7896	ditto by	10000
	157848936	ditto by	19991

Let us now multiply 7896 by 98884 :—

From	789600000		
Take	7896	× 1, 1, 1, and 5	
And from	789592104	product by	99,999
	7896	ditto by	1,000
	7896	ditto by	100
	7896	ditto by	10
	39480	ditto by	5
Take	8804040	ditto by	1115
There remains	780 | 788,064	ditto by	98884

More examples of this kind are unnecessary.

When there are ciphers on the right of either or both factors, they may be omitted in the multiplication, and afterwards annexed to the product:—

Thus, to multiply 42000 by 24000
24
168
84
1,008 | 000,000, the product.

It is evident, from what has been said in p. 32, that any number multiplied by 1000 is that number with three ciphers annexed. Since, then, the multiplier is 24,000, all that is necessary in this case is, to annex three additional ciphers to the above multiplicand, and multiply by 24; but it is obvious that, if the ciphers on the right of both factors be omitted in the work, and afterwards annexed, that the same result would be produced, as may be seen in the above example.

Under this head, a recent author says, that, "to attempt formal and scientific proofs of the several rules, in a work intended solely for *youth*, would be superfluous. It is needless to confirm," adds he, "what has already been established, or to prove truths to which all assent;" and "that mathematical demonstrations have been purposely omitted, as they could be of no advantage to the learner, and it is presumed the teacher does not want them."

It is painful for me to mention the faults of my neighbours, yet am I in duty bound, in the present instance, to say, that since the work alluded to is intended *solely for youth*, that it is *youth alone* that require scientific proofs of every step in their calculation; they will thereby better understand what they do, and will also retain it longer. As for those who have already been scientifically taught, they need no such proofs, because they are, or ought to be, familiar to them.

Are we, for instance, to take it for granted that smoke ascends, merely because it is a truth to which every one assents? or that a bullet will descend, if dropt from an eminence? Certainly not: we should examine them carefully for our own satisfaction, and know the reason of their ascending and descending: then, and not till then, should we be convinced of these, or of any other truths. He says that mathematical demonstrations have been omitted, as it is presumed the teacher does not want them; but I would ask what it avails the pupil that the master is possessed of necessary knowledge for the elucidation of any point in his profession, if he is told that they can be of no advantage to his pupil? "Do as I do" is by no means a satisfactory rule to an enquiring mind. In the preface to this work, he seems to be a little inconsistent; for a quotation runs thus:—"The great business of the preceptor, says a late writer, is to see that the learner *understands* the rules, as well as commits them to memory."

CHAPTER VII.

LECTURE III. CONTINUED.

WHEN the multiplier has significant figures on the right, with a mixture of significant and insignificant figures on the left, we disregard the ciphers, and multiply by the significant figures only; and the units' place of each product must be placed immediately under the multiplying figure: thus—

```
To multiply 3964285
         By 7004032
         ----------
            7928570
          11892855
        15857140
     27749995
     --------------
     27765978997120 the product.
```

When one part of the multiplier is a component part of another, a succeeding product may be more easily found from it than by repeating the multiplicand, and that in fewer figures: thus—

```
1st.    To multiply 546
                 By 328
                 ------
                   4368 × 4, for 8 × 4 = 32
                 17472
                 ------
                 179088 the product.
```

2dly. To multiply 546
By 546

```
         546
      By 546
      ------
        3276  × 9, for 6 × 9 = 54
       29484
      ------
      298116  the product.
```

3dly. To multiply 784329
By 108549

```
        784329
     By 108549
     ---------
       7058961  × 6, then by 12, or
      42353766  × 2
     84707532
   -----------
   85138128621  the product.
```

It is sometimes of advantage to multiply by a greater than the given multiplier, and afterwards to subtract a product by the difference between the real and assumed multiplier: thus—

1st. To multiply 5467 by 1988

```
  5467  × 12
     2
--------
10934000        product by        2000
   65604  which is too much by      12
--------                          ----
10868396    the product by        1988
```

2dly. To multiply 5467 by 5994

5467 × 6		
6		
32802000	the product by	6000
32802	which is too much by	6
32769198	the required prod. by	5994

There are many other advantages which may be taken in multiplying, but which cannot be exhibited till something has been said on the rule of Division.

The best method of proving multiplication, when the multiplier is small, is by addition. Thus, if I had to

Multiply 5467
By 5

I should have 27335 for the product. And, to prove by addition that this product is correct, 5467 must be repeated five times, and their sum will be the same as the product :—

Thus 5467
5467
5467
5467
5467

27335 as before

And when the multipliers are high, the proof may be found by making the multiplicand the multiplier, and the multiplier the multiplicand : (see p. 38,) thus—

```
Let us multiply    728  multiplicand
             By    369  multiplier
                  ----
                  6552  × 4, for 9 × 4 = 36
                 26208
                ------
                268632  the product.
```

Now, for the proof, let the factors be changed :—

```
Thus  369  multiplicand
  By  728  multiplier
     ----
     2952  × 9, for 8 × 9 = 72
    26568
   ------
   268632  the proof.
```

As additional proofs, the same example may be performed in various ways :—

```
Thus  728                    Or  728
      369                        369
   ------                     ------
   2184   × 2 and 3             4368
    4368                       2184    × 3
     6552                        6552
   ------                     ------
   268632 as before.          268632 as before.
```

Or 369		369
728		728
2583 × 4		2583
10332		738 × 4
		2952
268632		268632 as before, &c.

The proof by *Division* must be deferred till the pupil has heard something on that head. (See p. 73 et seq.)

The proof by casting out the nines is as false in Multiplication as it has been shown to be in proving Addition; but the above methods are infallible. (See p. 25.)

CHAPTER VIII.

LECTURE IV.

Simple Division is the method of finding how often one simple number is contained in another; or of dividing any given number into any proposed number of equal parts, which is the same as subtracting one number repeatedly from another, for one number is contained in another precisely the number of times that it can be subtracted from it:—thus $12 \div 4 = 3$; and $12 - 4 = 8$; $8 - 4 = 4$; and $4 - 4 = 0$. We have here subtracted 4 from 12 just 3 times; it is, therefore, contained in 12 exactly the same number of times by *Subtraction* as by *Division*. Division is, therefore, a short way of performing Subtraction. For, suppose a person had to divide 7300 by 365: this could be done in half a minute: the same person would take no less than ten minutes to do the same by Subtraction.

The number which is to be divided is the *Dividend*, that by which we divide the *Divisor*; the result is the *Quotient*; and what is over, the *remainder*: thus, in the above example, 12 is the dividend; 4, the divisor; 3, the quotient; and there is no remainder.

The quotient of any number divided by 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12, is the same as that found by subtracting the divisor as many times from the dividend as there are units in the divisor, which is evident from the above example.

Division by any number, from 2 to 12 inclusive, may be performed by multiplying and subtracting mentally, writing down only the quotient figures below the dividend. Thus, to divide 764 by 2:—

Divisor 2) 764 dividend

382 quotient.

We first inquire how often 2 is contained in the highest figure: thus—2 in 7, 3 times, and 1 over; we set down the 3 as the first quotient figure, and carry the 1 to the next figure in the dividend, which is 6; but when 1 is carried to the next position towards the right, it becomes 10 of the same (see p. 16.); we, therefore, next find how often 2 is contained in 16, and since there are 8 twos, and no remainder, we set down 8 as the second figure of the quotient: lastly, we divide the last figure of the dividend, which is 4, by 2, and we get the last quotient figure, which is 2: the division, therefore, of 764 by 2 is just 382.

If we divide all the constituent parts of a number, and add the several quotients, we get the same result as when we divide the number in its combined state. Thus, in the above example, the division of the number 764 in its combined state is 382, and the sum of the division of the constituent parts 700, 60, and 4, we will find the same: thus—

$$\begin{array}{rr} & 700 \div 2 = 350 \\ & 60 \div 2 = \ \ 30 \\ \text{And} & 4 \div 2 = \ \ \ \ 2 \\ \hline & 382 \text{ as before.} \end{array}$$

From this the process in Division is evident.

From having a perfect knowledge of the common multiplication table, division by any number under 13 can be no difficult task; and the division by any composite number, however high, may be as easily done by employing its component parts:—thus, to divide 864 by 24, we have but to divide by the component parts of 24, which are either 2 and 12, 3 and 8, or 4 and 6; now, by employing either, or all of these methods, we may find the quotient very readily by short Division; and, from what has been said in Multiplication, (pages 38 and 39,) it does not signify which component part we commence with : thus—

First Method.

$$24\left\{\begin{array}{r} 2)\ \underline{864} \\ 12)\ \underline{432} \end{array}\right. \quad \text{or} \quad 24\left\{\begin{array}{r} 12)\ \underline{864} \\ 2)\ \underline{\ \ 72} \end{array}\right.$$

36 the quotient. 36 as before.

Second Method.

$$24\left\{\begin{array}{r} 3)\ \underline{864} \\ 8)\ \underline{288} \end{array}\right. \quad \text{or} \quad 24\left\{\begin{array}{r} 8)\ \underline{864} \\ 3)\ \underline{108} \end{array}\right.$$

36 as before. 36 as before.

Third Method.

$$24\left\{\begin{array}{r} 4)\ \underline{864} \\ 6)\ \underline{216} \end{array}\right. \quad \text{or} \quad 24\left\{\begin{array}{r} 6)\ \underline{864} \\ 4)\ \underline{144} \end{array}\right.$$

36 as before. 36 as before.

In the above example, there happens to be no remainder in either division by the component parts; but, when there is a remainder in either, or both, the true remainder is found by the product of the first divisor and last remainder added to the first remainder, if any. (See Russell's Companion, Quest. 55.)

Let us divide 1279 by 36.

The component parts of this divisor are 3 and 12, 4 and 9, or 6 and 6 :—

First Method.

$$36\left\{\begin{array}{rl} 3) & 1279 \\ 12) & \underline{426\text{-}1} \end{array}\right.$$

35 quotient, and $6 \times 3 + 1 = 19$ rem.

Or,

$$36\left\{\begin{array}{rl} 12) & \underline{1279} \\ 3) & \underline{106\text{-}7} \end{array}\right.$$

35 quotient, and $1 \times 12 + 7 = 19$ rem.

Second Method.

$$36\left\{\begin{array}{rl} 4) & \underline{1279} \\ 9) & \underline{319\text{-}3} \end{array}\right.$$

35 and $4 \times 4 + 3 = 19$.

Or,

$$36\left\{\begin{array}{rl} 9) & \underline{1279} \\ 4) & \underline{142\text{-}1} \end{array}\right.$$

35 and $2 \times 9 + 1 = 19$.

Third Method.

$$36\left\{\begin{array}{rl} 6) & \underline{1279} \\ 6) & \underline{213\ 1} \end{array}\right.$$

35 and $3 \times 6 + 1 = 19$, as before.

In dividing by 3 and 12, our first remainder is 1, and second 6; now 6, the second remainder, multiplied by 3, the first divisor, = 18, and 18 added to the first remainder will give 19 as the true remainder.

In dividing by 12 and 3, our first remainder is 7, and second 1; now $1 \times 12 + 7 = 19$, the true remainder, as before.

In dividing by 4 and 9, our remainders are 3 and 4; then $4 \times 4 + 3 = 19$, true remainder, as before.

In dividing by 9 and 4, our remainders are 1 and 2; now $2 \times 9 + 1 = 19$, true remainder, as before.

And in dividing by 6 and 6, our remainders are 1 and 3; now $3 \times 6 + 1 = 19$, true remainder, as before.

Note:—when there is no remainder in the second division, the first is the true remainder.

No treatise of arithmetic that has ever come under my notice has condescended to give a rational reason for the above rule, except one or two, who explain it agreeably to the doctrine of fractions, which the pupil, at this early stage, cannot possibly comprehend.

But, as the method of dividing by component parts is generally recommended to the young pupil, and where the dividend is a high number, it should always have a decided preference to *Long Division;* but why authors have not deigned to satisfy the *young mind,* it is not my part to decipher, although I may here venture to say that *simplicity* in any work, and particularly in a scientific subject, is the *greatest recommendation* in its favour.

I shall now endeavour, and in as simple a manner as possible, to give a sufficient reason for multiplying the last remainder by the first divisor, adding the first remainder (if any), to the product, for the *true* remainder.

Any number, however high, is understood to be so many units, or *ones*, for when we say 100, 1000, 1000000, &c. we mean 100 ones, 1000 ones, and 1000000 ones, &c.

Now *ones*, divided by any number, gives us a quotient, which partakes of the name of our divisor; thus any number of *ones*, divided by 2, produces a quotient of so many *twos;* by 3, so many *threes;* and by 4, so many *fours*, &c.

Thus, if a school of 24 pupils were divided into classes, each containing 2 pupils, we should have 12 such classes, each containing 2; these 12 classes would, therefore, be 12 *twos:* if we put 3 in each class, we would have 8 classes, each containing 3; they would, therefore, be 8 *threes:* and, if we put 4 in each class, we shall have 6 classes, each containing 4 pupils; they would, therefore, be 6 *fours*, &c. and should there be any remainder after such divisions, it, of course, will be so many ones: thus, suppose the same number, 24, to be divided into classes, each containing 5, we should have 4 such classes, and 4 over: we have, in this case, 4 fives, and 4 ones over.

Now, *twos, threes, fours*, or *fives*, &c. divided by any number, will, from the above reasoning, give us remainders containing so many *twos, threes, fours*, or *fives*, &c. and, in order to show the true remainder in dividing by any two component parts, we must bring the second remainder, which, of course, will contain so many *twos*, *threes*, *fours*, or *fives*, &c. into *ones*, and add them to the first remainder of *ones* for the true remainder: now this, in fact, is precisely the same as multiplying the last remainder by the first divisor, and adding the first remainder.

In the above example, in dividing 1279 by 12 and 3, we have first, in the division by 12, which is the first component part, a quotient of 106 *twelves*, and 7 *ones* over; now these twelves, divided by the other component part, which is 3, gives us 35 as a quotient, and 1 *twelve* over; but 1 *twelve* and 7 *ones* are 19 *ones*, which is the proper remainder: this is evidently the same as multiplying 1 (the last remainder) by 12 (the first divisor), and adding 7 (the first remainder), which produces 19 for the true remainder, as before.

In dividing by 4 and 9, in the same example, we have first, in the division by 4, 426 *fours*, and 3 *ones* over; these *fours*, divided by the other component part, gives us 35 for the quotient, as before, and 4 *fours* over; now 4 fours, added to 3 ones, gives us a remainder of 19 ones, as before. It is therefore obvious, that when there is no second remainder, the first is the true remainder.

From the foregoing explanation, the reason of the rule, it is presumed, is made perfectly evident to the meanest capacity, and equally comprehensible to the youngest mind.

From what has been said on the division by *two* component parts, the division by *any number* of component parts may be very easily managed: thus, let us divide 74267 by 288: the component parts of this divisor may be 12, 12, and 2; 8, 12, and 3; or 9, 8, and 4, &c.

Now, by the first component parts we have:—

First.

288 { 144 { 12, 12 }, 2 }

12) 74267

12) 6188 (12's) and 11 (1's) over.

2) 515 (144's) and 8 (12's) over.

257 quotient, and 1 (144) over.

Now 1 (144)	=	144
8 (12's)	=	96
And 11 (1's)	=	11

Therefore 251 is the true remainder.

Then, by the second component parts:—

Second.

288 { 96 { 8, 12 }, 3 }

8) 74267

12) 9283 (8's) and 3 (1's) over.

3) 773 (96's) and 7 (8's) over.

257 quotient, and 2 (96's) over.

Now 2 (96's)	=	192
7 (8's)	=	56
And 3 (1's)	=	3

Therefore 251 the true remainder, as before.

Lastly, by the third component parts :—

Third.

$$288\begin{cases} 72\begin{cases} 9\,)\ \underline{74267} \\ 8\,)\ \underline{8251} \end{cases} \\ \end{cases}$$

9)	74267	
8)	8251	(9's) and 8 (1's) over.
	1031	(72's) and 3 (9's) over.
	257	quotient, and 3 (72's) over.

But 3 (72's)	=	216
3 (9's)	=	27
And 8 (1's)	=	8
Therefore		251 the true remainder, as before.

Therefore, when we have any number of divisors, we have but to multiply each remainder by all the preceding divisors, except the one which has produced it, and add them together for the true remainder. I have now sufficiently explained what is generally termed *Short Division*.

CHAPTER IX.

LECTURE IV. CONTINUED.

Long Division is precisely the same process as Short Division, with this difference, that, instead of multiplying and subtracting *mentally*, we show the whole operation at full length, which gives rise to the name *Long Division.* This method should, however, only be employed when the divisors are either incomposite, or very large.

An easy method of finding the quotient figures in Long Division is, first to see how often the left-hand figure of the divisor is contained in the left-hand figure or figures of the dividend, which we set down on the right of the dividend, as the first quotient figure: secondly, the product of this quotient figure and divisor we subtract from the first portion of the dividend; and, thirdly, to this remainder we annex the next figure in the dividend, and afterwards divide this number in the same way as we did the first for the second figure of the quotient: this process we continue till all the figures in the dividend are exhausted.

But when the product of the quotient figure multiplied by the divisor is more than that part of the dividend which we are dividing, we must make the quotient figure so much less as that the product may not more than equal that which we divide, because, in *arithmetical* calculations, we cannot take a greater number from a less.

And should our remainder at any time equal or exceed our divisor, it is evident, that, as the divisor is contained once in such remainder, that we must make our quotient figure 1 more than that which produced such remainder, from which it is evident that the greatest remainder that can possibly happen, in any division, is 1 less than the divisor. In dividing by 2, the greatest remainder we can have is 1; when 3 is the divisor, the greatest remainder will be 2; and, when 10 is the divisor, the greatest remainder will be 9, &c.

Example.

```
43) 79943 (1859 the quotient.
    43
    —
    369
    344
    —
     254
     215
     —
      393
      387
      —
        6 remainder.
```

Here we find the left-hand figure of the divisor, which is 4, contained in the left-hand figure of the dividend, which is 7, *once;* we, therefore, place 1 on the right of the dividend, as the first figure of the quotient; the product of this quotient figure and the divisor is just 43, and 43 subtracted from 79 leaves 36, which is precisely the same as saying 43 in 79,—once, and 36 over;* we now annex the next figure of the dividend, which is 9, to this remainder, and we have therefore 369 to be divided in the same manner; therefore, 4 in 36, 9 times; but 9 times 43 = 387, and 387 is more than 369, we must, therefore, (as we cannot take 387 from 369,) make our quotient figure one less; we, therefore, put 8 in the quotient; now 8 times 43 = 344, and 369 — 344 = 25; to this remainder we annex the next figure in the dividend, which is 4, and we have 254 as the next number to be divided; then 4 in 25, 6 times, and 6 times 43 being 258, which is greater than 254, we must put only 5 in the quotient; 5 times 43 = 215, and 254 — 215 = 39; we now annex the last figure of the dividend, which is 3, and we have 393 to divide by 43; then we say 4 in 39, 9 times; and 9 times 43 = 387, and 393 — 387 = 6; the quotient is, therefore, 1859, and 6 of a remainder.

* It is probable that the young pupil may enquire, how this method of finding the quotient figures gives the correct ones? Answer. —If the product of 43 × 1 (the first quotient figure,) be not more than 79, and if after subtracting 43 from 79, the remainder should be less than the divisor, the quotient figure is perfectly correct, and this will hold good with all the other quotient figures.

We are compelled to use the above method with incomposite numbers, because our common multiplication table goes no higher than 12; but if we had a table as high as 43, the above division might be done the short way, as follows :—

43) 79943

1859 quotient, 6 remainder.

43 in 79, once, and 36 over; in 369, 8 times, and 25 over; in 254, 5 times, and 39 over; and in 393, 9 times, and 6 over, which just gives the same quotient and remainder as by Long Division.

I am of opinion, that if a multiplication table raised to three or four places of figures, could be got by heart by our young arithmeticians, that Britain would have to boast, not of *one* Zerah Colburn only, as America does, but of *many* ; nay *all* her youth might soon, in this case, be justly esteemed the wonder of the world; for most of their calculations might, by this means, be performed *mentally*, and almost *instantaneously*, and the divisions in the above example might, therefore, be performed as easily as dividing by any number within the limits of our present table.

From the foregoing rule and example, it plainly appears that the division by any number, however high, is as simple as by the divisor which we there employed. Let us, then, divide 123456789 by 76325 :—

```
76325) 123456789 (1617 quotient
        76325
        ------
        471317
        457950
        ------
         133678
          76325
         ------
          573539
          534275
          ------
           39264 remainder.
```

Here, as in the former example, we see how often 7 is contained in 12; then in 47; then in 13; and, lastly, in 57; so that the division by this high number is equally simple as that by 43.

When, after any figure in the dividend is annexed, the divisor is not contained *once* in the number thus augmented, a cipher must be put in the quotient, and the following figure of the dividend annexed as a new dividual.*

```
Thus 57 ) 5923 (103 quotient.
          57
          --
           223
           171
           ---
            52 remainder.
```

* A *dividual* is the number produced at each annexation of the figure from the dividend.

Here we see that 57 is contained once in 59, and 2 over, but in 22 not any times; we, therefore, put a cipher in the quotient, and then 57 in 223, 3 times, with a remainder of 52; the quotient is, therefore, 103, and 52 over.

To divide by 10 is just the reverse operation of multiplying by 10; consequently, in a division by 10, the units' place is the remainder, and the other places the quotient. When 100 is the divisor, the units and tens are the remainder; and, to divide by 1000, the units, tens, and hundreds, are the remainder, and the other places the quotient.

In fact, to divide by any number whatever, composed of ciphers, with unity on their left, is merely to cut off from the right of the dividend as many places as there are ciphers in the divisor for the remainder, and the other places will be the quotient:—

	Dividends.		Divisors.		Quotients.	Remainders.
Thus	1234567	÷	. . . 10	=	123456, and	. . . 7
	1234567	÷	. . 100	=	12345, and	. . 67
	1234567	÷	. . 1000	=	1234, and	. . 567
	1234567	÷	1000000	=	1, and	234567

So that whatever significant figures we may have on the left of the ciphers in a divisor, we mark off the figures of the dividend, as stated above, and afterwards divide by the significant figures only; so that in dividing 7896473 by 12000, we cut off three figures from the right of the dividend, and divide by 12 only:—

Thus 12) 7896 | 473

658 quotient, and 473 remainder.

Let us now divide 789670000 by 270,000 :

Here there are four ciphers to the right of the significant figures in the divisor; we, therefore, cut off four figures from the right of the dividend, and divide by 27 only :—

Thus 27 { 9) 789670000
3) 8774-1

2924-2 × 9 + 1 = 19

But as this remainder happens to be at the fifth position from the right, it is, therefore, raised to its proper value, by annexing four ciphers; the proper remainder is, therefore, 190000, or, which is the same thing, annexing the figures cut off to the remainder, found as above, will produce the proper remainder.

This can be clearly seen from performing the same operation by Long Division, without any contraction:—

Thus 270000) 789670000 (2924 quotient, as before.
540000

2496700
2430000

667000
540000

1270000
1080000

And 190000 remainder, as before.

Let us next divide 176504 by 9000:—

9) 176 | 504

19 quotient, and 5 over.

To this 5 we now annex 504 (the figures cut off); the true remainder is, therefore, 5504.

But, should there be no remainder after the division, the figures cut off are the true remainder. Thus, divide 723246 by 80000:—

8) 72 | 3246

9 quotient, and nothing over: the true remainder is, therefore, 3246.

CHAPTER X.

LECTURE V.

Sequel to Multiplication and Division, with Proofs of Division, &c.

As the division by 10 has already been so clearly exhibited, that by 5 will, of course, be no difficult matter, by employing multiplication; for any number divided by 10 is that number wanting the units' place as the quotient, and the units' place as the remainder; double of this tenth part will give the fifth part: we have, therefore, but to cut off the units' place, and multiply by 2, for our division by 5:—

Thus 76 | 3÷5
2

152 quotient.

And, as the figure cut off is not 5, nor above 5, we have, therefore, 3 as the remainder in this case.

But when 5, or a number greater than 5, is cut off, since 5 is contained in it once, we add 1 to the product, and what is over, after such deduction, is the true remainder:—

Thus 76 | 7÷5
2
———
152

Here the figure cut off is 7, which is 2 above 5; we, therefore, add 1 to 152, which gives us 153 as the quotient, and 2 as the true remainder.

But this may also be managed as follows:—Multiply the dividend by 2, the units' place will be double the remainder, and the other places the quotient.

Let us take the last example:—

767÷5
2
———
153 | 4=153 quotient, and half of 4=2, the remainder, as before.

The multiplication by 5 may, therefore, be managed by annexing a cipher to the multiplicand, and afterwards dividing by 2, because 5 is the half of 10. Let us, therefore, multiply 767 by 5:—

Then 2) 7670
———
3835 the product.

For 767 × 5 = 3835, as before.

The division by 25 is the hundredth part multiplied by 4; we have, therefore, only to divide by 100, by cutting off two figures, and afterwards to multiply by 4, because 25 is the fourth part of 100, or rather because 100 is 4 times 25:—

Thus 79 | 60 ÷ 25
4

316
60 = 2-10

∴ 318 quotient, and 10 remainder.

Here we have cut off 60, but 60 is a remainder which we cannot possibly have in dividing by 25, for there are 2 twenty-fives in 60, and 10 over; we, therefore, add 2 to the product by 4, which produces 318 as the quotient, and 10 is the remainder.

Again, let us first multiply the given dividend by 4, and afterwards divide by 100, the figures cut off will just give 4 times the remainder, and the other places the quotient:—

Thus 7960 ÷ 25
4

318 | 40 = 318 quotient, and 40 ÷ 4 = 10 remainder, as before.

The multiplication by 25 may be performed by annexing two ciphers, and dividing by 4, since 25 is the fourth of 100: let us, therefore, multiply 327 by 25:—

Then 4) 32700

8175 the product.

For 327 × 25 = 8175, as before.

The division by 75, which is 3 times 25, may be performed by dividing the quotient of 25 by 3:—

Thus 7960 ÷ 75
4

3) 318 | 40 = 318 quotient, and $\frac{40}{4}$ = 10 remainder, as before.

106 quotient, and 10 remainder.

The multiplication by 75 may be performed thus: annex two ciphers, then multiply by 3, and divide by 4, for 300 divided by 4 is equal to 75.

Thus, to multiply 3275 by 75:—

327500
3

4) 982500

245625 the product.

For 3275 × 75 = 245625, as before.

The division by 125, which is 5 times 25, is done by dividing by 5, where 3 is used in the last case:—

Thus 7960 ÷ 125
4

5) 318 | 40 = 318 quotient, and 40 ÷ 4 = 10 rem.

63 required quot. and 3 (25's) over = 75;

And 75 + 10 = 85, the true remainder.

The remainder, in this case, requires some explanation: in the division by 5, we have a remainder of 3; but these 3 are certainly three 25's, for the other divisors, by the usual way of dividing by component parts, must have been 5 and 5, for $5 \times 5 \times 5 = 125$, our divisor; consequently three 25's, or 75, added to our first remainder, 10, gives us 85 as the true remainder, which may be seen by dividing by 125 by Long Division: thus—

```
125) 7960 (63 quotient.
     750
     ---
      460
      375
      ---
       85 remainder, as before.
```

But the same calculation may be performed thus:—the thousandth part of 8 times the dividend will give the quotient; and the eighth part of the figures cut off will be the true remainder: thus—

```
7960 ÷ 125
   8
-----
63 | 680
```

= 63 quotient, and $680 \div 8 = 85$, remainder, as before.

Annex three ciphers, and divide by 8, for the multiplication by 125; for annexing three ciphers is multiplying by 1000, and the eighth part of 1000 is 125 · t s—

```
8) 7960000
   -------
    995000. the product.
```

For $7960 \times 125 = 995000$, as before.

From what has been said on the above contractions, the multiples of 5 taken to any height may be easily managed by the same parity of reasoning.

To multiply and divide, therefore, by 625
3125
15625
and 78125, &c. I shall leave to the ingenuity of my readers.

CHAPTER XI.

LECTURE V. CONTINUED.

To divide by any number of 9's:—

When there is but one 9 in the divisor, begin at the units' place of the dividend, add to it all the digits on its left, carrying as in Addition; then commence at the tens' place, add it and all the digits on its left, then at the hundreds' place, adding it and the digits on its left in the same manner, and so on till in this way you have employed each figure in the dividend, dotting the figure at which you commence each addition, by which means you prevent the omission of an addition, as well as adding any twice:—

Example.

Divide 7426 by 9

$$\begin{array}{r|l} \dot{7}\dot{4}\dot{2} & \dot{6} \\ \hline 824 & 9 \end{array}$$

It is here necessary to observe, that if there be any carriage to the units' place of the quotient, that the same carriage must be added to the units' place of the remainder; and if, after this addition, all the places in the remainder be 9's, the quotient must be reckoned 1 more, with no remainder.

It will be seen from the following additions, (agreeably to the rule) that 1 was carried to the units' place of the quotient :—thus

		Amts.	
	$6+2+4+7$	$=19$	$\therefore 9$ and carry 1
	$1+2+4+7$	$=14$	$\therefore 4$ and carry 1
	$1+4+7$	$=12$	$\therefore 2$ and carry 1
And	$1+7$	$=8$	

The units' place of the several amounts gives us the places of the quotient and remainder; thus, commencing at the bottom of the column, we have 824 as the quotient, and 9 as the remainder. But the proper quotient is 825, with a remainder of 1, for in the above addition it appears that we carried 1 to the units' place of the quotient, namely, from 19; therefore 1 must be added to the remainder, 9, which gives us 10; but 10 is 1 more than our divisor, 1 must, therefore, be added to our former quotient, which gives us 825, with the true remainder, 1.

Let us now divide 6247 by 9 :—

$$\begin{array}{r|l} \dot{6}\dot{2}\dot{4} & \dot{7} \\ \hline 693 & 9 \end{array}$$

The following are the additions :—

Amts.
7 + 2 + 4 + 6 = 19, 9 and carry 1
1 + 4 + 2 + 6 = 13, 3 and carry 1
1 + 2 + 6 = 9
And 6

Here we carried 1 to the units' place of the quotient; 1 must, therefore, be added to the remainder, which gives us 10, as in the last example, for which we must add 1 more to the quotient, which gives us 694 as the true quotient, and 1 as the true remainder.

Another example will be sufficient.

Divide 2137 by 9 :—

$\dot{2}\;\dot{1}\;\dot{3} \mid \dot{7}$

2 3 7 | 3 = 237 quotient, and 4 remainder.

For 7 + 3 + 1 + 2 = 13, 3 and carry 1
1 + 3 + 1 + 2 = 7
1 + 2 = 3
And 2

Here we have 237 as the quotient, and, having carried 1 to the units of the quotient, the true remainder is, therefore, 3 + 1 = 4, which is less than 9; the units of the quotient must not, therefore, be increased.

When there are two 9's in the divisor, begin at the units' place, adding every alternate figure, doing the same at the tens' and hundreds' places, &c.

Let us divide 7632984 by 99:—

$$\begin{array}{l} \dot{7}\,\dot{6}\,\dot{3}\,\dot{2}\,\dot{9}\,|\,\dot{8}\,\dot{4} \\ \hline 7\,7\,1\,0\,0\,|\,8\,3 \end{array} = 77100 \text{ quotient, and 84 remainder.}$$

	Amts.	
For 4+9+3+7=23,	3	and carry 2
2+8+2+6=18,	8	and carry 1
1+9+3+7=20,	0	and carry 2
2+2+6 =10,	0	and carry 1
1+3+7 =11,	1	and carry 1
1+6 = 7		
And 7		

Here we carried 1 to the units' place of the quotient; the true remainder is, therefore, 84; and, as the remainder does not amount to 99, the quotient will be 77100.

In fact, whatever may be the number of 9's of which the divisor is composed, we have but at each addition to omit as many places less one on the left of each digit as there are 9's in the divisor, and then mark off with a perpendicular line as many figures from the units' place of this sum as there are 9's in the divisor.

Let us now divide 674854 by 999:—

$$\begin{array}{l} \dot{6}\,\dot{7}\,\dot{4}\,|\,\dot{8}\,\dot{5}\,\dot{4} \\ \hline 6\,7\,5\,|\,5\,2\,8 \end{array} = 675 \text{ quotient, and 529 remainder.}$$

For $4+4=8$
$5+7=12$, 2 and carry 1
$1+8+6=15$, 5 and carry 1
$1+4 \quad = 5$
7
And 6

Let us next divide 123456789 by 99999 :—

$$\begin{array}{c|c} \dot{1}\,\dot{2}\,\dot{3}\,\dot{4} & \dot{5}\,\dot{6}\,\dot{7}\,\dot{8}\,\dot{9} \\ \hline 1\,2\,3\,4 & 5\,8\,0\,2\,3 \end{array}$$

For $9+4=13$, 3 and carry 1
$1+8+3=12$, 2 and carry 1
$1+7+2=10$, 0 and carry 1
$1+6+1=8$
5
4
3
2
1

The quotient is, therefore, 1234, and 58023 as a remainder.

In like manner we may divide by any multiple of 9's, by first dividing by the number which produces such multiple, and then by the number of 9's; therefore, to divide 79832 by 198, we first divide by 2, then by 99, for $99 \times 2 = 198$:—

Thus 2) 79832÷198

399 | 16

403 | 18 = 403 quotient, and 19 over.

But 19 × 2 = 38, the true remainder.

And to divide by 39996, we first divide by 4, then by 9999, for 9999 × 4 = 39996:—

Thus 4) 79832

1 | 9958

1 | 9959 = 1 quotient, and 9959 over.

But 9959 × 4 = 39836, the true remainder.

Lastly, to divide 123456789 by 5999994, we first divide by 6, then by 999999, for 999999 × 6 = 5999994:—

Thus 6) 123456789÷5999994

20 | 576131 — 3

20 | 576151 = 20 quot. and 576151 over.

But 576151 × 6 = 3456909, true remainder.

(To find the different multiplying figures of any multiples of 9's, see p. 87.)

From what has been advanced on the division by any number of 9's, it is evident that the quotient of the division by any number of 1's, 2's, 3's, 4's, or 5's, &c. may be very easily found. Thus, it is obvious that the quotient by any number of 9's will just be the 9th part of the quotient by the same number of 1's, consequently 9 times the first quotient will give the quotient by such a number of 1's; half of this will be the quotient in a division by 2's; third of this for 3's, &c.

First, let us divide 79654 by 111

$$\begin{array}{l} \dot{7}\dot{9} \mid \dot{6}\dot{5}\dot{4} \\ \hline 79 \mid 733 = 79 \text{ quot. and } 733 \text{ rem. in dividing by } 999 \\ \therefore 9 \\ \hline 711 \\ \quad 6 \mid 67 = 733 \div 111 \\ \hline 717 \quad \text{quot. and } 67 \text{ rem. in dividing by } 111. \end{array}$$

Secondly, let us divide 79654 by 222:—

$$\begin{array}{l} \dot{7}\dot{9} \mid \dot{6}\dot{5}\dot{4} \\ \hline 79 \mid 733 = \text{quotient and remainder by } 999. \\ \quad 9 \\ \hline 711 \\ \quad 6 \mid 67 \\ \hline 2)717 \mid 67 = \text{quotient and remainder by } 111. \\ \hline \end{array}$$

358 quot. by 222, and 1 over; but this remainder is 1 (111) = 111; and this added to the former remainder, 67, gives 178 as the true remainder.

Thirdly, let us divide 79654 by 3333:—

$$\dot{7} \mid \dot{9}\dot{6}\dot{5}\dot{4}$$

7 | 9661 quotient and remainder by 9999.
9
—
63
8 | 773 = (9661 ÷ 1111)

3) 71 | 773 quotient and rem. in dividing by 1111.
—
23 quotient by 3333, and 2 over; but this remainder of 2 = 2 (1111's) = 2222, which, added to the former remainder, 773, gives us 2995 as the true remainder.

But the above quotient and remainder would have been found by multiplying the division by 9999 by 3, since 9999 ÷ 3 = 3333, our divisor:—

Thus 7 | 9661 = quot. and rem. of 79654 ÷ by 9999.
3
—
21
2 | 2995
—
23 quotient, as before, and 2995 remainder.

Here we have a remainder of 9661; but our divisor (3333) is contained in this *twice*, and 2995 over: this we have, therefore, added to the product by 3.

The division by any number of 6's may either be performed by the first rule, or—as the above sum is done—by the additional division by 2, because the division by any number of 6's is just half of the division by the same number of 3's :—

Thus 7 | 9661 quotient and remainder by 9999.
3
—
21
2 | 2995
2) 23 | 2995 = quotient and remainder by 3333.
—
11 quotient by 6666, and 1 over; but this remainder is 1 (3333), and, added to 2995, gives us 6328 as the true remainder.

As dividing by any number of 5's, 7's, or 8's, is managed precisely as directed in the rule, and in the same manner as the first three examples which follow the rule, it is unnecessary to give these divisions here.

When the multiplier (in finding the product of two numbers) is a multiple of any number of 9's, we multiply first by the number which produces this multiple, and afterwards find the product by as many 9's less one than there are places in the given multiplier. Thus, suppose it were required to multiply 765 by 29997 :—

As $9999 \times 3 = 29997$, our multiplier.
∴ 765
3
—
Then 22950000 (see p. 45.)
2295
—
22947705 = product by 29997.

When the multiple is produced by any of the nine digits, the multiplying digit may be very easily known, for the highest and lowest places of the multiple are the tens' and units' places of this digit multiplied by 9, therefore the 9th part of this product will produce our multiplying digit, and the number of 9's will just be 1 less than the number of places in the given multiplier, or as many as there are units in the multiplying digit, which may be observed from the following table :—

198	here	18÷9=2,	and	99×2=198
2997	—	27÷9=3,	—	999×3=2997
39996	—	36÷9=4,	—	9999×4=39996
499995	—	45÷9=5,	—	99999×5=499995
5999994	—	54÷9=6,	—	999999×6=5999994
69999993	—	63÷9=7,	—	9999999×7=69999993
799999992	—	72÷9=8,	—	99999999×8=799999992
8999999991	—	81÷9=9,	—	999999999×9=8999999991

Let us now multiply 765 by this last number, which is 8999999991 : its multiplying digit we know is 9 :—

∴ 7650
765
———
6885000000000
6885
—————
6884999993115 the product.

CHAPTER XII.

LECTURE V. CONTINUED.

We shall now proceed to the multiplication and division by mixed numbers, which are numbers composed of integers or whole numbers, and fractions.

1. When the multiplier is a mixed number, we first multiply by the integer, and to this we add the result produced by multiplying the multiplicand by the upper figure or numerator of the fraction divided by the lower figure or denominator.

Thus, to multiply 724 by $2\frac{3}{8}$:—

$$\begin{array}{r} 724 \\ 2 \\ \hline 1448 = 2 \\ 271\frac{4}{8} = \frac{3}{8} \\ \hline \end{array} \qquad \begin{array}{r} 724 \\ 3 \\ \hline 8\,)\,2172 \\ \hline 271\frac{4}{8} = \frac{3}{8} \end{array}$$

$\therefore$ $1719\frac{4}{8}$ the product by $2\frac{3}{8}$.

Or it may be done in this way :—multiply the integral part of the multiplier by the lower figure of the fraction; adding the upper; then multiply the multiplicand by this, and divide the result by the under figure of the fraction; this will give the desired product.

Let us take the above example :—

First $2\times 8+3=19$

Then 724
19

8) 13756

$1719\frac{4}{8}$ the product, as before.

It is evident, that since we have here multiplied our multiplier by 8, that the result will give us just 8 times the product which we want; the 8th part of this will, therefore, be the required product.

Let us now multiply 724 by $5\frac{7}{9}$:—

724
5

$3620=5$
$563\frac{1}{9}=\frac{7}{9}$

$4183\frac{1}{9}$ the product by $5\frac{7}{9}$.

724
7

9) 5068

$563\frac{1}{9}=\frac{7}{9}$.

Or,

First $5\times 9+7=52$

Then 724
52

1448
3620

9) 37648

$4183\frac{1}{9}$ as before.

And in Division, when the divisor or dividend contains a fraction, it may be removed by multiplying both by the denominator.

Thus, to divide 327 by $2\frac{1}{4}$:—

$$\begin{array}{rr} 2\frac{1}{4} & 327 \\ 4 & 4 \\ \hline 9)\ & 1308 \\ \hline & 145\frac{3}{9} \end{array}$$

the required quotient.

And, to divide $564\frac{3}{7}$ by 10 :—

$$\begin{array}{rr} 10 & 564\frac{3}{7} \\ 7 & 7 \\ \hline 70)\ & 395 \mid 1 \\ \hline & 56\frac{31}{70} \end{array}$$

the required quotient.

Had we, in the first example, only multiplied our divisor by 4, the result would have been but the 4th part of the required quotient; and had we only multiplied the dividend by 4, the result would have been 4 times the required quotient; but since we have multiplied both by the same number, the division by these new numbers must, of course, give the proper quotient. For instance, $8 \div 2 = 4$, and 4 times 8, or 32, divided by 4 times 2, or 8, is 4 also. Or, suppose that 2 persons each build 8 yards of a wall in 4 days, it is evident that 4 times these yards can be built by 4 times the number of persons in precisely the same time. It is clear, then, that when the divisor and dividend are increased in the same proportion, that the result must be the same as dividing the original dividend by the original divisor.

2. When both factors are mixed numbers, we multiply the integral part of each factor by the denominator of its fraction, adding the numerator; then the product of these new factors, divided by both denominators, or by the number of which they are the component parts, will give the product required.

Thus, to multiply $72\frac{1}{2}$ by $39\frac{3}{4}$:—

First $72 \times 2 + 1 = 145$, and $39 \times 4 + 3 = 159$

```
Then      145
          159
         ----
         1305
        2175
       ------
    2) 23055
       ------
    4) 11527 1
       ------
        2881 3×2+1=7=7/8.
```

The product of $72\frac{1}{2} \times 39\frac{3}{4}$ is, therefore, $2881\frac{7}{8}$; but, as 2 and 4 are the component parts of 8, the division of 23055 by 8 would evidently have produced the same result.

Let us now multiply $374\frac{1}{6}$ by $567\frac{4}{5}$:—

First $374 \times 6 + 1 = 2245$, and $567 \times 5 + 4 = 2839$.

```
        Then 2839
             2245
             ----
            14195
           11356
           5678
          5678
          ---------
Now6×5=3 | 0)637355 | 5
          ---------
          212451 25/30 the required product.
```

In dividing one mixed number by another, we first prepare them in the same way as we did in Multiplication, and afterwards multiply the new dividend by the denominator in the divisor, and the new divisor by the denominator in the dividend, and afterwards divide by these last results, in the usual way.

Let us divide $7563\frac{4}{5}$ by $54\frac{3}{7}$:—

```
First 7653×5+4=37819, and 54×7+3=381
            Then      7          Then  5
                 ------                ----
                 264733                1905
   Now  1905) 264733 ( 138 1843/1905 the quotient.
              1905
              ----
               7423
               5715
               -----
               17083
               15240
               -----
                1843 remainder.
```

It is obvious that the divisor is multiplied by 7 and 5, and the dividend by 5 and 7 (the component parts of 35); they are, therefore, increased in the same proportion,—the method of dividing is, consequently, correct.

One way of proving Division is to multiply the quotient by the number which we employed as a divisor, the result will be the dividend, if the quotient has been correctly ascertained; for in whatever proportion any number has been *decreased*, this has but to be *increased* in the same proportion to bring it back to what it was originally.

Thus 2) 79624
39812 the quotient.
2
79624 the proof.

Therefore, to prove Multiplication, we have but to *decrease* the product in the same proportion in which we *increased* the multiplicand :—

Thus 7564
2
2) 15128 the product.
7564 the proof.

Or if we subtract the multiplicand as many times as there are units in the multiplier, there will be no remainder, if the product be correct:—

```
Thus   1574
          4
       ----
       6296 the product.
       1574 (1)
       ----
       4722
       1574 (2)
       ----
       3148
       1574 (3)
       ----
       1574
       1574 (4)
       ----
       . . . the proof.
```

Two other methods of proving Division will close this part of the subject:—

1st. If we subtract the remainder from the dividend, and divide what is left by the quotient, the result will be equal to the former divisor:—

```
Thus   7) 5479
          ----
           782  5 quotient and remainder.

Now from 5479
 Subtract    5
          ----
    782) 5474 (7 the former divisor.
         5474
```

2dly. If we add together the remainder, and all the products of the several quotient figures, according to the order in which they stand in the work, the sum will be equal to the dividend :—

```
Thus    43 ) 6798 ( 158 quotient.
             43
             —
             249
             215
              —
              348
              344
               —
                4 remainder.

     Then   43
            215
             344
               4 remainder.
            ————
            6798 proof.
```

The Four fundamental Rules of the science of Arithmetic have now, it is presumed, been thoroughly investigated; and the Author now concludes by assuring his readers that a very little attention on their part is all that is necessary to acquire a solid knowledge of this very interesting and useful branch of education.

END OF THE ANALYSIS OF NOTATION, NUMERATION, AND THE FOUR FUNDAMENTAL RULES OF ARITHMETIC.

APPENDIX.

DOMESTIC CALCULATIONS, PERFORMED MENTALLY.

1. When the quantity is 12 :—

Reckon a shilling for every penny in the price: therefore, sixpence for a halfpenny, threepence for a farthing, and ninepence for three-farthings.

Examples.

1st. 12 yards at 1d. each = 1s. answer.
2d. 12 yards at $1\frac{1}{2}$d. each = 1s. 6d.
3d. 12 yards at $2\frac{3}{4}$d. each = 2s. 9d.
4th. 12 yards at 1s. 1d. each = 13s.
5th. 12 yards at 3s. 4d. each = 40s.
6th. 12 yards at 4s. $7\frac{3}{4}$d. each = 55s. 9d.

Reflections.

It is sometimes easier to make the quantity the price, and the price the quantity: thus—

1st. 12 yards, at 1d. or 1 yard, at 12d.=1s. ans.
2d. 12 yards, at $1\frac{1}{2}$d. or $1\frac{1}{2}$ yard, at 12d.=1s.6d.

Otherwise:—

3d. 12 yards at $2\frac{3}{4}$d. is 12 at 2d. = 2s. 0d.
12 at $\frac{1}{2}$d. = 6d.
And 12 at $\frac{1}{4}$d. = 3d.

∴ 12 yards at $2\frac{3}{4}$d. each = 2s. 9d.

2. When the quantity is any *multiple* of 12, or, which is the same thing, any exact number of twelves:—

Find the value of 12, as directed in the first case, and this multiplied by the number of twelves will be the required value.

Examples.

1st. 24 lbs. at 1d. = 1s. × 2 = 2s. ans.
2d. 36 lbs. at $1\frac{1}{2}$d. = 1s. 6d. × 3 = 4s. 6d.
3d. 48 lbs. at $2\frac{1}{4}$d. = 2s. 3d. × 4 = 9s.
4th. 60 lbs. at $3\frac{3}{4}$d. = 3s. 9d. × 5 = 18s. 9d.
5th. 72 lbs. at 1s. $1\frac{1}{2}$d. = 13s. 6d. × 6 = 81s.
6th. 84 lbs. at 2s. 2d. = 26s. × 7 = 182s.

Reflections.

1st. 24 lbs. at 1d. or 1 lb. at 2s. = 2s. ans.
2d. 36 lbs. at $1\frac{1}{2}$d. or $1\frac{1}{2}$ lb. at 3s. = 3s. + 1s. 6d. = 4s. 6d.
3d. 48 lbs. at $2\frac{1}{4}$d. or $2\frac{1}{4}$ lb. at 4s. = 8s. + 1s. = 9s.

3d. When the quantity is 20:—

The shillings in the price is the answer in pounds; therefore, if there be sixpence in the price, reckon 10s. more; if threepence, 5s.; if fourpence, 6s. 8d.; if two-pence, 3s. 4d.; and if one penny, 1s. 8d.: Or, *one-twelfth* of the pence will be the answer in pounds:—

Examples.

1st. 20 yards at 2s. = £2. ans.
2d. 20 yards at 3s. 6d. = £3. 10s.
3d. 20 yards at 4s. 3d. = £4. 5s.

Or,

4th. 20 yards at 5s. 4d. = 64 ÷ 12 = £5. 6s. 8d.
5th. 20 yards at 6s. 2d. = 74 ÷ 12 = £6. 3s. 4d.
6th. 20 yards at 7s. 5d. = 89 ÷ 12 = £7. 8s. 4d.

Reflections.

1st. 20 yards at 2s. or 2 yards at 20s. = £2. ans.
2d. 20 yards at 3s. 6d. or $3\frac{1}{2}$ yards at 20s. = £3. 10s.
3d. 20 yards at 4s. 3d. or $4\frac{1}{4}$ yards at 20s. = £4. 5s.

Note. The interest of any sum at 5 per cent. being given to find the *principal*, may be easily ascertained by this rule.

Example 1. What principal, lent out to interest at 5 per cent., would produce yearly the sum of £1?

£1 = 20s. ∴ £20. ans.

Example 2. What principal will produce £7. 10s. per annum?

£7. 10s. = 150s. ∴ £150. ans.

4. When the quantity is any multiple* of 20:—

Find the value of 20, as in case 3d.; this, multiplied by the number of twenties, will give the answer.

Examples.

1st.	40 gals. at 1s.	= £1	0	0 × 2	=	£2	0	0
2d.	60 gals. at 2s. 6d.	= 2	10	0 × 3	=	7	10	0
3d.	80 gals. at 3s. 3d.	= 3	5	0 × 4	=	13	0	0
4th.	100 gals. at 4s. $5\frac{1}{4}$d.†	= 4	8	9 × 5	=	22	3	9
5th.	180 gals. at 19s. 1d.	= 19	1	8 × 9	=	171	15	0
6th.	200 gals. at 1s. $2\frac{1}{2}$d.	= 1	4	2 × 10	=	12	1	8

Reflections.

1st.	40 gals. at 1s.	or 1 gal. at £2	=	£2	0	0
2d.	60 gals. at 2s. 6d.	or $2\frac{1}{2}$ gals. at 3	=	7	10	0
3d.	80 gals. at 3s. 3d.	or $3\frac{1}{4}$ gals. at 4	=	13	0	0

Or, 6th. 200 gals. at 1s. $2\frac{1}{2}$d. each is—

200 shillings	=	£10	0	0
200 twopences (400d.)	=	1	13	4
200 halfpence (100d.)	=	0	8	4
		£12	1	8 answer.

* The *multiple* of any number is that number repeated any integral number of times.

† 20 at 4s.	=	£4	0	0
20 at 5d.	=	0	8	4
20 at $\frac{1}{4}$d.	=	0	0	5
∴ 20 at 4s. $5\frac{1}{4}$d.	=	4	8	9

5. When the quantity is 24 :—

Reckon a pound for every 10d. in the price; therefore, 10s. for 5d.; 5s. for $2\frac{1}{2}$d,; 2s. 6d. for $1\frac{1}{4}$d.; 6d. for $\frac{1}{4}$; and, consequently, 2s. for every penny. Or, *twice* the pence is the answer in shillings: Or, *one-tenth* of the pence, Or *one-fortieth* of the farthings, is the answer in pounds.

Examples.

1st. 24 yds. at	10d.	=	£1	0	0
Or, 24 yds. at	10d.	= 10d. × 2= 20s.=	1	0	0
Or, 24 yds. at	10d.	= 10d.÷10=	1	0	0
Or, 24 yds. at	10d.	= 40d.÷40=	1	0	0
2d. 24 yds. at	$7\frac{1}{2}$d.	= $7\frac{1}{2}$s. × 2= 15s.=	0	15	0
3d. 24 yds. at	1s. 8d.	= 20 × 2= 40s.=	2	0	0
4th. 24 yds. at	3s.$9\frac{1}{2}$d.	= $45\frac{1}{2}$ × 2= 91s.=	4	11	0
5th. 24 yds. at	7s.$4\frac{3}{4}$d.	= $88\frac{3}{4}$ × 2=$177\frac{1}{2}$ =	8	17	6
6th. 24 yds. at	1s.$1\frac{1}{4}$d.	= $13\frac{1}{4}$ × 2= $26\frac{1}{2}$ =	1	6	6

Reflections.

1st. 24 yards at 10d.	or 10 yards at 2s.=	£1	0	0
2d. 24 yards at $7\frac{1}{2}$d.	or $7\frac{1}{2}$ yards at 2s.=	0	15	0
3d. 24 yards at 1s. 8d.	or 20 yards at 2s.=	2	0	0
	Or, 20 ÷ 10 =	2	0	0

6. When the quantity is 30:—

Reckon a pound for every 8d. in the price; therefore 10s. for 4d.; 5s. for 2d.; 2s. 6d. for 1d.; 1s. 3d.

for $\frac{1}{2}$; $7\frac{1}{2}$d. for $\frac{1}{4}$; and 1s. $10\frac{1}{2}$d. for $\frac{3}{4}$. Or, *one-eighth* of the pence is the answer in pounds.

Examples.

1st. 30 pen-knives at $8\frac{3}{4}$d. each = £1 1 $10\frac{1}{2}$, ans.
Or, £8 15 0 ÷ 8 = £1 1 $10\frac{1}{2}$.

2d. 30 scissors at 1s. 8d. each = £2 10 0.
Or, £20 ÷ 8 = £2 10 0.

3d. 30 hair-combs at 2s. $7\frac{1}{2}$ each = £3 18 9.
Or, £31 10 ÷ 8 = £3 18 9.

4th. 30 oranges at $1\frac{3}{4}$d. each = £0 4 $4\frac{1}{2}$.
Or, £1 15 ÷ 8 = £0 4 $4\frac{1}{2}$.

5th. 30 books at 3s. 9d. = £45÷8=£5 12 6.

6th. 30 thimbles at 2s. 4d.=£28÷8=£3 10 0.

Reflections.

1st.	30 at 8d. =	240d. =	£1	0	0	
	30 at $\frac{1}{2}$d. =	15d. =	0	1	3	
	30 at $\frac{1}{4}$d. =		0	0	$7\frac{1}{2}$	
	∴ 30 at $8\frac{3}{4}$d. =		1	1	$10\frac{1}{2}$	ans.
2d.	30 at 1s. =	30s. =	£1	10	0	
	30 at 8d. =	240d. =	1	0	0	
	∴ 30 at 1s. 8d. =		2	10	0	ans.

7. When the quantity is 40:—

Reckon a pound for every 6d. in the price; therefore, 10s. for 3d.; 5s. for $1\frac{1}{2}$d.; 2s. 6d. for $\frac{3}{4}$d.; 10d.

for $\frac{1}{4}$d.; and 1s. 8d. for $\frac{1}{2}$d.: Or, *one-sixth* of the pence is the answer in pounds.

Examples.

1st. 40 yards at 1s. 6d. each = £3 0 0, ans.
Or, £18 ÷ 6 = £3.

2d. 40 yards at 2s. 9d. each = 5 10 0
Or, £33 ÷ 6 = £5 10

3d. 40 yards at 7s. 6$\frac{1}{2}$d. each = 15 1 8

4th. 40 yards at 4s. 3$\frac{3}{4}$d. each = 8 12 6

5th. 40 yards at 1s. 1$\frac{1}{4}$d. each = 2 4 2

6th. 40 yards at 5$\frac{1}{2}$d. each = 0 18 4

Note. The above examples may also be managed as in Case 3.

Reflections.

1st. 40 at 1s. 6d. or 1$\frac{1}{2}$d. at 40s. = £3 0 0 ans.

2d. 40 at 2s. 9d. or 2$\frac{3}{4}$d. at 40s. = 5 10 0

Otherwise:—

1st.	40 yards at 1s.	= £2 0 0	ans.
	40 yards at 6d.	= 1 0 0	
∴	40 yards at 1s. 6d.	= 3 0 0	ans.
2d.	40 yards at 2s.	= £4 0 0	
	40 yards at 6d.	= 1 0 0	
	40 yards at 3d.	= 0 10 0	
∴	40 yards at 2s. 9d.	= 5 10 0	ans.

8. When the quantity is 48:—

Reckon a pound for every 5d. in the price; therefore, 10*s.* for 2*d*$\frac{1}{2}$.; 5*s.* for 1$\frac{1}{4}$*d.*; 1*s.* for $\frac{1}{4}$*d.*, &c.: Or, the farthings in the price divided by 20, Or the pence divided by 5, will be the answer in pounds.

Examples.

1st. 48 lbs. at 5$\frac{1}{2}$ = 1*l.* 2*s.* ans.

Or, 22*l.* ÷ 20 = 1*l.* 2*s.*

Or, 5*l.* 10*s.* ÷ 5 = 1*l.* 2*s.*

2d. 48 lbs. at 6*s.* 8$\frac{3}{4}$*d.* = 16*l.* 3*s.*

Or, 323*l.* ÷ 20 = 16*l.* 3*s.*

3d. 48 lbs. at 4*s.* 7$\frac{1}{4}$*d.* = 11*l.* 1*s.*

4th. 48 lbs. at 2*s.* 9*d.* = 33*l.* ÷ 5 = 6*l.* 12*s.*

5th. 48 lbs. at 5*s.*4$\frac{1}{2}$*d.* = 64*l.* 10*s.* ÷ 5 = 12*l.* 18*s.*

6th. 48 lbs. at 10*s.* 6*d.* = 126*l.* ÷ 5 = 25*l.* 4*s.*

Reflections.

1st. 48 at 5$\frac{1}{2}$*d.* or 5$\frac{1}{2}$ at 4*s.* = 1*l.* 2*s.* ans.

4th. 48 at 2*s.* 9*d.* or 2$\frac{3}{4}$ at 48*s.* = 6*l.* 12*s.*

6th. 48 at 10*s.* 6*d.* or 10$\frac{1}{2}$ at 48*s.* = 25*l.* 4*s.*

Or, 48 at 10*s.* = 24*l.*

48 at 6*d.* = 1*l.* 4*s.*

∴ 48 at 10*s.* 6*d.* = 25*l.* 4*s.* ans.

9. When the quantity is 60:—

Reckon a pound for every 4*d.* in the price; therefore, 10*s.* for 2*d.*; 5*s.* for 1*d.*: 2*s.* 6*d.* for $\frac{1}{2}$*d.*; 1*s.* 3*d.* for

¼*d.*: Or, *one-fourth* of the pence will be the answer in pounds.

Examples.

1st. 60 yards at 6*d.* = 1*l.* 10*s.* ans.
Or, 6*l.* ÷ 4 = 1*l.* 10*s.*

2d. 60 yards at 7¾*d.* = 1*l.* 18*s.* 9*d.*
Or, 7*l.* 15*s.* ÷ 4 = 1*l.* 18*s.* 9*d.*

3d. 60 yards at 1*s.* 1¼*d.* = 13*l.* 5*s.*÷4=3*l.* 6*s.* 3*d.*

4th. 60 yards at 2*s.* 3½*d.*=27*l.* 10*s.*÷4 = 6*l.* 17*s.* 6*d.*

5th. 60 yards at 5*s.* 4¾*d.* = 64*l.* 15*s.*÷ 4 =16*l.*3*s.*9*d.*

6th. 60 yards at 4*s.* 9*d.* = 57*l.* ÷ 4 = 14*l.* 5*s.*

Reflections.

1st. 60 at 6*d.* or 6 at 5*s.* = 1*l.* 10s. ans.
Or, 60 sixpences = 30*s.* = 1*l.* 10*s.*

2d. 60 at 7¾*d.* or 7¾ at 5*s.* = 1*l.* 18*s.* 9*d.*

Or, 60 sixpences	=	1*l.* 10*s.* 0*d.*
60 pence	=	5*s.* 0*d.*
60 halfpence	=	2*s.* 6*d.*
60 farthings	=	1*s.* 3*d.*
∴ 60 at 7¾*d.*		1*l.* 18*s.* 9*d.*

10. When the quantity is 80 :—

Reckon a pound for every 3*d.* in the price; therefore, 10*s.* for 1½*d.*; 5*s.* for ¾*d.*; 1*s.* 8*d.* for ¼*d.*; and 3*s.* 4*d.* for ½*d.*: Or, the pence divided by 3, Or the farthings by 12, will be the answer in pounds.

Examples.

1st. 80 lb. at 3*d*. = 1*l*. ans.

Or, 3*l*. ÷ 3 = 1*l*.

Or, 12*l*. ÷ 12 = 1*l*.

2d. 80 lb. at $7\frac{1}{2}$d. = 2*l*. 10*s*.

Or, 7*l*. 10*s*. ÷ 3, or 30*l*. ÷ 12 = 2*l*. 10*s*.

3d. 80 lb. at $9\frac{3}{4}$*d*. = 9*l*. 15*s*. ÷ 3 = 3*l*. 5*s*.

4th. 80 lb. at 1*s*. $1\frac{1}{4}$*d*. = 13*l*. 5*s*. ÷ 3 = 4*l*. 8*s*. 4*d*.

5th. 80 lb. at 2*s*. $2\frac{1}{2}$*d*. = 26*l*. 10*s*. ÷ 3 = 8*l*. 16*s*. 8*d*.

6th. 80 lb. at 3*s*. $3\frac{3}{4}$*d*. = 39*l*. 15*s*. ÷ 3 = 13*l*. 5*s*.

Reflections.

1st. 80 lbs. at 3*d*. or 3 lbs. at 6*s*. 8*d*. = 1*l*. ans.

Or, 80 threepences = 40 sixpences = 20*s*. = 1*l*.

2d. 80 lbs. at $7\frac{1}{2}$*d*. or $7\frac{1}{2}$ lbs. at 6*s*. 8*d*. = 2*l*. 10*s*.

Or, 80 sixpences = 2*l*. 0*s*. 0*d*.
80 pence = 6*s*. 8*d*.
80 halfpence = 3*s*. 4*d*.

∴ 80 lbs. at $7\frac{1}{2}$*d*. = 2*l*. 10*s*. 0*d*.

11. When the quantity is 96:—
One-tenth of the farthings is the answer in pounds.

Examples.

1st. 96 lbs. at $2\frac{1}{2}$*d*. = 10 ÷ 10 = 1*l*.

2d. 96 lbs. at $3\frac{3}{4}$*d*. = 15 ÷ 10 = 1*l*. 10*s*.

3d. 96 lbs. at 1s. $8\frac{1}{4}d.$ = 81 ÷ 10 = 8l. 2s.
4th. 96 lbs. at 2s. $6\frac{1}{2}d.$ = 122 ÷ 10 = 12l. 4s.
5th. 96 lbs. at 3s. 4d. = 160 ÷ 10 = 16l.
6th. 96 lbs. at $11\frac{3}{4}d.$ = 47 ÷ 10 = 4l. 14s.

Reflections.

As 96 pence are 8s.—
∴ 96 articles at 2*d.* each is the same as 2 articles at 8*s.* each.

12. To find the value of 100:—
Multiply the pence in the price by 5, and divide by 12, for the answer in pounds.

Examples.

1st. 100 yards at 5d. = 25 ÷ 12 = 2l. 1s. 8d. ans.
2d. 100 yards at 7d. = 35 ÷ 12 = 2l. 18s. 4d.
3d. 100 yards at 1s. = 60 ÷ 12 = 5l.
4th. 100 yards at 1s. 9d = 105 ÷ 12 = 8l. 15s.
5th. 100 yards at 2s. 6d. = 150 ÷ 12 = 12l. 10s.
6th. 100 yards at 3s. 4d. = 200 ÷ 12 = 16l. 13s. 4d.

13. To find the value of 1 cwt. at a given rate per lb.

To *twice* the number of farthings add *one-third* of the farthings: the amount will be the answer in shillings: Or, *one-third* of the farthings multiplied by 7 will give the same.

Example.

1 cwt. of sugar at $9\frac{3}{4}d.$ per lb.

Twice 39 = 78 shillings.
And $\frac{1}{3}$ of 39 = 13

91=£4 11*s.* ans.

Or, 39÷3=13, and 13×7=91, ans.

$$\text{Or, } \frac{39 \times 7}{3} = 91, \text{ ans.}$$

15. When the quantity is 120:—

Reckon a pound for every 2*d.* in the price; therefore 10*s.* for every 1*d.*; 5*s.* for $\frac{1}{2}d.$; and 2*s.* 6*d.* for $\frac{1}{4}d.$: Or, the pence divided by 2, Or the farthings by 8, will be the answer in pounds.

Examples.

1st. 120 yards at $5\frac{1}{2}d.$ = 2*l.* 15*s.* ans.
Or, 5*l.* 10*s.* ÷ 2 = 2*l.* 15*s.*
Or, 22*l.* ÷ 8 = 2*l.* 15*s.*

2d. 120 yards at $7\frac{3}{4}d.$ = 3*l.* 17*s.* 6*d.*

Or, 7*l.* 15*s.* ÷ 2, or 31*l.* ÷ 8 = 3*l.* 17*s.* 6*d.*

3d. 120 yards at 1*s.* $1\frac{1}{4}d.$ = 13*l.* 5*s.* ÷ 2 = 6*l.* 12*s.* 6*d.*

4th. 120 yards at 2*s.* $3\frac{1}{2}d.$ = 27*l.* 10*s.* ÷ 2 = 13*l.* 15*s.*

5th. 120 yards at 3*s.* $9\frac{3}{4}d.$ = 45*l.* 15*s.* ÷ 2 = 22*l.* 17*s.* 6*d.*

6th. 120 yards at 1*s.* 5*d.* = 17*l.* ÷ 2 = 8*l.* 10*s.*

Reflections.

1st. 120 yards at $5\frac{1}{2}d.$ or $5\frac{1}{2}$ yards at $10s. = 2l.\ 15s.$

2d. 120 yards at $7\frac{3}{4}d.$ or $7\frac{3}{4}$ yards at $10s. = 3l.17s.6d.$

Or,	120 sixpences	= 60s. =	3l. 0s. 0d.
	120 pence	=	10s. 0d.
	120 halfpence	= 60d. =	5s. 0d.
	120 farthings	= 30d. =	2s. 6d.
∴	120 yards at $7\frac{3}{4}d.$	=	3l. 17s. 6d.

15. When the quantity is 240:—

Reckon a pound for every penny in the price; therefore, 10*s.* for a halfpenny; 5*s.* for a farthing: Or, *one-fourth* of the farthings will be the pounds.

Examples.

1st. 240 lbs. at 1*d.* = 1*l.* ans.

2d. 240 lbs. at $2\frac{1}{4}d. = 2l.\ 5s.$

3d. 240 lbs. at $3\frac{3}{4}d. = 3l.\ 15s.$

4th. 240 lbs. at 4*d.* = 4*l.*

5th. 240 lbs. at $4\frac{1}{2}d. = 4l.\ 10s.$

6th. 240 lbs. at $11\frac{3}{4}d. = 11l.\ 15s.$

7th. 240 lbs. at $1s.8\frac{1}{2}d. = 20l.\ 10s.$

8th. 240 lbs. at 2*s.* 4*d.* = 28*l.*

Reflections.

As there are 240 pence in a pound—

∴ 240 articles at 2*d.* each is the same as 2 articles at 1*l.* each. And

240 at $5\frac{1}{2}d. = 5\frac{1}{2}$ at $1l. = 5l.\ 10s.$

16. To find the value of 313 articles, or, which is the same thing, the yearly outlay, reckoning the working-days only.

For every penny in the rate add together as many sovereigns, crowns, shillings, and pence:—

Example.

What should I give my gardner in the course of a year, at 3*s*. 4*d*. per day, Sundays excepted?

For	40*d*. say	£40	0	0
	40 crowns	10	0	0
	40 shillings	2	0	0
	40 pence		3	4
		52	3	4 ans.

17. When the quantity is 360—

One-half of the pence added to themselves will be the answer in pounds.

Examples.

1st.	360 yards at	2*d*.	= £3	0	0, ans.
2d.	360 yards at	3*d*.	= 4	10	0
3d.	360 yards at	$4\frac{1}{2}$*d*.	= 6	15	0
4th.	360 yards at	7*d*.	= 10	10	0
5th.	360 yards at	$8\frac{1}{4}$*d*.	= 12	7	6
6th.	360 yards at	2*s*.9*d*.	= 49	10	0

18. To find the value of 365 articles, or, which is the same thing, to find the yearly expenditure at a given rate per day.

For every penny in the rate add together as many sovereigns, half-sovereigns, groats, and pence.

Example.

If I give my char-woman 1*s.* 4*d.* per day, how much is that per year?

For	16*d.* say	£16	0	0
	16 half sov.	8	0	0
	16 groats		5	4
	16 pence		1	4
		24	6	8, ans.

19. When the quantity is 480—

Reckon 2*l.* for every penny in the price, Or 1*l.* for every halfpenny; Or, *one half* of the farthings will be the pounds—

Examples.

1st.	480 lbs. at 2*d.*	=£4	0	0, ans.
2d.	480 lbs. at 3½*d.*	= 7	0	0
3d.	480 lbs. at 7¼*d.*	= 14	10	0
4th.	480 lbs. at 1*s.* 6*d.*	= 36	0	0
5th.	480 lbs. at 1*s.* 7½*d.*	= 39	0	0
6th.	480 lbs. at 2*s.* 6*d.*	= 60	0	0

20. When the quantity is 960—

Reckon a pound for every farthing in the price; or 4 times the pence will be the answer in pounds.

Examples.

1st.	960 yards at 1*d*.	= £4	0	0, ans.
2d.	960 yards at 1½*d*.	= 6	0	0
3d.	960 yards at 2¾*d*.	= 11	0	0
4th.	960 yards at 7¼*d*.	= 29	0	0
5th.	960 yards at 1*s*. 8*d*.	= 80	0	0
6th.	960 yards at 2*s*. 4¼*d*.	= 113	0	0

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J. and C. Adlard, Printers, Bartholomew Close.